Representations of bicircular lift matroids

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Abstract

Bicircular lift matroids are a class of matroids defined on the edge set of a graph. For a given graph \( G \), the circuits of its bicircular lift matroid are the edge sets of those subgraphs of \( G \) that contain at least two cycles, and are minimal with respect to this property. The main result of this paper is a characterization of when two graphs give rise to the same bicircular lift matroid, which answers a question proposed by Irene Pivotto. In particular, aside from some appropriately defined “small” graphs, two graphs have the same bicircular lift matroid if and only if they are 2-isomorphic in the sense of Whitney.

Keywords: bicircular lift matroids, representation

1 Introduction

We assume the reader is familiar with fundamental definitions in matroid and graph theory. For a graph \( G \), a set \( X \subseteq E(G) \) is a cycle if \( G|X \) is a connected 2-regular graph. Bicircular lift matroids are a class of matroids defined on the edge set of a graph. For a given graph \( G \), the circuits of its bicircular lift matroid \( L(G) \) are the edge sets of those subgraphs of \( G \) that contain at least two cycles, and are minimal with respect to this property. That is, the circuits of \( L(G) \) consists of the edge sets of two edge-disjoint cycles with at most one common vertex, or three internally disjoint paths between a pair of distinct vertices. Bicircular lift matroids are a special class of lift matroids that arises from biased graphs. Biased graphs and lift matroids were introduced by Zaslavsky in [8, 9].


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graphs have isomorphic graphic frame matroids. Matthews [4] characterized which graphs give rise to isomorphic bicircular matroids that are graphic. Coullard, del Greco and Wagner [3, 7] characterized which graphs give rise to isomorphic bicircular matroids. In this paper, we characterize which graphs give rise to isomorphic bicircular lift matroids, which answers a question proposed by Pivotto in the Matroid Union blog [5]. In particular, except for some special graphs, each of which is a subdivision of a graph on at most four vertices, two graphs have the same bicircular lift matroid if and only if they are 2-isomorphic in the sense of Whitney [6]. The main result is used in [1] to prove that the class of matroids that are graphic or bicircular lift has a finite list of excluded minors.

To state our result completely we need more definitions. Let \( k, l, m \) be positive integers. We denote by \( K_m \) the complete graph with \( m \) vertices. We denote by \( K_2^m \) the graph obtained from \( K_2 \) with its unique edge replaced by \( m \) parallel edges. And we denote by \( K_3^{k,l,m} \) the graph obtained from \( K_3 \) with its three edges replaced by \( k, l, m \) parallel edges respectively. A graph obtained from graph \( G \) by replacing some edges of \( G \) with internally disjoint paths is a subdivision of \( G \). Note that \( G \) is a subdivision of itself. A path \( P \) of a connected graph \( G \) is an ear if each internal vertex of \( P \) has degree two and each end-vertex has degree at least three in \( G \), and \( P \) is contained in a cycle. A graph \( G \) is 2-edge-connected if each edge of \( G \) is contained in some cycle. Let \( M(G) \) denote the graphic matroid of a graph \( G \).

Given a set \( X \) of edges, we let \( G[X] \) denote the subgraph of \( G \) with edge set \( X \) and no isolated vertices. Let \((X_1, X_2)\) be a partition of \( E(G) \) such that \( V(G[X_1]) \cap V(G[X_2]) = \{u_1, u_2\} \). We say that \( G' \) is obtained by a Whitney Switching on \( G \) on \( \{u_1, u_2\} \) if \( G' \) is a graph obtained by identifying vertices \( u_1, u_2 \) of \( G[X_1] \) with vertices \( u_2, u_1 \) of \( G[X_2] \), respectively. A graph \( G' \) is 2-isomorphic to \( G \) if \( G' \) is obtained from \( G \) by a sequence of the operations: Whitney switchings, identifying two vertices from distinct components of a graph, or partitioning a graph into components each of which is a block of the original graph.

**Theorem 1.** (Whitney’s 2-Isomorphism Theorem) Let \( G_1 \) and \( G_2 \) be graphs. Then \( M(G_1) \cong M(G_2) \) if and only if \( G_1 \) and \( G_2 \) are 2-isomorphic.

It follows from Theorem 1 that if \( G_1 \) and \( G_2 \) are 2-isomorphic, then \( L(G_1) = L(G_2) \). The converse, however, is not true. This can be seen by choosing \( G_1 \) and \( G_2 \) to be isomorphic to \( K_4 \), but not to each other. Much of the remainder of the paper is aimed at characterizing when the converse to this statement is not true.

Let \( G_1 \) and \( G_2 \) be graphs with \( L(G_1) = L(G_2) \). Since \( E(G_i) \) is independent in \( L(G_i) \) if and only if \( G_i \) has at most one cycle, we may assume that \( G_1 \) and \( G_2 \) have at least two cycles. Moreover, since \( e \) is a cut-edge of \( G_1 \) if and only if \( e \) is a cut-edge of \( G_2 \) or \( G_2 \backslash e \) is a forest, an edge is a cut-edge of \( G_1 \) if and only if it is a cut-edge of \( G_2 \). Hence, to simplify the analysis below, it will be assumed for the remainder of the paper that \( G_1 \) and \( G_2 \) are 2-edge-connected. Observe that when \( L(G_1) \) has only one circuit, it is straightforward to characterize the structure of both \( G_1 \) and \( G_2 \). Thus, the remainder of the paper will further restrict the analysis to the case that \( L(G_1) \) has at least two circuits. In the paper, we prove
Theorem 2. Let $G_1$ be a 2-edge-connected graph such that $L(G_1)$ contains at least two circuits. Let $G_2$ be a graph with $L(G_1) = L(G_2)$. Then at least one of the following holds.

(1) $G_1$ and $G_2$ are 2-isomorphic.

(2) $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_4$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$.

(3) $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_3^{m,2,n}$ for some $m \in \{1, 2\}$ and $n \geq 2$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$. Moreover, when $n \geq 3$, the $n$ ears in $G_1$ having the same ends also have the same ends in $G_2$.

(4) $G_1$ and $G_2$ are 2-isomorphic to the graphs pictured in Figure 1.

![Figure 1: In (a) and (b), $n \geq 3$; and in (c), $n \geq 2$](image)

The following result, which is an easy consequences of Theorem 2, is used in [1] to prove that the class of matroids that are graphic or bicircular lift matroids has a finite list of excluded minors.

Two elements are a series pair of a graph $G$ if and only if each cycle can not intersect them in exactly one element. A series class is a maximal set $X \subseteq E(G)$ such that every two edges of $X$ form a series pair. Let co($G$) denote a graph obtained from $G$ by
contracting all cut-edges from $G$ and then, for each series class $X$, contracting all but one distinguished element of $X$.

**Corollary 3.** Let $G_1$ and $G_2$ be connected graphs with $L(G_1) = L(G_2)$ and such that $L(G_1)$ has at least two circuits. If $|V(\text{co}(G_1))| \geq 5$ then $G_1$ and $G_2$ are 2-isomorphic.

### 2 Proof of Theorem 2

Let $G$ be a graph, and $e, f \in E(G)$. We say that $e$ is a *link* if it has distinct end-vertices; otherwise $e$ is a *loop*. If $\{e, f\}$ is a cycle, then $e$ and $f$ are *parallel*. A *parallel class* of $G$ is a maximal subset $P$ of $E(G)$ such that any two members of $P$ are parallel and no member is a loop. Moreover, if $|P| \geq 2$ then $P$ is *non-trivial*; otherwise $P$ is trivial. Let $\text{si}(G)$ denote the graph obtained from $G$ by deleting all loops and all but one distinguished element of each non-trivial parallel class. Obviously, the graph we obtain is uniquely determined up to a renaming of the distinguished elements. If $G$ has no loops and no non-trivial parallel class, then $G$ is simple.

The following result is implied in ([9], Theorem 3.6.).

**Lemma 4.** Let $e$ be an edge of a graph $G$. Then we have

1. $L(G\setminus e) = L(G)\setminus e$;
2. when $e$ is a loop, $L(G)/e = M(G\setminus e)$;
3. when $e$ is a link, $L(G)/e = L(G/e)$.

**Corollary 5.** Let $G_1, G_2$ be graphs with $L(G_1) = L(G_2)$, and $e$ a loop of both $G_1$ and $G_2$. Then $G_1$ and $G_2$ are 2-isomorphic.

The idea used to prove the following Lemma was given by the referee.

**Lemma 6.** Let $G_1$ and $G_2$ be connected graphs without loops and with $|V(G_1)| = |V(G_2)|$ and $E(G_1) = E(G_2)$. Assume that for each edge $e \in E(G_1)$ the graphs $G_1/e$ and $G_2/e$ are 2-isomorphic. Then $G_1$ and $G_2$ are 2-isomorphic.

**Proof.** By Whitney’s 2-Isomorphism Theorem, to prove the result it suffices to show that each spanning tree of $G_1$ is also a spanning tree of $G_2$. Let $T_1$ be a spanning tree of $G_1$, and let $T_2$ be the subgraph of $G_2$ induced by $E(T_1)$. Assume that $T_2$ is not a spanning tree of $G_2$. Since $|V(G_1)| = |V(G_2)|$, the subgraph $T_2$ contains a cycle $C$. Let $e$ be an edge in $E(T_1)$. Then $T_1/e$ is acyclic and $T_2/e$ is not, and so $G_1/e$ and $G_2/e$ are not 2-isomorphic; a contradiction. 

**Lemma 7.** Let $G_1$ be a 2-edge-connected graph such that $L(G_1)$ contains at least two circuits. Let $G_2$ be a graph with $L(G_1) = L(G_2)$. Assume that $G_1$ has a link $e$ such that $e$ is a loop of $G_2$. Then $G_1$ and $G_2$ are 2-isomorphic to the graphs pictured in Figure 2.
Proof. Since \(L(G_1)\) contains at least two circuits and \(L(G_1) = L(G_2)\), the graph \(G_2 - \{e\}\) has cycles \(C_1\) and \(C_2\) such that \(C_1 \cup C_2\) is a circuit of \(L(G_2)\). Since \(e\) is a loop of \(G_2\), for some integer \(k \in \{2, 3\}\) there is a partition \((P_1, P_2, \ldots, P_k)\) of \(E(C_1 \cup C_2)\) such that when \(k = 2\) the sets \(P_1 \cup \{e\}\) and \(P_2 \cup \{e\}\) are circuits of \(L(G_1)\), and when \(k = 3\) the sets \(P_1 \cup P_2 \cup \{e\}\), \(P_2 \cup P_3 \cup \{e\}\) and \(P_1 \cup P_3 \cup \{e\}\) are circuits of \(L(G_1)\). Since \(E(C_1 \cup C_2)\) is also a circuit of \(L(G_1)\) and \(e\) is a link of \(G_1\), it is easy to verify that \(k = 3\) (that is, \(C_1 \cup C_2\) is a theta-subgraph of \(G_2\).) and (1) \(G_1|_{C_1 \cup C_2 \cup \{e\}}\) is 2-isomorphic to graphs pictured in Figure 3. Hence, by the arbitrary choice of \(C_1\) and \(C_2\), (2) no two cycles in \(G_2\) have at most one common vertex; and (3) each ear of a theta-subgraph of \(G_2\) is a cycle in \(G_1\) or a path connecting the end-vertices of \(e\) in \(G_1\).

For each edge \(f \in E(G_2) - (C_1 \cup C_2 \cup \{e\})\), there is a set \(X\) with \(f \in X \subseteq E(G_2) - (C_1 \cup C_2 \cup \{e\})\) such that \(G_2|_{C_1 \cup C_2 \cup X}\) is 2-edge-connected. By (2) \(G_2|_{C_1 \cup C_2 \cup X}\) is a subdivision of \(K_4\) or \(K_4^2\). (1) and (3) imply that \(G_2|_{C_1 \cup C_2 \cup X}\) is a subdivision of \(K_4^2\). Repeating the process several times, we have that \(G_2 - \{e\}\) is a \(K_n^2\)-subdivision for some integer \(n \geq 3\). Hence, \(G_1\) and \(G_2\) are 2-isomorphic to the graphs pictured in Figure 2.

By Lemma 7, to prove Theorem 2 we only need to consider the case that an edge is a
link in $G_1$ if and only if it is a link in $G_2$.

**Lemma 8.** Let $G_1$ and $G_2$ be connected and 2-edge-connected graphs with $L(G_1) = L(G_2)$ such that $L(G_1)$ has at least two circuits and such that each series class of $G_i$ is an ear of $G_i$ for each $i \in \{1, 2\}$. Then a set of edges is the edge set of an ear of $G_1$ if and only if it is the edge set of an ear of $G_2$.

*Proof.* Assume otherwise. Without loss of generality assume that $e$ and $f$ are contained in some ear of $G_1$, but not in the same ear of $G_2$. Evidently, $e$ is not in any cycle of $G_1 - \{f\}$ and $L(G_1 - \{f\})$ has a circuit as $L(G_1)$ has at least two circuits. Moreover, since $L(G_1 - \{f\}) = L(G_2 - \{f\})$, the edge $e$ is a coloop of $G_2 - \{f\}$; so $\{e, f\}$ is a bond of $G_2$. Then $e$ and $f$ are contained in the same ear of $G_2$ as each series class of $G_2$ is an ear of $G_2$, a contradiction. □

By possibly applying a sequence of Whitney’s switching we can assume that each series class in a graph $G$ is an ear of $G$. Furthermore, by Lemma 8 we can further assume that a set of edges is the edge set of an ear of $G_1$ if and only if it is the edge set of an ear of $G_2$. Hence, we only need consider cosimple graphs, where a graph is *cosimple* if it has no cut-edges or non-trivial series classes.

Let $\text{loop}(G)$ be the set consisting of loops of $G$.

\[ \begin{align*} G_1 & \quad \quad e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_n \\ G_2 & \quad \quad e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_{n-1} \end{align*} \]

\[ \begin{align*} (a) & \quad \quad e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_n \\ (b) & \quad \quad e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_{n-1} \end{align*} \]

Figure 4: $n \geq 3$.

**Lemma 9.** Let $G_1$ and $G_2$ be cosimple 2-edge-connected graphs with $2 \leq |V(G_1)| = |V(G_2)| \leq 3$. Assume that $L(G_1) = L(G_2)$ and $L(G_1)$ contains at least two circuits. Then exactly one of the following holds.

1. $G_1$ and $G_2$ are 2-isomorphic.
(2) \(|V(G_1)| = 2\), the graphs \(G_1\) and \(G_2\) are isomorphic to the graphs pictured in Figure 4.

(3) \(G_1\) and \(G_2\) are 2-isomorphic to \(K_{3,2}^m\) for some integers \(m \in \{1, 2\}\) and \(n \geq 2\), moreover, the \(n\) parallel edges in \(G_1\) are also the \(n\) parallel edges in \(G_2\) when \(n \geq 3\).

Proof. By Lemma 7 we may assume that \(\text{loop}(G_1) = \text{loop}(G_2)\). Then the lemma holds when \(|V(G_1)| = 2\). So assume that \(|V(G_1)| = 3\). Since \(\text{loop}(G_1) = \text{loop}(G_2)\), each non-trivial parallel class of \(G_1\) with at least three edges must be also a non-trivial parallel class of \(G_2\). Hence, when \(G_1\) has two parallel classes with at least three edges, (1) holds. So we may assume that \(G_1\) has at most one parallel class with at least three edges. On the other hand, since \(G_1\) and \(G_2\) are cosimple, \(G_1\) and \(G_2\) have three parallel classes and at least two of them are non-trivial. Hence, when \(G_1\) has no loops, (3) obviously holds; when \(G_1\) has a loop, since \(\text{loop}(G_1) = \text{loop}(G_2)\), Corollary 5 implies that \(G_1\) and \(G_2\) are 2-isomorphic, that is, (1) holds.

The star of a vertex \(v\) in a graph \(G\), denoted by \(st_G(v)\), is the set of edges of \(G\) incident with \(v\).

Lemma 10. Let \(G_1\) and \(G_2\) be 2-edge-connected cosimple graphs with exactly four vertices and without loops. Assume that \(L(G_1) = L(G_2)\) and \(L(G_1)\) has at least two circuits. Then at least one of the following holds.

(1) \(G_1\) and \(G_2\) are 2-isomorphic;

(2) \(G_1\) and \(G_2\) are isomorphic to \(K_4\);

(3) \(G_1\) and \(G_2\) are 2-isomorphic to the graphs pictured in Figure 5.

![Figure 5: n ≥ 7.](image-url)
three vertices, by Lemma 9 we have that (a) the graphs $G_1/f$ and $G_2/f$ are isomorphic to $K_3^{m,n}$ for some integers $m \in \{1, 2\}$ and $n \geq 2$; moreover, when $n \geq 3$ the $n$ parallel edges in $G_1$ are also the $n$ parallel edges in $G_2$.

10.1. Two edges are parallel in $G_1$ if and only if they are parallel in $G_2$.

*Subproof.* If two edges are parallel in $G_1$ but not $G_2$, then contracting one of the edges produces a counterexample to Lemma 7.

The simple proof of 10.1 is given by the referee. Since no non-trivial parallel classes in $G_1$ or $G_2$ contains $f$ by (a), 10.1 implies

10.2. Each 2-edge path joining the end-vertices of a non-trivial parallel class of $G_1$ is also a 2-edge path joining the end-vertices of the non-trivial parallel class of $G_2$.

10.3. Let $P_1, P_2$ be non-trivial parallel classes of $G_1$. Then $\text{si}(G_1|P_1 \cup P_2 \cup f)$ is a triangle.

*Subproof.* Since $G_1/f$ has no loop, neither $P_1$ nor $P_2$ contains $f$. If $P_1$ and $P_2$ are not contained in a parallel class of $G_1/f$, then $P_1$ and $P_2$ are contained in two different non-trivial parallel classes of $G_1/f$. Moreover, since $P_1$ and $P_2$ are also non-trivial parallel classes of $G_2$ by 10.1, by (a) we have that $G_1/f$ and $G_2/f$ are isomorphic, a contradiction. So $P_1$ and $P_2$ are contained in a parallel class of $G_1/f$. Then $\text{si}(G_1|P_1 \cup P_2 \cup f)$ is a triangle.

First we consider the case that $G_1/f$ is isomorphic to $K_3^{2,2,n}$. By 10.3, $G_1$ is obtained from $G_1/f$ by splitting a degree-4 vertex. Since $G_1$ is cosimple, $G_1$ is isomorphic to the graph pictured in Figure 5 with $e_5$ relabelled by $f$. Let $P$ be the unique non-trivial parallel class of $G_1$ with $n$ edges. Since $P$ is a also non-trivial parallel class of $G_2$ by 10.1 and the fact that $G_2/f$ is isomorphic to $K_3^{2,2,n}$, the graph $G_2$ is isomorphic to the graph pictured in Figure 5 with $e_5$ relabelled by $f$. So (3) holds.

Secondly we consider the case that $G_1/f$ is isomorphic to $K_3^{1,2,n}$. Let $e_i$ be the edge of $G_1/f$ that is not in a parallel class for $1 \leq i \leq 2$. Evidently, when $n \geq 3$, since $G_1/f$ and $G_2/f$ are not 2-isomorphic, $e_1 \neq e_2$. Since each vertex of $G_1$ has degree at least three, by 10.3 the graph $G_1$ is obtained from $G_1/f$ by splitting the vertex $v$ incident with two non-trivial parallel classes. When $|st_{G_1/f}(v)| = 4$, since $G_1$ is cosimple $G_1$ is isomorphic to $K_4$. By symmetry $G_2$ is also isomorphic to $K_4$. So (2) holds.

Assume that $|st_{G_1/f}(v)| \geq 5$, that is, a non-trivial parallel class $P$ incident with $v$ in $G_1/f$ has at least three edges. Then some proper subset $P'$ of $P$ is a non-trivial parallel class in $G_1$ as $G_1$ is cosimple. Let $\{f_1, f_2\}$ be the 2-edge parallel class in $G_1/f$. Since $\{f, f_1, f_2\}$ is a cycle in $G_1$ and $\{e_1, f_1, f_2\}$ is the neighbourhood of a degree-3 vertex in $G_1/f$ and $G_1$, by symmetry we may assume that $e_1, f_1$ is a 2-edge path joining the end-vertices of $P'$ in $G_1$ and $f_2$ is not incident with $P'$. On the other hand, by symmetry, $e_2$ is also contained in a 2-edge path joining the end-vertices of $P'$ in $G_2$. So $f_1 = e_2$ as $e_2 \in \{f_1, f_2\}$, consequently, $|P - P'| = 1$, for otherwise there are two such $P'$, which is not possible. Therefore, (3) holds.
Lemma 11. Let $G_1$ and $G_2$ be 2-edge-connected cosimple graphs with five vertices and without loops. Assume that $L(G_1) = L(G_2)$ and $L(G_1)$ has at least two circuits. Then $G_1$ and $G_2$ are 2-isomorphic.

Proof. By Lemma 4 (3), for each edge $e \in E(G_1)$ we have $L(G_1/e) = L(G_2/e)$. If $G_1/e$ and $G_2/e$ are 2-isomorphic for each edge $e \in E(G_1)$, then Lemma 6 implies that $G_1$ and $G_2$ are 2-isomorphic. So we may assume that for some edge $f \in E(G_1)$ we have $L(G_1/f) = L(G_2/f)$ but $G_1/f$ and $G_2/f$ are not 2-isomorphic.

We claim that $G_1/f$ and $G_2/f$ have no loops. Since $L(G_1/f)$ has at least two circuits, Lemma 7 implies that $\text{loop}(G_1/f) = \text{loop}(G_2/f)$. If $\text{loop}(G_1/f) \neq \emptyset$, then Corollary 5 implies that $G_1/f$ and $G_2/f$ are 2-isomorphic, a contradiction.

Since $G_1/f$ and $G_2/f$ are cosimple with four vertices and without loops, Lemma 10 implies that $G_1/f$ and $G_2/f$ are either 2-isomorphic to $K_4$ or to the graphs pictured in Figure 5. Since each vertex in $K_4$ has degree three and $G_1$ and $G_2$ are cosimple, neither $G_1/f$ nor $G_2/f$ is 2-isomorphic to $K_4$. So $G_1/f$ and $G_2/f$ are 2-isomorphic to the graphs pictured in Figure 5 with $G_1$ replaced by $G_1/f$ and all other labeling the same. Let $P$ be the non-trivial parallel class in $G_1/f$ and $G_2/f$. For each $i \in \{1, 2\}$, let $u_i$ and $v_i$ be the end-vertices of $f$ in $G_i$, let $x_i$ be the vertex of degree at least four in $G_i/f$ incident with $e_i$, and $y_i$ be the vertex of degree at least four in $G_i/f$ incident with $e_3$.

Since $|st_{G_i}(u_i)|, |st_{G_i}(v_i)| \geq 3$, the graph $G_i$ is obtained from $G_i/f$ by splitting $x_i$ or $y_i$. Without loss of generality we may assume that $G_1$ is obtained from $G_1/f$ by splitting $x_i$ for each $i \in \{1, 2\}$.

We claim that $|E(G_1/f)| = 7$, that is, $|P| = 2$. Assume otherwise. Then there is a subset $P'$ of $P$ with $|P'| \geq 2$ such that $P'$ is also a parallel class in $G_1$. Using a similar analysis to the one in the proof of 10.1 we have that $P'$ is also a parallel class in $G_2$. Assume that $e_1, e_2$ are adjacent in $G_1$. Since a union of any two edges in $P'$ and $\{e_1, e_2, e_3\}$ or $\{e_3, e_4, e_5\}$ is a circuit of $L(G_1)$, we deduce that $\{e_1, e_4, e\} \cup P'$ are contained in $st_{G_2}(u_2)$ or $st_{G_2}(v_2)$. Hence, $|P - P'| \leq 1$, implying that $(P - P') \cup \{f\}$ is a bond of $G_2$ with at most two edges, a contradiction as $G_2$ is cosimple. So $e_1, e_2$ are not adjacent in $G_1$. By symmetry we may assume that $st_{G_1}(v_i) = \{e_2, f\} \cup P'$. Since $P'$ is a parallel class of $G_2$ and the union of $\{e_3, e_4, e_5\}$ and any two edges in $P'$ is a circuit of $L(G_1)$, by symmetry we may assume that $\{e_4, f\} \cup P'$ are incident with $v_2$. Hence, $|P - P'| = 1$ and $st_{G_1}(u_1) = st_{G_2}(u_2) = (P - P') \cup \{e_1, f\}$. Set $\{e_6\} = P - P'$. See Figure 6. Then $\{e_1, e_2, e_3, e_5, e_6, f\}$ is a circuit of $L(G_1)$ but is not a circuit of $L(G_2)$, a contradiction. So $|E(G_1/f)| = 7$. Set $E(G_1/f) := \{e_1, e_2, \ldots, e_7\}$.

Since $G_1$ and $G_2$ are cosimple and $|E(G_1/f)| = 7$, we have $|st_{G_i}(u_i)| = |st_{G_i}(v_i)| = 3$ for each $i \in \{1, 2\}$. By symmetry, there are two cases to consider. First we consider the case $st_{G_1}(u_1) = \{f, e_1, e_2\}$. Since $\{e_1, e_2, e_3, e_4, e_5\}$ is a circuit of $L(G_1)$, by symmetry we can assume $st_{G_2}(v_2) = \{f, e_1, e_4\}$. Then $\{e_2, e_3, e_5, e_6, e_7, f\}$ is a circuit of $L(G_1)$ but is not a circuit of $L(G_2)$, a contradiction.

Secondly consider the case $st_{G_1}(u_1) = \{f, e_1, e_6\}$. Then $\{e_1, e_3, e_4, e_5, e_6\}$ is a circuit of $L(G_1)$. On the other hand, by symmetry and the analysis in the last paragraph we have $\{f, e_1, e_4\} \neq \{N_{G_2}(u_2), N_{G_2}(v_2)\}$. So $\{e_1, e_3, e_4, e_5, e_6\}$ is not a circuit of $L(G_2)$, a contradiction. □
Figure 6:

For convenience, Theorem 2 is restated here.

**Theorem 12.** Let $G_1$ be a 2-edge-connected graph such that $L(G_1)$ contains at least two circuits. Let $G_2$ be a graph with $L(G_1) = L(G_2)$. Then at least one of the following holds.

1. $G_1$ and $G_2$ are 2-isomorphic.
2. $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_4$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$.
3. $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_{3,m,2,n}^n$ for some $m \in \{1, 2\}$ and $n \geq 2$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$. Moreover, when $n \geq 3$, the $n$ ears in $G_1$ having the same ends also have the same ends in $G_2$.
4. $G_1$ and $G_2$ are 2-isomorphic to the graphs pictured in Figure 1.

**Proof.** If some loop $e$ of $G_1$ is also a loop of $G_2$, then by Corollary 5 we have that $G_1 \setminus e$ and $G_2 \setminus e$ are 2-isomorphic. So $G_1$ and $G_2$ are 2-isomorphic. Moreover, when some link of $G_1$ is a loop of $G_2$, Lemma 7 implies that (4) holds. Therefore, we may assume that neither $G_1$ nor $G_2$ has loops. By Whitney’s 2-Isomorphism Theorem we can further assume that $G_1$ and $G_2$ are connected, and each series class of $G_i$ is an ear of $G_i$ for each $i \in \{1, 2\}$. Using Lemma 8 we may assume that a subset of $E(G_i)$ is the edge set of an ear of $G_1$ if and only if it is the edge set of an ear of $G_2$. Therefore, we may assume that $G_1$ and $G_2$ are cosimple.

Since the rank of $L(G_i)$ is equal to $|V(G_i)|$, we have $|V(G_1)| = |V(G_2)|$. When $|V(G_1)| \leq 4$, Lemmas 9 and 10 imply that the result holds. We claim that when $|V(G_1)| \geq 5$ we have that $G_1$ and $G_2$ are 2-isomorphic. When $|V(G_1)| = 5$, the claim follows from Lemma 11. So we may assume that $|V(G_1)| \geq 6$. For each edge $e \in E(G_1)$, by Lemma 4 (3) we have $L(G_1/e) = L(G_2/e)$. By induction $G_1/e$ and $G_2/e$ are 2-isomorphic. So $G_1$ and $G_2$ are 2-isomorphic by Lemma 6.

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References

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