Decompositions of complete graphs into bipartite 2-regular subgraphs

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Abstract

It is shown that if G is any bipartite 2-regular graph of order at most $\frac{n}{2}$ or at least $n-2$, then the obvious necessary conditions are sufficient for the existence of a decomposition of the complete graph of order n into a perfect matching and edge-disjoint copies of G.

1 Introduction

A *decomposition* of a graph K is a set $\{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_t) = E(K)$ and $E(G_i) \cap E(G_j) = \varnothing$ for $1 \leqslant i < j \leqslant t$. If G is a fixed graph and $\mathcal{D} = \{G_1, G_2, \ldots, G_t\}$ is a decomposition such that G_i is isomorphic

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to G for $i = 1, 2, \ldots, t$, then $\mathcal{D}$ is called a *G-decomposition*. See [9] for a survey on G-decompositions, and see [5, 21] for general asymptotic existence results.

This paper concerns G-decompositions of complete graphs in the case where G is a 2-regular graph. See [1] for a survey of results on G-decompositions of complete graphs. The complete graph of order n is denoted by K_n, the cycle of order n is denoted by C_n, and the path of order n is denoted by P_n (so P_n has $n-1$ edges). If G is 2-regular and n is even, then there is no G-decomposition of K_n, and it is common to instead consider decompositions of $K_n - I$, where $K_n - I$ denotes the graph obtained from K_n by deleting the edges of a perfect matching. For each positive integer n, define K_n^* to be K_n if n is odd and $K_n - I$ if n is even. The number of edges in K_n^* is given by $n\lfloor\frac{n-1}{2}\rfloor$.

If G is a 2-regular graph of order k and there exists a G-decomposition of K_n^* ($n \geqslant 3$), then it is obvious that

$$3 \leqslant k \leqslant n \quad \text{and} \quad k \text{ divides } n\lfloor\tfrac{n-1}{2}\rfloor. \tag{1}$$

If G is a 2-regular graph of order k, then the conditions given in (1) are called *the obvious necessary conditions* for the existence of a G-decomposition of K_n^*. The following problem presents itself.

Problem 1: *For each 2-regular graph G and each positive integer n satisfying the obvious necessary conditions, determine whether there exists a G-decomposition of K_n^*.*

It is known that if G is a cycle, then the obvious necessary conditions are sufficient for the existence of a G-decomposition of K_n^* [4, 18]. However, when G is 2-regular but is not a cycle, there are cases where the obvious necessary condition are satisfied but no G-decomposition of K_n^* exists. There is no G-decomposition of K_n^* in each of the following cases (see [9] and [10]).

$$G = C_3 \cup C_3 \text{ and } n = 6, \qquad G = C_3 \cup C_3 \text{ and } n = 9, \qquad G = C_4 \cup C_5 \text{ and } n = 9,$$

$$G = C_3 \cup C_3 \cup C_5 \text{ and } n = 11, \qquad G = C_3 \cup C_3 \cup C_3 \cup C_3 \text{ and } n = 12.$$

If G has order n, then Problem 1 is precisely the well-known Oberwolfach Problem. See [10, 11, 20] for more information on the Oberwolfach Problem, and see [12] for a generalisation of the problem.

Problem 1 has been solved for every 2-regular graph of order at most 10 when n is odd [2], and various results on Problem 1 have been obtained via graph labellings. For example, in [3] it is shown that if G has order k and is 2-regular with at most three components, then there exists a G-decomposition of K_{2k+1}, and in [6] it is shown that if G is bipartite and 2-regular of order k, then there exists a G-decomposition of K_{2kx+1} for each positive integer x. Several strong results have also been obtained on Problem 1 for the case where G consists of disjoint 3-cycles [13, 14]. These results relate to *Kirkman signal sets* which are are used in devising codes for unipolar communication, see [15].

In [16], a simple but powerful idea is used to show that if both n and $n\lfloor\frac{n-1}{2}\rfloor/k$ are even, then there is a G-decomposition of K_n^* for every bipartite 2-regular graph G of order k. Our main result, see Theorem 10, extends this result to the case $n\lfloor\frac{n-1}{2}\rfloor/k$ is odd, except when $\frac{n}{2} < k < n-2$. The special case of this extension where $k = n$ (that is, the case corresponding to the Oberwolfach Problem) is the main result in [8].

2 Notation and Preliminary Results

For a given graph K, we define the graph $K^{(2)}$ by $V(K^{(2)}) = V(K) \times \mathbb{Z}_2$ and $E(K^{(2)}) = \{\{(x,a),(y,b)\} : \{x,y\} \in E(K),\ a,b \in \mathbb{Z}_2\}$. If $\mathcal{F} = \{G_1, G_2, \ldots, G_t\}$ is a set of graphs then we define $\mathcal{F}^{(2)} = \{G_1^{(2)}, G_2^{(2)}, \ldots, G_t^{(2)}\}$. Observe that if $\mathcal{F}$ is a decomposition of K, then $\mathcal{F}^{(2)}$ is a decomposition of $K^{(2)}$.

The following result of Häggkvist [16] is a critical ingredient in many of our constructions.

Theorem 1. (Häggkvist [16]) *If G is a bipartite 2-regular graph of order $2m$, then there is a G-decomposition of $P_{m+1}^{(2)}$.*

Parker [17] has completely settled the problem of decomposing complete bipartite graphs into paths of uniform length, and we need the following special case of her result.

Theorem 2. (Parker [17]) *If r and a are even with $r \leqslant 2a-2$, $r \leqslant 2b$, and r dividing ab, then there is a P_{r+1}-decomposition of $K_{a,b}$.*

We also need the following result of Tarsi on decompositions of complete graphs into isomorphic paths [19].

Theorem 3. (Tarsi [19]) *There is a P_{r+1}-decomposition of K_v if and only if $v \geqslant r+1$ and r divides $v(v-1)/2$.*

For each even $r \geqslant 2$, let Y_r denote any graph isomorphic to the graph with vertex set $\{v_1, v_2, \ldots, v_{r+1}\}$ and edge set

$$\{v_i v_{i+1} : i = 1, 2, \ldots, r\} \cup \{v_1 v_3\} \cup \{v_i v_{i+3} : i = 2, 4, \ldots, r-2\}$$

($E(Y_2) = \{v_1v_2, v_2v_3, v_1v_3\}$), and let X_{2r} denote the graph obtained from $Y_r^{(2)}$ by adding the edges $\{(v_1,0),(v_1,1)\}, \{(v_2,0),(v_2,1)\}, \ldots, \{(v_{r+1},0),(v_{r+1},1)\}$.

Lemma 4. *For each even $r \geqslant 2$, there exists a decomposition of K_{r+1} into $\frac{r-2}{2}$ Hamilton paths and a copy of Y_r.*

Proof. Let $r \geqslant 2$ be even and for $i = 0, 1, \ldots, r$ let M_i be the matching with edge set $\{\{x,y\} : x \neq y, x+y = i\}$ in the complete graph with vertex set $\mathbb{Z}_{r+1}$. Then

$$\{M_0 \cup M_1 \cup M_2, M_3 \cup M_4, M_5 \cup M_6, \ldots, M_{r-1} \cup M_r\}$$

is the required decomposition. □

Lemma 5. *If r is even, $2 \leqslant r \leqslant \frac{m-1}{2}$, and r divides $\frac{1}{2}m(m-1) - \frac{3r}{2}$, then there is a P_{r+1}-decomposition of $K_m - Y_r$.*

Proof. By Lemma 4, there is a P_{r+1}-decomposition of $K_{r+1} - Y_r$, so it suffices to show that there is a P_{r+1}-decomposition of $K_m - K_{r+1}$. But $K_m - K_{r+1}$ can be decomposed into $K_{r,m-r-1}$ and K_{m-r}, so it suffices to prove that $K_{r,m-r-1}$ and K_{m-r} each have P_{r+1}-decompositions. The former has a P_{r+1}-decomposition by Theorem 2, and the latter by Theorem 3. It is routine to check that the hypotheses of these two theorems are satisfied when r is even, $2 \leqslant r \leqslant \frac{m-1}{2}$ and r divides $\frac{1}{2}m(m-1) - \frac{3r}{2}$. □

For each even $r \geqslant 2$ we define the graph J_{2r} (see Figure 1) to be the graph with vertex set

$$V(J_{2r}) = \{u_1, u_2, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_{r+2}\}$$

and edge set

$$\begin{aligned} E(J_{2r}) = \; & \{\{u_i, v_i\} : i = 3, 4, \ldots, r+2\} \cup \\ & \{\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\}, \{u_i, v_{i+1}\}, \{v_i, u_{i+1}\} : i = 2, 3, \ldots, r+1\} \cup \\ & \{\{u_i, u_{i+3}\}, \{v_i, v_{i+3}\}\{u_i, v_{i+3}\}, \{v_i, u_{i+3}\} : i = 1, 3, \ldots, r-1\}. \end{aligned}$$

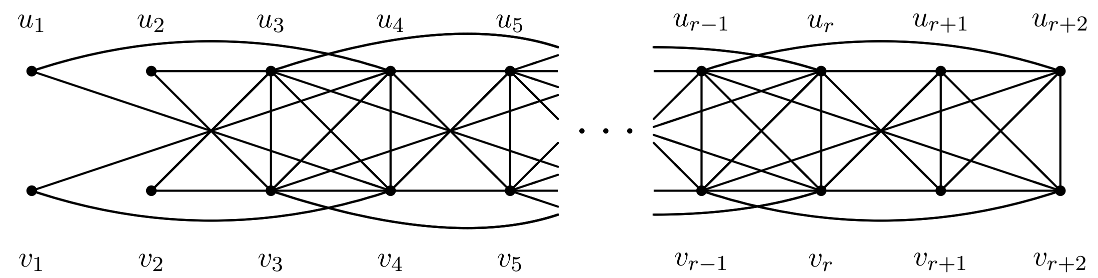

Figure 1: The graph J_{2r}

The following result is proved in [8], see Lemma 10 and the proof of Lemma 11.

Lemma 6. *If G is a bipartite 2-regular graph of order $2r$ where $r \geqslant 4$ is even, then there is a decomposition $\{H_1, H_2, H_3, H_4\}$ of J_{2r} such that*

(1) $V(H_1) = \{u_1, u_2, \ldots, u_r\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(2) $V(H_2) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_r\}$,

(3) $V(H_3) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(4) $V(H_4) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,

(5) each of H_1, H_2 and H_3 is isomorphic to G,

(6) H_4 is a 1-regular graph of order $2r$.

Lemma 7. *If $r \geqslant 2$ is even and G is any bipartite 2-regular graph of order $2r$, then there is a decomposition of X_{2r} into three copies of G and a 1-factor.*

Proof. If $r = 2$, then G is a 4-cycle, X_{2r} is isomorphic to K_6 and the result holds. So assume $r \geqslant 4$. Observe that if the edges $\{u_1, u_4\}, \{u_1, v_4\}, \{v_1, u_4\}, \{v_1, v_4\}$ of J_{2r} are replaced with $\{u_2, u_4\}, \{u_2, v_4\}, \{v_2, u_4\}, \{v_2, v_4\}$, the vertices u_1 and v_1 are deleted, and the edge $\{u_2, v_2\}$ is added, then the resulting graph is X_{2r}. Let $\{H_1, H_2, H_3, H_4\}$ be the decomposition of J_{2r} given by Lemma 6, let H_1' be the graph obtained from H_1 by replacing the edges $\{u_1, u_4\}$ and $\{u_1, v_4\}$ with $\{v_2, u_4\}$ and $\{v_2, v_4\}$, let H_2' be the graph obtained from H_2 by replacing the edges $\{v_1, u_4\}$ and $\{v_1, v_4\}$ with $\{u_2, u_4\}$ and $\{u_2, v_4\}$, let $H_3' = H_3$ and let H_4' be the graph obtained from H_4 by adding the edge $\{u_2, v_2\}$ (and the vertices u_2 and u_4). It is easy to see that $\{H_1', H_2', H_3', H_4'\}$ is the required decomposition of X_{2r}. □

3 Main Results

Lemma 8. *If $n \geqslant 6$ is even and G is any bipartite 2-regular graph of order $n-2$, then there is a G-decomposition of $K_n - I$.*

Proof. Let $m = \frac{n}{2}$. If m is even, then let $\mathcal{D}$ be a decomposition of K_m into $\frac{m}{2}$ Hamilton paths, and if m is odd, then let $\mathcal{D}$ be a decomposition of K_m into $\frac{m-3}{2}$ Hamilton paths and a copy of Y_{m-1}. The first of these decompositions exists by Theorem 3 and the second exists by Lemma 4. In either case let the vertex set of K_m be $\mathbb{Z}_m$ and let I be the 1-regular graph with $V(I) = \mathbb{Z}_m \times \mathbb{Z}_2$ and edge set $E(I) = \{(v, 0)(v, 1) : v \in \mathbb{Z}_m\}$.

Thus, $\mathcal{D}^{(2)} \cup \{I\}$ is a decomposition of K_n into $\frac{m}{2}$ copies of $P_m^{(2)}$ and the perfect matching I when m is even, and is a decomposition of K_n into $\frac{m-3}{2}$ copies of $P_m^{(2)}$, one copy of $Y_{m-1}^{(2)}$, and the perfect matching I when m is odd. Since the union of the copy of $Y_{m-1}^{(2)}$ and I is a copy of X_{n-2}, the result follows by Theorem 1 and Lemma 7. □

Lemma 9. *Let $r \geqslant 2$. If there is a P_{r+1}-decomposition of K_m or if r is even and there is a P_{r+1}-decomposition of $K_m - Y_r$, then there is a G-decomposition of $K_{2m} - I$ for every bipartite 2-regular graph of order $2r$.*

Proof. Let G be a bipartite 2-regular graph of order $2r$, let the vertex set of K_m be $\mathbb{Z}_m$ and let I be the 1-regular graph with $V(I) = \mathbb{Z}_m \times \mathbb{Z}_2$ and edge set $E(I) = \{(v, 0)(v, 1) : v \in \mathbb{Z}_m\}$.

If there is a P_{r+1}-decomposition $\mathcal{D}$ of K_m, then $\mathcal{D}^{(2)}$ is a $P_{r+1}^{(2)}$-decomposition of $K_{2m} - I$. By Theorem 1, we can decompose each copy of $P_{r+1}^{(2)}$ in $\mathcal{D}^{(2)}$ into two copies of G, thereby obtaining a G-decomposition of $K_{2m} - I$.

Thus, we can assume r is even and there is a P_{r+1}-decomposition of $K_m - Y_r$, and hence a decomposition $\mathcal{D}$ of K_m into one copy of Y_r and $(\binom{m}{2} - \frac{3r}{2})/r$ copies of P_{r+1}. It follows that $\mathcal{D}^{(2)} \cup \{I\}$ is a decomposition of K_{2m} into one copy of $Y_r^{(2)}$, $(\binom{m}{2} - \frac{3r}{2})/r$ copies of $P_{r+1}^{(2)}$, and a perfect matching. There are $r+1$ edges of I which form a 1-regular graph on the vertex set of the copy of $Y_r^{(2)}$, and the union of this 1-regular graph with the copy of $Y_r^{(2)}$ is a copy of X_{2r}. Thus, we have a decomposition of K_{2m} into one copy of X_{2r}, $(\binom{m}{2} - \frac{3r}{2})/r$ copies of $P^{(2)}_{r+1}$, and a matching M with $m-(r+1)$ edges (such that M and the copy of X_{2r} are vertex-disjoint).

By Theorem 1, we can decompose each copy of $P^{(2)}_{r+1}$ in $\mathcal{D}^{(2)}$ into two copies of G. Let $\mathcal{D}_P$ be the union of all of these decompositions. By Lemma 7, there is a decomposition $\mathcal{D}_X \cup \{M'\}$ of the copy of X_{2r} where $\mathcal{D}_X$ contains three copies of G and M' is a perfect matching in the copy of X_{2r}. This means that the union of M and M' is a perfect matching in K_{2m}. It follows that $\mathcal{D}_P \cup \mathcal{D}_X$ is a G-decomposition of $K_{2m} - I$. □

Theorem 10. *Let G be a bipartite 2-regular graph, let k be the order of G, and let $n \geqslant 4$ be even. There exists a G-decomposition of $K_n - I$ if and only if $3 \leqslant k \leqslant n$ and k divides $\frac{n(n-2)}{2}$, except possibly when $\frac{n}{2} < k < n-2$ and $\frac{n(n-2)}{2k}$ is odd both hold.*

Proof. The conditions $3 \leqslant k \leqslant n$ and k divides $\frac{n(n-2)}{2}$ are clearly necessary for the existence of a G-decomposition of $K_n - I$. The case $k = n$ is covered by the main theorem in [8] and the case $k = n-2$ is covered by Lemma 8. If $k = n-1$, then k does not divide $\frac{n(n-2)}{2}$ so there is nothing to prove. Thus, it remains only to show that there is a G-decomposition of $K_n - I$ when $3 \leqslant k \leqslant \frac{n}{2}$ and k divides $\frac{n(n-2)}{2}$.

Let $m = \frac{n}{2}$ and let $r = \frac{k}{2}$ (since G is bipartite, k is even and $r \geqslant 2$ is an integer). By Lemma 9, it suffices to show that there is a P_{r+1}-decomposition of K_m or that r is even and there is a P_{r+1}-decomposition of $K_m - Y_r$. If $2m(m-1)/k$ is even, then r divides $m(m-1)/2$ and so by Theorem 3, there is a P_{r+1}-decomposition of K_m. If $2m(m-1)/k$ is odd, then it follows that r is even, r divides $\frac{1}{2}m(m-1) - \frac{3r}{2}$, and $k \neq m$. So $r \leqslant \frac{m-1}{2}$ and there is a P_{r+1}-decomposition of $K_m - Y_r$ by Lemma 5. □

Lemma 9 gives a possible approach to settling the possible exceptions in Theorem 10. The missing ingredient is P_{r+1}-decompositions of $K_m - Y_r$ for $\frac{m}{2} < r < m-1$ where $r = \frac{k}{2}$ and $m = \frac{n}{2}$. The first few possible exceptions in Theorem 10 are $(k,n) = (12,20)$, $(20,30)$, $(24,42)$, $(28,44)$, $(36,54)$, $(40,72)$, $(44,68)$, $(48,80)$, $(52,78)$, $(56,72)$, $(60,92)$, $(60,102)$ and $(60,110)$.

It is worth remarking that the constructions used to prove Theorem 10 can be easily generalised as follows. In the proof of Lemma 9, each copy of $P^{(2)}_{r+1}$ can be decomposed independently, resulting in decompositions of $K_n - I$ into 2-regular graphs which are not all isomorphic. Although each copy of $P^{(2)}_{r+1}$ produces two isomorphic 2-regular graphs in the final decomposition, and the copy of X_{2r}, when it is present, produces three isomorphic 2-regular graphs in the final decomposition, this construction can produce a wide variety of different combinations of 2-regular graphs in the final decomposition.

The 2-regular graphs given by the construction of the preceding paragraph will all have the same order, namely $k = 2r$, but it is also possible to get around this constraint. Instead of using a P_{r+1}-decomposition of K_m or $K_m - Y_r$, one may use a decomposition of K_m or $K_m - Y_r$ into paths which are not necessarily all isomorphic. In [7] it is shown that the obvious necessary conditions are sufficient for the existence of a decomposition of K_m into paths of any specified lengths. This facilitates the construction of decompositions of $K_n - I$ into many combinations of 2-regular graphs of many different orders.

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