Strengthening \((a, b)\)-Choosability

Results to \((a, b)\)-Paintability

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Abstract

Let \(a, b \in \mathbb{N}\). A graph \(G\) is \((a, b)\)-choosable if for any list assignment \(L\) such that \(|L(v)| \geq a\), there exists a coloring in which each vertex \(v\) receives a set \(C(v)\) of \(b\) colors such that \(C(v) \subseteq L(v)\) and \(C(u) \cap C(w) = \emptyset\) for any \(uw \in E(G)\). In the online version of this problem, on each round, a set of vertices allowed to receive a particular color is marked, and the coloring algorithm chooses an independent subset of these vertices to receive that color. We say \(G\) is \((a, b)\)-paintable if when each vertex \(v\) is allowed to be marked \(a\) times, there is an algorithm to produce a coloring in which each vertex \(v\) receives \(b\) colors such that adjacent vertices receive disjoint sets of colors.

We show that every odd cycle \(C_{2k+1}\) is \((a, b)\)-paintable exactly when it is \((a, b)\)-choosable, which is when \(a \geq 2b + \lceil b/k \rceil\). In 2009, Zhu conjectured that if \(G\) is \((a, 1)\)-paintable, then \(G\) is \((am, m)\)-paintable for any \(m \in \mathbb{N}\). The following results make partial progress towards this conjecture. Strengthening results of Tuza and Voigt, and of Schauz, we prove for any \(m \in \mathbb{N}\) that \(G\) is \((5m, m)\)-paintable when \(G\) is planar. Strengthening work of Tuza and Voigt, and of Hladky, Kral, and Schauz, we prove that for any connected graph \(G\) other than an odd cycle or complete graph and any \(m \in \mathbb{N}\), \(G\) is \((\Delta(G)m, m)\)-paintable.

1 Introduction

Let \(G\) be a graph, and let \(g : V(G) \rightarrow \mathbb{N}\). A \(g\)-fold coloring of \(G\) assigns to each vertex \(v\) a set of \(g(v)\) distinct colors such that adjacent vertices have disjoint sets of colors. When \(g(v) = m\) for all \(v\), we call this an \(m\)-fold coloring. When all colors come from \(\{1, \ldots, k\}\), we call this a \(g\)-fold \(k\)-coloring and say that \(G\) is \((k, g)\)-colorable. When \(G\) is \((k, g)\)-colorable and \(g(v) = m\) for all \(v\), we say that \(G\) is \((k, m)\)-colorable. An ordinary proper \(k\)-coloring is also a 1-fold \(k\)-coloring.

More generally, let \(L\) be a list assignment specifying for each vertex \(v\) a set \(L(v)\) of available colors. A \(g\)-fold \(L\)-coloring of \(G\) is a \(g\)-fold coloring \(\phi\) of \(G\) such that \(\phi(v) \subseteq L(v)\) for each vertex \(v\). A graph \(G\) is \((f, g)\)-choosable if there is a \(g\)-fold \(L\)-coloring for any list assignment \(L\) such that \(|L(v)| \geq f(v)\) for all \(v \in V(G)\). Introduced by Erdős, Rubin,
and Taylor [2], when \( f(v) = k \) and \( g(v) = m \) for all \( v \in V(G) \) and \( G \) is \((f,g)\)-choosable, we say that \( G \) is \((k,m)\)-choosable. When \( g(v) = 1 \) for every \( v \in V(G) \), we shorten \((f,g)\)-choosable to \( f\)-choosable. Erdős, Rubin, and Taylor [2] conjectured that if \( G \) is \( k\)-choosable, then \( G \) is \((km,m)\)-choosable for all \( m \in \mathbb{N} \).

Schauz [5] introduced an online version of choosability, and Zhu [9] generalized \((f,g)\)-choosability. Vertices are colored in rounds, where the coloring algorithm must decide on round \( i \) which vertices will receive color \( i \), without knowing which colors will appear on any vertices later in the lists. This concept is formalized in the following game.

**Definition 1.1.** Let \( G \) be a graph where each vertex \( v \) is assigned a nonnegative number \( f(v) \) of tokens and a nonnegative number \( g(v) \) specifying how many times \( v \) must be colored. The \((f,g)\)-paintability game is played by two players: Lister and Painter. Each round, Lister marks a nonempty subset \( M \) of vertices that have been colored fewer than \( g(v) \) times; every vertex in \( M \) loses one token. Painter responds by coloring an independent subset \( D \) of \( M \); every vertex of \( D \) gains a color distinct from those used on earlier rounds. Lister wins the game by marking a vertex that has no tokens remaining, and Painter wins by coloring each vertex \( v \) on \( g(v) \) distinct rounds.

We say \( G \) is \((f,g)\)-paintable when Painter has a winning strategy on \( G \) in the \((f,g)\)-paintability game. If \( f(v) = a \) and \( g(v) = b \) for every \( v \in V(G) \) and Painter has a winning strategy, then we say \( G \) is \((a,b)\)-paintable. We say \( G \) is degree-\( m \)-paintable if \( G \) is \((f,m)\)-paintable where \( f(v) = d(v)m \) for all \( v \). When \( m = 1 \), we simply say “degree-paintable”.

Always, if \( G \) is not \((f,g)\)-choosable, then \( G \) is not \((f,g)\)-paintable since Lister can mimic a bad list assignment \( L \) by marking in round \( i \) the set \( \{ v \in V(G) : i \in L(v) \} \). Thus \((f,g)\)-paintability implies \((f,g)\)-choosability. In [9], Zhu made the following conjecture.

**Conjecture 1.2 ([9]).** If \( G \) is \( a\)-paintable, then \( G \) is \((am,m)\)-paintable for all \( m \in \mathbb{Z} \).

Mahoney, Meng, and Zhu [4] proved that Conjecture 1.2 is true for all 2-paintable graphs.

Thomassen [6] proved that every planar graph is 5-choosable. Tuza and Voigt [7] strengthened this result by proving that planar graphs are \((5m,m)\)-choosable for all \( m \in \mathbb{N} \). Schauz [5] strengthened Thomassen’s result in a different way by proving that planar graphs are 5-paintable.

In Section 2, we prove that planar graphs are \((5m,m)\)-paintable for all \( m \geq 1 \), which strengthens the previous results and makes partial progress towards Conjecture 1.2.

Let \( G \) be a connected graph other than an odd cycle or a complete graph, and let \( \Delta(G) \) denote the maximum degree of \( G \). Brooks’ Theorem [1] states that \( G \) is \( \Delta(G) \)-colorable. Stronger versions of Brooks’ Theorem are proved by Tuza and Voigt [7] and by Hladky, Kral, and Schauz [3]. In Section 3, we prove that \( G \) is \((\Delta(G)m,m)\)-paintable, strengthening both results and making partial progress towards Conjecture 1.2.

In [8], Voigt proved that if \( C_{2k+1} \) is \((a,b)\)-choosable, then \( a \geq 2b + \lceil b/k \rceil \). We conclude the introduction by strengthening this result, characterizing the \((a,b)\)-paintability and \((a,b)\)-choosability of odd cycles.

**Theorem 1.3.** For \( k \geq 1 \), the following are equivalent:

(a) \( C_{2k+1} \) is \((a,b)\)-paintable.
(b) \( C_{2k+1} \) is \((a, b)\)-choosable.

(c) \( a \geq 2b + \lceil b/k \rceil \).

**Proof.** (a) \( (a, b) \)-paintability implies \((a, b)\)-choosability.

(b) \( (c) \Rightarrow (b) \) Always proved in [8], we provide a short proof for completeness. Suppose \( C_{2k+1} \) is \((a, b)\)-choosable. Consider the list assignment where \( L(v) = \{1, \ldots, a\} \) for each vertex \( v \). Each color can be used on at most \( k \) vertices. Since each vertex must receive \( b \) colors, we have that the lists must have size at least \( (2k+1)b/k \).

(c) \( (b) \Rightarrow (a) \) Give the cycle a consistent orientation, and label the vertices \( v_0, \ldots, v_{2k} \). Consider all indices modulo \( 2k + 1 \). Let \( M \) be the set Lister marks. If \(|M| < 2k + 1\), then the graph induced by the marked set is a linear forest. Painter colors vertices greedily along each path starting at the tail.

If \(|M| = 2k + 1\), then we keep track of how many times moves of this type have occurred in the game. If a move of this type has been played \( i \) times before \((\text{mod} \ 2k + 1)\), then Painter colors \( \{v_i, v_{i+2}, \ldots, v_{i+2k-2}\} \). There are exactly \( 2k + 1 \) distinct independent sets of size \( k \) for \( C_{2k+1} \). In this strategy, Painter balances which of these independent sets is colored by cycling through all possible choices.

Suppose Lister can win against this particular Painter strategy when each vertex has \( 2b + \lceil b/k \rceil \) tokens. Let Lister’s marked sets be \( M_1, \ldots, M_t \), where Lister wins on round \( t \), and let Painter’s responses be \( D_1, \ldots, D_{t-1} \). Note that when Lister marks the set \( M_t \), some vertex will have been marked \( 2b + \lceil b/k \rceil + 1 \) times. In particular \( t \geq 2b + \lceil b/k \rceil \). If a vertex \( v_i \) is marked in the set \( M_j \), then Painter strategy implies that \( v_{i-1} \) was also marked in \( M_j \). Since \( v_{i-1} \) is colored at most \( b \) times, there must be at least \( \lceil b/k \rceil + 1 \) rounds where both \( v_i \) and \( v_{i-1} \) are marked and not colored. This only happens once every \( 2k + 1 \) rounds when both vertices are marked. Thus the number of rounds where all vertices are marked is at least \( \lceil b/k \rceil (2k+1)+1 \), which is greater than \( 2b + \lceil b/k \rceil \). So there are at least \( 2b + \lceil b/k \rceil + 1 \) rounds in which all vertices are marked. If any of these rounds were preceded by a round in which not all vertices were marked, then one vertex would be marked \( 2b + \lceil b/k \rceil + 1 \) times earlier than round \( t \). Thus the first \( 2b + \lceil b/k \rceil + 1 \) marked sets must all be \( V(C_{2k+1}) \). After marking all vertices just \( 2b + \lceil b/k \rceil \) times, Painter’s strategy ensures that every vertex is colored \( b \) times. Thus there is no vertex for Lister to mark on round \( 2b + \lceil b/k \rceil + 1 \), and thus there is no way for Lister to win the game against this Painter strategy. \( \square \)

## 2 Planar Graphs

The following lemma is a generalization of Lemma 2.2 in [5].

**Lemma 2.1** (Edge Lemma). If \( G \) is \((f, g)\)-paintable and \( uv \notin E(G) \), then \( G \cup uv \) is \((f', g)\)-paintable where \( f'(w) = \begin{cases} f(v) + f(u), & \text{if } w = v \\ f(u), & \text{otherwise} \end{cases} \).

**Proof.** Let \( S \) be a winning strategy for Painter in the \((f, g)\)-paintability game on \( G \). In the \((f', g')\)-paintability game on \( G \cup uv \), whenever Lister marks \( v \), we sacrifice a token on \( v \) by having Painter respond to the marked set \( M - v \). At most \( f(u) \) tokens are sacrificed on \( v \). In rounds when \( u \) is not marked, Painter may respond according to \( S \) because any
response in $G$ is an independent set in $G \cup uv$. At least $f(v)$ tokens are available for moves of this type, so $g(v)$ colors will be assigned to $v$ by playing according to $\mathcal{S}$. \hfill \Box

The following lemma is a generalization of Lemma 2.5 in [5].

**Lemma 2.2 (Merge Lemma).** Let $G = G_1 \cup G_2$, and let $T = V(G_1) \cap V(G_2)$. If $G_i$ is $(f_i, g_i)$-paintable and $f_2(v) = g_2(v) = g_1(v)$ for all $v \in T$, then $G$ is $(f, g)$-paintable where $f(v) = \begin{cases} f_1(v), & \text{if } v \in V(G_1) \\ f_2(v), & \text{otherwise} \end{cases}$ and $g(v) = \begin{cases} g_1(v), & \text{if } v \in V(G_1) \\ g_2(v), & \text{otherwise} \end{cases}$.

**Proof.** We use induction on $\sum g(v)$. For the basis step, if $\sum g(v) = 0$, then $G$ is trivially $(f, g)$-paintable. Now consider $\sum g(v) > 0$.

Let $M$ be the set marked by Lister. For $i \in \{1, 2\}$, let $\mathcal{S}_i$ be a winning strategy for Painter in $G_i$ under token assignment $f_i$, and let $M_i = M \cap V(G_i)$. Let $D_1$ be the response to $M_1$ in $G_1$ according to $\mathcal{S}_1$. In $G_2$, Painter responds to the marked set $M_2 - (T - D_1)$ according to $\mathcal{S}_2$. We interpret vertices of $(M - D_1) \cap T$ as having lost a token in $G_1$ but not in $G_2$. Because $f_2(v) = g_2(v)$ for all $v \in T$, it must be the case that $(D_1 \cap T) \subseteq D_2$.

Thus $D_1 \cup D_2$ is an independent set; Painter now colors $D_1 \cup D_2$.

Let $G^* = G - (D_1 \cup D_2)$. To make use of the induction hypothesis, we define the following functions:

$$f_1^i(v) = \begin{cases} f_1(v) - 1, & \text{if } v \in M \\ f_1(v), & \text{otherwise} \end{cases}$$

$$f_2^i(v) = \begin{cases} f_2(v) - 1, & \text{if } v \in M_i - (T - D_1) \\ f_2(v), & \text{otherwise} \end{cases}$$

For $i \in \{1, 2\}$, $g_i^i(v) = \begin{cases} g_i(v) - 1, & \text{if } v \in D_i \\ g_i(v), & \text{otherwise} \end{cases}$.

Because $D_1$ and $D_2$ were chosen according to a winning strategies in $G_1$ and in $G_2$, we have that $G_i$ is $(f_i^i, g_i^i)$-paintable for $i \in \{1, 2\}$ and $f_2^i(v) = g_2^i(v) = g_1^i(v)$ for all $v \in T$. Since $M \neq \emptyset$, we may assume that $D_1 \cup D_2 \neq \emptyset$. Thus $\sum g(v)$ decreases, and by induction this yields $G^*$ is $(f^*, g^*)$-paintable where $f^*(v) = \begin{cases} f_1^1(v), & \text{if } v \in V(G_1) \\ f_2^2(v), & \text{otherwise} \end{cases}$ and $g^*(v) = \begin{cases} g_1^1(v), & \text{if } v \in V(G_1) \\ g_2^2(v), & \text{otherwise} \end{cases}$. This proves that Painter’s response to $M$ is a winning move, and thus $G$ is $(f, g)$-paintable. \hfill \Box

We now prove the main theorem of this section.

**Theorem 2.3.** Planar graphs are $(5m, m)$-paintable for all $m \in \mathbb{N}$.

**Proof.** We proceed using an argument mirroring that of Thomassen [6] and of Schauz [5]. First, we restrict our attention to weak triangulations of planar graphs since adding edges only makes coloring the graph more difficult for Painter. Let $G$ be a planar graph of order $n$ with vertices $v_1, \ldots, v_p$ in clockwise order on the unbounded face. By induction on $n$, we prove a stronger result:

$$G \text{ is } (f, m)\text{-paintable when } f(v) = \begin{cases} m, & \text{if } v = v_p \\ 2m, & \text{if } v = v_1 \\ 3m, & \text{if } v = v_i \text{ for } 1 < i < p \\ 5m, & \text{otherwise} \end{cases}$$

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Case 1: There is a chord $v_i v_j$ connecting two vertices on the unbounded face. Let $G_1$ be the graph induced by the vertices of the cycle containing $v_1$ and $v_p$ and by the vertices on the interior of this cycle. Let $G_2$ have vertex set $(V(G) - V(G_1)) \cup \{v_i, v_j\}$ and edge set $E(G) - E(G_1)$. $f_1(v) = f(v)$ for all $v \in V(G_1)$, and $f_2(v) = \begin{cases} f(v), & \text{if } v \in V(G_2) - \{v_i, v_j\} \\ m, & \text{if } v \in \{v_i, v_j\} \end{cases}$.

By the induction hypothesis $G_1$ is $(f_1, m)$-paintable, and $G_2$ is $(f_2, m)$-paintable by first applying Lemma 2.1 to the edge $v_i v_j$ and then using the induction hypothesis. Lemma 2.2 then implies that $G$ is $(f, m)$-paintable.

Case 2: The unbounded face is chordless. Consider $N(v_2)$. Since all bounded faces are triangles, there exists a path $v_1, u_1, \ldots, u_t, v_3$. Let $U = \{u_1, \ldots, u_t\}$, and let $G' = G - v_2$. Applying the induction hypothesis to $G'$, we show that if each $u \in U$ is given $2m$ additional tokens, then we can extend a winning strategy for Painter on $G'$ to a winning strategy on $G$.

Suppose Lister marks a set $M$, and let $S$ be a winning strategy for Painter in $G'$. Let $D$ be Painter’s response to the marked set $M - \{v_2\}$ according to $S$. If $v_2 \not\in M$, then Painter colors $D$. If $v_2 \in M$ and $v_1 \in D$, then Painter colors $D$ and sacrifices a token on $v_2$. When $v_2 \in M$, and $v_1 \not\in D$, then Painter obtains the response $D'$ to the marked set $(M - \{v_2\}) - U$ according to $S$ and colors $v_2$ if $v_3 \not\in D'$. Each vertex of $U$ loses at most $2m$ tokens from moves of this type. Also, $v_2$ is marked and not colored at most $m$ times because of $v_3 \in D'$. Finally, $v_3$ never loses tokens because of $v_2$. Therefore, every vertex is colored $m$ times before it runs out of tokens. 

\section{Brooks’ Theorem}

Brooks’ Theorem [1] states that a connected graph $G$ is $\Delta(G)$-colorable except when $G$ is an odd cycle or a complete graph. Tuza and Voigt [7] strengthened this result by proving that such a graph $G$ is $(\Delta(G)m, m)$-choosable. Hladky, Kral, and Schauz [3] proved that such a graph $G$ is $\Delta(G)$-paintable. The following theorem strengthens both results.

**Theorem 3.1.** If $G$ is a connected graph other than an odd cycle or a complete graph, then $G$ is $(\Delta(G)m, m)$-paintable for all $m \in \mathbb{N}$.

Let $N_G[v]$ denote the closed neighborhood $N_G(v) \cup \{v\}$. We will make use of a degeneracy argument of Zhu [9] and a well-known structural lemma of Erdős, Rubin, and Taylor [2]. A block in a graph is a maximal 2-connected subgraph or a cut-edge.

**Proposition 3.2 ([9]).** Let $G$ be a graph with token-color assignments $f$ and $g$. If $f(v) \geq \sum_{u \in N_G[v]} g(u)$, then $G$ is $(f, g)$-paintable if and only if $G - v$ is $(f', g')$-paintable where $f'$ and $g'$ are the restrictions of $f$ and $g$ to $G - v$.

**Lemma 3.3 ([2]).** If $G$ is a 2-connected graph that is not an odd cycle or a complete graph, then $G$ contains an induced even cycle having at most one chord.

We now show that the induced subgraph obtained from the conclusion of Lemma 3.3 is degree-$m$-paintable for all $m \in \mathbb{N}$.

**Lemma 3.4.** An even cycle with at most one chord is degree-$m$-paintable for all $m \in \mathbb{N}$.
Proof. **Case 1:** Let $G$ be a chordless even cycle. Zhu [9] proved that $C_{2n}$ is $(2m, m)$-paintable for $n \geq 2, m \in \mathbb{N}$.

**Case 2:** Let $G$ be an even cycle with exactly one chord. Let $v_1, \ldots, v_n$ be the vertices of this cycle in clockwise order, and suppose $v_1v_2$ is the chord. Consider the graph $G'$ obtained from $G$ by removing the edge $v_nv_1$. Let $f'$ be a token assignment obtained from $f$ by removing $2m$ tokens from $v_1$. By Lemma 2.1, if $G'$ is $(f', g)$-paintable, then $G$ is degree-$m$-paintable. In $G'$, we repeatedly apply Proposition 3.2 to $V(G')$ in the order $v_n, v_{n-1}, \ldots, v_1$. At each step, the vertex being removed has at least as many tokens as the number of times it and its neighbors must be colored, therefore $G'$ is $(f', g)$-paintable. \qed

The next lemma allows us to extend good strategies on an induced subgraph to a larger graph.

**Lemma 3.5.** Given a connected graph $G$, if there exists an induced subgraph $H$ that is degree-$m$-paintable, then $G$ is degree-$m$-paintable for all $m \in \mathbb{N}$.

**Proof.** If $H = G$, there is nothing to show, so suppose $V(G) - V(H) \neq \emptyset$, and let $U = \{u_1, \ldots, u_t\} = V(G) - V(H)$. Let $S$ be a winning degree-$m$-paintability strategy for Painter on $H$.

Let $M$ be the set that Lister marks. Let $D$ be the independent subset of $M \cap U$ chosen greedily with respect to the ordering $u_1, \ldots, u_t$. According to $S$, Painter obtains a response $D'$ in $H$ to the marked set $(M \cap V(H)) - N(D)$. We sacrifice a token on each vertex of $M \cap V(H) \cap N(D)$, and Painter colors $D \cup D'$. Note that $D \cup D'$ is an independent set because we forbid coloring any neighbors of vertices in $D$.

Each $v \in V(H)$ sacrifices at most $m$ tokens for any neighbor outside of $H$, which guarantees that at least $d_H(v)m$ tokens are available for the strategy $S$. Each $u \in U$ is marked and not colored at most $m$ times for each earlier neighbor, which always leaves at least $m$ tokens available to color $u$ when it has no more incomplete earlier neighbors. Therefore $G$ is degree-$m$-paintable. \qed

Lemmas 3.3, 3.4, and 3.5 imply that every block of a non-degree-$m$-paintable connected graph must be an odd cycle or a clique. A connected graph in which every block is an odd cycle or a clique is called a **Gallai tree**.

**Theorem 3.6.** Given $m \in \mathbb{N}$, a connected graph $G$ is degree-$m$-paintable if and only if $G$ is not a Gallai tree.

**Proof.** If $G$ is a Gallai tree, it is not degree-$m$-choosable [7], and hence, not degree-$m$-paintable.

When $G$ is not a Gallai tree, there exists a block $B$ that is not a complete graph or an odd cycle. By Lemma 3.3, $B$ contains an induced even cycle with at most one chord. Lemma 3.4 implies that $B$ is degree-$m$-paintable. Lastly, Lemma 3.5 implies that $G$ is degree-$m$-paintable. \qed

We conclude by proving Theorem 3.1.

**Proof of Theorem 3.1.** If $G$ is not a Gallai tree, then Theorem 3.6 implies $(\Delta(G)m, m)$-paintability. We may assume that $G$ is a Gallai tree with at least two blocks. Thus $G$ is not $\Delta(G)$-regular, and every vertex of maximum degree is a cut-vertex. Thus every subgraph of $G$ contains a vertex of degree at most $\Delta(G) - 1$, so Proposition 3.2 implies $G$ is $(\Delta(G)m, m)$-paintable. \qed


References


