Nonexistence of certain singly even self-dual codes with minimal shadow

Stefka Bouyuklieva
Faculty of Mathematics and Informatics
Veliko Tarnovo University
5000 Veliko Tarnovo, Bulgaria
stefka@uni-vt.bg

Masaaki Harada Akihiro Munemasa
Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences
Tohoku University
Sendai 980–8579, Japan
mharada@m.tohoku.ac.jp, munemasa@math.is.tohoku.ac.jp

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Abstract

It is known that there is no extremal singly even self-dual \([n,n/2,d]\) code with minimal shadow for \((n,d) = (24m+2,4m+4),(24m+4,4m+4),(24m+6,4m+4),(24m+10,4m+4)\) and \((24m+22,4m+6)\). In this paper, we study singly even self-dual codes with minimal shadow having minimum weight \(d-2\) for these \((n,d)\). For \(n = 24m+2,24m+4\) and \(24m+10\), we show that the weight enumerator of a singly even self-dual \([n,n/2,4m+2]\) code with minimal shadow is uniquely determined and we also show that there is no singly even self-dual \([n,n/2,4m+2]\) code with minimal shadow for \(m \geq 155, m \geq 156\) and \(m \geq 160\), respectively. We demonstrate that the weight enumerator of a singly even self-dual code with minimal shadow is not uniquely determined for parameters \([24m+6,12m+3,4m+2]\) and \([24m+22,12m+11,4m+4]\).

Keywords: self-dual code, shadow, weight enumerator

1 Introduction

A (binary) code \(C\) of length \(n\) is a vector subspace of \(\mathbb{F}_2^n\), where \(\mathbb{F}_2\) denotes the finite field of order 2. The dual code \(C^\perp\) of \(C\) is defined as \(C^\perp = \{x \in \mathbb{F}_2^n \mid x \cdot y = 0 \text{ for all } y \in C\}\),
where $x \cdot y$ is the standard inner product. A code $C$ is called self-dual if $C = C^\perp$. Self-dual codes are divided into two classes. A self-dual code $C$ is doubly even if all codewords of $C$ have weight divisible by four, and singly even if there is at least one codeword of weight $\equiv 2 \pmod{4}$. Let $C$ be a singly even self-dual code and let $C_0$ denote the subcode of codewords having weight $\equiv 0 \pmod{4}$. Then $C_0$ is a subcode of codimension 1. The shadow $S$ of $C$ is defined to be $C_0^\perp \setminus C$. Shadows for self-dual codes were introduced by Conway and Sloane [5] in order to derive new upper bounds for the minimum weight of singly even self-dual codes. By considering shadows, Rains [9] showed that the minimum weight $d$ of a self-dual code of length $n$ is bounded by $d \leq 4 \left\lfloor \frac{n}{24} \right\rfloor + 6$ if $n \equiv 22 \pmod{24}$, $d \leq 4 \left\lfloor \frac{n}{24} \right\rfloor + 4$ otherwise. A self-dual code meeting the bound is called extremal.

Let $C$ be a singly even self-dual code of length $n$ with shadow $S$. Let $d(S)$ denote the minimum weight of $S$. We say that $C$ is a code with minimal shadow if $r = d(S)$, where $r = 4, 1, 2$ and 3 if $n \equiv 0, 2, 4$ and 6 $(\bmod{8})$, respectively. The concept of self-dual codes with minimal shadow was introduced in [2]. In that paper, different types of self-dual codes with the same parameters were compared with regard to the decoding error probability. In [3], the connection between singly even self-dual codes with minimal shadow of some lengths, combinatorial designs and secret sharing schemes was considered. It was shown in [4] that there is no extremal singly even self-dual code with minimal shadow for lengths $24m + 2, 24m + 4, 24m + 6, 24m + 10$ and $24m + 22$. In [3], it was shown that the weight enumerator of a (non-extremal) singly even self-dual $[24m + 2, 12m + 1, 4m + 2]$ code with minimal shadow is uniquely determined for each positive integer $m$. These motivate us to study singly even self-dual codes with minimal shadow having minimum weight two less than the hypothetical extremal case.

The main aim of this paper is to investigate singly even self-dual codes with minimal shadow having minimum weight $4m + 2$ for the lengths $24m + 2, 24m + 4$ and $24m + 10$. We show that the weight enumerator of a singly even self-dual code with minimal shadow having minimum weight $4m + 2$ is uniquely determined for lengths $24m + 2$ and $24m + 10$. For lengths $24m + 2, 24m + 4$ and $24m + 10$, nonnegativity of the coefficients of weight enumerators shows that there is no such code for $m$ sufficiently large. We also show that the uniqueness of the weight enumerator fails for the parameters $[24m + 6, 12m + 3, 4m + 2]$ and $[24m + 22, 12m + 11, 4m + 4]$.

The paper is organized as follows. In Section 2, we review the results given by Rains [9]. In Section 3, we show that there is no singly even self-dual $[24m + 2, 12m + 1, 4m + 2]$ code with minimal shadow for $m \geq 155$. In Sections 4 and 5, for parameters $[24m + 4, 12m + 2, 4m + 2]$ and $[24m + 10, 12m + 5, 4m + 2]$, we show that there is no singly even self-dual code with minimal shadow for $m \geq 156$ and for $m \geq 160$, respectively. Finally, in Section 6, we demonstrate that the weight enumerator of a singly even self-dual code with minimal shadow is not uniquely determined for parameters $[24m + 6, 12m + 3, 4m + 2]$ and $[24m + 22, 12m + 11, 4m + 4]$.

All computer calculations in this paper were done with the help of the algebra software Magma [1] and the mathematical software Maple and Mathematica.
2 Preliminaries

Let $C$ be a singly even self-dual code of length $n$ with shadow $S$. Write $n = 24m + 8l + 2r$, where $m$ is an integer, $l \in \{0, 1, 2\}$ and $r \in \{0, 1, 2, 3\}$. The weight enumerators $W_C(y)$ and $W_S(y)$ of $C$ and $S$ are given by ([5, (10), (11)])

$$W_C(y) = \sum_{i=0}^{12m+4l+r} a_i y^{2i} = \sum_{j=0}^{3m+l} c_j (1 + y^2)^{12m+4l+r-4j}(y^2(1-y^2)^2)^j,$$

$$W_S(y) = \sum_{i=0}^{6m+2l} b_i y^{4i} = \sum_{j=0}^{3m+l} (-1)^j c_j 2^{12m+4l+r-6j} y^{12m+4l+r-4j}(1-y^4)^2j,$$

respectively, for suitable integers $c_j$. Let

$$W_C(y) = (1+y^2)^{n/2-4j}(y^2(1-y^2)^2)^j = \sum_{i=0}^{12m+4l+r} \alpha_{i,j}' y^{2i} \quad (0 \leq j \leq 3m + l).$$

Then

$$\alpha_{i,j}' = \begin{cases} 0 & \text{if } 0 \leq i < j \leq 3m + l, \\ 1 & \text{if } 0 \leq i = j \leq 3m + l. \end{cases}$$

This implies that the $(3m + l + 1) \times (3m + l + 1)$ matrix $[\alpha_{i,j}']$ is invertible, since it is unitriangular. Let $[\alpha_{i,j}]$ be its inverse matrix. Then by (4), we have

$$\alpha_{i,j} = \begin{cases} 0 & \text{if } 0 \leq i < j \leq 3m + l, \\ 1 & \text{if } 0 \leq i = j \leq 3m + l, \end{cases}$$

and

$$y^{2i} = \sum_{j=0}^{3m+l} \alpha_{j,i}(1+y^2)^{n/2-4j}(y^2(1-y^2)^2)^j \quad (0 \leq i \leq 3m + l)$$

by (3). By (1), (5) and (6), we obtain

$$c_i = \sum_{j=0}^{i} \alpha_{i,j} a_j.$$ 

Lemma 1. For $1 \leq i \leq 3m + l$, we have

$$\alpha_{i,0} = -\frac{n}{2i} \sum_{0 \leq t \leq n/2+1-6i \atop t+i \text{ is odd}} (-1)^t \binom{n/2+1-6i}{t} \binom{n-7i-t-1}{i-t-1/2}.$$ 

Proof. For $1 \leq i$,

$$\alpha_{i,0} = -\frac{n}{2i} \text{[coeff. of } y^{i-1} \text{ in } (1+y)^{-n/2-1+4i}(1-y)^{-2i}].$$
[9]. Since
\[
(1 + y)^{-n/2-1+4i} (1 - y)^{-2i} \\
= (1 - y^2)^{-n/2-1+4i} (1 - y)^{n/2+1-6i} \\
= (1 - y^2)^{-n/2-1+4i} \sum_{t=0}^{n/2+1-6i} (-1)^t \left( \frac{n}{2} + 1 - 6i \right)_t y^t,
\]
we have
\[
\alpha_{i,0} = -\frac{n}{2i} \sum_{t=0}^{n/2+1-6i} (-1)^t \left( \frac{n}{2} + 1 - 6i \right)_t [\text{coeff. of } y^{i-1} \text{ in } (1 - y^2)^{-n/2-1+4i} y^t] \\
= -\frac{n}{2i} \sum_{0 \leq t \leq n/2+1-6i \atop i-t \text{ is odd}} (-1)^t \left( \frac{n}{2} + 1 - 6i \right)_t (1-t-i)/2 \left( -\frac{n}{2} - 1 + 4i \right)^{i-t-i/2}.
\]
The result follows by applying the formula
\[
(-1)^t \binom{-n}{j} = \binom{n + j - 1}{j}.
\]

Write
\[
(-1)^j 2^{n/2-6j} y^{n/2-4j} (1-y^4)^{2j} = \sum_{i=0}^{6m+2l} \beta'_{i,j} y^{4i+r} \quad (0 \leq j \leq 3m+l).
\]
Since \(n/2-4j = 4(3m+l-j) + r\), we have
\[
\beta'_{i,j} = \begin{cases} 
0 & \text{if } i < 3m+l-j, \\
(-1)^j 2^{n/2-6j} & \text{if } i = 3m+l-j.
\end{cases}
\]
This implies that the \((3m+l+1) \times (3m+l+1)\) matrix \([\beta'_{i,j}]\) is invertible, since it is lower triangular such that the diagonal elements are not zeros. Thus, the matrix \([\beta'_{i,j}]\) is also invertible. Let \([\beta_{i,j}]\) be its inverse matrix. Then
\[
y^{4i+r} = \sum_{j=0}^{3m+l} \beta_{j,i} (-1)^j 2^{n/2-6j} y^{n/2-4j} (1-y^4)^{2j} \quad (0 \leq i \leq 3m+l). \quad (9)
\]
Moreover, \([\beta_{3m+l-i,j}]\) is the inverse of the lower triangular matrix \([\beta'_{i,3m+l-j}]\), and so lower triangular as well, and
\[
\beta_{3m+l-i,j} = \beta_{j,3m+l-j}^{-1}.
\]
Thus
\[
\beta_{i,j} = \begin{cases} 
0 & \text{if } i > 3m+l-j, \\
(-1)^{3m+l-j} \beta_{j,3m+l-j}^{-1} & \text{if } i = 3m+l-j.
\end{cases}
\]
(10)
By (2), (9) and (10), we obtain
\[ c_i = \sum_{j=0}^{3m+l-i} \beta_{i,j} b_j. \]  
\hspace{1cm} (11)

**Lemma 2** (Rains [9]). For \( 1 \leq i \leq 3m + l \) and \( 0 \leq j \leq 3m + l \) with \( i + j \leq 3m + l \), we have
\[ \beta_{i,j} = (-1)^i 2^{-n/2+6i} \binom{3m + l + i - j - 1}{3m + l - i - j}. \]  
\hspace{1cm} (12)

From (7) and (11), we have
\[ c_i = \sum_{j=0}^{i} \alpha_{i,j} a_j = \sum_{j=0}^{3m+l-i} \beta_{i,j} b_j. \]  
\hspace{1cm} (13)

Now let \( C \) be a singly even self-dual \([24m + 8l + 2r, 12m + 4l + r, 4m + 2]\) code with minimal shadow. Suppose that \((l,r) \in \{(0, 1), (0, 2), (1, 1)\}\). Since the minimum weight of \( C \) is \( 4m + 2 \), we have
\[ a_0 = 1, a_1 = a_2 = \ldots = a_{2m} = 0. \]  
\hspace{1cm} (14)

Since the minimum weight of the shadow is 1 or 2, we have
\[
\begin{align*}
  b_0 &= 1, & \text{if } m = 1, \\
  b_0 &= 1, b_1 = b_2 = \ldots = b_{m-1} = 0 & \text{if } m \geq 2.
\end{align*}
\]  
\hspace{1cm} (15)

From (13), (14) and (15), we have
\[ c_i = \begin{cases} 
\alpha_{i,0} & \text{if } i = 0, 1, \ldots, 2m, \\
\beta_{i,0} & \text{if } i = 2m + l + 1, 2m + l + 2, \ldots, 3m + l.
\end{cases} \]  
\hspace{1cm} (16)

Suppose that \( l = 0 \). From (13), (14), (15) and (16), we obtain
\[
\begin{align*}
  c_{2m} &= \alpha_{2m,0} = \beta_{2m,0} + \beta_{2m,m} b_m, \\
  c_{2m-1} &= \alpha_{2m-1,0} = \beta_{2m-1,0} + \beta_{2m-1,m} b_m + \beta_{2m-1,m+1} b_{m+1}.
\end{align*}
\]  
\hspace{1cm} (17)  
\hspace{1cm} (18)

Suppose that \( l = 1 \). From (13), (14), (15) and (16), we obtain
\[
\begin{align*}
  c_{2m} &= \alpha_{2m,0} = \beta_{2m,0} + \beta_{2m,m} b_m + \beta_{2m,m+1} b_{m+1}, \\
  c_{2m+1} &= \alpha_{2m+1,0} + \alpha_{2m+1,2m+1} a_{2m+1} = \beta_{2m+1,0} + \beta_{2m+1,m} b_m.
\end{align*}
\]  
\hspace{1cm} (19)  
\hspace{1cm} (20)
3 Singly even self-dual $[24m + 2, 12m + 1, 4m + 2]$ codes with minimal shadow

It was shown in [3] that the weight enumerator of a singly even self-dual $[24m + 2, 12m + 1, 4m + 2]$ code with minimal shadow is uniquely determined for each length. In this section, we show that there is no singly even self-dual $[24m + 2, 12m + 1, 4m + 2]$ code with minimal shadow for $m \geq 155$.

Suppose that $m \geq 1$. Let $C$ be a singly even self-dual $[24m + 2, 12m + 1, 4m + 2]$ code with minimal shadow. The weight enumerators of $C$ and its shadow $S$ are written as in (1) and (2), respectively.

From (8),
\[ \alpha_{2m,0} = \frac{12m + 1}{m} \binom{5m}{m-1}. \]

From (10),
\[ \beta_{2m,m} = \frac{1}{2}. \]

From (12),
\[ \beta_{2m,0} = \frac{3}{2^2} \binom{5m-1}{m} = \frac{3(4m+1)}{5m} \binom{5m}{m-1}. \]

From (17),
\[ b_m = \frac{\alpha_{2m,0} - \beta_{2m,0}}{\beta_{2m,m}} = \frac{4(24m+1)}{5m} \binom{5m}{m-1}. \]

Remark 3. Unfortunately, $b_m$ was incorrectly reported in [12]. The correct formula for $b_m$ is given in [13]. We showed that $b_m$ is always a positive integer (see [13]).

From (8),
\[ \alpha_{2m-1,0} = -\frac{12m + 1}{2m - 1} \left( \binom{5m + 4}{m - 1} + 28 \binom{5m + 3}{m - 2} + 70 \binom{5m + 2}{m - 3} \right) + 28 \binom{5m + 1}{m - 4} + \binom{5m}{m - 5} \]
\[ = -\frac{8(12m + 1)(376m^3 - 4m^2 + 5m + 1)}{(4m + 2)(4m + 3)(4m + 4)(4m + 5)} \binom{5m}{m - 1}. \]

From (10),
\[ \beta_{2m-1,m+1} = -\frac{1}{2^7}. \]

From (12),
\[ \beta_{2m-1,0} = -\frac{1}{2^7} \frac{3m}{2m - 1} \binom{5m - 2}{m + 1} = -\frac{1}{2^7} \frac{3(4m - 1)(4m + 1)}{(5m - 1)(m + 1)} \binom{5m}{m - 1}, \]
\[ \beta_{2m-1,m} = -\frac{m}{2^5}. \]
From (18),
\[ b_{m+1} = \frac{\alpha_{2m-1,0} - \beta_{2m-1,0} - \beta_{2m-1,m} b_m}{\beta_{2m-1,m+1}} \]
\[ = -\frac{64(24m+1)f(m)}{(5m-1)(4m+2)(4m+3)(4m+4)(4m+5)} \left( \frac{5m}{m-1} \right), \]
where
\[ f(m) = 64m^5 - 14816m^4 + 2812m^3 + 46m^2 - 14m + 1. \]

**Theorem 4.** All coefficients in the weight enumerators of a singly even self-dual \([24m+2, 12m+1, 4m+2]\) code with minimal shadow and its shadow are nonnegative integers if and only if \(1 \leq m \leq 154\). In particular, for \(m \geq 155\), there is no singly even self-dual \([24m+2, 12m+1, 4m+2]\) code with minimal shadow.

**Proof.** We verified that the equation \(f(m) = 0\) has three solutions consisting of real numbers and the largest solution is in the interval \((231, 232)\). Thus, \(b_{m+1}\) is negative for \(m \geq 232\). Using (1) and (2), we determined numerically the weight enumerators of a singly even self-dual \([24m+2, 12m+1, 4m+2]\) code with minimal shadow and its shadow for \(m \leq 231\). The theorem follows from this calculation.

4 Singly even self-dual \([24m+4, 12m+2, 4m+2]\) codes with minimal shadow

**Proposition 5.** The weight enumerator of a singly even self-dual \([24m+4, 12m+2, 4m+2]\) code with minimal shadow is uniquely determined for each length.

**Proof.** The weight enumerator of a singly even self-dual \([4, 2, 2]\) code is uniquely determined. Suppose that \(m \geq 1\). Let \(C\) be a singly even self-dual \([24m+4, 12m+2, 4m+2]\) code with minimal shadow. The weight enumerators of \(C\) and its shadow \(S\) are written as using (1) and (2). Since \(\alpha_{i,0}\) \((i = 0, 1, \ldots, 2m)\) and \(\beta_{i,0}\) \((i = 2m+1, 2m+2, \ldots, 3m)\) are calculated by (8) and (12), respectively, from (16), \(c_i\) \((i = 0, 1, \ldots, 3m)\) depends only on \(m\). This means that the weight enumerator of \(C\) is uniquely determined for each length.

From (8),
\[ \alpha_{2m,0} = \frac{6m+1}{m} \left( 3 \binom{5m+1}{m-1} + \binom{5m}{m-2} \right) \]
\[ = \frac{(6m+1)(8m+1)}{m(2m+1)} \binom{5m}{m-1}. \]

From (10),
\[ \beta_{2m,m} = \frac{1}{2^m}. \]
From (12),
\[
\beta_{2m,0} = \frac{1}{2^2} \left( \binom{5m-1}{m} \right) = \frac{3(4m+1)}{10m} \left( \binom{5m}{m-1} \right).
\]

From (17),
\[
b_m = \frac{\alpha_{2m,0} - \beta_{2m,0}}{\beta_{2m,m}} = \frac{2(12m+1)(38m+7)}{5m(2m+1)} \binom{5m}{m-1}.
\]

**Remark 6.** Unfortunately, \(b_m\) was incorrectly reported in [12]. The correct formula for \(b_m\) is given in [13]. We showed that \(b_m\) is always a positive integer (see [13]).

From (8),
\[
\alpha_{2m-1,0} = -\frac{12m+2}{2m-1} \left( \binom{5m+5}{m-1} + 36 \binom{5m+4}{m-2} + 126 \binom{5m+3}{m-3} \right)
+84 \left( \binom{5m+2}{m-4} + 9 \binom{5m+1}{m-5} \right)
= -\frac{16(5m+1)(6m+1)(8m+1)(68m^2-m+3)}{(4m+2)(4m+3)(4m+4)(4m+5)(4m+6)} \left( \binom{5m}{m-1} \right).
\]

From (10),
\[
\beta_{2m-1,m+1} = \frac{1}{2^8}.
\]

From (12),
\[
\beta_{2m-1,0} = -\frac{1}{2^8} \frac{3m}{2m-1} \left( \binom{5m-2}{m+1} \right) = -\frac{1}{2^5} \frac{3(4m-1)(4m+1)}{5(5m-1)(m+1)} \left( \binom{5m}{m-1} \right),
\]
\[
\beta_{2m-1,m} = -\frac{m}{2^6}.
\]

From (18),
\[
b_{m+1} = \frac{\alpha_{2m-1,0} - \beta_{2m-1,0} - \beta_{2m-1,m} b_m}{\beta_{2m-1,m+1}}
= -\frac{128(12m+1)f(m)}{(5m-1)(4m+2)(4m+3)(4m+4)(4m+5)(4m+6)} \left( \binom{5m}{m-1} \right),
\]
where
\[
f(m) = 1216m^6 - 212096m^5 - 33020m^4 + 5440m^3 + 1171m^2 + 88m + 6.
\]

**Theorem 7.** All coefficients in the weight enumerators of a singly even self-dual \([24m+4,12m+2,4m+2]\) code with minimal shadow and its shadow are nonnegative integers if and only if \(1 \leq m \leq 155\). In particular, for \(m \geq 156\), there is no singly even self-dual \([24m+4,12m+2,4m+2]\) code with minimal shadow.
**Proof.** We verified that the equation \( f(m) = 0 \) has two solutions consisting of real numbers and the largest solution is in the interval \((174, 175)\). Thus, \( b_{m+1} \) is negative for \( m \geq 175 \). Using (1) and (2), we determined numerically the weight enumerators of a singly even self-dual \([24m+4, 12m+2, 4m+2]\) code with minimal shadow and its shadow for \( m \leq 174 \). The theorem follows from this calculation. \( \square \)

5 Singly even self-dual \([24m + 10, 12m + 5, 4m + 2]\) codes with minimal shadow

**Lemma 8** (Harada [7]). Suppose that \( n \equiv 2 \pmod{8} \). Let \( C \) be a singly even self-dual \([n, n/2, d]\) code with minimal shadow. If \( d \equiv 2 \pmod{4} \), then \( a_{d/2} = b_{(d-2)/4} \).

As a consequence, the weight enumerator of a singly even self-dual \([58, 29, 10]\) code with minimal shadow was uniquely determined in [7].

**Proposition 9.** The weight enumerator of a singly even self-dual \([24m + 10, 12m + 5, 4m + 2]\) code with minimal shadow is uniquely determined for each length.

**Proof.** The weight enumerator of a singly even self-dual \([10, 5, 2]\) code with minimal shadow is uniquely determined. Suppose that \( m \geq 1 \). Let \( C \) be a singly even self-dual \([24m + 10, 12m + 5, 4m + 2]\) code with minimal shadow. The weight enumerators of \( C \) and its shadow \( S \) are written as in (1) and (2), respectively. Since \( \alpha_{i,0} \) \((i = 0, 1, \ldots, 2m)\) and \( \beta_{i,0} \) \((i = 2m + 2, 2m + 3, \ldots, 3m + 1)\) are calculated by (8) and (12), respectively, from (16), \( c_i \) \((i = 0, 1, \ldots, 2m, 2m + 2, \ldots, 3m + 1)\) depends only on \( m \).

From (5) and (10), we have
\[
\alpha_{2m+1, 2m+1} = 1 \quad \text{and} \quad \beta_{2m+1, m} = -2,
\]
respectively. By Lemma 8, it holds that \( a_{2m+1} = b_m \). From (20), we obtain
\[
a_{2m+1} = \frac{\beta_{2m+1, 0} - \alpha_{2m+1, 0}}{3}.
\]
Therefore, from (20), \( c_{2m+1} \) depends only on \( m \). This means that the weight enumerator of \( C \) is uniquely determined for each length. \( \square \)

From (8), we have
\[
\alpha_{2m+1, 0} = -\frac{12m + 5}{2m + 1} \binom{5m + 1}{m}.
\]
From (12), we have
\[
\beta_{2m+1, 0} = -\frac{3m + 1}{2m + 1} \binom{5m + 1}{m}.
\]
Since $a_{2m+1} = b_m$, from (21), we have
\[ b_m = \binom{5m + 1}{m} = \frac{5m + 1}{4m + 1} \binom{5m}{m}. \]

From (8),
\[ \alpha_{2m,0} = \frac{12m + 5}{2m} 6 \binom{5m + 4}{m - 1} + 20 \binom{5m + 3}{m - 2} + 6 \binom{5m + 2}{m - 3} \]
\[ = \frac{4(12m + 5)(5m + 1)(5m + 2)(32m^2 + 19m + 3)}{(4m + 1)(4m + 2)(4m + 3)(4m + 4)(4m + 5)} \binom{5m}{m}. \]

From (10),
\[ \beta_{2m+1} = \frac{1}{2^5}. \]

From (12),
\[ \beta_{2m,0} = \frac{1}{2^5} \frac{3m + 1}{2m} \binom{5m}{m + 1} = \frac{1}{2^4} \frac{3m + 1}{m + 1} \binom{5m}{m}, \]
\[ \beta_{2m,m} = \frac{1}{2^5} \frac{2m + 1}{2m} \frac{4m}{4m + 1} = \frac{2m + 1}{2^4}. \]

From (19),
\[ b_{m+1} = \frac{\alpha_{2m,0} - \beta_{2m,0} - \beta_{2m,m} b_m}{\beta_{2m,m+1}} \]
\[ = -\frac{16(5m + 2)f(m)}{(4m + 1)(4m + 2)(4m + 3)(4m + 4)(4m + 5)} \binom{5m}{m}, \]
where
\[ f(m) = 64m^5 - 15040m^4 - 18036m^3 - 7924m^2 - 1511m - 105. \]

**Theorem 10.** All coefficients in the weight enumerators of a singly even self-dual $[24m + 10, 12m + 5, 4m + 2]$ code with minimal shadow and its shadow are nonnegative integers if and only if $1 \leq m \leq 159$. In particular, for $m \geq 160$, there is no singly even self-dual $[24m + 10, 12m + 5, 4m + 2]$ code with minimal shadow.

**Proof.** We verified that the equation $f(m) = 0$ has three solutions consisting of real numbers and the largest solution is in the interval (236, 237). Thus, $b_{m+1}$ is negative for $m \geq 237$. Using (1) and (2), we determined numerically the weight enumerators of a singly even self-dual $[24m + 10, 12m + 5, 4m + 2]$ code with minimal shadow and its shadow for $m \leq 236$. The theorem follows from this calculation. □

### 6 Remaining cases

For the remaining cases, we demonstrate that the weight enumerator of a singly even self-dual code with minimal shadow is not uniquely determined.
6.1 Singly even self-dual \([24m + 6, 12m + 3, 4m + 2]\) codes with minimal shadow

Using (1) and (2), the possible weight enumerators of a singly even self-dual \([30, 15, 6]\) code with minimal shadow and its shadow are given by

\[
1 + (35 - 8\beta)y^6 + (345 + 24\beta)y^8 + 1848y^{10} + \cdots
\]

\[
\beta y^3 + (240 - 6\beta)y^7 + (6720 + 15\beta)y^{11} + \cdots
\]

respectively, where \(\beta\) is an integer with \(1 \leq \beta \leq 4\). It is known that there is a singly even self-dual \([30, 15, 6]\) code with minimal shadow for \(\beta \in \{1, 2\}\) (see [5]).

Using (1) and (2), the possible weight enumerators of a singly even self-dual \([54, 27, 10]\) code with minimal shadow and its shadow are given by

\[
1 + (351 - 8\beta)y^{10} + (5543 + 24\beta)y^{12} + (43884 + 32\beta)y^{14} + \cdots
\]

\[
y^3 + (-12 + \beta)y^7 + (2874 - 10\beta)y^{11} + (258404 + 45\beta)y^{15} + \cdots
\]

respectively, where \(\beta\) is an integer with \(12 \leq \beta \leq 43\). It is known that there is a singly even self-dual \([54, 27, 10]\) code with minimal shadow for \(\beta \in \{12, 13, \ldots, 20, 21, 22, 24, 26\}\) (see [10]).

6.2 Singly even self-dual \([24m + 22, 12m + 11, 4m + 4]\) codes with minimal shadow

Using (1) and (2), the possible weight enumerators of a singly even self-dual \([22, 11, 4]\) code with minimal shadow and its shadow are given by

\[
1 + 2\beta y^4 + (77 - 2\beta)y^6 + (330 - 6\beta)y^8 + (616 + 6\beta)y^{10} + \cdots
\]

\[
\beta y^3 + (352 - 4\beta)y^7 + (1344 + 6\beta)y^{11} + \cdots
\]

respectively, where \(\beta\) is an integer with \(1 \leq \beta \leq 38\). It is known that there is a singly even self-dual \([22, 11, 4]\) code with minimal shadow for \(\beta \in \{2, 4, 6, 8, 10, 14\}\) (see [8]).

Using (1) and (2), the possible weight enumerators of a singly even self-dual \([46, 23, 8]\) code with minimal shadow and its shadow are given by

\[
1 + 2\beta y^8 + (884 - 2\beta)y^{10} + (10556 - 14\beta)y^{12} + (54621 + 14\beta)y^{14} + \cdots
\]

\[
y^3 + (-10 + \beta)y^7 + (6669 - 8\beta)y^{11} + (242760 + 28\beta)y^{15} + \cdots
\]

respectively, where \(\beta\) is an integer with \(10 \leq \beta \leq 442\). Let \(C_{46}\) be the code with generator matrix \(
\begin{bmatrix}
I_{23} & R
\end{bmatrix}
\)

where \(I_{23}\) denotes the identity matrix of order 23 and \(R\) is the 23 \(\times\) 23 circulant matrix with first row

\[
(01011101011100001111110).
\]

We verified that \(C_{46}\) is a singly even self-dual \([46, 23, 8]\) code. By considering self-dual neighbors of \(C_{46}\), we found singly even self-dual \([46, 23, 8]\) codes \(N_{46,i}\) with minimal shadow.
The possible weight enumerators of a singly even self-dual $[70, 35, 12]$ code with minimal shadow and its shadow are given by

$$1 + 2\beta y^{12} + (9682 - 2\beta)y^{14} + (173063 - 22\beta)y^{16} + \cdots,$$

$$y^3 + (-104 + \beta)y^{11} + (88480 - 12\beta)y^{15} + \cdots,$$

respectively, where $\beta$ is an integer $104 \leq \beta \leq 4841$ [6]. It is known that there is a singly even self-dual $[70, 35, 12]$ code with minimal shadow for many different $\beta$ [11, p. 1191].

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