Every triangulated 3-polytope of minimum degree 4 has a 4-path of weight at most 27

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Abstract

By \( \delta \) and \( w_k \) denote the minimum degree and minimum degree-sum (weight) of a \( k \)-vertex path in a given graph, respectively. For every 3-polytope, \( w_2 \leq 13 \) (Kotzig, 1955) and \( w_3 \leq 21 \) (Ando, Iwasaki, Kaneko, 1993), where both bounds are sharp. For every 3-polytope with \( \delta \geq 4 \), we have sharp bounds \( w_2 \leq 11 \) (Lebesgue, 1940) and \( w_3 \leq 17 \) (Borodin, 1997).

Madaras (2000) proved that every triangulated 3-polytope with \( \delta \geq 4 \) satisfies \( w_4 \leq 31 \) and constructed such a 3-polytope with \( w_4 = 27 \).

We improve the Madaras bound \( w_4 \leq 31 \) to the sharp bound \( w_4 \leq 27 \).

Keywords: Plane graph, structural property, normal plane map, 4-path.

1 Introduction

The degree of a vertex or face \( x \) in a graph \( G \), that is the number of edges incident with \( x \), is denoted by \( d(x) \). A \( k \)-vertex is a vertex \( v \) with \( d(v) = k \). By \( k^+ \) or \( k^- \) we denote any integer not smaller or not greater than \( k \), respectively. Hence, a \( k^+ \)-face \( f \) satisfies \( d(f) \geq k \), etc.

Let \( \delta(G) \) be the minimum vertex degree, and \( w_k(G) \) be the minimum degree sum (called weight) of a path on \( k \) vertices (\( k \)-path) in a graph \( G \).

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By $G_\delta$ denote the class of plane graphs $G$ with $\delta(G) \geq \delta$; it is trivial that $\delta \leq 5$. The subset of 3-connected graphs in $G_\delta$ is denoted by $P_\delta$. Given $k$, by $w_k$ denote the maximal weight of $k$-paths over $G \in G_\delta$ or $P \in P_\delta$.

Already in 1904, Wernicke [40] proved that every $P \in P_5$ satisfies $w_2(P) \leq 11$, which is tight. It follows from Lebesgue’s [31] results of 1940 that $P \in P_3$ implies $w_2(G) \leq 14$, which was improved in 1955 by Kotzig [30] to the tight bound $w_2 \leq 13$. In 1972, Erdős (see [20]) conjectured that Kotzig’s bound $w_2 \leq 13$ holds also in $G_3$. Barnette (see [20]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős’ conjecture is due to Borodin [4].

A number of sharp upper bounds on $w_2$ have been obtained as lemmas in numerous papers on coloring sparse planar graphs (for a survey see Borodin [10]). A traditional measure of sparseness of a planar graph $G$ is its girth $g(G)$, which is the length of the shortest cycle in $G$. Another measure, suggested by Erdős (see [20]), is the absence of cycles of length 4 to a certain constant. More generally, given a set $S$ of integers, a graph is $S$-free if it has no cycle with length from $S$.

The first results of this type were known already to Lebesgue’s [31] for $P_3$: if $g \geq 4$, then $w_2 \leq 8$, and if $g \geq 5$, then $w_2 = 6$.

As for $G_2$, we note that $\delta(K_{2,1}) = 1$ and $w_2(K_{2,1}) = t + 2$, so $w_2$ is unbounded in $G_2$ if $g \leq 4$. In addition to forbidding certain collections $S$ of cycle lengths, another way to find subclasses of $G_2$ with bounded $w_2$ is to impose restrictions on the set of 2-vertices in a graph. For example, forbidding 2-alternating cycles, which are cycles $v_1 \ldots v_{2k}$ with $d(v_1) = d(v_3) = \ldots = d(v_{2k-1}) = 2$, we have $w_2 \leq 15$ (Borodin [4]).

The first application of this fact was to show that the total choosability of planar graphs with maximum degree $\Delta$ at least 14 equals $\Delta + 1$ ( [4]). The notion of 2-alternating cycles, along with its more sophisticated analogues, have appeared in dozens of papers, since it sometimes provides crucial reducible configurations in coloring and partition problems (more often, on sparse graphs).

So far, not many precise results have been obtained on $w_k$ with $k \geq 3$. Back in 1922, Franklin [19] strengthened Wernicke’s bound $w_2 \leq 11$ for $P_5$ in [40] to $w_3(G) \leq 17$. Both bounds 11 and 17 are sharp, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

In 1993, Ando, Iwasaki, and Kaneko [3] proved that $w_3 \leq 21$ holds in $P_3$, which is sharp due to the Jendrol’ construction in [23]. In 1997, Borodin [8] showed that $P_3$ restricted to the graphs with $w_2 \geq 8$ (in particular, to $P_4$) satisfies $w_3 \leq 17$, which extends Franklin’s bound [19].

In 1996, Jendrol’ and Madaras [27] proved $w_4 \leq 23$ in $P_5$, which bound was extended in 2003 by Mohar, Skrekovski, and Voss [37] to the part of $P_4$ satisfying $w_2 \geq 9$. For the triangulations with $\delta = 5$, Madaras [33] proved in 2000 that $w_5 \leq 29$. Both bounds 23 and 29 are sharp.

For 4-connected planar graphs with at least $k$ vertices and any natural number $k,$
Mohar [36] nicely proved a sharp bound \( w_k \leq 6k - 1 \), using Tutte’s theorem [39] of 1956 that such graphs are hamiltonian.

We now consider sharp upper bounds on \( w_3 \) for graphs in \( P_2 \) with given girth \( g \). It is an old folkloric fact that \( g(G) \geq 16 \) implies \( w_3(G) = 6 \), which probably first appeared in print in Nešetřil, Raspau, and Sopena [38].

Recently, Jendrol’ and Maceková [25] proved \( w_3 \leq 7 \) if \( g \geq 10 \) and \( w_3 \leq 9 \) if \( g \geq 8 \). Aksenov, Borodin, and Ivanova [1] proved \( w_3 \leq 9 \) for \( g = 7 \).

As observed in [25], \( w_3 = \infty \) if \( g \leq 6 \), but if we forbid vertices with two neighbors of degree 2, that is consider \((2, \infty, 2)\)-free planar graphs, then \( w_3 \leq 10 \) for \( g \geq 5 \). The sharpness of this bound was later on confirmed in [1], where it was also proved that \( w_3 \leq 9 \) for \( g = 6 \). Jendrol’, Maceková, and Soták [26] soon after proved \( w_3 \leq 12 \) if \( g = 4 \) (also for \((2, \infty, 2)\)-free graphs in \( P_2 \)).

Some other structural results on \( 3^+ \)-paths in plane graphs can be found in [1, 3, 8, 9, 11–13, 16–28, 32–38] and the informative survey by Jendrol’ and Voss [29].

In 2000, Madaras [33] proved for triangulations with \( \delta = 4 \) that \( w_4 \leq 31 \) and gave a construction with \( w_4 = 27 \) by putting a 3-cycle \( A_1A_3A_3 \) into each face \( Z_1Z_2Z_3 \) of the icosahedron followed by adding the edges \( A_iZ_j \) whenever \( 1 \leq i, j \leq 3 \) and \( i \neq j \).

The purpose of our paper is to improve the bound \( w_4 \leq 31 \) by Madaras [33] to the sharp bound \( w_4 \leq 27 \).

**Theorem 1.** Every plane triangulation with \( \delta \geq 4 \) has a 4-path of weight at most 27, which is tight.

### 2 Proving Theorem 1

**Proof of Theorem 1.** Suppose that \( G \) is a counterexample to Theorem 1. In the course of the proof we should take care that a hypothetic 4-path \( P_4 = v_1v_2v_3v_4 \) with \( w(P_4) \leq 27 \) in \( G \) would not degenerate into a 3-cycle, which happens when \( v_1 \) coincides with \( v_4 \).

A \( k \)-component \( c_k \) is a maximal connected subgraph of \( G \) that consists of \( k \) vertices of degree at most 5. Clearly, any \( c_1 \) is simply a 5-vertex, \( c_2 \) consists of two adjacent vertices, and \( c_3 = uvw \) is either a 3-path (when there is no edge \( uw \) in \( G \)) or a 3-cycle. Furthermore, the following lemma shows, in particular, that if a 3-component \( c_3 \) is a cycle, then it is the boundary of a 3-face.

**Lemma 2.** No separating 3-cycle in \( G \) consists of three 5-vertices or has two 4-vertices.

**Proof of Lemma 2.** Suppose \( G \) has a separating 3-cycle \( S = uvw \), which means that at least one vertex of \( G \) lies inside \( S \) and at least one outside \( S \). Note that \( u \) has at least one neighbor inside \( S \) and at least one outside \( S \), for otherwise \( \{v, w\} \) is a separating set in \( G \), contrary to the 3-connectedness of \( G \). The same is true for \( v \) and \( w \).

First suppose that \( S \) consists of 5-vertices, which implies that at most nine edges join \( S \) to vertices not on \( S \). By symmetry, we can assume that at most four edges lead from \( S \) inside (rather than outside) \( S \). Again by symmetry, we can assume that each of
$u$ and $v$ has precisely one neighbor inside $S$. Since $G$ is a triangulation, there are 3-faces $uvx$, $vwx$, and $wxu$ inside $S$. This implies that there is precisely one vertex $x$, necessarily with $d(x) = 3$, inside $S$; a contradiction.

Now if $d(u) = d(v) = 4$, then the same argument works, which proves the second part of Lemma 2.

**Lemma 3.** $G$ has no $k$-component with $k \geq 4$.

*Proof of Lemma 3.* Suppose on the contrary that there is a connected subgraph $H$ of $G$ on four $5^-$-vertices. Since $G$ has no 4-path $P_4$ with $w(P_4) \leq 4 \times 5$, it follows that $H$ does not contain a 4-path. In turn, this means that $H$ is a star, with three rays and a $5^-$-vertex as a center. However, the center forms a 3-face with two consecutive neighbors in our triangulation $G$, which produces a 4-path on $H$, a contradiction. \qed

### 2.1 Discharging

Euler’s formula $|V| - |E| + |F| = 2$ for $G$ may be written as

$$\sum_{v \in V} (d(v) - 6) = -12,$$

where $V$, $E$, and $F$ are sets of vertices, edges, and faces of $G$, respectively. By $V_{6^+}$ and $V_{5^-}$ denote the sets of $6^+$-vertices and $5^-$-vertices in $V$, respectively, so (1) can be written as follows.

$$\sum_{v \in V_{6^+}} (d(v) - 6) + \sum_{v \in V_{5^-}} (d(v) - 6) = -12,$$

(2)

By $MC$ denote the set of minor components $[mc]$ of $G$. It follows from Lemma 3 that $V_{5^-}$ is split into minor components, each of which is a $k^-$-component with $k \leq 3$.

Every vertex $v$ contributes the *initial charge* $\mu(v) = d(v) - 6$ to (2), so only the charges of $5^-$-vertices are negative. The *initial charge* of a minor component $[mc]$ is the total initial charge of its vertices. Thus (2) can be rewritten as follows.

$$\sum_{v \in V_{6^+}} \mu(v) + \sum_{[mc] \in MC} \mu([mc]) = -12,$$

(3)

Using the properties of $M$ as a counterexample, we define a local redistribution of $\mu$’s, preserving their sum, such that the *new charge* $\mu'$ is non-negative for all $v \in V_{6^+}$ and $[mc] \in MC$. This will contradict the fact that the sum of the new charges is, by (3), equal to $-12$.

For $v \in V_{6^+}$, we put $\xi(v) = \frac{d(v)-6}{d(v)}$ and $\bar{\xi}(v) = \min\{\xi(v), \frac{1}{2}\}$. For an integer $k$ with $k \geq 6$, we put $\psi(k) = \frac{k-6}{k}$.

Our rules of discharging are as follows (see Fig. 1):
R1. For each 3-face \( f = xuv \) with \( d(u) \leq 5 \), \( d(v) \geq 6 \), and \( d(x) \geq 6 \), the vertex \( u \) receives \( \xi(v) \) and \( \xi(x) \) through \( f \) from \( v \) and \( x \), respectively.

R2. For each 3-face \( f = xuv \) with \( d(u) \leq 5 \), \( d(v) \leq 5 \), and \( d(x) \geq 6 \), each of \( u \) and \( v \) receives \( \frac{1}{2} \xi(x) \) from \( x \) through \( f \), with the following exception:

(E) if \( d(u) = d(v) = 4 \) and there is a 3-face \( uvw \) with \( d(w) = 4 \), then (the 16\(^+\)-vertex) \( x \) gives \( \frac{1}{4} \) to each of \( u \) and \( v \) through \( f \).

R3. For each 3-face \( f = xuv \) with \( d(u) = d(v) = 4 \) and \( 6 \leq d(x) \leq 7 \), each of \( u \) and \( v \) receives \( \frac{1}{4} \xi(x) \) from \( x \) through \( f \). In turn, \( x \) receives \( \frac{1}{12} \) from each adjacent 13\(^+\)-vertex \( y \) through each 3-face \( xyz \) such that \( d(z) \geq 13 \).

Concerning the second part of R3, we note that there are at least three 3-faces incident with \( x \) and two 13\(^+\)-vertices, so \( x \) receives at least \( 6 \times \frac{1}{12} \) through such faces.

![Figure 1: Rules of discharging.](image)

2.2 Checking \( \mu'(v) \geq 0 \) if \( d(v) \leq 5 \) and \( v \) forms a 1-component

By definition, \( v \) is surrounded by 6\(^+\)-neighbors, so only R1 is in action. Let \( S(v) = \{d_1, \ldots, d_{d(v)}\} \) be the multi-set of degrees of the neighbors of \( v \), where \( d_1 \leq \ldots \leq d_{d(v)} \).
For example, if \( v \) has one 6-neighbor, two 7-neighbors, and two 8-neighbors, then \( S(v) = \{6, 7, 8, 8\} \). We note that \( d_1 + d_2 + d_3 \geq 28 - d(v) \), for otherwise \( G \) would have a 4-path of weight at most 27, which is impossible.

**CASE 1.** \( d(v) = 4 \). If \( d_3 \geq 12 \), then \( \mu'(v) \geq 4 - 6 + 4 \times \psi(12) = 0 \) by R1.

If \( d_3 = 11 \), then \( d_2 \geq 7 \), since otherwise \( v \) has two 6-neighbors and an 11-neighbor, which implies a 4-path of weight at most \( 6 + 6 + 11 < 28 - 4 \), a contradiction. So, \( \mu'(v) \geq -2 + 2 \psi(7) + 4 \times \psi(11) = -2 + \frac{2}{7} + \frac{4 \times 5}{11} > -2 + \frac{22}{11} = 0 \), as desired.

If \( d_3 = 10 \), then \( d_1 + d_2 \geq 14 \), so \( v \) receives at least \( 2 \min\{\psi(6) + \psi(8), \psi(7) + \psi(7)\} = \frac{1}{2} \) from two neighbors of smallest degree and at least \( 4 \psi(10) = \frac{8}{5} \) from the other two neighbors. This implies that \( \mu'(v) \geq -2 + \frac{1}{2} + \frac{8}{5} > 0 \).

If \( d_3 = 9 \), then \( d_1 + d_2 \geq 15 \), so \( v \) receives at least \( 2 \min\{\psi(6) + \psi(9), \psi(7) + \psi(8)\} = \frac{2}{3} \) plus at least \( 4 \psi(9) = \frac{2}{3} \) by R1, whence \( \mu'(v) \geq 0 \).

Finally, \( d_3 \leq 8 \) is possible only if \( d_1 = \ldots = d_3 = 8 \), in which case \( \mu(v) \geq -2 + 8 \psi(8) = 0 \), as desired.

**CASE 2.** \( d(v) = 5 \). Note that \( d_3 \geq 8 \), for otherwise \( d_1 + d_2 + d_3 \leq 3 \times 7 < 28 - 5 \), a contradiction. Hence \( \mu'(v) \geq 5 - 6 + 6 \times \psi(8) > 0 \) by R1, and we are done.

### 2.3 Checking \( \mu'(c_2) \geq 0 \) for a 2-component \( c_2 = uv \)

**CASE 1.** \( d(u) = d(v) = 4 \). Denote the external neighbors of \( u \) and \( v \) by \( z_1, \ldots, z_4 \) as shown in Fig. 2a. Due to Lemma 3 applied to \( z_2 \) and \( z_4 \) combined with the 3-connectedness of \( G \), all \( z_i \) are distinct. Let \( z_k \) have the smallest degree in \( Z = \{z_1, \ldots, z_4\} \), where \( d(z_k) \geq 6 \) by the definition of 2-component. Note that each other vertex in \( Z \) has degree at least \( 20 - d(z_k) \) since \( w(P_4) \geq 28 \) by assumption.

![Figure 2: Notation for 2-components.](image)

**SUBCASE 1.1.** \( k \in \{1, 3\} \). If \( d(z_k) \leq 7 \), then the three 13⁺-vertices in \( Z - z_k \) together give \( c_2 \), through the incident faces, the charge \( 6 \times \frac{1}{2} \) by R1 and \( 2 \times \frac{1}{2} \) by R2, while \( z_k \) gives \( 2 \xi(v) \) by R1, \( 2 \times \frac{\xi(v)}{2} \) by R2, and \( 2 \times \frac{1}{4} \) by R3, so \( \mu'(c_2) \geq 2(4 - 6) + 6 \times \frac{1}{2} + 4 \times \frac{1}{4} = 0 \).

If \( d(z_k) = 8 \), then \( c_2 \) receives the following charges by R1 and R2: \( 2 \times \frac{1}{2} + 2 \times \frac{1}{2} \) from \( z_k \), \( 2 \times \frac{1}{2} + 2 \times \frac{1}{4} \) from \( z_4 - k \), and \( 4 \times \frac{1}{2} \) from \( z_2 \) and \( z_4 \) together, which yields \( \mu'(c_2) > 0 \).
If $d(z_k) = 9$, then we similarly have $\mu'(c_2) \geq -4 + 6 \times \frac{5}{11} + 2 \times \frac{5}{22} + 2 \times \frac{1}{3} + 2 \times \frac{1}{6} > 0$. Finally, if $d(z_k) \geq 10$, then $\mu'(c_2) \geq -4 + 8 \times \frac{2}{3} + 4 \times \frac{1}{5} = 0$.

**Subcase 1.2.** $k \in \{2, 4\}$. If $d(z_k) \leq 8$, then the three $12^+$-vertices in $Z$ together give $6 \times \frac{1}{2} + 4 \times \frac{1}{4}$ to $c_2$ by R1 and R2, whence $\mu'(c_2) \geq 0$.

If $d(z_k) = 9$, then we have $\mu'(c_2) \geq -4 + 6 \times \frac{5}{11} + 2 \times \frac{5}{22} + 2 \times \frac{1}{3} > 0$, and if $d(z_k) \geq 10$, then again $\mu'(c_2) \geq -4 + 8 \times \frac{2}{3} + 4 \times \frac{1}{5} = 0$.

**Case 2.** $d(u) = 4, d(v) = 5$. Here, $u$ and $v$ have neighbors $z_1, \ldots, z_5$ shown in Fig. 2b, and $c_2$ receives charge from $Z$ by R1 and R2 through twelve faces. However, now Lemma 2 is not applied, thus the edge $uv$ can belong to a separating 3-cycle $z_2uv$, which is only possible when $z_2 \in \{z_4, z_5\}$. If so, then $z_2$ appears twice in $Z$ and can have a relatively small degree, say 6 or 7, with no contradiction (since we are looking for a light 4-path rather than a light 3-cycle). Still, $c_2$ has at least three neighbors other than $z_k$.

This time, we can ignore the donations from $z_k$, since each neighbor other than $z_k$ has degree at least $19 - d(z_k) \geq 10$. Indeed, otherwise we would have a 4-path of weight at most $9 + 4 + 5 + 9$ on $\{z_k, u, v, z_l\}$ with $z_l \neq z_k$, which is impossible. Therefore, $c_2$ has at least three $10^+$-neighbors, and each of them gives at least $\frac{1}{2}$ to $c_2$ through each incident face by R1 and R2. Altogether, such donations happen at least eight times, which results in $\mu'(c_2) \geq 4 - 6 + 5 - 6 + 8 \times \frac{1}{3} > 0$.

**Case 3.** $d(u) = d(v) = 5$. Now $u$ and $v$ have six neighbors $z_1, \ldots, z_6$ shown in Fig. 2c. Again Lemma 2 is not applied, and it can happen that $\{z_2, z_3\} \cap \{z_5, z_6\} = z_k$. Since $z_k$ can appear in the multi-set $Z$ at most twice, it follows that $c_2$ has at least four neighbors different from $z_k$ and receives the charge by R1 and R2 from them at least ten times through the incident faces. Indeed, there are fourteen faces incident with $\{u, v\}$ and at most four of them may be incident with $z_k$. Since each vertex in $Z - z_k$ this time has degree at least $18 - d(z_k) \geq 9$, it follows that $\mu'(c_2) = 5 - 6 + 5 - 6 + 10 \times \frac{1}{3} > 0$, as desired.

### 2.4 Checking $\mu'(c_3) \geq 0$ for a 3-component $c_3 = xuv$

By definition, $c_3$ is surrounded by $6^+$-neighbors, say $z_i$.

**Case 1.** $d(x) = d(u) = d(v) = 4$. Now all $z_i$’s are $16^+$-vertices, regardless of the presence or absence of the edge $xv$ in $G$. If $xv \in E(G)$, then $c_3$ is a 3-face due to Lemma 2 and receives $12 \times \frac{1}{2}$ by R1 and R2E, whence $\mu'(c_3) = 3(4 - 6) + 6 = 0$. Otherwise, $c_3$ is a 3-path and hence receives $12 \times \frac{1}{2}$ by R2, so again $\mu'(c_3) = -6 + 6 = 0$.

**Case 2.** $d(x) = 5$. Now all $z_i$’s are $13^+$-vertices, for otherwise we would have a 4-path of weight at most $5 + 5 + 5 + 12$ on $\{u, v, x, z_i\}$. Hence $c_3$ receives at least $11 \times \frac{1}{2}$ by R1 and R2, whence $\mu'(c_3) \geq -1 - 2 - 2 + \frac{11}{2} > 0$.

### 2.5 Checking $\mu'(v) \geq 0$ for a 6$^+$-vertex $v$

**Case 1.** $d(v) = 6$. If $v$ does not participate in R3, then $\mu'(v) = \mu(v) = 0$. Otherwise, $v$ has four $14^+$-neighbors and receives $6 \times \frac{1}{12}$ from them by R3. Also, $v$ gives $\frac{1}{2}$ to the only 2-component by R3, so again $\mu'(v) = 0$. 

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CASE 2. $d(v) = 7$. If $v$ does not participate in R3, then $\mu'(v) \geq 1 - 7 \times \frac{1}{7} = 0$ by R1 and R2. Otherwise, $v$ has five 13*-neighbors, receives $8 \times \frac{1}{12}$ from them by R3, and gives $\frac{1}{2}$ to the 2-component by R3 and $3 \times \frac{1}{7}$ by R1 and R2, whence $\mu'(v) > 1 - \frac{1}{2} - \frac{3}{7} > 0$.

CASE 3. $8 \leq d(v) \leq 15$. Now $v$ can give charge away by R1, R2 and, if $d(v) \geq 13$, by R3. Since the donation through each incident face is at most $\xi(v)$, it follows that $\mu'(v) \geq d(v) - 6 - d(v) \times \xi(v) = 0$.

CASE 4. $d(v) \geq 16$. Now $v$ can participate in all the rules R1–R3. Since the donation through each incident face is at most $\frac{1}{2}$ except for R2E, and the donation of $4 \times \frac{1}{12}$ through the three consecutive faces in R2E combined with R1 can be regarded as giving $\frac{2}{3}$ per face, we have $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v) - 18}{3}$. Thus we are already done for $d(v) \geq 18$.

Finally, suppose $16 \leq d(v) \leq 17$. As we know, the (averaged) donation of $\frac{2}{3}$ through a face appears only in R2E, while the other, “individual,” donations by R1–R3 are of at most $\frac{1}{2}$. Furthermore, the faces receiving $\frac{2}{3}$ on the average form disjoint triples.

If $d(v) = 17$, then at most fifteen faces conduct $\frac{2}{3}$ from $v$ each after averaging, so $\mu'(v) \geq 17 - 6 - 15 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$.

Finally, suppose $d(v) = 16$. If at most twelve faces conduct $\frac{2}{3}$ each after averaging, then $\mu'(v) \geq 16 - 6 - 12 \times \frac{2}{3} - 4 \times \frac{1}{2} = 0$. If each of fifteen faces conducts $\frac{2}{3}$, then the sixteenth face consists of 16*-vertices and does not receive any charge from $v$ by R1–R3. This implies that $\mu'(v) \geq 16 - 6 - 15 \times \frac{2}{3} = 0$.

Thus we have proved that $\mu'(v) \geq 0$ for every $v \in V_{6+}$ and $\mu'([mc]) \geq 0$ for every $[mc] \in MC$, which contradicts (3):

$$0 \leq \sum_{v \in V_{6+}} \mu'(v) + \sum_{[mc] \in MC} \mu'([mc]) = \sum_{v \in V_{6+}} \mu(v) + \sum_{[mc] \in MC} \mu([mc]) = -12.$$ 

This completes the proof of Theorem 1. \hfill \Box

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