A better lower bound on average degree of 4-list-critical graphs

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Abstract

This short note proves that every non-complete $k$-list-critical graph has average degree at least $k - 1 + \frac{k-3}{k^2-2k+2}$. This improves the best known bound for $k = 4, 5, 6$. The same bound holds for online $k$-list-critical graphs.

1 Introduction

A graph $G$ is $k$-list-critical if $G$ is not $(k-1)$-choosable, but every proper subgraph of $G$ is $(k-1)$-choosable. For further definitions and notation, see [5, 2]. Table 1 shows some history of lower bounds on the average degree of $k$-list-critical graphs.

Main Theorem. Every non-complete $k$-list-critical graph has average degree at least

$$k - 1 + \frac{k-3}{k^2-2k+2}.$$ 

Main Theorem gives a lower bound of $3 + \frac{1}{10}$ for 4-list-critical graphs. This is the first improvement over Gallai’s bound of $3 + \frac{1}{13}$. The same proof shows that Main Theorem holds for online $k$-list-critical graphs as well. Our primary tool is a lemma proved with Kierstead [6] that generalizes a kernel technique of Kostochka and Yancey [8].

Definition. The maximum independent cover number of a graph $G$ is the maximum $\text{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets $I$ of $G$.

Kernel Magic (Kierstead and R. [6]). Every $k$-list-critical graph $G$ satisfies

$$2 \|G\| \geq (k-2)|G| + \text{mic}(G) + 1.$$
Table 1: History of lower bounds on the average degree $d(G)$ of $k$-critical and $k$-list-critical graphs $G$.

The previous best bounds in Table 1 for $k$-list-critical graphs hold for $k$-Alon-Tarsi-critical graphs as well. Since Kernel Magic relies on the Kernel Lemma, our proof does not work for $k$-Alon-Tarsi-critical graphs. Any improvement over Gallai’s bound of $3 + \frac{1}{13}$ for 4-Alon-Tarsi-critical graphs would be interesting.

## 2 The Proof

The connected graphs in which each block is a complete graph or an odd cycle are called **Gallai trees**. Gallai [4] proved that in a $k$-critical graph, the vertices of degree $k - 1$ induce a disjoint union of Gallai trees. The same is true for $k$-list-critical graphs [1, 3]. For a graph $T$ and $k \in \mathbb{N}$, let $\beta_k(T)$ be the independence number of the subgraph of $T$ induced on the vertices of degree $k - 1$ in $T$. When $k$ is defined in the context, put $\beta(T) := \beta_k(T)$.

**Lemma 1.** If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k - 1$, then

$$2 ||T|| \leq (k - 2)||T|| + 2\beta(T).$$

**Proof.** Suppose the lemma is false and choose a counterexample $T$ minimizing $|T|$. Plainly, $T$ has more than one block. Let $A$ be an endblock of $T$ and let $x$ be the unique cutvertex of $T$ with $x \in V(A)$. Consider $T' := T - (V(A) \setminus \{x\})$. By minimality of $|T|$,

$$2 ||T|| - 2 ||A|| \leq (k - 2)(|T| + 1 - |A|) + 2\beta(T').$$

Since $T$ is a counterexample, $2 ||A|| > (k - 2)(|A| - 1)$. So, if $k > 4$, then $A = K_{k-1}$ and if $k = 4$, then $A$ is an odd cycle. In both cases, $d_T(x) = k - 1$. Consider $T^* := T - V(A)$. By minimality of $|T|$,

$$2 ||T|| - 2 ||A|| - 2 \leq (k - 2)(|T| - |A|) + 2\beta(T^*).$$
Since \( T \) is a counterexample, \( 2 \| A \parallel + 2 > (k - 2) |A| + 2(\beta(T) - \beta(T^*)) \). In \( T^* \), all of \( x \)'s neighbors have degree at most \( k - 2 \). But \( d_T(x) = k - 1 \), so some vertex in \( \{x\} \cup N(x) \) is in a maximum independent set of degree \( k - 1 \) vertices in \( T \). Hence \( \beta(T^*) \leq \beta(T) - 1 \), which gives

\[
2 \| A \parallel > (k - 2) |A|
\]

a contradiction since \( k \geq 4 \). \( \square \)

**Proof of Main Theorem.** Let \( G \neq K_k \) be a \( k \)-list-critical graph. The theorem is trivially true if \( k \leq 3 \), so suppose \( k \geq 4 \). Let \( L \subseteq V(G) \) be the vertices with degree \( k - 1 \) and let \( H = V(G) \setminus L \). Put \( \|L\| := ||G[L]|| \) and \( \|H\| := ||G[H]|| \). By Lemma 1,

\[
2 \|L\| \leq (k - 2) |L| + 2\beta(L)
\]

Hence,

\[
2 \|G\| = 2 \|H\| + 2 \|H, L\| + 2 \|L\|
= 2 \|H\| + 2( (k - 1) |L| - 2 \|L\| ) + 2 \|L\|
= 2 \|H\| + 2(k - 1) |L| - 2 \|L\|
\geq 2 \|H\| + k |L| - 2\beta(L),
\]

which is

\[
\beta(L) \geq \|H\| + \frac{k}{2} |L| - \|G\|.
\]

Let \( M \) be the maximum of \( \|I, V(G) \setminus I\| \) over all independent sets \( I \) of \( G \) with \( I \subseteq H \). Since the vertices in \( L \) with \( k - 1 \) neighbors in \( L \) have no neighbors in \( H \),

\[
\text{mic}(G) \geq M + (k - 1)\beta(L).
\]

Applying Kernel Magic and using (1) gives

\[
2 \|G\| \geq (k - 2) |G| + M + (k - 1)\beta(L) + 1
\geq (k - 2) |G| + M + (k - 1) \left( \|H\| + \frac{k}{2} |L| - \|G\| \right) + 1
= (k - 2) |G| + M + (k - 1) \|H\| + \frac{k(k - 1)}{2} |L| - (k - 1) \|G\| + 1.
\]

Hence

\[
(k + 1) \|G\| \geq (k - 2) |G| + M + (k - 1) \|H\| + \frac{k(k - 1)}{2} |L| + 1
\]

(2)

Let \( C \) be the components of \( G[H] \). Then \( \alpha(C) \geq \frac{|C|}{\chi(C)} \) for all \( C \in C \). Whence

\[
M + (k - 1) \|H\| \geq \sum_{C \in C} k \frac{|C|}{\chi(C)} + (k - 1) \|C\|.
\]

(3)
If $\mathcal{L} = \emptyset$, then $G$ has average degree at least $k \geq k - 1 + \frac{k - 3}{k^2 - 2k + 2}$. So, assume $\mathcal{L} \neq \emptyset$. Then $G[H]$ is $(k-1)$-colorable by $k$-list-criticality of $G$. In particular, $\chi(C) \leq k - 1$ for every $C \in \mathcal{C}$. For every $C \in \mathcal{C}$,

$$k \frac{|C|}{\chi(C)} + (k - 1) ||C|| \geq \left( k - \frac{1}{2} \right) |C|.$$  

To see this, first suppose $C \in \mathcal{C}$ is not a tree. Then $||C|| \geq |C|$ and hence $k \frac{|C|}{\chi(C)} + (k - 1) ||C|| \geq k \frac{|C|}{k - 1} + (k - 1) |C| \geq (k - \frac{1}{2}) |C|$. If $C$ is a tree, then $\chi(C) \leq 2$ and hence $k \frac{|C|}{\chi(C)} + (k - 1) ||C|| \geq k \frac{|C|}{2} + (k - 1)(|C| - 1) \geq (k - \frac{1}{2}) |C|$ unless $|C| = 1$. This proves (4) since the bound is trivially satisfied when $|C| = 1$.

Now combining (2), (3) and (4) with the basic bound

$$|\mathcal{L}| \geq k |G| - 2 \|G||,$$

gives

$$(k + 1) \|G|| \geq (k - 2) |G| + \left( k - \frac{1}{2} \right) |\mathcal{H}| + \frac{k(k - 1)}{2} |\mathcal{L}| + 1$$

$$= \left( 2k - \frac{5}{2} \right) |G| + \frac{k^2 - 3k + 1}{2} |\mathcal{L}| + 1$$

$$\geq \left( 2k - \frac{5}{2} \right) |G| + \frac{k^2 - 3k + 1}{2} (k |G| - 2 \|G||) + 1.$$ 

After some algebra, this becomes

$$2 \|G|| \geq \left( k - 1 + \frac{k - 3}{k^2 - 2k + 2} \right) |G| + \frac{2}{k^2 - 2k + 2}.$$ 

That proves the theorem. \hfill \Box

The right side of equation (4) in the above proof can be improved to $k |C|$ unless $C$ is a $K_2$ where both vertices have degree $k$ in $G$. If these $K_2$’s could be handled, the average degree bound would improve to $k - 1 + \frac{k - 3}{(k - 1)^2}$.

**Conjecture.** Every non-complete (online) $k$-list-critical graph has average degree at least

$$k - 1 + \frac{k - 3}{(k - 1)^2}.$$ 

**References**


