On a special class of hyper-permutahedra

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Abstract

Minkowski sums of simplices in $\mathbb{R}^n$ form an interesting class of polytopes that seem to emerge in various situations. In this paper we discuss the Minkowski sum of the simplices Δ_{k-1} in $\mathbb{R}^n$ where k and n are fixed, their flags and some of their face lattice structure. In particular, we derive a closed formula for their *exponential generating flag function*. These polytopes are simple, include both the simplex Δ_{n-1} and the permutahedron Π_{n-1}, and form a Minkowski basis for more general permutahedra.

Keywords: polytope; permutahedron; Minkowski sum; flag polynomial; exponential flag function

1 Introduction and motivation

The Minkowski sum of simplices yields an important class of polytopes that includes and generalizes many known polytopes. For related references and some of the history of the significance of Minkowski sums of simplices we refer to the introduction in [2]. In [1] a closed formula for the ℓ-flag polynomial (See Definition 1.1) for an arbitrary Minkowski sum of k simplices is derived. In particular, this yields a closed formula for the f-vectors of generalized associahedra from [7]. This mentioned formula, however, is in terms of the *master polytope* $P(k)$, a (2^k-2)-dimensional polytope, the structure of which little is known about except when $k \leqslant 2$ [2], [1]. In this paper we focus on the family of Minkowski sum of the simplices of a fixed dimension. These polytopes are interesting for a variety of reasons. We mention a few here without attempting to be exhaustive: (i) The polytopes in this family are all simple, and so they have a nice enumeration of their flags of arbitrary length as we will see shortly (see Lemma 1.3). (ii) This family forms a chain, or an incremental bridge, between the simplex of a given dimension and the standard permutahedron of the same dimension, where each step, or link, is between two such simple polytopes that differ minimally, in the sense that the cardinality of the support of the vertices differs by one, as we will see in Proposition 2.3 in Section 2. (iii) Each polytope in this family is symmetric with respect to permutation of their coordinates, like the simplex and the standard permutahedron. In fact, they make up a subclass of the class of generalized permutohedra studied in [7] and [8], something we will discuss in more detail in Section 3. (iv) By contracting each face formed by vertices of identical positive support of any polytope of this family, one obtains a hypersimplex; a particular matroid base polytope (or matroid basis polytope) of the uniform matroid formed by all subsets of a fixed cardinality (the rank of the matroid) from a given ground set. In fact, each matroid base polytope of a matroid of a given rank is contained in a "mother"-hypersimplex, that is, its vertices are among the vertices of the "mother"-hypersimplex. The flags of matroid base polytopes have been studied in the literature, in particular in [5] and [6], in which a characterization of the faces of the matroid base polytopes is presented. Also, a formula for the **cd**-index of rank-two matroid base polytope is presented, describing the number of their flags in the most compact way possible, from a linear relations perspective. (v) Last but not least, this family forms a Minkowski basis for certain generalized permutahedra of the form $P_{n-1}(\tilde{x})$ as defined and discussed in [7, p. 13] in terms of non-negative integer combination as Minkowski sums. Indeed, matroid base polytopes form a subclass of the family of generalized permutahedra as shown in [3] where some of the work from [7] is generalized, especially the volume of a general matroid base polytope. This will discussed in Section 3.

The motivation for this paper stems from an observation on the enumeration of the flags of the standard permutahedron, presented in Proposition 1.6 here below, which we now will parse through and discuss.

Recall that the *permutahedron* Π_{n-1} is the convex hull of $\{(\pi(1), \pi(2), \ldots, \pi(n)) \in \mathbb{R}^n : \pi \in S_n\}$ where S_n is the symmetric group of degree n. The faces of Π_{n-1} have a nice combinatorial description as presented in Ziegler [10, p. 18]: each i-dimensional face of Π_{n-1} can be presented as an ordered partition of the set $[n] = \{1, \ldots, n\}$ into exactly $n - i$ distinct parts. In particular, Π_{n-1} has $\left\{ {n \atop n-i} \right\}(n-i)!$ faces of dimension i for each $i \in \{0, 1, \ldots, n\}$, where $\left\{ {n \atop k} \right\}$ denotes the Stirling number of the 2nd kind.

CONVENTIONS: (i) For an ℓ-tuple $\tilde{x} = (x_1, \ldots, x_\ell)$ of variables and an ℓ-tuple of numbers $\tilde{a} = (a_1, \ldots, a_\ell)$, let $\tilde{x}^{\tilde{a}} = x_1^{a_1} x_2^{a_2} \cdots x_\ell^{a_\ell}$. (ii) For $\tilde{a} = (a_1, \ldots, a_\ell)$ let $\partial(\tilde{a}) = (a_1, a_2 - a_1, a_3 - a_2, \ldots, a_\ell - a_{\ell-1})$. The following definition is from [1]:

Definition 1.1. *Let P be a polytope with $\dim(P) = d$ and $\ell \in \mathbb{N}$. For an ℓ-tuple of variables $\tilde{x} = (x_1, \ldots, x_\ell)$ the ℓ-*flag polynomial *of P is defined by*

$$\tilde{f}_P^\ell(\tilde{x}) := \sum_{\tilde{s}} f_{\tilde{s}}(P) \tilde{x}^{\partial(\tilde{s})},$$

where the sum is taken over all chains $\tilde{s} = (s_1, \ldots, s_\ell)$ with $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_\ell \leqslant d$ and $f_{\tilde{s}}(P)$ denotes the number of chains of faces $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\ell$ of P with $\dim(A_i) = s_i$ for each $i \in \{1, \ldots, \ell\}$.

Conventions: For a vector $\tilde{c} = (c_1, \ldots, c_n)$ we denote the linear functional $\tilde{x} \mapsto \tilde{c} \cdot \tilde{x}$ by $L_{\tilde{c}}$. For a given vector $\tilde{c}$ and a polytope P, we denote by $F_P(\tilde{c})$ or just $F(\tilde{c})$ the unique face of P determined by $\tilde{c}$ as the points that maximize $L_{\tilde{c}}$ when restricted to P. Further, we denote the set of all the faces of P by $\mathbf{F}(P)$. More specifically we denote the set of the i-dimensional faces of P by $\mathbf{F}_i(P)$, in particular, $\mathbf{F}_0(P)$ denotes the set of vertices of P. Finally, for a vector $\tilde{c}$ the set $\mathrm{supp}(\tilde{c}) = \{c_1, \ldots, c_n\}$ is the *support* of $\tilde{c}$.

Consider now the well-known description of the i-dimensional faces of Π_{n-1} as the ordered partitions of $[n]$ into $n-i$ parts: more explicitly, each functional $L_{\tilde{c}}$ where the support $\mathrm{supp}(\tilde{c})$ has exactly $n-i$ distinct values $c_1 < c_2 < \cdots < c_{n-i}$, when restricted to Π_{n-1}, takes its maximum value at exactly one i-dimensional face A. Here each value c_i of the support corresponds uniquely to one of the ordered parts defining the face A. Also, by "merging" two such consecutive values c_h and c_{h+1} (for example, by replacing both c_h and c_{h+1} by their average), we obtain a new functional $L_{\tilde{c}'}$ which is maximized at a face A' of dimension $i+1$ that contains the face A. So, by merging two consecutive parts into one part, we obtain a coarser ordered partition of $[n]$. This merging process can clearly be repeated. In this case we informally say that the first partition is a *refinement* of the last partition, or equivalently that the last partition is a *coarsening* of the first one. Although classical and trivial on its own, we state this as the following claim for reference purposes.

Claim 1.2. *For faces $A, B \in \mathbf{F}(\Pi_{n-1})$, then $A \subseteq B$ if, and only if, the ordered partition of $[n]$ corresponding to A is a refinement of the ordered partition of $[n]$ corresponding to B.*

Given a chain $\tilde{s} = (s_1, \ldots, s_\ell)$ with $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_\ell \leqslant n-1$, the number $f_{\tilde{s}}(\Pi_{n-1})$ of chains of faces $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\ell$ of Π_{n-1} with $\dim(A_i) = s_i$ for each $i \in \{1, \ldots, \ell\}$ can then by Claim 1.2 be obtained by first considering any of the $\left\{ {n \atop n-s_1} \right\}(n-s_1)!$ faces A_1 of dimension s_1, then merging $s_2 - s_1$ consecutive parts (in the ordered partition defining the face A_1) of the $n - s_1 - 1$ available consecutive pairs, then merging $s_3 - s_2$ consecutive parts of the $n - s_2 - 1$ available consecutive parts, and so on. Therefore the number $f_{\tilde{s}}(\Pi_{n-1})$ of chains of faces $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\ell$ where $\dim(A_i) = s_i$ for each i is given by

$$\begin{aligned} f_{\tilde{s}}(\Pi_{n-1}) &= \left\{ {n \atop n-s_1} \right\}(n-s_1)! \binom{n-s_1-1}{s_2-s_1} \binom{n-s_2-1}{s_3-s_2} \cdots \binom{n-s_{\ell-1}-1}{s_\ell - s_{\ell-1}} \\ &= \left\{ {n \atop n-s_1} \right\}(n-s_1)! \binom{n-s_1-1}{s_2-s_1 \ \cdots \ s_\ell - s_{\ell-1} \ \ n-s_\ell-1}. \end{aligned} \tag{1}$$

Such a simple formula for the number of $\tilde{s}$-chains of faces of Π_{n-1} as in (1) is not a coincidence, as it is solely the consequence of Π_{n-1} being a simple polytope: each vertex of a simple d-polytope has d neighboring vertices and is contained in d facets, and so each k-face containing a given vertex is uniquely determined by $\binom{d}{k}$ of its neighbors. Hence, for each $h \leqslant k$ every h-face is contained in exactly $\binom{d-h}{k-h}$ k-faces, and we obtain in general, as above, the following.

Lemma 1.3. *For any simple d-polytope P and a chain $\tilde{s} = (s_1, \ldots, s_\ell)$ with $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_\ell \leqslant d$, we have*

$$f_{\tilde{s}}(P) = f_{s_1}(P)\binom{d-s_1}{s_2 - s_1 \ \cdots \ s_\ell - s_{\ell-1} \ \ d - s_\ell},$$

where $f_{s_1}(P)$ is the number of s_1-faces of P.

Now, assume for a moment that for a d-polytope P we have a polynomial $\tilde{\phi}^\ell_P$ of the form

$$\tilde{\phi}^\ell_P(\tilde{x}) := \sum_{\tilde{s}} \frac{f_{\tilde{s}}(P)}{D(d, s_1)} \tilde{x}^{\partial(\tilde{s})}, \tag{2}$$

where D is a bivariate function on non-negative integers. If $D(d, s_1) = 1$ for all d, s_1 then $\tilde{\phi}^\ell = \tilde{f}^\ell$, the ℓ-flag polynomial from Definition 1.1. If P is simple, then by the above Lemma 1.3, the multinomial theorem and the definition of the f-polynomial ($f = \tilde{f}^1$ obtained by letting $\ell = 1$ in Definition 1.1), we obtain the following:

$$\begin{aligned}
\tilde{\phi}^\ell_P(\tilde{x}) &= \sum_{\tilde{s}} \frac{f_{\tilde{s}}(P)}{D(d, s_1)} \tilde{x}^{\partial(\tilde{s})} \\
&= \sum_{\tilde{s}} \frac{f_{s_1}(P)\binom{d-s_1}{s_2-s_1 \ \cdots \ s_\ell - s_{\ell-1} \ \ d-s_\ell}}{D(d, s_1)} x_1^{s_1} x_2^{s_2 - s_1} \cdots x_\ell^{s_\ell - s_{\ell-1}} \\
&= \sum_{s_1} \frac{f_{s_1}(P)}{D(d, s_1)} x_1^{s_1} \sum_{s_2, \ldots, s_\ell} \binom{d - s_1}{s_2 - s_1 \ \cdots \ s_\ell - s_{\ell-1} \ \ d - s_\ell} x_2^{s_2 - s_1} \cdots x_\ell^{s_\ell - s_{\ell-1}} \\
&= \sum_{s_1} \frac{f_{s_1}(P)}{D(d, s_1)} x_1^{s_1} (x_2 + \cdots + x_\ell + 1)^{d - s_1} \\
&= (x_2 + \cdots + x_\ell + 1)^d \sum_{s_1} \frac{f_{s_1}(P)}{D(d, s_1)} \left(\frac{x_1}{x_2 + \cdots + x_\ell + 1}\right)^{s_1} \\
&= (x_2 + \cdots + x_\ell + 1)^d \tilde{\phi}^1_P \left(\frac{x_1}{x_2 + \cdots + x_\ell + 1}\right),
\end{aligned}$$

showing that $\tilde{\phi}^\ell_P$ is uniquely determined by $\tilde{\phi}^1_P$ if P is a simple d polytope.

Corollary 1.4. *For a simple d-polytope P and $\tilde{\phi}^\ell_P$ from (2) we have*

$$\tilde{\phi}^\ell_P(\tilde{x}) = (x_2 + \cdots + x_\ell + 1)^d \tilde{\phi}^1_P \left(\frac{x_1}{x_2 + \cdots + x_\ell + 1}\right).$$

In particular, the ℓ-flag polynomial for any simple polytope is uniquely determined by its f-polynomial

$$\tilde{f}^\ell_P(\tilde{x}) = (x_2 + \cdots + x_\ell + 1)^d f_P \left(\frac{x_1}{x_2 + \cdots + x_\ell + 1}\right).$$

Remark: Despite this enumerative bonus for simple (and dually for simplicial) polytopes, the number of the flags do not yield much of the actual face lattice structure of simple or simplicial polytopes.

Going back to our motivating permutahedron Π_{n-1} and its number

$$f_i(\Pi_{n-1}) = \left\{ {n \atop n-i} \right\} (n-i)!$$

of faces, we see that for $n \geqslant 1$

$$\sum_{i=0}^{n-1} \frac{f_i(\Pi_{n-1})}{(n-i)!} x^{n-i} = \sum_{i=0}^{n-1} \left\{ {n \atop n-i} \right\} x^{n-i} = T_n(x),$$

where $T_n(x)$ is the Touchard polynomial of degree n, a.k.a. the Bell polynomial in one variable of degree n, as $T_n(x) = B_n(x, \dots, x)$ where $B_n(x_1, \dots, x_n)$ is the *complete Bell polynomial* of degree n in n variables denoted by $\phi_n(x_1, \dots, x_n)$ in [4, p. 263]), and we have the corresponding bivariate exponential generating function [4, p. 265]

$$T(x,y) = \sum_{n \geqslant 0} T_n(x) \frac{y^n}{n!} = \sum_{n,k \geqslant 0} \left\{ {n \atop k} \right\} x^k \frac{y^n}{n!} = e^{x(e^y - 1)}. \tag{3}$$

This suggests an exponential version of the ℓ-flag polynomial from Definition 1.1.

Definition 1.5. *Let P be a d-polytope and $\ell \in \mathbb{N}$. For an ℓ-tuple of variables $\tilde{x} = (x_1, \dots, x_\ell)$ define the* exponential ℓ-flag polynomial *of P by*

$$\tilde{\xi}_P^\ell(\tilde{x}) := \sum_{\tilde{s}} \frac{f_{\tilde{s}}(P)}{(d - s_1 + 1)!} \tilde{x}^{\partial(\tilde{s})},$$

where the sum is taken over all chains $\tilde{s} = (s_1, \dots, s_\ell)$ with $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_\ell \leqslant d$.

For each $a \geqslant 0$ define the exponential ℓ-generating function *of a given family of polytopes $\mathcal{P} = \{P_d\}_{d \geqslant 0}$, where each P_d is of dimension d, by*

$$\tilde{\xi}_{\mathcal{P};a}^\ell(\tilde{x}, y) := \sum_{d \geqslant a} \tilde{\xi}_{P_d}^\ell(\tilde{x}) \frac{y^{d+1}}{(d+1)!}.$$

In the case of $\ell = 1$ we call $\xi_P(x) := \tilde{\xi}_P^1(x)$ the *exponential face (or f-) polynomial* of P and for a family of polytopes $\mathcal{P} = \{P_d\}_{d \geqslant 0}$, each P_d a d-polytope, we call $\xi_{\mathcal{P};a}(x,y) := \tilde{\xi}_{\mathcal{P};a}^1(x,y)$ the *exponential face function* of $\mathcal{P}$. When there is not ambiguity and both the family $\mathcal{P}$ and the starting point a are clear, we omit the subscript in $\xi_{\mathcal{P};a}$ and simply write ξ.

For $a = 0$ and $\mathcal{P} = \{\Pi_{n-1}\}_{n\geqslant 1}$ we get by (3)

$$\begin{aligned}
\xi(x,y) &= \sum_{n\geqslant 1} \xi_{\Pi_{n-1}}(x)\frac{y^n}{n!} \\
&= \sum_{n\geqslant 1} \left(\sum_{i=0}^{n-1} \frac{f_i(\Pi_{n-1})}{(n-i)!} x^i \right) \frac{y^n}{n!} \\
&= \sum_{n\geqslant 1} \left(\sum_{i=0}^{n-1} \frac{f_i(\Pi_{n-1})}{(n-i)!} x^{-(n-i)} \right) \frac{(xy)^n}{n!} \\
&= \sum_{n\geqslant 1} T_n(x^{-1})\frac{(xy)^n}{n!} \\
&= T(x^{-1}, xy) - 1.
\end{aligned}$$

So by Corollary 1.4 applied to $\tilde{\xi}^{\ell}_{\Pi_{n-1}}$ we then get

$$\begin{aligned}
\tilde{\xi}^{\ell}(\tilde{x}, y) &= \frac{1}{x_2 + \cdots + x_\ell + 1} \xi\left(\frac{x_1}{x_2 + \cdots + x_\ell + 1}, (x_2 + \cdots + x_\ell + 1)y \right) \\
&= \frac{T\left(\frac{x_2 + \cdots + x_\ell + 1}{x_1}, x_1 y \right) - 1}{x_2 + \cdots + x_\ell + 1},
\end{aligned}$$

and again by (3) we get the following proposition.

Proposition 1.6. *The exponential generating function for all the ℓ-flags of all the permutahedra Π_{n-1} for $n \geqslant 1$ from Definition 1.5 is given by*

$$\tilde{\xi}^{\ell}(\tilde{x}, y) = \sum_{n\geqslant 1, \tilde{s}} \tilde{\xi}^{\ell}_{\Pi_{n-1}}(\tilde{x})\frac{y^n}{n!} = \frac{e^{\frac{x_2 + \cdots + x_\ell + 1}{x_1}(e^{x_1 y} - 1)} - 1}{x_2 + \cdots + x_\ell + 1}.$$

In particular, as the coefficient $[\tilde{x}^{\partial(\tilde{s})} y^n]\tilde{\xi}^{\ell}(\tilde{x}, y)$ of $\tilde{x}^{\partial(\tilde{s})} y^n$ in the expansion of $\tilde{\xi}^{\ell}_e(\tilde{x}, y)$ is given by

$$[\tilde{x}^{\partial(\tilde{s})} y^n]\tilde{\xi}^{\ell}_e(\tilde{x}, y) = \frac{f_{\tilde{s}}(\Pi_{n-1})}{(n - s_1)!n!},$$

then by Proposition 1.6 we have

$$f_{\tilde{s}}(\Pi_{n-1}) = (n - s_1)!n![\tilde{x}^{\partial(\tilde{s})} y^n] \left(\frac{e^{\frac{x_2 + \cdots + x_\ell + 1}{x_1}(e^{x_1 y} - 1)} - 1}{x_2 + \cdots + x_\ell + 1} \right).$$

Remarks: (i) Needless to say, there are many ways to define *an* exponential generating function for the ℓ-flags of the permutahedra; we chose one here that would yield nice formulae. (ii) Note that for $\ell = 1$ in Proposition 1.6, the sum $x_2 + \cdots + x_\ell$ is empty which yields the exponential face function

$$\xi(x, y) = e^{\left(\frac{e^{xy} - 1}{x}\right)} - 1.$$

Having presented our motivating example, a natural question arises whether formulae as in Proposition 1.6 can be generalized to a larger family of polytopes that include the permutahedron Π_{n-1}. This will be the subject of the rest of the paper, which is organized as follows. In Section 2 we formally define the polytopes $\Pi_{n-1}(k-1)$ for each $k \geqslant 1$ and $n \geqslant k$ and we present some basic properties. In Section 3 we describe how the *PI-family* $\mathcal{P}_n = \{\Pi_{n-1}(k-1)\}_{k=2,\ldots,n}$ fits in with various other families that generalize the standard permutahedron and we demonstrate how $\mathcal{P}_n$ forms a Minkowski basis for one such family of polytopes. The remaining two sections form the meat of this paper. In Section 4 we derive a formula for the f-polynomial of $\Pi_{n-1}(k-1)$ and describe its flags in terms of ordered pseudo-partitions of $[n] = \{1,\ldots,n\}$, in a similar way as we did in Claim 1.2 for the standard permutahedron Π_{n-1}. Finally, in Section 5 we derive a closed formula for the exponential ℓ-generating function $\tilde{\xi}^{\ell}_{\mathcal{P}_k^{\perp};k-1}(\tilde{x},y)$ for an arbitrary but fixed integer $k \geqslant 1$, where $\mathcal{P}_k^{\perp} = \{\Pi_{n-1}(k-1)\}_{n\geqslant k}$. Note that both families $\mathcal{P}_n$ and $\mathcal{P}_k^{\perp}$ cover all the polytopes $\Pi_{n-1}(k-1)$ when n and k roam respectively; $\bigcup_{n\geqslant 2}\mathcal{P}_n = \bigcup_{k\geqslant 2}\mathcal{P}_k^{\perp}$ are both partitions of the set of all $\Pi_{n-1}(k-1)$.

2 The PI-family of polytopes and basic properties

In this section we define the PI-family of polytopes we investigate and present some basic properties that naturally generalize those of the permutahedron Π_{n-1} and the simplex Δ_{n-1}. First we recall some basic definitions and notations we will be using.

For $n \in \mathbb{N}$ and $[n] = \{1,2,\ldots,n\}$, the *(standard) simplex* $\Delta_{n-1} = \Delta_{[n]}$ of dimension $n-1$ is given by $\Delta_{n-1} = \Delta_{[n]} = \{\tilde{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_i \geqslant 0 \text{ for all } i\ , x_1+\cdots+x_n = 1\}$. Each subset $F \subseteq [n]$ yields a face Δ_F of $\Delta_{[n]}$ given by $\Delta_F = \{\tilde{x} \in \Delta_{[n]} : x_i = 0 \text{ for } i \notin F\}$. Clearly Δ_F is itself a simplex embedded in $\mathbb{R}^n$. If $\mathcal{F}$ is a family of subsets of $[n]$, then we can form the *Minkowski sum* of simplices

$$P_{\mathcal{F}} = \sum_{F\in\mathcal{F}} \Delta_F = \left\{ \sum_{F\in\mathcal{F}} \tilde{x}_F : \tilde{x}_F \in \Delta_F \text{ for each } F \in \mathcal{F} \right\}. \tag{4}$$

In general, every nonempty face of any polytope $P \subseteq \mathbb{R}^n$ (in particular of $\Delta_{[n]}$) is given by the set of points that maximize a linear functional $L_{\tilde{c}} : \tilde{x} \mapsto \tilde{c}\cdot\tilde{x}$ restricted to P. We note that the permutahedron Π_{n-1} can be expressed as a *zonotope*, a Minkowski sum of simplices each of dimension one:

$$\Pi_{n-1} = \sum_{F\subseteq[n],\ |F|=2} \Delta_F.$$

In light of this we obtain a natural generalization

$$\Pi_{n-1}(k-1) := \sum_{F\subseteq[n],\ |F|=k} \Delta_F, \tag{5}$$

for each fixed $k \geqslant 2$, the Minkowski sum of all $(k-1)$-dimensional simplices in $\mathbb{R}^n$. We will refer to $\Pi_{n-1}(k-1)$ from (5) as the *hyper-permutahedron*. Note that $\Pi_{n-1}(1) = \Pi_{n-1}$, the standard permutahedron, and $\Pi_{n-1}(n-1) = \Delta_{n-1}$, the standard $(n-1)$-dimensional simplex.

REMARK: Clearly we could have denoted $\Pi_{n-1}(k-1)$ by the shifted and shorter $\Pi_n(k) \subseteq \mathbb{R}^{n+1}$, but to be consistent with the modern convention of denoting the standard $(n-1)$-dimensional permutahedron in $\mathbb{R}^n$ by Π_{n-1}, for example Ziegler's [10, p. 17], we have opted for $\Pi_{n-1}(k-1)$. Needless to say, this is entirely a matter of taste.

We now present some basic and standard facts about Minkowski sums of polytopes in general that we will be using. It's proof is standard and hence omitted.

Lemma 2.1. *Let $P_1, \ldots, P_k$ be polytopes in $\mathbb{R}^n$. Then $F \in \mathbf{F}\left(\sum_{i=1}^k P_i\right)$ iff (i) $F = \sum_{i=1}^k A_i$ where each $A_i \in \mathbf{F}(P_i)$ and (ii) there is a linear functional L on $\mathbb{R}^n$ such that each $L|_{P_i}$ is maximized at A_i and $L|_{\sum_{i=1}^k P_i}$ is maximized at F.*

Since $\max(L|_{\sum_{i=1}^k P_i}) = \sum_{i=1}^k \max(L|_{P_i})$ we can say a tad more.

Corollary 2.2. *The decomposition of $F \in \mathbf{F}\left(\sum_{i=1}^k P_i\right)$ as $F = \sum_{i=1}^k A_i$ from Lemma 2.1 is unique.*

Consider for a moment a functional $L_{\tilde{c}}$ where $c_1 > \cdots > c_n$. Clearly $L_{\tilde{c}}$ restricted to $\Pi_{n-1}(k-1)$ as defined in (5) will yield the vertex $(1, 0, \ldots, 0)$ of Δ_F from all the $\binom{n-1}{k-1}$ subsets $F \subseteq [n]$ with $1 \in F$. Further, $L_{\tilde{c}}$ will yield the vertex $(0, 1, 0, \ldots, 0)$ of Δ_F from all the $\binom{n-2}{k-1}$ subsets of $F \subseteq [n]$ with $1 \notin F$ and $2 \in F$ and so on. Hence, by (5) and Lemma 2.1 we see that $L_{\tilde{c}}$ will yield the unique vertex $\left(\binom{n-1}{k-1}, \binom{n-2}{k-2}, \ldots, \binom{k-1}{k-1}, 0, \ldots, 0\right)$ of $\Pi_{n-1}(k-1)$. By considering all permutations on n indices, we therefore have the following proposition.

Proposition 2.3. *Every vertex of $\Pi_{n-1}(k-1)$ has form $\tilde{u} = (u_1, \ldots, u_n)$ where the support is given by*

$$\operatorname{supp}(\tilde{u}) = \{u_1, \ldots, u_n\} = \left\{\binom{n-1}{k-1}, \binom{n-2}{k-1}, \ldots, \binom{k-1}{k-1}, 0\right\}.$$

There are exactly $k-1$ copies of 0 among $u_1, \ldots, u_n$ and hence exactly one copy of each nonzero integer from the above set. In particular $\Pi_{n-1}(k-1)$ has exactly $\frac{n!}{(k-1)!}$ vertices.

From Proposition 2.3 here above we see that $\Pi_{n-1}(k-1)$ is a degenerate case of the polytope $P_{n-1}(\tilde{v})$ from Postnikov [7], defined as the convex hull of $\{(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}) : \pi \in S_n\}$ for a fixed vector $\tilde{v} \in \mathbb{R}^n$. Namely, $\Pi_{n-1}(k-1) = P_{n-1}(\tilde{v})$ where $\tilde{v} = (v_1, \ldots, v_n) = (\binom{n-1}{k-1}, \binom{n-2}{k-1}, \ldots, \binom{k-1}{k-1}, 0, \ldots, 0)$. The combinatorial type of $P_n(\tilde{v})$, for any $\tilde{v} \in \mathbb{R}^n$ with $v_1 = v_2 = \cdots = v_{k-1} < v_k < \cdots < v_n$, is the same as that of $\Pi_{n-1}(k-1)$ (i.e. they have isomorphic face lattices) so, in particular, the combinatorial type of $P_n(\tilde{v})$ when all the v_i are distinct, is the same as that of the standard permutahedron Π_{n-1}. In [7] the volume $P_n(\tilde{v})$ is studied extensively, and it is shown to be a polynomial in the variables $v_1, \ldots, v_n$.

Note that every functional $L_{\tilde{c}}$ for which the coordinates $c_1, \ldots, c_n$ of $\tilde{c}$ are distinct will always yield a vertex of $\Pi_{n-1}(k-1)$, but not vice versa when $k \geqslant 3$. For such a

$\tilde{c}$ we can, as right before Claim 1.2, "merge" two consecutive values of the support of $\tilde{c}$ (i.e. replace both values by their average, say) and thereby obtain the unique edge of $\Pi_{n-1}(k-1)$, the endvertices of which form the max-set of this altered $L_{\tilde{c}}$. Note that the edges of $\Pi_{n-1}(k-1)$ are of two types or kinds: 1st kind having $k-1$ zeros among the coordinates of each generic point on the edge (i.e. edge points excluding the endpoints), and the 2nd kind with $k-2$ zeros among the coordinates of each generic point on the edge. The number e_1 of edges of the 1st kind is the same as the number of ordered partition of a chosen $(n-k+1)$-subset of $[n]$ into $n-k$ parts, and hence

$$e_1 = \binom{n}{n-k+1}\left\{ {n-k+1 \atop n-k} \right\}(n-k)! = \binom{n}{n-k+1}\binom{n-k+1}{2}(n-k)! = \frac{(n-k)n!}{2(k-1)!}.$$

The number e_2 of edges of the 2nd kind is the same as the number ways to choose an $(n-k+2)$-subset of $[n]$, partition it into $n-k+1$ parts and order the $n-k$ singletons of those parts, and so

$$e_2 = \binom{n}{n-k+2}\left\{ {n-k+2 \atop n-k+1} \right\}(n-k)! = \binom{n}{n-k+2}\binom{n-k+2}{2}(n-k)! = \frac{n!}{2(k-2)!}.$$

Hence, the total number of edges is given by $e_1 + e_2 = \frac{(n-1)n!}{2(k-1)!}$. We summarize in the following.

Proposition 2.4. *Every edge of $\Pi_{n-1}(k-1)$ is between a pair of vertices as given in Proposition 2.3 that differ in exactly two coordinates whose values are consecutive in the support of the vertices. Consequently the edges are of two types: (i) edges between two vertices, both with the same $k-1$ zero coordinates, and (ii) edges between two vertices, both with the same $k-2$ zero coordinates and one with its least nonzero entry where the other vertex has a zero. In particular, the number of edges of $\Pi_{n-1}(k-1)$ is $\frac{(n-1)n!}{2(k-1)!}$ and so $\Pi_{n-1}(k-1)$ is a simple polytope for all k and n.*

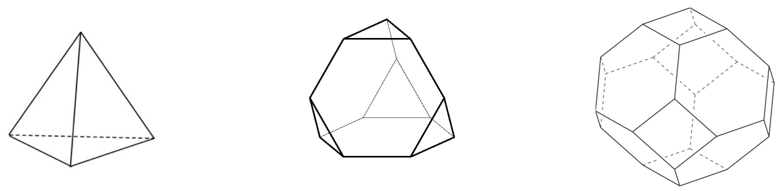
By the above Proposition 2.4 every $\Pi_{n-1}(k-1)$ is simple, so the sequence

$$\Delta_{n-1} = \Pi_{n-1}(n-1), \Pi_{n-1}(n-2), \ldots, \Pi_{n-1}(2), \Pi_{n-1}(1) = \Pi_{n-1},$$

can be viewed as discrete transition of simple polytopes from the simplex Δ_{n-1} to the standard permutahedron Π_{n-1}, see Figure 1. This is our first main reason to focus our study on the PI-family consisting of $\Pi_{n-1}(k-1)$ where $k = 2, \ldots, n-1$.

3 Comparing various types of generalizations of permutahedra

In this section we further promote the importance of the PI-family $\mathcal{P}_n = \{\Pi_{n-1}(k-1)\}_{k=2,\ldots,n}$ and briefly compare various families of polytopes from the literature, all generalizing the standard permutahedron in one form or another. We present some explicit characterizations of them and show that the PI-family $\mathcal{P}_n$ forms a Minkowski $\mathbb{Z}^+$-basis for a large family of polytopes that generalizes the standard permutahedron.



$\Pi_3(3)$ $\Pi_3(2)$ $\Pi_3(1)$

Figure 1: A simple transition between Δ_3 and Π_3 in the case $n = 4$.

There are, needless to say, many ways to generalize the standard permutahedron Π_{n-1}, and we have briefly mentioned two of them (namely, $\Pi_{n-1}(k-1)$ and $P_{n-1}(\tilde{v})$ from above). There are two other classes of important families of polytopes from [7] and from [8] we want to relate $\Pi_{n-1}(k-1)$ to. A good portion of the discussion immediately here below in this section, is, in one form or another, contained in [7] and [9], except for some minor observations, propositions, and examples toward the end of this section. We include it all in this short section though as it serves as a second main reason for our investigation, as well as for self-containment of the article.

First Class: For a collection $\tilde{Y} = \{y_I\}_{I\subseteq[n]}$ of non-negative real numbers $y_I \geqslant 0$ for each $I \subseteq [n]$, one can define a $P_{n-1}(\tilde{Y})$ as the Minkowski sum of simplices Δ_I scaled by y_I

$$P_{n-1}(\tilde{Y}) := \sum_{I\subseteq[n]} y_I \Delta_I,$$

and is referred to as a *generalized permutohedron* in [7]. Apriori this seems to be more general than the Minkowski sum in (4). However, if we consider a family $\mathcal{F}$ of subsets of $[n]$ containing (possibly) multiple copies of subsets of $[n]$, then $P_{\mathcal{F}}$ from (4) can be written as

$$P_{\mathcal{F}} = \sum_{I\subseteq[n]} n_I \Delta_I,$$

which can have the same combinatorial type as any $P_{n-1}(\tilde{Y})$. The class $P_{n-1}(\tilde{Y})$ (and hence also the class $P_{\mathcal{F}}$,) includes numerous classes of polytopes with highly interesting combinatorial structures, like the *associahedron*, the *cyclohedron*, etc. (see [7] for many more examples.)

Second Class: For a collection $\tilde{Z} = \{z_I\}_{I\subseteq[n]}$ of non-negative real numbers $z_i \geqslant 0$ for each $I \subseteq [n]$, one can define $P_{n-1}(\tilde{Z})$ by its bounding hyperplanes

$$P_{n-1}(\tilde{Z}) := \left\{ \tilde{x} \in \mathbb{R}^n : \sum_{i\in[n]} x_i = z_{[n]},\ \sum_{i\in I} x_i \leqslant z_I \text{ for } I \subset [n] \right\},$$

which is also refereed to as the generalized permutohedron in [7]. The following is a theorem of Rado [9].

Theorem 3.1. *The polytope $P_{n-1}(\tilde{v})$ where the coordinates are ordered $v_1 \geqslant \cdots \geqslant v_n$, can be presented as those $\tilde{t} \in \mathbb{R}^n$ satisfying $\sum_{i\in[n]} t_i = \sum_{i\in[n]} v_i$ and $\sum_{i\in I} t_i \leqslant \sum_{i\in[|I|]} v_i$ for each $I \subseteq [n]$.*

From Theorem 3.1 we see that if $z_I = z_J$ whenever $|I| = |J|$, then there are uniquely determined $v_1, \ldots, v_n$ such that $P_{n-1}(\tilde{Z}) = P_{n-1}(\tilde{v})$. Therefore, the class of $P_{n-1}(\tilde{Z})$ polytopes strictly includes all polytopes $P_{n-1}(\tilde{v})$.

Further, the following is from [7].

Proposition 3.2. *For a given collection $\tilde{Y} = \{y_I\}_{I\subseteq[n]}$ of non-negative real numbers $y_I \geqslant 0$, then $P_{n-1}(\tilde{Y}) = P_{n-1}(\tilde{Z})$ where $z_I = \sum_{J\subseteq I} y_J$ for each $I \subseteq [n]$.*

Hence, the class of $P_{n-1}(\tilde{Z})$ polytopes also includes the class of all $P_{n-1}(\tilde{Y})$ polytopes. By Observation 3.3 here below, this mentioned inclusion is also strict.

Suppose it is known that $P_{n-1}(\tilde{Z}) = P_{n-1}(\tilde{Y})$ for some $\tilde{Y} = \{y_I\}_{I\subseteq[n]}$. Then we may assume that $z_I = \sum_{J\subseteq I} y_J$ for each $I \subseteq [n]$. By a Möbius inversion we then get $y_I = \sum_{J\subseteq I}(-1)^{|I|-|J|} z_J$ for each $I \subseteq [n]$, so the y_I are uniquely determined in terms of the z_I. Hence we have the following.

Observation 3.3. *For $n \in \mathbb{N}$ and a collection $\tilde{Z} = \{z_I\}_{I\subseteq[n]}$ of non-negative real numbers, then $P_{n-1}(\tilde{Z}) = P_{n-1}(\tilde{Y})$ if and only if $y_I = \sum_{J\subseteq I}(-1)^{|I|-|J|} z_J \geqslant 0$ for each $I \subseteq [n]$.*

For $n \in \mathbb{N}$ we say that a collection $\tilde{T} = \{t_I\}_{I\subseteq[n]}$ is *symmetric* if $t_I = t_J$ whenever $|I| = |J|$. Hence, from Proposition 3.2 and the above Observation 3.3 we have the following.

Observation 3.4. *For any $n \in \mathbb{N}$ we have that $\tilde{Y} = \{y_I\}_{I\subseteq[n]}$ is symmetric if and only if $\tilde{Z} = \{z_I\}_{I\subseteq[n]}$ where $z_I = \sum_{J\subseteq I} y_J$ is symmetric. Further, if $\tilde{Y}$ is symmetric then*

$$P_{n-1}(\tilde{Y}) = \sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$$

for non-negative real numbers $y_1, \ldots, y_k$, where we interpret $\Pi_{n-1}(0)$ as a singleton point.

Remark: It is interesting to note that Proposition 3.2 from [7] has been generalized to include all real numbers $y_I \in \mathbb{R}$ for $I \subseteq [n]$, and not merely the non-negative ones, as stated in Proposition 2.4 in [3]. This implies, in particular, that with the right interpretation of a *Minkowski difference* of polytopes, and hence also a *signed Minkowski sum* of simplices, as defined and discussed in [7] and [3] (something we will not discuss further in this article), then both Observations 3.3 and 3.4 do hold for arbitrary real numbers $y_I = \sum_{J\subseteq I}(-1)^{|I|-|J|} z_J \in \mathbb{R}$ for each $I \subseteq [n]$ on one hand and for $y_1, \ldots, y_k \in \mathbb{R}$ on the other.

We now describe those permutahedra $P_{n-1}(\tilde{v})$ that can be written as $P_{n-1}(\tilde{Y})$ for some $\tilde{y} = \{y_I\}_{I\subseteq[n]}$.

For a sequence $(a_n)_{n\geqslant 0}$ of real numbers, recall the *(backward) difference* given by $\Delta(a_n) = a_n - a_{n-1}$ for each $n \geqslant 1$[1]. Iteratively we also have the *i-th order difference* by

[1]The *forward difference* is defined as $\Delta(a_n) = a_{n+1} - a_n$

$\Delta^i(a_n) = \Delta(\Delta^{i-1}(a_n))$ for each $i \geqslant 0$ and where $\Delta^0(a_n) = a_n$ for each n. Likewise, for an n-tuple $\tilde{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ we let $\Delta(\tilde{a}) = (\Delta(a_2), \ldots, \Delta(a_n)) = (a_2 - a_1, \ldots, a_n - a_{n-1}) \in \mathbb{R}^{n-1}$. Clearly, if $P_{n-1}(\tilde{v})$ can be written of the form $P_{n-1}(\tilde{Y})$ for some $\tilde{Y}$, then by Theorem 3.1 and Observation 3.3 we can assume $\tilde{Y}$ to be symmetric and so

$$P_{n-1}(\tilde{v}) = \sum_{I \subseteq [n]} y_{|I|} \Delta_I = y_1 \tilde{1} + \sum_{k=2}^{n} y_k \Pi_{n-1}(k-1),$$

where $\tilde{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. The following proposition is our main conclusion of this section, the proof of which follows thereafter.

Proposition 3.5. *For $n \in \mathbb{N}$ and $\tilde{v} \in \mathbb{R}^n$, $v_1 \leqslant \cdots \leqslant v_n$, then $P_{n-1}(\tilde{v}) = \sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ for non-negative $y_k \geqslant 0$ for each k, if and only if all the differences of $\tilde{v}$ are non-negative, that is, $\Delta^i(v_k) \geqslant 0$ for each $i \in \{0, 1 \ldots, n-1\}$ and $k \in \{i+1, \ldots, n\}$.*

Remark: If $\Delta^i(v_k) \geqslant 0$ for each i and k, then the y_k with $P_{n-1}(\tilde{v}) = \sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ are uniquely determined by $\tilde{v}$. Hence, $\{\Pi_{n-1}(k-1) : k \in \{1, \ldots, n\}\} = \mathcal{P}_n \cup \{\tilde{1}\}$ forms a *Minkowski basis* for such permutahedra $P_{n-1}(\tilde{v})$.

We will prove Proposition 3.5 in a few small steps. First a lemma in linear algebra.

Lemma 3.6. *For $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$ let*

$$\tilde{v}_{n-1}(k-1) = \left(0, \ldots, 0, \binom{k-1}{k-1}, \ldots, \binom{n-1}{k-1}\right).$$

Then $\tilde{u} \in \mathrm{span}_{\mathbb{R}^+}(\{\tilde{v}_{n-1}(k-1) : k \in \{1, \ldots, n\}\})$ iff all the differences of $\tilde{u}$ are non-negative.

The proof of the above Lemma 3.6 will use the following trivial fact.

Claim 3.7. *For any $n \in \mathbb{N}$ and $\tilde{v} \in \mathbb{R}^n$ then $\Delta(\tilde{v}) = \tilde{0} \Leftrightarrow \tilde{v} = v\tilde{1} = (v, v, \ldots, v) \in \mathbb{R}^{n-1}$.*

Proof. (Lemma 3.6) Note that $\Delta(\tilde{v}_{n-1}(k-1)) = \tilde{v}_{n-2}(k-2)$, so by induction on n, all the differences of $\tilde{v}_{n-1}(k-1)$ are non-negative. Since the difference operator Δ is linear, then all the differences of $\sum_{k=1}^{n} y_k \tilde{v}_{n-1}(k-1)$ are non-negative if each $y_k \geqslant 0$. Therefore if $\tilde{u} \in \mathrm{span}_{\mathbb{R}^+}(\{\tilde{v}_{n-1}(k-1) : k \in \{1, \ldots, n\}\})$, then it is necessary for all the differences of $\tilde{u}$ to be non-negative.

Conversely, let $\tilde{u} \in \mathbb{R}^n$ have all its differences non-negative. If $n = 1$ then clearly $\tilde{u} = u_1 \in \mathrm{span}_{\mathbb{R}^+}(\{\tilde{v}_0(0)\})$. Otherwise all the differences of $\Delta(\tilde{u})$ are non-negative, and hence by induction on n we can assume that $\Delta(\tilde{u}) = \sum_{k=2}^{n} y_k \tilde{v}_{n-2}(k-1)$ for some non-negative $y_2, \ldots, y_n$, and so

$$\Delta(\tilde{u}) = \sum_{k=2}^{n} y_k \Delta(\tilde{v}_{n-1}(k)) = \Delta\left(\sum_{k=2}^{n} y_k \tilde{v}_{n-1}(k-1)\right).$$

By Claim 3.7 we have $\tilde{u} - \sum_{k=2}^{n} y_k \tilde{v}_{n-1}(k) = y_1 \tilde{1}$ for some real y_1. Since all differences of $\tilde{u}$ are non-negative, in particular $\Delta^0(\tilde{u}) = \tilde{u}$, we have $y_1 = u_1 \geqslant 0$ and hence $\tilde{u} \in \mathrm{span}_{\mathbb{R}^+}(\{\tilde{v}_{n-1}(k-1) : k \in \{1, \ldots, n\}\})$. ☐

We now have what we need to prove Proposition 3.5.

Proof. (Proposition 3.5) We first note that if $\tilde{v}$ is as in Proposition 3.5, that is $P_{n-1}(\tilde{v}) = \sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ where $y_k \geqslant 0$ for each k, then $L_{\tilde{c}}$, where $\tilde{c} = (1, 2, \ldots, n) \in \mathbb{R}^n$, is maximized at $\tilde{v}$, when restricted to $P_{n-1}(\tilde{v})$, and is maximized at $\tilde{v}_{n-1}(k-1)$ when restricted to $\Pi_{n-1}(k-1)$ for each k. Hence, when restricted to $\sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ then $L_{\tilde{c}}$ is maximized at $\sum_{k=1}^{n} y_k \tilde{v}_{n-1}(k-1)$, and so $\tilde{v} = \sum_{k=1}^{n} y_k \tilde{v}_{n-1}(k-1)$. By Lemma 3.6, all the differences of $\tilde{v}$ must then be non-negative.

For the converse, if all the difference of $\tilde{v}$ are non-negative, then by Lemma 3.6 there are non-negative real coefficients $y_k \geqslant 0$ such that $\tilde{v} = \sum_{k=1}^{n} y_k \tilde{v}_{n-1}(k-1)$, at which $L_{\tilde{c}}$ where $\tilde{c} = (1, 2, \ldots, n) \in \mathbb{R}^n$ when restricted to both $P_{n-1}(\tilde{v})$ and $\sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ is maximized at. Similarly, for any permutation π of $\{1, \ldots, n\}$, the linear functional $L_{\pi(\tilde{c})}$ where $\pi(\tilde{c}) = (\pi(1), \ldots, \pi(n))$ when restricted to both $P_{n-1}(\tilde{v})$ and $\sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ is maximized at

$$\pi(\tilde{v}) = (v_{\pi(1)}, \ldots, v_{\pi(n)}) = \sum_{k=1}^{n} y_k \pi(\tilde{v}_{n-1}(k-1)).$$

By definition of $P_{n-1}(\tilde{v})$ we therefore see that every vertex of $P_{n-1}(\tilde{v})$ is also a vertex of $\sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$. But since every vertex of $\sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ has by Corollary 2.2 the unique form $\sum_{k=1}^{n} y_k \tilde{w}_k$ where each $\tilde{w}_k$ is a vertex of $\Pi_{n-1}(k-1)$, and each $\tilde{w}_k$ is the maximum set of the functional $L_{\pi(\tilde{c})}$ when restricted to $\Pi_{n-1}(k-1)$, then *every* vertex of $\sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ is indeed the maximum set of some $L_{\pi(\tilde{c})}$. Hence, the polytopes $P_{n-1}(\tilde{v})$ and $\sum_{k=1}^{n} y_k \Pi_{n-1}(k-1)$ have the same set of vertices, and so must be the same polytope. This completes the proof. ◻

EXAMPLE: Consider the polytope $P_3(0, 1, 2, 2)$, and assume it can be written as $P_3(\tilde{Y})$ for some $\tilde{Y} = \{y_I\}_{I \subseteq [4]}$. By Theorem 3.1 and Observation 3.3 we can assume $\tilde{Y}$ to be symmetric and so $P_3(0, 1, 2, 2) = \sum_{k=1}^{4} y_k \Pi_3(k-1)$. Looking at the differences of $(0, 1, 2, 2)$ we get

$$\begin{bmatrix} \Delta^0 \\ \Delta^1 \\ \Delta^2 \\ \Delta^3 \end{bmatrix} = \begin{bmatrix} 0 & & 1 & & 2 & & 2 \\ & 1 & & 1 & & 0 & \\ & & 0 & & -1 & & \\ & & & -1 & & & \end{bmatrix}$$

containing two negative entries in the differences of $\tilde{v} = (0, 1, 2, 2)$. By Proposition 3.5 $P_3(0, 1, 2, 2)$ cannot be written in the form of $P_3(0, 1, 2, 2) = \sum_{k=1}^{4} y_k \Pi_3(k-1)$. However, $P_3(0, 1, 2, 2)$ is still a symmetric polytope and has dimension 3 by Lemma 4.1 here below.

By Proposition 3.5 we have the following.

Corollary 3.8. *The PI-family $\mathcal{P}_n = \{\Pi_{n-1}(k-1)\}_{k=2,\ldots,n}$ forms a Minkowski $\mathbb{Z}^+$-basis for those polytopes $P_{n-1}(\tilde{v})$ that are of the form $P_{n-1}(\tilde{Y})$ for some family $\tilde{Y} = \{y_I\}_{I \subseteq [n]}$ of non-negative real numbers.*

4 The flag polynomial of the hyper-permutahedron

Having briefly compared three types of polytopes, $P_{n-1}(\tilde{v})$, $P_{n-1}(\tilde{Y})$, and $P_{n-1}(\tilde{Z})$, each of which can be viewed as generalizations of the standard permutahedron, we see that the polytopes $P_{n-1}(k-1)$ for $k \in \{1,\ldots,n\}$ form a Minkowski basis for those polytopes $P_{n-1}(\tilde{v})$ that can be expressed as $P_{n-1}(\tilde{Y})$. Hence, this can be viewed as a further justification for studying them, and so we will in this section focus on the PI-family $\mathcal{P}_n = \{\Pi_{n-1}(k-1)\}_{k=2,\ldots,n}$ for a given $n \in \mathbb{N}$. We will discuss the face lattice and its flag polynomial. Since many formal statements are the same for $\Pi_{n-1}(k-1)$ as with the more general $P_{n-1}(\tilde{v})$ and are, in fact, more transparent, we will consider the polytope $P_{n-1}(\tilde{v})$ in many cases, and then derive corollaries about $\Pi_{n-1}(k-1)$.

First, we will derive some facts from linear algebra that will come in handy in this section.

Consider two points $\tilde{a}, \tilde{c} \in \mathbb{R}^n$ where neither of them has all its coordinates the same. In this case there is a proper partition $A \cup B = [n]$ such that $c_i > c_j$ for all $(i,j) \in A \times B$. As neither A nor B is empty, we cannot have $a_i = a_j$ for all $(i,j) \in A \times B$, since that would imply $a_i = a_j$ for all $i,j \in [n]$. Hence, there is an $(i,j) \in A \times B$ with $a_i \neq a_j$. If $\tau = (i,j) \in S_n$ then

$$L_{\tilde{c}}(\tilde{a}) - L_{\tilde{c}}(\tau(\tilde{a})) = c_i a_i + c_j a_j - (c_i a_j + c_j a_i) = (c_i - c_j)(a_i - a_j) \neq 0.$$

Hence, we have the following.

Lemma 4.1. *Let $\tilde{a}, \tilde{c} \in \mathbb{R}^n$, neither of which have all its coordinates the same. Then there is a transposition $\tau \in S_n$ such that $L_{\tilde{c}}(\tau(\tilde{a})) \neq L_{\tilde{c}}(\tilde{a})$.*

In particular, for $\tilde{a}$ and $\tilde{c}$ as in Lemma 4.1, $P_{n-1}(\tilde{a}) \not\subseteq \ker(L_{\tilde{c}})$ and so $\dim(P_{n-1}(\tilde{a})) = n-1$. Now, since $\Pi_{n-1}(k-1) = P_{n-1}(\tilde{v}_{n-1}(k-1))$, where

$$\tilde{v}_{n-1}(k-1)) = \left(0,\ldots,0,\binom{k-1}{k-1},\ldots,\binom{n-1}{k-1}\right)$$

is as in Lemma 3.6, we then have the following.

Corollary 4.2. *Let $\tilde{v} \in \mathbb{R}^n$. Then*

$$\dim(P_{n-1}(\tilde{v})) = \begin{cases} 0 & \text{if } v_1 = \cdots = v_n, \\ n-1 & \text{otherwise.} \end{cases}$$

In particular $\dim(\Pi_{n-1}(k-1)) = n-1$ *for every* $k \in \{2,\ldots,n\}$.

We now generalize Corollary 4.2 slightly. As the symmetric group S_n denotes the group of bijections $[n] \to [n]$, we can adopt the notation $S(X)$ for the group of bijections $X \to X$, where X is a given set. With this convention $S_n = S([n])$ and clearly $S(X) \cong S_{|X|}$ for any finite set X. For any collection $X_1,\ldots,X_k$ of disjoint subsets of $[n]$ we then have the internal product $S(X_1)S(X_2)\cdots S(X_k)$, a subgroup of $S([n])$ which is isomorphic to the direct product $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ where $|X_i| = n_i$. For a vector $\tilde{v} \in \mathbb{R}^n$ and a subset X of $[n]$ we let $\mathrm{Proj}_X : \mathbb{R}^n \to \mathbb{R}^{|X|}$ denote the projection onto all the coordinate in X. If $X = \{i\}$ is a singleton set, we let $\mathrm{Proj}_i = \mathrm{Proj}_{\{i\}}$ be the projection onto the i-th coordinate. Further we let $\delta_X(\tilde{v})$ denote the indicator function

$$\delta_X(\tilde{v}) = \begin{cases} 0 & \text{if } |\mathrm{supp}(\mathrm{Proj}_X(\tilde{v}))| = 1, \\ 1 & \text{otherwise .} \end{cases} \tag{6}$$

We now have by Corollary 4.2 the following more general statement.

Proposition 4.3. *For disjoint subsets $X_1, \ldots, X_h$ of $[n]$ and $\tilde{v} \in \mathbb{R}^n$ we have*

$$\dim(\mathrm{conv}(\{\pi(\tilde{v}) : \pi \in S(X_1) \cdots S(X_h)\})) = \sum_{i=1}^{h} \delta_{X_i}(\tilde{v})(|X_i| - 1).$$

Note that the above Proposition 4.3 holds in particular for every partition $X_1 \cup \cdots \cup X_h = [n]$ of $[n]$.

We seek to describe the face lattice of the polytope $P_{n-1}(\tilde{v})$ where $\tilde{v}$ has non-negative real entries, in a similar fashion as was done when describing the faces of the standard permutahedron Π_{n-1} earlier, namely by considering the max set of a linear functional restricted to the polytope. In [8] the combinatorial structure of classes of polytopes that include those of $P_{n-1}(\tilde{v})$ is studied in great depth. In particular, the f-, h- and γ-vectors of these classes of polytopes are studied. Many explicit formulae for the h- and γ-vectors involving descent statistics of permutations are given. Here we take a different OR-like (operations research) approach, involving linear functionals, that more directly relates to the characterization of the faces as presented in Claim 1.2.

We say that two vectors $\tilde{a}$ and $\tilde{c}$ have the same *order type* if $a_i \leqslant a_j \Leftrightarrow c_i \leqslant c_j$ for all $i, j \in \{1, \ldots, n\}$. The order type defines an equivalence relation among vectors $\tilde{c} \in \mathbb{R}^n$, and clearly all vectors of the same type yield the same face of $P_{n-1}(\tilde{v})$, as the set of maximum points of $L_{\tilde{c}}$ when restricted to $P_{n-1}(\tilde{v})$. Denote by $[\tilde{c}]$ the order type equivalence class of the vector $\tilde{c} \in \mathbb{R}^n$. So, if $F(\tilde{c})$ denotes the unique face as the set of maximum points of $L_{\tilde{c}}$ restricted to $P_{n-1}(\tilde{v})$, then, by the above, $F(\tilde{c}) = F(\tilde{c}')$ whenever $[\tilde{c}] = [\tilde{c}']$, and hence the face $F([\tilde{c}])$ is well defined. Also note that $P_{n-1}(\tilde{v})$ and $P_{n-1}(\tilde{v}')$ have the same combinatorial type iff $[\tilde{v}] = [\tilde{v}']$. Finally, if $\delta_X(\tilde{c})$ denote the indicator function from (6), then clearly $\delta_X([\tilde{c}])$ is well defined.

As real addition is commutative, then for any permutation $\pi \in S_n$ we have

$$L_{\pi(\tilde{c})}(\pi(\tilde{x})) = \pi(\tilde{c}) \cdot \pi(\tilde{x}) = \tilde{c} \cdot \tilde{x} = L_{\tilde{c}}(\tilde{x}).$$

Hence, if let $\pi(F) = \{\pi(\tilde{x}) : \tilde{x} \in F\}$, then clearly we have the following.

Observation 4.4. *For any permutation $\pi \in S_n$ we have $\pi(F([\tilde{a}])) = F([\pi(\tilde{a})])$, and, in particular, $P_{n-1}(\tilde{v}) = P_{n-1}(\pi(\tilde{v}))$.*

Consider the polytope $P_{n-1}(\tilde{v})$ for a given vector $\tilde{v}$ with non-negative real entries. To describe the face $F(\tilde{c})$ of $P_{n-1}(\tilde{v})$, we first note that $\tilde{c}$ yields a unique ordered partition of $[n]$

$$[n] = X_1(\tilde{c}) \cup \cdots \cup X_h(\tilde{c}), \tag{7}$$

where $c_i = c_j$ for all $i, j \in X_\ell(\tilde{c})$ and $c_i < c_j$ if $i \in X_\ell(\tilde{c})$ and $j \in X_{\ell'}(\tilde{c})$ where $\ell < \ell'$. Note that $L_{\tilde{c}}$ restricted to the set of vertices $\mathbf{F}_0(P_{n-1}(\tilde{v}))$ takes its maximum value on those vertices $\tilde{u}$, the order of whose entries are in agreement with the order of the entries of $\tilde{c}$, that is $c_i < c_j \Rightarrow u_i \leqslant u_j$. This is clearly a necessary and sufficient condition $\tilde{u} \in \mathbf{F}_0(P_{n-1}(\tilde{v}))$ must satisfy in order for $L_{\tilde{c}}(\tilde{u})$ to be a maximum value of $L_{\tilde{c}}$ when restricted to $P_{n-1}(\tilde{v})$. Formally we have a following description.

Observation 4.5. *For a given $\tilde{c} \in \mathbb{R}^n$ the face of $P_{n-1}(\tilde{v})$ determined by $[\tilde{c}]$ is given by*

$$\begin{aligned} F([\tilde{c}]) &= \operatorname{conv}(\{\tilde{u} \in \mathbf{F}_0(P_{n-1}(\tilde{v})) : c_i < c_j \Rightarrow u_i \leqslant u_j\}) \\ &= \operatorname{conv}(\{\tilde{u} \in \mathbf{F}_0(P_{n-1}(\tilde{v})) : u_i < u_j \Rightarrow c_i \leqslant c_j\}). \end{aligned}$$

Clearly by Observation 4.4, we can assume $\tilde{v}$ to be ordered in any way convenient for our purposes. In particular, when describing the face $F([\tilde{c}])$ of $P_{n-1}(\tilde{v})$, we can for simplicity assume that the order of $\tilde{v}$ agrees with that of $\tilde{c}$, so $v_i \leqslant v_j$ whenever $c_i < c_j$, that is we can assume $\tilde{v} \in F(\tilde{c})$ by Observation 4.5. In terms of the partition from (7), we then obtain another equivalent form by Proposition 4.3.

Proposition 4.6. *For a given $\tilde{c} \in \mathbb{R}^n$ the face of $P_{n-1}(\tilde{v})$ determined by $[\tilde{c}]$ that contains $\tilde{v}$ is given by*

$$F([\tilde{c}]) = \operatorname{conv}(\{\mu_1 \cdots \mu_h(\tilde{v}) : \mu_i \in S(X_i(\tilde{c})) \subseteq S([n]), \quad i \in \{1, \ldots, h\}\}).$$

In particular we have

$$\dim(F([\tilde{c}])) = \sum_{i=1}^{h} \delta_{X_i(\tilde{c})}(\tilde{v})(|X_i(\tilde{c})| - 1).$$

We note that if $\tilde{c}$ and $\tilde{v}$ are both ordered, $c_1 \leqslant \cdots \leqslant c_n$ and $v_1 \leqslant \cdots \leqslant v_n$, and $\delta_{X_\ell(\tilde{c}) \cup X_{\ell+1}(\tilde{c})}(\tilde{v}) = 0$ for some ℓ, then we can replace each c_i where $i \in X_\ell(\tilde{c})$ and c_j where $j \in X_{\ell+1}(\tilde{c})$ with a single value between c_i and c_j, say $(c_i + c_j)/2$, and thereby obtaining a vector $\tilde{c}'$ with a strictly smaller support than $\tilde{c}$ such that $F([\tilde{c}]) = F([\tilde{c}'])$. In this case we have merged the two consecutive intervals $X_\ell(\tilde{c})$ and $X_{\ell+1}(\tilde{c})$ into one interval without altering the corresponding face of $P_{n-1}(\tilde{v})$ that these vectors determine.

Definition 4.7. *For ordered vectors $\tilde{c}, \tilde{v} \in \mathbb{R}^n$ we say that $\tilde{c}$ is $\tilde{v}$-*reduced *if for every $\ell \in \{1, \ldots, h\}$ from (7) we have $\delta_{X_\ell(\tilde{c}) \cup X_{\ell+1}(\tilde{c})}(\tilde{v}) = 1$.*

Turning our attention now back to the more specific PI-family $\mathcal{P}_n = \{\Pi_{n-1}(k-1)\}_{k=2,\ldots,n}$ we note that vectors of distinct order type can yield the same face of $\Pi_{n-1}(k-1)$ when $k \geqslant 3$, but for $k = 2$ (when $\Pi_{n-1}(k-1) = \Pi_{n-1}$, the standard permutahedron) then each face corresponds uniquely to the order type of the vector yielding it.

Observation 4.8. *For every $k \geqslant 2$ the map $[\tilde{c}] \mapsto F([\tilde{c}]) \in \mathbf{F}(\Pi_{n-1}(k-1))$ is always surjective, and it is injective (and hence bijective) iff $k = 2$. In particular, the total number of order types $[\tilde{c}]$ where $\tilde{c} \in \mathbb{R}^n$ is the same as $|\mathbf{F}(\Pi_{n-1})|$, the total number of faces (including the polytope itself) of Π_{n-1}.*

Since $\Pi_{n-1}(k-1) = P_{n-1}(\tilde{v}_{n-1}(k-1))$ where $\tilde{v}_{n-1}(k-1) = \left(0, \ldots, 0, \binom{k-1}{k-1}, \ldots, \binom{n-1}{k-1}\right)$ from Lemma 3.6, then when considering a face $F([\tilde{c}])$ of $\Pi_{n-1}(k-1)$ we can assume $\tilde{c}$ to be $\tilde{v}_{n-1}(k-1)$-reduced. Therefore we can assume the partition (or rather the disjoint union) of $[n]$ induced by $\tilde{c}$ from (7) to have the form $[n] = Z(\tilde{c}) \cup X_0(\tilde{c}) \cup \cdots \cup X_p(\tilde{c})$ where $Z(\tilde{c})$ consists of those indices from $[n]$ whose coordinates of $F(\tilde{c})$ all are zero, which could potentially be empty. In fact, letting $n_\ell = |X_\ell(\tilde{c})|$, we see that $F(\tilde{c})$ is the unique face that is the convex combination of those vertices of $\Pi_{n-1}(k-1)$ where the n_p largest entries occur in coordinates from $X_p(\tilde{c})$, the next largest entries occur in coordinates from $X_{p-1}(\tilde{c})$, etc., the n_0 2nd smallest entries, not all zero (but where some could be zero), occur in coordinates from $X_0(\tilde{c})$ and lastly the smallest entries, all zero, occur in coordinates from $Z(\tilde{c})$. As noted, with this setup $Z(\tilde{c})$ could be empty. We therefore must relax the notion of partition in order to obtain a description of the face $F(\tilde{c}) = F([\tilde{c}])$.

Definition 4.9. *For $n \in \mathbb{N}$ call a tuple $(Z, X_0, \ldots, X_p)$ an* ordered pseudo-partition *of $[n]$ (or an* OPP *for short) if $Z \cup X_0 \cup \cdots \cup X_p = [n]$ is a disjoint union, $X_0, \ldots, X_p$ are all non-empty and Z might possibly be empty.*

Remark: Although the above Definition 4.9 is motivated by a vector in $\tilde{c} \in \mathbb{R}^n$, and its dot-product with a vertex $\tilde{v} \in \Pi_{n-1}(k-1)$, the definition of an OPP does not depend on it.

Theorem 4.10. *For $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, then every d-face of $\Pi_{n-1}(k-1)$ is in one-to-one correspondence with an OPP $(Z, X_0, \ldots, X_p)$ of $[n]$ where (i) $0 \leqslant |Z| \leqslant k-1$, (ii) $k \leqslant |Z| + |X_0| \leqslant n$, and (iii) $n - |Z| - p - 1 = d$.*

Proof. From an OPP $\mathcal{P} = (Z, X_0, \ldots, X_p)$ of $[n]$ satisfying the conditions (i) – (iii) in Theorem 4.10 above, we obtain a vector $\tilde{c} = \tilde{c}(\mathcal{P})$ with $\mathrm{Proj}_l(\tilde{c}) = 0$ if $l \in Z$ and $\mathrm{Proj}_l(\tilde{c}) = i + 1$ if $l \in X_i$. In this case the face $F(\tilde{c})$ is exactly the convex combination of those vertices of $\Pi_{n-1}(k-1)$ where the largest $|X_p|$ entries occur in coordinates from X_p, the largest $|X_{p-1}|$ entries of the remaining $n - |X_p|$ ones occur in coordinates from X_{p-1} etc, the largest $|X_0|$ entries of the remaining $n - (|X_1| + \cdots + |X_p|)$ ones occur in coordinates from X_0, and finally, all the coordinates from Z contain only zeros. Hence, each OPP $\mathcal{P}$ yields a unique face $F(\tilde{c}(\mathcal{P}))$.

On the other hand, every (proper) face F of $\Pi_{n-1}(k-1)$ has the form $F([\tilde{c}])$ for some $\tilde{c} \in \mathbb{R}^n$ where $\mathrm{supp}(\tilde{c}) = [h]$ for some $h \in [n]$ Viewing $\tilde{c}$ as a function $c : [n] \to [h]$ with $c(i) = c_i$ for each $i \in [n]$, we obtain a OPP $\mathcal{P} = \mathcal{P}(\tilde{c})$ as in the following way.

Letting $p \geqslant 0$ be the least integer with $|X_0| + \cdots + |X_p| \geqslant n - k + 1$, where $X_i = c^{-1}(h - p + i)$ for each $i = 0, \ldots, p$, and $Z = c^{-1}([h - p - 1])$, will give us our desired OPP $\mathcal{P}(\tilde{c}) = (Z, X_0, \ldots, X_p)$.

Clearly we have $\mathcal{P}(F(\tilde{c}(\mathcal{P}))) = \mathcal{P}$ and $F(\tilde{c}(\mathcal{P}(F(\tilde{c})))) = F(\tilde{c})$. Note that in general $\tilde{c}(\mathcal{P}(F(\tilde{c}))) \neq \tilde{c}$, but they yield the same face. This proves the one-to-one correspondence between OPPs and (proper) faces of $\Pi_{n-1}(k-1)$.

Finally, by Proposition 4.6 if $F = F(\tilde{c}(\mathcal{P}))$ is the unique face obtained from the OPP $\mathcal{P}$, then

$$\dim(F) = \sum_{i=0}^{p}(|X_i| - 1) = |X_0| + \cdots + |X_p| - p - 1 = n - |Z| - p - 1,$$

which completes the proof. $\square$

By Theorem 4.10 we can derive the f-polynomial of $\Pi_{n-1}(k-1)$ by enumerating all OPP $\mathcal{P}$ satisfying (i) and (ii) in Theorem 4.10 with $d = n - |Z| - p - 1$ being a given fixed number. For disjoint $Z, X_0 \subseteq [n]$ there are $\left\{{n-|Z|-|X_0| \atop p}\right\}p!$ ordered partitions $(X_1, \ldots, X_p)$ of the remaining elements of $[n] \setminus (Z \cup X_0)$.

Letting $i = |Z| \in \{0, \ldots, k-1\}$ and $j = |X_0|$, we get by Theorem 4.10 that $i \in \{0, \ldots, k-1\}$ and $i + j \in \{k, \ldots, n\}$. Hence, each ordered $(X_1, \ldots, X_p)$ of the remaining $n - i - j$ elements from $[n] \setminus (Z \cup X_0)$ will by Theorem 4.10 yield a face of dimension $n - i - p - 1$. As there are $\binom{n}{i}\binom{n-i}{j}$ ways of choosing a legitimate pair (Z, X_0), we have the following Proposition.

Proposition 4.11. *The f-polynomial $f_{\Pi_{n-1}(k-1)}(x) = \sum_{i=0}^{n-1} f_i(\Pi_{n-1}(k-1))x^i$ of $\Pi_{n-1}(k-1)$ is given by*

$$f_{\Pi_{n-1}(k-1)}(x) = \sum_{\substack{0 \leqslant i \leqslant k-1 \\ k \leqslant i+j \leqslant n}} \binom{n}{i}\binom{n-i}{j} \sum_{p=0}^{n-i-j} \left\{{n-i-j \atop p}\right\} p! x^{n-i-p-1}.$$

Remarks: (i) Note that the coefficients $[x^0]f_{\Pi_{n-1}(k-1)}(x)$ and $[x^1]f_{\Pi_{n-1}(k-1)}(x)$ agree with previous Propositions 2.3 and 2.4 on the number of vertices and edges respectively. (ii) When $k = n$ we obtain

$$f_{\Pi_{n-1}(k-1)}(x) = f_{\Pi_{n-1}(n-1)}(x) = \frac{(x+1)^n - 1}{x},$$

the f-polynomial of the $(n-1)$-dimensional simplex.

By Propositions 4.11 and 2.4 and Corollary 1.4 we obtain the ℓ-flag polynomial of $\Pi_{n-1}(k-1)$ in the following.

Corollary 4.12. *For each $k \in \{2, \ldots, n\}$ the ℓ-flag polynomial $\tilde{f}^{\ell}_{\Pi_{n-1}(k-1)}(\tilde{x})$ of $\Pi_{n-1}(k-1)$ is given by*

$$\tilde{f}^{\ell}_{\Pi_{n-1}(k-1)}(\tilde{x}) = (x_2 + \cdots + x_\ell + 1)^{n-1} f_{\Pi_{n-1}(k-1)}\left(\frac{x_1}{x_2 + \cdots + x_\ell + 1}\right),$$

where $f_{\Pi_{n-1}(k-1)}$ is the f-polynomial of $\Pi_{n-1}(k-1)$ given in Proposition 4.11.

We complete this section on the face lattice of $\Pi_{n-1}(k-1)$ by describing the faces of $\Pi_{n-1}(k-1)$ in terms of OPPs of $[n]$ and when one face contains another in a similar fashion as in Claim 1.2.

From Observation 4.5 we immediately obtain the following.

Proposition 4.13. *If $\tilde{a}, \tilde{c} \in \mathbb{R}^n$ are such that $a_i \leqslant a_j \Rightarrow c_i \leqslant c_j$, then for the corresponding faces of $P_{n-1}(\tilde{v})$ we have $F([\tilde{a}]) \subseteq F([\tilde{c}])$.*

Proposition 4.13 yields a sufficient condition for the vectors $\tilde{a}$ and $\tilde{c}$ that implies $F([\tilde{a}]) \subseteq F([\tilde{c}])$. We will now describe exactly the relationship between $\tilde{a}$ and $\tilde{c}$ such that for faces of $P_{n-1}(\tilde{v})$ we have $F([\tilde{a}]) \subseteq F([\tilde{c}])$.

Assume $\tilde{a}, \tilde{c} \in \mathbb{R}^n$ are such for their corresponding faces of $P_{n-1}(\tilde{v})$ we have that $F([\tilde{a}]) \subseteq F([\tilde{c}])$. Since there is a permutation $\alpha \in S_n$ with $\alpha(\tilde{a})$ ordered, i.e. $a_{\alpha(1)} \leqslant \cdots \leqslant a_{\alpha(n)}$, we can, for simplicity, assume $\tilde{a}$ is ordered $a_1 \leqslant \cdots \leqslant a_n$. In this case the partition $[n] = X_1(\tilde{a}) \cup \cdots \cup X_h(\tilde{a})$ of $[n]$ induced by $\tilde{a}$ as in (7) is a union of consecutive intervals. By Observation 4.4 we can assume $\tilde{v}$ is ordered in the same way as $\tilde{a}$ is, so $v_1 \leqslant \cdots \leqslant v_n$. If $\delta_{X_\ell(\tilde{a})}(\tilde{v}) = 1$, then for any transposition $\tau \in S(X_\ell(\tilde{a})) \subseteq S_n$ we have $\tau(\tilde{v}) \in F([\tilde{a}]) \subseteq F([\tilde{c}])$ and hence $\tilde{c} \cdot \tilde{v} = \tilde{c} \cdot \tau(\tilde{v})$. By Lemma 4.1 we must have the following.

Claim 4.14. *$c_i = c_j$ for all $i, j \in X_\ell(\tilde{a})$ with $\delta_{X_\ell(\tilde{a})}(\tilde{v}) = 1$.*

Assume now $i \in X_\ell(\tilde{a})$ and $j \in X_{\ell+1}(\tilde{a})$ where $\tilde{a}$ is $\tilde{v}$-reduced. In this case one of the following three conditions hold: (i) $v_i < v_j$, (ii) $v_i = v_j$ and $\delta_{X_\ell(\tilde{a})}(\tilde{v}) = 1$, and hence there is an $i' \in X_\ell(\tilde{a})$ with $v_{i'} < v_i = v_j$, or (iii) $v_i = v_j$ and $\delta_{X_{\ell+1}(\tilde{a})}(\tilde{v}) = 1$, and hence there is an $j' \in X_{\ell+1}(\tilde{a})$ with $v_i = v_j < v_{j'}$.

In case (i) consider the transposition $\tau = (i, j)$. Since $\tilde{v} \in F([\tilde{a}]) \subseteq F([\tilde{c}])$ we have $\tilde{c} \cdot \tilde{v} \geqslant \tilde{c} \cdot \tau(\tilde{v})$, and hence $c_i v_i + c_j v_j \geqslant c_i v_j + c_j v_i$ or $(c_j - c_i)(v_j - v_i) \geqslant 0$. Therefore $c_i \leqslant c_j$ must hold.

In case (ii) consider the transposition $\tau = (i', j)$. As in previous case we have $\tilde{c} \cdot \tilde{v} \geqslant \tilde{c} \cdot \tau(\tilde{v})$, and hence $c_{i'} v_{i'} + c_j v_j \geqslant c_{i'} v_j + c_j v_{i'}$ or $(c_j - c_{i'})(v_j - v_{i'}) \geqslant 0$. Therefore $c_{i'} \leqslant c_j$ must hold, and so by Claim 4.14 $c_i = c_{i'} \leqslant c_j$ must hold.

Finally, in case (iii) consider the transposition $\tau = (i, j')$. As in previously we have $\tilde{c} \cdot \tilde{v} \geqslant \tilde{c} \cdot \tau(\tilde{v})$, and hence $c_i v_i + c_{j'} v_{j'} \geqslant c_i v_{j'} + c_{j'} v_i$ or $(c_{j'} - c_i)(v_{j'} - v_i) \geqslant 0$. Therefore $c_i \leqslant c_{j'}$ must hold, and so by Claim 4.14 $c_i \leqslant c_{j'} = c_j$ must hold. Hence, we have the following.

Claim 4.15. *For a $\tilde{v}$-reduced $\tilde{a}$, if $i \in X_\ell(\tilde{a})$ and $j \in X_{\ell'}(\tilde{a})$ with $\ell < \ell'$, then $c_i \leqslant c_j$.*

By Observation 4.4 and the previous two Claims 4.14 and 4.15, noting that the ordering of both $\tilde{a}$ and $\tilde{v}$ was assumed for the sake of argument, we have the following summarizing theorem.

Theorem 4.16. *Let $\tilde{a}, \tilde{c} \in \mathbb{R}^n$ where $\tilde{a}$ is $\tilde{v}$-reduced, and $F([\tilde{a}]), F([\tilde{c}])$ be the corresponding induced faces of $P_{n-1}(\tilde{v})$. Assume $\tilde{v} \in F([\tilde{a}])$ and let $[n] = X_1(\tilde{a}) \cup \cdots \cup X_h(\tilde{a})$ be the partition of $[n]$ induced by $\tilde{a}$ as in (7). With this setup we have $F([\tilde{a}]) \subseteq F([\tilde{c}])$ if and only if we have the following.*

1. $a_i < a_j \Rightarrow c_i \leqslant c_j$.

2. For every part $X_\ell(\tilde{a})$ *with* $\delta_{X_\ell(\tilde{a})}(\tilde{v}) = 1$, *we have* $\delta_{X_\ell(\tilde{a})}(\tilde{c}) = 0$.

Finally in this section, we further seek a description of the faces and flags of $\Pi_{n-1}(k-1)$ as described in Claim 1.2 for the standard permutahedron Π_{n-1}. To do so, we apply Theorem 4.16 to describe when exactly one face $F([\tilde{a}])$ of $\Pi_{n-1}(k-1)$ is contained in another $F([\tilde{c}])$ in terms of the characterization given in Theorem 4.10. We can assume both $\tilde{a}$ and $\tilde{c}$ to be $\tilde{v}_{n-1}(k-1)$-reduced.

Note that, trivially, if $|X_\ell(\tilde{a})| = 1$, then clearly $\delta_{X_\ell(\tilde{a})}(\tilde{c}) = 0$. Assume that $F([\tilde{a}])$ and $F([\tilde{c}])$ correspond to the OPPs $(Z, X_0, \ldots, X_p)$ and $(Z', X'_0, \ldots, X'_{p'})$ of $[n]$ respectively, and that $\tilde{v} \in F([\tilde{a}]) \subseteq F([\tilde{c}])$ is a vertex of $\Pi_{n-1}(k-1)$. By Theorems 4.10 and 4.16 and the above note, we have that $\delta_{X_i}(\tilde{c}) = 0$ for all $i \in \{0, \ldots, p\}$, and hence $X_0 \subseteq X'_0$, and further each part from $\{X_1, \ldots, X_p\}$ is contained in a part from $\{X'_0, \ldots, X'_{p'}\}$. So, as a direct consequence of Theorem 4.16 We have the following.

Corollary 4.17. *For two faces* $F([\tilde{a}])$ *and* $F([\tilde{c}])$ *of* $\Pi_{n-1}(k-1)$, *where both* $\tilde{a}$ *and* $\tilde{c}$ *are* $\tilde{v}_{n-1}(k-1)$*-reduced, corresponding to the OPPs* $(Z, X_0, \ldots, X_p)$ *and* $(Z', X'_0, \ldots, X'_{p'})$ *respectively, we have* $F([\tilde{a}]) \subseteq F([\tilde{a}])$ *if, and only if, the disjoint union* $D \cup X_0 \cup \cdots \cup X_p$ *is a refinement of* $X'_0 \cup \cdots \cup X'_{p'}$ *where* $X_0 \subseteq X'_0$ *and* $D = X'_0 \setminus X_0$ *is the difference.*

Remark: Note that $D \subseteq [n] \setminus (X_0 \cup \cdots \cup X_p) \subseteq Z$.

Note that we have a well defined map $\{0, 1, \ldots, p\} \ni i \mapsto i' \in \{0, 1, \ldots, p'\}$ where i' is the unique index with $X_i \subseteq X'_{i'}$. In this way we have.

Observation 4.18. *The above map* $i \mapsto i'$ *is an increasing surjection.*

5 A closed formula for the exponential generating function

In this final section we derive a closed formula for the exponential ℓ-generating function $\tilde{\xi}^{\ell}_{\mathcal{P}^{\perp}_k;k-1}(\tilde{x}, y)$ from Definition 1.5 of the family $\mathcal{P}^{\perp}_k = \{\Pi_{n-1}(k-1)\}_{n \geqslant k}$ of all the hyper-permutahedra, which we will henceforth denote by $\tilde{\xi}^{\ell}_k(\tilde{x}, y)$, analogous to the result of Proposition 1.6. Unless otherwise stated $k \geqslant 1$ is an arbitrary but fixed integer throughout.

If we let

$$\tilde{g}^{\ell}_n(\tilde{x}) := \sum_{\tilde{s}} \frac{f_{\tilde{s}}(\Pi_{n-1}(k-1))}{(n-s_1)!} x_1^{n-s_1} x_2^{s_2-s_1} \cdots x_\ell^{s_\ell - s_{\ell-1}},$$

then by Definition 1.5 we have

$$x_1^n \tilde{g}^{\ell}_n(x_1^{-1}, x_2, \ldots, x_\ell) = \tilde{\xi}^{\ell}_{\Pi_{n-1}(k-1)}(\tilde{x}) \tag{8}$$

and so for

$$\tilde{g}^{\ell}(\tilde{x}, y) := \sum_{n \geqslant 1} \tilde{g}^{\ell}_n(\tilde{x}) \frac{y^n}{n!} = \sum_{n \geqslant k} \tilde{g}^{\ell}_n(\tilde{x}) \frac{y^n}{n!}$$

we have

$$\tilde{g}^{\ell}(x_1^{-1}, x_2, \ldots, x_\ell, x_1 y) = \sum_{n \geqslant k} x_1^n \tilde{g}_n^{\ell}(x_1^{-1}, x_2, \ldots, x_\ell) \frac{y^n}{n!} = \tilde{\xi}_k^{\ell}(\tilde{x}, y). \tag{9}$$

Hence, it suffices to obtain a closed formula for $\tilde{g}^{\ell}(\tilde{x}, y)$. Further we note that for $\ell = 1$ we obtain by (8) that $x_1^n \tilde{g}_n^1(x_1^{-1}) = \tilde{\xi}^1(x_1)$ and hence $\tilde{g}_n^1(x_1) = x_1^n \tilde{\xi}^1(x_1^{-1})$ and so by Proposition 4.11 that

$$g_n(x) := \tilde{g}_n^1(x) = \sum_{\substack{0 \leqslant i \leqslant k-1 \\ k \leqslant i+j \leqslant n}} \binom{n}{i} \binom{n-i}{j} \sum_{p=0}^{n-i-j} \left\{ {n-i-j \atop p} \right\} p! \frac{x^{i+p+1}}{(i+p+1)!}. \tag{10}$$

Further, by (8) for general ℓ and $\ell = 1$, and Corollary 1.4 applied to $\tilde{\xi}^{\ell}$ and (10) we obtain

$$\begin{aligned}
\tilde{g}_n^{\ell}(\tilde{x}) &= \frac{1}{S} g_n(x_1(x_2 + \cdots + x_\ell + 1)) \\
&= \frac{1}{S} \left[\sum_{\substack{0 \leqslant i \leqslant k-1 \\ k \leqslant i+j \leqslant n}} \binom{n}{i} \binom{n-i}{j} \sum_{p=0}^{n-i-j} \left\{ {n-i-j \atop p} \right\} p! \frac{(x_1(x_2 + \cdots + x_\ell + 1))^{i+p+1}}{(i+p+1)!} \right],
\end{aligned}$$

where $S = x_2 + \cdots + x_\ell + 1$. Hence, it suffices to obtain a closed formula for

$$g(x, y) := \tilde{g}^1(x, y) = \sum_{n \geqslant 1} g_n(x) \frac{y^n}{n!} = \sum_{n \geqslant k} g_n(x) \frac{y^n}{n!}, \tag{11}$$

since then

$$\tilde{g}_n^{\ell}(\tilde{x}, y) = \sum_{n \geqslant k} \tilde{g}_n^{\ell}(\tilde{x}) \frac{y^n}{n!} = \frac{g(x_1(x_2 + \cdots + x_\ell + 1), y)}{x_2 + \cdots + x_\ell + 1}. \tag{12}$$

By (10) we have

$$g_n(x) = \sum_{i=0}^{k-1} g_{n;i}(x) \tag{13}$$

where for each $i \in \{0, 1, \ldots, k-1\}$

$$\begin{aligned}
g_{n;i}(x) &= \sum_{j=k-i}^{n-i} \binom{n}{i} \binom{n-i}{j} \sum_{p=0}^{n-i-j} \left\{ {n-i-j \atop p} \right\} p! \frac{x^{i+p+1}}{(i+p+1)!} \\
&= \binom{n}{i} \sum_{j=k-i}^{n-i} \binom{n-i}{j} \sum_{p=0}^{n-i-j} \left\{ {n-i-j \atop p} \right\} p! \frac{x^{i+p+1}}{(i+p+1)!},
\end{aligned} \tag{14}$$

and so its $(i+1)$-th derivative w.r.t. x is

$$g_{n;i}^{(i+1)}(x) = \binom{n}{i} \sum_{j=k-i}^{n-i} \binom{n-i}{j} \sum_{p=0}^{n-i-j} \left\{ {n-i-j \atop p} \right\} x^p := \binom{n}{i} \gamma_{n;i}(x). \tag{15}$$

From this we deduce that for

$$g_i(x,y) := \sum_{n\geqslant 1} g_{n;i}(x)\frac{y^n}{n!} \tag{16}$$

where $g_{n;i}(x)$ is given in (14), we have

$$\begin{aligned} g_i^{(i+1)}(x,y) &:= \frac{\partial^{i+1}}{\partial x^{i+1}} g_i(x,y) \\ &= \sum_{n\geqslant 1} g_{n;i}^{(i+1)}(x)\frac{y^n}{n!} \\ &= \sum_{n\geqslant 1} \binom{n}{i}\gamma_{n;i}(x)\frac{y^n}{n!} \\ &= \sum_{n\geqslant i} \binom{n}{i}\gamma_{n;i}(x)\frac{y^n}{n!} \\ &= \frac{y^i}{i!}\sum_{n\geqslant i}\gamma_{n;i}(x)\frac{y^{n-i}}{(n-i)!}. \end{aligned}$$

Before we continue, we need the following.

Lemma 5.1. *For $m, N \in \mathbb{N}$ we have*

$$\sum_{i=1}^{N-m+1}\binom{N}{i}\left\{ {N-i \atop m-1} \right\} = m\left\{ {N \atop m} \right\}.$$

Proof. Call an unordered partition of $[N]$ *rooted* if it has one distinguished part, the *root* r. For each of the unordered partitions of $[N]$ into m parts, we have m possible roots, and so the number of rooted partitions of $[N]$ into m parts is, on one hand, given by $m\left\{ {N \atop m} \right\}$.

On the other hand, we can start by choosing a root r of valid cardinality $i \in \{1, \ldots, N-m+1\}$, and then consider the $\left\{ {N-i \atop m-1} \right\}$ unordered partitions of the remaining subset of $[N] \subseteq r$. For each i this can be done in $\binom{N}{i}\left\{ {N-i \atop m-1} \right\}$ possible ways. Adding these ways for all possible i will give us the expression on the left in the stated equation. □

The coefficient $[x^h](\gamma_{n;i}(x))$ of x^h in the polynomial $\gamma_{n;i}(x)$ defined in (15) is by direct tallying and the above Lemma 5.1 given by

$$\begin{aligned} [x^h](\gamma_{n;i}(x)) &= \sum_{q=k-i}^{n-i-h}\binom{n-i}{q}\left\{ {n-i-q \atop h} \right\} \\ &= \sum_{q=1}^{n-i-h}\binom{n-i}{q}\left\{ {n-i-q \atop h} \right\} - \sum_{q=1}^{k-i-1}\binom{n-i}{q}\left\{ {n-i-q \atop h} \right\} \\ &= \left\{ {n-i \atop h+1} \right\}(h+1) - \sum_{q=1}^{k-i-1}\binom{n-i}{q}\left\{ {n-i-q \atop h} \right\}. \end{aligned}$$

In light of this, we can further write $\gamma_{n;i}(x)$ as $\gamma_{n;i}(x) = \alpha_{n;i}(x) - \sum_{q=1}^{k-i-1} \alpha_{n;i,q}(x)$, where

$$\alpha_{n;i}(x) := \sum_{h \geqslant 0} \left\{ {n-i \atop h+1} \right\} (h+1)x^h, \quad \alpha_{n;i,q}(x) := \sum_{h \geqslant 0} \binom{n-i}{q} \left\{ {n-i-q \atop h} \right\} x^h.$$

Defining the corresponding exponential series

$$\alpha_i(x,y) := \sum_{n \geqslant i} \alpha_{n;i}(x) \frac{y^{n-i}}{(n-i)!}, \quad \alpha_{i,q}(x,y) := \sum_{n \geqslant i} \alpha_{n;i,q}(x) \frac{y^{n-i}}{(n-i)!},$$

we then get

$$\begin{aligned} g_i^{(i+1)}(x,y) &= \frac{y^i}{i!} \sum_{n \geqslant i} \gamma_{n;i}(x) \frac{y^{n-i}}{(n-i)!} \\ &= \frac{y^i}{i!} \sum_{n \geqslant i} \left(\alpha_{n;i}(x) - \sum_{q=1}^{k-i-1} \alpha_{n;i,q}(x) \right) \frac{y^{n-i}}{(n-i)!} \\ &= \frac{y^i}{i!} \left(\alpha_i(x,y) - \sum_{q=1}^{k-i-1} \alpha_{i;q}(x,y) \right). \end{aligned}$$

Up to a constant we obtain $\int \alpha_{n;i}(x)\, dx = \sum_{h \geqslant 0} \left\{ {n-i \atop h+1} \right\} x^{h+1}$, and so by (3) we get

$$\int \alpha_i(x,y)\, dx = \sum_{h,n \geqslant i} \left\{ {n-i \atop h+1} \right\} x^{h+1} \frac{y^{n-i}}{(n-i)!} = \sum_{h,m \geqslant 0} \left\{ {m \atop h+1} \right\} x^{h+1} \frac{y^m}{m!} = e^{x(e^y-1)} - 1,$$

and so by differentiating

$$\alpha_i(x,y) = e^{x(e^y-1)}(e^y - 1). \tag{17}$$

Similarly, but with neither integration nor differentiation, we obtain by direct manipulation and by again (3)

$$\begin{aligned} \alpha_{i;q}(x,y) &= \sum_{n \geqslant i} \alpha_{n;i,q}(x) \frac{y^{n-i}}{(n-i)!} \\ &= \sum_{n \geqslant i} \left(\sum_{h \geqslant 0} \binom{n-i}{q} \left\{ {n-i-q \atop h} \right\} x^h \right) \frac{y^{n-i}}{(n-i)!} \\ &= \sum_{m \geqslant 0} \left(\sum_{h \geqslant 0} \binom{m}{q} \left\{ {m-q \atop h} \right\} x^h \right) \frac{y^m}{m!} \\ &= \sum_{m,h \geqslant 0} \binom{m}{q} \left\{ {m-q \atop h} \right\} \frac{y^m}{m!} x^h \\ &= \frac{y^q}{q!} \sum_{m,h \geqslant 0} \left\{ {m-q \atop h} \right\} \frac{y^{m-q}}{(m-q)!} x^h \\ &= \frac{y^q}{q!} e^{x(e^y-1)}. \end{aligned}$$

Consequentially, by the above and (17) we then get

$$\begin{aligned} g_i^{(i+1)}(x,y) &= \frac{y^i}{i!}\left(\alpha_i(x,y) - \sum_{q=1}^{k-i-1}\alpha_{i;q}(x,y)\right) \\ &= \frac{y^i}{i!}\left(e^y - E_{k-i-1}(y)\right)e^{x(e^y-1)}, \end{aligned}$$

where $E_m(x) = 1 + x + \cdots + \frac{x^m}{m!}$ is the m-th degree polynomial approximation of e^y. By the above and the defining sum of $g_i(x,y)$ in (16) we have

$$g_i^{(i+1)}(x,y) = \frac{y^i}{i!}\left(e^y - E_{k-i-1}(y)\right)e^{x(e^y-1)},\ \ g_i(0,y) = g_i'(0,y) = \cdots = g_i^{(i)}(0,y) = 0 \text{ for all } y. \tag{18}$$

The closed expression for $g_i(x,y)$ is uniquely determined by (18) and by integrating $i+1$ times we get

$$g_i(x,y) = \frac{y^i(e^y - E_{k-i-1}(y))}{i!(e^y-1)^{i+1}}\left(e^{x(e^y-1)} - E_i(x(e^y-1))\right). \tag{19}$$

By (11), (13), (16) and (19) we then get

$$g(x,y) = \sum_{n\geqslant 1} g_n(x)\frac{y^n}{n!} = \sum_{i=0}^{k-1} g_i(x,y) = \sum_{i=0}^{k-1}\frac{y^i(e^y - E_{k-i-1}(y))}{i!(e^y-1)^{i+1}}\left(e^{x(e^y-1)} - E_i(x(e^y-1))\right),$$

and so by the above expression, (12) and (9) we have the following main theorem of this section.

Theorem 5.2. *The exponential ℓ-generating function for all the ℓ-flags of all the hyper-permutahedra $\mathcal{P}_k^{\perp} = \{\Pi_{n-1}(k-1)\}_{n\geqslant k}$ from Definition 1.5 is given by*

$$\begin{aligned} \tilde{\xi}_k^{\ell}(\tilde{x},y) &= \sum_{n\geqslant 1,\tilde{s}} \tilde{\xi}^{\ell}_{\Pi_{n-1}(k-1)}(\tilde{x})\frac{y^n}{n!} \\ &= \frac{1}{S}\sum_{i=0}^{k-1}\frac{(x_1y)^i(e^{x_1y} - E_{k-i-1}(x_1y))}{i!(e^{x_1y}-1)^{i+1}}\left(e^{\frac{S}{x_1}(e^{x_1y}-1)} - E_i\left(\frac{S}{x_1}(e^{x_1y}-1)\right)\right). \end{aligned}$$

where $S = x_2 + \cdots + x_\ell + 1$ and $E_m(x) = 1 + x + \cdots + \frac{x^m}{m!}$ is the m-th polynomial approximation of e^x.

Remarks: (i) The question remains, whether one could possible further simplify the expression given in Theorem 5.2. Since, however, we are dealing with enumeration of OPP as described in Theorem 4.10, in which the cardinalities of the parts depend both on n and k, it seems unlikely to the author that a substantial simplification exists. (ii) Letting $k = 2$ we obtain the exponential ℓ-generating function for all the ℓ-flags of all the hyper-permutahedra $\{\Pi_{n-1}\}_{n\geqslant 2}$ as the following

$$\tilde{\xi}_2^{\ell}(\tilde{x},y) = \frac{1}{x_2 + \cdots + x_\ell + 1}\left(e^{\frac{x_2+\cdots+x_\ell+1}{x_1}(e^{x_1y}-1)} - 1 - y(x_2 + \cdots + x_\ell + 1)\right),$$

which is consistent with Proposition 1.6, when we note that the family $\{\Pi_{n-1}\}_{n\geqslant 1}$ there contains Π_0 for $n = 1$, whereas in Theorem 5.2 the family $\{\Pi_{n-1}(k-1)\}_{n\geqslant k}$ becomes $\{\Pi_{n-1}\}_{n\geqslant 2}$ omitting the singleton Π_0.

When $\ell = 1$ in Theorem 5.2 the sum $x_2 + \cdots + x_\ell$ is empty and so we obtain the face function of all the hyper-permutahedra in the following.

Corollary 5.3. *The exponential generating function for the faces of the hyper-permutahedra $\{\Pi_{n-1}(k-1)\}_{n\geqslant k}$ from Definition 1.5 is given by*

$$\xi_k(x,y) = \tilde{\xi}_k^1(x,y) = \sum_{i=0}^{k-1} \frac{(xy)^i(e^{xy} - E_{k-i-1}(xy))}{i!(e^{xy}-1)^{i+1}} \left(e^{\left(\frac{e^{xy}-1}{x}\right)} - E_i\left(\frac{e^{xy}-1}{x}\right)\right).$$

where $E_m(x) = 1 + x + \cdots + \frac{x^m}{m!}$ is the m-th polynomial approximation of e^x.

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References

[1] Geir Agnarsson, The flag polynomial of the Minkowski sum of simplices. *Annals of Combinatorics*, no. 3, **17**, 401 – 426, (2013).

[2] Geir Agnarsson and Walter Morris, On Minkowski sum of simplices. *Annals of Combinatorics*, **13**, 271 – 287, (2009).

[3] Federico Ardila; Carolina Benedetti; Jeffrey Doker, Matroid polytopes and their volumes. *Discrete Comput. Geom.*, **43**, no. 4, 841 – 854. (2010).

[4] E. T. Bell, Exponential polynomials. *Ann. of Math. (2)*, **35**, no. 2, 258 – 277, (1934).

[5] Sangwook Kim, Flag enumerations of matroid base polytopes. 20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), 283 – 294, *Discrete Math. Theor. Comput. Sci. Proc.*, AJ, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, (2008).

[6] Sangwook Kim, Flag enumerations of matroid base polytopes. *J. Combin. Theory Ser. A*, **117** no. 7, 928 – 942, (2010).

[7] Alexander Postnikov, Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, no. 6, 1026 – 1106, (2009).

[8] Alex Postnikov, Victor Reiner, Lauren Williams, Faces of generalized permutohedra. *Doc. Math.*, **13**, 207 – 273, (2008).

[9] Richard Rado, An inequality. *J. London Math. Soc.*, **27**, 1 – 6, (1952).

[10] Günter M. Ziegler, Lectures on polytopes, *Graduate Texts in Mathematics*, GTM – 152, Springer-Verlag Inc., New York, (1995).