A new construction of non-extendable intersecting families of sets

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Abstract
In 1975, Lovász conjectured that any maximal intersecting family of \( k \)-sets has at most \( \lfloor (e - 1)k! \rfloor \) blocks, where \( e \) is the base of the natural logarithm. This conjecture was disproved in 1996 by Frankl and his co-authors. In this short note, we reprove the result of Frankl et al. using a vastly simplified construction of maximal intersecting families with many blocks. This construction yields a maximal intersecting family \( G_k \) of \( k \)-sets whose number of blocks is asymptotic to \( e^2 \left( \frac{k}{2} \right)^{k-1} \) as \( k \to \infty \).

Keywords: Intersecting family of \( k \)-sets, Maximal \( k \)-cliques.

1 Introduction
For positive integers \( k \), by a \( k \)-set we mean a set of size \( k \). The members of a family \( \mathcal{F} \) of \( k \)-sets are usually called the blocks of \( \mathcal{F} \). Such a family is said to be an intersecting family if any two of its blocks have a non-empty intersection. A family \( \mathcal{F} \) of \( k \)-sets is called a maximal intersecting family of \( k \)-sets if (a) \( \mathcal{F} \) is an intersecting family of \( k \)-sets, and (b) there is no intersecting family \( \mathcal{G} \) of \( k \)-sets such that \( \mathcal{G} \supseteq \mathcal{F} \). Maximal intersecting families of \( k \)-sets are also known as \( k \)-uniform maximal cliques.

This notion was introduced by Erdős and Lovász in [1]. In this paper they proved the amazing result that any maximal intersecting family of \( k \)-sets has at most \( k^k \) blocks, and hence for any given \( k \), there are only finitely many maximal intersecting families of \( k \)-sets. (This result may be viewed as a special case of [4, Theorem 2.3].) Therefore, Erdős and Lovász initiated the problem of finding or estimating the function \( M(k) \) defined by

\[
M(k) := \max \{|\mathcal{F}| : \mathcal{F} \text{ is a maximal intersecting family of } k \text{ - sets}\}.
\]

In [1], Erdős and Lovász also proved:
Lemma 1. For $k \geq 2$, $M(k) \geq 1 + k \cdot M(k-1)$.

Proof. Let $\mathcal{F}$ be a maximal intersecting family of $(k-1)$-sets with $M(k-1)$ blocks. Choose a $k$-set $B$ disjoint from all the blocks of $\mathcal{F}$, and consider the family

$$\hat{\mathcal{F}} := \{B\} \cup \{A \cup \{x\} : A \in \mathcal{F}, \ x \in B\}$$

Then it is easy to verify that $\hat{\mathcal{F}}$ is a maximal intersecting family of $k$-sets with $1+k\cdot M(k-1)$ blocks. \qed

Since $M(1) = 1$, using Lemma 1 and an easy induction on $k$, one deduces that

$$M(k) \geq k! \left( \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} \right) = [(e-1)k!].$$

Thus Erdős and Lovász showed

$$[(e-1)k!] \leq M(k) \leq k^k. \quad (1)$$

In [3], Lovász conjectured that the lower bound in (1) is sharp, i.e. $M(k) = [(e-1)k!]$ for all $k$. In [2], Frankl et al. disproved this conjecture (for all $k \geq 4$) by an extremely elegant but complicated family of counterexamples. Indeed, it is hard to verify that their construction actually yields a maximal intersecting family of $k$-sets. (We addressed this question in a recent paper [5] with Mukherjee.) There appears to be a gap in the proof sketched in [2]. Specifically, the Claim 2 in [2] seems to be incorrect. Therefore, it seems desirable to present a simpler construction (with short and complete proof) reproving this result.

In this note we prove:

**Theorem 2.**

$$M(k) \geq \begin{cases} \left( \frac{k}{2} + 1 \right)^{k-1} + (k-1)\left( \frac{k}{2} + 1 \right)^{\frac{k}{2}-1} & \text{for even } k, \\ 1 + k\left( \frac{k+1}{2} \right)^{k-2} + k(k-2)\left( \frac{k+1}{2} \right)^{\frac{k}{2}-3} & \text{for odd } k. \end{cases}$$

Note that this lower bound is asymptotic to $e^2(\frac{k}{2})^{k-1}$ through even $k$ and $2e(\frac{k}{2})^{k-1}$ through odd $k$. Since $k!$ grows roughly like $\left( \frac{k}{2} \right)^k$, the bound in Theorem 2 beats the lower bound of (1) by the exponential factor $\left( \frac{k}{2} \right)^k$ (roughly). This bound is comparable to the explicit bound of [2], but the latter is somewhat better. Our bound beats the lower bound of (1) for $k \geq 8$, while the bound in [2] is better than (1) for $k \geq 4$. In a final remark in section 2, we indicate how the lower bound of Theorem 2 may be sharpened.

## 2 The construction

In this section, $k$ is an even number. We construct a family $\mathcal{G}_k$ as follows.
Construction. Let $X_n$, $0 \leq n \leq k - 2$ be pairwise disjoint sets, where each $X_n$ is of size $1 + \frac{k}{2}$. Let $\alpha$ be a new symbol which does not belong to any of the $X_i$'s. The blocks of $\mathbb{G}_k$ are of two types. The blocks of type 1 are the sets $X_n \cup \{x_i : 1 \leq i \leq \frac{k}{2} - 1\}$, where $0 \leq n \leq k - 2$, and $x_i \in X_{n+i}$ for $1 \leq i \leq \frac{k}{2} - 1$. Here the addition in the suffix is modulo $k - 1$. The blocks of type 2 are the sets $\{\alpha\} \cup \{x_i : 0 \leq i \leq k - 2\}$, where $x_i \in X_i$ for $0 \leq i \leq k - 2$.

To appreciate the following proof, it may be helpful to visualize the sets $X_n$ as $k - 1$ equally spaced blobs arranged on a circle.

**Theorem 3.** For each even number $k$, $\mathbb{G}_k$ is a maximal intersecting family of $k$-sets.

**Proof.** Clearly $\mathbb{G}_k$ is a family of $k$-sets. For indices $m$, $n$ in the range $0 \leq m, n \leq k - 2$, let us write $m \to n$ to denote that $n \equiv m + i \pmod{k - 1}$, for some $i$ in the range $1 \leq i \leq \frac{k}{2} - 1$. Notice that, for $m \neq n$, exactly one of the relations $m \to n$ and $n \to m$ holds true.

Clearly any block of type 2 intersects all the blocks of $\mathbb{G}_k$. Let $B_1$ and $B_2$ be two blocks of type 1. Then there are two indices $m$, $n$ such that $B_1 \supseteq X_m$ and $B_2 \supseteq X_n$. If $m = n$, then $B_1$ and $B_2$ intersect at least in $X_m$. Otherwise, we may assume without loss of generality that $m \to n$. Then every block containing $X_m$ intersects $X_n$. Therefore $B_1$ and $B_2$ intersect in this case also. Thus, $\mathbb{G}_k$ is an intersecting family of $k$-sets.

Let $C$ be a set of size $k$ which intersects all the blocks of $\mathbb{G}_k$. To prove that $\mathbb{G}_k$ is maximal, it suffices to show that $C$ must be a block.

Case A : $\alpha \notin C$. Since $C$ intersects all the blocks of type 2, it follows that there is an index $m$ such that $C \supseteq X_m$. Since $C$ is a $k$-set and the $X$'s are pairwise disjoint sets of size $\frac{k}{2} + 1$, this index $m$ is unique. Suppose, if possible, that there is an index $n$ such that $C \cap X_n = \emptyset$ and $m \to n$. Since $C$ intersects all the blocks containing $X_n$, it follows there must exist an index $l$ such that $X_l \subset C$ and $n \to l$. By the uniqueness of the index $m$, we get $l = m$. Therefore, $m \to n$ and $n \to m$, a contradiction. Thus, $C$ intersects all the $(\frac{k}{2} - 1)$ sets $X_n$ such that $m \to n$. Since, also, $C \supseteq X_m$ it follows that there is a block $B \supseteq X_m$ such that $C \supseteq B$. Since $|C| = k = |B|$, we get that $C = B$ is a block of type 1 in this case.

Case B : $\alpha \in C$. Let $T = C \setminus \{\alpha\}$. Thus $T$ is a set of size $k - 1$ which intersects all the type 1 blocks. Suppose, if possible, that there is an index $n$ such that $T \cap X_n = \emptyset$. Then arguing as in the previous case, we see that there is a unique index $m$ such that $T \supseteq X_m$. Also, $n \to m$ for all indices $n$ for which $T \cap X_n = \emptyset$. Contrapositively, $X_n \cap T \neq \emptyset$ for all the $\frac{k}{2} - 1$ indices $n$ such that $m \to n$. Since $T \supseteq X_m$ and $|X_m| = \frac{k}{2} + 1$, it follows that $|T| \geq \frac{k}{2} + 1 + \frac{k}{2} - 1 = k$, and hence $|C| > k$, a contradiction. Thus, $T \cap X_n \neq \emptyset$ for all $n$. Since there are $k - 1 = |T|$ pairwise disjoint sets $X_n$, it follows that $|T \cap X_n| = 1$ for all $n$. Hence $C$ is a block of type 2 in this case. \(\square\)

**Proof of Theorem 2.** First let $k$ be an even positive integer. Note that $\mathbb{G}_k$ has $(\frac{k}{2} + 1)^{k-1}$ blocks of type 2 and $(k - 1)(\frac{k}{2} + 1)^{\frac{k}{2}-1}$ blocks of type 1. Therefore the total number of blocks in $\mathbb{G}_k$ is $(\frac{k}{2} + 1)^{k-1} + (k - 1)(\frac{k}{2} + 1)^{\frac{k}{2}-1}$. Since by Theorem 3, $\mathbb{G}_k$ is a maximal intersecting family of $k$-sets, this number is a lower bound on $M(k)$.
Next let \( k > 1 \) be an odd integer. (The result is trivial for \( k = 1 \).) Using Lemma 1 and the above bound (with \( k - 1 \) in place of \( k \)) we get \( M(k) \geq 1 + k \cdot M(k - 1) \geq 1 + k\left(\frac{k+1}{2}\right)^{k-2} + k(k - 2)\left(\frac{k+1}{2}\right)^{\frac{k+1}{2}-1} \).

**Remark.** When \( k \) is an odd positive integer, we may modify the above construction by putting \( |X_n| = \left\{ \begin{array}{ll}
\frac{k+1}{2} & \text{if } 0 \leq n \leq \frac{k-1}{2} - 1 \\
\frac{k+1}{2} + 1 & \text{if } \frac{k-1}{2} \leq n \leq k - 2
\end{array} \right. \) and taking the type 1 blocks of \( G_k \) to be the sets \( X_m \cup \{x_i : 1 \leq i \leq k - |X_m|\} \), where \( 0 \leq m \leq k - 2 \) and \( x_i \in X_{m+i} \) for all \( i \). Here addition in the suffix is modulo \( k - 1 \). The type 2 blocks are as before. Then it can be shown that the resulting family \( G_k \) is again a maximal intersecting family of \( k \)-sets. The proof is similar, but a little more complicated. Using this construction (together with the preceding construction for even positive integer \( k \)) we can prove the following estimate, which improves upon Theorem 2.

**Theorem 4.**

\[
M(k) \geq |G_k| = \left\{ \begin{array}{ll}
(k - 1)(\frac{k}{2} + 1)^{\frac{k-1}{2}} + (\frac{k}{2} + 1)^{k-1} & \text{if } k \text{ is an even integer} \\
\frac{k+3}{2}\left(\frac{k+3}{2}\right)^{\frac{k-1}{2}} - (\frac{k+1}{2})^{\frac{k-1}{2}} + (\frac{k+1}{2})^{\frac{k+1}{2}}(\frac{k+3}{2})^{\frac{k-1}{2}} & \text{if } k \text{ is an odd integer}
\end{array} \right.
\]

This lower bound is asymptotic to \( e^2(\frac{k}{2})^{k-1} \) as \( k \to \infty \). It seems safe to propose:

**Conjecture.** \( M(k) \) is asymptotic to \( e^2(\frac{k}{2})^{k-1} \) as \( k \to \infty \).

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**References**


