A note on non-$\mathbb{R}$-cospectral graphs

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Submitted: Mar 10, 2016; Accepted: Feb 23, 2017; Published: Mar 17, 2017
Mathematics Subject Classifications: 05C50

Abstract
Two graphs $G$ and $H$ are called $\mathbb{R}$-cospectral if $A(G) + yJ$ and $A(H) + yJ$ (where $A(G)$, $A(H)$ are the adjacency matrices of $G$ and $H$, respectively, $J$ is the all-one matrix) have the same spectrum for all $y \in \mathbb{R}$. In this note, we give a necessary condition for having $\mathbb{R}$-cospectral graphs. Further, we provide a sufficient condition ensuring only irrational orthogonal similarity between certain cospectral graphs. Some concrete examples are also supplied to exemplify the main results.

Keywords: $\mathbb{R}$-cospectral graphs; Walk generating function; Irrational orthogonal matrix

1 Introduction
Throughout this paper, we are concerned with undirected simple graphs (loops and multiple edges are not allowed). Let $G$ be a simple graph with $(0, 1)$-adjacency matrix $A(G)$.
The spectrum of $G$ consists of all the eigenvalues (together with their multiplicities) of $A(G)$. Two graphs $G$ and $H$ are called cospectral if $G$ and $H$ share the same spectrum. If $A(G) + yJ$ and $A(H) + yJ$ (where $J$ is the all-one matrix) have the same spectrum for all $y \in \mathbb{R}$, then $G$ and $H$ are called $\mathbb{R}$-cospectral (also generalized cospectral). By taking $y = -1$ we see that $\mathbb{R}$-cospectral graphs have cospectral complements.

A graph $G$ is said to be determined by its (generalized) spectrum (DGS) if any graph (generalized) cospectral with $G$ is isomorphic with $G$. A fundamental problem in spectral graph theory is “which graphs are determined by their spectrum?” Up to now, only a small number of graphs with special structures are known to be DGS, simplify because DGS is a property difficult to prove. Haemers conjectured that almost all graphs are determined by their spectrum [5], however, it is still open. For excellent surveys on this subject, see [5, 7]. Recently, several new results are published such as some almost complete graphs [2] are shown to be DGS as well as the spectral characterization of pineapple graphs [10] are corrected. When it comes to the generalized spectrum, a necessary and sufficient condition for the corona of some graphs to be DGS is established [12], moreover, the DGS property of a large family of graphs $\mathcal{F}_n$ [16] is also obtained [17].

Non-isomorphic cospectral graphs are easily made, early in the last century, Schwenk [14] proved that almost all trees are cospectral. Godsil and McKay [8] invented a powerful and productive method called GM-switching, which can produce numerous pairs of $\mathbb{R}$-cospectral graphs. If $G$ and $H$ are cospectral, then their adjacency matrices $A(G)$ and $A(H)$ are similar. Since both are real symmetric, $A(G)$ and $A(H)$ have a real orthogonal similarity, that is, there exists an orthogonal matrix $Q$ such that $A(G) = Q^T A(H)Q$. An orthogonal matrix $Q$ is regular if it has all row and column sums 1. A regular orthogonal matrix $Q$ has level $\ell$ if $\ell$ is the smallest positive integer such that $\ell Q$ is an integral matrix. $Q$ has level $\ell = \infty$ if it has irrational entries. It easily follows that $G$ and $H$ are $\mathbb{R}$-cospectral if $Q$ is regular. The following result, due to Johnson and Newman [11], gives some equivalent conditions for $\mathbb{R}$-cospectral graphs.

\textbf{Theorem 1.} [1, 11] Let $G$ and $H$ be two graphs with adjacency matrices $A(G)$ and $A(H)$, respectively, then the following are equivalent:

(i) $G$ and $H$ are cospectral with cospectral complements.

(ii) $G$ and $H$ are $\mathbb{R}$-cospectral.

(iii) There exists a regular orthogonal matrix $Q$, such that $A(G) = Q^T A(H)Q$.

It is well known that many simple necessary conditions are achieved for cospectral graphs such as equal number of vertices; edges; triangles [5] etc. However, except for that, such similar necessary conditions for having $\mathbb{R}$-cospectral graphs are seldom considered. In this note, we show that $\mathbb{R}$-cospectral graphs have the same number of cycles $C_4$; paths $P_3$; paths $P_4$ (all the subgraphs $C_4$, $P_3$, $P_4$ above are not necessarily induced).

The rationality property of the entries of $Q$ in Theorem 1 was firstly considered in [15]. If the eigenvalues of graph $G$ are restricted to be all simple (its multiplicity is one) and main (it has an associated eigenvector the sum of whose entries is not equal to zero), then Wang and Xu showed that the regular orthogonal matrix $Q$ must be rational.
Theorem 2. [15, 17] Let $G$ be a graph whose eigenvalues are all main and simple (i.e., controllable graph), $A(G)$ and $A(H)$ be the adjacency matrices of graphs $G$ and $H$, respectively. Then $G$ and $H$ are $\mathbb{R}$-cospectral if and only if there exists a unique regular rational orthogonal matrix $Q$ such that $A(G) = Q^T A(H) Q$.

Motivated by this theorem, another two questions arise.

(i) Can we give some families of cospectral graphs such that there only exists real irrational orthogonal similarity between their adjacency matrices?

(ii) Can we find a pair of $\mathbb{R}$-cospectral graphs $G$ and $H$ such that there exists two regular rational orthogonal matrices $Q_1$, $Q_2$ and $A(G) = Q_1^T A(H) Q_1$, $A(H) = Q_2^T A(H) Q_2$.

Theorem 2 states that if the walk matrix of a graph $G$ is nonsingular, then there exists a graph $H$ that is cospectral with $G$ w.r.t. the generalized spectrum if and only if there exists a rational regular orthogonal matrix such that the adjacency matrices are similar. So it basically means that the regular orthogonal matrix has a level. Nevertheless, we provide a sufficient condition which ensures only irrational orthogonal similarity between certain cospectral graphs (or equivalently when there is no condition on the walk matrix being nonsingular). Moreover, S. O’rourke and B. Touriz lately [13] shows that the relative number of controllable graphs compared to the total number of simple graphs on $n$ vertices approaches one as $n$ tends to infinity, which gives even more evidence of the importance of the result of item (i) presented in this note. In addition, we supply a pair of $\mathbb{R}$-cospectral graphs, which can also be found in [1], satisfying the above statement (ii).

2 Main results

By a walk of length $k$ in a graph we mean any sequence of (not necessarily different) vertices $v_1v_2 \ldots v_kv_{k+1}$ such that for each $i = 1, 2, \ldots, k$ there is an edge from $v_i$ to $v_{i+1}$. The walk is closed if $v_{k+1} = v_1$. We start with the renowned walk generating function.

Lemma 3. [3] Let $G$ be a graph with complement $\overline{G}$, and $H_G(t) = \sum_{k=0}^{\infty} N_k(G) t^k$ be the generating function of the number $N_k(G)$ of walks of length $k$ in $G$ ($k = 0, 1, 2, \ldots$). Then $H_G(t) = \frac{1}{2} [(-1)^n P_G(-\frac{i\pi}{2})/P_G(\frac{i\pi}{2}) - 1]$, where $P_G(t)$ and $P_{\overline{G}}(t)$ are characteristic polynomials of $G$ and $\overline{G}$, respectively.

Since $\mathbb{R}$-cospectral graphs are cospectral with cospectral complements, they have the same characteristic polynomial and complement characteristic polynomial. It easily follows a corollary.

Corollary 4. Let $G$ and $H$ be $\mathbb{R}$-cospectral graphs. Then $G$ and $H$ have the same walk generating function and the same number of walks of any length $k$.

Consider the number of walks and closed walks of length no more than four, then we can prove the following necessary condition for $\mathbb{R}$-cospectral graphs.
Theorem 5. For the adjacency matrix, the following can be deduced from the generalized spectrum:

1. the number of subgraphs \(C_4\).
2. the number of subgraphs \(P_3\).
3. the number of subgraphs \(P_4\), all the subgraphs \(C_4, P_3, P_4\) are not necessarily induced.

Proof. Suppose \(G\) and \(H\) are \(\mathbb{R}\)-cospectral, then \(G\) and \(H\) have the same number of closed walks and walks of any given length \(k\). It is well known that the numbers of closed walks of length 2, 3, 4 are equal to twice number of edges; six times the number of triangles; twice number of edges plus four times the numbers \(P_3\) and eight times the number of cycles \(C_4\) in a graph; respectively. Thus \(G\) and \(H\) have the same number of edges, triangles. For (closed or non-closed) walks, it is clear that the numbers of walks of length 2, 3 are equal to twice number of edges and \(P_3\) contained in a graph; the sum of twice number of edges, four times the number of \(P_3\), twice number of \(P_4\) and six times the number of triangles, respectively, in a graph. Therefore, \(\mathbb{R}\)-cospectral graphs have the same number of cycles \(C_4\) and paths \(P_3, P_4\).

Theorem 5 can be easily used to determine some cospectral graphs that are not \(\mathbb{R}\)-cospectral, see an example below.

Example 6. Let \(W_n\) \((n \geq 2)\) be the tree obtained from the path \(P_{n+2}\) by appending a pendant vertex to the two end 2-degree vertices, respectively, and \(W_1 \cong K_{1,4}\). Let \(X_n\) \((n \geq 1)\) be the disjoint union of the cycle \(C_4\) and path \(P_n\). It is known from [4] that \(W_n\) and \(X_n\) are cospectral. Since \(W_n\) contains no cycle \(C_4\) and \(X_n\) has one, it follows from Theorem 5 that \(W_n\) and \(X_n\) are non-\(\mathbb{R}\)-cospectral.

Next we present another main result of the note.

Theorem 7. Let \(G\) and \(H\) be a pair of cospectral graphs with a common simple eigenvalue \(\lambda_0\). If \(A(G)\) possesses a irrational normal eigenvector \(x_0\) while \(A(H)\) has a rational normal eigenvector \(y_0\) corresponding to \(\lambda_0\), then there exists no rational orthogonal matrix \(Q\) such that \(A(G) = Q^T A(H)Q\).

Proof. Assume to the contrary that there exists a rational orthogonal matrix \(Q = (q_{ij})\), where \(q_{ij} \in \mathbb{Q}\), such that \(A(G) = Q^T A(H)Q\), then \(QA(G) = A(H)Q\). It follows that

\[
\lambda_0 Qx_0 = Q(A(G)x_0) = A(H)(Qx_0),
\]

namely, \(Qx_0\) is also an eigenvector of \(H\) corresponding to the simple eigenvalue \(\lambda_0\). So

\[
y_0 = kQx_0 \quad (k \neq 0, \ k \in \mathbb{R}).
\]

From \(1 = y_0^T y_0 = k^2 x_0^T x_0 = k^2\), we see that \(k = \pm 1\) and \(x_0 = \pm Q^T y_0\). Since \(x_0\) is an irrational eigenvector while \(\pm Q^T y_0\) is rational, we get a contradiction. \(\square\)
In order to exemplify Theorem 7, we give an example below. In fact, one may find other such cospectral pairs.

**Example 8.** If \( n \neq k^2 - 1 \) \((k = 2, 3, \ldots)\), then there exists no rational orthogonal matrix \( Q \) such that \( A(W_n) = Q^T A(X_n) Q \).

Label the vertices of \( W_n \) as follows: the vertices on the path \( P_{n+2} \) be consecutively marked as \( u_1, \ldots, u_n, u_{n+3}, u_{n+4} \) are adjacent to \( u_{n+1} \) and \( u_2 \), respectively. Label the path \( P_n \) of \( X_n \) by \( v_1, \ldots, v_n \) and the four vertices on the cycle \( C_4 \) by \( v_{n+1}, \ldots, v_{n+4} \). Then the adjacency matrices \( A(W_n) \) and \( A(X_n) \) are of the form:

\[
A(W_n) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & \ddots & \ddots & 0 & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\
0 & 0 & 0 & \ddots & \ddots & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{bmatrix},
\]

(3)

\[
A(X_n) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 1 \\
0 & 0 & 1 & 0 \\
& & & \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}.
\]

Since graphs \( W_n \) and \( X_n \) share the common adjacency spectrum \( \{2, 0(2), -2\} \cup \{2 \cos \frac{j \pi}{n+1} \} \) \((j = 1, \ldots, n)\), where the exponent indicates the multiplicity of the eigenvalue. We see that the spectral radius 2 is a simple eigenvalue. Solving the characteristic equation \( A(W_n)x = 2x \) gives that \( x = (1, 2, 3, \ldots, 2, 1, 1, 1)^T \). Similarly, from \( A(X_n)y = 2y \), we obtain \( y = (0, \ldots, 0, 1, 1, 1, 1)^T \). Normalizing the above two eigenvectors gives \( x_0 = 2\sqrt{n + 1}x \) and \( y_0 = \frac{1}{2}y \). Obviously, \( x_0 \) is irrational while \( y_0 \) is rational, it follows our deduction by Theorem 7

Furthermore, we calculate the concrete irrational orthogonal matrix \( Q \) in Example 8 for \( n = 1, 2, 3 \).

**Example 9.** Suppose \( A(W_n) \) and \( A(X_n) \) be shown in Eqs. (3) and (4), define \( Q_1, Q_2 \)
and $Q_3$ as follows:

$$Q_1 = \begin{bmatrix}
-\frac{1}{2\sqrt{3}} & 0 & \sqrt{\frac{3}{2}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\
\frac{1}{2} + \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2} + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
-\frac{1}{2} + \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2} + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
\end{bmatrix},$$

$$Q_2 = \begin{bmatrix}
0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \\
\frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 & \frac{1}{2} + \frac{1}{2\sqrt{3}} & -\frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 \\
-\frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 & \frac{1}{2} + \frac{1}{2\sqrt{3}} & 0 & 0 & \frac{1}{2} + \frac{1}{2\sqrt{3}} \\
\end{bmatrix},$$

$$Q_3 = \begin{bmatrix}
\frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{5}} & 0 & \frac{1}{2\sqrt{5}} & 0 & -\frac{1}{2\sqrt{2}} - \frac{\sqrt{5}}{4} & -\frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{5}} & -\frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{5}} \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
\frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{5}} & 0 & -\frac{1}{2\sqrt{5}} & 0 & -\frac{1}{2\sqrt{2}} + \frac{\sqrt{5}}{4} & -\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{5}} & -\frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{5}} \\
-\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{5}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
\frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{5}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\
-\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\end{bmatrix}.$$  

Then a direct calculation shows that $A(W_n) = Q_n^T A(X_n) Q_n$ ($n = 1, 2, 3$).

Finally, we provide a pair of $\mathbb{R}$-cospectral graphs on eight vertices to answer the question of the statement (ii).

**Example 10.** [1] Let $\Gamma_1$ and $\Gamma_2$ be a pair of $\mathbb{R}$-cospectral graphs with adjacency matrices

$$A(\Gamma_1) = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}, \quad A(\Gamma_2) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\end{bmatrix}.$$
Suppose that
\[
\tilde{Q}_1 = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}.
\]

It is straightforward to verify \( A(\Gamma_1) = \tilde{Q}_i^T A(\Gamma_2) \tilde{Q}_i \) \((i = 1, 2)\). In addition, simple calculation shows that \( A(\Gamma_1) \) and \( A(\Gamma_2) \) have the identical characteristic polynomial
\[
P(x) = x^2(x^6 - 14x^4 - 14x^3 + 21x^2 + 14x - 11),
\]
so it follows that \( \Gamma_1 \) (\( \Gamma_2 \)) has a multiple eigenvalue 0 and only seven main eigenvalues.

Acknowledgements

The authors would like to express their gratitude to the anonymous reviewers for their careful reading of the manuscript and the detailed suggestions which led to a great improvement of the presentation of the paper.

References


