4-Factor-criticality of vertex-transitive graphs

Wuyang Sun*  
Center for Discrete Mathematics  
Fuzhou University  
Fuzhou, Fujian 350108, China  
swywuyang@163.com

Heping Zhang  
School of Mathematics and Statistics  
Lanzhou University  
Lanzhou, Gansu 730000, China  
zhanghp@lzu.edu.cn

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Abstract

A graph of order \( n \) is \( p \)-factor-critical, where \( p \) is an integer of the same parity as \( n \), if the removal of any set of \( p \) vertices results in a graph with a perfect matching. 1-factor-critical graphs and 2-factor-critical graphs are well-known factor-critical graphs and bicritical graphs, respectively. It is known that if a connected vertex-transitive graph has odd order, then it is factor-critical, otherwise it is elementary bipartite or bicritical. In this paper, we show that a connected vertex-transitive non-bipartite graph of even order at least 6 is 4-factor-critical if and only if its degree is at least 5. This result implies that each connected non-bipartite Cayley graph of even order and degree at least 5 is 2-extendable.

Keywords: Vertex-transitive graph; 4-Factor-criticality; Matching; Connectivity

1 Introduction

Only finite and simple graphs are considered in this paper. A matching of a graph is a set of its mutually nonadjacent edges. A perfect matching of a graph is a matching covering all its vertices. A graph is called factor-critical if the removal of any one of its vertices results in a graph with a perfect matching. A graph is called bicritical if the removal of any pair of its distinct vertices results in a graph with a perfect matching. The concepts of factor-critical and bicritical graphs were introduced by Gallai [9] and by Lovász [12], respectively. In matching theory, factor-critical graphs and bicritical graphs are two basic bricks in matching structures of graphs [17]. Later on, the two concepts were generalized to the concept of \( p \)-factor-critical graphs by Favaron [7] and Yu [21], independently. A graph of order \( n \) is said to be \( p \)-factor-critical, where \( p \) is an integer of the same parity as \( n \), if the removal of any \( p \) vertices results in a graph with a perfect matching.

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q-extendable graphs was proposed by Plummer [17] in 1980. A connected graph of even order $n$ is \textit{q-extendable}, where $q$ is an integer with $0 \leq q < n/2$, if it has a perfect matching and every matching of size $q$ is contained in one of its perfect matchings. Favaron [8] showed that for $q$ even, every connected non-bipartite $q$-extendable graph is $q$-factor-critical. In 1993 Yu [21] introduced an analogous concept for graphs of odd order. A connected graph of odd order is $q_{1/2}$-extendable, if the removal of any one of its vertices results in a $q$-extendable graph.

A graph $G$ is said to be \textit{vertex-transitive} if for any two vertices $x$ and $y$ in $G$ there is an automorphism $\varphi$ of $G$ such that $y = \varphi(x)$. A graph with a perfect matching is \textit{elementary} if the union of its all perfect matchings forms a connected subgraphs. In [13], there is a following classic result about the factor-criticality and bicriticality of vertex-transitive graphs.

\textbf{Theorem 1} ([13]). \textit{Let $G$ be a connected vertex-transitive graph of order $n$. Then 
(a) $G$ is factor-critical if $n$ is odd; 
(b) $G$ is either elementary bipartite or bicritical if $n$ is even.}

A question arises naturally: Does a vertex-transitive non-bipartite graph has larger $p$-factor-criticality?

In fact, the $q$-extendability and $q_{1/2}$-extendability of Cayley graphs, an important class of vertex-transitive graphs, have been investigated in literature. It was proved in papers [3, 4, 16] that a connected Cayley graph of even order on an abelian group, a dihedral group or a generalized dihedral group is 2-extendable except for several circulant graphs of degree at most 4. Miklavčič and Šparl [16] also showed that a connected Cayley graph on an abelian group of odd order $n \geq 3$ either is a cycle or is $1_{1/2}$-extendable. Chan et al. [3] raised the problem of characterizing 2-extendable Cayley graphs.

In [22], we showed that a connected vertex-transitive graph of odd order $n \geq 3$ is 3-factor-critical if and only if it is not a cycle. This general result is stronger than $1_{1/2}$-extendability of Cayley graphs. In this paper, we obtain the following main result which gives a simple characterization of 4-factor-critical vertex-transitive non-bipartite graphs.

\textbf{Theorem 2.} \textit{Let $G$ be a connected vertex-transitive non-bipartite graph of degree $k$ and of even order at least 6. Then $G$ is 4-factor-critical if and only if $k \geq 5$.}

By Theorem 2, we know that all connected non-bipartite Cayley graphs of even order and of degree at least 5 is 2-extendable.

The necessity of Theorem 2 is clear. Our main task is to show the sufficiency of Theorem 2 by contradiction. Suppose that $G$ is a connected non-bipartite vertex-transitive graph $G$ of even order at least 6 and of degree at least 5 but $G$ is not 4-factor-critical. By the $s$-restricted edge-connectivity of $G$, we find that in most cases a suitable integer $s$ can be chosen such that every $\lambda_s$-atom of $G$ is an imprimitive block. Then we can deduce contradictions. Some preliminary results are presented in Section 2 and some properties of $\lambda_s$-atoms of $G$ which are used to show their imprimitivity are proved in Section 3. Finally, we complete the proof of Theorem 2 in Section 4.
2 Preliminaries

In this section, we introduce some notations and results needed in this paper.

Let $G = (V(G), E(G))$ be a graph. For $X \subseteq V(G)$, let $\overline{X} = V(G) \backslash X$. For $Y \subseteq \overline{X}$, denote by $[X, Y]$ the set of edges of $G$ with one end in $X$ and the other in $Y$. In particular, we denote $[X, \overline{X}]$ by $\nabla(X)$ and denote $|\nabla(X)|$ by $d_G(X)$. Denote by $N_G(v)$ the set of vertices in $\overline{X}$ which are ends of some edges in $\nabla(X)$. If $X = \{v\}$, then $X$ is usually written to $v$. Vertices in $N_G(v)$ are called the neighbors of $v$. If no confusion exists, the subscript $G$ are usually omitted. Denote by $G[X]$ the subgraph induced by $X$ and denote by $G - X$ the subgraph induced by $\overline{X}$. The set of edges in $G[X]$ is denoted by $E(X)$. Denote by $c_0(G)$ the number of the components of $G$ which have odd order. For a subgraph $H$ of $G$, we denote $d_G(V(H_i))$ and $\nabla(V(H_i))$ by $d_G(H_i)$ and $\nabla(H_i)$, respectively.

For a connected graph $G$, a subset $F \subseteq E(G)$ is said to be an edge-cut of $G$ if $G - F$ is disconnected, where $G - F$ is the graph with vertex-set $V(G)$ and edge-set $E(G) \backslash F$. The edge-connectivity of $G$ is the minimum cardinality over all the edge-cuts of $G$, denoted by $\lambda(G)$. A subset $X \subseteq V(G)$ is called a vertex-cut of $G$ if $G - X$ is disconnected. The vertex-connectivity of $G$ of order $n$, denoted by $\kappa(G)$, is $n - 1$ if $G$ is the complete graph $K_n$ and is the minimum cardinality over all the vertex-cuts of $G$ otherwise. It is well known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum vertex-degree of $G$.

There are two properties of $p$-factor-critical graphs.

**Theorem 3** ([7, 21]). A graph $G$ is $p$-factor-critical if and only if $c_0(G - X) \leq |X| - p$ for all $X \subseteq V(G)$ with $|X| \geq p$.

**Theorem 4** ([7]). If a graph $G$ is $p$-factor-critical with $1 \leq p < |V(G)|$, then $\kappa(G) \geq p$ and $\lambda(G) \geq p + 1$.

Let $X$ be a subset of $V(G)$. Denoted by $\mathfrak{C}_{G - X}$ the set of the components of $G - X$. $X$ is called to be matchable to $\mathfrak{C}_{G - X}$ if the bipartite graph $G_X$, which arises from $G$ by contracting the components in $\mathfrak{C}_{G - X}$ to single vertices and deleting all the edges in $E(X)$, contains a matching covering $X$. The following general result will be used.

**Theorem 5** ([5]). Every graph $G$ contains a set $X$ of vertices with the following properties:
(a) $X$ is matchable to $\mathfrak{C}_{G - X}$;
(b) Every component of $G - X$ is factor-critical.

Given any such set $X$, the graph $G$ contains a perfect matching if and only if $|X| = |\mathfrak{C}_{G - X}|$.

The girth of a graph $G$ with a cycle is the length of a shortest cycle in $G$ and the odd girth of a non-bipartite graph $G$ is the length of a shortest odd cycle in $G$. The girth and odd girth of $G$ are denoted by $g(G)$ and $g_0(G)$, respectively. $l$-cycle means a cycle of length $l$. We present two useful lemmas as follows.

**Lemma 6** ([15]). Let $G$ be a graph with $g_0(G) > 3$. Then $|E(G)| \leq \frac{1}{4}|V(G)|^2$.

**Lemma 7** ([1]). Let $G$ be a $k$-regular graph. If $g_0(G) > 3$, then $|V(G)| \geq kg_0(G)/2$.

Now we list some useful properties of vertex-transitive graphs as follows.
Theorem 8 ([14]). Let $G$ be a connected vertex-transitive $k$-regular graph. Then $\lambda(G) = k$.

Theorem 9 ([19]). Let $G$ be a connected vertex-transitive $k$-regular graph. Then $\kappa(G) > \frac{2}{3}k$.

Lemma 10 ([19]). Let $G$ be a connected vertex-transitive $k$-regular graph. If $\kappa(G) < k$, then $\kappa(G) = m\tau(G)$ for some integer $m \geq 2$, where

\[ \tau(G) = \min\{\min\{|V(P)| : P \text{ is a component of } G - X\} : X \text{ is a minimum vertex-cut of } G\} \]

Lemma 11 ([19]). Let $G$ be a connected vertex-transitive $k$-regular graph with $k = 4$ or $6$. Then $\kappa(G) = k$.

An imprimitive block of $G$ is a proper non-empty subset $X$ of $V(G)$ such that for any automorphism $\varphi$ of $G$, either $\varphi(X) = X$ or $\varphi(X) \cap X = \emptyset$.

Lemma 12 ([18]). Let $G$ be a vertex-transitive graph and $X$ be an imprimitive block of $G$. Then $G[X]$ is also vertex-transitive.

Theorem 13 ([10]). Let $G$ be a connected vertex-transitive $k$-regular graph of order $n$. Let $S$ be a subset of $V(G)$ chosen such that $\frac{1}{2}(k + 1) \leq |S| \leq \frac{1}{2}n$, $d(S)$ is as small as possible, and, subject to these conditions, $|S|$ is as small as possible. If $d(S) < \frac{2}{3}(k + 1)^2$, then $S$ is an imprimitive block of $G$.

Corollary 14 ([10]). Let $G$ be a connected vertex-transitive $k$-regular graph of order $n$. Let $S$ be a subset of $V(G)$ chosen such that $1 < |S| \leq \frac{1}{2}n$, $d_G(S)$ is as small as possible, and, subject to these conditions, $|S|$ is as small as possible. If $d_G(S) < 2(k - 1)$, then $d_G(S) = |S| \geq k$ and $d_G[S](v) = k - 1$ for all $v \in S$.

Corollary 15. Let $G$ be a connected vertex-transitive $k$-regular graph. Suppose $g(G) > 3$ or $|V(G)| < 2k$. Then $d_G(X) \geq 2k - 2$ for every $X \subseteq V(G)$ with $2 \leq |X| \leq |V(G)| - 2$.

Proof. If $k = 2$, then it is trivial. Now suppose $k \geq 3$ and that there is a subset $X \subseteq V(G)$ with $2 \leq |X| \leq |V(G)| - 2$ such that $d_G(X) < 2k - 2$. Let $S$ be a subset of $V(G)$ chosen such that $1 < |S| \leq \frac{1}{2}|V(G)|$, $d_G(S)$ is as small as possible, and, subject to these conditions, $|S|$ is as small as possible. Then $d_G(S) \leq d_G(X) < 2k - 2$. By Corollary 14, $d_G(S) = |S| \geq k$ and $d_G[S](v) = k - 1$ for all $v \in S$. As $2k - 3 < \frac{3}{2}(k + 1)^2$, $S$ is an imprimitive block of $G$ by Theorem 13. Then $|S|$ is a divisor of $|V(G)|$, which implies $|V(G)| \geq 2|S| \geq 2k$. Thus $g(G) > 3$. Noting that $|E(S)| = \frac{1}{2}(k - 1)|S| \leq \frac{1}{4}|S|^2$ by Lemma 6, we have $d_G(S) = |S| \geq 2k - 2$, a contradiction.

A subset $X$ of $V(G)$ is called an independent set of $G$ if any two vertices in $X$ are not adjacent. The maximum cardinality of independent sets of $G$ is the independent number of $G$, denoted by $\alpha(G)$. 

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Lemma 16. Let $G$ be a non-bipartite vertex-transitive $k$-regular graph. Then $\alpha(G) \leq \frac{1}{2}|V(G)| - \frac{k}{4}$ if $g_0(G) > 3$, and $\alpha(G) \leq \frac{1}{2}|V(G)|$ if $g_0(G) = 3$.

Proof. Let $X$ be a maximum independent set of $G$ and set $g_0 := g_0(G)$. Noting that $G$ is regular and non-bipartite, we have $|X| < |X|$. Set $t = |X| - |X|$. Since $G$ is vertex-transitive, the number of $g_0$-cycles of $G$ containing any given vertex in $G$ is constant. Let $q$ be this constant number and let $m$ be the number of all the $g_0$-cycles of $G$. Note that each $g_0$-cycle of $G$ contains at most $(g_0 - 1)/2$ vertices in $X$ and at least $(g_0 + 1)/2$ vertices in $X$. We have $q|X| \leq \frac{1}{2}m(g_0 - 1)$ and $q|X| \geq \frac{1}{2}m(g_0 + 1)$, which implies $qt = q(|X| - |X|) \geq m$.

We know $q|V(G)| = mg_0$ by the vertex-transitivity of $G$. Then $qt \geq m = q\frac{2}{g_0}|V(G)|$, implying $t \geq \frac{|V(G)|}{g_0}$. If $g_0 = 3$, then $\alpha(G) = \frac{1}{2}(|V(G)| - t) \leq \frac{1}{3}|V(G)|$. If $g_0 > 3$, then $|V(G)| \geq kg_0/2$ by Lemma 7, which implies $\alpha(G) = \frac{1}{2}(|V(G)| - t) \leq \frac{1}{2}|V(G)| - \frac{k}{4}$.

A graph $G$ is called trivial if $|V(G)| = 1$.

Lemma 17. Let $G$ be a connected vertex-transitive non-bipartite graph. Let $X$ be an independent set of $G$. Suppose that $G - X$ has $|X| - t$ trivial components, where $t$ is a positive integer. Then $g_0(G) \geq \frac{2|X|}{t} + 1$.

Proof. Let $Y$ be the set of vertices in the trivial components of $G - X$ and set $g_0 := g_0(G)$. Let $n_{i,j}$ be the number of $g_0$-cycles of $G$ which contain exactly $i$ vertices in $X$ and $j$ vertices in $Y$. Set $s = \frac{1}{2}(g_0 - 1)$. Since $X$ and $Y$ are independent sets of $G$, each $g_0$-cycle of $G$ contains at most $s$ vertices in $X$ and contains less vertices in $Y$ than in $X$. Let $q$ be the number of $g_0$-cycles of $G$ containing any given vertex in $G$. We have $\sum_{0 \leq j < i \leq s} in_{i,j} = q|X|$ and $\sum_{0 \leq j < i \leq s} jn_{i,j} = q|Y| = q(|X| - t)$. Then $q|X| = \sum_{0 \leq j < i \leq s} in_{i,j} \leq \sum_{0 \leq j < i \leq s} s(i - j)n_{i,j} = s(\sum_{0 \leq j < i \leq s} in_{i,j} - \sum_{0 \leq j < i \leq s} jn_{i,j}) = sqt = s\frac{1}{2}(g_0 - 1)qt$, which implies $g_0 \geq \frac{2|X|}{t} + 1$.

Lemma 18. Let $G$ be a vertex-transitive graph with a triangle. Then the number of trivial components of $G - X$ is not larger than $|E(X)|$ for each subset $X \subseteq V(G)$.

Proof. Let $Y$ be the set of vertices in the trivial components of $G - X$. Suppose $|Y| > |E(X)|$. Let $q$ be the number of triangles of $G$ containing any given vertex in $G$. Note that there are $q|Y|$ triangles of $G$ containing vertices in $Y$. As $|Y| > |E(X)|$, it implies that $G[X]$ has an edge $e$ which is contained in more than $q$ triangles. This means that more than $q$ triangles containing both ends of $e$, a contradiction.

Lemma 19. Let $G$ be a connected triangle-free vertex-transitive 6-regular graph of even order. Suppose that there are 3 distinct vertices with the same neighbors. Then $G$ is bipartite.

Proof. Suppose, to the contrary, that $G$ is non-bipartite. Then $g_0 := g_0(G) \geq 5$. Let $C = u_0u_1 \ldots u_{g_0-1}u_0$ be a $g_0$-cycle of $G$. For any pair of vertices $u$ and $v$ in $V(C)$, we know $N(u) \neq N(v)$. So for each $u_i \in V(C)$ there are two distinct vertices $u_i'$ and $u_i''$ in $V(C)$ such that $N(u_i) = N(u_i') = N(u_i'')$ by the vertex-transitivity of $G$. Set
Let $U_i = \{u_i, u'_i, u''_i\}$. Then $U_i$ is an independent set of $G$ and $U_i \cap U_j = \emptyset$ for $j \neq i$. Noting that $u_i$ and $u_{i+1}$ are adjacent, we have that every vertex in $U_i$ is adjacent to every vertex in $U_{i+1}$, where $i + 1$ is an arithmetic on modular $g_0$. Since $G$ is 6-regular and connected, $|V(G)| = |\bigcup_{i=0}^{g_0-1} U_i| = 3g_0$, which implies that $|V(G)|$ is odd, a contradiction.\qed

\section{$\lambda_s$-atoms of vertex-transitive graphs}

In this section, we will introduce some properties of the $\lambda_s$-atoms of vertex-transitive graphs. The concept of $\lambda_s$-atoms [11, 20] of graphs is used in investigating the $s$-restricted edge-connectivity of graphs. The $s$-restricted edge-connectivity of graphs was proposed by Fàbrega and Fiol [6].

For a connected graph $G$ and some positive integer $s$, an edge-cut $F$ of $G$ is said to be an $s$-restricted edge-cut of $G$ if every component of $G - F$ has at least $s$ vertices. The minimum cardinality of $s$-restricted edge-cuts of $G$ is the $s$-restricted edge-connectivity of $G$, denoted by $\lambda_s(G)$. By the definition of $\lambda_s(G)$, we can see that $\lambda(G) = \lambda_1(G) \leq \lambda_2(G) \leq \lambda_3(G) \cdots$ as long as these parameters exists.

A proper subset $X$ of $V(G)$ is called a $\lambda_s$-fragment of $G$ if $\nabla(X)$ is an $s$-restricted edge-cut of $G$ with minimum cardinality. We can see that for every $\lambda_s$-fragment $X$ of $G$, $G[X]$ and $G[X]$ are connected graphs of order at least $s$. A $\lambda_s$-fragment of $G$ with minimum cardinality is called a $\lambda_s$-atom of $G$.

Lemma 20. Let $G$ be a connected triangle-free vertex-transitive graph of degree $k \geq 5$. For an integer $s$ with $4 \leq s \leq 8$, suppose $\lambda_s(G) \leq 3k$. Let $S$ be a $\lambda_s$-atom of $G$.

(a) For $X \subseteq V(G)$ with $|X| \geq s$ and $|\overline{X}| \geq s$, we have $d_G(X) \geq \lambda_s(G)$. Furthermore, $d_G(X) > \lambda_s(G)$ if $G[X]$ or $G[\overline{X}]$ is disconnected.

(b) For $A \subseteq S$ with $1 \leq |A| \leq |S| - s$, we have $d_{G[S]}(A) > \frac{1}{2}d_G(A)$.

c) For each $\lambda_s$-atom $T$ of $G$ with $S \neq T$ and $S \cap T \neq \emptyset$, we have $d_G(S \cap T) + d_G(S \cup T) \leq 2\lambda_s(G)$, $d_G(S \cap T) + d_G(T \setminus S) \leq 2\lambda_s(G)$, $|S \cap T| \leq s - 1$ and $|S \setminus T| \leq s - 1$.

Proof. (a) If $G[X]$ and $G[\overline{X}]$ are connected, then $\nabla(X)$ is an $s$-restricted edge-cut of $G$ and hence $d_G(X) > \lambda_s(G)$. Thus it only needs to show $d_G(X) > \lambda_s(G)$ if $G[X]$ or $G[\overline{X}]$ is disconnected.

Suppose that $G[X]$ is disconnected. If each component of $G[X]$ has less than 4 vertices, then $d_G(X) = k|X| - 2|E(X)| \geq k|X| - 2(|X| - 2) \geq (k - 2)s + 4 > 3k > \lambda_s(G)$. Then we assume that $G[X]$ has a component $H_1$ with at least 4 vertices. If each component of $G[\overline{V}(H_1)]$ has less than 4 vertices, then $d_G(X) > d_G(H_1) = d_G(\overline{V}(H_1)) > \lambda_s(G)$. Then we assume further that $G[\overline{V}(H_1)]$ has a component $H_2$ with at least 4 vertices. Noting that both $G$ and $H_1$ are connected, we have that $G[\overline{V}(H_2)]$ is connected, which implies that $\nabla(H_2)$ is a 4-restricted edge-cut of $G$. Noting that $\lambda(G) = k$ by Theorem 8, we have $d_G(X) > \lambda(G) + d_G(H_1) \geq k + d(V(H_2)) > k + \lambda_4(G)$.

So $d(X) > \lambda_4(G)$. Next we consider the case that $5 \leq s \leq 8$. Set $\tau_s(G) = \min\{d(A) : A \subseteq V(G), 4 \leq |A| \leq s - 1\}$. Then $\lambda_4(G) \geq \min\{\lambda_s(G), \tau_s(G)\}$. For each subset $A \subseteq V(G)$ with $4 \leq |A| \leq 7$, noting that $|E(A)| \leq \frac{1}{2}|A|^2$ by Lemma 6, we have $d(A) = k|A| - 2|E(A)| \geq k|A| - \frac{1}{2}|A|^2 > 2k$. Hence $\tau_s(G) > 2k$. If $\lambda_s(G) > 2k$, then $d(X) > \lambda_s(G)$.
Proof. Suppose, to the contrary, that vertex-transitive graph of degree \( k \). If \( \lambda_s(G) \leq 2k \), then, noting \( \min\{\lambda_s(G), \tau_s(G)\} \leq \lambda_4(G) \leq \lambda_s(G) \), we have \( d(X) \geq k + \lambda_4(G) = k + \lambda_s(G) > \lambda_s(G) \).

(b) To the contrary, suppose \( d_{G[S]}(A) \leq \frac{1}{2}d_{G}(A) \). Then \( d_{G}(S \setminus A) = d_{G}(S) - (d_{G}(A) - 2d_{G[S]}(A)) \leq d_{G}(S) = \lambda_s(G) \). By (a), \( G[S \setminus A] \) and \( G[\overline{S} \cup A] \) are connected. Hence \( \nabla(S \setminus A) \) is an s-restricted edge-cut of \( G \). By the minimality of \( \lambda_s \)-atoms of \( G \), we have \( d_{G}(S \setminus A) > \lambda_s(G) \), a contradiction.

(c) By the well-known submodular inequality (see [2] for example), we have that \( d_{G}(S \cap T) + d_{G}(S \cup T) \leq d_{G}(S) + d_{G}(T) = 2\lambda_s(G) \) and \( d_{G}(S \cap T) + d_{G}(T \setminus S) = d_{G}(S \cap \overline{T}) + d_{G}(S \cup \overline{T}) \leq d_{G}(S) + d_{G}(\overline{T}) = 2\lambda_s(G) \). Next we show \( |S \cap T| \leq s - 1 \) and \( |S \setminus T| \leq s - 1 \). Clearly, they hold if \( |S| = s \). So we may assume \( |S| > s \).

Suppose \( |S \cap T| \geq s \). Then \( d_{G}(S \cap T) = d_{G}(S) + 2d_{G[S]}(S \setminus T) - d_{G}(S \setminus T) > d_{G}(S) = \lambda_s(G) \) by (b). Noting \( |S \cup T| \geq |V(G)| - |S| - |T| + |S \cap T| \geq s \), we have \( d_{G}(S \cup T) \geq \lambda_s(G) \) by (a). Hence \( d_{G}(S \cap T) + d_{G}(S \cup T) > 2\lambda_s(G) \), a contradiction. Thus \( |S \cap T| \leq s - 1 \).

If \( |S \cap T| = |T \setminus S| \geq s \), then we can similarly obtain \( d_{G}(S \cap T) > \lambda_s(G) \) and \( d_{G}(T \setminus S) > \lambda_s(G) \) by (b), which implies \( d_{G}(S \cap T) + d_{G}(T \setminus S) > 2\lambda_s(G) \), a contradiction. Thus \( |S \setminus T| \leq s - 1 \).

Lemma 21. Let \( G \) be a connected triangle-free vertex-transitive 5-regular graph of even order. For \( s = 5 \) or \( 6 \), suppose \( \lambda_s(G) = s + 9 \). Then \( |S| \geq s + 5 \) for a \( \lambda_s \)-atom \( S \) of \( G \).

Proof. Suppose, to the contrary, that \( |S| < s + 5 \). As \( s + 9 = d_{G}(S) = 5|S| - 2|E(S)| \), \( |S| \) and \( s \) have different parities. Hence \( |S| \geq s + 1 \). By Lemma 20(b), \( \delta(G[S]) \geq 3 \). If \( |S| = s + 1 \), then \( 2|E(S)| \geq \delta(G[S]) |S| \geq 3 |S| \), which implies \( d_{G}(S) = 5|S| - 2|E(S)| \leq 2|S| = 2s + 2 < s + 9 \), a contradiction. Thus \( |S| = s + 3 \). Let \( R \) be the set of vertices \( u \) in \( S \) with \( d_{G[S]}(u) = 3 \). By Lemma 20(b), \( E(R) = \emptyset \). Noting \( 3s + 9 \leq \sum_{u \in S} d_{G[S]}(u) = 2|E(S)| = 5|S| - \lambda_s(G) = 4s + 6 \), we have \( |R| \geq |S| - (4s + 6 - 3s - 9) = 6 \). Since \( s = 5 \) or \( 6 \), \( d_{G[S]}(R) = 3|R| \geq 18 > 5(s - 3) \geq d_{G[S]}(S \setminus R) \), a contradiction.

Lemma 22. Let \( G \) be a bicritical graph. If \( G \) is not 4-factor-critical, then there is a subset \( X \subseteq V(G) \) with \( |X| \geq 4 \) such that \( c_0(G - X) = |X| - 2 \) and every component of \( G - X \) is factor-critical.

Proof. Since \( G \) is not 4-factor-critical, there is a set \( X_1 \) of \( k \) vertices of \( G \) such that \( G - X_1 \) has no perfect matchings. By Theorem 5, \( G - X_1 \) has a vertex set \( X_2 \) such that \( X_2 \) is matchable to \( \mathcal{C}_{G - X_1 - X_2} \) and every component of \( G - X_1 - X_2 \) is factor-critical. Set \( X = X_1 \cup X_2 \). Then \( c_0(G - X) = |\mathcal{C}_{G - X}| - |X_2| = |X| - 4 \). Since \( G \) is bicritical, we have \( c_0(G - X) \leq |X| - 2 \) by Theorem 3. Hence \( |X| - 4 < c_0(G - X) \leq |X| - 2 \). Noting that \( c_0(G - X) \) and \( |X| \) have the same parity, we have \( c_0(G - X) = |X| - 2 \).

In the rest of this section, we always suppose that \( G \) is a connected non-bipartite vertex-transitive graph of degree \( k \geq 5 \) and even order, but \( G \) is not 4-factor-critical. Also we always use the following notation. Let \( X \) be a subset of \( V(G) \) with \( |X| \geq 4 \) such that \( c_0(G - X) = |X| - 2 \) and every component of \( G - X \) is factor-critical. By Theorem 1 and Lemma 22, such subset \( X \) exists. Let \( H = H_1, H_2, \ldots, H_p \) be the nontrivial components of \( G - X \). For a positive integer \( m \), let \( [m] \) denote the set \( \{1, 2, \ldots, m\} \).
Lemma 23. We have $p \geq 1$. Furthermore, if $g(G) > 3$, then

(a) $p = 1$ if $\lambda_5(G) > 2k$,

(b) $|X| \geq 7$ and $|V(H)| \geq 9$ if $\lambda_5(G) > 4k - 8$ and $5 \leq k \leq 6$, and

(c) $|X| \geq 10$ and $|V(H)| \geq 15$ if $\lambda_6(G) \geq 14$ and $k = 5$.

Proof. If $p = 0$, then \(|V(G)| = 2|X| - 2 \geq 2k - 2 \geq 8\) and $\alpha(G) \geq |X| = \frac{1}{2}|V(G)| - 1 > \max\{\frac{1}{3}|V(G)|, \frac{1}{2}|V(G)| - \frac{1}{2}\}$, which contradicts Lemma 16. Thus $p \geq 1$.

Next we suppose $g(G) > 3$. For each $i \in [p]$, we have $|V(H_i)| \geq 5$ as $H_i$ is triangle-free and factor-critical.

Suppose $\lambda_5(G) > 2k$. By Lemma 20(a), $d(H_i) \geq \lambda_5(G)$ for each $i \in [p]$. We have $2pk < p\lambda_5(G) \leq \sum_{i=1}^{p} d(H_i) = d(X) - k(\alpha(G) - X) - p) \leq k(p + 2)$, which implies $p < 2$.

Thus $p = 1$. (a) is proved.

Suppose $\lambda_5(G) > 4k - 8$ and $5 \leq k \leq 6$. We know $p = 1$ by (a). Assume $k = 6$. Notice that $G$ is non-bipartite. It follows from Lemma 19 that $|X| \geq 7$. As $d(H) \leq 3k$ and $H$ is triangle-free and factor-critical, we have $|V(H)| \geq 9$. Assume next $k = 5$. Notice that $|V(G)| = |V(H)| + 2|X| - 3 \geq 12$. By Lemma 20(a), $d(A) \geq \lambda_5(G) > 12$ for every subset $A \subseteq V(G)$ with $|A| = 6$, which implies that $G$ has no subgraphs which are isomorphic to the complete bipartite graph $K_{3,3}$. By the vertex-transitivity of $G$, it follows that $G$ has also no subgraphs which are isomorphic to $K_{2,5}$. So $|X| \geq 7$. If $E(X) = \emptyset$, then $g_0(G) \geq 7$ by Lemma 17, which implies $|V(H)| \geq 13$. If $E(X) \neq \emptyset$, then $d(H) = 13$, which implies $|V(H)| \geq 9$. Hence the statement (b) holds.

Now we suppose $\lambda_6(G) \geq 14$ and $k = 5$. Then $\lambda_5(G) \geq \min\{\lambda_6(G), 5k - 12\} = 13$. We know $p = 1$ by (a). By the above argument, we know $|X| \geq 7$, $|V(H)| \geq 9$ and that $G$ has no subgraphs which are isomorphic to $K_{2,5}$ or $K_{3,3}$. By Lemma 20(a), $d(V(H) \cup A) \geq \lambda_6(G)$ and $d(V(H) \setminus A) \geq \lambda_6(G)$ for every subset $A \subseteq V(G)$ with $|A| \leq 2$. It implies that $E(X) = \emptyset$, $|\nabla(v) \cap \nabla(H)| \leq 3$ for each $u \in V(G)$ and each of $X$ and $V(H)$ has at most one vertex $v$ with $|\nabla(v) \cap \nabla(H)| = 3$. Set $Y = V(H) \cup X$.

Suppose $|X| = 7$. Then $X$ has one vertex $u_1$ with 3 neighbors in $V(H)$ and other vertices in $X$ has exactly two neighbors in $V(H)$. Choose a vertex $u_2 \in X \setminus \{u_1\}$ and a vertex $u_3 \in Y \setminus N(u_1)$. Since $G$ is vertex-transitive, there is an automorphism $\varphi_1$ of $G$ such that $\varphi_1(u_3) = u_2$. Noting that $|N(v) \cap N(u_2)| \geq 3$ for each $v \in Y$, we have $\varphi_1(Y) \subseteq X$, which implies $|\nabla(v) \cap \nabla(H)| = 3$ for each $v \in N(u_2) \cap V(H)$, a contradiction.

Suppose $8 \leq |X| \leq 9$. Then there are two vertices $u_4$ and $u_5$ in $X$ with $|N(u_4) \cap V(H)| = 2$ and $|N(u_5) \cap V(H)| \leq 1$. Since $G$ is vertex-transitive, there is an automorphism $\varphi_2$ of $G$ such that $\varphi_2(u_3) = u_4$. Then $\varphi_2(Y) \cap V(H) \neq \emptyset$ and $|\varphi_2(Y) \cap Y| = 2$. As $G$ has no subgraphs which are isomorphic to $K_{2,5}$ or $K_{3,3}$, it follows that $|\varphi_2(X) \cap X| \geq 6$. Hence $\varphi_2(Y) \subseteq V(H) \cup Y$ and $\varphi_2(X) \subseteq V(H) \cup X$. Noting that $|\nabla(u) \cap \nabla(H)| \leq 3$ and $N(u) \subseteq \varphi_2(X)$ for each $u \in \varphi_2(Y) \cap V(H)$, we have $|\varphi_2(X) \cap V(H)| = 3$. Notice that each of $X$ and $V(H)$ has at most one vertex $v$ with $|\nabla(v) \cap \nabla(H)| = 3$. We know $d_{G[\varphi_2(X) \cup Y]}(\varphi_2(X) \cap V(H)) \geq 3$, which implies $|\varphi_2(Y) \cap V(H)| \geq 3$. It follows that $N_H(\varphi_2(Y) \cap V(H)) \geq 3$. Now we have $|\varphi_2(X) \cap X| = 6$ and $|\varphi_2(X) \cap V(H)| = 3$ as $|\varphi_2(X)| = |X| \leq 9$. It follows that $G[\varphi_2(X \cup Y) \cap V(H)]$ contains a subgraph isomorphic to $K_{3,3}$ if $|\varphi_2(Y) \cap V(H)| \geq 4$ and $G[\varphi_2(X \cup Y) \cap V(H)]$ contains a subgraph isomorphic to $K_{3,3}$ otherwise, a contradiction.
Thus $|X| \geq 10$. Then $g_0(G) \geq 9$ by Lemma 17. Let $C$ be a $g_0(H)$-cycle of $H$. Then $g_0(H) \geq g_0(G) \geq 9$ and $|N_H(v) \cap V(C)| \leq 2$ for each $v \in V(H) \setminus V(C)$. Noting $15 = d(V(H)) = 5|V(H)| - 2|E(H)|$, we can easily verify $|V(H)| \geq 15$. (c) is proved. □

**Lemma 24.** Suppose $k = 5$, $\lambda_6(G) = \lambda_7(G) = 12$ and $g(G) > 3$. For a $\lambda_7$-atom $S$ of $G$, we have that $S$ is an imprimitive block of $G$.

**Proof.** Suppose, to the contrary, that $S$ is not an imprimitive block of $G$. Then there is an automorphism $\varphi_1$ of $G$ such that $\varphi_1(S) \neq S$ and $\varphi_1(S) \cap S \neq \emptyset$. Set $T = \varphi_1(S)$. By Lemma 20(c), we have $|S \cap T| \leq 6$ and $|S \setminus T| \leq 6$, which implies $|S| \leq 12$. As $12 = d(S) = 5|S| - 2|E(S)|$, $|S|$ is an even integer. By Lemma 20(b), $\delta(G[S]) \geq 3$. For each $u \in S$, we have $d_{G[S]}(u) \geq \lambda_6(G)$ by Lemma 20(a), which implies $|N_G(u) \cap S| \leq 2$. Noting that $\lambda_6(G) \geq \lambda_5(G) \geq \lambda_4(G) \geq \min\{4k - 8, 5k - 12, \lambda_6(G)\} = 12$, we have $\lambda_S(G) = \lambda_4(G) = 12$. By Lemma 23, $p = 1$. By Lemma 20(a), we have $d_G(H) \geq \lambda_5(G) = 12$. Then either $d_G(H) = 13$ and $|E(X)| = 1$, or $d_G(H) = 15$ and $E(X) = \emptyset$.

Figure 1. Some possible cases of $G[S]$. In each $G_i$, $2 \leq i \leq 5$, the two graphs in the virtual boxes correspond to $G[S \cap T]$ and $G[S \setminus T]$.

**Case 1.** $|S| = 8$.

We have $|E(S)| = \frac{1}{2}(5|S| - \lambda_6(G)) = 14$. It is easy to verify that $G[S]$ is isomorphic to $G_1$ in Figure 1. Label $G[S]$ as in $G_1$ and set $W = \{w_1, w_2, w_3, w_4\}$. As $|N_G(u) \cap S| \leq 2$ for each $u \in S$, $G$ has no vertex $v$ different from $w_1$ such that $N_G(v) = N_G(w_1)$. Hence $G$ has no subgraphs isomorphic to $K_{2,5}$ by the vertex-transitivity of $G$.

**Claim 1.** Each edge in $G$ is contained in a 4-cycle of $G$.

Suppose that $G$ has an edge contained in no 4-cycles of $G$. Since $G$ is vertex-transitive, each vertex in $G$ is incident with an edge contained in no 4-cycles of $G$ and there is an automorphism $\varphi_2$ of $G$ such that $\varphi_2(w_1) = w_2$. As each edge in $G[S]$ is contained in a 4-cycle, we have $\varphi_2(N_{G[S]}(w_1)) \subseteq N_{G[S]}(w_2)$ and $N_{G[S]}(\varphi_2(z_i)) \subseteq \varphi_2(S)$ for each $i \in \{2, 3\}$. It implies $|S \cap \varphi_2(S)| \geq 7$. On the other hand, noting $\varphi_2(S) \neq S$, we have $|S \cap \varphi_2(S)| \leq 6$ by Lemma 20(c), a contradiction. Thus Claim 1 holds.

**Claim 2.** For any vertex $x \in V(G)$ with $2 \leq |\nabla(x) \cap \nabla(H)| \leq 3$ such that $d_{G[X]}(u) = 0$ for each $u \in \{x\} \cup N_G(x) \setminus X$, there is a subset $A \subseteq N_G(x)$ with $|A| \geq |\nabla(x) \cap \nabla(H)| - 1$ and a vertex $y \in V(G) \setminus \{x\}$ such that $\{x, y\} \subseteq \nabla(H)$ and $|\nabla(u) \cap \nabla(H)| \geq 3$ for each $u \in A$.

Since $G$ is vertex-transitive, there is an automorphism $\varphi_3$ of $G$ such that $\varphi_3(w_2) = x$. Let $T_1$ be one of $X$ and $V(H)$ such that $x \in T_1$, and let $T_2$ be the other of $X$ and $V(H)$.
Then $\varphi_3(w_3) \in T_1$ and $|\varphi_3(N_{G[S]}(w_2)) \cap T_2| \geq |\nabla(x) \cap \nabla(H)| - 1$. If $|\varphi_3(N_{G[S]}(w_2)) \cap T_2| \leq 2$ or $\varphi_3(W) \subseteq T_1$, then we choose $A$ to be $\varphi_3(N_{G[S]}(w_2)) \cap T_2$. If $|\varphi_3(N_{G[S]}(w_2)) \cap T_2| = 3$ and $\varphi_3(W) \setminus T_1 \neq \emptyset$, then $\varphi_3(W) \cap T_1 = 3$ and $\{\varphi_3(z_2), \varphi_3(z_3)\} \subseteq T_2$. In the second case, we choose $A$ to be $\{\varphi_3(z_2), \varphi_3(z_3)\}$. Then $A$ and $\varphi_3(w_3)$ are a subset and a vertex which satisfy the condition. Thus Claim 2 holds.

**Subcase 1.1.** $d_G(H) = 13$.

Let $x_1x_2$ be the edge in $E(X)$. We know $|X| \geq 6$ and $|V(H)| \geq 7$. By Lemma 20(a), $d_G(V(H) \cup A) \geq \lambda_3(G)$ and $d_G(V(H) \setminus A) \geq \lambda_4(G)$ for each subset $A \subseteq V(G)$ with $|A| \leq 2$, which implies that $|\nabla(u) \cap \nabla(H)| \leq 3$ for each $u \in V(G)$ and each of $X$ and $V(H)$ has at most one vertex $v$ with $|\nabla(v) \cap \nabla(H)| = 3$. Hence it follows from Claim 2 that $|\nabla(u) \cap \nabla(H)| \leq 2$ for each $u \in X \setminus \{x_1, x_2\}$. This together with Claim 2 implies that $|\nabla(u) \cap \nabla(H)| \leq 1$ for each $u \in V(H) \setminus N_G(\{x_1, x_2\})$.

We claim $|\nabla(u) \cap \nabla(H)| \leq 2$ for each $u \in N_G(\{x_1, x_2\}) \cap V(H)$. Otherwise, suppose that there is a vertex $u_0 \in N_G(\{x_1, x_2\}) \cap V(H)$ with $|\nabla(u_0) \cap \nabla(H)| = 3$. Since $G$ is vertex-transitive, there is an automorphism $\varphi_4$ of $G$ such that $\varphi_4(u_0) = u_0$. It implies that there is a vertex $u_1 \in \varphi_4(N_{G[S]}(w_2) \cap (X \setminus \{x_1, x_2\}))$ such that $|\nabla(u_1) \cap \nabla(H)| = 3$, a contradiction.

Thus it follows from Claim 2 that $|\nabla(u) \cap \nabla(H)| \leq 1$ for each $u \in X \setminus \{x_1, x_2\}$. Noting $|N_G(\{x_1, x_2\}) \cap V(H)| \leq 5$, we have $|\nabla(N_G(\{x_1, x_2\}) \cap V(H)) \cap \nabla(H)| \leq 10$ by the claim in the previous paragraph. Hence there is an edge $x_3x_4 \in \nabla(H)$ such that $x_3 \in X \setminus \{x_1, x_2\}$ and $|\nabla(x_3) \cap \nabla(H)| = |\nabla(x_4) \cap \nabla(H)| = 1$. Then $x_3x_4$ is contained in no 4-cycles of $G$, contradicting Claim 1. Hence Subcase 1.1 cannot occur.

**Subcase 1.2.** $d_G(H) = 15$.

Notice that $G$ has no subgraphs which are isomorphic to $K_{2,5}$. We know $|X| \geq 6$. Next we show $|V(H)| \geq 9$. Let $O_i$ be the set of vertices $u$ in $G$ with $|\nabla(u) \cap \nabla(H)| = i$ for $1 \leq i \leq 5$. If $|X| \geq 7$, then $g_0(G) \geq 7$ by Lemma 17, which implies $|V(H)| \geq 13$. Then we assume $|X| = 6$. As $G$ has no subgraphs which are isomorphic to $K_{2,5}$, we have $|O_3 \cap X| = 3$ and $|O_2 \cap X| = 3$. Noting $g(G) > 3$, we can obtain $|V(H)| \neq 5$. By Claim 2, $|O_3 \cap V(H)| \geq 2$, which implies $|V(H)| \geq 7$. Hence $|V(H)| \geq 9$.

By Lemma 20(a), $d_G(V(H) \cup A) \geq \lambda_3(G)$ and $d_G(V(H) \setminus A) \geq \lambda_4(G)$ for each subset $A \subseteq V(G)$ with $|A| \leq 4$. It implies $O_5 = \emptyset$, $|O_4 \cap X| \leq 1$, $|O_3 \cap X| \leq 3$, $|O_3 \cap V(H)| \leq 3$ and $|O_4 \cap X| \cdot |O_3 \cap X| = 0$.

We claim $O_4 = \emptyset$. Otherwise, suppose $O_4 \neq \emptyset$. Noting that $\delta(H) \geq 2$ as $H$ is factor-critical, we have $O_4 \subseteq X$. Now we know $|O_4| = 1$ and $O_3 \cap X = \emptyset$. It follows from Claim 2 that $O_3 \cap V(H) = \emptyset$ and $O_3 \subseteq N_G(O_4)$. As $|\nabla(N_G(O_4) \cap V(H))| \leq 8$, there is an edge $x_5x_6 \in \nabla(H)$ with $\{x_5, x_6\} \subseteq O_1$. Then $x_5x_6$ is contained in no 4-cycles of $G$, contradicting Claim 1.

Let $F_1$ be the subgraph of $G$ with vertex set $\bigcup_{i=1}^3 O_i$ and edge set $\nabla(H)$ and let $F_2$ be the subgraph of $F_1$ which is induced by $O_3$. By Claim 2, $\delta(F_2) \geq 2$. Hence $F_2$ is connected. Then $F_1$ is connected by Claims 1 and 2. Let $t$ be the number of vertices $u$ in $F_2$ with $d_{F_2}(u) = 2$. We have $15 = |E(F_1)| \leq |E(F_2)| + 2t = 6|O_3| - 3|E(F_2)|$ by Claim 2. It follows that $|O_3| = 6$ and $6 \leq |E(F_2)| \leq 7$.

Assume $|E(F_2)| = 6$. Then $F_2$ is a 6-cycle. For each $u \in O_3 \cap X$, there is a vertex
$y_u \in X \setminus O_3$ such that $N_{F_2}(u) \subseteq N_G(y_u)$ by Claim 2. It implies that there is a vertex $y \in X \setminus O_3$ such that $O_3 \cap V(H) \subseteq N_G(y)$, which contradicts $|O_3 \cap X| \leq 3$.

Assume $|E(F_2)| = 7$. As $|E(F_1)\setminus E(F_2)| = 8$, it follows from Claim 2 that there is a vertex $u_2 \in V(F_1) \setminus O_3$ with $d_{F_1}(u_2) = 2$ and we know $|N_{F_1}(u_2) \cap O_3| = 1$ and $d_{F_1}(N_{F_1}(u_2) \setminus O_3) = 1$. It is easy to see that there is no vertex $u'$ in $G$ such that $|N_G(u') \cap N_G(u_2)| = 4$. Noting $|N_G(u_2) \cap N_G(w_3)| = 4$, we have that there is no automorphism $\varphi$ of $G$ such that $\varphi(u_2) = u_2$, which contradicts the vertex-transitivity of $G$.

Case 2. $|S| = 10$ or 12.

Claim 3. For any given two distinct $\lambda_7$-atoms $S_1$ and $S_2$ of $G$ with $S_1 \cap S_2 \neq \emptyset$, $G[S_1 \cap S_2]$ and $G[S_1 \setminus S_2]$ are isomorphic to $K_{3,3}$ or $K_{2,2}$.

By Lemma 20(e), we have $d_G(S_1 \cap S_2) + d_G(S_1 \cup S_2) \leq 2\lambda_7(G)$, $d_G(S_1 \setminus S_2) + d_G(S_2 \setminus S_1) \leq 2\lambda_7(G)$, $|S_1 \cap S_2| \leq 6$ and $|S_1 \setminus S_2| \leq 6$. Then $|S_1 \cap S_2| \geq 4$ and $|S_1 \setminus S_2| \geq 4$. By Lemma 20(a), each of $d_G(S_1 \cap S_2)$, $d_G(S_1 \cup S_2)$, $d_G(S_1 \setminus S_2)$ and $d_G(S_2 \setminus S_1)$ is not less than $\lambda_4(G)$.

Noting $\lambda_4(G) = 12$, we have $d_G(S_1 \cap S_2) = d_G(S_1 \setminus S_2) = 12$. Then $G[S_1 \cap S_2]$ and $G[S_1 \setminus S_2]$ are isomorphic to $K_{3,3}$ or $K_{2,2}$. So Claim 3 holds.

Let $R_i$ be the set of vertices $u$ in $S$ with $d_{G[S]}(u) = i$ for $3 \leq i \leq 5$. By Lemma 20(b), $E(G[R_3]) = \emptyset$.

Claim 4. $R_5 = \emptyset$, or $G[R_5]$ is a 6-cycle and $|S| = 12$.

Suppose $R_5 \neq \emptyset$. It only needs to show that $|S| = 12$ and $G[R_5]$ is a 6-cycle. Assume $R_4 \neq \emptyset$. Choose a vertex $u \in R_4$ and a vertex $v \in R_5$. Let $\varphi_5$ be an automorphism of $G$ such that $\varphi_5(u) = v$. Then $\varphi_5(N_{G[S]}(u)) \subseteq N_{G[S]}(v)$, which contradicts that $G[\varphi_5(S) \cap S]$ is isomorphic to $K_{3,3}$ or $K_{2,2}$ by Claim 3. Thus $R_4 = \emptyset$. Noting $|R_3| + |R_5| = |S|$ and $3|R_3| + 5|R_5| = 2|E(S)| = 5|S| - 12$, we have $|R_3| = 6$. For any two vertices $u', u'' \in R_5$, it follows from Claim 3 that $\varphi(S) = S$ for every automorphism $\varphi$ of $G$ with $\varphi(u') = u''$. Hence $G[R_5]$ is $r$-regular, for some integer $r$. Then $18 = 3|R_3| = d_{G[S]}(R_3) = d_{G[S]}(R_5) = (5 - r)(|S| - 6)$, which implies $|S| = 12$ and $r = 2$. Hence $G[R_5]$ is a 6-cycle and Claim 4 is proved.

By Claim 3, $G[S \cap T]$ and $G[S \setminus T]$ are isomorphic to $K_{3,3}$ or $K_{2,2}$. Noting $E(G[R_3]) = \emptyset$, we have by Claim 4 that $G[S]$ is isomorphic to $G_2$, $G_3$, $G_4$ or $G_5$ in Figure 1.

Claim 5. Each vertex in $G$ is contained in exactly two distinct $\lambda_7$-atoms of $G$.

By the vertex-transitivity of $G$, it only needs to show that $S' = S$ or $S' = T$ for a $\lambda_7$-atom $S'$ of $G$ with $S' \cap S \cap T \neq \emptyset$. Suppose $S' \neq S$ and $S' \neq T$. From Figure 1, we can see that $S$ has no subset $A$ different from $S \cap T$ and $S \setminus T$ such that $G[A]$ is isomorphic to $K_{3,3}$. Hence it follows from Claim 3 that $S' \cap S = S \cap T = S' \cap T$. Then $12 = d_G(S \cap T) \geq d_{G[S]}(S \cap T) + d_{G[T]}(S \cap T) + d_{G[S]}(S \cap T) = 18$, a contradiction. So Claim 5 holds.

Suppose $|S| = 10$. Then $G[S]$ is isomorphic to $G_2$. By Claims 3 and 5, there is a $\lambda_7$-atom $S''$ of $G$ such that $S'' \cap S = S' \cap T$. Choose a vertex $u_3 \in S \setminus T$ and a vertex $u_4 \in S \cap T$. Noting that $G[S \setminus T]$ is not isomorphic to $G[S \cap T]$, we know by Claim 5 that there is no automorphism $\varphi$ of $G$ such that $\varphi(u_3) = u_4$, a contradiction.
Suppose next $|S| = 12$. Then $G[S]$ is isomorphic to $G_3$, $G_4$ or $G_5$. Let $V_1, V_2, \ldots, V_m$ be all subsets of $V(G)$ which induce subgraphs of $G$ isomorphic to $K_{3,3}$. Noting that $G[S \cap T]$ and $G[S \setminus T]$ are isomorphic to $K_{3,3}$, we can obtain by Claims 3 and 5 that $V_1, V_2, \ldots, V_m$ form a partition of $V(G)$ and for each $V_i$ there are exactly two elements $j_1, j_2 \in \{1, 2, \ldots, m\} \setminus \{i\}$ such that $G[V_i \cup V_{j_1}]$ and $G[V_i \cup V_{j_2}]$ are isomorphic to $G[S]$. We denote $V_i \sim V_j$ if $G[V_i \cup V_j]$ is isomorphic to $G[S]$, and assume $V_1 \sim V_2 \sim \cdots \sim V_m \sim V_1$. If $G[S]$ is isomorphic to $G_3$, then it is easy to verify that $G$ is bipartite, a contradiction. Thus $G[S]$ is isomorphic to $G_4$ or $G_5$.

Assume that there is some $V_q \subseteq V(H)$. If $G[S]$ is isomorphic to $G_4$, then $|V_q \cap X| = 3$ and $N_G(V_q \setminus X) \cap V_{q-1} \subseteq X$, which implies $|E(X)| \geq |E(V_{q-1}) \cap E(X)| \geq 2$, a contradiction. Thus $G[S]$ is isomorphic to $G_5$. Let $V_j$ be chosen such that $V_j \cap V(H) \neq \emptyset$ and $|j - q|$ is as small as possible. Then $|V_j \cap X| = 3$ and $|N(u) \cap X| \geq 4$ for each $u \in V_j \cap V(H)$, which contradicts that $\delta(H) \geq 2$.

We now assume that $V_i \cap V(H) \neq \emptyset$ for $1 \leq i \leq m$. Then $|V_i \cap X| > |V_i \setminus (V(H) \cup X)|$ if $V_i \cap X \neq \emptyset$. Choose some $V_q'$ which contains vertices in $V(G) \setminus (V(H) \cup X)$. Then $V_{q-1} \cap X \neq \emptyset$ and $V_{q+1} \cap X \neq \emptyset$. Noting $c_0(G - X) = |X| - 2$, we can obtain that for each $i \in [m]$, $|V_i \cap X| = |V_i \setminus (V(H) \cup X)| + 1$ if $i \in \{q' - 1, q', q' + 1\}$ and $|V_i \cap X| = 0$ otherwise. Then $|V_q' \setminus (V(H) \cup X)| = 2$. Hence $|V_{q-1} \cap X| = |V_{q+1} \cap X| = 3$. Now we have $V_{q-1} \sim V_q' \sim V_{q+1} \sim V_{q'-1}$, which implies $V(G) = V_{q-1} \cup V_q' \cup V_{q+1}$ and $|V(H)| = 3$. It follows that $g(G) = 3$, a contradiction. 

Lemma 25. Suppose $k = 5$, $\lambda_5(G) = \lambda_b(G) = 13$ and $g(G) > 3$. For a $\lambda_b$-atom $S$ of $G$, we have $|S| \geq 11$.

Proof. To the contrary, suppose $|S| < 11$. As $13 = d(S) = 5|S| - 2|E(S)|$, $|S|$ is odd. Then $|S| \geq 7$. By Lemma 20(b), $\delta(G[S]) \geq 3$. By Lemma 23, we have $p = 1$, $|X| \geq 7$ and $|V(H)| \geq 9$. Hence $|V(G)| \geq 20$.

Assume $|S| = 7$. Then $|E(S)| = \frac{1}{2}(5|S| - 13) = 11$. If $G[S]$ is bipartite, then $|E(S)| \geq \frac{1}{2}(|S| + 1)\delta(G[S]) \geq 12$, a contradiction. Thus $G[S]$ is non-bipartite. Let $C$ be a shortest cycle of odd length in $G[S]$. Then $5 \leq |V(C)| \leq 7$. Noting that $|N_{G[S]}(u) \cap V(C)| \leq 2$ for each $u \in S \setminus V(C)$, we have $|E(S)| \leq 10$, a contradiction.

So $|S| = 9$. Let $R_i$ be the set of vertices $u$ in $S$ with $d_{G[S]}(u) = i$ for $3 \leq i \leq 5$.

Claim 1. For any automorphism $\varphi$ of $G$ with $\varphi(R_4 \cup R_5) \cap (R_4 \cup R_5) \neq \emptyset$, either $\varphi(S) = S$ or $G[S \cap \varphi(S)]$ is isomorphic to $K_{2,3}$.

Suppose $\varphi(S) \neq S$. By Lemma 20(c), $|S \cap \varphi(S)| \leq 5$, $|S \setminus \varphi(S)| \leq 5$ and $d(S \cap \varphi(S)) + d(S \cup \varphi(S)) \leq 2\lambda_b(G)$. Then $4 \leq |S \cap \varphi(S)| \leq 5$ and $|S \cup \varphi(S)| = |S| + |\varphi(S)| - |S \cap \varphi(S)| \leq 14$. As $|V(G)| \geq 20$, we have $d(S \cup \varphi(S)) \geq \lambda_b(G)$ by Lemma 20(a). Hence $d(S \cap \varphi(S)) \leq \lambda_b(G) = 13$. This, together with the fact that $|N_{G[\varphi(S)]}(u) \cap N_{G[S]}(u)| \geq 3$ for each $u \in \varphi(R_4 \cup R_5) \cap (R_4 \cup R_5)$, implies that $G[S \cap \varphi(S)]$ is isomorphic to $K_{2,3}$. So Claim 1 holds.

By Claim 1, it follows that $G$ has no automorphism $\varphi$ such that $\varphi(R_4) \cap R_5 \neq \emptyset$. It implies $R_4 = \emptyset$ or $R_5 = \emptyset$. Noting $\sum_{i=3}^{5}|R_i| = 2|E(S)| = 32$ and $\sum_{i=3}^{5}|R_i| = |S| = 9$, we have $|R_3| = 4$, $|R_4| = 5$ and $R_5 = \emptyset$. By Lemma 20(b), $E(R_3) = \emptyset$. Hence $|E(R_4)| = 4$.
As \( g(G[S]) \geq g(G) > 3 \), it is easy to verify that either \( G[R_4] \) has a 4-cycle or \( G[R_4] \) is isomorphic to \( K_{1,4} \). Let \( u_1 \) and \( u_2 \) be two vertices in \( R_4 \) with \( d_{G[R_4]}(u_1) < d_{G[R_4]}(u_2) \). Since \( G \) is vertex-transitive, there is an automorphism \( \psi \) of \( G \) such that \( \psi(u_2) = u_1 \). By Claim 1, \( G[\psi(S) \cap S] \) is isomorphic to \( K_{2,3} \). As \( u_1, u_2 \in R_4 \), we know \( d_{G[\psi(S) \cap S]}(u_1) = 3 \). Notice that \( |N_{G[S]}(u) \cap N_{G[S]}(u_1)| \leq 2 \) for each \( u \in S \setminus \{u_1\} \) if \( G[R_4] \) has a 4-cycle. It follows that \( G[R_4] \) is isomorphic to \( K_{1,4} \). Since \( d_{G[\psi(S) \cap S]}(v) = 2 \) for each \( v \in N_{G[\psi(S) \cap S]}(u_1) \), it follows that \( N_{G[\psi(S) \cap S]}(u_1) \subseteq R_3 \). It implies that the vertex in \( R_3 \setminus N_{G[S]}(u_1) \) has only two neighbors in \( S \), which contradicts \( \delta(G[S]) \geq 3 \).

**Lemma 26.** Suppose \( k = 5 \), \( \lambda_6(G) = \lambda_7(G) = 14 \) and \( g(G) > 3 \). For a \( \lambda_7 \)-atom \( S \) of \( G \), we have \( |S| \geq 14 \).

**Proof.** By Lemma 23, we have \( p = 1 \), \( |X| \geq 10 \) and \( |V(H)| \geq 15 \). Hence \( |V(G)| \geq 32 \). For \( 1 \leq i \leq 5 \), let \( O_i \) be the set of vertices \( u \) in \( G \) with \( |\{u \cap (V(H))| = i \), and set \( m_i = |O_i \cap X| \) and \( n_i = |O_i \cap V(H)| \). By Lemma 20(a), \( d_G(V(H) \cup A) \geq \lambda_6(G) \) and \( d_G(V(H) \setminus A) \geq \lambda_6(G) \) for each subset \( A \) of \( V(G) \) with \( |A| \leq 2 \). This, together with the fact that \( d_G(H) \) is odd, implies that \( d_G(H) = 15 \), \( O_4 \cup O_5 = \emptyset \), \( m_3 \leq 1 \) and \( n_3 \leq 1 \). Hence \( E(X) = \emptyset \). Then \( g_0(G) \geq 9 \) by Lemma 17.

Suppose \( |S| < 14 \). As \( 5|S| - 2|E(G[S])| = 14 \), \( |S| \) is an even integer with \( 8 \leq |S| \leq 12 \). By Lemma 20(b), \( \delta(G[S]) \geq 3 \). As \( g_0(G) \geq 9 \), it follows that \( G[S] \) is bipartite. By Lemma 20(a), \( d_G(S \cup \{u\}) \geq \lambda_6(G) \) for each \( u \in S \) and \( d_G(A) \geq \lambda_6(G) \) for each subset \( A \subseteq V(G) \) with \( |A| = 6 \). Hence \( |N_G(u) \cap S| \leq 2 \) for each \( u \in S \) and \( G \) has no subgraphs which are isomorphic to \( K_{3,3} \).

**Claim 1.** For any two distinct \( \lambda_7 \)-atoms \( S_1 \) and \( S_2 \) of \( G \) with \( S_1 \cap S_2 \neq \emptyset \), we have \( d_G(S_1 \cap S_2) \leq 14 \) and furthermore, \( G[S_1 \cap S_2] \) and \( G[S_1 \setminus S_2] \) are isomorphic to \( K_{2,4} \) or \( K_{3,3} - e \) if \( |S| = 12 \), where \( K_{3,3} - e \) is a subgraph of \( K_{3,3} \) obtained by deleting an edge \( e \) from \( K_{3,3} \).

By Lemma 20(c), we have \( |S_1 \cap S_2| \leq 6 \), \( |S_1 \setminus S_2| \leq 6 \), \( d_G(S_1 \cap S_2) + d_G(S_1 \setminus S_2) \leq 2\lambda_7(G) \), and \( d_G(S_1 \setminus S_2) + d_G(S_2 \setminus S_1) \leq 2\lambda_7(G) \). Noting \( |V(G)| \geq 32 \), we have \( d_G(S_1 \cup S_2) \geq \lambda_7(G) \) by Lemma 20(a). Hence \( d_G(S_1 \cap S_2) \leq \lambda_7(G) = 14 \). Next assume \( |S| = 12 \). Then \( |S_1 \cap S_2| = |S_1 \setminus S_2| = 6 \). By Lemma 20(a), each of \( d_G(S_1 \cap S_2) \), \( d_G(S_1 \setminus S_2) \) and \( d_G(S_2 \setminus S_1) \) is not less than \( \lambda_6(G) \). Hence \( d_G(S_1 \cap S_2) = d_G(S_1 \setminus S_2) = 14 \). It implies that \( G[S_1 \cap S_2] \) and \( G[S_1 \setminus S_2] \) are isomorphic to \( K_{2,4} \) or \( K_{3,3} - e \). So Claim 1 holds.

![Figure 2. The illustration in the proof of Lemma 26.](image-url)
As $G[S]$ is a bipartite graph with $|E(S)| = 13$ and $\delta(G[S]) \geq 3$, $G[S]$ is isomorphic to $G_6$ in Figure 2. Let $v_1, v_2$ be the two vertices in $S$ with $d_{G[S]}(v_1) = d_{G[S]}(v_2) = 4$ and choose a vertex $v_3 \in N_{G[S]}(v_1)\backslash\{v_2\}$.

We claim that each edge in $G$ is contained in a 4-cycle of $G$. Otherwise, suppose that $G$ has an edge contained in no 4-cycles of $G$. Since $G$ is vertex-transitive, each vertex in $G$ is incident with an edge contained in no 4-cycles of $G$ and there is an automorphism $\varphi_1$ of $G$ such that $\varphi_1(v_3) = v_2$. Clearly, $\varphi_1(S) \neq S$. Noting that each edge in $G[S]$ is contained in a 4-cycle of $G[S]$, we have $d_{G[\varphi_1(S)\cup S]}(u) \leq 4$ for each $u \in \varphi_1(S) \cup S$. Then $\varphi_1(N_{G[S]}(v_3) \subseteq N_{G[S]}(v_2)$ and $N_{G[S]}(\varphi_1(v_1)) \subseteq \varphi_1(N_{G[S]}(v_1))$. Noting that $|\varphi_1(S) \cap S| \leq 6$ by Lemma 20(c) and $d_G(\varphi_1(S) \cap S) \leq 14$ by Claim 1, $G[S \cap \varphi_1(S)]$ is isomorphic to $K_{3,3}$. As $d_G(S \cup \varphi_1(S)) \geq \lambda_6(G) = 14$ by Lemma 20(a), it follows that $G[S \cup \varphi_1(S)]$ is isomorphic to $G_7$ in Figure 2, where the graph in the virtual box corresponds to $G[S \cap \varphi_1(S)]$. Choose a vertex $v_4 \in S \cap \varphi_1(S)$ with $d_{G[S \cap \varphi_1(S)]}(v_4) = 2$. Let $\varphi_2$ be an automorphism of $G$ such that $\varphi_2(\varphi_1(v_1)) = v_4$. Then $\varphi_2(N_{G[S]}(v_1)) = N_{G[S \cap \varphi_1(S)]}(v_4)$ and $\varphi_2(N_{G[S]}(v_2)) \backslash (S \cup \varphi_1(S)) \neq \emptyset$. Then $d_G(S \cup \varphi_2(S) \cup \varphi_2(N_{G[S]}(v_2))) \leq 14 = \lambda_6(G)$, contradicting Lemma 20(a). So this claim holds.

For each $uv \in \nabla(H)$, noting that $uv$ is contained in a 4-cycle of $G$ by the previous claim, we have $|\nabla(u) \cap H| + |\nabla(v) \cap H| \geq 3$. For each $u \in O_3 \cup O_3$, there is an automorphism $\varphi_3$ of $G$ such that $\varphi_3(u) = u$, which implies that there is a vertex $v \in \varphi_3(N_{G[S]}(v_1))$ such that $uv \in \nabla(H)$ and $|\nabla(v) \cap H| = 3$. Hence $m_1 \leq n_2 + 2n_3$, $m_2 \leq 3n_3$ and $n_2 \leq 3m_4$. Noting $m_3 \leq 1$ and $n_3 \leq 1$, we have $15 + \sum_{i=1}^{3} i m_i \leq n_2 + 2n_3 + 6n_3 + 3m_3 \leq 6m_3 + 8n_3 \leq 14$, a contradiction.

**Case 2.** $|S| = 10$ or 12.

Let $R_i$ be the set of vertices $u$ in $S$ with $d_{G[S]}(u) = i$ for $3 \leq i \leq 5$. Then $E(R_3) = \emptyset$ by Lemma 20(b). Let $Z$ and $W$ be the bipartition of $G[S]$ such that $|Z| \leq |W|$. Noting $\frac{1}{2}(5|S| - 14) = |E(S)| \geq \delta(G[S])|W| \geq 3|W|$, we have $|W| < \frac{1}{2} |S| + 2$.

**Claim 2.** If $R_3 \neq \emptyset$, then, for each $v \in R_4$, there is exactly one vertex $w$ in $S \backslash \{v\}$ with $N_{G[S]}(v) \subseteq N_{G[S]}(w)$.

Suppose $R_3 \neq \emptyset$. Choose a vertex $u \in R_3$ and a vertex $v \in R_4$. Let $\varphi_4$ be an automorphism of $G$ such that $\varphi_4(u) = v$. Then $N_{G[S]}(v) \subseteq \varphi_4(N_{G[S]}(u))$. Noting that $|S \cap \varphi_4(S)| \leq 6$ by Lemma 20(c) and $d_G(S \cap \varphi_4(S)) \leq 14$ by Claim 1, we have that $G[S \cap \varphi_4(S)]$ is isomorphic to $K_{2,4}$. It implies that $S$ has a vertex $w$ different from $v$ with $N_{G[S]}(v) \subseteq N_{G[S]}(w)$. As $G$ has no subgraphs isomorphic to $K_{3,3}$, such vertex $w$ is unique. So Claim 2 holds.

**Claim 3.** $|W| = |Z|$ and $R_5 = \emptyset$.

Suppose, to the contrary, that $|W| > |Z|$, or $|W| = |Z|$ and $R_5 \neq \emptyset$. As $E(R_3) = \emptyset$, it follows that $|W| = 6$ if $|S| = 10$.

Assume $|W| = |Z| + 2 = 7$. Noting $|E(S)| = 23$, there is a vertex $v_5 \in (R_4 \cup R_5) \cap W$ and a vertex $v_6 \in R_5 \cap Z$. Let $\varphi_5$ be an automorphism of $G$ such that $\varphi_5(v_5) = v_6$. Then $\varphi_5(S) \neq S$ and $\varphi_5(N_{G[S]}(v_5)) \subseteq N_{G[S]}(v_6)$. Hence $G[S \cap \varphi_5(S)]$ is isomorphic to $K_{2,4}$ by Claim 1. It implies $|\varphi_5(W) \backslash S| = 5$, contradicting that $G[\varphi_5(S) \backslash S]$ is isomorphic to $K_{2,4}$.
or $K_{3,3} - e$ by Claim 1.

Assume $|W| = 6$. If $|S| = 10$, we know $|R_4 \cap Z| = |R_5 \cap Z| = 2$ as $E(R_3) = \emptyset$ and $|E(S)| = 18$. If $|S| = 12$, we know either $|R_5 \cap Z| = 2 = |R_4 \cap Z| + 1$ or $|R_5 \cap Z| = 1 = |R_4 \cap Z| - 2$ as $|E(S)| = 23$. It follows from Claim 2 that there is a vertex $v_7 \in R_4 \cap Z$ and a vertex $v_8 \in (R_4 \cup R_5) \setminus \{v_7\}$ such that $N_G[S](v_7) \subseteq N_G[S](v_8)$ and $(R_5 \cap Z) \setminus \{v_8\} \neq \emptyset$. It implies that $G[S]$ has a subgraph which is isomorphic to $K_{3,3}$, a contradiction. So Claim 3 holds.

**Subcase 2.1.** $|S| = 10$.
Noting $E(R_3) = \emptyset$, we have by Claim 3 that $G[S]$ is isomorphic to $G_8$ in Figure 2. We label $G[S]$ as in $G_8$ and assume $x_1 \in Z$ and $y_1 \in W$.

**Claim 4.** $|N_G(u) \cap N_G(v)| \leq 3$ for any two distinct vertices $u$ and $v$ in $G$.

Suppose that there are two distinct vertices $u$ and $v$ in $G$ with $|N_G(u) \cap N_G(v)| \geq 4$. Notice that $|N_G(u) \cap S| \leq 2$ for each $u \in S$. By the vertex-transitivity of $G$, for each $y_i \in \{y_1, y_2, y_3\}$ there is a vertex $y_j \in \{y_1, y_2, y_3\} \setminus \{y_i\}$ such that $|N_G(y_i) \cap N_G(y_j)| \geq 4$. It follows that there is a vertex $w \in S$ such that $\{y_1, y_2, y_3\} \subseteq N_G(w)$, a contradiction. So Claim 4 holds.

Let $\varphi_6$ be an automorphism of $G$ such that $\varphi_6(y_5) = y_1$. Then $\varphi_6(S) \neq S$ and $|\varphi_6(N_G(y_5)) \cap N_G(y_1)| \geq 2$. Then $|\varphi_6(S) \cap S| \leq 6$ by Lemma 20(c) and $d_G(\varphi_6(S) \cap S) \leq 14$ by Claim 1. By Claim 4, $G[S]$ has no subgraphs isomorphic to $K_{2,4}$. It follows that $|\varphi_6(S) \cap W| \leq 3$ and $|\varphi_6(S) \cap Z| \leq 3$.

Assume that $\varphi_6(N_G(y_5)) \cap \{x_1, x_2\} = \emptyset$ and $\varphi_6(N_G(y_5)) \cap \{x_4, x_5\} = \emptyset$. Then $|N_G(y_1) \cap N_G(y_5)| = 3$ for each $u \in \varphi_6(N_G(y_5)) \cap \{x_1, x_2\}$ and $|N_G(y_1) \cap N_G(y_5)| \geq 2$ for each $v \in \varphi_6(N_G(y_5)) \cap \{x_4, x_5\}$. It follows that $|\varphi_6(S) \cap W| = 3$ and $|\varphi_6(S) \cap \{y_4, y_5\}| = 1$. Noting $2 \leq |\varphi_6(S) \cap Z| \leq 3$, we can see $d_G(\varphi_6(S) \cap S) > 14$, a contradiction.

Assume $\varphi_6(N_G(y_5)) \cap N_G(y_1) = \{x_4, x_5\}$. Then $\varphi_6(y_4) \in \{y_2, y_3\} \cup S$, which implies that $|N_G(y_1) \cap N_G(y_4)| \geq 4$ or $|N_G(x_4) \cap N_G(x_5)| \geq 4$. It contradicts Claim 4.

Thus $\varphi_6(N_G(y_5)) \cap N_G(y_1) = \{x_1, x_2\}$. By Claim 4, we have $\varphi_6(y_4) \in \{y_4, y_5\}$ and $\varphi_6((y_1, y_2, y_3)) \cap W = \{y_4, y_5\}, \varphi_6(y_4)$. Then $\{\varphi_6(x_4), \varphi_6(x_5)\} \subseteq S$. Assume $\varphi_6(y_4) = y_4$. Set $\{y_6, y_7\} = \varphi_6((y_1, y_2, y_3)) \setminus W, \{x_6\} = \varphi_6(N_G(y_5)) \setminus N_G(y_1)$ and $\{x_7, x_8\} = \{\varphi_6(x_4), \varphi_6(x_5)\}$. Then the graph $G_9$ showed in Figure 2 is a subgraph of $G$.

We can see that each edge incident with $x_1$ is contained in a 4-cycle of $G$. Then, by the vertex-transitivity of $G$, each edge $uv \in \nabla(H)$ is contained in a 4-cycle of $G$, which implies $|\nabla(u) \cap \nabla(H)| \geq 2$ or $|\nabla(v) \cap \nabla(H)| \geq 2$. Hence there is a vertex $u' \in G$ with $2 \leq |\nabla(u') \cap \nabla(H)| \leq 3$. Let $\varphi_7$ be an automorphism of $G$ such that $\varphi_7(y_4) = u'$. It is easy to verify that either $\varphi_7(N_G(y_4) \cup S(y_4))$ has a vertex $u$ with $|\nabla(u) \cap \nabla(H)| \geq 4$ or it has two vertices $v'$ and $v''$ with $\{u'v', u'v''\} \subseteq \nabla(H)$ and $|\nabla(v') \cap \nabla(H)| = |\nabla(v'') \cap \nabla(H)| = 3$, contradicting the fact that $O_4 \cup O_5 = \emptyset, m_3 \leq 1$ and $n_3 \leq 1$.

**Subcase 2.2.** $|S| = 12$.

As $|E(G[S])| = 23$, $G[S]$ is not regular. Let $\varphi_8$ be an automorphism of $G$ such that $\varphi_8(S) \neq S$ and $\varphi_8(S) \cap S \neq \emptyset$. Set $T = \varphi_8(S)$. It follows from Claims 1 and 3 that...
Suppose $d_{G[S\setminus T]}(u) = 5$ for each $u \in S \cap T$, each of $G[S\setminus T]$, $G[S \cap T]$ and $G[T\setminus S]$ is isomorphic to $K_{3,3} - e$ and $d_{G[S]}(v) = d_{G[T]}(v) = 4$ for each $v \in S \cap T$ with $d_{G[S\cap T]}(v) = 3$.

Let $v_9$ and $v_{10}$ be two vertices in $W \cap T$ with $d_{G[S\setminus T]}(v_9) = 3 = d_{G[S\setminus T]}(v_{10}) + 1$. We know either $d_{G[S]}(v_{10}) = 4$ or $d_{G[T]}(v_{10}) = 4$ and assume, without loss of generality, that $d_{G[S]}(v_{10}) = 4$. Let $\varphi_0$ be an automorphism of $G$ such that $\varphi_0(v_9) = v_{10}$. Let $Q$ be one of $\varphi_0(S)$ and $\varphi_0(T)$ such that $Q \neq S$. Since $d_{G[G]}(v_{10}) = 4$, we know $Q \neq T$. By Claims 1 and 3, each of $G[G\setminus S]$, $G[Q\setminus S]$, $G(Q \cap T)$ and $G[T\setminus Q]$ is isomorphic to $K_{3,3} - e$. Noting $d_{G[G]}(v_{10}) = d_{G[G]}(v_{10}) = 4$, we have $|N_{G[G]}(v_{10}) \cap S| = 3$, which implies $2 \leq |Q \cap S \cap T| \leq 5$.

Assume $2 \leq |Q \cap S \cap T| \leq 3$. Noting that $G[Q \cap T]$ is isomorphic to $K_{3,3} - e$, we have $d_{G[Q\cap T]}(Q \cap S \cap T) > |Q \cap S \cap T|$ isomorphic to $K_{3,3} - e$, a contradiction.

Assume $4 \leq |Q \cap S \cap T| \leq 5$. Then $|N_{G[G]}(v_{10}) \cap S \cap T| = 2$. If $E(Q \cap S \cap T) = \emptyset$, then $d_{G[Q\cap S]}(Q \cap S \cap \overline{T}) + d_{G[Q\cap T]}(Q \cap S \cap \overline{T}) \geq 2|Q \cap S \cap \overline{T}| = 2$ and $|E(Q \cap S \cap \overline{T})| = 1$. Then $d_{G[Q\cap S]}(Q \cap S \cap \overline{T}) + d_{G[Q\cap T]}(Q \cap S \cap \overline{T}) \geq 3 + 3 > 2 \geq |(|Q \cap S \cap \overline{T}, S \cup T)|$, a contradiction.

\begin{proof}

By Lemma 23, we have $p = 1$, $|X| \geq 10$ and $|V(H)| \geq 15$. By Lemma 20(a), $d_G(A) \geq \lambda_6(G) \geq 14$, $d_G(V(H) \cup B) \geq \lambda_6(G)$ and $d_G(V(H) \setminus B) \geq \lambda_8(G)$ for any two subsets $A$ and $B$ of $V(G)$ with $|A| = 6$ and $|B| \leq 1$. It implies that $G$ has no subgraphs isomorphic to $K_{3,3}$, $d_G(H) = 15$ and $|\nabla(u) \cap \nabla(H)| \leq 2$ for each $u \in V(G)$. Hence $E(X) = \emptyset$ and there is an edge $u_1u_2 \in \nabla(H)$ such that $N_G(u_1) \cap X = \{u_2\}$. By Lemma 17, $g_0(G) \geq 9$.

Suppose $|S| < 15$. As $g_0(G) \geq 9$ and $15 = \lambda_6(G) = d_G(S) = 5|S| - 2|E(G[S])|$, it follows that $|S|$ is odd and $G[S]$ is bipartite. By Lemma 20(b), $\delta(G[S]) \geq 3$. Let $W$ and $Z$ be the bipartition of $G[S]$ such that $|W| > |Z|$. We have $|W| = \frac{1}{2}(|S| + 1)$ if $|S| \leq 11$, and $7 \leq |W| \leq 8$ if $|S| = 13$.

\textbf{Case 1.} There is a vertex $v_1$ in $S$ with $d_{G[S]}(v_1) = 5$.

Let $R$ be one of $W$ and $Z$ such that $v_1 \in R$. As $\delta(G[S]) \geq 3$ and $|E(S)| = \frac{1}{2}(|S| - 2|E(G[S])|)$, it follows that $N_{G[S]}(N_{G[S]}(v_1)) = R$. Since $G$ is vertex-transitive, there is an automorphism $\varphi_1$ of $G$ such that $\varphi_1(v_1) = u_2$. Then $\varphi_1(R) \subseteq X \cup V(H)$. Noting that $|\nabla(u) \cap \nabla(H)| \leq 2$ for each $u \in V(G)$, we have $\varphi_1(S \setminus R) \cap X = \emptyset$. Notice that $G$ has no subgraphs isomorphic to $K_{3,3}$. We have $|\varphi_1(R) \cap X| \geq 4$ as $|N_G(u_2) \setminus V(H)| \geq 3$ and $\delta(G[S]) \geq 3$. Then $|\varphi_1(S) \cap V(H)| \leq 6$ as $|S| \leq 13$. It follows that $d_{G[\varphi_1(S)]}(u_1) = 3$. Then $d_{G[\varphi_1(S)]}(u) \geq 4$ for each $u \in N_{G[\varphi_1(S)]}(u_1)$ by Lemma 20(b). Now we know $|S| = 13$, $|\varphi_1(R) \cap X| = 4 = |\varphi_1(R) \cap V(H)| + 2$ and $|\varphi_1(S \setminus R) \cap V(H)| = 4 = |\varphi_1(S \setminus R) \cap V(H)| + 1$. Then $R = Z$ and $|N_G(u_2) \setminus V(H)| = 2$.

Noting that $|\nabla(u) \cap \nabla(H)| \leq 2$ for each $u \in V(G)$, we have $d_{G[\varphi_1(S)]}(u) \leq 4$ for each $u \in \varphi_1(W)$. Since $\delta(G[S]) \geq 3$ and $G$ has no subgraphs isomorphic to $K_{3,3}$, two vertices in $\varphi_1(W) \setminus V(H)$ has exactly 3 neighbors in $\varphi_1(Z) \cap X$. So $d_{G[\varphi_1(S)]}(u) \geq 4$ for each $u \in \varphi_1(W) \setminus N_G(u_2)$ as $|E(S)| = 25$. Then there is a vertex $u_3 \in \varphi_1(Z) \cap X$ such that $\varphi_1(W) \setminus N_G(u_2) \subseteq \varphi_1(W) \setminus N_G(u_3)$.

\end{proof}
Assume $\varphi_1(Z) \cap V(H) = \{u_4, u_5\}$. Let $\varphi_2$ be an automorphism of $G$ such that $\varphi_2(u_4) = u_2$. Then $u_1 \notin \varphi_2(N_{G_2}(u_4))$ and $\varphi_2(\{u_2, u_3, u_4\}) \subseteq X$, which implies $|\nabla(u) \cap \nabla(H)| \geq 3$ for the vertex $u \in (N_G(u_2) \cap V(H)) \setminus \{u_1\}$, a contradiction.

**Case 2.** $d_{G[S]}(u) \leq 4$ for each $u \in S$.

If $|S| = 13$, then, noting $|E(G[S])| = 25$ and $5 \leq |Z| \leq 6$, we have that there is a vertex $u \in Z$ with $d_{G[S]}(u) = 5$, a contradiction. Thus $|S| \leq 11$. There is a vertex $w \in W$ with $d_{G[S]}(w) = |W| - 2$ such that $d_{G[S]}(u) = 4$ for each $u \in N_{G[S]}(w)$. Choose a vertex $z \in N_{G[S]}(w)$.

We claim that the edge $u_1u_2$ is contained in a 4-cycle of $G$. Suppose not. Since $G$ is vertex-transitive, each vertex in $G$ is incident with an edge contained in no 4-cycles of $G$ and there is an automorphism $\varphi_3$ of $G$ such that $\varphi_3(w) = z$. We know $\varphi_3(S) \neq S$. Noting that $|N_{G[S]}(u) \cap N_{G[S]}(v)| \geq 2$ for every subset $\{u, v\} \subseteq Z$, each edge in $G[S]$ is contained in a 4-cycle of $G$. Hence $\varphi_3(N_{G[S]}(u)) \subseteq N_{G[S]}(z)$ and $N_{G[S]}(u) \subseteq \varphi_3(S)$ for each $u \in \varphi_3(N_{G[S]}(w))$. By Lemma 20(c), $|S \cap \varphi_3(S)| \leq 7$ and $d_G(S \cap \varphi_3(S)) + d_G(S \cup \varphi_3(S)) \leq 2\lambda_8(G)$. If $|S| = 11$, then $|S \cap \varphi_3(S)| \geq |\varphi_3(N_{G[S]}(u)) \cap N_{G[S]}(w)| = 8$, a contradiction. Thus $|S| = 9$. As $\delta(G[S]) \geq 3$, we have $Z = \bigcup_{u \in \varphi_3(N_{G[S]}(u))} N_{G[S]}(u) \subseteq \varphi_3(S)$. Hence $|S \cap \varphi_3(S)| = 7$ and $d_G(S \cap \varphi_3(S)) = 17$. Noting that $d_G(S \cup \varphi_3(S)) \geq \lambda_5(G)$ by Lemma 20(a), we have $d_G(S \cap \varphi_3(S)) + d_G(S \cup \varphi_3(S)) < 2\lambda_8(G)$, a contradiction.

Thus $|N_G(u_2) \cap V(H)| = 2$. Let $\varphi_4$ be an automorphism of $G$ such that $\varphi_4(u) = u_2$ if $|S| = 9$, and $\varphi_4(u) = u_2$ if $|S| = 11$. If $u_1 \in \varphi_4(S)$, then $|Z| \geq d_{G[\varphi_4(S)]}(u_1) - 1 + |N_{G[\varphi_4(S)]}(N_{G[\varphi_4(S)]}(u_2) \cap V(H))| \geq 2 + 3 = 5$ if $|S| = 9$, and $|W| \geq 7$ if $|S| = 11$, a contradiction. Thus $u_1 \notin \varphi_4(S)$. Then $\varphi_5([Z]) \subseteq X$ if $|S| = 9$ and $\varphi_5([W]) \subseteq X$ if $|S| = 11$, which implies $|\nabla(u) \cap \nabla(H)| \geq 3$ for the vertex $u \in (N_G(u_2) \cap V(H)) \setminus \{u_1\}$, a contradiction.

**Lemma 28.** Suppose $k = 6$, $\lambda_5(G) = 16$ and $g(G) > 3$. For a $\lambda_5$-atom $S$ of $G$, we have $|S| \geq 9$.

**Proof.** To the contrary, suppose $|S| \leq 8$. As $\frac{1}{2}(6|S| - \lambda_5(G)) = |E(S)| \leq \frac{1}{2}|S|^2$ by Lemma 6, we have $|S| \geq 8$. Hence $|S| = 8$ and $G[S]$ is isomorphic to $K_{4,4}$.

By Lemma 23, $p = 1$. Then $|X| \geq 7$ by Lemma 19. Noting that $d(H) \leq 18$ and $H$ is triangle-free and factor-critical, we have $|V(H)| \geq 11$. Let $O_i$ be the set of vertices $u$ in $G$ with $|\nabla(u) \cap \nabla(H)| = i$ for $2 \leq i \leq 6$. By Lemma 20(a), we have $d(V(H) \cup A) \geq \lambda_5(G)$ and $d(V(H) \setminus A) \geq \lambda_5(G)$ for each subset $A \subseteq V(G)$ with $|A| \leq 3$, which implies $d(H) \geq 16$, $O_5 \cup O_6 = \emptyset$, $|O_4 \cap X| \leq 1$ and $|O_4 \cap V(H)| \leq 1$.

Suppose that $S$ is an imprimitive block of $G$. Then the orbits $S = S_1, S_2, \ldots, S_m$ of $S$ under the automorphism group of $G$ form a partition of $V(G)$. If $E(S_i) \cap E(X) \neq \emptyset$ for some $S_i$, then $d(H) = 16$ and $|S_i \cap V(H)| = 6$, which implies $d(V(H) \cup S_i) \leq 14 < \lambda_5(G)$, a contradiction. Thus $E(S_j) \cap E(X) = \emptyset$ for each $S_j$. As $|G - X| = |X| - 2$, it follows that $|O_4| \geq 3$, which contradicts the fact that $|O_4| = |O_4 \cap X| + |O_4 \cap V(H)| \leq 2$.

Suppose next that $S$ is not an imprimitive block of $G$. Then there is an automorphism $\varphi_1$ of $G$ such that $\varphi_1(S) \neq S$ and $\varphi_1(S) \cap S \neq \emptyset$. Let $T = \varphi_1(S)$. As $G$ is 6-regular, we have $\delta(G[S \cap T]) \geq 2$. By Lemma 20(c), $|S \cap T| \leq 4$. Hence $G[S \cap T]$ is a 4-cycle of $G$. Assume $S \cap T = \{v_1, v_2, v_3, v_4\}$, where $N(v_1) = N(v_2)$ and $N(v_3) = N(v_4)$.
By the vertex-transitivity of $G$, for each $u \in V(G)$ there is a vertex $u'$ different from $u$ such that $N(u') = N(u)$. Assume $E(X) \neq \emptyset$. Then $|E(X)| = 1$ and let $u_1 u_2$ be the edge in $E(X)$. We know that there is a vertex $u'_1$ in $V(H)$ with $N(u'_1) = N(u_1)$, which implies $|N(u_1) \cap V(H)| = 5$. Then $O_5 \neq \emptyset$, a contradiction. Thus $E(X) = \emptyset$. As for each $u \in V(G)$ there is a vertex $u'$ different from $u$ such that $N(u') = N(u)$, it follows that there is a vertex $u_3 \in X$ with $2 \leq |N(u_3) \cap V(H)| \leq 4$. Let $\varphi_2$ be an automorphism of $G$ such that $\varphi_2(v_1) = u_3$. If $\varphi_2(\{v_3, v_4\}) \backslash V(H) \neq \emptyset$, then $N(u_3) \cap V(H) = \varphi_2(N(v_1)) \cap V(H) \subseteq \bigcup_{i=4}^{6} O_i$. If $\varphi_2(\{v_3, v_4\}) \subseteq V(H)$, then $\varphi_2(\{v_3, v_4\}) \subseteq \bigcup_{i=4}^{6} O_i$. So $|\bigcup_{i=4}^{6} O_i \cap V(H)| \geq 2$, a contradiction. □

Lemma 29. Suppose $k = 6$, $\lambda_5(G) = \lambda_8(G) = 18$ and $g(G) > 3$. For a $\lambda_8$-atom $S$ of $G$, we have $|S| \geq 15$.

Proof. To the contrary, suppose $8 \leq |S| \leq 14$. By Lemma 23, we have $p = 1$, $|X| \geq 7$ and $|V(H)| \geq 9$. By Lemma 20(a), we have $d_G(V(H) \cup A) \geq \lambda_5(G)$ and $d_G(V(H) \cup A) \geq \lambda_5(G)$ for each subset $A \subseteq V(G)$ with $|A| \leq 1$, which implies $d_G(H) = 18$ and $|\nabla(u) \cap \nabla(H)| \leq 3$ for each $u \in V(G)$. Then $g_0(G) \geq 7$ by Lemma 17. It follows that $G[A]$ is bipartite for each subset $A \subseteq V(G)$ with $|A| \leq 13$ and $d_G(A) = 18$. Hence $|V(H)| \geq 15$, and $G[S]$ is bipartite if $|S| \leq 13$. Then $|V(G)| \geq 26$.

Case 1. $|S| = 8$.

By Lemma 20(a), $d_G(A) \geq \lambda_5(G)$ for every subset $A \subseteq V(G)$ with $7 \leq |A| \leq 8$, which implies $\delta(G[S]) \geq 3$ and $G$ has no subgraphs isomorphic to $K_{4,4}$. As $|E(G[S])| = \frac{1}{2}(6|S| - 18) = 15$ and $G[S]$ is bipartite, there is a vertex $u_0 \in S$ with $d_G[S](u_0) = 3$ and $G[S \backslash \{u_0\}]$ is isomorphic to $K_{3,5}$.

![Figure 3. The illustration in the proof of Lemma 29.](image)

Claim 1. There are no two distinct vertices $u$ and $v$ in $G$ with $N_G(u) = N_G(v)$.

Suppose that $u_1$ and $u_2$ are two distinct vertices in $G$ with $N_G(u_1) = N_G(u_2)$. Let $x$, $y$ and $z$ be the three vertices in $S$ which have 4 neighbors in $S \backslash \{u_0\}$. Noting that $G$ has no subgraphs isomorphic to $K_{4,4}$, we have, by the definition of the vertex-transitivity of $G$, that for each vertex $u \in \{x, y, z\}$ there is a vertex $u' \in \{x, y, z\} \backslash \{u\}$ such that $N_G(u) = N_G(u')$. It follows that $N_G(x) = N_G(y) = N_G(z)$. Then $G$ is bipartite by Lemma 19, a contradiction. So Claim 1 holds.

Claim 2. $G$ has no subgraphs isomorphic to $K_{3,5}$.

Suppose that $u_3$, $u_4$ and $u_5$ are three distinct vertices in $G$ with $|N_G(u_3) \cap N_G(u_4) \cap N_G(u_5)| = 5$. By Claim 1 and the vertex-transitivity of $G$, it follows that for each $u \in$
there are two distinct vertices $u', u'' \in (N_G(u_3) \cap N_G(u_4)) \setminus \{u\}$ such that $|N_G(u) \cap N_G(u') \cap N_G(u'')| = 5$. It implies that there is a vertex $v \in V(G) \setminus \{u_3, u_4, u_5\}$ such that $|N_G(v) \cap N_G(u_3) \cap N_G(u_4)| \geq 4$. So $G$ has a subgraph isomorphic to $K_{4,4}$, a contradiction. Claim 2 is proved.

**Claim 3.** $G$ has no subgraphs isomorphic to $G_{10}$ in Figure 3.

Suppose that $G_{10}$ is a subgraph of $G$. Let $\varphi_1$ be an automorphism of $G$ such that $\varphi_1(a_2) = a_1$. Noting that $d_G(V(G_{10}) \cup A) \geq \lambda_5(G)$ for each subset $A \subseteq V(G)$ with $|A| \leq 1$ by Lemma 20(a), we have $G_{10} = G[V(G_{10})]$ and $|N_G(u) \cap V(G_{10})| \leq 3$ for each $u \in V(G_{10})$. We know $\varphi_1(a_3) \in \{a_2, a_3\}$ if $|\varphi_1(N_{G_{10}}(a_2)) \cap N_{G_{10}}(a_1)| = 4$. Hence either each edge in $\nabla(a_1)$ or each edge in $\nabla(\varphi_1(a_3))$ is contained in a 4-cycle of $G$. By the vertex-transitivity of $G$, each edge in $G$ is contained in a 4-cycle of $G$. It follows that $|\nabla(u) \cap \nabla(H)| \geq 3$ for each edge $uv \in \nabla(H)$.

We claim that $|\nabla(u) \cap \nabla(H)| \geq 2$ for each $u \in V(G)$. Otherwise, noting that $|\nabla(u) \cap \nabla(V(G))| \leq 3$ for each $u \in V(G)$, we suppose that there is a vertex $u_6$ in $G$ with $|\nabla(u_6) \cap \nabla(H)| = 3$. Let $\varphi_2$ be an automorphism of $G$ such that $\varphi_2(b_2) = u_6$. By considering the definition of $\varphi_2(V(G_{10}))$, we can obtain that there is a vertex $u \in \varphi_2(N_{G_{10}}(b_2))$ with $|\nabla(u) \cap \nabla(H)| \geq 4$, a contradiction.

Thus there a vertex $u_7 \in V(G)$ with $|\nabla(u_7) \cap \nabla(H)| = 2$. Let $\varphi_3$ be an automorphism of $G$ such that $\varphi_3(a_2) = u_7$. Then there is a vertex $u \in \varphi_3(N_{G_{10}}(a_2))$ with $|\nabla(u) \cap \nabla(H)| \geq 3$, a contradiction. So Claim 3 holds.

By Claim 2, it follows that $G[S]$ is isomorphic to $G_{11}$ in Figure 3 and we label $G[S]$ as in $G_{11}$. Then $|N_G(u) \cap S| \leq 2$ for each $u \in \overline{S}$ by Claims 2 and 3. Let $\varphi_4$ be an automorphism of $G$ such that $\varphi_4(z_1) = z_4$. If $\varphi_4(N_{G[S]}(z_1)) \subseteq N_{G[S]}(z_1)$, then there is a vertex $u \in \varphi_4(S) \setminus S$ with $|N_G(u) \cap S| \geq 3$, a contradiction. Thus $\varphi_4(N_{G[S]}(z_1)) \setminus S \neq \emptyset$.

Assume $\varphi_4(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \{w_1, w_j\}$. As $|N_G(u) \cap S| \leq 2$ for each $u \in \varphi_4(N_{G[S]}(z_1)) \setminus S$, it follows that $|\varphi_4(\{z_2, z_3, z_4\}) \setminus S| = 2$. Then $N_G(w_i) = N_G(w_j)$, contradicting Claim 1.

Assume $\varphi_4(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \{w_i\}$. Then $|\varphi_4(\{z_2, z_3, z_4\}) \setminus S| = 2$, which implies that each edge in $\nabla(w_i)$ is contained in a 4-cycle of $G$. Then each edge in $G$ is contained in a 4-cycle of $G$ by the vertex-transitivity of $G$. Thus there is a vertex $u_8 \in V(G)$ with $2 \leq |\nabla(u_8) \cap \nabla(H)| \leq 3$. Let $\varphi_5$ be an automorphism of $G$ such that $\varphi_5(z_4) = u_8$. As $|N_G(w_1) \cap N_G(w_2) \cap N_G(w_3)| = 4$ and $|N_G(\varphi_4(w_1)) \cap N_G(\varphi_4(w_2)) \cap N_G(\varphi_4(w_3))| = 4$, it follows that there is a vertex $u \in \varphi_5(N_{G[S, \varphi_4(S)]}(z_4))$ with $|\nabla(u) \cap \nabla(H)| \geq 4$, a contradiction.

Thus $\varphi_4(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \emptyset$. By Claim 1, it follows that $\varphi_4(\{z_2, z_3, z_4\}) = N_G(w_4) \setminus S$. Let $\varphi_6$ be an automorphism of $G$ such that $\varphi_6(z_1) = z_3$. Similarly, we have $\varphi_6(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \emptyset$ and $\varphi_6(\{z_2, z_3, z_4\}) = N_G(w_4) \setminus S$. It implies that $G[N_G(w_4) \cup \varphi_4(N_{G[S]}(z_1)) \cup \varphi_6(N_{G[S]}(z_1))]$ has a subgraph isomorphic to $K_{3,5}$ or $G_{10}$, contradicting Claim 2 or Claim 3.

**Case 2.** $9 \leq |S| \leq 14$.

By Lemma 20(b), $\delta(G[S]) \geq 4$. If $|S| = 9$, then $18 = \frac{1}{2}(6|S| - \lambda_8(G)) = |E(S)| \geq \frac{1}{2}(|S| + 1)\delta(G[S]) \geq 20$, a contradiction. Thus $|S| \geq 10$. If $|S| \leq 13$, then let $W$ and $Z$
be the bipartition of $G[S]$ with $|Z| \leq |W|$ and we have $|W| = |Z| + \frac{1}{2}(1 - (-1)^{|S|})$.

**Subcase 2.1.** $10 \leq |S| \leq 12$.

We claim that $d_{G[S]}(u) \leq 5$ for each $u \in S$. Otherwise, suppose that there is a vertex $v_1 \in S$ with $d_{G[S]}(v_1) = 6$. Choose a vertex $u_9 \in X$ with $\nabla(u_9) \cap \nabla(H) \neq \emptyset$. Let $\varphi_7$ be an automorphism of $G$ such that $\varphi_7(v_1) = u_9$. As $\delta(G[S]) \geq 4$, it follows that $\varphi_7(S \setminus N_G(v_1)) \subseteq X$, which implies that $|\nabla(u) \cap \nabla(V(H))| \geq 4$ for each $u \in \varphi_7(N_G(v_1)) \cap V(H)$, a contradiction.

Noting that $4 \leq d_{G[S]}(u) \leq 5$ for each $u \in S$, and recalling $|E(S)| = 3|S| - 9$ and $|W| = |Z| + \frac{1}{2}(1 - (-1)^{|S|})$, we know that there is a vertex $v_2 \in Z$ and $v_3 \in N_{G[S]}(v_2)$ such that $d_{G[S]}(v_2) = d_{G[S]}(v_3) = 5$.

Now we claim that each edge in $G$ is contained in a 4-cycle of $G$. Otherwise, suppose that $G$ has an edge contained in no 4-cycles. By the definition of the vertex-transitivity of $G$, each vertex in $G$ is incident with an edge contained in no 4-cycles of $G$. Let $\varphi_8$ be an automorphism of $G$ such that $\varphi_8(v_3) = v_2$. Then $\varphi_8(S) \neq S$. Notice that each edge in $G[S]$ is contained in a 4-cycle of $G[S]$. We have $\varphi_8(N_{G[S]}(v_3)) \subseteq N_{G[S]}(v_2)$ and $N_{G[S]}(\varphi_8(v_2)) \subseteq \varphi_8(N_{G[S]}(v_2))$. It implies $|\varphi_8(S) \cap S| \geq 8$, contradicting Lemma 20(e).

Thus $|\nabla(u) \cap \nabla(H)| + |\nabla(v) \cap \nabla(H)| \geq 3$ for each edge $uv \in \nabla(H)$. Then there is a vertex $u_{10} \in V(G)$ with $|\nabla(u_{10}) \cap \nabla(H)| \geq 2$.

Suppose $|S| = 10$. Then $|W| = |Z| = 5$. Let $\varphi_9$ be an automorphism of $G$ such that $\varphi_9(v_2) = v_{10}$. Then there is a vertex $u \in \varphi_9(N_{G[S]}(v_2))$ with $|\nabla(u) \cap \nabla(H)| \geq 4$, a contradiction.

Thus $11 \leq |S| \leq 12$. Let $R_i$ be the set of vertices $u$ in $S$ with $d_{G[S]}(u) = i$ for $i = 4, 5, 6$. Then $|R_0| = |R_5 \cap Z| = 4$ if $|S| = 11$, and $|R_3 \cap W| = |R_5 \cap Z| = 3$ if $|S| = 12$.

Suppose that there is a vertex $u_{11} \in V(G)$ with $|\nabla(u_{11}) \cap \nabla(H)| = 3$. For a vertex $v \in S$, let $\psi$ be an automorphism of $G$ such that $\psi(v) = u_{11}$. Then $\psi(S) \cap \nabla(H) \neq \emptyset$ and $\psi(S) \setminus \nabla(H) \neq \emptyset$. As $\delta(G[S]) \geq 4$ and $|\nabla(u) \cap \nabla(H)| \leq 3$ for each $u \in V(G)$, it follows that $|\psi(S) \cap X| = 4$ and $G[\psi(S) \cap V(H)]$ is isomorphic to $K_{1,4}$ or $K_{2,4}$. It implies $|N_{G[S]}(v) \cap R_4| \geq \frac{|S|}{6}$ and that there are two vertices $v', v'' \in R_4$ with $N_{G[S]}(v') = N_{G[S]}(v'')$. If $|S| = 11$, then $N_{G[S]}(u) \cap R_4 = \emptyset$ for each $u \in W \setminus N_{G[S]}(R_4 \cap Z)$, a contradiction. Thus $|S| = 12$. Then $|N_{G[S]}(u) \cap R_4| \geq 2$ for each $u \in S$. So $\delta(G[R_4]) \geq 2$.

Noting $|R_4| = |R_5| = 6$, we have $12 \geq 4|R_4| - \delta(G[R_4])|R_4| \geq 4|R_4| - 2|E(R_4)| = |[R_4, R_5]| = 5|R_3| - 2|E(R_3)| \geq 30 - 18$, which implies $d_{G[R_4]}(u) = 2$ for each $u \in R_4$. Then $G[R_4]$ is a 6-cycle of $G$, which contradicts that $R_4$ has two vertices $v'$ and $v''$ with $N_{G[S]}(v') = N_{G[S]}(v'')$.

So $|\nabla(u) \cap \nabla(H)| \leq 2$ for each $u \in V(G)$. Then $|\nabla(u_{10}) \cap \nabla(H)| = 2$. We can see that there is no automorphism $\varphi$ of $G$ such that $\varphi(v_2) = u_{10}$, contradicting that $G$ is vertex-transitive.

**Subcase 2.2.** $13 \leq |S| \leq 14$.

**Claim 4.** For two distinct $\lambda_8$-atoms $S_1$ and $S_2$ of $G$ with $S_1 \cap S_2 \neq \emptyset$, $G[S_1 \setminus S_2]$ and $G[S_1 \cap S_2]$ are isomorphic to $K_{3,3}$ or $K_{3,4}$.

By Lemma 20(c), we have $|S_1 \setminus S_2| \leq 7$, $|S_1 \cap S_2| \leq 7$, $d_G(S_1 \setminus S_2) + d_G(S_2 \setminus S_1) \leq 2\lambda_8(G)$ and $d_G(S_1 \cap S_2) + d_G(S_1 \cup S_2) \leq 2\lambda_8(G)$. Then $|S_1 \setminus S_2| \geq 6$ and $|S_1 \cap S_2| \geq 6$. By Lemma
20(a), each of $d_G(S_1 \setminus S_2)$, $d_G(S_2 \setminus S_1)$, $d_G(S_1 \cap S_2)$, and $d_G(S_1 \cup S_2)$ is not less than $\lambda_5(G)$. Noting $\lambda_5(G) = \lambda_6(G) = 18$, we have $d_G(S_1 \setminus S_2) = d_G(S_1 \cap S_2) = 18$. Hence $G[S_1 \setminus S_2]$ and $G[S_1 \cap S_2]$ are isomorphic to $K_{3,3}$ or $K_{3,4}$. So Claim 4 holds.

Since $G[S]$ is not a regular graph, there is an automorphism $\varphi_{10}$ of $G$ such that $\varphi_{10}(S) \neq S$ and $\varphi_{10}(S) \cap S \neq \emptyset$. Then $G[S \setminus \varphi_{10}(S)]$ and $G[S \cap \varphi_{10}(S)]$ are isomorphic to $K_{3,3}$ or $K_{3,4}$ by Claim 4. Set $B = S \cap \varphi_{10}(S)$.

**Claim 5.** $S$ has no subset $A$ different from $S \setminus B$ and $B$ such that $G[A]$ is isomorphic to $K_{3,4}$ and $G[S \setminus A]$ is isomorphic to $K_{3,3}$ or $K_{3,4}$.

Suppose, to the contrary, that $S$ has a subset $A$ satisfying the above condition. Assume $|S| = 13$. As $|W| = |Z| + 1 = 7$, we know $|A \cap W| = 4$. It follows that there is a vertex $v_4 \in S$ with $d_G[S](v_4) = 6$. Choose a vertex $v_5 \in S$ such that $d_G[S](v_5) \geq 5$ and $\{|v_4, v_5| \cap W| = 1$. Let $\varphi_{11}$ be an automorphism of $G$ such that $\varphi_{11}(v_5) = v_4$. Then $\varphi_{11}(S) \neq S$ and $\varphi_{11}(N_G[S](v_5)) \subseteq N_G[S](v_4)$, contradicting that $G[S \cap \varphi_{11}(S)]$ is isomorphic to $K_{3,3}$ or $K_{3,4}$ by Claim 4. Assume next $|S| = 14$. Then each of $G[S \setminus B], G[B], G[A]$ and $G[S \setminus A]$ is isomorphic to $K_{3,4}$. As $|E(S)| = 33$, we know $d_G[S](B) = 9$. If $|A \cap B| = 1$, then $d_G[S](B \setminus A) = 5 + 5, a contradiction. Thus Claim 6 holds.

**Claim 6.** Each vertex in $G$ is contained in exactly two distinct $\lambda_8$-atoms of $G$.

By the vertex-transitivity of $G$, it only needs to show that $S' = S$ or $\varphi_{10}(S)$ for a $\lambda_8$-atom $S'$ of $G$ with $S' \cap B \neq \emptyset$. Suppose $S' \neq S$ and $S' \neq \varphi_{10}(S)$. By Claims 4 and 5, we have $S' \cap S = B = S' \cap \varphi_{10}(S)$. Then $18 = d_G(B) \geq d_G[S](B) + d_G[S](\varphi_{10}(S))(B) + d_G[S'](B) \geq 3 \times 9$, a contradiction. Thus Claim 6 holds.

Let $D$ be one of $S \setminus B$ and $B$ such that $G[D]$ is isomorphic to $K_{3,4}$. Choose two vertices $v_6$ and $v_7$ in $D$ such that $d_G[D](v_6) = d_G[D](v_7) = 1 = 3$. By Claim 6, there is only one $\lambda_8$-atom $T$ of $G$ which is different from $S$ and contains $v_6$. By Claims 4 and 5, we have $S \cap T = D$. By Claim 6, $S$ and $T$ are the only $\lambda_8$-atoms of $T$ which contain $v_7$. It implies that there is no automorphism $\varphi$ of $G$ such that $\varphi(v_6) = v_7$, a contradiction.

### 4 Proof of Theorem 2

In this section we complete the proof of Theorem 2.

**Proof of Theorem 2.** If $G$ is 4-factor-critical, then by Theorem 8 and Theorem 4 we have $k = \lambda(G) \geq 5$. So we consider the sufficiency. Suppose $k \geq 5$. We will prove that $G$ is 4-factor-critical.

Suppose, to the contrary, that $G$ is not 4-factor-critical. We know by Theorem 1 that $G$ is bicritical. By Lemma 22, there is a subset $X \subseteq V(G)$ with $|X| \geq 4$ such that $c_0(G - X) = |X| - 2$ and every component of $G - X$ is factor-critical. Let $H_1, H_2, \ldots, H_p, H_{p+1}, \ldots, H_t$ be the components of $G - X$, where $t = |X| - 2$ and $H_1, H_2, \ldots, H_p$
are the nontrivial components of $G - X$. We know $p \geq 1$ by Lemma 23. For each $i \in [p]$, since $H_i$ is factor-critical, $\delta(H_i) \geq 2$. For every subset $J \subseteq [t]$, we have

$$\sum_{i \in J} d_G(H_i) + \lambda(G)(t - |J|) \leq \sum_{i=1}^t d_G(H_i) \leq d_G(X) = k(t + 2) - 2|E(X)|,$$

which implies

$$\sum_{i \in J} d_G(H_i) + 2|E(X)| \leq k(|J| + 2). \quad (1)$$

Hence $|E(X)| \leq k$. Set $Y = \bigcup_{j=p+1}^t V(H_j)$.

**Case 1.** $g(G) = 3$.

By Lemma 18, $|E(X)| \geq t - p = |X| - 2 - p$.

**Subcase 1.1.** $d_G(A) \geq 2k - 2$ for all $A \subseteq V(G)$ with $2 \leq |A| \leq |V(G)| - 2$.

For each $i \in [p]$, we have $d_G(H_i) \geq 2k - 2$. If $k$ is odd, then $d_G(H_i)$ is odd and hence $d_G(H_i) \geq 2k - 1$. So $d_G(H_i) \geq 2k - \frac{1}{2}(3 + (-1)^k)$ for each $i \in [p]$. Now we have

$$(2k - \frac{1}{2}(3 + (-1)^k))p + 2(|X| - 2 - p) \leq \sum_{i=1}^p d_G(H_i) + 2|E(X)| \leq k(p + 2), \quad (2)$$

which implies $(k - 2 - \frac{1}{2}(3 + (-1)^k))p + 2(|X| - 2 - k) \leq 0$. Hence $|X| \leq k + 1$.

Suppose $|X| < k$. Then $p = t = |X| - 2$. By Theorem 9, $|X| \geq \kappa(G) > \frac{2}{3}k$. Hence we know from (2) that $2k \geq (k - \frac{1}{2}(3 + (-1)^k))p > (k - \frac{1}{2}(3 + (-1)^k))(\frac{2}{3}k - 2)$. That is, $k^2 - 7k + 3 < 0$ if $k$ is odd and $k^2 - 8k + 6 < 0$ otherwise. It follows that $k \leq 6$. If $k = 6$, then $|X| \geq \kappa(G) = k$ by Lemma 11, a contradiction. Thus $k = 5$. Then $\kappa(G) = |X| = 4$. By Lemma 10, $\tau(G) = 2$. It implies that there is an edge $x_0y_0 \in E(G)$ such that $|N_G(x_0) \cap N_G(y_0)| = 4$.

Noting $k = 5$, we know from (2) that $|E(X)| \leq 1$. Choose a vertex $u \in X$ with $d_G(x_0)(u) = 0$. Since $G$ is vertex-transitive, there is an automorphism $\varphi_1$ of $G$ such that $\varphi_1(x_0) = u$. Assume, without loss of generality, that $\varphi_1(y_0) \in V(H_1)$. Noting $|N_G(x_0) \cap N_G(y_0)| = 4$, we have $N_G(u) \subseteq V(H_1)$. Then $d_G(V(H_1) \cup \{u\}) = d_G(X) - d_G(H_2) - 5 \leq 20 - 9 - 5 < 2k - 2$, a contradiction.

Thus $k \leq |X| \leq k + 1$. Noting $(k - 2 - \frac{1}{2}(3 + (-1)^k))p + 2(|X| - 2 - k) \leq 0$, we have $p \leq 2$ and $k \leq 8$. Then $|Y| = |X| - 2 - p \geq k - 4 \geq 1$. For any given vertex $v$, let $q$ be the number of triangles containing $v$ in $G$. By the vertex-transitivity of $G$, each vertex in $G$ is contained in $q$ triangles of $G$, which implies that each edge in $G$ is contained in at most $q$ triangles of $G$.

**Claim 1.** $E(X)$ is a matching of $G$.

Assume $p = 2$ or $|X| = k + 1$. Then we know from (2) that $|E(X)| = |X| - 2 - p = |Y|$. Since there are $q|Y|$ triangles of $G$ containing one vertex in $Y$, each edge in $E(X)$ is contained in $q$ triangles of $G$. It implies that $E(X)$ is a matching of $G$. Next we assume $p = 1$ and $|X| = k$. If two edges in $E(X)$ are adjacent, then $|E(X)| = q \geq 2|Y| = 2(k - 3)$.
and hence \( d_G(H_1) + 2|E(X)| \geq 2k - 2 + 4(k - 3) > 3k \), which contradicts the inequality (1). So Claim 1 holds.

By Claim 1, it follows that each edge incident with a vertex in \( Y \) is contained in at most one triangle of \( G \). Then, by the vertex-transitivity of \( G \), each edge in \( E(X) \) is contained in at most one triangle of \( G \).

Suppose \( |X| = k + 1 \). From (2), we know \( k \leq 6, p = 1 \) and \( |E(X)| = |Y| = k - 2 \). Since \( G \) has \( q|Y| \) triangles containing one vertex in \( Y \), each edge in \( E(X) \) is contained in \( q \) triangles of \( G \). Noting that each edge in \( E(X) \) is contained in at most one triangle of \( G \), we have \( q = 1 \). Then \( |E(N_G(u))| = 1 \) for each \( u \in Y \), which implies \( |X| \geq 2|E(X)| + (k - |E(X)| - 1) = 2k - 3 > k + 1 \), a contradiction.

Thus \( |X| = k \). Then for each \( e \in E(X) \) and each \( u \in Y \), \( G \) has a triangle containing \( e \) and \( u \). As each edge in \( E(X) \) is contained in at most one triangle of \( G \), it follows that \( |Y| = 1 \), which implies \( p = 2 \) and \( k = 5 \). From (2), we know \( d_G(H_1) = d_G(H_2) = 9 \) and \( |E(X)| = 1 \). Assume \( |V(H_1)| \leq |V(H_2)| \). Let \( u_1 \) be the vertex in \( Y \). For a vertex \( u_2 \in V(H_1) \) with \( N_G(u_2) \cap X \neq \emptyset \), we have \( |N_G(u_2) \cap X| \leq 3 \) as \( \delta(H_1) \geq 2 \). As \( H_2 \) is a component of \( G - N_G(u_1) \) with maximum cardinality, we have, by the definition of the vertex-transitivity of \( G \), that \( H_2 \) also is a component of \( G - N_G(u_2) \) with maximum cardinality. Then \( N_G(X \setminus N_G(u_2)) \cap V(H_2) = \emptyset \). It implies \( d_G(V(H_1) \cup (X \setminus N_G(u_2))) < 8 = 2k - 2 \), a contradiction. Hence Subcase 1.1 cannot occur.

**Subcase 1.2.** There is a subset \( A \subseteq V(G) \) with \( 2 \leq |A| \leq |V(G)| - 2 \) such that \( d_G(A) < 2k - 2 \).

We choose a subset \( S \) of \( V(G) \) such that \( 1 < |S| \leq \frac{1}{2}|V(G)|, d(S) \) is as small as possible, and, subject to these conditions, \( |S| \) is as small as possible. Then \( d_G(S) \leq d_G(A) \leq 2k - 3 \). By Corollary 14, \( d_G(S) = |S| \geq k \) and \( G[S] \) is \((k - 1)\)-regular. As \( 2k - 3 < \frac{2}{5}(k + 1)^2 \), \( S \) is an imprimitive block of \( G \) by Theorem 13. Thus \( G[S] \) is vertex-transitive by Lemma 12. We also know that the orbits \( S = S_1, S_2, \ldots, S_m \) of \( S \) under the automorphism group of \( G \) form a partition of \( V(G) \) and each \( G[S_i] \) is \((k - 1)\)-regular.

Set \( I_i = \{ j \in \{ 1, 2, \ldots, m_1 \} : S_j \cap V(H_i) \neq \emptyset \} \) for each \( i \in [t] \) and set \( \mathcal{M} = \bigcup_{i \in I_i} S_j : i \in [t] \). If any two sets in \( \mathcal{M} \) are disjoint, then \( 2|X| \geq 2|\bigcup_{U \in \mathcal{M}} \nabla(U)| \geq \sum_{U \in \mathcal{M}} d_G(U) \geq |\mathcal{M}|d_G(S) \).

Suppose \( |S| = k \). Then each \( G[S] \) is isomorphic to \( K_k \) and hence \( G[S_j] \) has common vertices with at most one component of \( G - X \). Hence \( |\mathcal{M}| = c_0(G - X) = |X| - 2 \) and any two sets in \( \mathcal{M} \) are disjoint. Then \( 2|X| \geq |\mathcal{M}|d_G(S) = (|X| - 2)k > 2|X|, \) a contradiction.

Suppose \( |S| = k + 1 \). As \( \delta(H_j) \geq 2 \) for each \( j \in [p] \), we have that for each \( S_i \), \( |S_i \setminus X| = |S_i \setminus X| = 2 \) or \( S_i \setminus X \subseteq V(H_{i'}) \) for some \( i' \in [t] \). Hence \( |\mathcal{M}| \geq p + \frac{1}{2}(t - p) = \frac{1}{2}(t + p) \geq \frac{1}{2}(t + 1) = \frac{1}{2}(|X| - 1) \) and any two sets in \( \mathcal{M} \) are disjoint. Then \( 2|X| \geq |\mathcal{M}|d_G(S) \geq \frac{1}{2}(|X| - 1)(k + 1) > 2|X|, \) a contradiction.

Thus \( |S| \geq k + 2 \). Noting that \( (k - 1)|S| \) is even and \( k + 2 \leq |S| \leq 2k - 3 \), we have \( |S| = k + 2 \) if \( 5 \leq k \leq 6 \). For each \( i \in [p] \), if \( V(H_i) \cap S_j \neq \emptyset \), then \( |V(H_i) \cap S_j| \geq 2 \) as \( \delta(H_i) \geq 2 \).

**Claim 2.** For each \( S_i \), there is an element \( a_i \in [p] \) such that \( V(H_{a_i}) \cap S_i \neq \emptyset \).
Suppose $S_i \subseteq X \cup Y$. By Lemma 16, $|S_i \cap Y| \leq \frac{1}{3}|S_i|$. If $k \geq 6$, then $|E(X)| \geq |E(S_i \cap X)| = \frac{1}{2}(k-1)|S_i| \geq \frac{1}{2}(k-1)(k+2) > k$, a contradiction. Thus $k = 5$. Then $|S_i| = k + 2$ and $|S_i \cap Y| \leq \frac{1}{3}|S_i| = 2$. Hence $|E(X)| \geq |E(S_i \cap X)| \geq \frac{1}{2}(k-1)(|S_i| - 4) = \frac{1}{2}(k-1)(k-2) > k$, a contradiction. So Claim 2 holds.

Claim 3. $X \setminus S_i \neq \emptyset$ for each $S_i$.

Suppose $X \subseteq S_i$. Choose a component $H_j$ of $G-X$ such that $H_j \neq H_{a_i}$. Then $|V(H_j)\cap S_i| = |N_G(V(H_j)\cap S_i)\setminus S_i| \leq |V(H_j)\setminus S_i|$. Hence $V(H_j)\setminus S_i \neq \emptyset$. Then there is some $S_j \subseteq V(H_j)\setminus S_i$. Now we know $d_G(V(H_j)\setminus S_i) \geq d_G(S_i) = |S_i|$. On the other hand, we have $d_G(V(H_j)\setminus S_i) \leq |S_i|\setminus V(H_{a_i})| \leq |S_i|$, a contradiction. So Claim 3 holds.

Claim 4. For each $i \in [p]$, we have $d_G(H_i) \geq 2k-2$ if there is some $S_j$ such that $S_j \cap V(H_i) \neq \emptyset$ and $S_j \setminus V(H_i) \neq \emptyset$.

Suppose $S_j \cap V(H_i) \neq \emptyset$ and $S_j \setminus V(H_i) \neq \emptyset$. By Claim 3, $X \setminus S_j \neq \emptyset$. Suppose $|V(H_i) \cup S_j| = 1$. Then $V(H_i) \cup S_j = X \setminus S_j$, which implies $|V(H_i) \cup X| = 1$. Hence $t = 2$ and $p = 1$, implying $|X| - 2 \geq k - 2 > 2$, a contradiction. Thus $|V(H_i) \cup S_j| \geq 2$. Then $|S_j| = d_G(S) \leq d_G(V(H_i) \cup S_j) \leq |V(H_i), V(H_i) \cup S_j| \cup |S_j| \setminus V(H_i)|$, which implies $|V(H_i), V(H_i) \cup S_j| \cup |S_j| \setminus V(H_i)| \geq d_G(S_j \cap V(H_i)) + |V(H_i), V(H_i) \cup S_j| \cup |S_j| \setminus V(H_i)|$. Hence $d_G(H_i) \geq d_G(S_j \cap V(H_i)) + |V(H_i), V(H_i) \cup S_j| \cup |S_j| \setminus V(H_i)| \geq d_G(S_j \cap V(H_i)) + |S_j| \setminus V(H_i)| \geq 2k-4$ by Corollary 15, which implies $d_G(H_i) \geq 2k-4 + |S_j| \setminus V(H_i)| \geq 2k-2$. If $|S_j| \setminus V(H_i)| = 1$, then $d_G(H_i) \geq k - 1 + |S_j| \setminus V(H_i)| \geq 2k$. Claim 4 holds.

Claim 5. $S_i \subseteq V(H_{a_i}) \cup X$ for each $S_i$.

Suppose, to the contrary, that $G-X$ has a component $H_b$ with $V(H_b) \cap (S_i \setminus V(H_{a_i})) \neq \emptyset$. Let $\theta$ be an integer such that $\theta = 1$ if $|V(H_b) = 1$ and $\theta = 0$ otherwise. As $X \setminus S_i \neq \emptyset$ by Claim 2, there is some $S_j$ with $S_j \cap (X \setminus S_i) \neq \emptyset$. Set $J = \{a_i, b\} \cup \{a_j\}$. For each $i' \in [p]$, we have $d_G(H_{i'}) \geq d_G(S) \geq k+2$ and furthermore $d_G(H_{i'}) \geq 2k-2$ by Claim 4 if $i' \in [p] \cap J$. If $|J| = 2$, then, noting that $d_G(S_j \setminus V(H_{a_i}) \cap S_i) \geq 2k-4$ by Corollary 15 and $\lambda(G[S_j]) = k-1$ by Theorem 8, we have $d_G(H_{a_i}) \geq d_G(S_j \setminus V(H_{a_i}) \cap S_i) + d_G(S_j \setminus V(H_{a_i}) \cap S_j) \geq 2k-4 + k = 3k-5$.

Assume $5 \leq k \leq 6$. We know that $|S| = k + 2$ and $G[S \cap V(H_{i'})]$ is isomorphic to $K_2$ for each $i' \in \{a_i, b\} \cap [p]$. Hence $S_i \subseteq V(H_{a_i}) \cup V(H_b) \cup X$. If $\theta = 1$, then $|E(G[S \cap X])| = \frac{1}{2}((k-1)|S_i \cap X| - (k-1) - (2k-4)) \geq \frac{1}{2}(k^2 - 5k + 6) \geq 3$. If $\theta = 0$, then $k = 6$ as $G[S_i]$ is vertex-transitive, which implies $|E(S_i \cap X)| = 2$. Now we have

\[
\sum_{i' \in J} d_G(H_{i'}) + 2|E(X)|
\geq (3k-5)(3 - |J|) + 2(2k-2)(|J| - 2) + \theta k + (1 - \theta)(2k-2) + 2|E(S_i \cap X)|
= k(|J| + 2) + |J| + k - 9 - \theta(k-2) + 2|E(S_i \cap X)| > k(|J| + 2),
\]

which contradicts the inequality (1).

Assume $k \geq 7$. If $\theta = 1$, then $t = |X| - 2 \geq k - 2 \geq 5$. If $\theta = 0$, then $t = |X| - 2 \geq
\[ [2k/3] - 2 \geq 3 \text{ by Theorem 9. Now we have} \]
\[
\sum_{\nu \in \mathcal{H}} \delta_G(H_{\nu}) + 2|E(X)| \\
\geq (3k - 5)(3 - |J|) + 2(2k - 2)(|J| - 2) + \theta(p - |J| + 1)(k + 2) + \\
(1 - \theta)(2k - 2 + (p - |J|)(k + 2)) + (t - p)k + 2(t - p) \\
= k(t + 2) + 2t + \theta(k + 2) + (1 - \theta)(2k - 2) - |J| - k - 7 > k(t + 2),
\]
which contradicts the inequality (1). So Claim 5 holds.

By Claims 2 and 5, it follows that \(|\mathcal{M}| = p = t \) and any two sets in \( \mathcal{M} \) are disjoint. Then \( 2|X| \geq |\mathcal{M}|d_G(S) \geq (|X| - 2)(k + 2) > 2|X|, \) a contradiction.

**Case 2.** \( g(G) \geq 4. \)

For each \( j \in [p], \) we know from (1) that \( \delta_G(H_j) \leq 3k. \) Let \( F_j \) be a component of \( G[V(H_j)] \) which contains a vertex in \( V(G \setminus (V(H_j) \cup X)). \) Then \( \nabla(F_j) \) is a 5-restricted edge-cut of \( G. \) Hence \( \lambda_5(G) \leq \delta_G(F_j) \leq \delta_G(H_j) \leq 3k. \) As it follows from Corollary 15 that \( \lambda_4(G) \geq 2k - 2, \) we have \( 2k - 2 \leq \lambda_4(G) \leq \lambda_5(G) \leq 3k. \)

**Claim 6.** If \( \lambda_5(G) \geq 4k - 8 \) and \( k \leq 6, \) then \( p = 1, |V(H_1)| \geq 7, \lambda_7(G) \leq 3k \) and furthermore, \( \lambda_8(G) \leq 3k: \) if \( \lambda_5(G) > 4k - 8. \)

Suppose \( \lambda_5(G) \geq 4k - 8 \) and \( k \leq 6. \) Then \( p = 1 \) by Lemma 23. We claim that \( G[V(H_1)] \) is connected. Otherwise, \( d_G(H_1) \geq \lambda(G) + \delta_G(F_1) \geq k + \lambda_5(G) > 3k, \) a contradiction. Suppose \( |V(H_1)| = 5. \) As \( g(G) \geq 4 \) and \( H_1 \) is factor-critical, \( H_1 \) is a 5-cycle of \( G. \) It follows that \( k = 5, E(X) = \emptyset \) and \( |X| \geq 8. \) Then \( g_9(G) \geq 7 \) by Lemma 17, a contradiction. Thus \( |V(H_1)| \geq 7. \) Then \( \nabla(H_1) \) is a 7-restricted edge-cut of \( G \) and \( \lambda_7(G) \leq d_G(V(H_1)) \leq 3k. \) If \( \lambda_5(G) > 4k - 8, \) then \( |X| \geq 7 \) and \( |V(H_1)| \geq 9 \) by Lemma 23, which implies \( \lambda_8(G) \leq d_G(H_1) \leq 3k. \) So Claim 6 holds.

By Claim 6, we can discuss Case 2 in the following two subcases.

**Subcase 2.1.** \( k = 5, \lambda_5(G) = 12 \) and \( \lambda_7(G) \geq 13. \)

We have \( \lambda_4(G) = 12. \) As \( \lambda_7(G) \) exists, \( |V(G)| \geq 14. \) Then, by Lemma 20(a), \( d_G(A) \geq \lambda_5(G) \) for each subset \( A \subseteq V(G) \) with \( |A| = 7, \) which implies that \( G \) has no subgraphs isomorphic to \( K_{3,4}. \) By the definition of the vertex-transitivity of \( G, \) we can obtain that \( G \) has no subgraphs isomorphic to \( K_{2,5}. \) By Claim 6, \( p = 1 \) and \( |V(H_1)| \geq 7. \) Hence \( |X| \geq 6 \) and \( |V(G)| \geq 16. \) By Lemma 20(a), \( d_G(V(H_1) \cup A) \geq \lambda_7(G) \) for each subset \( A \subseteq X \) with \( |A| \leq 1, \) which implies \( d_G(H_1) \geq 13 \) and \( |N_G(u) \cap V(H_1)| \leq 3 \) for each \( u \in X. \) Noting \( \delta(H_1) \geq 2, \) we have \( \mid \nabla(u) \cap \nabla(H_1) \mid \leq 3 \) for each \( u \in V(G). \)

**Claim 7.** There is no subset \( A \subseteq V(G) \) with \( |A| \leq 3 \) such that \( A \cap V(H_1) \neq \emptyset, \mid \nabla(A) \cap \nabla(H_1) \mid = 3 \mid A \mid \) and \( d_G((V(H_1) \cup A) \setminus (V(H_1) \cap A)) \leq 12. \)

Suppose, to the contrary, that such subset \( A \) of \( V(G) \) exists. Set \( B = (V(H_1) \cup A) \setminus (V(H_1) \cap A). \) Then \( |B| \geq 4 \) and \( |B| \geq 7. \) By Lemma 20(a), we have \( d_G(B) \geq \lambda_4(G) \) and furthermore, \( d_G(B) \geq \lambda_7(G) \) if \( |B| \geq 7. \) As \( d_G(B) \leq 12, \) we know \( |B| \leq 6 \) and \( d_G(B) = 12. \) It implies that \( E(V(H_1) \cap A) = \emptyset \) and \( G[B] \) is isomorphic to \( k_{2,2} \) or \( K_{3,3}. \)
Hence $G[V(H_1) \cup A]$ is bipartite. Then $H_1$ is bipartite, contradicting the fact that $H_1$ is factor-critical. So Claim 7 holds.

As $\lambda_5(G) = 12 < \lambda_7(G)$ and $k = 5$, each $\lambda_5$-atom of $G$ induces a subgraph which is isomorphic to $K_{3,3}$. Let $T_1, T_2, \ldots, T_{m_2}$ be all the subsets of $V(G)$, which induce subgraphs isomorphic to $K_{3,3}$. Let $R_i$ be the set of vertices in $X$ with $i$ neighbors in $V(H_1)$ for $1 \leq i \leq 3$ and let $Q$ be the set of vertices in $V(H_1)$ with 3 neighbors in $X$.

**Subcase 2.1.1.** There are two distinct $T_i$ and $T_j$ with $T_i \cap T_j \neq \emptyset$.

Noting that $G$ has no subgraphs isomorphic to $K_{3,3}$ or $K_{2,5}$, we have $|T_i \cap T_j| = 2$ or 4. If $|T_i \cap T_j| = 4$, then $d_G(T_i \cap T_j) \leq 12 < \lambda_7(G)$, which contradicts Lemma 20(a). Thus $|T_i \cap T_j| = 2$. Assume $T_i \cap T_j = \{v_1, v_2\}$.

**Claim 8.** For each $u \in X$ with $d_G(X)(u) = 0$ and $N_G(u) \cap V(H_1) \neq \emptyset$, we have $N_G(u) \cap V(H_1) \subseteq Q$ if $u \in R_1 \cup R_2$, and $|N_G(u) \cap V(H_1) \cap Q| \geq 1$ if $u \in R_3$.

Since $G$ is vertex-transitive, there is an automorphism $\varphi_2$ of $G$ such that $\varphi_2(v_1) = u$. If $u \in R_1 \cup R_2$, then $\varphi_2(N_G(v_2)) \subseteq X$, which implies $N_G(u) \cap V(H_1) \subseteq Q$. If $u \in R_3$, then $|\varphi_2(N_G(v_2)) \cap X| \geq 3$, which implies $|N_G(u) \cap V(H_1) \cap Q| \geq 1$. So Claim 8 holds.

Assume $E(X) \neq \emptyset$. Then $|E(X)| = 1$ and $\sum_{i=1}^3 |R_i| = 13$, which implies $\sum_{i=1}^3 |R_i| \geq 5$. By Claim 8, $Q \neq \emptyset$. We have $d_G(V(H_1) \setminus \{u\}) \leq 12$ for each $u \in Q$, contradicting Claim 7.

Thus $E(X) = \emptyset$. As $d_G(V(H_1) \cup A) \geq \lambda_4(G)$ for each subset $A \subseteq X$ with $|A| = 4$ by Lemma 20(a), we have $|R_3| \leq 3$. By Claim 8, $|\nabla(Q) \cap \nabla(H_1)| \geq |R_3| + 2|R_2| + |R_1| = 15 - 2|R_3| \geq 9$, which implies $|Q| \geq 3$. Choose a subset $Q' \subseteq Q$ with $|Q'| = 3$. Then $d_G(V(H_1) \setminus Q') \leq 12$, contradicting Claim 7. Hence Subcase 2.1.1 cannot occur.

**Subcase 2.1.2.** Any two distinct $T_i$ and $T_j$ are disjoint.

By the vertex-transitivity of $G$, each vertex in $G$ is contained in a $\lambda_5$-atom of $G$. Hence $T_1, T_2, \ldots, T_{m_2}$ form a partition of $V(G)$.

Assume $E(X) \neq \emptyset$. As $c_0(G - X) = |X| - 2$ and $|E(X)| = 1$, it follows that there is some $T_i$ such that $T_i \cap X \neq \emptyset$, $T_i \cap V(H_1) \neq \emptyset$ and $E(T_i) \cap E(X) = \emptyset$. Then there is a vertex $u_1 \in T_i \cap (R_3 \cup Q)$. By Claim 7, it follows that $u_1 \in X$. We have $d_G(V(H_1) \cup \{u_1\}) = 12 < \lambda_7(G)$, contradicting Lemma 20(a).

Thus $E(X) = \emptyset$. Set $\mathcal{B}_1 = \{T_j : |T_j \cap X| = 3, j \in [m_2]\}$ and $\mathcal{B}_2 = \{T_j : |T_j \cap X| < 3, j \in [m_2]\}$. Let $D = \bigcup_{A \in \mathcal{B}_1} A \cap V(H_1) \cup \bigcup_{A \in \mathcal{B}_2} A \cap X$. Noting $c_0(G - X) = |X| - 2$ and $p = 1$, we have $|D| = 3$. By Claim 7, we have $D \subseteq X$. If $|X| \geq 7$, then $d_G(H_1 + D) = 12 < \lambda_7(G)$, which contradicts Lemma 20(a). Thus $|X| = 6$. As $G$ has no subgraphs isomorphic to $K_{2,5}$, we know that $|R_2| = |R_3| = 3$ and $G[Y \cup R_2]$ is isomorphic to $K_{3,3}$. Choose a vertex $u_3 \in R_3$ and a vertex $u_4 \in Y$. Let $\varphi_3$ be an automorphism of $G$ such that $\varphi_3(u_4) = u_3$. Noting $\varphi_3(Y \cup R_2) \cap (Y \cup R_2) = \emptyset$, we have $\varphi_3(Y) = R_3$ and $\varphi_3(R_2) \subseteq V(H_1)$. It implies $D \subseteq V(H_1)$ by the choice of $D$, a contradiction. Hence Subcase 2.1 cannot occur.

**Subcase 2.2.** $k \neq 5$, $\lambda_5(G) \neq 12$ or $\lambda_5(G) = \lambda_7(G) = 12$. 

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Let $S'$ be a $\lambda_s$-atom of $G$, where

$$s = \begin{cases} 
4, & \text{if } k \leq 6 \text{ and } \lambda_5(G) < 4k - 8; \\
7, & \text{if } k = 5 \text{ and } \lambda_5(G) = \lambda_7(G) = 12; \\
6, & \text{if } k = 5 \text{ and } \lambda_5(G) = \lambda_6(G) = 13; \\
7, & \text{if } k = 5, \lambda_5(G) = 13 \text{ and } \lambda_6(G) = \lambda_7(G) = 14; \\
8, & \text{if } k = 5, \lambda_5(G) = 13, \lambda_6(G) \geq 14 \text{ and } \lambda_8(G) = 15; \\
5, & \text{if } k = 5 \text{ and } \lambda_5(G) = 14; \\
6, & \text{if } k = 5 \text{ and } \lambda_5(G) = \lambda_6(G) = 15; \\
5, & \text{if } k = 6 \text{ and } \lambda_5(G) = 4k - 8; \\
8, & \text{if } k = 6 \text{ and } \lambda_5(G) = 18; \\
5, & \text{if } k \geq 7.
\end{cases}$$

**Claim 9.** $S'$ is an imprimitive block of $G$ such that $|S'| > \frac{1}{2}\lambda_s(G)$ if $k \leq 6$ and $|S'| > \frac{1}{3}\lambda_s(G)$ otherwise.

If $k = 5$ and $\lambda_5(G) = \lambda_7(G) = 12$, then, by Lemma 24, Claim 9 holds. So we assume $k > 5$ or $\lambda_5(G) \neq 12$. By Lemma 6, $\frac{1}{2}|S'|^2 \geq 2|E(S')| = k|S| - \lambda_s(G)$. If $5 \leq k \leq 6$ and $\lambda_5(G) < 4k - 8$, then $\frac{1}{2}|S'|^2 \geq k|S' - \lambda_s(G)| > k|S'| - 4k + 8$, which implies $|S'| > 2k - 4 \geq \max\{2(s-1), \frac{1}{2}\lambda_s(G)\}$. If $5 \leq k \leq 6$ and $\lambda_5(G) \geq 4k - 8$, then $|S'| > 2(s-1)$ and $2|S'| > \lambda_s(G)$ by Lemmas 21 and 25-29. If $k \geq 7$, then $\frac{1}{2}|S'|^2 \geq k|S'| - \lambda_s(G) \geq k|S'| - 3k$ and hence $|S'| > k + 2 > \max\{2(s-1), \frac{1}{3}\lambda_s(G)\}$. Suppose $S'$ is not an imprimitive block of $G$. Then there is an automorphism $\varphi$ of $G$ such that $\varphi(S') \neq S'$ and $\varphi(S') \cap S' \neq \emptyset$. By Lemma 20(c), $|S'| = |S' \cap \varphi(S')| + |S' \setminus \varphi(S')| \leq 2(s-1)$, a contradiction. So Claim 9 holds.

By Claim 9 and Lemma 12, $G[S']$ is vertex-transitive and hence it is $(k-1)$-regular if $k \leq 6$ and is $(k-1)$-regular or $(k-2)$-regular otherwise. From Claim 9, we also know that the orbits $S' = S'_1, S'_2, \ldots, S'_{m_3}$ of $S'$ under the automorphism group of $G$ form a partition of $V(G)$.

**Claim 10.** $G[S']$ is $(k-1)$-regular.

Suppose that $G[S']$ is $(k-2)$-regular. Then $k \geq 7$, $s = 5$ and $2|S'| = \lambda_s(G) \leq 3k$, which implies $|S'| \leq \frac{3k}{2}$. By Lemma 6, $\frac{1}{2}|S'|^2 \geq |E(S')| = \frac{1}{2}k(k-2)|S'|$, which implies $|S'| \geq 2(k-2)$. Now $2(k-2) \leq |S'| \leq \frac{3k}{2}$, which implies $k \leq 8$ and $|S'| = 2(k-2)$. Hence $G[S']$ is isomorphic to $K_{k-2,k-2}$. For each $i \in [p]$, noting $3k \geq d_G(H_i) \geq \lambda_s(G) = 4(k-2)$ and that $d_G(H_i)$ has the same parity with $k$, we have $d_G(H_i) = 3k$. Hence $p = 1$, $E(X) = \emptyset$, $|V(H_1)| > 5$ and $|X| \geq k$. As $c_0(G - X) = |X| - 2$, there is some $S'_i$ with $S'_i \cap X \neq \emptyset$ and $S'_i \cap V(H_1) \neq \emptyset$. Then there is a vertex $u \in S'_i$ with $|\nabla(u) \cap \nabla(H_1)| \geq k-2$. Then $d_G(V(H_1) \cup \{u\}) \leq d_G(H_1) - (k-4) = 2k + 4 < 4(k-2) = \lambda_s(G)$ if $u \in X$ and $d_G(V(H_1) \setminus \{u\}) < \lambda_s(G)$ otherwise, contradicting Lemma 20(a). So Claim 10 holds.

As $\delta(H_j) \geq 2$ for each $i \in [p]$, it follows from Claim 10 that $\delta(G[V(H_j) \cap S'_i]) \geq 1$ if $V(H_j) \cap S'_i \neq \emptyset$. 

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Claim 11. For each $S'_i$, $S'_i \setminus (X \cup Y) \neq \emptyset$ or $|S'_i \cap X| = |S'_i \cap Y|$.

Suppose $|S'_i \cap X| > |S'_i \cap Y|$ for some $S'_i \subseteq X \cup Y$. If $G[S'_i]$ is bipartite, then $|S'_i \cap Y| \leq |S'_i \cap X| - 2$. If $G[S'_i]$ is non-bipartite, then $|S'_i \cap Y| \leq \alpha(G[S'_i]) \leq \frac{1}{4}|S'_i| - \frac{k-1}{2}$ by Lemma 16, which implies $|S'_i \cap Y| \leq |S'_i \cap X| - \frac{k-1}{2} \leq |S_i \cap X| - 2$. Thus $|E(S'_i \cap X)| = \frac{1}{2}(k-1)(|S'_i \cap X| - |S'_i \cap Y|) \geq k-1$. Noting $d_G(H_i) \geq \lambda_5(G) \geq 2k-2$, we have $d_G(H_i) + 2|E(X)| \geq 2k-2 + 2(k-1) > 3k$, a contradiction. So Claim 11 holds.

Subcase 2.2.1. $|S''| \leq 2k - 1$.

Claim 12. If $S'_i \cap V(H_j) \neq \emptyset$ for some $j \in [p]$, then $S'_i \subseteq V(H_j) \cup X$.

Suppose $S'_i \cap V(H_j) \neq \emptyset$ for some $j \in [p]$ and $S'_i \cap V(H_j') \neq \emptyset$ for some $j' \in [t] \setminus \{j\}$. As $\delta(G[S'_i \cap V(H_j)]) \geq 1$, there is an edge $x_i y_1 \in E(S'_i \cap V(H_j))$. Then $|S'_i \cap V(H_j) \cup X| \geq |N_G[S'_i](x_i) \cup N_G[S'_i](y_1)| = 2k - 2$. It implies $|S'_i \cap V(H_j')| = 1$ and $|S'_i| = 2k - 1$. Then $|V(H_j')| = 1$ and $|X| \geq |N_G(V(H_j'))| = 2$. Hence $|V(H_j) \cap S''| = |N_G(V(H_j)) \setminus S'_i| + (c_0(G-X) - 2) \geq 1 + k - 4 \geq 2$. By Corollary 15, we have

$$2k-2 \leq d_G(V(H_j) \cup S'_i)$$
$$\leq d_G(H_j) - d_{G[S'_i]}(S'_i \cap V(H_j)) + |S'_i \setminus V(H_j)|$$
$$= d_G(H_j) - ((k-1)|S'_i \cap X| - 2|E(S'_i \cap X)| - (k-1)) + |S'_i \cap X| + 1$$
$$= d_G(H_j) + 2|E(S'_i \cap X)| - (k-2)|S'_i \cap X| + k$$
$$\leq 3k - (k-2)(k-1) + k = -k^2 + 7k - 2,$$

which implies $k = 5$. It is easy to verify that there is no triangle-free non-bipartite 4-regular graph of order 9, which implies $|S''| \neq 9 = 2k - 1$, a contradiction. So Claim 12 holds.

Set $I'_i = \{j \in [m_3] : S'_i \cap V(H_i) \neq \emptyset\}$ for each $i \in [t]$ and $\mathcal{M}' = \{\bigcup_{j \in I'_i} S'_j : i \in [t]\}$. Then any two sets in $\mathcal{M}'$ are disjoint by Claim 12. By Lemma 20(a), $d_G(U) \geq \lambda_s(G)$ for each $U \in \mathcal{M}'$. Then, by Claim 11, we have

$$2(p + 2 + (k-1)(|\mathcal{M}'| - p))$$
$$= 2|X| \geq 2|\bigcup_{U \in \mathcal{M}'} \bigcup \{\mathcal{M}' \setminus |\mathcal{M}'| + 2\} - d_G(U) \geq |\mathcal{M}'| \lambda_s(G) \geq |\mathcal{M}'|(2k-2),$$

which implies $p \leq \frac{2k}{k-2} < 1$, a contradiction. Hence Subcase 2.2.1 cannot occur.

Subcase 2.2.2 $|S''| \geq 2k$.

We have $\lambda_s(G) = |S''| \geq 2k$. If $s = 4$, then $\lambda_5(G) \geq \lambda_s(G) \geq 2k$. If $s \geq 5$, then $\lambda_5(G) \geq 2k$ by the choice of $s$. Then $2kp \leq p\lambda_5(G) \leq \sum_{i=1}^p d_G(H_i) + 2|E(X)| \leq (k+2)p$, which implies $p \leq 2$.

Let $\mathcal{N} = \{S'_i : S'_i \cap X \neq \emptyset \text{ and } S'_i \setminus (X \cup Y) \neq \emptyset, i \in [m_3]\}$.

By Claim 11, $\sum_{A \in \mathcal{N}} |A \cap X| - |A \cap Y| = \sum_{i=1}^{m_3} (|S'_i \cap X| - |S'_i \cap Y|) = |X| - |Y| = p + 2$. Noting $|A \cap X| > |A \cap Y|$ for each $A \in \mathcal{N}$, we have $1 \leq |\mathcal{N}| \leq p + 2$. Choose a set $S'_{j_1} \in \mathcal{N}$. Without loss of generality, we assume $S'_{j_1} \cap V(H_1) \neq \emptyset$. 

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Suppose $p = 2$. Then $E(X) = \emptyset$ and $2k = \lambda_3(G) = d_G(H_1) = d_G(H_2)$. Hence $\lambda_4(G) = \lambda_3(G) = 2k = |S'|$. For each $u \in V(G)$ and each $i \in [p]$, we have $d_G(V(H_i) \cup \{u\}) \geq \lambda_4(G)$ and $d_G(V(H_i) \setminus \{u\}) \leq \lambda_4(G)$ by Lemma 20(a), which implies $|\nabla(u) \cap \nabla(H_i)| \leq k - 3$. Hence $|S'_j \setminus V(H_1)| \geq 2$ and $\delta(G[S'_j \cap V(H_1)]) \geq 2$, which implies $|S'_j \setminus V(H_1)| \geq 4$. Choose an edge $x_2y_2 \in E(S'_j \cap V(H_1))$. Then $|S'_j \setminus (V(H_1) \cup X)| \leq |S'_j \setminus (N_G[S'_j]\{x_2\} \cup N_G[S'_j]\{y_2\})| = 2$. It follows that $S'_j \cap V(H_2) = \emptyset$. Noting that $d_G(S'_j)(S'_j \cap V(H_1)) \geq k_4 - 4$ by Corollary 15, we have $|S'_j \setminus X| \geq |S'_j \setminus Y| + 2$. Now
\[
d_G(V(H_1) \cup S'_j) \leq d_G(H_1) - d_G[S'_j](V(H_1) \cap S'_j) + |S'_j \setminus V(H_1)|
= 2k - (k - 1)(|S'_j \setminus X| - |S'_j \setminus Y|) + |S'_j \setminus V(H_1)|
\leq 2k - 2(k - 1) + 2k - 4 = 4 = \lambda_4(G),
\]
contradicting Lemma 20(a).

Thus $p = 1$. Suppose $|\mathcal{N}| = 1$. Then $|S'_j \cap X| = |S'_j \cap Y| + 3$ and there is some $S'_j \subseteq V(H_1) \cup S'_j$. We know by Claim 11 that $G[S'_j]$ is bipartite. Hence there is some $S'_j \subseteq V(H_1) \setminus S'_j$. By Lemma 20(a), we have
\[
|S'| = \lambda_s(G) \leq d_G(V(H_1) \cup S'_j) \leq d_G(H_1) - d_G[S'_j](S'_j \setminus V(H_1)) + |S'_j \setminus V(H_1)|
= d_G(H_1) + 2|E(S'_j \cap X)| - 3(k - 1) + |S'_j \setminus V(H_1)|
\leq 3k - 3(k - 1) + |S'_j \setminus V(H_1)|.
\]
Similarly, we can obtain $|S'| \leq d_G(H_1 - S'_j) \leq 3 + |S'_j \setminus V(H_1)|$. Then $2|S'| \leq 6 + |S'_j|$, which implies $|S'| \leq 6 < 2k$, a contradiction.

Thus $|\mathcal{N}| \geq 2$. For each $S'_j \in \mathcal{N}$, noting $|S'_j \cap V(H_1)| \geq 2$, if $|S'_j \setminus V(H_1)| \geq 2$, then, by Corollary 15, we have $d_G[S'_j(S'_j \cap V(H_1)) \geq k_4 - 4$, which implies that $|S'_j \cap X| = 1$ if $|S'_j \cap X| = |S'_j \cap Y| + 1$. If $|\mathcal{N}| = 3$, then $|S'_j \cap X| = 1$ for each $S'_j \in \mathcal{N}$ and hence $d_G(V(H_1) \cup (\bigcup_{S'_j \in \mathcal{N}} S'_j)) \leq d_G(H_1) - 3(k - 2) \leq 6 < \lambda_s(G)$, which contradicts Lemma 20(a). Thus $|\mathcal{N}| = 2$. Assume $\mathcal{N} = \{S'_j, S'_j\}$ and $|S'_j \cap X| = 1$. We know that there is some $S'_j \subseteq V(G) \setminus (V(H_1) \cup S'_j \cup S'_j)$. By Lemma 20(a),
\[
|S| = \lambda_s(G) \leq d_G(V(H_1) \cup S'_j \cup S'_j) 
\leq d_G(H_1) - d_G[S'_j](S'_j \cap V(H_1)) + |S'_j \setminus V(H_1)| - (k - 2)
= d_G(H_1) + 2|E(S'_j \cap X)| - 2(k - 1) + |S'_j \setminus V(H_1)| - (k - 2)
\leq 3k - 3(k - 1) + |S'_j \setminus V(H_1)|.
\]
Similarly, we can obtain $|S'| \leq d_G(V(H_1) \cap S'_j \setminus S'_j) \leq 4 + |S'_j \cap V(H_1)|$. Then $2|S'| \leq 8 + |S'_j|$, which implies $|S'| \leq 8 < 2k$, a contradiction.

References