A Flag Whitney number formula for matroid Kazhdan-Lusztig polynomials

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Abstract

For a representation of a matroid the combinatorially defined Kazhdan-Lusztig polynomial computes the intersection cohomology of the associated reciprocal plane. However, these polynomials are difficult to compute and there are numerous open conjectures about their structure. For example, it is unknown whether or not the coefficients are non-negative for non-representable matroids. The main result in this note is a combinatorial formula for the coefficients of these matroid Kazhdan-Lusztig polynomials in terms of flag Whitney numbers. This formula gives insight into some vanishing behavior of the matroid Kazhdan-Lusztig polynomials.

1 Introduction

Flag Whitney numbers of the second kind are key players in the study of the cd-index and are usually grouped together into one function called the flag f-vector (see [23]). They count chains in a ranked partially ordered set with prescribed ranks. There are many open conjectures about classical Whitney numbers of the first and second kind (for example [17] and [5]). Some parts of these conjectures have just been recently resolved. In particular, Adiprasito, Huh, and Katz in [1] proved the log-concavity conjecture for arbitrary matroids (see 15.3 exercise 5 in [24]) and Huh and Wang in [9] proved the top heaviness conjecture (see [12]) for representable matroids. There are also some classical results, like the generalized "Hyperplane Theorem" (see [5]). However, there is still much unknown about Whitney numbers, like the points-lines-planes conjecture (see [17]).

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In this note, flag Whitney numbers are used to understand the Kazhdan-Lusztig polynomial of a matroid or more generally of a finite ranked poset. The Kazhdan-Lusztig polynomial of a matroid, first studied in [13], is a single variable polynomial defined by a recursion which mimics the classical Kazhdan-Lusztig polynomials (see [11]). In the combinatorial recursion the characteristic polynomial takes the place of the classical "R-polynomials". It turns out that these matroid Kazhdan-Lusztig polynomials sit in the general framework of Kazhdan-Lusztig-Stanley polynomials as developed by Stanley in [22] and refined by Brenti in [3]. If the matroid is realizable over $\mathbb{C}$ then the matroid Kazhdan-Lusztig polynomial is the intersection cohomology Poincaré polynomial of the associated reciprocal plane (again see [13]). These polynomials are notoriously difficult to compute in general. For example, the main focus in [14] is the intersection cohomology for uniform matroids of rank $n-1$ on n elements. In [6] an equivariant matroid Kazhdan-Lusztig polynomial is defined and impressively used to find a formula for the ordinary matroid Kazhdan-Lusztig polynomial for all uniform matroids.

Our main result, Theorem 11, is a formula for the coefficients of the posets Kazhdan-Lusztig polynomial in terms of sums of "top-heavy pairs" of flag Whitney numbers of the second kind (see Section 2). The crux of this formula is developing the index set to which we sum over. At this point this index set is written down recursively, however probably with a little more work one can write it down in a closed form. In [13] the degree 1 and 2 coefficients of the Kazhdan-Lusztig polynomial of a matroid are written in terms of doubly-indexed Whitney numbers of the second kind (see [7]). Theorem 11 extends those formulas to arbitrary coefficients. This "top-heavy" formula gives a clear picture of why these polynomial's positive degree coefficients vanish when the poset is modular. Recently, Proudfoot, Xu, and Young in [15] have found a formula similar to Theorem 11 using a completely different technique. Their result is not decomposed in terms of top-heavy pairs, but is stated much more concretely and in closed form.

The paper is organized as follows. In Section 2 some basic facts concerning flag Whitney numbers are collected. The central focus there is the celebrated Theorem of Phillip Hall [8] on the Euler characteristic of a finite lattice. Then in Section 3 we review the definition of matroid Kazhdan-Lusztig polynomials and some basic combinatorial results. In section 4 we create the index set used to state Theorem 11. Finally in Section 5 we state the main theorem and note a consequence and write down the formulas for coefficients up to degree 5.

2 Whitney numbers

In this section we record combinatorial results on Whitney numbers for use on the matroid Kazhdan-Lusztig polynomials. Let $\mathcal{P}$ be a ranked locally finite poset. For the remainder of the paper we use $I = \{i_1, \ldots, i_k\}$ (or other k-tuples) to denote an ordered (by usual $\leqslant$) k-tuple such that for all j, $i_j \in \{0, 1, 2, \ldots, \mathrm{rk}\mathcal{P}\}$. The set of partial flags in $\mathcal{P}$ indexed by I is

$$\mathcal{P}_I = \{(X_1, \ldots, X_k) \in \mathcal{P}^k \mid \forall\ 1 \leqslant j \leqslant k, \mathrm{rk}(X_j) = i_j\}.$$

Definition 1. The *partial flag Whitney numbers of the second kind indexed by I* are

$$W_I(\mathcal{P}) = |\mathcal{P}_I|.$$

When the context of the poset is clear we will just write W_I instead of $W_I(\mathcal{P})$.

The classical Whitney numbers of the second kind are those with $k = 1$ and in this case we just write W_i. A good reference for classical Whitney numbers is [2] and these were generalized to 2 subscripts in [7]. These flag Whitney numbers of the second kind have the property that some indices are "trivial". For example, if $\mathcal{P}$ is a lattice then $W_{0,i} = W_i$.

Remark 2. What we call flag Whitney numbers here were studied by Richard Stanley in multiple papers [19], [18], [20], and [21] where they were denoted by $\alpha(\mathcal{P}, I)$, but they were only considered for strictly increasing flags. Here we study flags where some coordinates may be equal.

Example 3. Let $\mathcal{B}_n$ be the rank n Boolean lattice. For just this example we let $\mathcal{B}_n(j) = (\mathcal{B}_n)_j$ the rank j component of $\mathcal{B}_n$. If $I = (i_1, \ldots, i_k)$ then

$$W_I(\mathcal{B}_n) = \binom{n}{i_1, i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k}.$$

Proof. We start at the bottom of the chain $i_1 \leqslant \cdots \leqslant i_k$. There are exactly $\binom{n}{i_1}$ elements of rank i_1 in $\mathcal{B}_n$ (i.e. $|\mathcal{B}_n(j)| = \binom{n}{j}$). Then for any $X \in \mathcal{B}_n(i_1)$ the restriction to X is $\mathcal{B}_n^X \cong \mathcal{B}_{n-i_1}$ and the elements above X of rank i_2 in $\mathcal{B}_n$ are now of rank $i_2 - i_1$ in $\mathcal{B}_n^X$. So, for every $X \in \mathcal{B}_n(i_1)$ the number of elements above it is $\binom{n-i_1}{i_2-i_1}$. In general for every $Y \in \mathcal{B}_N(i_j)$ there are $\binom{n-i_j}{i_{j+1}-i_j}$ above it in $\mathcal{B}_n(i_{j+1})$. Hence

$$W_I(\mathcal{B}_n) = \binom{n}{i_1}\binom{n-i_2}{i_2-i_1}\cdots\binom{n-i_{k-1}}{i_k-i_{k-1}} = \binom{n}{i_1, i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n - i_k}.$$
□

Now we work towards a formula that relates the multi-indexed Whitney numbers of second kind to that of the first. The Whitney numbers of the first kind are built using the Möbius function on the poset. The Möbius function on a poset $\mathcal{P}$ is $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ defined recursively by $\mu(X, X) = 1$ and

$$\mu(X, Y) = -\sum_{X \leqslant Z < Y} \mu(X, Z).$$

For a locally finite ranked poset with minimal element $\hat{0}$ the classical Whitney numbers of the first kind are the numbers

$$w_i = \sum_{X \in \mathcal{P}_i} \mu(\hat{0}, X).$$

Let $\mathcal{P}$ be a locally finite ranked poset with smallest element $\hat{0}$. All the elements of $\mathcal{P}$ of rank k we denote by $\mathcal{P}_k := \{X \in \mathcal{P} \mid \mathrm{rk}X = k\}$ and for $I = \{i_1, \ldots, i_s\}$ we set $\mathcal{P}(I) = \{\vec{X} = (X_1, \ldots, X_s) \mid \forall 1 \leqslant i \leqslant s,\ X_i \in \mathcal{P}(i)\}$. Also, we call $\mathcal{P}_X = \{Y \in \mathcal{P} \mid Y \leqslant X\}$ the localization of $\mathcal{P}$ at X and $\mathcal{P}^X = \{Y \in \mathcal{P} \mid Y \geqslant X\}$ the restriction of $\mathcal{P}$ at X. Using these new posets we record a few basic lemmas which are foundational for computing various Whitney numbers.

Lemma 4. *If $I \subseteq \{1, \ldots, n-1\}$ then*

$$\sum_{X \in \mathcal{P}_n} W_I(\mathcal{P}_X) = W_{I \cup \{n\}}(\mathcal{P}).$$

Proof. By definition

$$W_I(\mathcal{P}_X) = \sum_{\substack{\vec{X} \in \mathcal{P}(I) \\ \vec{X} \leqslant X}} \zeta(\vec{X}).$$

Thus

$$\sum_{X \in \mathcal{P}_n} W_I(\mathcal{P}_X) = \sum_{X \in \mathcal{P}_n} \sum_{\substack{\vec{X} \in \mathcal{P}(I) \\ \vec{X} \leqslant X}} \zeta(\vec{X}) = \sum_{X \in \mathcal{P}_n} \sum_{\vec{X} \in \mathcal{P}(I)} \zeta(\vec{X}, X). \tag{1}$$

Since the right hand side of 1 is exactly $W_{I \cup \{n\}}$ we are done. □

We add the "dual" of Lemma 4 for later use whose proof is very similar.

Lemma 5. *Let $r = \mathrm{rk}(\mathcal{P})$ and $I \subseteq \{1, \ldots, t-1\}$. For $I = \{i_1, \ldots, i_k\}$ set $I[t] = \{i_1 + t, i_2 + t, \ldots, i_k + t\}$ and assume that $i + t \leqslant r$ for all $i \in I$. With this notation we have*

$$\sum_{X \in \mathcal{P}_t} W_I(\mathcal{P}^X) = W_{\{t\} \cup I[t]}(\mathcal{P}).$$

Proof. In this case the indices I must be shifted to be accounted for in $\mathcal{P}$ because $\mathcal{P}^X$ is all elements above X. So,

$$\sum_{X \in \mathcal{P}_t} W_I(\mathcal{P}^X) = \sum_{X \in \mathcal{P}_t} \sum_{\vec{X} \in \mathcal{P}^X(I)} \zeta(\vec{X})$$

$$= \sum_{X \in \mathcal{P}_t} \sum_{\substack{\vec{X} \in \mathcal{P}(I+t) \\ X \leqslant \vec{X}}} \zeta(\vec{X}) = \sum_{X \in \mathcal{P}_t} \sum_{\vec{X} \in \mathcal{P}(I+t)} \zeta(X, \vec{X}) = \sum_{\vec{Y} \in (\{t\} \cup (I+t))} \zeta(\vec{Y})$$

which is exactly $W_{\{t\} \cup (I+t)}$. □

We add another lemma for use on understanding the Kazhdan-Lusztig polynomial, which is really a combination of Lemma 4 and Lemma 5.

Lemma 6. *Let $r = \mathrm{rk}(\mathcal{P})$, $k \in [r]$ and $I, J \subseteq [r]$ such that for all $i \in I$, $i \leqslant k$ and for all $j \in J$, $j + k \leqslant r$. In this case we can define $J[k] = \{j + k \mid j \in J\}$. Then*

$$\sum_{F \in L_k} W_I(\mathcal{P}_F) W_J(\mathcal{P}^F) = W_{I \cup \{k\} \cup J[k]}(\mathcal{P}).$$

Proof. Let $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_t\}$. Look at the sum

$$\sum_{F \in L_k} W_I(\mathcal{P}_F) W_J(\mathcal{P}^F) = \sum_{F \in \mathcal{P}_k} \left(\sum_{\mathscr{X}} 1 \right) \left(\sum_{\mathscr{Y}} 1 \right)$$

where the summation condition $\mathscr{X}$ correspond to $X_u \in \mathcal{P}_{i_u}$ for $1 \leqslant u \leqslant s$ and $X_{i_1} \leqslant \cdots X_{i_s} \leqslant F$ and the summation condition $\mathscr{Y}$ corresponds to $Y_v \in \mathcal{P}_{j_v}$ for $1 \leqslant v \leqslant t$ and $F \leqslant Y_{j_1} \leqslant \cdots \leqslant Y_{j_t}$. Then we switch the sums and we have the result. $\square$

Now we present the main lemma of this section. This is proved on page 154 of [21] and can follow by using the previous Lemmas. It can also be thought of as a consequence of Phillip Hall's Theorem (see [8] and [16], Proposition 6).

Lemma 7. *If $\mathcal{P}$ is a locally finite, ranked poset with a minimal element and $1 \leqslant n \leqslant \mathrm{rk}(\mathcal{P})$ then*

$$w_{0,n} = \sum_{I \subseteq \{1, \ldots, n-1\}} (-1)^{|I|+1} W_{I \cup \{n\}}.$$

3 The Kazhdan-Lusztig polynomial of a matroid

The aim of this section is to develop formulas for certain coefficients of the Kazhdan-Lusztig polynomial of a matroid. This result gives some hint that these polynomials may be more tractable to understand than the classical Kazhdan-Lusztig polynomials. These matroid Kazhdan-Lusztig polynomials were originally defined for matroids or equivalently for geometric lattices. However they can be defined for any finite ranked poset with bottom and top elements. To do this we need a little notation. Let $\mathcal{P}$ be a finite ranked poset with a top element $\hat{0}$ and bottom element $\hat{1}$. For $F \in \mathcal{P}$ the restriction of $\mathcal{P}$ to F is

$$\mathcal{P}^F = \{E \in \mathcal{P} \mid E \geqslant F\}$$

and the localization of $\mathcal{P}$ at F is

$$\mathcal{P}_F = \{E \in P \mid E \leqslant F\}.$$

The last ingredient we need is the infamous characteristic polynomial.

$$\chi(\mathcal{P}, t) = \sum_{X \in \mathcal{P}} \mu(\hat{0}, X) t^{\mathrm{rk}(X)}.$$

The following definition of matroid Kazhdan-Lusztig polynomials requires a fascinating property of the characteristic polynomial, that it is a $\mathcal{P}$-kernel in the sense of [3] and [22].

Definition 8 ([13], Theorem 2.2)**.** Let $\mathcal{P}$ be a finite ranked poset with top and bottom. The *Kazhdan-Lusztig polynomial* of $\mathcal{P}$, $P(\mathcal{P},t)$ is the polynomial recursively defined which satisfies

1. If $\mathrm{rk}(\mathcal{P}) = 0$ then $P(\mathcal{P},t) = 1$.
2. If $\mathrm{rk}(\mathcal{P}) > 0$ then $\deg(P(\mathcal{P},t)) < .5\mathrm{rk}(\mathcal{P})$.
3. For all $\mathcal{P}$,
$$t^{\mathrm{rk}(\mathcal{P})}P(\mathcal{P},t^{-1}) = \sum_{F\in\mathcal{P}} \chi(\mathcal{P}_F,t)P(\mathcal{P}^F,t).$$

Now we gather some basic results on the first few coefficients from [13].

Proposition 9 ([13], Propositions 2.11, 2.12, and 2.16)**.** *Let L be a geometric lattice with rank r. Then*

1. *The constant coefficient of $P(L,t)$ is 1.*
2. *The linear coefficient of $P(L,t)$ is $W_{r-1} - W_1$.*
3. *The quadratic coefficient of $P(L,t)$ is*
$$-[W_{1,r-1} - W_{1,2}] + [W_{r-3,r-1} - W_{r-3,r-2}] + [W_{r-2} - W_2].$$

Using Proposition 9 for any matroid of rank less than 6 we can quickly calculate the Kazhdan-Lusztig polynomials. However, a central aim in this field is to compute the Kazhdan-Lusztig polynomials for certain infinite families of arrangements with higher ranks. For example, a closed formula of the Kazhdan-Lusztig polynomials for the family of braid matroids or reflection types is unknown. Quite surprisingly the only infinite family of non-trivial matroids with a known closed formula are the uniform matroids (see [14] and [6]). One aim of this note is to generalize Proposition 9 in order to compute the Kazhdan-Lusztig polynomials for some families of matroids.

4 The index set

In this section we develop some notation to state a formula for any coefficient of the Kazhdan-Lusztig polynomial. We are going to compute the degree k term. Developing the index set to sum over is the crux. Throughout we denote $\{1,\dots,n\}$ by $[n]$. We are going to define the index set, which we will call S_k, recursively. Eventually S_k will be a set of subsets of $\mathbb{Z}[r]$ where we will view r as a variable. The elements in each subset will either be an integer or of the form $r - n$ for an integer n. Then later when we want to use the formula the aim will be to substitute the rank of the matroid or poset for r. The base is $S_1 = \{\{1\}\}$. For $1 \leqslant t$ put

$$A_t = \left\{ I \in 2^{[t]} \middle| t \in I \right\}$$

and for $t \leqslant 0$ we set $A_t = \emptyset$. So, for example $A_1 = \{\{1\}\}$, $A_2 = \{\{2\}, \{1, 2\}\}$, and $A_3 = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Next for $3 \leqslant s \leqslant 2k - 1$ we need a function $f_s : \mathbb{Z}[r] \to \mathbb{Z}[r]$ defined by

$$f_s(p) = \mathrm{eval}(p, s) + r - s$$

where $\mathrm{eval}(p, s)$ means evaluating the polynomial p at s. Then for a finite subset I of $\mathbb{Z}[r]$ put $F_s(I) = \{f_s(i) \mid i \in I\}$. For example, $F_4(\{1, r-2, r-1\}) = \{r-3, r-2, r-1\}$. Then for $k > 1$ set

$$S_k = A_k \cup \bigcup_{s=3}^{2k-1} T_k^s \tag{2}$$

where

$$T_k^s = \bigcup_{u \leqslant i < s/2} T_k^s(i) \tag{3}$$

with $u = \max\{1, s - k\}$ and

$$T_k^s(i) = \left\{ \alpha \sqcup \{r - s\} \sqcup F_s(\beta) \middle| \alpha \in A_{k-s+i},\ \beta \in S_i \right\}. \tag{4}$$

It is important to make sure S_k is well defined. Note that $i < s/2 < k$ implies the sets $T_k^s(i)$ are well defined and hence means S_k is well defined.

For example, with $k = 2$ we have $S_2 = A_2 \cup T_2^3$, $u = 1$, $T_2^3 = T_3^3(1)$, and

$$T_2^3(1) = \left\{ \alpha \sqcup \{r - 3\} \sqcup F_3(\beta) \middle| \alpha \in A_0 = \emptyset,\ \beta \in S_1 \right\} = \{\{r-3, r-2\}\}. \tag{5}$$

Hence $S_2 = \{\{2\}, \{1, 2\}, \{r-3, r-2\}\}$ and note that these are exactly the index sets of the right hand terms in each bracket of (3) in Proposition 9. This is the aim of this index set. It will give the “bottom terms” in the “top heavy pairs” for our main formula.

Now we need a technical lemma to finish the remainder of the construction. This lemma is a key piece of the entire formula. To prove this lemma we will need a little notation. For $I \in T_k^s$ let

$$\max{}^r(I) = \max\{n \in \mathbb{Z} \mid r - n \in I\}$$

and similarly

$$\min{}^r(I) = \min\{n \in \mathbb{Z} \mid r - n \in I\}.$$

Lemma 10. *For each $I = \alpha \sqcup \{r - s\} \sqcup F_s(\beta) \in T_k^s$ there does not exist a different α' and β' such that $I = \alpha' \sqcup \{r - s\} \sqcup F_s(\beta')$.*

Proof. First we need a few facts about the elements of $I \in T_k^s$. By induction on $k \geqslant 2$ we show that

$$\max\{\max{}^r(I) \mid I \in S_k\} = 2k - 1. \tag{6}$$

The base case is when $k = 2$ and we computed above that $S_2 = \{\{2\}, \{1, 2\}, \{r-3, r-2\}\}$. Hence, $\max\{\max^r(I) \mid I \in S_2\} = 3 = 2 \cdot 2 - 1$. Now suppose that $k > 2$. Since

$\max^r(I) = 0$ for $I \in 2^{[k-1]}$ we may assume that $I \in T_k^s$ for some $3 \leqslant s \leqslant 2k-1$. So, $I = \alpha \sqcup \{r-s\} \sqcup F_s(\beta)$ where $\alpha \in 2^{[k-s+i]}$ and $\beta \in S_i$ for some $\max\{1, k-s\} \leqslant i \leqslant s/2$. Since α has no r variables we only need to consider $F_s(\beta)$. By induction since $i < k$ we have $\max^r(\beta) \leqslant 2i-1 < s$. Hence $\max^r(I) = s$ and since the maximum that s can be is $2k-1$ we have finished proving (6). Notice that within this proof we have also concluded that

$$\max^r(I) = s \tag{7}$$

for $I \in T_k^s$. If $I \in T_k^s$ is of the form $I = \alpha \sqcup \{r-s\} \sqcup F_s(\beta)$, since the elements of α are all integers and $F_{s'}(I) = F_{s'}(\alpha) \sqcup \{r-s\} \sqcup F_s(\beta)$, then we can also conclude that

$$\max^r(F_{s'}(I)) = \max\{s, s' - \min(\alpha)\}. \tag{8}$$

Now we show directly that for each $I = \alpha \sqcup \{r-s\} \sqcup F_s(\beta) \in T_k^s$ there does not exist another α' and β' such that $I = \alpha' \sqcup \{r-s\} \sqcup F_s(\beta')$. The base case is when $k = 2$ and in this case there is only $T_2^3 = \{\{r-3, r-2\}\}$ which only has one set as seen above. Now suppose $k > 2$ and there was such an α' and β'. Since the α and α' sets only have integer elements (i.e. no r variables) and all other elements contain the variable r then definitely $\alpha = \alpha'$. Also, $\alpha = \alpha' = J \cup \{k-s+i\}$ for some $J \in 2^{[k-s+i-1]}$. This implies that $|\beta| = |\beta'|$ and $\beta, \beta' \in S_i$ for some $\max\{1, s-k\} \leqslant i < s/2$. Pairing with (7) we have that $F_s(\beta) = F_s(\beta')$ with $\beta, \beta' \in S_i$. If $\beta, \beta' \in A_i$ then clearly $\beta = \beta'$ since f_s is injective when restricted to just integers.

Suppose $\beta \in A_i$ and $\beta' \in T_i^{s'}$ where $3 \leqslant s' \leqslant 2i-1$. Then $\min^r(F_s(\beta)) = s - i$. Also, there exists $\bar{\alpha} \in A_{i-s'+i'}$ and $\bar{\beta} \in S_{i'}$ such that $\beta' = \bar{\alpha} \sqcup \{r-s'\} \sqcup F_{s'}(\bar{\beta})$ where $\max\{1, s'-i\} \leqslant i' \leqslant s'/2$. Note that the function F_s is the identity on elements that are outputs from another function $F_{s'}$ because elements in the image set of $F_{s'}$ are of the form $r - d$ where d is an integer and the definition of F_s is to take these elements plug in to $r - s$ and then subtract off the s. Hence $F_s(\beta') = F_s(\bar{\alpha}) \sqcup \{r-s'\} \sqcup F_{s'}(\bar{\beta})$. Then $\min^r(F_s(\bar{\alpha})) = s - (i - s' + i') > s/2 + s'/2 > i + i' > i' \geqslant \min^r(F_{s'}(\bar{\beta}))$ by induction. Hence by induction on i, with the base $i = 1$ clear from the fact that $S_1 = \{\{1\}\}$ and so $\beta = \{1\}$, we have that $\min^r(F_{s'}(\beta')) \leqslant i$. Since $s - i > i$ we have concluded that it is impossible in this case to have $F_s(\beta) = F_{s'}(\beta')$.

In order to treat this next case we need another general inequality. Suppose $\beta \in T_i^{s'}$ with $\beta = \alpha \sqcup \{r-s'\} \sqcup F_{s'}(\lambda)$ where $\alpha \in A_{i-s'-i'}$, $\lambda \in S_{i'}$, and $\max\{1, s'-i\} \leqslant i' < s'/2$. Then picking s such that $\max\{1, s-k\} \leqslant i < s/2$ we will compute $F_s(\beta)$. Note that $\min^r(F_s(\alpha)) = s - (i - s' + i') = s - i + s_1 - i_1 > s/2 + s'/2 > s'$ and $\max^r\{F_s(\beta \backslash \alpha)\} = s'$. Hence

$$\min^r(F_s(\alpha)) > \max^r(F_s(\beta \backslash \alpha)). \tag{9}$$

Combining this with (8) we get

$$\max^r(F_s(\beta)) = s - \min(\alpha). \tag{10}$$

Now we can deal with the next case directly. Suppose that $\beta \in T_i^{s_1}$ and $\beta' \in T_i^{s_2}$ where $F_s(\beta) = F_s(\beta')$. So, there exists i_1 and i_2 satisfying $\max\{1, s_1 - i\} \leqslant i_1 \leqslant s_1/2$ and

$\max\{1, s_2-i\} \leqslant i_2 \leqslant s_2/2$ with $\beta = \alpha_1 \sqcup \{r-s_1\} \sqcup F_{s_1}(\lambda_1)$ and $\beta' = \alpha_2 \sqcup \{r-s_2\} \sqcup F_{s_2}(\lambda_2)$ where $\alpha_1 \in A_{i-s_1+i_1}$, $\alpha_2 \in A_{i-s_2+i_2}$, $\lambda_1 \in S_{i_1}$, and $\lambda_2 \in S_{i_2}$. Now assume that $s_1 \leqslant s_2$. So, by (9) we have that $\alpha_1 \supseteq \alpha_2$. Then we can consider $\beta \backslash \alpha_1$ and $\beta \backslash \alpha_2$. If $\alpha_2 \neq \emptyset$ then by a second induction on $|\beta| = |\beta'|$ (with the base case being $|\beta| = 1$ is trivial) we are done. If $\alpha_2 = \emptyset$ then $F_s(\beta') = \beta' = F_s(\alpha_1) \sqcup \{r-s_1\} \sqcup F_{s_1}(\lambda_1)$. If $|\alpha_1| \geqslant 1$ then $\beta' \backslash \{r-s_2\} = F_s(F_{s_2}(\lambda_2)) = F_s(\alpha_1 \backslash \{\min\{\alpha_1\}) \sqcup \{r-s_1\} \sqcup F_{s_1}(\lambda_1)$ and again by induction we are done. If $|\alpha_1| = \emptyset$ then $s_1 = s_2$ and $F_{s_1}(\lambda_2) = F_{s_1}(\lambda_1)$ and again by induction we are done. $\square$

The set S_k will be the index set which we will sum over. But we need to create a "top heavy" partner for each index set I to get the full formula. To do this we need a function $d : S_k \to \mathbb{Z}[r]$ defined as follows. For $I \in A_k$ define

$$d(I) = \begin{cases} k & \text{if } I = \{k\} \\ k - \max\{I \backslash \{k\}\} & \text{if } I \neq \{k\} \end{cases}$$

and for $I \in T_k^s$ define

$$d(I) = \min{}^r(I \backslash \{r - \min{}^r(I)\}) - \min{}^r(I).$$

For example, $d(\{2\}) = 2$, $d(\{1,2\}) = 1$, and $d(\{r-3, r-2\}) = \min^r(\{r-3\}) - 2 = 1$. Finally the "top heavy" partner for $I \in S_k$ is

$$t(I) = \begin{cases} I \backslash \{k\} \cup \{r - d(I)\} & \text{if } I \in A_k \\ I \backslash \{r - \min^r(I)\} \cup \{r - d(I)\} & \text{otherwise.} \end{cases}$$

For example, $t(\{2\}) = \{r-2\}$, $t(\{1,2\}) = \{1, r-1\}$, and $t(\{r-3, r-2\}) = \{r-3, r-1\}$ and again notice that these are exactly the index sets for the "top terms" (left hand) of the "top heavy" pairs in the brackets of (3) in Proposition 9.

The last piece of the formula we need is a sign function $s_k : S_k \to \mathbb{Z}$. We also do this recursively. The base is $k = 1$ and we set $s_1(\{1\}) = 0$. For $k > 1$ again we split this up differently for $I \in A_k$ and $I \in \bigcup_{s=3}^{2k-1} T_k^s$. For $I \in A_k$ set $s_k(I) = |I| - 1$. For $I \in T_k^s(i)$ there exists $\alpha \in A_{k-s+i}$ and $\beta \in S_i$ such that $I = \alpha \sqcup \{r-s\} \sqcup F_s(\beta)$ where $s_i(\beta)$ is already defined in the context of S_i. Then set $s_k(I) = |\alpha| + s_i(\beta)$. This makes sense because of Lemma 10. For example, $s_2(\{2\}) = 0$, $s_2(\{1,2\}) = 1$. For $\{r-3, r-2\}$ we get that $\alpha = \emptyset$ and $\beta = \{1\}$ as seen in 5. Hence $s_2(\{r-3, r-2\}) = 0 + 0$ and these signs again match the "top heavy" decomposition of the Kazhdan-Lusztig coefficient of the $k = 2$ quadratic term in (3) of Proposition 9. These are all the ingredients we need for the main formula.

5 Top-heavy formula for Kazhdan-Lusztig polynomials

In this section we state and prove the main formula which is the following theorem.

Theorem 11. *For any finite, ranked lattice $\mathcal{P}$ with rank r the degree k coefficient of the Kazhdan-Lusztig polynomial of $\mathcal{P}$ for $1 \leqslant k < r/2$ is*

$$\sum_{I \in S_k} (-1)^{s_k(I)} \Big(W_{t(I)}(\mathcal{P}) - W_I(\mathcal{P}) \Big).$$

Proof. Induct on k. The base $k = 1$ is done in Proposition 9. Now we compute the degree k term where $k > 1$. In (3) of the recursion in Definition 8 the left hand side has the degree k coefficient on the t^{r-k} term. This is the terms that we will examine on the right hand side. First we split the right hand side up in terms of rank so that we rewrite it as

$$\sum_{s=0}^{r} \sum_{F \in \mathcal{P}_{r-s}} \chi(\mathcal{P}_F, t) P(\mathcal{P}^F, t). \tag{11}$$

Now we reduce this further. Suppose that $s > 2k-1$. Then for $F \in \mathcal{P}_{r-s}$ $\deg(\chi(\mathcal{P}_F, t)) = r-s$ and $\deg(P(\mathcal{P}^F, t)) < s/2$. So, $deg(\chi(\mathcal{P}_F, t)) + \deg(P(\mathcal{P}^F, t)) < r-s+s/2 = r-s/2 \leqslant r-k$. Hence we can reduce (11) to

$$\sum_{s=0}^{2k-1} \sum_{F \in \mathcal{P}_{r-s}} \chi(\mathcal{P}_F, t) P(\mathcal{P}^F, t). \tag{12}$$

Note that since $k < r/2$ and $\deg(P(\mathcal{P}^F, t)) = s$ we know that the coefficients $P(\mathcal{P}^F, t)$ will all be computed by induction. For any polynomial p let $u(i, p)$ denote the coefficient of the i^{th} term and $d(i, p)$ be the i^{th} term down from the top term (i.e. if $\deg p = d$ then $d(i, p) = u(d-i, p)$). Then for each term in (12) the possible products which will yield a degree $r-k$ term are of the form

$$d(k-s+i, \chi(\mathcal{P}_F, t)) u(i, P(\mathcal{P}^F, t))$$

where $\max\{0, s-k\} \leqslant i < s/2$. Hence the total coefficient we are seeking is

$$\sum_{s=0}^{2k-1} \sum_{F \in \mathcal{P}_{r-s}} \sum_{\substack{\max\{0,s-k\} \\ \leqslant i < s/2}} d(k-s+i, \chi(\mathcal{P}_F, t)) u(i, P(\mathcal{P}^F, t)). \tag{13}$$

We first focus on the terms where $i = 0$. For these terms we have by Proposition 9

$$u(0, P(\mathcal{P}^F, t)) = 1$$

and

$$d(k-s, \chi(\mathcal{P}_F, t)) = w_{0,k-s}(\mathcal{P}_F). \tag{14}$$

Then we use Theorem 7 on (14) to get

$$d(k-s, \chi(\mathcal{P}_F, t)) = \sum_{I \subseteq \{1, \ldots, k-s-1\}} (-1)^{|I|+1} W_{I \cup \{k-s\}}(\mathcal{P}_F). \tag{15}$$

Notice that $0 \leqslant s \leqslant k$. Summing over $F \in \mathcal{P}_{r-s}$ and applying Lemma 4 to (15) we get

$$\sum_{F \in \mathcal{P}_{r-s}} d(k-s, \chi(\mathcal{P}_F, t)) = \sum_{I \subseteq \{1,\ldots,k-s-1\}} (-1)^{|I|+1} W_{I \cup \{k-s\} \cup \{r-s\}}(\mathcal{P}) \tag{16}$$

as long as $s \neq 0$. In that case since $\mathcal{P}$ is a lattice the sum only contains one term where $\mathcal{P}_F = \mathcal{P}$. Hence

$$d(k, \chi(\mathcal{P}, t)) = \sum_{I \subseteq \{1,\ldots,k-1\}} (-1)^{|I|+1} W_{I \cup \{k\}}(\mathcal{P}). \tag{17}$$

The subscripts of (17) give exactly all the terms of A_k as well as the signs $s_k(I)$ for $I \in A_k$. Finally for each $I \subseteq A_k$ with $I = J \cup \{k-s\} \cup \{k\}$ for $1 \leqslant s < k$ there is exactly one term in (16), that being $J \cup \{k-s\} \cup \{r-s\}$ which is the top heavy pair to I. Also note that this exactly covers all the terms of (16) and (17). Hence we have verified the formula for $i = 0$ and equivalently A_k.

Next we focus on the case where $1 \leqslant i$. Since $i < s/2 \leqslant k$ we have by induction that

$$u(i, P(\mathcal{P}^F, t)) = \sum_{I \in S_i} (-1)^{s_i(I)} \Big(W_{t(I)}(\mathcal{P}^F) - W_I(\mathcal{P}^F) \Big). \tag{18}$$

Because $F \in \mathcal{P}_{r-s}$ we know $\mathrm{rk}(\mathcal{P}^F) = s$ and so when using this induction all the subscripts in the formula have r replaced with s. The terms coming from the characteristic polynomial are

$$d(k-s+i, \chi(\mathcal{P}_F, t)) = w_{0,k-s+i}(\mathcal{P}_F). \tag{19}$$

Again using Theorem 7 (19) becomes

$$d(k-s+i, \chi(\mathcal{P}_F, t)) = \sum_{\alpha \in A_{k-s+i}} (-1)^{|\alpha|} W_\alpha(\mathcal{P}_F). \tag{20}$$

Next putting (18) and (20) together for each $i > 0$ term of (13) we get

$$\sum_{s=0}^{2k-1} \left[\sum_{\alpha \in A_{k-s+i}} (-1)^{|\alpha|} W_\alpha(\mathcal{P}_F) \right] \left[\sum_{\beta \in S_i} (-1)^{s_i(\beta)} \Big(W_{t(\beta)}(\mathcal{P}^F) - W_\beta(\mathcal{P}^F) \Big) \right]. \tag{21}$$

Moving sums together (21) becomes

$$\sum_{s=0}^{2k-1} \sum_{\alpha \in A_{k-s+i}} \sum_{\beta \in S_i} (-1)^{|\alpha| + s_i(\beta)} \Big(W_\alpha(\mathcal{P}_F) W_{t(\beta)}(\mathcal{P}^F) - W_\alpha(\mathcal{P}_F) W_\beta(\mathcal{P}^F) \Big). \tag{22}$$

Then summing over $F \in \mathcal{P}_{r-s}$ and applying Lemma 6 to (22) we get

$$\sum_{s=0}^{2k-1} \sum_{\alpha \in A_{k-s+i}} \sum_{\beta \in S_i} (-1)^{|\alpha| + s_i(\beta)} \Big(W_{\alpha \cup \{r-s\} \cup t(\beta)[r-s]}(\mathcal{P}) - W_{\alpha \cup \{r-s\} \cup \beta[r-s]}(\mathcal{P}) \Big). \tag{23}$$

This all makes sense because inside $\mathcal{P}^F$ the rank is s and all the elements of every β above are $< s$. Hence in the total lattice $\mathcal{P}$ we can add $r-s$ and satisfy the hypothesis of Lemma 6. Finally to finish the proof note that $t(\beta)[r-s] = t(F_s(\beta))$ and $\beta[r-s] = F_s(\beta)$. $\square$

In Table 1 we print the formula from Theorem 11 for $k = \{1, 2, 3, 4, 5\}$. For $k = 6$ the formula takes up too much space. This was calculated using Sage (see [4]). There we only list the index set $S(k)$ and the corresponding sign $s_k(I)$.

k	$s_k(I)I$ for I in $S(k)$
1	+[1]
2	+[2], +[r - 3, r - 2], -[1, 2]
3	+[3], -[1, 3], -[2, 3], +[1, 2, 3], -[1, r - 3, r - 2], +[r - 4, r - 3], +[r - 5, r - 3], +[r - 5, r - 3, r - 2], -[r - 5, r - 4, r - 3]
4	+[4], -[1, 4], -[2, 4], +[1, 2, 4], -[3, 4], +[1, 3, 4], +[2, 3, 4], -[1, 2, 3, 4], -[2, r - 3, r - 2], +[1, 2, r - 3, r - 2], -[1, r - 4, r - 3], +[r - 5, r - 4], -[1, r - 5, r - 3], -[1, r - 5, r - 3, r - 2], +[1, r - 5, r - 4, r - 3], +[r - 6, r - 4], +[r - 6, r - 3, r - 2], -[r - 6, r - 5, r - 4], +[r - 7, r - 4], -[r - 7, r - 6, r - 4], -[r - 7, r - 5, r - 4], +[r - 7, r - 6, r - 5, r - 4], -[r - 7, r - 6, r - 3, r - 2], +[r - 7, r - 4, r - 3], +[r - 7, r - 5, r - 3], +[r - 7, r - 5, r - 3, r - 2], -[r - 7, r - 5, r - 4, r - 3]
5	+[5], -[1, 5], -[2, 5], +[1, 2, 5], -[3, 5], +[1, 3, 5], +[2, 3, 5], -[1, 2, 3, 5], -[4, 5], +[1, 4, 5], +[2, 4, 5], -[1, 2, 4, 5], +[3, 4, 5], -[1, 3, 4, 5], -[2, 3, 4, 5], +[1, 2, 3, 4, 5], -[3, r - 3, r - 2], +[1, 3, r - 3, r - 2], +[2, 3, r - 3, r - 2], -[1, 2, 3, r - 3, r - 2], -[2, r - 4, r - 3], +[1, 2, r - 4, r - 3], -[1, r - 5, r - 4], +[r - 6, r - 5], +[r - 7, r - 5], +[r - 7, r - 3, r - 2], -[r - 7, r - 6, r - 5], +[r - 8, r - 5], -[r - 8, r - 7, r - 5], -[r - 8, r - 6, r - 5], +[r - 8, r - 7, r - 6, r - 5], -[r - 8, r - 7, r - 3, r - 2], +[r - 8, r - 4, r - 3], +[r - 8, r - 5, r - 3], +[r - 8, r - 5, r - 3, r - 2], -[r - 8, r - 5, r - 4, r - 3], +[r - 9, r - 5], -[r - 9, r - 8, r - 5], -[r - 9, r - 7, r - 5], +[r - 9, r - 8, r - 7, r - 5], -[r - 9, r - 6, r - 5], +[r - 9, r - 8, r - 6, r - 5], +[r - 9, r - 7, r - 6, r - 5], -[r - 9, r - 8, r - 7, r - 6, r - 5], -[r - 9, r - 7, r - 3, r - 2], +[r - 9, r - 8, r - 7, r - 3, r - 2], -[r - 9, r - 8, r - 4, r - 3], +[r - 9, r - 5, r - 4], +[r - 9, r - 6, r - 4], +[r - 9, r - 6, r - 3, r - 2], -[r - 9, r - 6, r - 5, r - 4], +[r - 9, r - 7, r - 4], -[r - 9, r - 7, r - 6, r - 4], -[r - 9, r - 7, r - 5, r - 4], +[r - 9, r - 7, r - 6, r - 5, r - 4], -[r - 9, r - 7, r - 6, r - 3, r - 2], +[r - 9, r - 7, r - 4, r - 3], +[r - 9, r - 7, r - 5, r - 3], +[r - 9, r - 7, r - 5, r - 3, r - 2], -[r - 9, r - 7, r - 5, r - 4, r - 3]

Table 1: Low degree coefficient formulas for the matroid KL polynomial

Remark 12. Note that Lemma 10 shows that there is exactly one term of each type in the formula of Theorem 11.

The top heavy pairs decomposition of Theorem 11 together with the "Hyperplane Theorem" ([2], Proposition 8.5.1) gives an easy proof of the "hard" implication of Proposition 2.14 of [13].

Corollary 13. *If L is a modular lattice then $P(L, t) = 1$.*

Remark 14. Recently in [9] June Huh and Botong Wang proved that if the poset is the lattice of flats of a realizable matroid then each term in Theorem 11 is positive (this was called the "top heaviness conjecture", see [12]). Also, it is conjectured that each of the coefficients themselves are positive for matroids (see [13]). However many of the signs s_k are negative and at the moment we do not see a general relationship between these conjectures other than the formula of Theorem 11 itself. A nice line of future research would be to investigate how Theorem 11 and Huh and Wang's result could prove the non-negativity of the coefficients of the matroid Kazhdan-Lusztig polynomials.

Remark 15. The function $\alpha : [n] \to \mathbb{Z}$ defined by $\alpha(I) = W_I$ is called the *flag f-vector* in the literature (see [10] and [23]). The flag f-vector is used in the original definition of the so called *cd-index* of the poset $\mathcal{P}$. One possible problem for future research could be to find a relationship between the cd-index and the matroid Kazhdan-Lusztig polynomial using Theorem 11.

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