On $t$-common list-colorings

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Abstract

In this paper, we introduce a new variation of list-colorings. For a graph $G$ and for a given nonnegative integer $t$, a $t$-common list assignment of $G$ is a mapping $L$ which assigns each vertex $v$ a set $L(v)$ of colors such that given set of $t$ colors belong to $L(v)$ for every $v \in V(G)$. The $t$-common list chromatic number of $G$ denoted by $ch_t(G)$ is defined as the minimum positive integer $k$ such that there exists an $L$-coloring of $G$ for every $t$-common list assignment $L$ of $G$, satisfying $|L(v)| \geq k$ for every vertex $v \in V(G)$. We show that for all positive integers $k, \ell$ with $2 \leq k \leq \ell$ and for any positive integers $i_1, i_2, \ldots, i_{k-2}$ with $k \leq i_{k-2} \leq \cdots \leq i_1 \leq \ell$, there exists a graph $G$ such that $\chi(G) = k$, $ch(G) = \ell$ and $ch_t(G) = i_t$ for every $t = 1, \ldots, k-2$. Moreover, we consider the $t$-common list chromatic number of planar graphs. From the four color theorem [1, 2] and the result of Thomassen [9], for any $t = 1$ or 2, the sharp upper bound of $t$-common list chromatic number of planar graphs is 4 or 5. Our first step on $t$-common list chromatic number of planar graphs is to find such a sharp upper bound. By constructing a planar graph $G$ such that $ch_1(G) = 5$, we show that the sharp upper bound for 1-common list chromatic number of planar graphs is 5. The sharp upper bound of 2-common list chromatic number of planar graphs is still open. We also suggest several questions related to $t$-common list chromatic number of planar graphs.

Keywords: graph coloring, list coloring, planar graph

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1 Introduction

Throughout this paper, all graphs are finite, undirected, and simple. For a graph $G$, let $V(G)$ and $E(G)$ be the vertex set and the edge set of $G$, respectively. The neighborhood of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to $v$.

For a given graph $G$, a proper $k$-coloring $\phi : V(G) \to \{1, 2, \ldots, k\}$ of a graph $G$ is an assignment of colors to the vertices of $G$ so that any two adjacent vertices receive distinct colors. The chromatic number $\chi(G)$ of a graph $G$ is the least positive integer $k$ such that there exists a proper $k$-coloring of $G$. If $G$ has a proper $k$-coloring, namely $\chi(G) \leq k$, then we say that $G$ is $k$-colorable. A list assignment of a graph $G$ is a mapping $L$ which assigns each vertex $v$ a set $L(v)$ of colors. An $L$-coloring of $G$ is a proper vertex coloring $\phi$ of $G$ such that $\phi(v) \in L(v)$ for each $v$. We say $G$ is $L$-colorable if there exists an $L$-coloring of $G$. For a positive integer $k$, we say $G$ is $k$-choosable if $G$ has an $L$-coloring for every list assignment $L$ satisfying $|L(v)| \geq k$ for every $v \in V(G)$. The list chromatic number or choice number $ch(G)$ of $G$ is the minimum positive integer $k$ such that $G$ is $k$-choosable. Clearly $\chi(G) \geq ch(G)$ for every graph $G$.

For a graph $G$ and for a given nonnegative integer $t$, a $t$-common list assignment of a graph $G$ is a mapping $L$ which assigns each vertex $v$ a set $L(v)$ of colors such that given set of $t$ colors belong to every $L(v)$, namely $|\cap_{v \in V(G)} L(v)| \geq t$. Note that 0-common list assignment is just a list assignment. The $t$-common list chromatic number of $G$ denoted by $ch_t(G)$ is defined as the minimum positive integer $k$ such that $G$ is $L$-colorable for every $t$-common list assignment $L$ of $G$ satisfying $|L(v)| \geq k$ for every vertex $v$. Clearly, $ch_t(G) = t$ for every integer $t \geq \chi(G)$.

Before exploring this topic, we describe an application of $t$-common list-coloring. A company has $n$ chemicals they have manufactured that need to be stored. Some pairs of chemicals are incompatible. For this reason, such pairs should be kept in distinct storage vessels. Say $t$ storage vessels can keep all chemicals while other storage vessels can only keep certain chemicals because of storage vessel’s conditions. Determine minimum positive integer $k$ such that all chemicals can be stored if the number of possible storage vessels for each chemical is at least $k$. In other words, each chemical can potentially be stored in at least $k$ vessels, given the restriction that some storage vessels can only store certain chemicals. We can convert this storage problem into a $t$-common list-coloring problem on a graph. Consider a graph $G = (V, E)$ with all chemicals as a vertex set, and an edge between chemicals $x$, $y$ if and only if $x$ and $y$ are incompatible. For every vertex $v \in V$, let $L(v)$ be the set of all storage vessels which can keep the chemical corresponding to $v$. Now the list assignment $L$ is a $t$-common list assignment and the above question corresponds to find $t$-common list chromatic number $ch_t(G)$ of $G$.

In the next section of this paper, we investigate several properties of the $t$-common list chromatic numbers. Furthermore we show that for all positive integers $k, \ell$ with $2 \leq k \leq \ell$
and for any positive integers \(i_1, i_2, \ldots, i_{k-2}\) with \(k \leq i_{k-2} \leq \cdots \leq i_1 \leq \ell\), there exists a graph \(G\) such that \(\chi(G) = k\), \(ch(G) = \ell\) and \(ch_t(G) = i_t\) for all \(t = 1, \ldots, k-2\). In Section 3, we consider the \(t\)-common list chromatic number of planar graphs. By constructing a planar graph \(G\) such that \(ch_1(G) = 5\), we show that the sharp upper bound for \(1\)-common list chromatic number of planar graphs is 5. Furthermore, we suggest several questions related to \(t\)-common list chromatic number of planar graphs.

\section{Some properties of \(t\)-common list colorings}

In this section, we consider several properties of the \(t\)-common list chromatic number. For a graph \(G\) with connected components \(G_1, G_2, \ldots, G_i\) and for every nonnegative integer \(t\), one can easily see that \(ch_t(G) = \max\{ch_t(G_j) \mid j = 1, \ldots, i\}\). So from now on, we will consider the \(t\)-common list chromatic number of a connected graph. As with other graph coloring parameters, it holds that for every subgraph \(H\) of \(G\) and for every nonnegative integer \(t\), \(ch_t(H) \leq ch_t(G)\). The next lemma gives some relationships among \(\chi(G), ch_t(G)\) and \(ch_t(G)\).

\textbf{Lemma 1.} Let \(G\) be a connected graph with \(V(G) = \{v_1, \ldots, v_n\}\) and let \(\chi(G) = k\). The following properties hold.

1. \(\chi(G) = ch_{k-1}(G) \leq ch_{k-2}(G) \leq \cdots \leq ch_1(G) \leq ch(G)\).
2. For every nonnegative integer \(t\) with \(t \geq k\), \(ch_t(G) = t\).

\textbf{Proof.}\n
(1) Let \(t\) be a positive integer such that \(t \leq k - 1\). Note that the chromatic number \(\chi(G)\) is the minimum \(i\) such that \(G\) has an \(L\)-coloring for \(L(v_1) = \cdots = L(v_n) = \{c_1, c_2, \ldots, c_i\}\). Since the above list assignment \(L\) is a special \(t\)-common list assignment of \(G\), we have \(\chi(G) \leq ch_t(G)\). Note that every \(t\)-common list assignment is a \((t-1)\)-common list assignment. So \(\chi(G) \leq ch_{k-1}(G) \leq ch_{k-2}(G) \leq \cdots \leq ch_1(G) \leq ch(G)\).

Let \(L\) be a \((k-1)\)-common list assignment such that \(c_1, \ldots, c_{k-1} \in L(v_i)\) and \(|L(v_i)| = k\) for all \(i = 1, \ldots, n\). Since \(\chi(G) = k\), the vertex set \(V(G)\) can be partitioned into \(k\) independent sets \(I_1, \ldots, I_k\). For all \(j = 1, \ldots, k-1\), assign the color \(c_j\) to every vertex in \(I_j\) and for every \(v \in I_k\), assign the color \(c_v \in L(v) \setminus \{c_1, \ldots, c_{k-1}\}\) to \(v\). This assignment is an \(L\)-coloring, and so \(ch_{k-1}(G) \leq \chi(G)\). This implies that \(ch_{k-1}(G) = \chi(G)\).

(2) By definition of \(ch_t(G)\), one can easily show that \(ch_t(G) = t\) for every \(t\) with \(t \geq k\). \(\square\)

By Lemma 1, we have the following corollary.

\textbf{Corollary 2.} For a connected graph \(G\) and a nonnegative integer \(t\), \(ch_t(G) = t + 1\) if and only if \(t = \chi(G) - 1\).

For a graph \(G\), a list assignment \(L\) of \(G\) is called a \textit{maximal unavailable list assignment} of \(G\) if \(G\) has no \(L\)-coloring and \(|L(v)| = ch(G) - 1\) for every \(v \in V(G)\). For example, a
cycle $C_3$ of length 3 with vertex set $\{v_1, v_2, v_3\}$ has a maximal unavailable list assignment $L$ with $L(v_1) = L(v_2) = L(v_3) = \{a, b\}$. Note that $\chi(C_3) = 3$ and $L$ is the unique maximal unavailable list assignment up to permutation of colors.

It is known that for all positive integers $k$ and $\ell$ with $2 \leq k \leq \ell$, there exists a graph $G$ such that $\chi(G) = k$ and $\chi(G) = \ell$ [5]. In the remaining part of this section, we generalize this result as follows: for all positive integers $k, \ell$ with $2 \leq k \leq \ell$ and for any positive integers $i_1, i_2, \ldots, i_{k-2}$ with $k \leq i_{k-2} \leq \cdots \leq i_1 \leq \ell$, there exists a graph $G$ such that $\chi(G) = k$, $\chi(G) = \ell$ and $\chi_t(G) = i_t$ for all $t = 1, \ldots, k - 2$. For this purpose, we introduce two graph operations. The first graph operation is defined here, and the second is defined later. For every graph $G$ with $V(G) = \{v_1, \ldots, v_n\}$, the duplication $D(G)$ of $G$ is defined as follows:

$$V(D(G)) = V(G) \cup \{v_{i,j} \mid i, j = 1, \ldots, n\} \text{ and }$$

$$E(D(G)) = E(G) \cup \{\{v_{i,r}, v_{i,s}\} \mid i = 1, \ldots, n, \ \{v_r, v_s\} \in E(G)\}$$

$$\cup \{\{v_i, v_{i,j}\} \mid i, j = 1, \ldots, n\}.$$ 

Namely $D(G)$ is obtained by the following ways: With $G$, construct $n$ more copies $G_1, G_2, \ldots, G_n$ of $G$, which correspond to vertices of $G$, and add edges between $v_i$ and every vertex in the corresponding copy $G_i$ for all $i = 1, \ldots, n$. For convenience, let $G_i$ be the induced subgraph of $D(G)$ with vertex set $\{v_{i,j} \mid j = 1, \ldots, n\}$ for each $i \in \{1, \ldots, n\}$. Note that $G_i$ is isomorphic to $G$.

**Lemma 3.** Let $G$ be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$ and let $\chi(G) = k$. Now the following properties hold.

1. $\chi(D(G)) = \chi(G) + 1$.
2. $\chi(D(G)) = \chi(G) + 1$.
3. For every nonnegative integer $t$ with $1 \leq t \leq k$, $\chi_t(D(G)) = \chi_{t-1}(G) + 1$.

**Proof.** (1) Let $H$ be the induced subgraph of $D(G)$ with $V(H) = \{v_1\} \cup \{v_{1,j} \mid j = 1, \ldots, n\}$. Now $H$ is isomorphic to a graph join of a trivial graph and $G$. So $\chi(H) = k + 1$, which implies that $\chi(D(G)) \geq \chi(G) + 1$.

Let $\phi : V(G) \to \{c_1, \ldots, c_k\}$ be a proper $k$-coloring of $G$. For all $i = 1, \ldots, n$, let $C_i = \{c_1, \ldots, c_k, c_{k+1}\} - \{\phi(v_i)\}$. Now $G_i$ has a proper $k$-coloring $\phi_i$ with the color set $C_i$. These proper $k$-colorings define a proper $(k+1)$-coloring of $D(G)$. Hence $\chi(D(G)) \leq k + 1$ and so $\chi(D(G)) = \chi(G) + 1$.

(2) Let $L_1$ be a maximal unavailable list assignment of $G$. Choose a color $c$ which does not belong to $L_1(v)$ for any $v \in V(G)$. Let $L_2$ be a list assignment of $D(G)$ defined by $L_2(v_i) = L_2(v_{i,j}) = L_1(v_j) \cup \{c\}$ for all $i, j = 1, \ldots, n$. Suppose that $D(G)$ has an $L_2$-coloring. Now, $c$ should be assigned to at least one of $v_1, \ldots, v_n$, say $v_1$, because $L_1$ is a maximal unavailable list assignment of $G$, and hence $G_1$ has a proper coloring $\phi$ such
that $\phi(v_{i,j}) \in L_1(v_j)$. Since $G_i$ is isomorphic to $G$, this implies that $G$ has an $L_1$-coloring. This is a contradiction. So $\text{ch}(D(G)) \geq \text{ch}(G) + 1$.

Let $L$ be a list assignment of $D(G)$ such that $|L(u)| = \text{ch}(G) + 1$ for every $u \in V(D(G))$. Consider the induced subgraph $G$ of $D(G)$. Now, $G$ has an $L$-coloring $\phi$. For all $i, j = 1, \ldots, n$, let $L_3(v_{i,j}) = L(v_{i,j}) - \{\phi(v_i)\}$. Since for all $i, j = 1, \ldots, n$, $|L_3(v_{i,j})| \geq \text{ch}(G)$, $G_i$ has an $L_3$-coloring, and hence $D(G)$ has an $L$-coloring. So $\text{ch}(D(G)) \leq \text{ch}(G) + 1$. Therefore $\text{ch}(D(G)) = \text{ch}(G) + 1$.

(3) Let $L_4$ be a $(t - 1)$-common list assignment of $G$ such that $G$ has no $L_4$-coloring and $|L_4(v)| = \text{ch}_{t-1}(G) - 1$ for every $v \in V(G)$. Choose a color $c$ which does not belong to $L_4(v)$ for any $v \in V(G)$. Let $L_5$ be a $t$-common list assignment of $D(G)$ defined by $L_5(v_j) = L_5(v_{i,j}) = L_4(v_j) \cup \{c\}$ for all $i, j = 1, \ldots, n$. Suppose that $D(G)$ has an $L_5$-coloring. Then $c$ should be assigned to some $v_i \in \{v_1, \ldots, v_n\}$, and hence $G_i$ has a proper coloring $\psi$ such that $\psi(v_{i,j}) \in L_4(v_j)$. This implies that $G$ has an $L_4$-coloring, a contradiction. So $\text{ch}_{t}(D(G)) \geq \text{ch}_{t-1}(G) + 1$.

Let $L'$ be a $t$-common list assignment of $D(G)$ such that $|L'(u)| = \text{ch}_{t-1}(G) + 1$ for every $u \in V(D(G))$. Since a $t$-common list assignment of $G$ is also a $(t - 1)$-common list assignment, $G$ has an $L'$-coloring $\phi'$. For all $i, j = 1, \ldots, n$, let $L_6(v_{i,j}) = L'(v_{i,j}) - \{\phi'(v_i)\}$. Now the restriction of $L_6$ onto $G_i$ is a $(t - 1)$-common list assignment such that $|L_6(v_{i,j})| \geq \text{ch}_{t-1}(G)$. This implies that for all $i = 1, \ldots, n$, $G_i$ has an $L_6$-coloring, and hence $D(G)$ has an $L'$-coloring. So $\text{ch}_{t}(D(G)) \leq \text{ch}_{t-1}(G) + 1$. Therefore $\text{ch}_{t}(D(G)) = \text{ch}_{t-1}(G) + 1$. \qed

For complete bipartite graph $K_{n,n}$, $\chi(K_{n,n}) = 2$ and $\text{ch}(K_{n,n})$ approaches infinity as $n$ goes to the infinity. In particular, $\text{ch}(K_{n,n}) \geq k + 1$ for $n = \binom{2k-1}{k}$. One can easily show that $\text{ch}(K_{n+1,n+1}) = \text{ch}(K_{n,n})$ or $\text{ch}(K_{n,n}) + 1$. So for any integer $k$ with $k \geq 2$, there exists a smallest positive integer $n$ such that $\text{ch}(K_{n,n}) = k$. We denote such an integer by $\gamma(k)$.

Now, we introduce the second graph operation. Let $G$ be a connected graph and let $k$ be a positive integer. Let $H$ be a complete bipartite graph with $\gamma(k)$ vertices on each part. For a vertex $v \in V(G)$, an attachment $A(G,v,k)$ is a graph defined by

$$V(A(G,v,k)) = V(G) \cup V(H) \cup \{x\}$$

and

$$E(A(G,v,k)) = E(G) \cup E(H) \cup \{{v,x}\}, \{x,u\},$$

where $u$ is a vertex in $H$. Namely $A(G,v,k)$ is obtained by connecting $G$ and $H$ with a path of length 2 whose ends are $v$ and a vertex $u$ in $H$. For convenience, we use $U$ and $V$, where $u \in U$, to refer to the vertex sets in the bipartition of $V(H)$.

The following lemma gives the chromatic number, the list chromatic number, and the $t$-common list chromatic number of $A(G,v,k)$ for a connected graph $G$ with $\chi(G) \geq 2$.

**Lemma 4.** Let $G$ be a connected graph with $\chi(G) \geq 2$. For every $v \in V(G)$ and for every positive integer $k$, the following properties hold.
(1) $\chi(A(G, v, k)) = \chi(G)$.
(2) $ch(A(G, v, k)) = \max\{ch(G), k\}$.
(3) For every nonnegative integer $t$, $ch_t(A(G, v, k)) = ch_t(G)$.

Proof. (1) By the definition of $A(G, v, k)$, the chromatic number of $A(G, v, k)$ is the maximum of $\chi(G)$ and $\chi(K_{\gamma(k), \gamma(k)})$. Since the chromatic number of a complete bipartite graph is 2, $\chi(A(G, v, k)) = \chi(G)$.

(2) If $k = 2$, then $\gamma(2) = 1$, namely $A(G, v, k)$ is a graph obtained by attaching a path of length 3 to $v$. So $ch(A(G, v, k)) = ch(G)$, which is the maximum of $ch(G)$ and 2. Assume that $k \geq 3$. Since both $G$ and $K_{\gamma(k), \gamma(k)}$ are subgraphs of $A(G, v, k)$, $ch(A(G, v, k)) \geq \max\{ch(G), k\}$. Let $L$ be a list assignment of $A(G, v, k)$ such that $|L(w)| = \max\{ch(G), k\}$ for every $w \in V(A(G, v, k))$. Now both $G$ and $K_{\gamma(k), \gamma(k)}$ as subgraphs of $A(G, v, k)$ have $L$-colorings $\phi_1$ and $\phi_2$, respectively. By assigning a color $c \in L(x) - \{\phi_1(v), \phi_2(u)\}$ to $x$, we have $L$-coloring of $A(G, v, k)$. So $ch(A(G, v, k)) \leq \max\{ch(G), k\}$. Therefore $ch(A(G, v, k)) = \max\{ch(G), k\}$.

(3) Since $G$ is a subgraph of $A(G, v, k)$, $ch_t(A(G, v, k)) \geq ch_t(G)$. Let $L_1$ be a $t$-common list assignment of $A(G, v, k)$ such that $|L_1(w)| = ch_t(G)$ for every $w \in V(A(G, v, k))$. Let $c$ be a color belonging to $L_1(w)$ for every vertex $w \in V(A(G, v, k))$. Now $G$ has an $L_1$-coloring $\phi_2$. If $\phi_2(v) = c$, then for every $u' \in U$, let $\phi_3(u') = c$ and for every $y \in V \cup \{x\}$, choose a color $c'$ in $L_1(y) - \{c\}$ and let $\phi_3(y) = c'$. Now $\phi_2$ and $\phi_3$ give an $L_1$-coloring of $A(G, v, k)$. When $\phi_2(v) \neq c$, assign $c$ to every vertex in $V \cup \{x\}$ and for every $u' \in U$, assign an arbitrary color $c'$ in $L_1(u') - \{c\}$. Now $\phi_2$ and this assignment give an $L_1$-coloring of $A(G, v, k)$. So $ch_t(A(G, v, k)) \leq ch_t(G)$, and hence $ch_t(A(G, v, k)) = ch_t(G)$.

Finally, we have the following theorem.

Theorem 5. For all positive integers $k, \ell$ with $2 \leq k \leq \ell$ and for any positive integers $i_1, i_2, \ldots, i_{k-2}$ with $k \leq i_{k-2} \leq \cdots \leq i_1 \leq \ell$, there exists a graph $G$ such that $\chi(G) = k$, $ch(G) = \ell$ and $ch_t(G) = i_t$ for every $t = 1, \ldots, k - 2$.

Proof. Let $k, \ell$ be positive integers satisfying $2 \leq k \leq \ell$ and let $i_1, i_2, \ldots, i_{k-2}$ be positive integers such that $k \leq i_{k-2} \leq \cdots \leq i_1 \leq \ell$. Let $H_0 = K_{\gamma(i_{k-2} - k + 2), \gamma(i_{k-2} - k + 2)}$ and choose a vertex $v$ in $H_0$. For every $j = 1, \ldots, k - 3$, let $H_j = A(D(H_{j-1}), v, i_{k-j-2} - k + j + 2)$ and let $G = A(D(H_{k-3}), v, \ell)$. The rest is to prove that $\chi(G) = k$, $ch(G) = \ell$ and $ch_t(G) = i_t$ for every $t = 1, \ldots, k - 2$.

By Lemmas 3 and 4,

$\chi(G) = \chi(D(H_{k-3})) = \chi(H_{k-3}) + 1 = \chi(D(H_{k-4})) + 1 = \cdots = \chi(H_0) + k - 2 = k.$

Note that $ch(H_0) = i_{k-2} - k + 2$ and

$ch(H_1) = \max\{ch(D(H_0)), i_{k-3} - k + 3\} = \max\{ch(H_0) + 1, i_{k-3} - k + 3\} = i_{k-3} - k + 3.$
It can also be shown that for every \( j \leq t \) \((1 \leq t \leq k - 4)\), \( ch(H_j) = i_{k-j-2} - k + j + 2 \). It follows that
\[
ch(H_{t+1}) = \max\{ch(D(H_t)), i_{k-t-3} - k + t + 3\} \\
= \max\{ch(H_t) + 1, i_{k-t-3} - k + t + 3\} = i_{k-t-3} - k + t + 3.
\]
Therefore for every \( j = 1, \ldots, k - 3 \), \( ch(H_j) = i_{k-j-2} - k + j + 2 \). Furthermore we have
\[
ch(G) = \max\{ch(D(H_{k-3})), \ell\} = \max\{ch(H_{k-3}) + 1, \ell\} = \max\{i_1, \ell\} = \ell.
\]
Now \( ch_1(G) = ch_1(D(H_{k-3})) = ch(H_{k-3}) + 1 = i_1 \) and for every \( t = 2, \ldots, k - 2 \),
\[
ch_t(G) = ch_{t-1}(H_{k-3}) + 1 = ch_{t-2}(H_{k-4}) + 2 = \cdots = ch(H_{k-t-2}) + t = i_t. \tag*{\blacksquare}
\]

3 On \( t \)-common list colorings of planar graphs

By the famous four color theorem, every planar graph is known to be 4-colorable [1, 2]. Voigt [10] gave an example of a non-4-choosable planar graph and Thomassen [9] showed that every planar graph is 5-choosable. So for every planar graph \( G \), \( ch_2(G) \leq ch_1(G) \leq 5 \). From this inequality, one can ask whether there is a planar graph with \( ch_1(G) = 5 \). We prove that 5 is the sharp upper bound for 1-common list chromatic number of planar graphs. To this end, we first introduce the following lemma.

\textbf{Lemma 6.} Let \( G_1 \) be the graph drawn in Figure 1. Suppose that \( L \) is a list assignment with \( L(x) = L(y) = L(u_1) = L(u_2) = \{1, 2, 3, 4\} \), \( L(v_1) = L(w_1) = \{1, 3, 4, 5\} \), \( L(v_2) = L(w_2) = \{2, 3, 4, 5\} \), and \( L(v_3) = L(w_3) = \{1, 2, 4, 5\} \). Then \( G_1 \) has no \( L \)-coloring \( \phi \) with \( \phi(x) = 1, \phi(y) = 2 \).

\textit{Proof.} Suppose that \( G_1 \) has such an \( L \)-coloring \( \phi \). By simple observation, we can check that \( \{\phi(u_1), \phi(u_2)\} = \{3, 4\} \), and we may assume that \( \phi(u_1) = 4 \) by symmetry. This implies that the cycle with vertices \( v_1, v_2, \) and \( v_3 \) is 2-colorable, which is a contradiction. Therefore, \( G_1 \) is not \( L \)-colorable with \( x, y \) being colored 1, 2, respectively. \tag*{\blacksquare}

Now, the following theorem provides a planar graph \( G \) such that \( ch_1(G) = 5 \) and \( ch_2(G) = 4 \).

\textbf{Theorem 7.} Let \( G \) be the graph drawn in Figure 2, while dashed arrows are copies of \( G_1 \) as mentioned in Figure 1. The graph \( G \) is a planar graph satisfying \( ch_1(G) = 5 \) and \( ch_2(G) = 4 \).

\textit{Proof.} One can easily check that \( G \) is a planar graph. Note that \( ch_1(G) \leq 5 \) by Lemma 1. Let \( L \) be a list assignment of \( G \) with \( L(x_i) = \{1, 2, 3, 4\} \) for all \( i \), and as defined in Lemma 6 for the vertices on the copies of \( G_1 \). Note that the color 4 belongs to all lists.
Figure 1: The graph $G_1$ is on the left. For two fixed vertices $x, y$ of $G_1$, we simply draw a dashed arrow from $x$ to $y$ to represent the graph $G_1$, as on the right.

Suppose that $G$ has an $L$-coloring $\phi$. There exist $x_i$ and $x_j$ such that $\phi(x_i) = 1$ and $\phi(x_j) = 2$. By Lemma 6, the copy of $G_1$ corresponding to the dashed arrow from $x_i$ to $x_j$ has no $L$-coloring, which is a contradiction. Therefore $G$ is not $L$-colorable and hence $\chi_1(G) = 5$.

Now, we prove that for every 2-common list assignment $L$ of $G$ with $L(v) \geq 4$ for every vertex $v$, there exists an $L$-coloring of $G$. To this end, it is enough to find a bipartition of the vertex set of $G$ into two sets $U, V$ such that $U$ induces a bipartite subgraph of $G$ and $V$ induces a 2-choosable subgraph.

For clarity, we use $G_{i,j}$ to refer to a copy of $G_1$ corresponding to the dashed arrow from $x_i$ to $x_j$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Moreover, for each $i, j$, and $k$, we call vertices of $G_{i,j}$ corresponding to $u_k, v_k$, and $w_k$ by $u_{i,j}^k$, $v_{i,j}^k$, and $w_{i,j}^k$, respectively.

Let

\[
S_0 = \{x_1, x_2\}, \quad S_1 = \{v^{1,2}_1, v^{1,2}_2, w^{1,2}_1, w^{1,2}_2\} \cup \{v^{2,1}_1, v^{2,1}_2, w^{2,1}_1, w^{2,1}_2\},
\]

\[
S_2 = \bigcup_{s \in \{1,2\}, t \in \{3,4\}} \left( \{u^{s,t}_1, v^{s,t}_2, v^{s,t}_3, w^{s,t}_1, w^{s,t}_2\} \cup \{u^{t,s}_1, v^{t,s}_2, v^{t,s}_3, w^{t,s}_1, w^{t,s}_2\} \right),
\]

\[
S_3 = \{u^{3,4}_1, u^{3,4}_2, v^{3,4}_1, v^{3,4}_2, v^{3,4}_3, w^{3,4}_1, w^{3,4}_2, w^{3,4}_3\} \cup \{u^{4,3}_1, u^{4,3}_2, v^{4,3}_1, v^{4,3}_2, v^{4,3}_3, w^{4,3}_1, w^{4,3}_2, w^{4,3}_3\}.
\]

Let $S = S_0 \cup S_1 \cup S_2 \cup S_3$. Now the subgraph $G[S]$ induced by $S$ is a bipartite graph, and $G \setminus S$ is a forest, which is always 2-choosable. Therefore $\chi_2(G) = 4$. □

It is unknown whether there exists a planar graph $G$ satisfying $\chi_2(G) = 5$. So we propose the following question.

**Question 1.** Is there a planar graph $G$ such that $\chi_2(G) = 5$ or does it hold that $\chi_2(G) \leq 4$ for every planar graph $G$?

The well-known theorem of Grötzsch [6] states that every planar triangle-free graph is 3-colorable. This theorem was later slightly sharpened by Grünbaum [7] and Aksenov...
Figure 2: The graph $G$. Each dashed arrow represents a copy of $G_1$.

[3], who showed that every planar graph with at most 3 triangles is 3-colorable. The case of list coloring is different. Voigt [11], Gutner [8], Glebova et. al [4] gave examples of triangle-free planar graphs that are not 3-choosable. One can check that for each such example $G$, there exists an independent set $S$ such that $G - S$ is a forest. This implies the 1-common list chromatic number $\chi_1(G)$ is 3. Hence we propose the following question.

**Question 2.** Is there a triangle-free planar graph $G$ such that $\chi_1(G) = 4$ or does it hold that $\chi_1(G) \leq 3$ for every triangle-free planar graph $G$?

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**References**


