Lower Bounds for Cover-Free Families

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Abstract

Let $F$ be a set of blocks of a $t$-set $X$. A pair $(X,F)$ is called an $(w,r)$-cover-free family $((w,r)-CFF)$ provided that, the intersection of any $w$ blocks in $F$ is not contained in the union of any other $r$ blocks in $F$.

We give new asymptotic lower bounds for the number of minimum points $t$ in a $(w,r)$-CFF when $w \leq r = |F|^\epsilon$ for some constant $\epsilon \geq 1/2$.

Keywords: Cover-Free Family, Lower Bound.

1 Introduction

Let $F$ be a set of blocks (subsets) of a $t$-set $X$. A pair $(X,F)$ is called a $(w,r)$-cover-free family $((w,r)-CFF)$ provided that, for any $w$ blocks $A_1,A_2,\ldots,A_w \in F$ and any other $r$ blocks $B_1,B_2,\ldots,B_r \in F$ we have

$$\bigcap_{i=1}^w A_i \nsubseteq \bigcup_{j=1}^r B_j.$$ 

Since using De Morgan, a $(w,r)-CFF$ can be turned into $(r,w)-CFF$, throughout the paper we assume that $w \leq r$. Cover-free families were first introduced in 1964 by Kautz and Singleton [5].

Let $N(n,(w,r))$ denote the minimum number of points $|X|$ in any $(w,r)$-CFF having $|F| = n$ blocks. The best known lower bound for $N(n,(1,r))$ is [2, 4, 7]

$$N(n,(1,r)) = \Omega \left( \frac{r^2}{\log r} \log n \right)$$

when $r \leq \sqrt{n}$, and, $\Omega(n)$ when $r > \sqrt{n}$. The constant of the $\Omega()$ is asymptotically $1/2$, $1/4$ and $1/8$, respectively. Stinson et. al. [8], proved that

$$N(n,(w,r)) \geq N(n-1,(w-1,r)) + N(n-1,(w,r-1)).$$  

(2)
They then use it with (1) to prove two bounds. The first bound is

$$N(n,(w,r)) \geq \Omega \left( \frac{(w+r)(w+r)}{\log \left( \frac{w+r}{w} \right)} \log n \right)$$  \hspace{1cm} (3)

when \( r \leq \sqrt{n} \), [8, 6], and

$$N(n,(w,r)) \geq \Omega \left( \frac{(w+r)}{\log (w+r)} \log n \right)$$  \hspace{1cm} (4)

for any \( r \leq n \), [8]. To the best of our knowledge (4) is the best bound known when \( \sqrt{n} \leq r \leq n \). D’yachkov et. al. breakthrough result, [3], implies that for \( r \leq \sqrt{n} \) and \( r, n \to \infty \)

$$N(n,(w,r)) = \Theta \left( \frac{(w+r)(w+r)}{\log \left( \frac{w+r}{w} \right)} \log n \right)$$  \hspace{1cm} (5)

and for \( r \geq \sqrt{n} \) and \( r, n \to \infty \)

$$N(n,(w,r)) \leq O \left( \frac{r}{w} \cdot \frac{(w+r)}{\log (w+r)} \log n \right).$$  \hspace{1cm} (6)

In this paper we give a new lower bound for \((w,r)\)-CFF when \( r > \sqrt{n} \). We combine the two techniques used in [8, 6] and [1] to give the following asymptotic lower bound.

**Theorem 1.** For any \( 2 \leq k \leq w < r \leq n/2 \) we have

$$N(n,(w,r)) \geq \frac{k^w k!}{2(k+1)^{2k}} \cdot \frac{r^{w+1}}{(w+1)! \ln^k r} = \Omega \left( \frac{\sqrt{k}}{e^k} \cdot \frac{r^{w+1}}{(w+1)! \ln^{k+1} r} \log n \right)$$

for

$$ (n+k-1-w)^{\frac{k+1}{k+1}} \leq r \leq (n+k-w)^{\frac{k}{k+1}} $$

and

$$N(n,(w,r)) = \Theta \left( \binom{n}{w} \right)$$

for

$$r = \Omega \left( (n \log n)^{\frac{w}{w+1}} \right).$$

Our bound is

$$\Theta \left( \frac{\sqrt{k} \cdot r}{w(e \ln r)^k} \right).$$
times greater than the previous bound in (4). In particular, when \( k \) is constant, our lower bound improves the bound in (4) to

\[
N(n, (w, r)) \geq \Omega \left( \frac{r}{w \log^k r} \cdot \frac{(w + r)^{w}}{w \log (w + r) \log n} \right).
\]

(7)

A slightly better bound can be achieved when

\[
(n + k - w)^{\frac{k}{k+1}} \leq r \leq (n + k - w)^{\frac{k}{k+1}} \ln^{1/(k+1)} n.
\]

For example, let \( w = 4 \). Table 1 compares our results with the previous results (asymptotic values).

<table>
<thead>
<tr>
<th>( r )</th>
<th>Previous Lower Bounds (3), (4)</th>
<th>Upper Bound [3]</th>
<th>Our Lower Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r \leq n^{1/2} )</td>
<td>( r^4 \log n \log r \log r )</td>
<td>( r^5 \log n \log r \log r )</td>
<td>( r^6 \log n \log^2 r \log r )</td>
</tr>
<tr>
<td>( n^{1/2} \leq r \leq n^{2/3} )</td>
<td>( r^4 \log n \log r \log r )</td>
<td>( r^5 \log n \log r \log r )</td>
<td>( r^6 \log n \log^2 r \log r )</td>
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<tr>
<td>( n^{2/3} \leq r \leq n^{3/4} )</td>
<td>( r^4 \log n \log r \log r )</td>
<td>( r^5 \log n \log r \log r )</td>
<td>( r^6 \log n \log^2 r \log r )</td>
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<td>( n^{3/4} \leq r \leq n^{4/5} )</td>
<td>( r^4 \log n \log r \log r )</td>
<td>( r^5 \log n \log r \log r )</td>
<td>( r^6 \log n \log^2 r \log r )</td>
</tr>
<tr>
<td>( n &gt; r \geq (n \log n)^{4/5} )</td>
<td>( r^4 )</td>
<td>( n^4 )</td>
<td>( n^4 )</td>
</tr>
</tbody>
</table>

Table 1: Results for \( w = 4 \).

## 2 First Lower Bound

In this section, we prove

**Lemma 2.** Let \( w \leq r \leq n/2 \). If

\[
\begin{align*}
  r &= \Omega \left( (n \log n)^{\frac{w}{w+1}} \right) \\
  \end{align*}
\]

then

\[
N(n, (w, r)) = \Theta \left( \binom{n}{w} \right).
\]

(8)

Otherwise,

\[
N(n, (w, r)) \geq \Omega \left( \left( \frac{r}{(w + 1) \ln r} \right)^{w+1} \log n \right).
\]

(9)

Lemma 2 follows from the following.
Lemma 3. Let $\epsilon < 1$ be any constant. For $w \leq r \leq n/2$ we have

$$N(n,(w,r)) \geq \min \left( (1-\epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r}, \epsilon \left( \frac{n}{w} \right) \right). \tag{10}$$

Proof. Let $(X,F)$ be an optimal $(w,r)$-CFF. Let $F = \{F_1, \ldots, F_n\}$,

$$|X| = N = N(n,(w,r))$$

and assume without loss of generality that $X = [N] := \{1, \ldots, N\}$. Define $v^{(i)} \in \{0,1\}^n$, $i = 1, \ldots, N$ where $v^{(i)}_j = 1$ if and only if $i \in F_j$. Let $V = \{v^{(i)} | i = 1, \ldots, N\}$. Let $V_0$ be the set of $v^{(i)}$ of weight $w^t(v^{(i)})$ (i.e., $\sum_j v^{(i)}_j$) equal to $w$. Let

$$m = \frac{(w+1)^2 n \ln r}{wr}$$

and consider the two sets $V_1 = \{v^{(i)} \mid w < w^t(v^{(i)}) < m\}$ and $V_2 = \{v^{(i)} \mid w^t(v^{(i)}) \geq m\}$. Obviously, $V = V_0 \cup V_1 \cup V_2$ is a partition of $V$. Suppose by contradiction that

$$|V_0| \leq \epsilon \left( \frac{n}{w} \right)$$

and

$$\max(|V_1|,|V_2|) \leq (1-\epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r}.$$ 

Consider $W = \{(j_1, \ldots, j_w) \mid 1 \leq j_1 < \cdots < j_w \leq n\}$ and $W' \subset W$ the set of all $(j_1, \ldots, j_w)$ where no $v^{(i)} \in V_0$, $i = 1, \ldots, N$, satisfies $v^{(i)}_{j_1} = \cdots = v^{(i)}_{j_w} = 1$. Obviously,

$$|W'| = \left( \frac{n}{w} \right) - |V_0| \geq (1-\epsilon) \left( \frac{n}{w} \right).$$

Fix an element $v \in V_1$ and randomly and uniformly choose $j = (j_1, \ldots, j_w) \in W'$. We have

$$\Pr_{j \in W'}[v_{j_1} = \cdots = v_{j_w} = 1] \leq \frac{\text{wt}(v)}{|W'|} \leq \frac{m}{(1-\epsilon) \left( \frac{n}{w} \right)}.$$

Therefore, the expectation of the number of $v \in V_1$ for which $v_{j_1} = \cdots = v_{j_w} = 1$ is at most

$$\frac{m}{(1-\epsilon) \left( \frac{n}{w} \right)} |V_1| \leq \frac{1}{1-\epsilon} \left( \frac{m}{n} \right)^w |V_1| \leq \frac{1}{1-\epsilon} \left( \frac{w+1}{w} \right)^{2w} \frac{\ln r}{r} \cdot (1-\epsilon) \left( \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r} \right) = \frac{w^w}{w+1}.$$
Therefore, there is \( j' = (j'_1, \ldots, j'_w) \in W' \) such that the number of \( v \in V_1 \) that satisfies \( v_{j'_1} = \cdots = v_{j'_w} = 1 \) is \( r_1 \leq r/(w+1) \). Since the weight of every \( v \in V_1 \) is greater than \( w \), we can choose \( r_1 \) new entries \( j''_1, \ldots, j''_w \not\in \{j'_1, \ldots, j'_w\} \) such that for every \( v \in V_1 \) where \( v_{j'_1} = \cdots = v_{j'_w} = 1 \) there is \( j''_{\ell} \) such that \( v_{j''_{\ell}} = 1 \).

Now randomly and uniformly choose

\[
 r_2 := \left\lceil \frac{wr}{w+1} \right\rceil
\]

distinct \( k_1, \ldots, k_{r_2} \in [n] \). Let \( A \) be the event that \( \{k_1, \ldots, k_{r_2}\} \cap \{j'_1, \ldots, j'_w\} \neq \emptyset \). The probability that \( A \) does not happen is

\[
\frac{\binom{n-w}{r_2}}{\binom{n}{r_2}} \geq \frac{\binom{n-w}{r_2}}{2^w \binom{n}{r_2}} = 1 - \frac{1}{2^w}.
\]

Then

\[
\Pr[A \cup (\exists v \in V_2) \ v_{k_1} = \cdots = v_{k_{r_2}} = 0] \leq 1 - \frac{1}{2^w} + |V_2| \frac{\binom{n-m}{r_2}}{\binom{n}{r_2}} \\
\leq 1 - \frac{1}{2^w} + |V_2| \left( \frac{n-m}{n} \right)^{r_2} \\
\leq 1 - \frac{1}{2^w} + |V_2| e^{-\frac{mr_2}{w}}
\]

and

\[
|V_2| e^{-\frac{mr_2}{w}} \leq (1 - \epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{p^{w+1}}{\ln^w r} \cdot e^{-(w+1)^2 \ln r} \\
\leq (1 - \epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{p^{w+1}}{\ln^w r} \cdot e^{-(w+1) \ln r} \\
= (1 - \epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{1}{\ln^w r} \\
< \frac{1}{2^w}.
\]

Therefore,

\[
\Pr[A \cup (\exists v \in V_2) \ v_{k_1} = \cdots = v_{k_{r_2}} = 0] < 1.
\]

Therefore, there is \( \{k_1, \ldots, k_{r_2}\} \) such that \( \{k_1, \ldots, k_{r_2}\} \cap \{j'_1, \ldots, j'_w\} = \emptyset \) and for every \( v \in V_2 \) there is \( k_{\ell} \in \{k_1, \ldots, k_{r_2}\} \) where \( v_{k_{\ell}} = 1 \).

Now it is easy to see that there is no \( v \in V \) where \( v_{j'_1} = \cdots = v_{j'_w} = 1 \), \( v_{j''_{\ell}} = \cdots = v_{j''_{w_1}} = 0 \) and \( v_{k_1} = \cdots = v_{k_{r_2}} = 0 \). This implies that

\[
\bigcap_{i=1}^w F_{j'_i} \subseteq \bigcup_{i=1}^{r_1} F_{j''_i} \cup \bigcup_{i=1}^{r_2} F_{k_i},
\]

which is a contradiction.  \( \square \)
3 The Second Bound

In this section we prove Theorem 1.

**Lemma 4.** For any $2 \leq k \leq w \leq r \leq n/2$ and

$$2 \leq r \leq (n + k - w)^{\frac{k}{r + 1}}$$

$$N(n,(w,r)) \geq \frac{k^kk!}{2(k + 1)^{2k}} \frac{r^{w+1}}{(w + 1)! \ln^k r} = \Omega \left( \frac{r^{w+1}}{(w + 1)! \ln^k r} \right).$$

**Proof.** We prove the lemma by induction on $w$.

From Lemma 3 the lemma holds for $w = k$. Now assume the bound holds for some $w$ and every $r$ that satisfies $r \leq (n + k - w)^{\frac{k}{r + 1}}$. We now prove the bound for $w + 1$ and $r \leq (n + k - w - 1)^{\frac{k}{r + 1}}$. We have

$$N(n,(w+1,r)) \geq N(n-1,(w,r)) + N(n-1,(w+1,r-1)) \quad (11)$$

$$\geq \sum_{j=1}^{r} N(n - r + j - 1,(w,j)) \quad (12)$$

$$\geq N(n - r,(w,1)) + \sum_{j=2}^{r} \frac{k^kk!}{2(k + 1)^{2k}} \frac{j^{w+1}}{(w + 1)! \ln^k j} \quad (13)$$

$$\geq \frac{k^kk!}{2(k + 1)^{2k}(w + 1)! \ln^k r} \sum_{j=1}^{r} j^{w+1}$$

$$\geq \frac{k^kk!}{2(k + 1)^{2k}(w + 1)! \ln^k r} \int_{0}^{r} x^{w+1} dx$$

$$\geq \frac{k^kk!}{2(k + 1)^{2k}(w + 2)! \ln^k r} r^{w+2}.$$

Here, inequality (11) comes from [8]. Inequality (12) follows from the fact that $N(n - r + 1,(w+1,1)) \geq N(n - r,(w,1))$. Inequality (13) follows from the induction hypothesis since

$$j = r - (r - j)$$

$$\leq (n + k - w - 1)^{\frac{k}{r + 1}} - (r - j)$$

$$\leq (n + k - w - 1 - (r - j))^{\frac{k}{r + 1}}$$

$$= ((n - r + j - 1) + k - w)^{\frac{k}{r + 1}}. \quad \square$$

**Lemma 5.** Let $w \leq r \leq n/2$. If

$$r = \Omega \left( (n \log n)^{\frac{w}{w + 1}} \right)$$

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then

\[ N(n, (w, r)) = \Theta \left( \binom{n}{w} \right). \]

**Proof.** Let \( r > c(n \log n) \frac{w}{w+1} \) for large enough constant \( c > 2e \) and \( c' = c^{(w+1)/w} \). Since

\[
\frac{1}{2} \frac{w^w}{(w + 1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r} \geq 1 \frac{w^w}{2 (w + 1)^{2w+1}} \cdot \frac{c^{w+1} n^w \log^w n}{(\frac{w}{w+1})^w \ln^w (c'n \log n)}
\]

\[
\geq 1 \frac{1}{2 (w + 1) \left( 1 + \frac{1}{w} \right)^w} \cdot \frac{c^{w+1} n^w \ln^w n}{w^w \ln^w (n^2)}
\]

\[
\geq \frac{c}{2e} \frac{1}{(w + 1)} \left( \frac{c}{2e} \right)^w \frac{n}{w} \geq \binom{n}{w},
\]

by Lemma 3, the result follows. \( \square \)

**References**


