Covering k-uniform hypergraphs by monochromatic loose paths

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Abstract

A conjecture of Gyárfás and Sárközy says that in every 2-coloring of the edges of the complete k-uniform hypergraph $\mathcal{K}_n^k$, there are two disjoint monochromatic loose paths of distinct colors such that they cover all but at most $k-2$ vertices. Recently, the authors affirmed the conjecture. In the note we show that for every 2-coloring of $\mathcal{K}_n^k$, one can find two monochromatic paths of distinct colors to cover all vertices of $\mathcal{K}_n^k$ such that they share at most $k-2$ vertices. Omidi and Shahsiah conjectured that $R(\mathcal{P}_t^k, \mathcal{P}_t^k) = t(k-1) + \lfloor \frac{t+1}{2} \rfloor$ holds for $k \geqslant 3$ and they affirmed the conjecture for $k = 3$ or $k \geqslant 8$. We show that if the conjecture is true, then $k-2$ is best possible for our result.

Keywords: Complete uniform hypergraphs; monochromatic loose path; covering

1 Introduction

A hypergraph $\mathcal{H} = (V, E)$ consists of a set V of vertices and a set E of edges, where each edge is a subset of V. If all the edges of $\mathcal{H}$ have same size k, then the hypergraph $\mathcal{H}$ is said to be k-uniform. Let $\mathcal{K}_n^k$ denote the complete k-uniform hypergraph on n vertices (the family of all k-element subsets of an n-element set). For any integer $m \geqslant 1$, let $[m] = \{1, 2, \ldots, m\}$. A k-uniform loose (or linear) path of length ℓ, denoted by $\mathcal{P}_\ell^k$, is a k-uniform hypergraph with edges $e_1, e_2, \ldots, e_\ell$ such that for any $i \in [\ell-1]$, $|e_i \cap e_{i+1}| = 1$ and $|e_i \cap e_j| = 0$ for all other pairs $\{i, j\}$, $i \neq j$. For a vertex $v \in V(\mathcal{P}_\ell^k)$, we call v an

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endpoint of $\mathcal{P}^k_\ell$ if $v \in e_i$ for $i \in \{1, \ell\}$ and $v \notin e_j$ for each $j \neq i$. For $k = 2$ we obtain the usual definition of a path P_ℓ with ℓ edges.

The subject of edge-coloring of graphs has been a classical topic of graph theory. A famous result which is called Ramsey's theorem showed that: Given an integer $n \geqslant 2$, whenever the edges of a sufficiently large complete graph are colored with two colors, red and blue, there is a monochromatic copy of the complete graph K_n. Then many authors focused on the related research and expanded several branches in the subject. For example, in generalized graph Ramsey theory, it is of interest and well motivated to find desired monochromatic structures (cycles, paths, trees and so on) in edge-colored graphs. In this paper, r-coloring always means edge-coloring with r colors (traditionally red and blue when $r = 2$).

A simple and basic proposition, introduced by Gerencsér and Gyárfás in [8], says that

Proposition 1. ([8]) *In any 2-coloring of a finite complete graph the vertices can be partitioned into a red and a blue path. Here the empty set and a single vertex are accepted as a path of any color.*

In fact, Proposition 1 subsequently gave birth to the area of partitioning edge-colored complete graphs into monochromatic subgraphs. In 1979, Lehel conjectured that Proposition 1 remains true if paths are replaced by cycles. Here the empty set, a single vertex and a single edge are accepted as a cycle of any color. Gyárfás [9] proved the following weaker statement: The vertices of a 2-coloring of K_n can be covered by a red and a blue cycle such that the two cycles have at most one common vertex. It took a long time to get rid of the common vertex. Using the Regularity Lemma and some new techniques, Łuczac, Rödl and Szemerédi [16] first succeeded to prove the conjecture for large n. Then Bessy and Thomassé [1] found an elementary and elegant proof of Lehel's conjecture. Another famous conjecture (usually called Ryser's conjecture), appeared in the Ph.D. thesis of Henderson [5], states that for any r-coloring of the edges of a graph G, the vertex set $V(G)$ can be covered by at most $\alpha(G)(r-1)$ monochromatic trees, where $\alpha(G)$ is the independence number of G. For the case $\alpha(G) = 1$, i.e. for complete graphs, the cases $r = 3, 4, 5$ were proved by Gyárfás [6], Duchet [3] and Tuza [22], respectively. However, all remaining cases of Ryser's conjecture are still open. There have been many further results, questions and conjectures in this area, many of which generalize Proposition 1 in graphs. We refer to two surveys [10, 14].

However, in contrast to the graph case, there are only a few results on covering the vertices of hypergraphs with monochromatic pieces. Note that for any 2-coloring of complete graph K_n, we cannot be guaranteed to obtain a monochromatic cycle of length greater than $\lceil \frac{2n}{3} \rceil$. However, in the case for hypergraphs, Gyárfás et al. [13] proved that every 2-coloring of $\mathcal{K}^3_n$ contains a monochromatic $\mathcal{C}^3_n$ and conjectured that for $k \geqslant 2$ and sufficiently large n, every $(k-1)$-coloring of $\mathcal{K}^k_n$ contains a monochromatic $\mathcal{C}^k_n$, where the Hamiltonian Berge cycle $\mathcal{C}^k_n$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$, together with a set of distinct edges $e_1, e_2, \ldots, e_n$ such that $v_i, v_{i+1} \in e_i$ ($v_{n+1} = v_1$). Bustamante, Hàn and Stein [2] showed that for every $\eta > 0$ there exists an integer n_0 such that every 2-coloring of the 3-uniform complete hypergraph on $n \geqslant n_0$ vertices contains two disjoint monochromatic tight cycles of distinct colors that together cover all but at most ηn vertices. The same result holds if we replace tight cycles with loose cycles. Gyárfás and Sárközy [12] conjectured that for every k-uniform hypergraph $\mathcal{H}$, the vertex set $V(\mathcal{H})$ can be partitioned into at most $\alpha(\mathcal{H})$ loose cycles. Ergemlidze, Györi and Methuku [4] recently affirmed a weaker version of the conjecture for $k = 3$. If the conjecture is true then it extends Pósa theorem to k-uniform hypergraphs. To the best of our knowledge, even for $\alpha(\mathcal{H}) = k - 1$, that is, $\mathcal{H}$ is a complete k-uniform hypergraph, the conjecture is still open. Somewhat surprisingly, Király [15] proved an analogue of Ryser's conjecture for hypergraphs: For $k \geqslant 3$ and every r-coloring of $\mathcal{K}_n^k$, the vertex set can be covered by at most $\lceil \frac{r}{k} \rceil$ monochromatic connected subhypergraphs. Gyárfás and Sárközy [11] also proved that for every $(k-1)$-coloring of $\mathcal{K}_n^k$, there is a partition of the vertex set into monochromatic loose cycles such that their number depends only on r and k. In [12] they proved that in any r-coloring of a k-uniform hypergraph with independence number α, there is a partition of the vertex set into monochromatic loose cycles such that their number depends only on r, k and α. There are other results about partition and covering of hypergraphs, many of the proofs use the hypergraph Regularity Lemma, we refer to [7, 11, 20, 21].

In the note, we only consider covering hypergraphs by loose paths. A loose path is proper if the path contains at least one edge. Similar to the graph case, a set of less than k vertices in an edge-colored k-uniform hypergraph is accepted as a loose path of any color. However, it is not always possible to extend Proposition 1 to loose paths of hypergraphs. In 2013, Gyárfás and Sárközy [11] posed the following conjecture:

Conjecture 2. *([11])* In every 2-coloring of the complete k-uniform hypergraph $\mathcal{K}_n^k$, there are two disjoint monochromatic loose paths of distinct colors such that they cover all but at most $k-2$ vertices. This estimate is sharp for sufficiently large n.

Recently, the authors affirmed the conjecture in [17]. Based on Conjecture 2 and the proof, we focus on the following natural question: For every 2-coloring of $\mathcal{K}_n^k$, can we find two monochromatic paths of distinct colors to cover all vertices of $V(\mathcal{K}_n^k)$ such that the two paths share as few vertices as possible?

Note that any result on covering the vertices of edge-colored hypergraphs by monochromatic loose paths will imply a Ramsey-type result as a corollary (even though these Ramsey-type corollaries are often not sharp). Conversely, the values of Ramsey numbers of loose paths also imply that we can cover all vertices of edge-colored graphs hypergraphs by a small number of monochromatic loose paths.

Omidi and Shahsiah posed the following conjecture on the values of Ramsey numbers of loose paths for $k \geqslant 3$ in [19].

Conjecture 3. *([19])* For every $s \geqslant t \geqslant 3$, $R(\mathcal{P}_s^k, \mathcal{P}_t^k) = s(k-1) + \lfloor \frac{t+1}{2} \rfloor$.

For $k = 3$, Omidi and Shahsiah affirmed the conjecture in [18]. For $k \geqslant 8$ and $s = t$ (the diagonal case), they also gave a positive answer in [19].

Our main result is as follows:

Theorem 4. *For each 2-coloring of $\mathcal{K}_n^k$, all the vertices can be covered by two monochromatic loose paths of distinct colors such that the two paths share at most $k-2$ vertices. Besides, if Conjecture 3 is true for diagonal case then $k-2$ is best possible.*

First of all we show that if Conjecture 3 is true then $k-2$ is best possible. By Proposition 1, we only need to consider $k \geqslant 3$. Let $n = R(\mathcal{P}_t^k, \mathcal{P}_t^k)$, where $t \geqslant 3$ is an integer satisfying that $\lfloor \frac{t+1}{2} \rfloor = k+2$. Then there is a 2-coloring of $\mathcal{K}_n^k$ such that it contains no copy of a monochromatic $\mathcal{P}_{t+1}^k$. Let $\mathcal{P}_R$ and $\mathcal{P}_B$ be two paths of distinct colors covering all vertices of the colored $\mathcal{K}_n^k$. To show the sharpness of the result, it suffices to prove $|\mathcal{P}_R \cap \mathcal{P}_B| \geqslant k-2$. It is obvious that the two paths are proper, since the longest monochromatic path covers exactly $t(k-1)+1$ vertices and there are $k+1$ vertices remained. Note that $n = t(k-1) + \lfloor \frac{t+1}{2} \rfloor = (t+1)(k-1)+3$ and $|\mathcal{P}_R| + |\mathcal{P}_B| = r(k-1)+2$ for some positive integer $r > t+1$. It means that $|\mathcal{P}_R \cap \mathcal{P}_B| = |\mathcal{P}_R| + |\mathcal{P}_B| - |\mathcal{P}_R \cup \mathcal{P}_B| = |\mathcal{P}_R| + |\mathcal{P}_B| - n \geqslant (r-t-1)(k-1) - 1 \geqslant k-2$.

2 Some Lemmas

The greatest common divisor (gcd) of two or more positive integers is the largest positive integer that is a divisor of them. We first give two basic and well-known lemmas of number theory. We modify slightly the latter lemma such that it is more suitable for us, so we present its proof for the completeness.

Lemma 5. *Let a_1, a_2 and n be three positive integers. Then the indeterminate equation*

$$a_1 x + a_2 y = n \tag{1}$$

has an integer solution if and only if $\gcd(a_1, a_2) | n$.

Lemma 6. *Let a_1 and a_2 be two co-prime positive integers. For any integer $n > a_1 a_2$, the indeterminate equation (1) has a positive integer solution (x, y) with $y \leqslant a_1$.*

Proof. Let (x_0, y_0) be any integer solution of equation (1). Then $(x_0 + a_2 t, y_0 - a_1 t)$ is also an integer solution of equation (1), where t is an arbitrary integer. Then there exists an integer t_0 satisfying $0 < y_0 - a_1 t_0 \leqslant a_1$. Hence $x_0 + a_2 t_0 > 0$ since $(x_0 + a_2 t_0) a_1 = n - (y_0 - a_1 t) a_2 > a_1 a_2 - a_1 a_2 = 0$. Now $(x, y) = (x_0 + a_2 t_0, y_0 - a_1 t_0)$ is a desired positive integer solution of (1). □

By Lemma 6 we have the following corollary:

Corollary 7. *Let $k \geqslant 3$ and $n \geqslant k(k-2)+4$ be positive integers. Then there are nonnegative integers $x_1 \geqslant 2$ and $x_2 \leqslant k-2$ such that $n-2 = x_1(k-1) + x_2(k-2)$.*

Proof. By Lemma 6, the indeterminate equation $n-2 = x_1(k-1) + x_2(k-2)$ has a positive integer solution (x_1', x_2') with $x_2' \leqslant k-1$. If $x_2' = k-1$ then let $(x_1, x_2) = (x_1' + k - 2, 0)$, otherwise let $(x_1, x_2) = (x_1', x_2')$. Using $n \geqslant k(k-2)+4$ and $x_2 \leqslant k-2$, we have $x_1(k-1) = n - 2 - x_2(k-2) \geqslant k(k-2) + 2 - (k-2)(k-2) = 2(k-1)$. Hence $x_1 \geqslant 2$ and (x_1, x_2) is desired nonnegative integer solution of the indeterminate equation. □

The authors proved the following slightly stronger result than Conjecture 2 in [17].

Lemma 8. ([17]) *Suppose that the edges of the complete k-uniform hypergraph $\mathcal{K}_n^k$ are colored with two colors, where $n \equiv 2 \bmod (k-1)$. Then $V(\mathcal{K}_n^k)$ can be partitioned into monochromatic loose paths of distinct colors.*

In contrast to the graph case, there are few known results about the Ramsey numbers of hypergraphs. Concerning to exact values of diagonal Ramsey numbers of loose paths we have the following result, which is useful to our main proof of Theorem 4.

Lemma 9. ([7]) *For $t = 2, 3, 4$ and $k \geqslant 3, R(\mathcal{P}_t^k, \mathcal{P}_t^k) = t(k-1) + \lfloor \frac{t+1}{2} \rfloor$.*

Lemma 10. ([19]) *For every $t \geqslant 3$ and $k \geqslant 8$, $R(\mathcal{P}_t^k, \mathcal{P}_t^k) = t(k-1) + \lfloor \frac{t+1}{2} \rfloor$.*

3 The proof of Theorem 4

In this section we will prove Theorem 4. For simplicity, we partition the proof into the following lemmas.

Lemma 11. *Let $n-2 = x_1(k-1)+x_2(k-2)$, where $x_1 \geqslant 2$, $x_2 \leqslant k-2$ are two nonnegative integers. For any 2-coloring of $\mathcal{K}_n^k$, the vertices can be covered by two monochromatic loose paths of distinct colors such that the two paths have at most $k-2$ common vertices. In particular, the result holds for all $n \geqslant k(k-2)+4$.*

Proof. If all the edges of $\mathcal{K}_n^k$ are colored with the same color, then the result is trivial. So we assume that both colors are used at least once.

We partition $V(\mathcal{K}_n^k)$ into two sets V_1, V_2 such that $|V_1| = x_1(k-1)+2$, $|V_2| = x_2(k-2)$, where $x_1 \geqslant 2$ and $x_2 \leqslant k-2$. Note that the initial case for $x_2 = 0$ is easy to check by Lemma 8. Now we consider the case for $x_2 \geqslant 1$.

Denote by $\mathcal{H}_1$ the complete k-uniform subhypergraph induced by V_1. Then by Lemma 8, $V(\mathcal{H}_1)$ can be partitioned into a red path $\mathcal{P}_R$ and a blue path $\mathcal{P}_B$.

Case 1 Both $\mathcal{P}_R$ and $\mathcal{P}_B$ are proper.

We partition V_2 into x_2 $(k-2)$-element vertex subsets, say $V_2^1, \ldots, V_2^{x_2}$. For the first $(k-2)$-element vertex subset V_2^1, we consider the edge consisting of V_2^1 together with any two endpoints picked from $\mathcal{P}_R$ and $\mathcal{P}_B$ respectively. If the edge is red, then adding it to $\mathcal{P}_R$ will form a longer red path, otherwise adding it to $\mathcal{P}_B$ will form a longer blue path.

For $2 \leqslant i \leqslant x_2 - 1$, suppose that there is a red path $\mathcal{P}'_R$ and a blue path $\mathcal{P}'_B$ such that they cover exactly $V_1 \cup V_2^1 \cup \cdots \cup V_2^i$. Then consider the edge consisting of the $(i+1)$th $(k-2)$-element vertex subset V_2^{i+1} of V_2 and two new endpoints picked from $\mathcal{P}'_R$ and $\mathcal{P}'_B$ respectively. Note that $x_2 \leqslant k-2$ ensures the two endpoints distinct from any endpoint picked before. For any case (the edge is red or blue), we can obtain a longer red or blue path. Then the two paths of distinct colors will cover $V_1 \cup V_2^1 \cup \cdots \cup V_2^{i+1}$. By induction on i we complete the proof.

Case 2 One of $\mathcal{P}_R$ and $\mathcal{P}_B$ is not proper.

Without loss of generality suppose that $\mathcal{P}_B$ is not proper. Then $\mathcal{P}_B$ is induced by a singleton vertex, say v, since $\mathcal{P}_R$ contains $x(k-1)+1$ vertices for some integer x and

$|V_1| = x_1(k-1)+2$. If $x_2 = 1$, then we are done since a set consisting of any $k-1$ vertices can be regarded as an edge of any color. Now suppose that $x_2 \geqslant 2$. Let $e = \{v_1, \ldots, v_k\}$ be the last edge and $v_2, \ldots, v_k$ the $k-1$ endpoints of $\mathcal{P}_R$. Denote by $\mathcal{H}_2$ the complete subhypergraph induced by $V_2 \cup \{v, v_2, \ldots, v_k\}$.

Subcase 2.1 $\mathcal{H}_2$ has a blue edge f.

Let V_1' consist of all vertices of the red path $\mathcal{P}_R - e$ and the blue edge f. Let $V_2' = V(\mathcal{K}_n^k)\backslash V_1'$. Then V_1' and V_2' form a new partition of $V(\mathcal{K}_n^k)$ such that V_1' can be partitioned into two proper paths of distinct colors. The rest discussion is similar to Case 1.

Subcase 2.2 $\mathcal{H}_2$ is a red complete subhypergraph.

Denote $\mathcal{H}_3 = \mathcal{H}_2 - \{v_2, \ldots, v_{k-1}\}$. Then we can choose a red path $\mathcal{P}_R'$ (as long as possible) in $\mathcal{H}_3$ starting at v_k such that the number of uncovered vertices by $\mathcal{P}_R'$ in $\mathcal{H}_3$ is at most $k-2$. Then $\mathcal{P}_R$ together with $\mathcal{P}_R'$ form a new red path, such that the path covers all but at most $k-2$ vertices of $\mathcal{K}_n^k$.

By Corollary 7, the result holds for all $n \geqslant k(k-2)+4$. This completes the proof. $\square$

Lemma 12. *Let $n \leqslant 5k-3$ be a positive integer. For any 2-coloring of $\mathcal{K}_n^k$, the vertices can be partitioned into two monochromatic loose paths of distinct colors.*

Proof. If $n \leqslant 2k-2$ then the result is trivial. Now consider the case for $2k-1 \leqslant n \leqslant 5k-4$. Using Lemma 9, there is a monochromatic loose path $\mathcal{P}_t^k$ of length $t \in \{2,3,4\}$ such that $\mathcal{P}_t^k$ covers all but at most $k-1$ vertices of $\mathcal{K}_n^k$. Then $\mathcal{P}_t^k$ together with the path consisting of all remaining vertices will cover all vertices of $\mathcal{K}_n^k$.

If $n = 5k-3 = 5(k-1)+2$, then by Lemma 8, $V(\mathcal{K}_n^k)$ can be covered by two disjoint monochromatic loose paths of distinct colors. This completes the proof. $\square$

The following result is easy to verify, but plays an important role in our proofs.

Lemma 13. *For each 2-coloring of $\mathcal{K}_n^k$, either $V(\mathcal{K}_n^k)$ can be covered by two monochromatic disjoint loose paths of distinct colors, or for each integer $1 \leqslant x \leqslant k-1$ there are two edges of distinct colors such that they have exactly x common vertices.*

Proof. By Lemma 12 we only need to consider the case for $n \geqslant 5k-2$. We prove the result by induction on x. We first show that if two colors are used at least once for the 2-coloring of $\mathcal{K}_n^k$, then there are two edges of distinct colors such that they have exactly one common vertex. Otherwise, we can order randomly the vertices of $\mathcal{K}_n^k$ such that all but at most $k-2$ ordered vertices forms a monochromatic path, but this means that $\mathcal{K}_n^k$ can be partitioned into two disjoint paths of distinct colors.

Now suppose that the result holds for smaller x, that is, there are two edges f and g of distinct colors such that they have exactly x common vertices.

Let $f = \{u_1, u_2, \ldots, u_x, u_{x+1}, \ldots, u_k\}$ and $g = \{u_1, u_2, \ldots, u_x, v_{x+1}, \ldots, v_k\}$. Consider the edge e such that $e \cap f = \{u_1, u_2, \ldots, u_x, u_{x+1}\}$ and $e \cap g = \{u_1, u_2, \ldots, u_x, v_{x+1}\}$. Then either e and f are of distinct colors or e and g are. This completes the proof. $\square$

Lemma 14. *Let $n = 2k-x_1+x_2(k-2)$, where $0 \leqslant x_1 \leqslant k-1$, $x_2 \leqslant \min\{k-x_1+1, k-2\}$ are two positive integers. Then for every 2-coloring of $\mathcal{K}_n^k$, the vertices can be covered by two monochromatic loose paths of distinct colors such that they share at most $k-2$ vertices.*

Proof. Note that the case for $x_1 = 0$ is proved in Lemma 11. Now consider $x_1 \geqslant 1$. By Lemma 13, we pick two edges f and g of distinct colors and $|f \cap g| = x_1$. Similar to the proof of Lemma 11, either we find two desired paths or we fail in the final step. If we fail, then all preceding $x_2 - 1$ new edges are of common color, and the $x_2 - 1$ edges together with f or g, say f, form a monochromatic path $\mathcal{P}$. Since we fail in the final step, $x_2 = k - x_1 + 1$. Note that now all k vertices of g are in $\mathcal{P}$ (x_1 vertices are in f and the other $k - x_1$ are in $x_2 - 1$ new edges). Hence $\mathcal{P}$ and the path consisting of the last $(k-2)$-element vertex subset are two desired paths. $\square$

For two positive integers n and $3 \leqslant k \leqslant 7$, we have the following corollary.

Corollary 15. *Let $3 \leqslant k \leqslant 7$ be an integer. For any 2-coloring of $\mathcal{K}_n^k$, the vertices can be covered by two monochromatic loose paths of distinct colors such that the two paths have at most $k-2$ common vertices.*

Proof. By Lemmas 11 and 12, the statement is straight-forward for $k \leqslant 6$. We now consider the case for $k = 7$. Note that $5k - 2 = 33$ and $k(k-2) + 3 = 38$. By Lemmas 11 and 12 again, we only consider the case for $33 \leqslant n \leqslant 38$. For each $34 \leqslant n \leqslant 38$, there are two integers $x_1 \geqslant 2$ and $0 \leqslant x_2 \leqslant k-2$ such that $n = x_1(k-1) + x_2(k-2) + 2$. By Lemma 11, we are done. If $n = 33$ then $n = 2k - 1 + 4(k-2)$. By Lemma 14, we are done as well. This completes the proof. $\square$

Lemma 16. *Let k, s and t be three positive integers, where $k \geqslant 8, 2 \leqslant s \leqslant k-2, 2 \leqslant t \leqslant k-1$ and $t(k-1) + s \leqslant k(k-2) + 3$. Then one of the indeterminate equations*

$$t(k-1) + s = a(k-1) + b(k-2) + 2 \tag{2}$$

$$t(k-1) + s = 2k - a + b(k-2) \tag{3}$$

has a nonnegative integer solution (a, b) such that $a \geqslant 2, b \leqslant k-2$ if (2) holds, and $0 \leqslant a \leqslant k-1, b \leqslant \min\{k-a+1, k-2\}$ if (3) holds.

Proof. For each given k we consider all possible couples (s,t). For a couple (s,t), if (2) or (3) has a desired solution, then we say that the couple (s,t) fits (2) or (3) for short. It is easy to check that the initial couple $(s,t) = (2,2)$ fits (2).

If $t \leqslant k-2$ then let $(s, t+1)$ be the successor of (s,t). If $t = k-1$ then let $(s+1, 2)$ be the successor of (s,t). Now suppose that (s,t) fits (2) or (3) for some $2 \leqslant s \leqslant k-2, 2 \leqslant t \leqslant k-2$. We shall prove that its successor also fits (2) or (3). We partition our discussion into two cases.

Case 1 The successor is $(s, t+1)$.

If the couple (s,t) fits (2), then $(t+1)(k-1) + s = (a+1)(k-1) + b(k-2) + 2$ follows by adding $(k-1)$ on both sides of (2) and hence $(s, t+1)$ fits (2) as well. If the couple (s,t) fits (3), then $(a-1, b+1)$ solves (3) as desired, unless $a = 0$ or $b = k-2$. If $a = 0$ then $(t+1)(k-1) + s = 2k + b(k-2) + (k-1) = 3(k-1) + b(k-2) + 2$. If $b = k-2$ then $k - a + 1 \geqslant k-2$ since $b \leqslant \min\{k-a+1, k-2\}$. That is, $a \in \{1,2,3\}$. For each a, $(t+1)(k-1) + s = (k-a+1)(k-1) + (a-1)(k-2) + 2$. For the both cases, we have $(s, t+1)$ fits (2). So $(s, t+1)$ fits (2) or (3) as well.

Case 2 The successor is $(s+1, 2)$.

Note that for all $2 \leqslant s \leqslant k-2$, we have $2(k-1)+s+1 = 2k-(k-s-1)+(k-2)$. That is, $(a, b) = (k-s-1, 1)$ solves (3) as desired and $(s+1, 2)$ fits (3).

This completes the proof. □

Lemma 17. *Let $k \geqslant 8$. For any 2-coloring of $\mathcal{K}_n^k$, the vertices can be covered by two monochromatic loose paths such that they share at most $k-2$ vertices.*

Proof. Let $n = t(k-1)+s$, where $t \geqslant 0$ and $0 \leqslant s \leqslant k-2$. If $t \leqslant 2$ then $n \leqslant 2(k-1)+s < 5k-3$. By Lemma 12, we are done. If $t \geqslant k$, then either $k = 4$ or $n \geqslant k(k-1)+s > k(k-2)+4$. By Corollary 15 or Lemma 11, we are done as well. So we only need to consider the case for $3 \leqslant t \leqslant k-1$. By Lemma 10, if $s \geqslant \lfloor \frac{t+1}{2} \rfloor$, then there is a monochromatic path $\mathcal{P}_t^k$ covering $n-s+1$ vertices of $\mathcal{K}_n^k$. Hence $\mathcal{P}_t^k$ and the path consisting of the $s-1$ uncovered vertices cover all vertices of $\mathcal{K}_n^k$. If $s < \lfloor \frac{t+1}{2} \rfloor$, then there is a monochromatic path $\mathcal{P}_{t-1}^k$ covering $n-s-k+2$ vertices of $\mathcal{K}_n^k$. Let $U = V(\mathcal{K}_n^k) \setminus V(\mathcal{P}_{t-1}^k)$ and W be the endpoint set of $\mathcal{P}_{t-1}^k$. Then $|U| = s+k-2$ and $|W| = 2(k-1)$. If $s = 0$ or $s = 1$, then $\mathcal{P}_{t-1}^k$ and the path consisting of U cover all vertices of $\mathcal{K}_n^k$. For $s \geqslant 2$, by Lemma 16, we have $n = a(k-1)+b(k-2)+2$ or $n = 2k-a+b(k-2)$ for desired (a, b). Using Lemmas 11 and 14, we complete the proof. □

Now combining Corollary 15 and Lemma 17 together, we complete the proof of Theorem 4.

Remark. We will spend the remainder of this paper discussing some problems on covering edge-colored hypergraphs.

We proved Theorem 4 by partitioning the vertex set of $\mathcal{K}_n^k$ into two subsets of desired sizes such that the resulting two paths covering $\mathcal{K}_n^k$ share at most $k-2$ vertices. However, for each edge of the two paths, the edge contains at most one common vertex. That is, it is possible that the two paths intersect at many edges. It would be interesting to know if there are two paths covering $\mathcal{K}_n^k$ such that the two paths intersect at most one edge? Lemma 8 says that if $n \equiv 2 \bmod (k-1)$ then the answer is yes.

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