A Short Proof of Moll’s Minimal Conjecture

Lun Lv
School of Sciences
Hebei University of Science and Technology
Shijiazhuang 050018, P.R. China
klunlv@163.com
Submitted: Mar 1, 2017; Accepted: Sep 25, 2017; Published: Oct 6, 2017
Mathematics Subject Classifications: 05A20, 11B83, 33F99

Abstract
We give a short proof of Moll’s minimal conjecture, which has been confirmed by Chen and Xia.

Keywords: Boros-Moll polynomial; Moll’s minimal conjecture; spiral property

1 Introduction
The Boros-Moll polynomials, denoted by $P_m(a)$, arise in the evaluation of the following quartic integral, see [2–6,12]. For any $a > -1$ and any nonnegative integer $m$,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a),$$

where

$$P_m(a) = 2^{-2m} \sum_{k=0}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k. \tag{1.1}$$

Let $d_l(m)$ be the coefficient of $a^l$ in $P_m(a)$. Then (1.1) gives

$$d_l(m) = 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{l}. \tag{1.2}$$

Much progress has been made since Boros and Moll [1] proved the positivity of the coefficients of $P_m(a)$. Boros and Moll [4] have proved that the sequence $\{d_l(m)\}_{0 \leq l \leq m}$ is unimodal. The log-concavity of the sequence $\{d_l(m)\}_{1 \leq l \leq m-1}$ was conjectured by Moll [12], and it was proved by Kauers and Paule [11] based on recurrence relations. Chen and Xia [9] showed that the sequence $d_l(m)$ satisfies the strongly ratio monotone property...
which implies the log-concavity and the spiral property. Chen and Gu [7] proved the reverse ultra log-concavity of the Boros-Moll polynomials. By introducing the structure of partially 2-colored permutations, Chen, Pang and Qu [8] found a combinatorial proof of the log-concavity of the Boros-Moll polynomials. Moll also posed a conjecture that is stronger than the log-concavity of the polynomials \( P_m(a) \). This conjecture was called Moll’s minimum conjecture, and has been confirmed by Chen and Xia [10].

The main objective of this paper is to give a short proof of the following equivalent form of Moll’s minimal conjecture, which was confirmed by Chen and Xia [10].

\[ \text{Theorem 1.1 (Theorem 2.1 [10]).} \text{ Given } m \geq 2, \text{ for } 1 \leq l \leq m, \text{ } l(l+1)(d_l^2(m) - d_{l+1}(m)d_{l-1}(m)) \text{ attains its minimum at } l = m \text{ with } m(m+1)d_m^2(m). \]

2 The Proof of Theorem 1.1

Chen and Gu [7] proved the following theorem, which gave a lower bound of \( \frac{d_l^2(m)}{d_{l+1}(m)d_{l-1}(m)} \).

\[ \text{Theorem 2.1 (Theorem 1.2 [7]).} \text{ For } m \geq 2 \text{ and } 1 \leq l \leq m-1, \text{ we have} \]

\[ \frac{d_l^2(m)}{d_{l+1}(m)d_{l-1}(m)} > \frac{(m-l+1)(m+l)(l+1)}{l(m-l)(m+l+1)}. \]  

Multiplying both sides of (2.1) by \( l \) and then plusing \( ld_l^2(m) \) to the two sides gives the following result.

\[ \text{Theorem 2.2.} \text{ For } m \geq 2 \text{ and } 1 \leq l \leq m-1, \text{ we have} \]

\[ l(l+1) \left( d_l^2(m) - d_{l+1}(m)d_{l-1}(m) \right) > \left( l + \frac{2l^3}{(m+l)(m-l+1)} \right) d_l^2(m). \]  

On the other hand, Chen and Xia [9] have shown the spiral property of sequence \( \{d_i(m)\}_{1 \leq i \leq m-1} \), that is

\[ d_{m-1}(m) < d_1(m) < d_{m-2}(m) < d_2(m) < \cdots < d_{\lfloor m/2 \rfloor}(m). \]  

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( f(l) = l + \frac{2l^3}{(m+l)(m-l+1)} \). Then for \( 1 \leq l \leq m-1, \)

\[ f'(l) = 1 + \frac{6l^2}{(m+l)(m-l+1)} + \frac{2l^3(2l-1)}{(m+l)^2(m-l+1)^2} > 0. \]

Restricting \( l \in N^+ \), we see that the sequence \( \{l + \frac{2l^3}{(m+l)(m-l+1)}\}_{1 \leq l \leq m-1} \) is strictly monotone increasing.
Combining (2.2) and (2.3), we get
\[
\begin{align*}
l(l+1)(d_l^2(m) - d_{l+1}(m)d_{l-1}(m)) &> (l + \frac{2l^3}{(m + l)(m - l + 1)})d_l^2(m) \\
&\geq \min\{(1 + \frac{2}{(m + 1)m})d_l^2(m), (m - 1 + \frac{(m - 1)^3}{2m - 1})d_{m-1}^2(m)\}. \tag{2.4}
\end{align*}
\]

By direct computation we may deduce from (1.2) that
\[
\begin{align*}
(1 + \frac{2}{(m + 1)m})d_l^2(m) &\geq m(m + 1)d_m^2(m), \\
(m - 1 + \frac{(m - 1)^3}{2m - 1})d_{m-1}^2(m) &\geq m(m + 1)d_m^2(m).
\end{align*}
\]

It follows by (2.4) that
\[
l(l+1)(d_l^2(m) - d_{l+1}(m)d_{l-1}(m)) > m(m + 1)d_m^2(m), \quad 1 \leq l \leq m - 1.
\]

This completes the proof. \hfill \Box

**Acknowledgements**

This work was supported by the National Natural Science Foundation of China, the Natural Science Foundation of Hebei Province (A2014208152) and the Top Young-aged Talents Program of Hebei Province.

**References**


