Decompositions of complete graphs into bipartite 2-regular subgraphs

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Submitted: Aug 26, 2014; Accepted: Mar 11, 2016; Published: Apr 1, 2016
Mathematics Subject Classifications: 05C51, 05C70, 05B30

Abstract

It is shown that if \( G \) is any bipartite 2-regular graph of order at most \( \frac{n}{2} \) or at least \( n - 2 \), then the obvious necessary conditions are sufficient for the existence of a decomposition of the complete graph of order \( n \) into a perfect matching and edge-disjoint copies of \( G \).

1 Introduction

A decomposition of a graph \( K \) is a set \( \{G_1, G_2, \ldots, G_t\} \) of subgraphs of \( K \) such that \( E(G_1) \cup E(G_2) \cup \cdots \cup E(G_t) = E(K) \) and \( E(G_i) \cap E(G_j) = \emptyset \) for \( 1 \leq i < j \leq t \). If \( G \) is a fixed graph and \( \mathcal{D} = \{G_1, G_2, \ldots, G_t\} \) is a decomposition such that \( G_i \) is isomorphic

∗Supported by the Australian Research Council via grants DP120103067, DP120100790, DP150100506, DP150100530.
†Supported by NSERC Canada through the Discovery Grant program.
‡Supported by NSERC Canada through the Discovery Grant program.
to $G$ for $i = 1, 2, \ldots, t$, then $D$ is called a $G$-decomposition. See [9] for a survey on $G$-decompositions, and see [5, 21] for general asymptotic existence results.

This paper concerns $G$-decompositions of complete graphs in the case where $G$ is a 2-regular graph. See [1] for a survey of results on $G$-decompositions of complete graphs. The complete graph of order $n$ is denoted by $K_n$, the cycle of order $n$ is denoted by $C_n$, and the path of order $n$ is denoted by $P_n$ (so $P_n$ has $n-1$ edges). If $G$ is 2-regular and $n$ is even, then there is no $G$-decomposition of $K_n$, and it is common to instead consider decompositions of $K_n - I$, where $K_n - I$ denotes the graph obtained from $K_n$ by deleting the edges of a perfect matching. For each positive integer $n$, define $K_n^*$ to be $K_n$ if $n$ is odd and $K_n - I$ if $n$ is even. The number of edges in $K_n^*$ is given by $n\lfloor \frac{n-1}{2}\rfloor$.

If $G$ is a 2-regular graph of order $k$ and there exists a $G$-decomposition of $K_n^*$ ($n \geq 3$), then it is obvious that

$$3 \leq k \leq n \quad \text{and} \quad k \text{ divides } n\lfloor \frac{n-1}{2}\rfloor.$$ (1)

If $G$ is a 2-regular graph of order $k$, then the conditions given in (1) are called the obvious necessary conditions for the existence of a $G$-decomposition of $K_n^*$. The following problem presents itself.

**Problem 1:** For each 2-regular graph $G$ and each positive integer $n$ satisfying the obvious necessary conditions, determine whether there exists a $G$-decomposition of $K_n^*$.

It is known that if $G$ is a cycle, then the obvious necessary conditions are sufficient for the existence of a $G$-decomposition of $K_n^*$ [4, 18]. However, when $G$ is 2-regular but is not a cycle, there are cases where the obvious necessary condition are satisfied but no $G$-decomposition of $K_n^*$ exists. There is no $G$-decomposition of $K_n^*$ in each of the following cases (see [9] and [10]).

- $G = C_3 \cup C_3$ and $n = 6$, $G = C_3 \cup C_3$ and $n = 9$, $G = C_4 \cup C_5$ and $n = 9$,
- $G = C_3 \cup C_3 \cup C_5$ and $n = 11$, $G = C_3 \cup C_3 \cup C_3 \cup C_3$ and $n = 12$.

If $G$ has order $n$, then Problem 1 is precisely the well-known Oberwolfach Problem. See [10, 11, 20] for more information on the Oberwolfach Problem, and see [12] for a generalisation of the problem.

Problem 1 has been solved for every 2-regular graph of order at most 10 when $n$ is odd [2], and various results on Problem 1 have been obtained via graph labellings. For example, in [3] it is shown that if $G$ has order $k$ and is 2-regular with at most three components, then there exists a $G$-decomposition of $K_{2k+1}$, and in [6] it is shown that if $G$ is bipartite and 2-regular of order $k$, then there exists a $G$-decomposition of $K_{2k+1}$ for each positive integer $x$. Several strong results have also been obtained on Problem 1 for the case where $G$ consists of disjoint 3-cycles [13, 14]. These results relate to *Kirkman signal sets* which are are used in devising codes for unipolar communication, see [15].

In [16], a simple but powerful idea is used to show that if both $n$ and $n\lfloor \frac{n-1}{2}\rfloor/k$ are even, then there is a $G$-decomposition of $K_n^*$ for every bipartite 2-regular graph $G$ of order $k$. Our main result, see Theorem 10, extends this result to the case $n\lfloor \frac{n-1}{2}\rfloor/k$ is
Lemma 4. paths and a copy of \( \{v_1,v_2,\ldots,v_t\} = \{v_1,v_2,v_3,v_1v_3\} \), and let \( X_r \) denote the graph obtained from \( Y_r \) by adding the edges \( \{(v_1,0),(v_1,1)\}, \{(v_2,0),(v_2,1)\}, \ldots, \{(v_{r+1},0),(v_{r+1},1)\} \).

Lemma 4. For each even \( r \geq 2 \), there exists a decomposition of \( K_{r+1} \) into \( \frac{r-2}{2} \) Hamilton paths and a copy of \( Y_r \).

Proof. Let \( r \geq 2 \) be even and for \( i = 0,1,\ldots,r \) let \( M_i \) be the matching with edge set \( \{x,y\}: x \neq y, x+y = i \} \) in the complete graph with vertex set \( Z_{r+1} \). Then

\[
\{M_0 \cup M_1 \cup M_2, M_3 \cup M_4, M_5 \cup M_6, \ldots, M_{r-1} \cup M_r\}
\]

is the required decomposition. \( \square \)

Lemma 5. If \( r \) is even, \( 2 \leq r \leq \frac{m-1}{2} \), and \( r \) divides \( \frac{1}{2}m(m-1) - \frac{3r}{2} \), then there is a \( P_{r+1} \)-decomposition of \( K_m - Y_r \).

2 Notation and Preliminary Results

For a given graph \( K \), we define the graph \( K^{(2)} \) by \( V(K^{(2)}) = V(K) \times \mathbb{Z}_2 \) and \( E(K^{(2)}) = \{(x,a),(y,b) : (x,y) \in E(K), a,b \in \mathbb{Z}_2\} \). If \( F = \{G_1,G_2,\ldots,G_t\} \) is a set of graphs then we define \( F^{(2)} = \{G_1^{(2)},G_2^{(2)},\ldots,G_t^{(2)}\} \). Observe that if \( F \) is a decomposition of \( K \), then \( F^{(2)} \) is a decomposition of \( K^{(2)} \).

The following result of Häggkvist [16] is a critical ingredient in many of our constructions.

Theorem 1. (Häggkvist [16]) If \( G \) is a bipartite 2-regular graph of order \( 2m \), then there is a \( G \)-decomposition of \( P_{m+1}^{(2)} \).

Parker [17] has completely settled the problem of decomposing complete bipartite graphs into paths of uniform length, and we need the following special case of her result.

Theorem 2. (Parker [17]) If \( r \) and \( a \) are even with \( r \leq 2a - 2 \), \( r \leq 2b \), and \( r \) dividing \( ab \), then there is a \( P_{r+1} \)-decomposition of \( K_{a,b} \).

We also need the following result of Tarsi on decompositions of complete graphs into isomorphic paths [19].

Theorem 3. (Tarsi [19]) There is a \( P_{r+1} \)-decomposition of \( K_v \) if and only if \( v \geq r + 1 \) and \( r \) divides \( v(v-1)/2 \).

For each even \( r \geq 2 \), let \( Y_r \) denote any graph isomorphic to the graph with vertex set \( \{1,2,\ldots,r\} \) and edge set

\[
\{v_{i+1}v_i : i = 1,2,\ldots,r\} \cup \{v_1v_3\} \cup \{v_{i+3}v_i : i = 2,4,\ldots,r-2\}
\]

\( \sum_{i=1}^{r} \) and \( \sum_{i=1}^{r} \). Then

\[
M_0 \cup M_1 \cup M_2, M_3 \cup M_4, M_5 \cup M_6, \ldots, M_{r-1} \cup M_r
\]

is the required decomposition.
Proof. By Lemma 4, there is a $P_{r+1}$-decomposition of $K_{r+1} - Y_r$, so it suffices to show that there is a $P_{r+1}$-decomposition of $K_m - K_{r+1}$. But $K_m - K_{r+1}$ can be decomposed into $K_{r,m-r-1}$ and $K_{m-r}$, so it suffices to prove that $K_{r,m-r-1}$ and $K_{m-r}$ each have $P_{r+1}$-decompositions. The former has a $P_{r+1}$-decomposition by Theorem 2, and the latter by Theorem 3. It is routine to check that the hypotheses of these two theorems are satisfied when $r$ is even, $2 \leq r \leq \frac{m-1}{2}$ and $r$ divides $\frac{1}{2}m(m - 1) - \frac{3}{2}$.

For each even $r \geq 2$ we define the graph $J_{2r}$ (see Figure 1) to be the graph with vertex set

$$V(J_{2r}) = \{u_1, u_2, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_{r+2}\}$$

and edge set

$$E(J_{2r}) = \{(u_i, v_i) : i = 3, 4, \ldots, r + 2\} \cup \{(u_i, u_{i+1}), (v_i, v_{i+1}), (u_i, v_{i+1}), (v_i, u_{i+1}) : i = 2, 3, \ldots, r + 1\} \cup \{(u_i, u_{i+3}), (v_i, v_{i+3}) : i = 1, 3, \ldots, r - 1\}.$$

![Figure 1: The graph $J_{2r}$](image)

The following result is proved in [8], see Lemma 10 and the proof of Lemma 11.

**Lemma 6.** If $G$ is a bipartite 2-regular graph of order $2r$ where $r \geq 4$ is even, then there is a decomposition $\{H_1, H_2, H_3, H_4\}$ of $J_{2r}$ such that

1. $V(H_1) = \{u_1, u_2, \ldots, u_r\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,
2. $V(H_2) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_1, v_2, \ldots, v_r\}$,
3. $V(H_3) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,
4. $V(H_4) = \{u_3, u_4, \ldots, u_{r+2}\} \cup \{v_3, v_4, \ldots, v_{r+2}\}$,
5. each of $H_1$, $H_2$ and $H_3$ is isomorphic to $G$,
6. $H_4$ is a 1-regular graph of order $2r$.

**Lemma 7.** If $r \geq 2$ is even and $G$ is any bipartite 2-regular graph of order $2r$, then there is a decomposition of $X_{2r}$ into three copies of $G$ and a 1-factor.
Proof. If \( r = 2 \), then \( G \) is a 4-cycle, \( X_{2r} \) is isomorphic to \( K_6 \) and the result holds. So assume \( r \geq 4 \). Observe that if the edges \( \{u_1, u_4\}, \{u_1, v_4\}, \{v_1, u_4\}, \{v_1, v_4\} \) of \( J_2r \) are replaced with \( \{u_2, u_4\}, \{u_2, v_4\}, \{v_2, u_4\}, \{v_2, v_4\} \), the vertices \( u_1 \) and \( v_1 \) are deleted, and the edge \( \{u_2, v_2\} \) is added, then the resulting graph is \( X_{2r} \). Let \( \{H_1, H_2, H_3, H_4\} \) be the decomposition of \( J_2r \) given by Lemma 6, let \( H'_1 \) be the graph obtained from \( H_1 \) by replacing the edges \( \{u_1, u_4\} \) and \( \{u_1, v_4\} \) with \( \{v_2, u_4\} \) and \( \{v_2, v_4\} \), let \( H'_2 \) be the graph obtained from \( H_2 \) by replacing the edges \( \{v_1, u_4\} \) and \( \{v_1, v_4\} \) with \( \{u_2, u_4\} \) and \( \{u_2, v_4\} \), let \( H'_3 = H_3 \) and let \( H'_4 \) be the graph obtained from \( H_4 \) by adding the edge \( \{u_2, v_2\} \) (and the vertices \( u_2 \) and \( u_4 \)). It is easy to see that \( \{H'_1, H'_2, H'_3, H'_4\} \) is the required decomposition of \( X_{2r} \). \( \square \)

3 Main Results

Lemma 8. If \( n \geq 6 \) is even and \( G \) is any bipartite 2-regular graph of order \( n - 2 \), then there is a \( G \)-decomposition of \( K_n - I \).

Proof. Let \( m = \frac{n}{2} \). If \( m \) is even, then let \( D \) be a decomposition of \( K_m \) into \( \frac{m}{2} \) Hamilton paths, and if \( m \) is odd, then let \( D \) be a decomposition of \( K_m \) into \( \frac{m-1}{2} \) Hamilton paths and a copy of \( Y_{m-1} \). The first of these decompositions exists by Theorem 3 and the second exists by Lemma 4. In either case let the vertex set of \( K_m \) be \( \mathbb{Z}_m \) and let \( I \) be the 1-regular graph with \( V(I) = \mathbb{Z}_m \times \mathbb{Z}_2 \) and edge set \( E(I) = \{(v, 0)(v, 1) : v \in \mathbb{Z}_m\} \).

Thus, \( D^{(2)} \cup \{I\} \) is a decomposition of \( K_n \) into \( \frac{m}{2} \) copies of \( P_m^{(2)} \) and the perfect matching \( I \) when \( m \) is even, and is a decomposition of \( K_n \) into \( \frac{m-1}{2} \) copies of \( P_m^{(2)} \), one copy of \( Y_{m-1}^{(2)} \), and the perfect matching \( I \) when \( m \) is odd. Since the union of the copy of \( Y_{m-1}^{(2)} \) and \( I \) is a copy of \( X_{n-2} \), the result follows by Theorem 1 and Lemma 7. \( \square \)

Lemma 9. Let \( r \geq 2 \). If there is a \( P_{r+1} \)-decomposition of \( K_m \) or if \( r \) is even and there is a \( P_{r+1} \)-decomposition of \( K_m - Y_r \), then there is a \( G \)-decomposition of \( K_{2m} - I \) for every bipartite 2-regular graph of order \( 2r \).

Proof. Let \( G \) be a bipartite 2-regular graph of order \( 2r \), let the vertex set of \( K_m \) be \( \mathbb{Z}_m \) and let \( I \) be the 1-regular graph with \( V(I) = \mathbb{Z}_m \times \mathbb{Z}_2 \) and edge set \( E(I) = \{(v, 0)(v, 1) : v \in \mathbb{Z}_m\} \).

If there is a \( P_{r+1} \)-decomposition \( D \) of \( K_m \), then \( D^{(2)} \) is a \( P_{r+1}^{(2)} \)-decomposition of \( K_{2m} - I \). By Theorem 1, we can decompose each copy of \( P_{r+1}^{(2)} \) in \( D^{(2)} \) into two copies of \( G \), thereby obtaining a \( G \)-decomposition of \( K_{2m} - I \).

Thus, we can assume \( r \) is even and there is a \( P_{r+1} \)-decomposition of \( K_m - Y_r \), and hence a decomposition \( D \) of \( K_m \) into one copy of \( Y_r \) and \( (\binom{m}{2} - \frac{3r}{2})/r \) copies of \( P_{r+1} \). It follows that \( D^{(2)} \cup \{I\} \) is a decomposition of \( K_{2m} \) into one copy of \( Y_r^{(2)} \), \( (\binom{m}{2} - \frac{3r}{2})/r \) copies of \( P_{r+1}^{(2)} \), and a perfect matching. There are \( r+1 \) edges of \( I \) which form a 1-regular graph on the vertex set of the copy of \( Y_r^{(2)} \), and the union of this 1-regular graph with the copy of \( Y_r^{(2)} \) is a copy of \( X_{2r} \). Thus, we have a decomposition of \( K_{2m} \) into one copy...
of $X_{2r}$, $(m^2/2 - 3r)/r$ copies of $P_{r+1}$, and a matching $M$ with $m - (r + 1)$ edges (such that $M$ and the copy of $X_{2r}$ are vertex-disjoint).

By Theorem 1, we can decompose each copy of $P_{r+1}^{(2)}$ in $D^{(2)}$ into two copies of $G$. Let $D_P$ be the union of all of these decompositions. By Lemma 7, there is a decomposition $D_X \cup \{M\}$ of the copy of $X_{2r}$, where $D_X$ contains three copies of $G$ and $M$ is a perfect matching in the copy of $X_{2r}$. This means that the union of $M$ and $M'$ is a perfect matching in $K_{2m}$. It follows that $D_P \cup D_X$ is a $G$-decomposition of $K_{2m} - I$. \hfill \Box

**Theorem 10.** Let $G$ be a bipartite 2-regular graph, let $k$ be the order of $G$, and let $n \geq 4$ be even. There exists a $G$-decomposition of $K_n - I$ if and only if $3 \leq k \leq n$ and $k$ divides $\frac{n(n-2)}{2}$, except possibly when $\frac{n}{2} < k < n - 2$ and $\frac{n(n-2)}{2k}$ is odd both hold.

**Proof.** The conditions $3 \leq k \leq n$ and $k$ divides $\frac{n(n-2)}{2}$ are clearly necessary for the existence of a $G$-decomposition of $K_n - I$. The case $k = n$ is covered by the main theorem in [8] and the case $k = n - 2$ is covered by Lemma 8. If $k = n - 1$, then $k$ does not divide $\frac{n(n-2)}{2}$ so there is nothing to prove. Thus, it remains only to show that there is a $G$-decomposition of $K_n - I$ when $3 \leq k \leq \frac{n}{2}$ and $k$ divides $\frac{n(n-2)}{2}$.

Let $m = \frac{n}{2}$ and let $r = \frac{k}{2}$ (since $G$ is bipartite, $k$ is even and $r \geq 2$ is an integer). By Lemma 9, it suffices to show that there is a $P_{r+1}$-decomposition of $K_m$ or that $r$ is even and there is a $P_{r+1}$-decomposition of $K_m - Y_r$. If $2m(m-1)/k$ is even, then $r$ divides $m(m-1)/2$ and so by Theorem 3, there is a $P_{r+1}$-decomposition of $K_m$. If $2m(m-1)/k$ is odd, then it follows that $r$ is even, $r$ divides $\frac{1}{2}m(m-1) - \frac{3r}{2}$, and $k \neq m$. So $r \leq \frac{m-1}{2}$ and there is a $P_{r+1}$-decomposition of $K_m - Y_r$ by Lemma 5. \hfill \Box

Lemma 9 gives a possible approach to settling the possible exceptions in Theorem 10. The missing ingredient is $P_{r+1}$-decompositions of $K_m - Y_r$ for $m/2 < r < m - 1$ where $r = \frac{k}{2}$ and $m = \frac{n}{2}$. The first few possible exceptions in Theorem 10 are $(k, n) = (12, 20), (20, 30), (24, 42), (28, 44), (36, 54), (40, 72), (44, 68), (48, 80), (52, 78), (56, 72), (60, 92), (60, 102)$ and $(60, 110)$.

It is worth remarking that the constructions used to prove Theorem 10 can be easily generalised as follows. In the proof of Lemma 9, each copy of $P_{r+1}^{(2)}$ can be decomposed independently, resulting in decompositions of $K_n - I$ into 2-regular graphs which are not all isomorphic. Although each copy of $P_{r+1}^{(2)}$ produces two isomorphic 2-regular graphs in the final decomposition, and the copy of $X_{2r}$, when it is present, produces three isomorphic 2-regular graphs in the final decomposition, this construction can produce a wide variety of different combinations of 2-regular graphs in the final decomposition.

The 2-regular graphs given by the construction of the preceding paragraph will all have the same order, namely $k = 2r$, but it is also possible to get around this constraint. Instead of using a $P_{r+1}$-decomposition of $K_m$ or $K_m - Y_r$, one may use a decomposition of $K_m$ or $K_m - Y_r$ into paths which are not necessarily all isomorphic. In [7] it is shown that the obvious necessary conditions are sufficient for the existence of a decomposition of $K_m$ into paths of any specified lengths. This facilitates the construction of decompositions of $K_n - I$ into many combinations of 2-regular graphs of many different orders.
References


