On well-covered, vertex decomposable and Cohen-Macaulay graphs

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Abstract

Let $G = (V, E)$ be a graph. If $G$ is a König graph or if $G$ is a graph without 3-cycles and 5-cycles, we prove that the following conditions are equivalent: $\Delta_G$ is pure shellable, $R/I_{\Delta}$ is Cohen-Macaulay, $G$ is an unmixed vertex decomposable graph and $G$ is well-covered with a perfect matching of König type $e_1, \ldots, e_g$ without 4-cycles with two $e_i$’s. Furthermore, we study vertex decomposable and shellable (non-pure) properties in graphs without 3-cycles and 5-cycles. Finally, we give some properties and relations between critical, extendable and shedding vertices.

Keywords: Cohen-Macaulay, shellable, well-covered, unmixed, vertex decomposable, König, girth

1 Introduction

Let $G$ be a simple graph (without loops and multipies edges) whose vertex set is $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. The edge ideal of $G$, denoted by $I(G)$, is the ideal of $R$ generated by all monomials $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$. $G$ is a Cohen-Macaulay graph if $R/I(G)$ is a Cohen-Macaulay
ring (see [3], [20]). A subset $F$ of $V(G)$ is a stable set or independent set if $e \not\in F$ for each $e \in E(G)$. The cardinality of the maximum stable set is denoted by $\beta(G)$. $G$ is called well-covered if every maximal stable set has the same cardinality. On the other hand, a subset $D$ of $V(G)$ is a vertex cover of $G$ if $D \cap e \neq \emptyset$ for every $e \in E(G)$. The number of vertices in a minimum vertex cover of $G$ is called the covering number of $G$ and it is denoted by $\tau(G)$. This number coincide with $\text{ht}(I(G))$, the height of $I(G)$. If the minimal vertex covers have the same cardinality, then $G$ is called an unmixed graph. Notice that, $D$ is a vertex cover if and only if $V(G) \setminus D$ is a stable set. Hence, $\tau(G) = n - \beta(G)$ and $G$ is well-covered if and only if $G$ is unmixed. The Stanley-Reisner complex of $I(G)$, denoted by $\Delta_G$, is the simplicial complex whose faces are the stable sets of $G$. Recall that a simplicial complex $\Delta$ is called pure if every facet has the same number of elements. Thus, $\Delta_G$ is pure if and only if $G$ is well-covered.

Some properties of $G$, $\Delta_G$ and $I(G)$ allow an interaction between Commutative Algebra and Combinatorial Theory. Examples of these properties are: Cohen-Macaulayness, shellability, vertex decomposability and well-coveredness. These properties have been studied in ([3], [4], [6], [7], [11], [12], [13], [16], [17], [18], [20], [22]). In general, we have the following implications (see [3], [16], [20], [22])

$$\text{Unmixed vertex decomposable } \Rightarrow \text{ Pure shellable } \Rightarrow \text{ Cohen-Macaulay } \Rightarrow \text{ Well-covered}$$

The equivalence between the Cohen-Macaulay property and the unmixed vertex decomposable property has been studied for some families of graphs: bipartite graphs (in [7] and [11]); very well-covered graphs (in [5] and [13]); graphs with girth at least 5, block-cactus (in [12]); and graphs without 4-cycles and 5-cycles (in [2]). For this paper, a cycle $C = (z_1, z_2, \ldots, z_n)$ can have chords (edges between non-consecutive vertices in $C$) in $G$. A cycle without chords is called an induced cycle.

If a bipartite graph is well-covered, pure shellable or Cohen-Macaulay, then it is König and has a perfect matching. The perfect matching is important because it allowed Hibi and Herzog to characterize Cohen-Macaulay bipartite graph (see [11]). Similarly, the existence of a perfect matching allows one to find a classification of well-covered bipartite graphs (see [15] and [19]). However, a 3-cycle and a 5-cycle are Cohen-Macaulay graphs, but they does not have a perfect matching. This is the motivation for the study of Cohen-Macaulay graphs without 3-cycles and 5-cycles. In particular, we are interested in knowing if these graphs have a perfect matching. In this paper we prove that it is affirmative.

The paper is organized as follow: in section 2 we give some properties and relations between critical, shedding and extendable vertices that we will use in the following sections. In section 3 we prove some results about well-covered graphs. In section 4 we prove the equivalences of unmixed vertex decomposable and Cohen-Macaulay properties for König graphs and graphs without 3-cycles and 5-cycles. We prove that theses properties are equivalent to the following condition: $G$ is an unmixed König graph with a perfect matching $e_1, \ldots, e_g$ without 4-cycles with two $e_i$’s. This result extends the criterion of
Herzog-Hibi for Cohen-Macaulay bipartite graphs, given in [11]. In [17] Van Tuyl proved that the vertex decomposable property, the shellable (non-pure) property and the sequentially Cohen-Macaulay property are equivalent in bipartite graphs. Furthermore, in [18] Van Tuyl and Villarreal give a criterion that characterize shellable bipartite graphs. These results and results obtained in section 4, motivate us to study the vertex decomposable property and the shellable (non-pure) property for graphs without 3-cycles and 5-cycles. In section 5, we prove that the neighborhood of a 2-connected block of $G$ has a free vertex, if $G$ is a bipartite shellable graph or if $G$ is a vertex decomposable graph without 3-cycles and 5-cycles. Also, we prove that the criterion of Van Tuyl-Villarreal can be extended to vertex decomposable graphs without 3-cycles and 5-cycles and shellable graphs with girth at least 11. The equivalence between the shellable property and the vertex decomposable property for graphs without 3-cycles and 5-cycles is an open problem.

2 Critical, extendable and shedding vertices.

Let $X$ be a subset of $V(G)$. The subgraph induced by $X$ in $G$, denoted by $G[X]$ is the graph with vertex set $X$ and whose edge set is $\{\{x, y\} \in E(G) \mid x, y \in X\}$. Furthermore, let $G \setminus X$ denote the induced subgraph $G[V(G) \setminus X]$. Now, if $v \in V(G)$, then the set of neighbors of $v$ (in $G$) is denoted by $N_G(v)$ and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v$ in $G$ is $\deg_G(v) = |N_G(v)|$.

**Definition 1.** $G$ is vertex decomposable if $G$ is a totally disconnected graph or there is a vertex $v$ such that

(a) $G \setminus v$ and $G \setminus N_G[v]$ are both vertex decomposable, and

(b) each stable set in $G \setminus N_G[v]$ is not a maximal stable set in $G \setminus v$.

A shedding vertex of $G$ is any vertex which satisfies the condition (b). Equivalently, $v$ is a shedding vertex if for every stable set $S$ contained in $G \setminus N_G[v]$, there is some $x \in N_G(v)$ such that $S \cup \{x\}$ is stable.

**Lemma 2.** If $x$ is a vertex of $G$, then $x$ is a shedding vertex if and only if $|N_G(x) \setminus N_G(S)| \geq 1$ for every stable set $S$ of $G \setminus N_G[x]$.

**Proof.** $\Rightarrow$ We take a stable set $S$ of $G \setminus N_G[x]$. Since $x$ is a shedding vertex, then there is a vertex $z \in N_G(x)$ such that $S \cup \{z\}$ is a stable set of $G \setminus x$. Thus, $z \notin N_G[S]$. Therefore, $|N_G(x) \setminus N_G(S)| \geq 1$.

$\Leftarrow$ We take a stable set $S$ of $G \setminus N_G[x]$. Thus, there exists a vertex $z \in N_G(x) \setminus N_G(S)$. Since $z \in N_G(x)$, we have that $z \notin S$. Furthermore, $z \notin N_G(S)$, then $S \cup \{z\}$ is a stable set of $G \setminus x$. Consequently, $S$ is not a maximal stable set of $G \setminus x$. Therefore, $x$ is a shedding vertex. $\square$

**Definition 3.** Let $S$ be a stable set of $G$. If $x$ is of degree zero in $G \setminus N_G[S]$, then $x$ is called isolated vertex in $G \setminus N_G[S]$, or we say that $S$ isolates to $x$. 

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By Lemma 2, we have that \( x \) is not a shedding vertex if and only if there exists a stable set \( S \) of \( G \setminus N_G(x) \) such that \( N_G(x) \subseteq N_G(S) \), i.e. \( x \) is an isolated vertex in \( G \setminus N_G[S] \).

**Corollary 4.** Let \( S \) be a stable set of \( G \). If \( S \) isolates \( x \) in \( G \), then \( x \) is not a shedding vertex in \( G \setminus N_G[y] \) for all \( y \in S \).

**Proof.** Since \( S \) isolates \( x \), then \( \deg_{G \setminus N_G[S]}(x) = 0 \) and in particular \( x \in V(G \setminus N_G[S]) \). Thus, \( N_G(x) \subseteq N_G[S] \setminus S \). Hence, if \( y \in S \) and \( G' = G \setminus N_G[y] \), then \( x \in V(G') \).

Furthermore, since \( S \cap N_G[x] = \emptyset \), then \( S' = S \setminus y \) is a stable set in \( G' \setminus N_G'[x] \). Now, since \( S \) isolates \( x \), if \( a \in N_G(x) \), then there exists \( s \in S \) such that \( \{a, s\} \in E(G) \). But \( a \in N_G(x) \), then \( a \notin N_G[y] \), consequently \( s \in S' \) and \( \{a, z\} \in E(G') \). This implies \( |N_{G'}(x) \setminus N_{G'}(S')| = 0 \). Therefore, by Lemma 2, \( x \) is not a shedding vertex in \( G' \).

**Theorem 5.** If \( x \) is a shedding vertex of \( G \), then one of the following conditions hold:

(a) There is \( y \in N_G(x) \) such that \( N_G[y] \subseteq N_G[x] \).

(b) \( x \) is in a 5-cycle with at most one chord.

**Proof.** We take \( N_G(x) = \{y_1, y_2, \ldots, y_k\} \). If \( G \) does not satisfy (a), then there is \( \{z_1, \ldots, z_k\} \subseteq V(G) \setminus N_G[x] \) such that \( \{y_i, z_i\} \in E(G) \) for \( i \in \{1, \ldots, k\} \). We denote by \( L = \{z_1, \ldots, z_q\} = \{z_1, \ldots, z_k\} \) and suppose that \( z_i \neq z_j \) for \( 1 \leq i < j \leq q \). By Lemma 2, if \( L \) is a stable set of \( G \), then \( |N_G(x) \setminus N_G(L)| \geq 1 \). But \( N_G(x) = \{y_1, \ldots, y_k\} \subseteq N_G(L) \), then \( L \) is not a stable set. Hence, \( q \geq 2 \) and there exist \( z_{i_1}, z_{i_2} \in L \) such that \( \{z_{i_1}, z_{i_2}\} \in E(G) \). Thus, there exist \( y_{j_1} \) and \( y_{j_2} \) such that \( y_{j_1} \neq y_{j_2} \) and \( \{y_{j_1}, z_{i_1}\}, \{y_{j_2}, z_{i_2}\} \in E(G) \). Furthermore, \( \{z_{i_1}, y_{j_2}\}, \{z_{i_2}, y_{j_1}\}, \{z_{i_1}, x\}, \{z_{i_2}, x\} \notin E(G) \). Therefore, \( (x, y_{j_1}, z_{i_1}, z_{i_2}, y_{j_2}) \) is a 5-cycle of \( G \) with at most one chord.

**Definition 6.** A vertex \( v \) is called simplicial if the induced subgraph \( G[N_G(v)] \) is a complete graph (or clique). Equivalently, a simplicial vertex is a vertex that appears in exactly one clique.

**Remark 7.** If \( v, w \in V(G) \) such that \( N_G[v] \subseteq N_G[w] \), then \( w \) is a shedding vertex of \( G \) (see Lemma 6 in [22]). In particular, if \( v \) is a simplicial vertex, then any \( w \in N_G(v) \) is a shedding vertex (see Corollary 7 in [22]).

**Corollary 8.** Let \( G \) be graph without 4-cycles. If \( x \) is a shedding vertex of \( G \), then \( x \) is in a 5-cycle or there exists a simplicial vertex \( z \) such that \( \{x, z\} \in E(G) \) with \( |N_G[z]| \leq 3 \).

**Proof.** By Theorem 5, if \( x \) is not in a 5-cycle, then there is \( z \in N_G(x) \) such that \( N_G[z] \subseteq N_G[x] \). If \( \deg_G(z) = 1 \), then \( z \) is a simplicial vertex. If \( \deg_G(z) = 2 \), then \( N_G(z) = \{x, w\} \). Consequently, \( (z, x, w) \) is a 3-cycle since \( N_G[z] \subseteq N_G[x] \). Thus, \( z \) is a simplicial vertex. Now, if \( \deg_G(z) \geq 3 \), then there are \( w_1, w_2 \in N_G(z) \setminus x \). Since \( N_G[z] \subseteq N_G[x] \), we have that \( (w_1, z, w_2, x) \) is a 4-cycle of \( G \). This is a contradiction. Therefore, \( |N_G[z]| \leq 3 \) and \( z \) is a simplicial vertex.
Remark 9. If $G$ is a 5-cycle with $V(G) = \{x_1, x_2, x_3, x_4, x_5\}$, then each $x_i$ is a shedding vertex.

Proof. We can assume that $i = 1$, then $\{x_3\}$ and $\{x_4\}$ are the stable sets in $G \setminus N_G[x_1]$. Furthermore, $\{x_3, x_5\}$ and $\{x_2, x_4\}$ are stable sets in $G \setminus x_1$. Hence, each stable set of $G \setminus N_G[x_1]$ is not a maximal stable set in $G \setminus x_1$. Therefore, $x_1$ is a shedding vertex. □

Definition 10. A vertex $v$ of $G$ is critical if $\tau(G \setminus v) < \tau(G)$. Furthermore, $G$ is called a vertex critical graph if each vertex of $G$ is critical.

Lemma 11. If $\tau(G \setminus v) < \tau(G)$, then $\tau(G) = \tau(G \setminus v) + 1$. Moreover, $v$ is a critical vertex if and only if $\beta(G) = \beta(G \setminus v)$.

Proof. If $t$ is a minimal vertex cover such that $|t| = \tau(G \setminus v)$, then $t \cup \{v\}$ is a vertex cover of $G$. Thus, $\tau(G) \leq |t \cup \{v\}| = \tau(G \setminus v) + 1$. Consequently, if $\tau(G) > \tau(G \setminus v)$, then $\tau(G) = \tau(G \setminus v) + 1$.

Now, we have that $\tau(G) + \beta(G) = |V(G)| = |V(G \setminus v)| + 1 = \tau(G \setminus v) + \beta(G \setminus v) + 1$. Hence, $\beta(G) = \beta(G \setminus v)$ if and only if $\tau(G) = \tau(G \setminus v) + 1$. Therefore, $v$ is a critical vertex if and only if $\beta(G) = \beta(G \setminus v)$. □

Definition 12. A vertex $v$ of $G$ is called an extendable vertex if $G$ and $G \setminus v$ are well-covered graphs with $\beta(G) = \beta(G \setminus v)$.

Note that if $v$ is an extendable vertex, then every maximal stable set $S$ of $G \setminus v$ contains a vertex of $N_G(v)$.

Corollary 13. Let $G$ be an unmixed graph and $x \in V(G)$. The following conditions are equivalent:

(a) $x$ is an extendable vertex.

(b) $|N_G(x) \setminus N_G(S)| \geq 1$ for every stable set $S$ of $G \setminus N_G[x]$.

(c) $x$ is a shedding vertex.

(d) $x$ is a critical vertex and $G \setminus x$ is unmixed.

Proof. (a) $\iff$ (b) ([8], Lemma 2).

(b) $\iff$ (c) By Lemma 2.

(a) $\iff$ (d) Since $G$ is unmixed, then by Lemma 11, $x$ is extendable if and only if $x$ is a critical vertex and $G \setminus x$ is unmixed. □
3 König and well-covered graphs

In this paper we denoted by $Z_G$ the set of the isolated vertices of $G$, that is,

$$Z_G = \{ x \in V(G) \mid \deg_G(x) = 0 \}.$$

**Definition 14.** $G$ is a König graph if $\tau(G) = \nu(G)$ where $\nu(G)$ is the maximum number of pairwise disjoint edges. A perfect matching of König type of $G$ is a collection $e_1, \ldots, e_g$ of pairwise disjoint edges whose union is $V(G)$ and $g = \tau(G)$.

**Proposition 15.** Let $G$ be a König graph and $G' = G \setminus Z_G$. Then the following are equivalent:

(a) $G$ is unmixed.

(b) $G'$ is unmixed.

(c) If $V(G') \neq \emptyset$, then $G'$ has a perfect matching $e_1, \ldots, e_g$ of König type such that for any two edges $f_1 \neq f_2$ and for two distinct vertices $x \in f_1$, $y \in f_2$ contained in some $e_i$, one has that $(f_1 \setminus x) \cup (f_2 \setminus y)$ is an edge.

**Proof.** (a)$\iff$(b) Since $V(G) \setminus V(G') = Z_G$, then $C$ is a vertex cover of $G$ if and only if $C$ is a vertex cover of $G'$. Therefore, $G$ is unmixed if and only if $G'$ is unmixed.

(b)$\iff$(c) By ([14], Lemma 2.3 and Proposition 2.9).

**Definition 16.** A graph $G$ is called very well-covered if it is well-covered without isolated vertices and $|V(G)| = 2\cdot ht(I(G))$.

**Lemma 17.** $G$ is an unmixed König graph if and only if $G$ is totally disconnected or $G' = G \setminus Z_G$ is very well-covered.

**Proof.** $\Rightarrow$ If $G$ is not totally disconnected, then from Proposition 15, $G'$ has a perfect matching $e_1, \ldots, e_g$ of König type. Hence, $|V(G')| = 2g = 2\tau(G') = 2\cdot ht(I(G'))$. Furthermore, $G'$ is unmixed, therefore $G'$ is very well-covered.

$\Leftarrow$) If $G$ is totally disconnected, then $\nu(G) = 0$ and $\tau(G) = 0$. Hence, $G$ is an unmixed König graph. Now, if $G$ is not totally disconnected, then $G'$ is very well-covered. Consequently, by ([10], Corollary 3.7) $G'$ has a perfect matching. Thus, $\nu(G') = |V(G')|/2 = \text{ht}(G') = \tau(G')$. Hence, $G'$ is König. Furthermore, $\nu(G) = \nu(G')$ and $\tau(G) = \tau(G')$, then $G$ is König. Finally, since $G'$ is unmixed, by Proposition 15, $G$ is also unmixed.

**Definition 18.** A subgraph $H$ of $G$ is called a c-minor (of $G$) if there exists a stable set $S$ of $G$, such that $H = G \setminus N_G[S]$.

**Remark 19.** Each connected component of a graph $G$ is a c-minor of $G$.

**Remark 20.** The unmixed property is closed under c-minors. That is, each c-minor of $G$ has the same property (see [20]).
Definition 21. A vertex of degree one is called leaf or free vertex. Furthermore, an edge which is incident with a leaf is called pendant.

Lemma 22. If $G$ is an unmixed graph and $x \in V(G)$, then $N_G(x)$ does not contain two free vertices.

Proof. We suppose that there exists $x \in V(G)$ such that $y_1, \ldots, y_s$ are free vertices in $N_G(x)$. Hence, $G_1 = G \setminus N_G[y_1, \ldots, y_s] = G \setminus \{x, y_1, \ldots, y_s\}$ is unmixed. Now, we take a maximal stable set $S$ of $G_1$. Thus, $|S| = \beta(G_1)$ since $G_1$ is unmixed. Consequently, $S_1 = S \cup \{y_1, \ldots, y_s\}$ is a stable set in $G$. We take $S_2$ a maximal stable in $G$ such that $x \in S_2$. Since $G$ is unmixed, we have that $|S_2| \geq |S_1| = |S| + s$. Furthermore, $S_2 \setminus x$ is a stable set in $G_1$, then $|S_2| \leq \beta(G_1) + 1$. This implies $\beta(G_1) + 1 \geq |S| + s$. But, $|S| = \beta(G_1)$, therefore, $s \leq 1$. \hfill $\square$

Definition 23. If $v, w \in V(G)$, then the distance $d(u, v)$ between $u$ and $v$ in $G$ is the length of the shortest path joining them, otherwise $d(u, v) = \infty$. Now, if $H \subseteq G$, then the distance from a vertex $v$ to $H$ is $d(v, H) = \min\{d(v, u) \mid u \in V(H)\}$. Furthermore, if $W \subseteq V(G)$, then we define $d(v, W) = d(v, G[W])$ and $D_i(W) = \{v \in V(G) \mid d(v, W) = i\}$.

Proposition 24. Let $G$ be an unmixed connected graph without 3-cycles and 5-cycles. If $C$ is a 7-cycle and $H$ is a c-minor of $G$ with $C \subseteq H$ such that $C$ has three non-adjacent vertices of degree 2 in $H$, then $C$ is a c-minor of $G$.

Proof. We take a minimal c-minor $H$ of $G$ such that $C \subseteq H$ and $C$ has three non-adjacent vertices of degree 2 in $H$. We can suppose that $C = (x, z_1, w_1, a, b, w_2, z_2)$ with $\deg_H(x) = \deg_H(w_1) = \deg_H(w_2) = 2$. If $\{z_1, b\} \in E(H)$, then $(z_1, b, w_2, z_2, x)$ is a 5-cycle of $G$. Thus, $\{z_1, b\} \notin E(H)$, similarly $\{z_2, a\} \notin E(H)$. Furthermore, since $G$ does not have 3-cycles, then $\{z_1, z_2\}, \{z_1, a\}, \{z_2, b\} \notin E(H)$. Hence, $C$ is an induced cycle in $H$. On the other hand, if there exists $v \in V(H)$ such that $d(v, C) \geq 2$, then $H' = H \setminus N_G[v]$ is a c-minor of $G$ and $C \subseteq H' \subseteq H$. This is a contradiction by the minimality of $H$. Therefore, $d(v, C) \leq 1$ for each $v \in V(H)$.

Now, if $\deg_H(b) \geq 3$, then there exists $c \in V(H) \setminus V(C)$ such that $\{b, c\} \in E(H)$. If $\{c, z_2\} \notin E(G)$ implies that $N_{H_b}(z_2)$ has two free vertices, $w_2$ and $x$, in $H_1 = H \setminus N_H[w_1, c]$, this is a contradiction by Lemma 22. Thus $\{c, z_2\} \in E(H)$. Furthermore, $\{a, c\}, \{z_1, c\} \notin E(H)$ since $(a, b, c)$ and $(z_1, w_1, a, b, c)$ are not cycles in $G$. Hence, if $\deg_H(c) \geq 3$, then there exists $d \in V(H) \setminus V(C)$ such that $\{c, d\} \in E(H)$. Also, $\{d, b\}, \{d, z_2\}, \{d, z_1\} \notin E(H)$ since $(c, b, d), (z_2, d, c)$ and $(z_1, x, z_2, c, d)$ are not cycles of $G$. But $d(d, C) \leq 1$, so $\{a, d\} \in E(H)$. Consequently, $N_{H_b}(z_1)$ has two free vertices, $w_1$ and $x$, in $H_2 = H \setminus N_H[d, w_2]$, a contradiction by Lemma 22, then $\deg_H(c) = 2$. This implies, $N_{H_b}(z_2)$ has two free vertices, $w_2$ and $c$, in $H_3 = H \setminus N_H[a]$. This is not possible, therefore $\deg_H(b) = 2$. Similarly, $\deg_H(a) = 2$.

Now, if $\deg_H(z_2) \geq 3$ we have that there exists $c' \in V(H) \setminus V(C)$ such that $\{c', z_2\} \in E(H)$. If there exists $d' \in V(H) \setminus V(C)$ such that $\{c', d'\} \in E(H)$, then $\{d', z_1\}$ or $\{d', z_2\} \in E(G)$, since $d(d', C) \leq 1$. But $(c', d', z_2)$ and $(x, z_2, c', d', z_1)$ are not cycles.
of $H$, thus, $N_H(c') \subseteq \{z_1, z_2\}$. Consequently, $N_{H'}(z_2)$ has two free vertices, $x$ and $c'$, in $H = H \setminus N_H[w_1]$, a contradiction. Hence $\deg_H(z_2) = 2$. Similarly, $\deg_H(z_1) = 2$. Furthermore, since $H$ is minimal, then it is connected. Therefore, $H = C$ and $C$ is a $c$-minor of $G$.

4 König and Cohen-Macaulay graphs without 3-cycles and 5-cycles

**Definition 25.** A simplicial complex $\Delta$ is shellable if the facets (maximal faces) of $\Delta$ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j - 1\}$ with $F_j \setminus F_l = \{v\}$. In this case, $F_1, \ldots, F_s$ is called a shelling of $\Delta$. A graph $G$ is called shellable if $\Delta_G$ is shellable. Furthermore, the facet set of $\Delta$ is denoted by $F(\Delta)$.

**Remark 26.** The following properties: shellable, Cohen-Macaulay, sequentially Cohen-Macaulay and vertex decomposable are closed under $c$-minors (see [1], [20]).

**Remark 27.** If $G$ is very well-covered with a perfect matching $e_1, \ldots, e_g$, then the following conditions are equivalent:

2. There are no 4-cycles with two $e_i$’s.

**Proof.** By ([5], Theorem 3.4).

**Proposition 28.** Let $G$ be a König graph where $G' = G \setminus Z_G$. Then the following properties are equivalent:

1. $G$ is unmixed vertex decomposable.
2. $\Delta_G$ is pure shellable.
3. $R/I(G)$ is Cohen-Macaulay.
4. $V(G') = \emptyset$ or $G'$ is an unmixed graph with a perfect matching $e_1, \ldots, e_g$ of König type without 4-cycles with two $e_i$’s.
5. $V(G') = \emptyset$ or there exists a relabelling of the vertices $V(G') = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ such that $\{x_i, y_i\}, \ldots, \{x_h, y_h\}$ is a perfect matching, $X = \{x_1, \ldots, x_h\}$ is a minimal vertex cover of $G'$ and the following conditions holds:
   a. If $a_i \in \{x_i, y_i\}$ and $\{a_i, x_j\}, \{y_j, x_k\} \in E(G')$, then $\{a_i, x_k\} \in E(G')$ for $i \neq j$ and $j \neq k$;
   b. If $\{x_i, y_j\} \in E(G')$, then $i \leq j$. 

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Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) In each case $G$ is unmixed and König. Hence, by Lemma 17, $G$ is totally disconnected or $G'$ is very well-covered. If $G$ is totally disconnected, then we obtain the equivalences. Now, if $G'$ is very well-covered, then by ([13], Theorem 1.1) we obtain the equivalences.

(iv)$\Rightarrow$(iii) We can assume that $V(G') \neq \emptyset$. Thus, by Lemma 17, $G'$ is very well-covered. Hence, by Remark 27 $G'$ is Cohen-Macaulay. Therefore, $G$ is Cohen-Macaulay.

(iii)$\Rightarrow$ (v) Since $R/J(G)$ is Cohen-Macaulay, then $G$ is unmixed. Consequently, by Lemma 17, we can assume that $G'$ is very well-covered. Hence, by ([13], Lemma 3.1), $G'$ satisfies (v).

(v)$\Rightarrow$(iv) We can assume that $V(G') \neq \emptyset$. Since, $e_1 = \{x_1, y_1\}, \ldots, e_h = \{x_h, y_h\}$ is a perfect matching, then $\nu(G') = h$. Furthermore, $X$ is a minimal vertex cover, so $\tau(G') = h$. Hence, $e_1, \ldots, e_h$ is a perfect matching of König type. Thus, from (a) and Proposition 15, $G'$ is unmixed. On the other hand $\{y_1, \ldots, y_h\}$ is a stable set. Therefore, from (b), there are no 4-cycles with two $e_i$’s.

Corollary 29. Let $G$ be a connected König graph. If $G$ is Cohen-Macaulay, then $G$ is an isolated vertex or $G$ has at least one free vertex.

Proof. By Proposition 28, if $G$ is not an isolated vertex, then $G$ has a perfect matching $e_1 = \{x_1, y_1\}, \ldots, e_h = \{x_h, y_h\}$ where $\{x_1, \ldots, x_h\}$ is a minimal vertex cover. Thus, $\{y_1, \ldots, y_h\}$ is a maximal stable set. Furthermore, if $\{x_i, y_j\}$, then $i \leq j$. Hence, $N_G(y_1) = \{x_1\}$. Therefore, $y_1$ is a free vertex.

Lemma 30. Let $G$ be an unmixed connected graph with a perfect matching $e_1, \ldots, e_g$ of König type without 4-cycles with two $e_i$’s and $g \geq 2$. For each $z \in V(G)$ we have that:

(a) If $\deg_G(z) \geq 2$, then there exist $\{z, w_1\}, \{w_1, w_2\} \in E(G)$ such that $\deg_G(w_2) = 1$. Furthermore, $e_1 = \{w_1, w_2\}$ for some $i \in \{1, \ldots, g\}$.

(b) If $\deg_G(z) = 1$, then there exist $\{z, w_1\}, \{w_1, w_2\}, \{w_2, w_3\} \in E(G)$ such that $\deg_G(w_3) = 1$. Moreover, $e_1 = \{z, w_1\}$ and $e_j = \{w_2, w_3\}$ for some $i, j \in \{1, \ldots, g\}$.

Proof. Since $e_1 = \{x_1, y_1\}, \ldots, e_g = \{x_g, y_g\}$ is a perfect matching of König type we can assume $D = \{x_1, \ldots, x_g\}$ is a minimal vertex cover. Thus, $F = \{y_1, \ldots, y_g\}$ is a maximal stable set. By Proposition 28, we can assume that if $\{x_i, y_j\} \in E(G)$, then $i \leq j$. Now, we take a vertex $z \in V(G)$.

(a) First, we suppose that $z = x_k$ and there is a vertex $x_j$ in $N_G(x_k)$. If $y_j$ is a free vertex, then we take $w_1 = x_j$ and $w_2 = y_j$, and $e_j = \{w_1, w_2\}$. Now, we can assume $N_G(y_j) \setminus x_j = \{x_{p_1}, \ldots, x_{p_r}\}$ with $p_1 < \cdots < p_r < j$. If $y_{p_1}$ is not a free vertex, then there is a vertex $x_p$ with $p < p_1$ such that $\{x_p, y_{p_1}\} \in E(G)$. Since $G$ is unmixed, from Proposition 15, we obtain that $\{x_p, y_j\} = (\{x_p, y_{p_1}\} \setminus y_{p_1}) \cup (\{y_j, x_{p_1}\} \setminus x_{p_1}) \in E(G)$. But $p < p_1$, a contradiction since $p_1$ is minimal. Consequently, $\deg_G(y_{p_1}) = 1$. Also, from Proposition 15, we have that $\{x_k, x_{p_1}\} = (\{x_k, x_j\} \setminus x_j) \cup (\{x_{p_1}, y_j\} \setminus y_j) \in E(G)$. Hence, we take $w_1 = x_{p_1}$ and $w_2 = y_{p_1}$, and we have that $e_{p_1} = \{w_1, w_2\}$. Now, we
assume that $z = x_k$ and $N_G(x_k) \setminus y_k = \{y_{j1}, \ldots, y_{jt}\}$ with $k < j_1 < \cdots < j_t$. We suppose that $\deg_G(x_j) \geq 2$. If there is a vertex $y_r$ such that $\{x_{j1}, y_r\} \in E(G)$, then $r > j_t$.

Since $G$ is unmixed, $\{x_j, y_{j1}\} \setminus y_{j1} \cup \{y_r, x_{j1}\} \setminus x_{j1} \in E(G)$, a contradiction since $j_t$ is maximal. Thus, there exists a vertex $x_p$ such that $\{x_{j1}, x_p\} \in E(G)$. But, since $G$ is unmixed, then $\{x_k, x_p\} = (\{x_k, y_{j1}\} \setminus y_{j1}) \cup \{x_p, x_{j1}\} \setminus x_{j1} \in E(G)$. This is a contradiction since $N_G(x_k) \setminus y_k = \{y_{j1}, \ldots, y_{jt}\}$. Consequently, $\deg_G(x_{j1}) = 1$. Therefore, we take $w_1 = y_{j1}$ and $w_2 = x_{j1}$, with $e_{j1} = \{w_1, w_2\}$.

Finally, we assume that $z = y_k$ since $y_k$ is not a free vertex, then $N_G(y_k) \setminus x_k = \{x_{j1}, \ldots, x_{jt}\}$ with $j_1 < \cdots < j_t < k$. If $y_{j1}$ is not a free vertex, then there is a vertex $x_q$ such that $\{x_q, y_{j1}\} \in E(G)$ with $q < j_1$. This implies $\{x_q, y_k\} = (\{x_q, y_{j1}\} \setminus y_{j1}) \cup (\{x_{j1}, y_k\} \setminus x_{j1}) \in E(G)$. But $q < j_1$, a contradiction. Therefore, $\deg_G(y_{j1}) = 1$ and we take $w_1 = x_{j1}$ and $w_2 = y_{j1}$. Hence, $e_{j1} = \{w_1, w_2\}$.

(b) Since $e_1, \ldots, e_g$ is a perfect matching, then there exists $i \in \{1, \ldots, g\}$ such that $e_i = \{z, z'\}$. Since $G$ is connected, $z$ is a free vertex and $g \geq 2$, then $\deg_G(z') \geq 2$. Thus, by (a) there exist $w'_1, w'_2 \in V(G)$ such that $\{z', w'_1\}, \{w'_1, w'_2\} \in E(G)$ where $\deg_G(w'_2) = 1$ and $\{w'_1, w'_2\} = e_j$ for some $j \in \{1, \ldots, g\}$. Therefore, we take $w_1 = z'$, $w_2 = w'_1$, $w_3 = w'_2$. Consequently, $e_i = \{z, w_1\}$ and $e_j = \{w_2, w_3\}$.

Remark 31. If $C_n$ is a $n$-cycle, then $C_n$ is vertex decomposable, shellable or sequentially Cohen-Macaulay if and only if $n = 3$ or $5$ (see [9] and [22]). Furthermore, a chordal graph, which is a graph whose induced cycles are 3-cycles, is vertex decomposable (see Corollary 7 in [22]). In particular trees are vertex decomposable.

Theorem 32. Let $G$ be a graph without 3-cycles and 5-cycles. If $G_1, \ldots, G_k$ are the connected components of $G$, then the following conditions are equivalent:

(a) $G$ is unmixed vertex decomposable.

(b) $G$ is pure shellable.

(c) $G$ is Cohen-Macaulay

(d) $G$ is unmixed and if $G_i$ is not an isolated vertex, then $G_i$ has a perfect matching $e_1, \ldots, e_g$ of König type without 4-cycles with two $e_i$’s.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) (see [16], [20], [22]).

(d) $\Rightarrow$ (a) Since each component $G_i$ is König, then $G$ is König. Therefore, from Proposition 28, $G$ is unmixed vertex decomposable.

(c) $\Rightarrow$ (d) Since $G$ is Cohen-Macaulay, then $G$ is unmixed. We proceed by induction on $|V(G)|$. We take $x \in V(G)$ such that $\deg_G(x)$ is minimal and we suppose that $N_G(x) = \{z_1, \ldots, z_r\}$. By Remark 26, $G' = G \setminus N_G(x)$ is a Cohen-Macaulay graph. We take $G'_1, \ldots, G'_s$, the connected components of $G'$. We can assume that $V(G'_j) = \{y_i\}$ for $i \in \{1, \ldots, s'\}$. Since $\deg_G(x)$ is minimal, this implies $\{y_i, z_j\} \in E(G)$ for all $i \in \{1, \ldots, s'\}$ and $j \in \{1, \ldots, r\}$. Since $G$ does not contain 3-cycles, we have that $N_G(x)$ is a stable
set. If \( s' = s \), then the only maximal stable sets of \( G \) are \( \{y_1, \ldots, y_{s'}, x\} \) and \( \{z_1, \ldots, z_r\} \). Thus, \( G \) is a bipartite graph. So, \( G \) is König. Hence, by Proposition 28, \( G \) satisfies (d).

Consequently, we can assume \( s > s' \), implying that there is a component \( G'_i \) with an edge \( e = \{w, w'\} \).

Now, we suppose that \( r \geq 2 \). Since \( \deg_G(x) \) is minimal there exist \( a, b \in V(G) \) such that \( \{a, w\}, \{b, w'\} \in E(G) \). If \( a = b \), then \( \{a, w, w'\} \) is a 3-cycle in \( G \). Hence, \( a \neq b \). If \( a, b \in N_G(x) \), then \( \{x, a, w, w', b\} \) is a 5-cycle in \( G \). Thus, \( |\{w, w', a, b\} \cap V(G'_i)| \geq 3 \).

By induction hypothesis, \( G'_i \) satisfies (d). So, \( G'_i \) has a perfect matching and \( \tau(G'_i) \geq 2 \).

Furthermore, by Corollary 29, \( G'_i \) has a free vertex \( a' \). Then, by Lemma 30 (b), there exist edges \( \{a', w_1\}, \{w_1, w_2\}, \{w_2, b'\} \in E(G'_i) \) such that \( \deg_{G'_i}(a') = \deg_{G'_i}(b') = 1 \).

By the minimality of \( \deg_G(x) \) we have that \( a' \) and \( b' \) are adjacent with at least \( r - 1 \) neighbor vertices of \( x \). If \( r \geq 3 \), then there exists \( z_j \) such that \( z_j \in N_G(a') \cap N_G(b') \). This implies that \( \{a', w_1, w_2, b', z_j\} \) is a 5-cycle of \( G \). But \( G \) does not have 5-cycles, consequently, \( r = 2 \).

We can assume that \( \{a', z_1\}, \{b', z_2\} \in E(G) \), implying \( C = \{x, z_1, a', w_1, w_2, b', z_2\} \) is a 7-cycle with \( \deg_G(a') = \deg_G(b') = \deg_G(x) = 2 \). Hence, by Proposition 24, \( C \) is a c-minor of \( G \). Thus, by Remark 26, \( C \) is Cohen-Macaulay. This is a contradiction by Remark 31.

Therefore, \( \deg_G(x) = r \leq 1 \).

If \( r = 0 \), then the result is clear. Now, if \( r = 1 \) we can assume that \( G_1, \ldots, G_k \) are the connected components of \( G \) and \( z_1 \in V(G_1) \). Consequently, the connected components of \( G \setminus N_G(x) \) are \( F_1, \ldots, F_l, G_2, \ldots, G_k \) where \( F_1, \ldots, F_l \) are the connected components of \( G_1 \setminus N_G(x) \). By induction hypothesis \( G_2, \ldots, G_k \) satisfy (d). If \( F_j = \{d_j\} \), then \( N_G(z_1) \) has two free vertices, \( d_j \) and \( x \), a contradiction by Lemma 22. Hence, \( |V(F_j)| \geq 2 \) for \( i \in \{1, \ldots, l\} \). By induction hypothesis, we have that \( F_i \) has a perfect matching \( M_i = \{e'_1, \ldots, e'_g_i\} \) of König type. Thus, \( \{e\} \cup \bigcup_{i=1}^l M_i \) is a perfect matching of \( G_1 \), where \( e = \{x, z_1\} \).

Also, \( \{z_1\} \cup \bigcup_{i=1}^l X_i \) is a vertex cover of \( G_1 \), where \( X_i \) is a minimal vertex cover of \( F_i \). Consequently, \( \nu(G_1) \geq 1 + \sum_{i=1}^l |M_i| = 1 + \sum_{i=1}^l g_i = 1 + \sum_{i=1}^l |X_i| \geq \tau(G_1) \).

This implies that \( G_1 \) is König. Furthermore, by Remark 26, we have that \( G_1 \) is Cohen-Macaulay. Therefore, by Proposition 28, \( G_1 \) satisfies (d).

\[ \square \]

**Corollary 33.** Let \( G \) be a connected graph without 3-cycles and 5-cycles. If \( G \) is Cohen-Macaulay, then \( G \) has at least one extendable vertex adjacent to a free vertex.

**Proof.** From Theorem 32, \( G \) is König. Thus, by Corollary 29 there exists a free vertex \( x \). If \( N_G(x) = \{y\} \), then by Remark 7, \( y \) is a shedding vertex. Therefore, from Corollary 13, \( y \) is an extendable vertex, since \( G \) is unmixed.

\[ \square \]

**Definition 34.** \( G \) is called whisker graph if there exists an induced subgraph \( H \) of \( G \) such that \( V(H) = \{x_1, \ldots, x_s\}, V(G) = V(H) \cup \{y_1, \ldots, y_s\} \) and \( E(G) = E(H) \cup W(H) \) where \( W(H) = \{\{x_1, y_1\}, \ldots, \{x_s, y_s\}\} \). The edges of \( W(H) \) are called whiskers and they form a perfect matching.

**Definition 35.** The girth of \( G \) is the length of the smallest cycle or infinite if \( G \) is a forest.
Corollary 36. Let $G$ be a connected graph of girth 6 or more. If $G$ is not an isolated vertex, then the following conditions are equivalent:

(i) $G$ is unmixed vertex decomposable.

(ii) $\Delta_G$ is pure shellable.

(iii) $R/I(G)$ is Cohen-Macaulay.

(iv) $G$ is an unmixed König graph.

(v) $G$ is very well-covered.

(vi) $G$ is unmixed with $G \neq C_7$.

(vii) $G$ is a whisker graph.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) (see [16], [20], [22]). (iii) $\Rightarrow$ (iv) $G$ is unmixed and from Theorem 32, $G$ is König. (iv) $\Rightarrow$ (v) From Lemma 17. (v) $\Rightarrow$ (vi) It is clear, since $C_7$ is not very well-covered.

(vi) $\Rightarrow$ (vii) By ([8], Corollary 5), the pendant edges $\{x_1,y_1\}, \ldots, \{x_g,y_g\}$ of $G$ form a perfect matching. Since $\{x_i,y_i\}$ is a pendant edge, we can assume that $\deg_G(y_i) = 1$ for each $1 \leq i \leq g$. We take $H = G[x_1,\ldots,x_n]$. Therefore, $G$ is a whisker graph with $W(H) = \{\{x_1,y_1\}, \ldots, \{x_g,y_g\}\}$.

(vii) $\Rightarrow$ (i) By ([6], Theorem 4.4).

5 Vertex decomposable and shellable properties in graphs without 3-cycles and 5-cycles

Definition 37. A 5-cycle $C$ of $G$ is called basic if $C$ does not contain two adjacent vertices of degree three or more in $G$.

Lemma 38. If $G$ is a graph, then any vertex of degree at least 3 in a basic 5-cycle is a shedding vertex.

Proof. Let $C = (x_1,x_2,x_3,x_4,x_5)$ be a basic 5-cycle. We suppose that $\deg_G(x_1) \geq 3$, since $C$ is a basic 5-cycle, then $\deg_G(x_2) = \deg_G(x_5) = 2$. Also, we can assume that $\deg_G(x_3) = 2$. We take a stable set $S$ of $G \setminus N_G[x_1]$. Since $\{x_3,x_4\} \in E(G)$, then $|S \cap \{x_3,x_4\}| \leq 1$. Hence, $x_3 \notin S$ or $x_4 \notin S$. Consequently, $S \cup \{x_2\}$ or $S \cup \{x_5\}$ is a stable set of $G \setminus x_1$. Therefore, $x_1$ is a shedding vertex.

Remark 39. If $G$ has a shedding vertex $v$ where $G \setminus v$ and $G \setminus N_G[v]$ are shellable with shelling $F_1,\ldots,F_k$ and $G_1,\ldots,G_q$, respectively, then $G$ is shellable with shelling $F_1,\ldots,F_k,G_1 \cup \{v\},\ldots,G_q \cup \{v\}$ (see Lemma 6 in [21]).
Therefore, \( H \) is a maximal stable set of \( F \). \( x \) is shellable. Since \( G \) and \( \deg \) are shellable graphs. Now, we suppose \( G \neq C \). We can assume that \( \deg_{G}(x_{1}) \geq 3 \). Since \( C \) is a basic 5-cycle, then \( \deg_{G}(x_{2}) = \deg_{G}(x_{3}) = 2 \). Also, we can suppose \( \deg_{G}(x_{3}) = 2 \) and \( \deg_{G}(x_{4}) \geq 2 \). By Lemma 38, \( x_{1} \) is a shedding vertex. Furthermore by Remark 26, we have that \( G \) is a shellable graph. Now, we will prove that \( G_{1} = G \setminus x_{1} \) is shellable. Since \( G \) is shellable and since shellability is closed under c-minors, then \( G_{2} = G \setminus N_{G}[x_{2}] \) is shellable. We assume that \( F_{1}, \ldots, F_{r} \) is a shelling of \( \Delta_{G_{1}} \). Also, \( G_{3} = G \setminus N_{G}[x_{3}, x_{5}] \) is shellable. We suppose that \( H_{1}, H_{2}, \ldots, H_{k} \) is shelling of \( \Delta_{G_{3}} \). We take \( F \in \mathcal{F} \left( \Delta_{G_{1}} \right) \). If \( x_{2} \notin F \), then \( F \setminus x_{2} \in \mathcal{F} \left( \Delta_{G_{2}} \right) \) and there exists \( F_{i} \) such that \( F = F_{i} \cup \{ x_{2} \} \). If \( x_{3} \notin F \), then \( x_{3} \notin F \). Thus, \( x_{3} \notin F \). Hence, \( F \setminus \{ x_{3}, x_{5} \} \in \mathcal{F} \left( \Delta_{G_{4}} \right) \), then there exists \( H_{j} \) such that \( F = H_{j} \cup \{ x_{3}, x_{5} \} \). This implies, \( \mathcal{F} \left( \Delta_{G_{4}} \right) = \{ F_{1} \cup \{ x_{2} \}, \ldots, F_{r} \cup \{ x_{2} \}, H_{1} \cup \{ x_{3}, x_{5} \}, \ldots, H_{k} \cup \{ x_{3}, x_{5} \} \} \). Furthermore, \( F_{1} \cup \{ x_{2} \}, \ldots, F_{r} \cup \{ x_{2} \} \) and \( H_{1} \cup \{ x_{3}, x_{5} \}, \ldots, H_{k} \cup \{ x_{3}, x_{5} \} \) are shellings. Now, \( x_{3} \in \left( H_{j} \cup \{ x_{3}, x_{5} \} \right) \setminus \left( F_{i} \cup \{ x_{2} \} \right) \) and \( H_{j} \) is a stable set of \( G \) without vertices of \( C \). So, \( H_{j} \cup \{ x_{3}, x_{5} \} \) is a maximal stable set of \( G \) since \( N_{G}[x_{2}, x_{3}] = V(C) \) and \( \{ x_{2}, x_{5} \} \notin E(G) \). Consequently, \( H_{j} \cup \{ x_{2}, x_{5} \} = F_{i} \cup \{ x_{2} \} \) for some \( l \in \{ 1, \ldots, r \} \) and \( H_{j} \cup \{ x_{3}, x_{5} \} \setminus (F_{i} \cup \{ x_{2} \}) = \{ x_{3} \} \). Therefore, \( G_{1} \) is a shellable graph.

\[ \Rightarrow \] By Remark 39.

**Definition 41.** A cut vertex of a graph is one whose removal increases the number of connected components. A block of a graph is a maximal subgraph without cut vertices. A connected graph without cut vertices with at least three vertices is called 2-connected graph.

In the following result \( P \) is a property closed under c-minors.

**Theorem 42.** Let \( G \) be a graph without 3-cycles and 5-cycles with a 2-connected block \( B \). If \( G \) satisfies the property \( P \) and \( B \) does not satisfy \( P \), then there exists \( x \in D_{1}(B) \) such that \( \deg_{G}(x) = 1 \).

**Proof.** By contradiction, we assume that if \( x \in D_{1}(B) \), then \( |N_{G}(x)| > 1 \). Thus, there exist \( a, b \in N_{G}(x) \) with \( a \neq b \). We can suppose that \( a \in V(B) \). If \( b \notin V(B) \), then \( G[\{ x \} \cup V(B)] \) is 2-connected. But \( B \subseteq G[\{ x \} \cup V(B)] \). This is a contradiction since \( B \) is a block. Consequently, \( V(B) \cap N_{G}(x) = \{ a \} \). Now, we suppose that \( b \in D_{1}(B) \). Since there is no 3-cycle in \( G \), then \( a \notin N_{G}(b) \). Hence, there exists \( c \in N_{G}(b) \cap V(B) \) such that \( c \neq a \). This implies \( G[\{ x, b \} \cup V(B)] \) is 2-connected. But \( B \subseteq G[\{ x, b \} \cup V(B)] \), a contradiction. Then \( D_{1}(B) \cap N_{G}(x) = \emptyset \). Thus, \( N_{G}(x) \cap (V(B) \cup D_{1}(B)) = \{ a \} \) and \( b \in D_{2}(B) \). Now, if \( D_{1}(B) = \{ x_{1}, \ldots, x_{r} \} \), then there exists an \( a_{i} \) such that \( V(B) \cap N_{G}(x_{i}) = \{ a_{i} \} \). Also, there exists \( b_{l} \) such that \( b_{l} \in N_{G}(x_{i}) \cap D_{2}(B) \). We can suppose that \( L = \{ b_{1}, \ldots, b_{s} \} = \emptyset \).
Therefore, there exists a free vertex in $P$ not satisfy $P$. Furthermore, $G$ since $G$ and $G$ is 2-connected. But $B$ is a block, then $\{b_i, b_j\} \not\in E(G)$. Therefore, $L$ is a stable set. Furthermore, $G' = G \setminus N_G[L]$ is a $c$-minor of $G$, implying that $G'$ satisfies the property $P$. Since $D_1(B) \subset N_G(L)$, we have that $B$ is a connected component of $G'$. But, $B$ does not satisfy $P$. This is a contradiction since each connected component of $G$ is a $c$-minor. Therefore, there exists a free vertex in $D_1(B)$. 

Corollary 43. Let $G$ be a graph without 3-cycles and 5-cycles and $B$ a 2-connected block. If $G$ is shellable (unmixed, Cohen-Macaulay, sequentially Cohen-Macaulay or vertex decomposable) and $B$ is not shellable (unmixed, Cohen-Macaulay, sequentially Cohen-Macaulay or vertex decomposable), then there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.

Proof. From Remark 20, Remark 26 and Theorem 42.

Corollary 44. Let $G$ be a bipartite graph and $B$ a 2-connected block. If $G$ is shellable, then there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.

Proof. Since $G$ is bipartite, then $B$ is bipartite. If $H$ is a shellable bipartite graph, then $H$ has a free vertex (see [18], Lemma 2.8), and so $H$ is not 2-connected. Hence, $B$ is not shellable. Therefore, by Corollary 43, there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.

Lemma 45. Let $G$ be a graph without 3-cycles and 5-cycles. If $G$ is vertex decomposable, then $G$ has a free vertex.

Proof. Since $G$ is vertex decomposable, then there is a shedding vertex $x$. Furthermore, there are no 5-cycles in $G$. Hence, by Theorem 5, there exists $y \in N_G(x)$ such that $N_G[y] \subseteq N_G[x]$. If $z \in N_G(y) \setminus x$, then $(x, y, z)$ is a 3-cycle. This is a contradiction. Therefore, $N_G(y) = \{x\}$, implying that $y$ is a free vertex.

Theorem 46. Let $G$ be a graph without 3-cycles and 5-cycles. $G$ is vertex decomposable if and only if there exists a free vertex $x$ with $N_G(x) = \{y\}$ such that $G_1 = G \setminus N_G[x]$ and $G_2 = G \setminus N_G[y]$ are vertex decomposable.

Proof. $\Rightarrow$ By Lemma 45 there exists a free vertex $x$. Furthermore, by Remark 26, $G_1$ and $G_2$ are vertex decomposable.

$\Leftarrow$ By Remark 7, $y$ is a shedding vertex. Moreover, $G \setminus y = G_1 \cup \{x\}$. Furthermore, since $G_1$ is vertex decomposable, then $G_1 \setminus y$ is also it. Therefore, $G$ is vertex decomposable, since $G_2$ is vertex decomposable.

Corollary 47. If $G$ is a 2-connected graph without 3-cycles and 5-cycles, then $G$ is not a vertex decomposable.

Proof. Since $G$ is 2-connected, then $G$ does not have a free vertex. Therefore, by Lemma 45, $G$ is not vertex decomposable.
Theorem 48. Let $G$ be a vertex decomposable graph without 3-cycles and 5-cycles. If $B$ is a 2-connected block of $G$, then $D_1(B)$ has a free vertex.

Proof. By Corollary 47, $B$ is not vertex decomposable. Therefore, by Theorem 42, $D_1(B)$ has a free vertex. \qed

Definition 49. Let $G_1,G_2$ be graphs. If $K = G_1 \cap G_2$ is a complete graph with $|V(K)| = k$, then $G = G_1 \cup G_2$ is called the $k$-clique-sum (or clique-sum) of $G_1$ and $G_2$ in $K$.

Corollary 50. If $G$ is the $2$-clique-sum of the cycles $C_1$ and $C_2$ with $|V(C_1)| = r_1 \leq r_2 = |V(C_2)|$, then $G$ is vertex decomposable if and only if $r_1 = 3$ or $r_1 = r_2 = 5$.

Proof. $\Leftarrow$) First, we suppose that $r_1 = 3$. Consequently, we can assume $C_1 = (x_1, x_2, x_3)$ and $x_2, x_3 \in V(C_1) \cap V(C_2)$. Thus, $x_1$ is a simplicial vertex. Hence, by Remark 7, $x_2$ is a shedding vertex. Furthermore, $G \setminus x_2$ and $G \setminus N_G[x_2]$ are trees. Consequently, by Remark 31, $G \setminus x_2$ and $G \setminus N_G[x_2]$ are vertex decomposable graphs. Therefore, $G$ is vertex decomposable.

Now, we assume that $r_1 = r_2 = 5$ with $C_1 = (x_1, x_2, x_3, x_4, x_5)$ and $C_2 = (y_1, y_2, x_3, y_4, y_5)$. We take a stable set $S$ in $G \setminus N_G[x_5]$. If $x_2 \in S$, then $S \cup \{x_1\}$ is a stable set in $G_1 = G \setminus x_5$. If $x_2 \notin S$, then $S \cup \{x_1\}$ is a stable set in $G_1$. Consequently, by Lemma 2, $x_5$ is a shedding vertex. Since $x_2$ is a neighbor of a free vertex in $G_1$, then $x_2$ is a shedding vertex in $G_1$. Furthermore, since $G_1 \setminus x_2$ and $G_1 \setminus N_G[x_2]$ are forests, then they are vertex decomposable graphs, by Remark 31. Thus, $G_1$ is vertex decomposable. Since $G \setminus N_G[x_5] = C_2$, it is vertex decomposable by Remark 31. Therefore, $G$ is vertex decomposable.

$\Rightarrow$) By Corollary 47, we have that $\{r_1, r_2\} \cap \{3, 5\} \neq \emptyset$. We suppose $r_1 \neq 3$. So $r_1 = 5$ or $r_2 = 5$. Consequently, we can assume that $\{C_1, C_2\} = \{C, C'\}$ where $C = (x_1, x_2, x_3, x_4, x_5)$ and $x_2, x_3 \in V(C) \cap V(C')$. Thus, $G \setminus N_G[x_5] = C'$ is vertex decomposable. Hence, from Remark 31, $|V(C')| \in \{3, 5\}$. But $r_1 \neq 3$, then $|V(C')| = 5$ and $r_1 = r_2 = 5$. Therefore, $r_1 = 3$ or $r_2 = 5$. \qed

Lemma 51. Let $G$ be a 2-connected graph with girth at least 11. Then $G$ is not shellable.

Proof. Since $G$ is 2-connected, then $G$ is not a forest. Consequently, if $r$ is the girth of $G$, then there exists a cycle $C = (x_1, x_2, \ldots, x_r)$. If $G = C$, then $G$ is not shellable. Hence, $G \neq C$ implying $D_1(C) \neq \emptyset$. We take $y \in D_1(C)$, without loss of generality we can assume that $\{x_1, y\} \in E(G)$. If $\{x_i, y\} \in E(G)$ for some $i \in \{2, \ldots, r\}$, then we take the cycles $C_1 = (y, x_1, x_2, \ldots, x_i)$ and $C_2 = (y, x_1, x_r, x_{r-1}, \ldots, x_i)$. Thus, $|V(C_1)| = i + 1$ and $|V(C_2)| = r - i + 3$. Since $r$ is the girth of $G$, then $i + 1 \geq r$ and $r - i + 3 \geq r$. Consequently, $3 \geq i$ implies $4 \geq r$. But $r \geq 11$, this is a contradiction. This implies that $|N_G(y) \cap V(C)| = 1$. Now, we suppose that there exist $y_1, y_2 \in D_1(C)$ such that $\{y_1, y_2\} \in E(G)$. We can assume that $\{x_1, y_1\}, \{x_i, y_2\} \in E(G)$. Since $r \geq 11$, then there are no 3-cycles in $G$. In particular, $x_1 \neq x_i$. Now, we take the cycles $C' = (y_1, x_1, \ldots, x_i, y_2)$ and $C'' = (y_1, x_1, x_r, x_{r-1}, \ldots, x_i, y_2)$. So, $|V(C')| = i + 2$ and $|V(C'')| = r - i + 4$. Since $r$ is the girth, we have that $i + 2 \geq r$ and $r - i + 4 \geq r$. Hence, $4 \geq i$ and...
6 ≥ r, this is a contradiction. Then $D_1(C)$ is a stable set. Now, since $G$ is 2-connected, then for each $y ∈ D_1(C)$ there exists $z ∈ N_G(y) ∩ D_2(C)$. If there exist $z_1, z_2 ∈ D_2(C)$ such that $\{z_1, z_2\} ∈ E(G)$, then there exist $y_1, y_2 ∈ D_1(C)$ such that $\{z_1, y_1\}, \{z_2, y_2\} ∈ E(G)$. Since there are no 3-cycles in $G$, we have that $y_1 ≠ y_2$. We can assume that $\{x_1, y_1\}, \{x_i, y_j\} ∈ E(G)$. Since there are no 5-cycles, then $i ≠ 1$. Consequently, there exist cycles $C'_1 = (x_1, \ldots, x_i, y_j, z_1, y_1)$ and $C'_2 = (x_1, \ldots, x_r, x_1, y_1, z_1, z_2, y_j)$. This implies $r ≤ |V(C'_1)| = i + 4$ and $r ≤ |V(C'_2)| = r - i + 6$. Hence, $i ≤ 6$ and $r ≤ 10$, this is a contradiction. Then $D_2(C)$ is a stable set. Furthermore, $C$ is a connected component of $G \setminus N_G[D_2(C)]$. But $C$ is not shellable, therefore $G$ is not shellable, from Remark 26.

**Theorem 52.** If $G$ has girth at least 11, then $G$ is shellable if and only if there exists $x ∈ V(G)$ with $N_G(x) = \{y\}$ such that $G \setminus N_G[x]$ and $G \setminus N_G[y]$ are shellable.

**Proof.** $\Leftarrow$) By ([18], Theorem 2.9).

$\Rightarrow$) By Remark 26, shellability is closed under c-minors. Consequently, it is only necessary to prove that $G$ has a free vertex. If every block of $G$ is an edge or a vertex, then $G$ is a forest and there exists $x ∈ V(G)$ with $\deg_G(x) = 1$. Hence, we can assume that there exists a 2-connected block $B$ of $G$. The girth of $B$ is at least 11, since $B$ is an induced subgraph of $G$. Thus, by Lemma 51, $B$ is not shellable. Therefore, by Theorem 42, there exists $x ∈ D_1(B)$ such that $\deg(x) = 1$.

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**References**


