On the multicolor Ramsey number for 3-paths of length three

Tomasz Łuczak* and Joanna Polcyn
A. Mickiewicz University
Faculty of Mathematics and Computer Science
ul. Umultowska 87,
61-614 Poznań, Poland

tomasz@amu.edu.pl

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Abstract

We show that if we color the hyperedges of the complete 3-uniform hypergraph on \(2n + \sqrt{18n+1} + 2\) vertices with \(n\) colors, then one of the color classes contains a loose path of length three.

Let \(P\) denote the 3-uniform path of length three by which we mean the only connected 3-uniform hypergraph on seven vertices with the degree sequence \((2,2,1,1,1,1,1)\). By \(R(P; n)\) we denote the multicolored Ramsey number for \(P\) defined as the smallest number \(N\) such that each coloring of the hyperedges of the complete 3-uniform hypergraph \(K_N^{(3)}\) with \(n\) colors leads to a monochromatic copy of \(P\). It is easy to check that \(R(P; n) \geq n + 6\) (see [2, 5]), and it is believed that in fact equality holds, i.e.

\[ R(P; n) = n + 6. \]

Gyárfás and Raeisi [2] proved, among many other results, that \(R(P; 2) = 8\). Their theorem was extended by Omidi and Shahsiah [9] to loose paths of arbitrary lengths, but still only for the case of two colors. On the other hand, in a series of papers [5, 7, 12, 10] it was verified that \(R(P; n) = n + 6\) for all \(3 \leq n \leq 10\).

Note that from the fact that for \(N \geq 8\) the largest \(P\)-free 3-uniform hypergraph on \(N\) vertices contains at most \(\binom{N-1}{2}\) hyperedges (see [6]), it follows that for \(n \geq 3\) we have (see [2])

\[ R(P; n) \leq 3n + 1. \]

Our main goal is to improve the above bound.

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Theorem. \( R(P; n) \leq 2n + \sqrt{18n + 1} + 2. \)

Let \( C \) denote the (loose) 3-uniform 3-cycle, i.e. the only 3-uniform linear hypergraph with six vertices and three hyperedges. Furthermore, let \( F \) be the 3-uniform hypergraph on vertices \( v_1, v_2, v_3, v_4, v_5 \) such that the first four of these vertices span a clique, and \( v_5 \) is contained in the following three hyperedges: \( v_1v_2v_5, v_2v_3v_5, \) and \( v_3v_4v_5. \) The following fact will be crucial for our argument.

Lemma. Let \( H \) be a 3-uniform \( P \)-free hypergraph on \( n \geq 5 \) vertices. Then we can delete from \( H \) fewer than \( 3n \) hyperedges in such a way that the resulting hypergraph contains no copies of \( C \) and \( F. \)

Proof. Let us first consider components containing \( C. \) Jackowska, Polcyn and Ruciński [7] showed that each such component of \( H \) on \( n_i \) vertices has at most \( 3n_i - 8 < 3n_i \) hyperedges, provided \( n_i \geq 7. \) Furthermore, from the complete 3-uniform hypergraph on \( n_i = 6 \) vertices it is enough to delete \( 10 < 3n_i \) hyperedges to get a star, which clearly contains no copies of \( C \) and \( F. \) Hence, to get rid of all copies of \( C \) (and \( F \) in components containing \( C \)) it is enough to remove fewer than \( 3n' \) hyperedges from components containing them, where \( n' \) denotes the number of vertices in these components combined. Now let us consider components containing \( F \) but not \( C. \) It is easy to check by direct inspection that any hyperedge \( e \) which shares with \( F \) just one vertex would create a copy of \( P. \) Moreover, any hyperedge \( e' \) which shares with \( F \) two vertices would create a copy of \( C. \) Consequently, each copy of \( F \) in a \( P \)-free, \( C \)-free 3-uniform hypergraph is contained in a component on 5 vertices. Note that each such component has at most \( \binom{5}{3} = 10 \) hyperedges and we can destroy \( F \) by removing just \( 4 < 3 \cdot 5 \) of them. Thus, one can delete from \( H \) fewer than \( 3n \) hyperedges to destroy all copies of \( C \) and \( F. \)

Proof of Theorem. Consider a coloring of the hyperedges of the complete 3-uniform hypergraph on \( 2n + m \) vertices with \( n \) colors. Assume that no color class contains a copy of \( P. \) Then, by the lemma, we can mark as ‘blank’ fewer than \( r = 3(2n + m)n \) hyperedges of the hypergraph in such a way that when we ignore blank hyperedges no color class contains a monochromatic copy of \( C \) and \( F. \)

Let us color a pair of vertices \( vw \) with a color \( s, s = 1, 2, \ldots, n, \) if there exist at least three hyperedges of color \( s, s = 1, 2, \ldots, n, \) which contain this pair. If there are many such colors we choose any of them; if there are none we leave \( vw \) uncolored. Note that every uncolored pair must be contained in at least \( m - 2 \) blank hyperedges. Consequently, fewer than \( 3r/(m - 2) \) pairs remain uncolored. But then there exists a color \( t, t = 1, 2, \ldots, n, \) such that there are more than

\[
\left\lfloor \binom{2n+m}{2} - \frac{3r}{m-2} \right\rfloor / n = 2n + m + (2n + m)\left[\frac{m-1}{2n} - \frac{9}{m-2}\right]
\]

pairs colored with \( t. \) If \( m \geq \sqrt{18n + 1} + 2, \) then

\[
\frac{m-1}{2n} - \frac{9}{m-2} > 0,
\]
and the graph $G_t$ spanned by these pairs has more edges than vertices. But this means that $G_t$ contains a path of length 3, i.e. there are vertices $v_1, v_2, v_3, v_4$ and a color $t$ such that each of the three pairs $v_1v_2$, $v_2v_3$, $v_3v_4$ is contained in at least three different hyperedges colored with $t$. We shall show that this leads to a contradiction.

Indeed, let $H_t$ be a hypergraph spanned by hyperedges colored with the $t$th color. Observe first that since $v_2v_3$ is contained in three different hyperedges of $H_t$ there must be one which is different from $v_1v_2v_3$ and $v_2v_3v_4$; let us call it $v_2v_3v_5$ where $v_5 \neq v_1, v_2, v_3, v_4$. Furthermore, $v_1v_2$ must be contained in a hyperedge $v_1v_2w$ of $H_t$ where $w \neq v_3, v_5$, while $v_3v_4$ is contained in some $v_3v_4u$, where $u \neq v_2, v_5$. Note now that if $w \neq v_4$ and $u \neq w, v_1$, then $H_t$ contains a copy of $P$ which contradicts the fact that it is $P$-free. The case $w = u \neq v_1, v_4$, as well as the cases $w = v_4, u \neq v_1$, and $u = v_1, w \neq v_4$, would lead to a cycle $C$. Finally, if the only possible choices for $w$ and $u$ are $w = v_4$ and $u = v_1$, then the vertices $v_1, \ldots, v_5$ span a copy of $F$, so we arrive at a contradiction again.

**Remark** The bound $3n$ given in the lemma is rather crude. Using the fact that each component of $H$ on $n_i$ vertices containing $C$ and two disjoint hyperedges has at most $n_i + 5$ hyperedges, provided $n_i \geq 7$ (see [10]) and by careful analysis of 3-uniform intersecting families (see [1, 3, 4, 8, 11]) one can improve it by a constant factor and, consequently, improve by a constant factor the second order term in the estimate for $R(P; n)$. Nonetheless our method, based on the reduction of the hypergraph problem to the analogous problem for graphs, clearly cannot be used to produce an upper bound better than $2n$. However, we still believe that $n + 6$ could be the correct value for $R(P; n)$.

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**References**


