A Brooks type theorem for the maximum local edge connectivity

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Abstract

For a graph $G$, let $\chi(G)$ and $\lambda(G)$ denote the chromatic number of $G$ and the maximum local edge connectivity of $G$, respectively. A result of Dirac implies that every graph $G$ satisfies $\chi(G) \leq \lambda(G) + 1$. In this paper we characterize the graphs $G$ for which $\chi(G) = \lambda(G) + 1$. The case $\lambda(G) = 3$ was already solved by Aboulker, Brettell, Havet, Marx, and Trotignon. We show that a graph $G$ with $\lambda(G) = k \geq 4$ satisfies $\chi(G) = k + 1$ if and only if $G$ contains a block which can be obtained from copies of $K_{k+1}$ by repeated applications of the Hajós join.

Keywords: graph coloring; connectivity; critical graphs; Brooks' theorem

1 Introduction and main result

The paper deals with the classical vertex coloring problem for graphs. The term graph refers to a finite undirected graph without loops and without multiple edges. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the least number of colors needed to color the vertices of $G$ such that each vertex receives a color and adjacent vertices receive different colors. There are several degree bounds for the chromatic number. For a graph $G$, let $\delta(G) = \min_{v \in V(G)} d_G(v)$ and $\Delta(G) = \max_{v \in V(G)} d_G(v)$ denote the minimum degree and the maximum degree of $G$, respectively. Furthermore, let

$$\text{col}(G) = 1 + \max_{H \subseteq G} \delta(H)$$

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denote the coloring number of $G$, and let

$$\text{mad}(G) = \max_{\varnothing \neq H \subseteq G} \frac{2|E(H)|}{|V(H)|}$$

denote the maximum average degree of $G$. By $H \subseteq G$ we mean that $H$ is a subgraph of $G$. If $G$ is the empty graph, that is, $V(G) = \varnothing$, we briefly write $G = \varnothing$ and define $\delta(G) = \Delta(G) = \text{mad}(G) = 0$ and $\text{col}(G) = 1$. A simple sequential coloring argument shows that $\chi(G) \leq \text{col}(G)$, which implies that every graph $G$ satisfies

$$\chi(G) \leq \text{col}(G) \leq \lfloor \text{mad}(G) \rfloor + 1 \leq \Delta(G) + 1.$$

These inequalities were discussed in a paper by Jensen and Toft [10]. Brooks’ famous theorem provides a characterization for the class of graphs $G$ satisfying $\chi(G) = \Delta(G) + 1$. Let $k \geq 0$ be an integer. For $k \neq 2$, let $B_k$ denote the class of complete graphs having order $k + 1$; and let $B_2$ denote the class of odd cycles. A graph in $B_k$ has maximum degree $k$ and chromatic number $k + 1$. Brooks’ theorem [2] is as follows.

**Theorem 1** (Brooks 1941). Let $G$ be a non-empty graph. Then $\chi(G) \leq \Delta(G) + 1$ and equality holds if and only if $G$ has a connected component belonging to the class $B_{\Delta(G)}$.

In this paper we are interested in connectivity parameters of graphs. Let $G$ be a graph with at least two vertices. The local connectivity $\kappa_G(v, w)$ of distinct vertices $v$ and $w$ is the maximum number of internally vertex disjoint $v$-$w$ paths of $G$. The local edge connectivity $\lambda_G(v, w)$ of distinct vertices $v$ and $w$ is the maximum number of edge-disjoint $v$-$w$ paths of $G$. The maximum local connectivity of $G$ is

$$\kappa(G) = \max \{\kappa_G(v, w) \mid v, w \in V(G), v \neq w\},$$

and the maximum local edge connectivity of $G$ is

$$\lambda(G) = \max \{\lambda_G(v, w) \mid v, w \in V(G), v \neq w\}.$$

For a graph $G$ with $|G| \leq 1$, we define $\kappa(G) = \lambda(G) = 0$. Clearly, the definition implies that $\kappa(G) \leq \lambda(G)$ for every graph $G$. By a result of Mader [11] it follows that $\delta(G) \leq \kappa(G)$. Since $\kappa$ is a monotone graph parameter in the sense that $H \subseteq G$ implies $\kappa(H) \leq \kappa(G)$, it follows that every graph $G$ satisfies $\text{col}(G) \leq \kappa(G) + 1$. Consequently, every graph $G$ satisfies

$$\chi(G) \leq \text{col}(G) \leq \kappa(G) + 1 \leq \lambda(G) + 1 \leq \Delta(G) + 1. \quad (1)$$

Our aim is to characterize the class of graphs $G$ for which $\chi(G) = \lambda(G) + 1$. For such a characterization we use the fact that if we have an optimal coloring of each block of a graph $G$, then we can combine these colorings to an optimal coloring of $G$ by permuting colors in the blocks if necessary. For every non-empty graph $G$, we thus have

$$\chi(G) = \max \{\chi(H) \mid H \text{ is a block of } G\}. \quad (2)$$
We also need a famous construction, first used by Hajós [9]. Let \( G_1 \) and \( G_2 \) be two vertex-disjoint graphs and, for \( i = 1, 2 \), let \( e_i = v_i w_i \) be an edge of \( G_i \). Let \( G \) be the graph obtained from \( G_1 \) and \( G_2 \) by deleting the edges \( e_1 \) and \( e_2 \) from \( G_1 \) and \( G_2 \), respectively, identifying the vertices \( v_1 \) and \( v_2 \), and adding the new edge \( w_1 w_2 \). We then say that \( G \) is the Hajós join of \( G_1 \) and \( G_2 \) and write \( G = (G_1, v_1, w_1) \triangle (G_2, v_2, w_2) \) or briefly \( G = G_1 \triangle G_2 \).

For an integer \( k \geq 0 \) we define a class \( \mathcal{H}_k \) of graphs as follows. If \( k \leq 2 \), then \( \mathcal{H}_k = \mathcal{B}_k \). The class \( \mathcal{H}_3 \) is the smallest class of graphs that contains all odd wheels and is closed under taking Hajós joins. Recall that an odd wheel is a graph obtained from an odd cycle by adding a new vertex and joining this vertex to all vertices of the cycle. If \( k \geq 4 \), then \( \mathcal{H}_k \) is the smallest class of graphs that contains all complete graphs of order \( k + 1 \) and is closed under taking Hajós joins. Our main result is the following counterpart of Brooks’ theorem. In fact, Brooks’ theorem may easily be deduced from it.

**Theorem 2.** Let \( G \) be a non-empty graph. Then \( \chi(G) \leq \lambda(G) + 1 \) and equality holds if and only if \( G \) has a block belonging to the class \( \mathcal{H}_{\lambda(G)} \).

For the proof of this result, let \( G \) be a non-empty graph with \( \lambda(G) = k \). By (1), we obtain \( \chi(G) \leq k + 1 \). By an observation of Hajós [9] it follows that every graph in \( \mathcal{H}_k \) has chromatic number \( k + 1 \). Hence if some block of \( G \) belongs to \( \mathcal{H}_k \), then (2) implies that \( \chi(G) = k + 1 \). So it only remains to show that if \( \chi(G) = k + 1 \), then some block of \( G \) belongs to \( \mathcal{H}_k \). For proving this, we shall use the critical graph method, see [12].

A graph \( G \) is critical if every proper subgraph \( H \) of \( G \) satisfies \( \chi(H) < \chi(G) \). We shall use the following two properties of critical graphs. As an immediate consequence of (2) we obtain that if \( G \) is a critical graph, then \( G = \emptyset \) or \( G \) contains no separating vertex, implying that \( G \) is its only block. Furthermore, every graph contains a critical subgraph with the same chromatic number.

Let \( G \) be a non-empty graph with \( \lambda(G) = k \) and \( \chi(G) = k + 1 \). Then \( G \) contains a critical subgraph \( H \) with chromatic number \( k + 1 \), and we obtain that \( \lambda(H) \leq \lambda(G) = k \). So the proof of Theorem 2 is complete if we can show that \( H \) is a block of \( G \) which belongs to \( \mathcal{H}_k \). For an integer \( k \geq 0 \), let \( \mathcal{C}_k \) denote the class of graphs \( H \) such that \( H \) is a critical graph with chromatic number \( k + 1 \) and with \( \lambda(H) \leq k \). We shall prove that the two classes \( \mathcal{C}_k \) and \( \mathcal{H}_k \) are the same.

## 2 Connectivity of critical graphs

In this section we shall review known results about the structure of critical graphs. First we need some notation. Let \( G \) be an arbitrary graph. For an integer \( k \geq 0 \), let \( \mathcal{C}O_k(G) \) denote the set of all colorings of \( G \) with color set \( \{1, 2, \ldots, k\} \). Then a function \( f : V(G) \rightarrow \{1, 2, \ldots, k\} \) belongs to \( \mathcal{C}O_k(G) \) if and only if \( f^{-1}(c) \) is an independent vertex set of \( G \) (possibly empty) for every color \( c \in \{1, 2, \ldots, k\} \). A set \( S \subseteq V(G) \cup E(G) \) is called a separating set of \( G \) if \( G - S \) has more components than \( G \). A vertex \( v \) of \( G \) is called a separating vertex of \( G \) if \( \{v\} \) is a separating set of \( G \). An edge \( e \) of \( G \) is called a bridge of \( G \) if \( \{e\} \) is a separating set of \( G \). For a vertex set \( X \subseteq V(G) \), let \( \partial_G(X) \)
denote the set of all edges of $G$ having exactly one end in $X$. Clearly, if $G$ is connected and $\emptyset \neq X \subset V(G)$, then $F = \partial_G(X)$ is a separating set of edges of $G$. The converse is not true. However if $F$ is a minimal separating edge set of a connected graph $G$, then $F = \partial_G(X)$ for some vertex set $X$. As a consequence of Menger’s theorem about edge connectivity, we obtain that if $v$ and $w$ are distinct vertices of $G$, then

$$\lambda_G(v, w) = \min\{|\partial_G(X)| \mid X \subseteq V(G), v \in X, w \notin X\}.$$  

Color critical graphs were first introduced and investigated by Dirac in the 1950s. He established the basic properties of critical graphs in a series of papers [3], [4] and [5]. Some of these basic properties are listed in the next theorem.

**Theorem 3** (Dirac 1952). Let $G$ be a critical graph with chromatic number $k + 1$ for an integer $k \geq 0$. Then the following statements hold:

(a) $\delta(G) \geq k$.

(b) If $k \in \{0, 1\}$, then $G$ is a complete graph of order $k + 1$; and if $k = 2$, then $G$ is an odd cycle.

(c) No separating vertex set of $G$ is a clique of $G$. As a consequence, $G$ is connected and has no separating vertex, i.e., $G$ is a block.

(d) If $v$ and $w$ are two distinct vertices of $G$, then $\lambda_G(v, w) \geq k$. As a consequence $G$ is $k$-edge-connected.

Theorem 3(a) leads to a very natural way of classifying the vertices of a critical graph into two classes. Let $G$ be a critical graph with chromatic number $k + 1$. The vertices of $G$ having degree $k$ in $G$ are called low vertices of $G$, and the remaining vertices are called high vertices of $G$. So any high vertex of $G$ has degree at least $k + 1$ in $G$. Furthermore, let $G_L$ be the subgraph of $G$ induced by the low vertices of $G$, and let $G_H$ be the subgraph of $G$ induced by the high vertices of $G$. We call $G_L$ the low vertex subgraph of $G$ and $G_H$ the high vertex subgraph of $G$. This classification is due to Gallai [8] who proved the following theorem. Note that statements (b) and (c) of Gallai’s theorem are simple consequences of statement (a), which is an extension of Brooks’ theorem.

**Theorem 4** (Gallai 1963). Let $G$ be a critical graph with chromatic number $k + 1$ for an integer $k \geq 1$. Then the following statements hold:

(a) Every block of $G_L$ is a complete graph or an odd cycle.

(b) If $G_H = \emptyset$, then $G$ is a complete graph of order $k + 1$ if $k \neq 2$, and $G$ is an odd cycle if $k = 2$.

(c) If $|G_H| = 1$, then either $G$ has a separating vertex set of two vertices or $k = 3$ and $G$ is an odd wheel.
As observed by Dirac, a critical graph is connected and contains no separating vertex. Dirac [3] and Gallai [8] characterized critical graphs having a separating vertex set of size two. In particular, they proved the following theorem, which shows how to decompose a critical graph having a separating vertex set of size two into smaller critical graphs.

**Theorem 5** (Dirac 1952 and Gallai 1963). Let $G$ be a critical graph with chromatic number $k + 1$ for an integer $k \geq 3$, and let $S \subseteq V(G)$ be a separating vertex set of $G$ with $|S| \leq 2$. Then $S$ is an independent vertex set of $G$ consisting of two vertices, say $v$ and $w$, and $G - S$ has exactly two components $H_1$ and $H_2$. Moreover, if $G_i = G[V(H_i) \cup S]$ for $i \in \{1, 2\}$, we can adjust the notation so that for some coloring $f_1 \in CO_k(G_1)$ we have $f_1(v) = f_1(w)$. Then the following statements hold:

(a) Every coloring $f \in CO_k(G_1)$ satisfies $f(v) = f(w)$ and every coloring $f \in CO_k(G_2)$ satisfies $f(v) \neq f(w)$.

(b) The subgraph $G'_1 = G'_1 + vw$ obtained from $G_1$ by adding the edge $vw$ is critical and has chromatic number $k + 1$.

(c) The vertices $v$ and $w$ have no common neighbor in $G_2$ and the subgraph $G'_2 = G_2/S$ obtained from $G_2$ by identifying $v$ and $w$ is critical and has chromatic number $k + 1$.

Dirac [6] and Gallai [8] also proved the converse theorem, that $G$ is critical and has chromatic number $k + 1$ provided that $G'_1$ is critical and has chromatic number $k + 1$ and $G_2$ obtained from the critical graph $G'_2$ with chromatic number $k + 1$ by splitting a vertex into $v$ and $w$ has chromatic number $k$.

Hajós [9] invented his construction to characterize the class of graphs with chromatic number at least $k + 1$. Another advantage of the Hajós join is the well known fact that it not only preserve the chromatic number, but also criticality. It may be viewed as a special case of the Dirac–Gallai construction, described above.

**Theorem 6** (Hajós 1961). Let $G = G_1 \triangle G_2$ be the Hajós join of two graphs $G_1$ and $G_2$, and let $k \geq 3$ be an integer. Then $G$ is critical and has chromatic number $k + 1$ if and only if both $G_1$ and $G_2$ are critical and have chromatic number $k + 1$.

If $G$ is the Hajós join of two graphs that are critical and have chromatic number $k + 1$, where $k \geq 3$, then $G$ is critical and has chromatic number $k + 1$. Moreover, $G$ has a separating set consisting of one edge and one vertex. Theorem 5 implies that the converse statement also holds.

**Theorem 7.** Let $G$ be a critical graph graph with chromatic number $k + 1$ for an integer $k \geq 3$. If $G$ has a separating set consisting of one edge and one vertex, then $G$ is the Hajós join of two graphs.

Next we will discuss a decomposition result for critical graphs having chromatic number $k + 1$ an having an separating edge set of size $k$. Let $G$ be an arbitrary graph. By an edge cut of $G$ we mean a triple $(X,Y,F)$ such that $X$ is a non-empty proper subset of
\(V(G), Y = V(G) \setminus X, \) and \(F = \partial_G(X) = \partial_G(Y).\) If \((X, Y, F)\) is an edge cut of \(G,\) then we denote by \(X_F (\text{respectively } Y_F)\) the set of vertices of \(X (\text{respectively, } Y)\) which are incident to some edge of \(F.\) An edge cut \((X, Y, F)\) of \(G\) is non-trivial if \(|X_F| \geq 2\) and \(|Y_F| \geq 2.\) The following decomposition result was proved independently by T. Gallai and Toft [13].

**Theorem 8** (Toft 1970). Let \(G\) be a critical graph with chromatic number \(k + 1\) for an integer \(k \geq 3,\) and let \(F \subseteq E(G)\) be a separating edge set of \(G\) with \(|F| \leq k.\) Then \(|F| = k\) and there is an edge cut \((X, Y, F)\) of \(G\) satisfying the following properties:

\(a\) Every coloring \(f \in \mathcal{C}O_k(G[X])\) satisfies \(|f(X_F)| = 1\) and every coloring \(f \in \mathcal{C}O_k(G[Y])\) satisfies \(|f(Y_F)| = k.\)

\(b\) The subgraph \(G_1\) obtained from \(G[X \cup Y_F]\) by adding all edges between the vertices of \(Y_F,\) so that \(Y_F\) becomes a clique of \(G_1,\) is critical and has chromatic number \(k + 1.\)

\(c\) The subgraph \(G_2\) obtained from \(G[Y]\) by adding a new vertex \(v\) and joining \(v\) to all vertices of \(Y_F\) is critical and has chromatic number \(k + 1.\)

A particular nice proof of this result is due to T. Gallai (oral communication to the second author). Recall that the clique number of a graph \(G,\) denoted by \(\omega(G),\) is the largest cardinality of a clique in \(G.\) A graph \(G\) is perfect if every induced subgraph \(H\) of \(G\) satisfies \(\chi(H) = \omega(H).\) For the proof of the next lemma, due to Gallai, we use the fact that complements of bipartite graphs are perfect.

**Lemma 9.** Let \(H\) be a graph and let \(k \geq 3\) be an integer. Suppose that \((A, B, F')\) is an edge cut of \(H\) such that \(|F'| \leq k\) and \(A\) as well as \(B\) are cliques of \(H\) with \(|A| = |B| = k.\) If \(\chi(H) \geq k + 1,\) then \(|F'| = k\) and \(F' = \partial_H(\{v\})\) for some vertex \(v\) of \(H.\)

**Proof.** The graph \(H\) is perfect and so \(\omega(H) = \chi(H) \geq k + 1.\) Consequently, \(H\) contains a clique \(X\) with \(|X| = k + 1.\) Let \(s = |A \cap X|\) and hence \(k + 1 - s = |B \cap X|.\) Since \(|A| = |B| = k,\) this implies that \(s \geq 1\) and \(k + 1 - s \geq 1.\) Since \(X\) is a clique of \(H,\) the set \(E'\) of edges of \(H\) joining a vertex of \(A \cap X\) with a vertex of \(B \cap X\) satisfies \(|E'| \leq k'\) and \(|E'| = s(k + 1 - s).\) The function \(g(s) = s(k + 1 - s)\) is strictly concave on the real interval \([1, k]\) as \(g''(s) = -2.\) Since \(g(1) = g(k) = k,\) we conclude that \(g(s) > k\) for all \(s \in (1, k).\) Since \(g(s) = |E'| \leq |F'| \leq k,\) this implies that \(s = 1\) or \(s = k.\) In both cases we obtain that \(|E'| = |F'| = k,\) and hence \(E' = F' = \partial_H(\{v\})\) for some vertex \(v\) of \(H.\)

Based on Lemma 9 it is easy to give a proof of Theorem 8, see also the paper by Dirac, Sorensen, and Toft [7]. Theorem 8 is a reformulation of a result by Toft [13, Chapter 4] in his Ph.D thesis. Toft gave a complete characterization of the class of critical graphs, having chromatic number \(k + 1\) and containing a separating edge set of size \(k.\) The characterization involves critical hypergraphs.
3 Proof of the main result

Theorem 10. Let \( k \geq 0 \) be an integer. Then the two graph classes \( \mathcal{C}_k \) and \( \mathcal{H}_k \) coincide.

Proof. That the two classes \( \mathcal{C}_k \) and \( \mathcal{H}_k \) coincide if \( 0 \leq k \leq 2 \) follows from Theorem 3(b). In this case both classes consists of all critical graphs with chromatic number \( k + 1 \).

In what follows we therefore assume that \( k \geq 3 \). The proof of the following claim is straightforward and left to the reader.

Claim 1. The odd wheels belong to the class \( \mathcal{C}_3 \) and the complete graphs of order \( k + 1 \) belong to the class \( \mathcal{C}_k \).

Claim 2. Let \( k \geq 3 \) be an integer, and let \( G = G_1 \triangle G_2 \) the Hajós join of two graphs \( G_1 \) and \( G_2 \). Then \( G \) belongs to the class \( \mathcal{C}_k \) if and only if both \( G_1 \) and \( G_2 \) belong to the class \( \mathcal{C}_k \).

Proof: We may assume that \( G = (G_1, v_1, w_1) \triangle (G_2, v_2, w_2) \) and \( v \) is the vertex of \( G \) obtained by identifying \( v_1 \) and \( v_2 \). First suppose that \( G_1, G_2 \in \mathcal{C}_k \). From Theorem 6 it follows that \( G \) is critical and has chromatic number \( k + 1 \). So it suffices to prove that \( \lambda(G) \leq k \). To this end let \( u \) and \( u' \) be distinct vertices of \( G \) and let \( p = \lambda_G(u, u') \). Then there is a system \( \mathcal{P} \) of \( p \) edge disjoint \( u-u' \) paths in \( G \). If \( u \) and \( u' \) belong both to \( G_1 \), then only one path \( P \) of \( \mathcal{P} \) may contain vertices not in \( G_1 \). In this case \( P \) contains the vertex \( v \) and the edge \( w_1w_2 \). If we replace in \( P \) the subpath \( uPw_1 \) by the edge \( v_1w_1 \), we obtain a system of \( p \) edge disjoint \( u-u' \) paths in \( G_1 \), and hence \( p \leq \lambda_{G_1}(u, u') \leq k \). If \( u \) and \( u' \) belong to \( G_2 \), a similar argument shows that \( p \leq k \). It remains to consider the case that one vertex, say \( u \), belongs to \( G_1 \) and the other vertex \( u' \) belongs to \( G_2 \). By symmetry we may assume that \( u \neq v \). Again at most one path \( P \) of \( \mathcal{P} \) uses the edge \( w_1w_2 \) and the remaining paths of \( \mathcal{P} \) all uses the vertex \( v (= v_1 = v_2) \). If we replace \( P \) by the path \( uPw_1 + w_1v_1 \), then we obtain \( p \) edge disjoint \( u-v_1 \) path in \( G_1 \), and hence \( p \leq \lambda_{G_1}(u, v_1) \leq k \). This shows that \( \lambda(G) \leq k \) and so \( G \in \mathcal{C}_k \).

Suppose conversely that \( G \in \mathcal{C}_k \). From Theorem 6 it follows that \( G_1 \) and \( G_2 \) are critical graphs, both with chromatic number \( k + 1 \). So it suffices to show that \( \lambda(G_i) \leq k \) for \( i = 1, 2 \). By symmetry it suffices to show that \( \lambda(G_1) \leq k \). To this end let \( u \) and \( u' \) be distinct vertices of \( G_1 \) and let \( p = \lambda_{G_1}(u, u') \). Then there is a system \( \mathcal{P} \) of \( p \) edge disjoint \( u-u' \) paths in \( G_1 \). At most one path \( P \) of \( \mathcal{P} \) can contain the edge \( v_1w_1 \). Since \( k \geq 3 \), there is a \( v_2w_2 \) path \( P' \) in \( G_2 \) not containing the edge \( v_2w_2 \). So if we replace the edge \( v_1w_1 \) of \( P \) by the path \( P' + w_2v_1 \), we get \( p \) edge disjoint \( u-u' \) paths of \( G \), and hence \( p \leq \lambda_G(u, u') \leq k \). This shows that \( \lambda(G_1) \leq k \) and by symmetry \( \lambda(G_2) \leq k \). Hence \( G_1, G_2 \in \mathcal{C}_k \). \( \triangle \)

As a consequence of Claim 1 and Claim 2 and the definition of the class \( \mathcal{H}_k \) we obtain the following claim.

Claim 3. Let \( k \geq 3 \) be an integer. Then the class \( \mathcal{H}_k \) is a subclass of \( \mathcal{C}_k \).

Claim 4. Let \( k \geq 3 \) be an integer, and let \( G \) be a graph belonging to the class \( \mathcal{C}_k \). If \( G \) is \( 3 \)-connected, then either \( k = 3 \) and \( G \) is an odd wheel, or \( k \geq 4 \) and \( G \) is a complete graph of order \( k + 1 \).
Proof: The proof is by contradiction, where we consider a counterexample $G$ whose order $|G|$ is minimum. Then $G \in C_k$ is a 3-connected graph, and either $k = 3$ and $G$ is not an odd wheel, or $k \geq 4$ and $G$ is not a complete graph of order $k + 1$. First we claim that $|G_H| \geq 2$. If $G_H = \emptyset$, then Theorem 4(b) implies that $G$ is a complete graph of order $k + 1$, a contradiction. If $|G_H| = 1$, then Theorem 4(c) implies that $k = 3$ and $G$ is an odd wheel, a contradiction. This proves the claim that $|G_H| \geq 2$. Then let $u$ and $v$ be distinct high vertices of $G$. Since $G \in C_k$, Theorem 3(d) implies that $\lambda_G(u, v) = k$ and, therefore, $G$ contains a separating edge set $F$ of size $k$ which separates $u$ and $v$. From Theorem 8 it then follows that there is an edge cut $(X, Y, F)$ satisfying the three properties of that theorem. Since $F$ separates $u$ and $v$, we may assume that $u \in X$ and $v \in Y$. By Theorem 8(a), $|Y_F| = k$ and hence each vertex of $Y_F$ is incident to exactly one edge of $F$. Since $Y$ contains the high vertex $v$, we conclude that $|Y_F| < |Y|$. Now we consider the graph $G'$ obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of $Y_F$, so that $Y_F$ becomes a clique of $G'$. By Theorem 8(b), $G'$ is a critical graph with chromatic number $k + 1$. Clearly, every vertex of $Y_F$ is a low vertex of $G'$ and every vertex of $X$ has in $G'$ the same degree as in $G$. Since $X$ contains the high vertex $u$ of $G$, this implies that $|X_F| < |X|$. Since $G$ is 3-connected, we conclude that $|X_F| \geq 3$ and that $G'$ is 3-connected.

Now we claim that $\lambda(G') \leq k$. To prove this, let $x$ and $y$ be distinct vertices of $G'$. If $x$ or $y$ is a low vertex of $G'$, then $\lambda_{G'}(x, y) \leq k$ and there is nothing to prove. So assume that both $x$ and $y$ are high vertices of $G'$. Then both vertices $x$ and $y$ belong to $X$. Let $p = \lambda_G(x, y)$ and let $P$ be a system of $p$ edge disjoint $x$-$y$ paths in $G'$. We may choose $P$ such that the number of edges in $P$ is minimum. Let $P_1$ be the paths in $P$ which uses edges of $F$. Since $|Y_F| = k$ and each vertex of $Y_F$ is incident with exactly one edge of $F$, this implies that each path $P$ in $P_1$ contains exactly two edges of $F$. Since $|X_F| < |X|$ and $|Y_F| < |Y|$, there are vertices $u' \in X \setminus X_F$ and $v' \in Y \setminus Y_F$. By Theorem 3(d) it follows that $\lambda_{G'}(u', v') = k$ and, therefore, there are $k$ edge disjoint $u'$-$v'$ paths in $G$. Since $|Y_F| = k$, for each vertex $z \in Y_F$, there is a $v'$-$z$ path $P_z$ in $G[Y]$ such that these paths are edge disjoint. Now let $P$ be an arbitrary path in $P_1$. Then $P$ contains exactly two vertices of $Y_F$, say $z$ and $z'$, and we can replace the edge $z z'$ of the path $P$ by a $z$-$z'$ path contained in $P_z \cup P_{z'}$. In this way we obtain a system of $p$ edge disjoint $x$-$y$ paths in $G$, which implies that $p \leq \lambda_G(x, y) \leq k$. This proves the claim that $\lambda(G') \leq k$. Consequently $G' \in C_k$. Clearly, $|G'| < |G|$ and either $k = 3$ and $G'$ is not an odd wheel, or $k \geq 4$ and $G$ is not a complete graph of order $k + 1$. This, however, is a contradiction to the choice of $G$. Thus the claim is proved.

Claim 5. Let $k \geq 3$ be an integer, and let $G$ be a graph belonging to the class $C_k$. If $G$ has a separating vertex set of size 2, then $G = G_1 \triangle G_2$ is the Hajós sum of two graphs $G_1$ and $G_2$, both belong to $C_k$.

Proof: If $G$ has a separating set consisting of one edge and one vertex, then Theorem 7 implies that $G$ is the Hajós join of two graphs $G_1$ and $G_2$. By Claim 2 it then follows that both $G_1$ and $G_2$ belong to $C_k$ and we are done. It remains to consider the case that $G$ does not contain a separating set consisting of one edge and one vertex. By assumption,
there is a separating vertex set of size 2, say $S = \{u, v\}$. Then Theorem 5 implies that $G - S$ has exactly two components $H_1$ and $H_2$ such that the graphs $G_i = G[V(H_i) \cup S]$ with $i \in \{1, 2\}$ satisfies the three properties of that theorem. In particular, we have that $G' = G_1 + uv$ is critical and has chromatic number $k + 1$. By Theorem 3(d), it then follows that $\lambda_{G_1}(u, v) \geq k$ implying that $\lambda_{G_1}(u, v) \geq k - 1$. Since $G \in C_k$, we then conclude that $\lambda_{G_2}(u, v) \leq 1$. Since $G_2$ is connected, this implies that $G_2$ has a bridge $e$. Since $k \geq 3$, we conclude that $\{u, e\}$ or $\{v, e\}$ is a separating set of $G$, a contradiction. $\triangle$

As a consequence of Claim 4 and Claim 5, we conclude that the class $\mathcal{C}_k$ is a subclass of the class $\mathcal{H}_k$. Together with Claim 3 this yields $\mathcal{H}_k = \mathcal{C}_k$ as wanted.

Proof of of Theorem 2: For the proof of this theorem let $G$ be a non-empty graph with $\lambda(G) = k$. By inequality (1) we obtain that $\chi(G) \leq k + 1$. If one block $H$ of $G$ belongs to $\mathcal{H}_k$, then $H \in \mathcal{C}_k$ (by Theorem 10) and hence $\chi(G) = k + 1$ (by (2)).

Assume conversely that $\chi(G) = k + 1$. Then $G$ contains a subgraph $H$ which is critical and has chromatic number $k + 1$. Clearly, $\lambda(H) \leq \lambda(G) \leq k$, and, therefore, $H \in \mathcal{C}_k$. By Theorem 3(c), $H$ contains no separating vertex. We claim that $H$ is a block of $G$. For otherwise, $H$ would be a proper subgraph of a block $G'$ of $G$. This implies that there are distinct vertices $u$ and $v$ in $H$ which are joined by a path $P$ of $G$ with $E(P) \cap E(H) = \emptyset$. Since $\lambda_H(u, v) \geq k$ (by Theorem 3(c)), this implies that $\lambda_{G'}(u, v) \geq k + 1$, which is impossible. This proves the claim that $H$ is a block of $G$. By Theorem 10, $\mathcal{C}_k = \mathcal{H}_k$ implying that $H \in \mathcal{H}_k$. This completes the proof of the theorem.

The case $\lambda = 3$ of Theorem 2 was obtained earlier by Aboulker, Brettell, Havet, Marx, and Trotignon [1]; their proof is similar to our proof. Let $\mathcal{L}_k$ denote the class of graphs $G$ satisfying $\lambda(G) \leq k$. It is well known that membership in $\mathcal{L}_k$ can be tested in polynomial time. It is also easy to show that there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, decides whether $G$ or one of its blocks belong to $\mathcal{H}_k$. So it can be tested in polynomial time whether a graph $G \in \mathcal{L}_k$ satisfies $\chi(G) \leq k$. Moreover, the proof of Theorem 2 yields a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring of $\mathcal{CO}_k(G)$ when such a coloring exists. This result provides a positive answer to a conjecture made by Aboulker et al. [1, Conjecture 1.8]. The case $k = 3$ was solved by Aboulker et al. [1].

Theorem 11. For fixed $k \geq 1$, there is a polynomial-time algorithm that, given a graph $G \in \mathcal{L}_k$, finds a coloring in $\mathcal{CO}_k(G)$ or a block of $G$ belonging to $\mathcal{H}_k$.

Sketch of Proof: The Theorem is evident if $k \in \{1, 2\}$; and the case $k = 3$ was solved by Aboulker et al. [1]. Hence we assume that $k \geq 4$ and $G \in \mathcal{L}_k$. If we find for each block $H$ of $G$ a coloring in $\mathcal{CO}_k(H)$, we can piece these colorings together by permuting colors to obtain a coloring in $\mathcal{CO}_k(G)$. Hence we may assume that $G$ is a block. Since $\lambda(G) \leq k$ and $\lambda(H) = k$ for every graph $H \in \mathcal{H}_k$, it then follows that no proper subgraph of $G$ belongs to $\mathcal{H}_k$.

First, we check whether $G$ has a separating set $S$ consisting of one vertex and one edge. If we find such a set, say $S = \{v, e\}$ with $v \in V(G)$ and $e \in E(G)$, then $G - e$ is the union of two connected graphs $G_1$ and $G_2$ having only vertex $v$ in common where $e = w_1w_2$ and $w_1, w_2 \in V(G_1) \cap V(G_2)$.
$w_i \in V(G_i)$ for $i = 1, 2$. Both blocks $G_1' = G_1 + vw_1$ and $G_2' = G_2 + vw_2$ belong to $\mathcal{L}_k$. Now we check whether these blocks belong to $\mathcal{H}_k$. If both blocks $G_1'$ and $G_2'$ belong to $\mathcal{H}_k$, then $vw_i \notin E(G_i)$ for $i = 1, 2$, and hence $G$ belongs to $\mathcal{H}_k$ and we are done. If one of the blocks, say $G_1'$ does not belong to $\mathcal{H}_k$, we can construct a coloring $f_1 \in \mathcal{CO}_k(G_1')$. Since no block of $G_2$ belongs to $\mathcal{H}_k$, we can construct a coloring $f_2 \in \mathcal{CO}_k(G_2')$. Then $f_1 \in \mathcal{CO}_k(G_1')$ and $f_1(v) \neq f_1(w_1)$. Since $k \geq 4$, we can permute colors in $f_2$ such that $f_1(v) = f_2(v)$ and $f_1(w_1) \neq f_1(w_2)$. Consequently, $f = f_1 \cup f_2$ belongs to $\mathcal{CO}_k(G)$ and we are done.

It remains to consider the case that $G$ contains no separating set consisting of one vertex and one edge. Then let $p$ denote the number of vertices of $G$ whose degree is greater than $k$. If $p \leq 1$, then let $v$ be a vertex of maximum degree in $G$. Color $v$ with color 1 and let $L$ be a list assignment for $H = G - v$ satisfying $L(u) = \{2, 3, \ldots, k\}$ if $vu \in E(G)$ and $L(u) = \{1, 2, \ldots, k\}$ otherwise. Then $H$ is connected and $|L(u)| \geq d_H(u)$ for all $u \in V(H)$. Now we can use the degree version of Brooks’ theorem, see [12, Theorem 2.1]. Either we find a coloring $f$ of $H$ such that $f(u) \in L(u)$ for all $u \in V(H)$, yielding a coloring of $\mathcal{CO}_k(G)$, or $|L(u)| = d_H(u)$ for all $u \in V(H)$ and each block of $H$ is a complete graph or an odd cycle. In this case, $d_H(u) \in \{k, k - 1\}$ for all $u \in V(H)$ and, since $k \geq 4$, each block of $H$ is a $K_k$ or a $K_2$. Since $G$ contains no separating set consisting of one vertex and one edge, this implies that $H = K_k$ and so $G = K_{k+1} \in \mathcal{H}_k$ and we are done.

If $p \geq 2$, then we choose two vertices $u$ and $u'$ whose degrees are greater than $k$. Then we construct an edge cut $(X, Y, F)$ with $u \in X$, $u' \in Y$, and $|F| = \lambda_G(u, u')$. We may assume that $a = |X_F|$ and $b = |Y_F|$ satisfies $a \leq b \leq k$.

If $b \leq k - 1$, then both graphs $G[X]$ and $G[Y]$ belong to $\mathcal{L}_k$ and there are colorings $f_X \in \mathcal{CO}_k(G[X])$ and $f_Y \in \mathcal{CO}_k(G[Y])$. Note that no block of these two graphs can belong to $\mathcal{H}_k$. By permuting colors in $f_Y$, we can combine the two colorings $f_X$ and $f_Y$ to obtain a coloring $f \in \mathcal{CO}_k(G)$. To see this, we apply Lemma 9 to the auxiliary graph $H = H(f_X, f_Y)$ obtained from two disjoint complete graphs of order $k$, one with vertex set $A = \{a_1, a_2, \ldots, a_k\}$ and the other one with vertex set $B = \{b_1, b_2, \ldots, b_k\}$, by adding all edges of the form $a_ib_j$ for which there exists an edge $e = vv' \in E$ such that $f_X(v) = i$ and $f_Y(v') = j$. By the assumption on the edge cut $(X, Y, F)$ it follows from Lemma 9 that $\chi(H) \leq k$, which leads to to the desired coloring $f$.

If $a < b = k$, then we consider the graph $G_1$ obtained from $G[X \cup Y_F]$ by adding all edges between the vertices of $Y_F$, so that $Y_F$ becomes a clique of $G_1$. Then $G_1$ belongs to $\mathcal{L}_k$ (see the proof of Claim 4) and, since $G$ contains no separating set consisting of one vertex and one edge, the block $G_1$ does not belongs to $\mathcal{H}_k$. Hence there are colorings $f_1 \in \mathcal{CO}_k(G_1)$ and $f_Y \in \mathcal{CO}_k(G[Y])$. Then the restriction of $f_1$ to $X$ yields a coloring $f_X \in \mathcal{CO}_k(G[X])$ such that $|f_X(X_F)| \geq 2$. By permuting colors in $f_Y$, we can combine the two colorings $f_X$ and $f_Y$ to obtain a coloring $f \in \mathcal{CO}_k(G)$ (by applying Lemma 9 to the auxiliary graph $H = H(f_X, f_Y)$ as in the former case).

It remains to consider the case $a = b = k$. Then let $G_2$ be the graph obtained from $G[Y \cup X_F]$ by adding all edges between the vertices of $X_F$, so that $X_F$ becomes a clique of $G_2$. Then we find colorings $f_1 \in \mathcal{CO}_k(G_1)$ and $f_2 \in \mathcal{CO}_k(G_2)$ and, hence, colorings $f_X \in \mathcal{CO}_k(G[X])$ and $f_Y \in \mathcal{CO}_k(G[Y])$ such that $|f_X(X_F)| \geq 2$ and $|f_Y(Y_F)| \geq 2$. By
permuting colors in $f_Y$, we can combine the two colorings $f_X$ and $f_Y$ to obtain a coloring $f \in CO_k(G)$ (by using Lemma 9).

\section*{References}


