A Euclidean Ramsey result in the plane

Sergei Tsaturian
Department of Mathematics
University of Manitoba
Winnipeg, Canada
s.tsaturian@gmail.com

Submitted: Jul 9, 2017; Accepted: Nov 12, 2017; Published: Nov 24, 2017
Mathematics Subject Classification: 05D10

Abstract

An old question in Euclidean Ramsey theory asks, if the points in the plane are red-blue coloured, does there always exist a red pair of points at unit distance or five blue points in line separated by unit distances? An elementary proof answers this question in the affirmative.

1 Introduction

Many problems in Euclidean Ramsey theory ask, for some \( d \in \mathbb{Z}^+ \), if the \( d \)-dimensional Euclidean space \( \mathbb{E}^d \) is coloured with \( r \geq 2 \) colours, does there exist a colour class containing some desired geometric structure? Research in Euclidean Ramsey theory was surveyed in [4–6] by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus; for a more recent survey, see Graham [7].

Say that two geometric configurations are congruent iff there exists an isometry (distance preserving bijection) between them. For \( d \in \mathbb{Z}^+ \), and geometric configurations \( F_1, F_2 \), let the notation \( \mathbb{E}^d \to (F_1, F_2) \) mean that for any red-blue coloring of \( \mathbb{E}^d \), either the red points contain a congruent copy of \( F_1 \), or the blue points contain a congruent copy of \( F_2 \).

For a positive integer \( i \), denote by \( \ell_i \) the configuration of \( i \) collinear points with distance 1 between consecutive points. One of the results in [5] states that

\[
\mathbb{E}^2 \to (\ell_2, \ell_4).
\]

(1)

In the same paper, it was asked if \( \mathbb{E}^2 \to (\ell_2, \ell_5) \), or perhaps a weaker result holds: \( \mathbb{E}^3 \to (\ell_2, \ell_5) \).

The result (1) was generalised by Juhász [10], who proved that if \( T_4 \) is any configuration of 4 points, then \( \mathbb{E}^2 \to (\ell_2, T_4) \). Juhász [9] informed me that Iván’s thesis [8] contains
a proof that for any configuration $T_5$ of 5 points, $\mathbb{E}^3 \rightarrow (\ell_2, T_5)$ (which implies that $\mathbb{E}^3 \rightarrow (\ell_2, \ell_5)$). Arman and Tsaturian [1] proved that $\mathbb{E}^3 \rightarrow (\ell_2, \ell_6)$.

In this paper, it is proved that $\mathbb{E}^2 \rightarrow (\ell_2, \ell_5)$:

**Theorem 1.** Let the Euclidean space $\mathbb{E}^2$ be coloured in red and blue so that there are no two red points distance 1 apart. Then there exist five blue points that form an $\ell_5$.

The existence of a $k$, such that $\mathbb{E}^2 \not\rightarrow (\ell_2, \ell_k)$, was first noted by Erdős and Graham [3], who mention the upper bound of “10000000, more or less”. A more precise bound for $k = 10^{10}$ follows from a recent result of Conlon and Fox [2], who showed that for all $n \geq 2$, $\mathbb{E}^n \not\rightarrow (\ell_2, \ell_{10^n})$.

**2 Proof of Theorem 1**

The proof is by contradiction; it is assumed that there are no five blue points forming an $\ell_5$. The following lemmas are needed.

**Lemma 2.** Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. If there is no blue $\ell_5$, then there are no three blue points forming an equilateral triangle with side length 3 and with a red centre.

**Proof.** Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Suppose that blue points $A$, $B$ and $C$ form an equilateral triangle with side length 3 and with red centre $O$. Consider the part of the unit triangular lattice shown in Figure 1(a). The points $D$, $E$, $F$, $G$ are blue, since they are distance 1 apart from $O$. The point $X$ is red; otherwise $XADEB$ is a red $\ell_5$. Similarly, $Y$ is red (to prevent red $YAFGC$). Then $X$ and $Y$ are two red points distance 1 apart, which contradicts the assumption. □

Figure 1: Red points are denoted by diamonds, blue points are denoted by discs.
Lemma 3. Let \( \mathbb{E}^2 \) be coloured in red and blue so that there is no red \( \ell_2 \). If there is no blue \( \ell_5 \), then there are no three red points forming an equilateral triangle with side length 3 and with a red centre.

Proof. Suppose that \( \mathbb{E}^2 \) is coloured in red and blue so that there is no red \( \ell_2 \) and no blue \( \ell_5 \). Suppose that blue points \( A, B \) and \( C \) form an equilateral triangle with side length 3 and with red centre \( O \). Let \( A', B', C' \) be the images of \( A, B \) and \( C \), respectively, under a rotation about \( O \) so that \( AA' = BB' = CC' = 1 \) (see Figure 1(b)). Then \( A', B', C' \) are blue and form an equilateral triangle with side length 3 and red center \( O \), which contradicts the result of Lemma 2. \( \square \)

Define \( \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7 \) to be the configurations of three, four, five, six and seven points (respectively), depicted in Figure 2 (all the smallest distances between the points are equal to \( \sqrt{3} \)).

![Figure 2](image)

Lemma 4. Let \( \mathbb{E}^2 \) be coloured in red and blue so that there is no red \( \ell_2 \). If there is no blue \( \ell_5 \), then there are no seven red points forming a \( \mathcal{T}_7 \).

Proof. Suppose that \( \mathbb{E}^2 \) is coloured in red and blue so that there is no red \( \ell_2 \) and no blue \( \ell_5 \). Suppose that \( A, B, C, D, E, F \) and \( G \) are red points forming a \( \mathcal{T}_7 \) (as in Figure 3). Let \( X \) be the reflection of \( F \) in \( BC \). Let \( X', A', F' \) be the images of \( X, A, F \), respectively, under the clockwise rotation about \( B \) such that \( XX' = AA' = FF' = 1 \). Since \( A \) and \( F \) are red, \( A' \) and \( F' \) are blue. If \( X' \) is blue, then \( X'A'F' \) is a blue equilateral triangle with side length 3 and red center \( B \), which contradicts the result of Lemma 2. Therefore, \( X' \) is red.

Let \( X'', D'', F'' \) be the images of \( X, D, F \), respectively, under the clockwise rotation about \( C \) such that \( XX'' = DD'' = FF'' = 1 \). Since \( D \) and \( F \) are red, \( D'' \) and \( F'' \) are blue. If \( X'' \) is blue, then \( X''D''F'' \) is a blue equilateral triangle with side length 3 and red center \( C \), which contradicts the result of Lemma 2. Therefore, \( X'' \) is red. Consider the clockwise rotation through 60° about \( X \). This rotation sends \( C \) to \( B \), and so every
point on the circle with radius $\sqrt{3}$ centered at $C$ is sent to the corresponding point on the circle with radius $\sqrt{3}$ centered at $B$; in particular, $X'$ can be viewed as the image of $X''$. Therefore $XX'X''$ is a unit equilateral triangle, hence $X'X''$ is a red $\ell_2$, which contradicts the assumption of the lemma.

\[ \square \]

**Figure 3**

**Lemma 5.** Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. Let $A$, $B$, $C$ be three red points forming a $\Theta_3$. If there is no blue $\ell_5$, then there exists a red $\Theta_6$ that contains $\{A, B, C\}$ as a subset.

**Proof.** Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Let $A$, $B$, $C$ be three red points forming a $\Theta_3$. Consider the unit triangular lattice depicted in Figure 4.

Suppose that there is no red point $D$ such that $A$, $B$, $C$, $D$ form a $\Theta_4$. Then points $X$, $Y$, $Z$ are blue. Points $E$, $F$, $G$, $H$, $I$, $J$ are blue, since each of them is distance 1 apart from a red point. If the point $K$ is red, then the points $L$ and $M$ are blue and $LMYGH$ is a blue $\ell_5$. Therefore, $K$ is blue. Then $N$ is red (otherwise $KJIZN$ is a blue $\ell_5$), hence $P$ and $Q$ are blue, which leads to a blue $\ell_5$ $PQFEX$. A contradiction is obtained, therefore there exists a red point $D$ such that $A$, $B$, $C$, $D$ form a $\Theta_4$.

\[ \square \]

**Figure 4**
Let $A, B, C, D$ form a red $\mathfrak{T}_4$. Consider the part of the unit triangular lattice depicted in Figure 5. Suppose that there is no red point $E$ such that $A, B, C, D, E$ form a $\mathfrak{T}_5$. Then the points $X, F$ and $G$ are blue. Points $H, I, K, L, M, N$ are blue, since each of them is distance 1 apart from a red point. Point $P$ is red (otherwise $FHIGP$ is a blue $\ell_5$), therefore $Q$ and $R$ are blue. Then $X, N, M, Q, R$ form a blue $\ell_5$, which gives a contradiction. Hence, there exists a red point $E$ such that $A, B, C, D, E$ form a $\mathfrak{T}_5$.

Figure 5

Let $A, B, C, D, E$ form a $\mathfrak{T}_5$ (Figure 6). Suppose that $F$ is blue. By Lemma 3, points $X$ and $Y$ are blue (otherwise $X, E, C$ (or $Y, A, D$) form a red triangle with side length 3 and red center $B$). Points $G, H, I, J, K, L, M, N$ are blue, since each one of them is at distance 1 from a red point. If point $P$ is blue, then $Q$ is red (otherwise $QPKLF$ is a blue $\ell_5$), $U$ and $T$ are blue and form a blue $\ell_5$ with points $G, H$ and $X$. Therefore, $P$ is red. Similarly, $R$ is red (otherwise $S$ is red and $VWJIY$ is a blue $\ell_5$). Then $A, B, C, D, E, P, R$ form a red $\mathfrak{T}_7$, which is not possible by Lemma 4. Therefore, $F$ is red and $A, B, C, D, E, F$ form a red $\mathfrak{T}_6$.

Figure 6

Lemma 6. Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. Let $\mathcal{L}$ be a unit triangular lattice that contains three red points forming a $\mathfrak{T}_3$. If there is no blue $\ell_5$ in $\mathbb{E}^2$, then the colouring of $\mathcal{L}$ is unique (up to translation or rotation by a multiple of 60°), and is depicted in Figure 7.
Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$. Suppose there exist three red points of $\mathcal{L}$ that form a $\mathcal{T}_3$. By Lemma 5, it may be assumed that there is a red $\mathcal{T}_6$. Denote its points by $A, B, C, D, E, F$ (see Figure 8). It will be proved that the translate $A'B'C'D'E'F'$ of $ABCDEF$ by the vector of length 5 collinear to $\overrightarrow{AD}$ is red.

Consider the points shown in Figure 8. Since $A, D$ and $F$ are red, by Lemma 3, $I$ is blue. Since $C, F$ and $D$ are red, by Lemma 3, $J$ is blue. Points $K, L, M, N$ are blue, since each one is distance 1 apart from a red point. If $R$ is red, then both $P$ and $Q$ are blue and form a blue $\ell_5$ with $K, L$ and $I$. Therefore $R$ is blue. Then the point $A'$ is red (otherwise $A'JNMR$ is a blue $\ell_5$).

Since $S_1, S_2, S_3, S_4$ are blue (as distance 1 apart from red points $D$ and $A'$), $B'$ is red. Similarly, $F'$ is red. Points $V$ and $W$ are blue as they are distance 1 apart from $C$. Points $U$ is blue by Lemma 3 (since $A, D$ and $B$ are red). If $X$ is red, then $X_1$ and $X_2$ are blue and a blue $\ell_5$ $UVWX_1X_2$ is formed. Therefore, $X$ is blue. Similarly, $Y$ is blue. By Lemma 5, $A'B'F'$ must be contained in a red $\mathcal{T}_6$, and since $X$ and $Y$ are blue, the only possible such $\mathcal{T}_6$ is $A'B'C'D'E'F'$. Hence, $A', B', C', D', E', F'$ are blue.

Similarly, the translates of $ABCDEF$ by vectors of length 5 collinear to $\overrightarrow{EB}$ and $\overrightarrow{CF}$ are red. By repeatedly applying the same argument to the new red translates, it can be seen that all the translates of $ABCDEF$ by a multiple of 5 in $\mathcal{L}$ are red. All the other points are blue, as each one is distance 1 apart from a red point. Hence, the colouring as in Figure 7 is obtained. \qed
Lemma 7. Let $\mathbb{E}^2$ be coloured in red and blue so that there is no red $\ell_2$. Let $\mathcal{L}$ be a unit triangular lattice that does not contain three red points forming a $\mathcal{T}_3$. If there is no blue $\ell_5$ in $\mathbb{E}^2$, then the colouring of $\mathcal{L}$ is unique (up to translation or rotation by a multiple of $60^\circ$), and is depicted in Figure 9.

![Figure 8](image-url)

**Figure 8**

Proof. Suppose that $\mathbb{E}^2$ is coloured in red and blue so that there is no red $\ell_2$ and no blue $\ell_5$.

If $\mathcal{L}$ does not contain a red point, then any $\ell_5$ is blue, therefore $\mathcal{L}$ contains a red point $A$. By Lemma 2, one of the points of $\mathcal{L}$ at distance $\sqrt{3}$ to $A$ is red (otherwise the three such points form a blue triangle with side length 3 and red centre $A$). Denote this point by $B$ (Figure 10). Since $\mathcal{L}$ does not contain a red $\mathcal{T}_3$, the points $D$ and $G$ are blue. Points $E$, $F$, $I$, $H$, $K$, $J$ are blue, since they are distance 1 apart from $B$. Then the point $B'$
is red (otherwise blue $\ell_5$ $DEFGB'$ is formed). Point $N$ is 1 apart from $B'$, hence blue. Then $C$ and $A'$ are red (otherwise a blue $\ell_5$ is formed).

By repeating the same argument for points $B$ and $C$, $B$ and $A$ (instead of $A$ and $B$), and so on, it can be shown that any node of $L$ on the line $AB$ is red. Similarly, since $A'$ and $B'$ are both red, any node of $L$ on the line $A'B'$ is red. By the same argument, $A''$, $B''$ and any node on the line containing them is red; $A'''$, $B'''$ and any node on the line containing them is red, and so on. By colouring all point distance 1 apart form red points blue, the colouring in Figure 9 is obtained.

\[\text{Figure 10}\]

Proof of Theorem 1. Let the Euclidean space $E^2$ be coloured in red and blue so that there are no two red points distance 1 apart. Suppose that there are no five blue points that form an $\ell_5$. Then there is a red point $A$. Consider two points $B$ and $C$, both distance 5 apart from $A$, such that $|BC| = 1$. At least one of the points $B$ and $C$ (say, $B$) is blue. Consider the unit triangular lattice $L$ that contains $A$ and $B$. By Lemma 6 and Lemma 7, $L$ is coloured either as in Figure 7 or as in Figure 9. But neither one of the colourings contains two points of different colour distance 5 apart, which gives a contradiction. Therefore, there exist five blue points that form an $\ell_5$.

Acknowledgements

I would like to thank Ron Graham and Rozália Juhász for providing information about the current state of the problem. I would like to thank Andrii Arman and David Gunderson for valuable comments and suggestions.

References


