On bipartite $Q$-polynomial distance-regular graphs with diameter 9, 10, or 11

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Submitted: Sep 20, 2017; Accepted: Feb 19, 2018; Published: Mar 2, 2018
Mathematics Subject Classifications: 05C50, 05E30

Abstract

Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D$. In [J. S. Caughman, Bipartite $Q$-polynomial distance-regular graphs, Graphs Combin. 20 (2004), 47–57], Caughman showed that if $D \geq 12$, then $\Gamma$ is $Q$-polynomial if and only if one of the following (i)-(iv) holds: (i) $\Gamma$ is the ordinary $2D$-cycle, (ii) $\Gamma$ is the Hamming cube $H(D,2)$, (iii) $\Gamma$ is the antipodal quotient of the Hamming cube $H(2D,2)$, (iv) the intersection numbers of $\Gamma$ satisfy $c_i = (q^i - 1)/(q - 1)$, $b_i = (q^D - q^i)/(q - 1)$ ($0 \leq i \leq D$), where $q$ is an integer at least 2. In this paper we show that the above result is true also for bipartite distance-regular graphs with $D \in \{9,10,11\}$.

Keywords: bipartite distance-regular graph; $Q$-polynomial property

1 Introduction

As a classification of all distance-regular graphs is currently beyond our reach, classifications of some subclasses of distance-regular graphs are also very important projects. One such subclass is the class of $Q$-polynomial bipartite distance-regular graphs. This paper is part of an effort to understand and classify $Q$-polynomial bipartite distance-regular graphs (see [3, 4, 5, 6, 7] for relevant literature). A crucial step towards a classification of this class was made by Caughman, who proved the following result.

*This work is supported in part by the Slovenian Research Agency (research program P1-0285 and research projects N1-0032, N1-0038, N1-0062, J1-6720, and J1-7051).
Theorem 1.1 ([7, Theorem 1.1]) Let \( \Gamma \) denote a bipartite distance-regular graph with diameter \( D \geq 12 \). Then \( \Gamma \) is \( Q \)-polynomial if and only if one of the following (i)-(iv) holds:

(i) \( \Gamma \) is the ordinary \( 2D \)-cycle.

(ii) \( \Gamma \) is the Hamming cube \( H(D, 2) \).

(iii) \( \Gamma \) is the antipodal quotient of the Hamming cube \( H(2D, 2) \).

(iv) The intersection numbers of \( \Gamma \) satisfy

\[
\begin{align*}
   c_i &= q^i - 1, \\
   b_i &= \frac{q^D - q^i}{q - 1} \quad (0 \leq i \leq D),
\end{align*}
\]

where \( q \) is an integer at least 2.

In this paper we prove an analogue of Theorem 1.1 for bipartite distance-regular graphs with diameter \( D \in \{9, 10, 11\} \). We follow the ideas of Caughman [7] and use the Terwilliger algebra of \( \Gamma \) to prove our result. Generalization of Theorem 1.1 to bipartite distance-regular graphs with diameter less than 12 is also mentioned as an open problem in the recent survey paper Distance-regular graphs by van Dam, Koolen and Tanaka, see [10, Section 18.3].

The paper is organized as follows. In Sections 2, 3, 4 we review some basic definitions and results about distance-regular graphs, the \( Q \)-polynomial property of distance-regular graphs, and the Terwilliger algebra of distance-regular graphs. In Section 5 we review and prove some results concerning multiplicities of irreducible modules of the Terwilliger algebra. In Sections 6 and 7 we prove our main result.

2 Preliminaries

In this section we review some definitions and basic concepts. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Throughout this paper, \( \Gamma = (X, R) \) will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set \( X \), edge set \( R \), path length distance function \( \partial \), and diameter \( D := \max \{ \partial(x, y) | x, y \in X \} \). For a vertex \( x \in X \) define \( \Gamma_i(x) \) to be the set of vertices at distance \( i \) from \( x \). We abbreviate \( \Gamma(x) := \Gamma_1(x) \). Let \( k \) denote a nonnegative integer. Then \( \Gamma \) is said to be regular with valency \( k \) whenever \( |\Gamma(x)| = k \) for all \( x \in X \). The graph \( \Gamma \) is said to be distance-regular whenever for all integers \( h, i, j \) \( (0 \leq h, i, j \leq D) \), and all \( x, y \in X \) with \( \partial(x, y) = h \), the number

\[
p^h_{ij} := |\{ z | z \in X, \partial(x, z) = i, \partial(y, z) = j \}|
\]

is independent of \( x, y \). The constants \( p^h_{ij} \) are known as the intersection numbers of \( \Gamma \). For convenience, set \( c_i := p^1_{ii-1} \) for \( 1 \leq i \leq D \), \( a_i := p^i_{ii} \) for \( 0 \leq i \leq D \), \( b_i := p^i_{ii+1} \) for \( 0 \leq i \leq D - 1 \), \( k_i := p^0_{ii} \) for \( 0 \leq i \leq D \), and \( c_0 = b_D = 0 \). We observe \( a_0 = 0 \), \( c_1 = 1 \).
Moreover, $\Gamma$ is regular with valency $k = b_0$, and $c_i + a_i + b_i = k$ for $0 \leq i \leq D$. It is well-known that

$$k_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i} \quad (0 \leq i \leq D). \quad (2)$$

Observe that $\Gamma$ is bipartite if and only if $a_i = 0$ for $0 \leq i \leq D$. In this case $b_i + c_i = k$ for $0 \leq i \leq D$.

From now on we assume $\Gamma$ is distance-regular with diameter $D \geq 3$ and valency $k \geq 3$. We recall the Bose-Mesner algebra of $\Gamma$. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices over $\mathbb{C}$ which have rows and columns indexed by $X$. For $0 \leq i \leq D$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $x,y$ entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \quad (x,y \in X). \quad (3)$$

We call $A_i$ the $i$-th distance matrix of $\Gamma$. We abbreviate $A = A_1$ and call $A$ the adjacency matrix of $\Gamma$. The matrices $A_0, A_1, \ldots, A_D$ form a basis for a commutative semi-simple $\mathbb{C}$-algebra $M$, known as the Bose-Mesner algebra, see for example [11, Lemma 11.2.2]. The algebra $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that $E_i E_j = \delta_{ij} E_i \ (0 \leq i, j \leq D)$, see [2, Theorem 2.6.1]. The $E_0, E_1, \ldots, E_D$ are known as the primitive idempotents of $\Gamma$, and $E_0$ is the trivial idempotent.

For $0 \leq i \leq D$ define a real number $\theta_i$ by $A = \sum_{i=0}^D \theta_i E_i$. Then $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. The scalars $\theta_0, \theta_1, \ldots, \theta_D$ are distinct, since $A$ generates $M$ [1, p. 197]. The $\theta_0, \theta_1, \ldots, \theta_D$ are known as the eigenvalues of $\Gamma$. We remark $k \geq \theta_i \geq -k$ for $0 \leq i \leq D$, and $\theta_0 = k$ [2, p. 45].

Let $\theta$ denote an eigenvalue of $\Gamma$, and let $E$ denote the associated primitive idempotent. For $0 \leq i \leq D$ define a real number $\theta_i^*$ by $E = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i$. We call the sequence $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ the dual eigenvalue sequence associated with $\theta, E$.

### 3 The $Q$-polynomial property

In this section we recall the $Q$-polynomial property of distance-regular graphs. Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$, and let $A_0, A_1, \ldots, A_D$ denote the distance matrices of $\Gamma$. Observe that $A_i \circ A_j = \delta_{ij} A_i \ (0 \leq i, j \leq D)$, where $\circ$ denotes entrywise multiplication, and so algebra $M$ is closed under $\circ$. Let $E_0, E_1, \ldots, E_D$ denote the primitive idempotents of $\Gamma$. The Krein parameters $q_{ij}^h \ (0 \leq h, i,j \leq D)$ of $\Gamma$ are defined by

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D). \quad (4)$$

We say $\Gamma$ is $Q$-polynomial (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents), whenever for all distinct integers $i, j \ (0 \leq i, j \leq D)$ the following
Let \( \theta \) denote the \((i)–(iii)\) such that \( \theta \) respects to Lemma 3.1. We have the following important result about bipartite Q-polynomial distance-regular graphs, see [7, Lemma 3.2, Lemma 3.3].

**Lemma 3.1** Let \( \Gamma \) denote a bipartite distance-regular graph with diameter \( D \geq 4 \), valency \( k \geq 3 \), and intersection numbers \( b_i, c_i \) \((0 \leq i \leq D)\). We assume \( \Gamma \) is Q-polynomial with respect to \( E_0, E_1, \ldots, E_D \). For \( 0 \leq i \leq D \) let \( \theta_i \) denote the eigenvalue associated with \( E_i \). Let \( \theta_0^*, \ldots, \theta_D^* \) denote the dual eigenvalue sequence associated with \( E_1 \). Assume \( \Gamma \) is not the \( D \)-cube or the antipodal quotient of the \( 2D \)-cube. Then there exist scalars \( q, s^* \in \mathbb{R} \) such that (i)–(iii) hold below.

(i) \( |q| > 1 \), \( s^*q^i \neq 1 \) \((2 \leq i \leq 2D + 1)\);

(ii) \( \theta_i = h((q^{D-i} - q^i), \quad \theta_i = \theta_0^* + h^*(1 - q^i)(1 - s^*q^i+1)q^{-i} \) for \( 0 \leq i \leq D \), where

\[
h = \frac{1 - s^*q^3}{(q - 1)(1 - s^*q^{D+2})}, \quad h^* = \frac{(q^D + q^2)(q^D + q)}{q(q^2 - 1)(1 - s^*q^{2D})}, \quad \theta_0^* = \frac{h^*(q^D - 1)(1 - s^*q^2)}{q(q^{D-1} + 1)};
\]

(iii) \( k = c_D = h(q^D - 1) \), and

\[
c_i = \frac{h(q^i - 1)(1 - s^*q^{D+i+1})}{1 - s^*q^{2i+1}}, \quad b_i = \frac{h(q^D - q^i)(1 - s^*q^i+1)}{1 - s^*q^{2i+1}} \quad (1 \leq i \leq D - 1).
\]

4 **The Terwilliger algebra**

In this section we recall the Terwilliger algebra of a distance-regular graph. Let \( \Gamma \) denote a distance-regular graph with diameter \( D \geq 3 \), valency \( k \geq 3 \), and distance matrices \( A_0, A_1, \ldots, A_D \). Fix any vertex \( x \in X \). For \( 0 \leq i \leq D \) let \( E_i^x = E_i^x(x) \) denote the diagonal matrix in \( \text{Mat}_X(\mathbb{C}) \) with \( y, y \) entry \( (A_i)_{xy} \) \( (y \in X) \). Let \( T = T(x) \) denote the subalgebra of \( \text{Mat}_X(\mathbb{C}) \) generated by \( A \) and \( E_0^x, \ldots, E_D^x \). We call \( T \) the Terwilliger algebra of \( \Gamma \) with respect to \( x \). We remark that \( T \) is finite dimensional and semisimple.

Let \( V \) denote the \( \mathbb{C} \)-vector space consisting of the column vectors over \( \mathbb{C} \) which have rows indexed by \( X \). Observe that \( \text{Mat}_X(\mathbb{C}) \) acts on \( V \) by left multiplication. We refer to \( V \) as the standard module of \( T \). By a \( T \)-module we mean a subspace \( W \) of the standard module \( V \) such that \( BW \subseteq W \) for all \( B \in T \). Let \( W \) denote a \( T \)-module. Then \( W \) is said to be irreducible whenever \( W \) is nonzero and \( W \) contains no \( T \)-modules other than zero and \( W \). Let \( W \) and \( W' \) denote \( T \)-modules. By a \( T \)-isomorphism from \( W \) to \( W' \), we mean a vector space isomorphism \( \sigma : W \rightarrow W' \) such that \( (\sigma B - B\sigma)W = 0 \) for all \( B \in T \). The modules \( W \) and \( W' \) are said to be isomorphic whenever there exists a \( T \)-isomorphism from \( W \) to \( W' \).
Let $W$ denote a $T$-module and let $W'$ denote a $T$-module contained in $W$. Then the orthogonal complement of $W'$ in $W$ is a $T$-module. From this we find $W$ is an orthogonal direct sum of irreducible $T$-modules. Taking $W = V$ we find $V$ is an orthogonal direct sum of irreducible $T$-modules. Let $W$ denote an irreducible $T$-module. By the multiplicity with which $W$ appears in $V$, we mean the number of irreducible $T$-modules in this sum which are isomorphic to $W$. It is known that the multiplicity of $W$ is independent of the decomposition of $V$.

Let $W$ denote an irreducible $T$-module. We define the endpoint $r$ and the diameter $d$ of $W$ by $r = \min \{i \mid 0 \leq i \leq \dim E_i^* W \neq 0\}$ and $d = |\{i \mid 0 \leq i \leq d, E_i^* W \neq 0\}| - 1$. Similarly, the dual endpoint $t$ and dual diameter $d^*$ of $W$ are defined by $t = \min \{i \mid 0 \leq i \leq \dim E_i W \neq 0\}$ and $d^* := |\{i \mid 0 \leq i \leq d, E_i W \neq 0\}| - 1$. We say $W$ is thin, whenever $\dim(E^*_i W) \leq 1$ for every $0 \leq i \leq D$.

Assume now that our distance-regular graph $\Gamma$ is $Q$-polynomial. Let $W$ denote an irreducible $T$-module with endpoint $r$, dual endpoint $t$, diameter $d$ and dual diameter $d^*$. Then $W$ is thin by [4, Theorem 9.3]. We comment on $r, t, d$ and $d^*$. By [4, Lemma 5.1(ii)] we have $2r + d^* \geq D$, and by [4, Lemma 9.2(ii)] we have that $d = d^*$. It follows that $(D - d)/2 \leq r$. It is also clear that $r + d \leq D$, and so we have that

$$\frac{D - d}{2} \leq r \leq D - d.$$ 

We have $2t + d = D$ by [4, Theorem 9.4(ii)], and so $D - d$ is even. By [4, Theorem 13.1], the isomorphism class of $W$ is determined by $r$ and $d$. For the rest of the paper we will consider the following situation.

**Notation 4.1** Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, intersection numbers $b_i, c_i$, and distance matrices $A_i (0 \leq i \leq D)$. We fix $x \in X$ and let $E^*_i = E^*_i(x)$ ($0 \leq i \leq D$) and $T = T(x)$ denote the dual idempotents and the Terwilliger algebra of $\Gamma$ with respect to $x$, respectively. Let $V$ denote the standard module for $\Gamma$. Let $H(D, 2)$ denote the $D$-dimensional hypercube, and let $\overline{H}(2D, 2)$ denote the antipodal quotient of $H(2D, 2)$.

## 5 Multiplicities of the irreducible $T$-modules

With reference to Notation 4.1, assume that $\Gamma$ is $Q$-polynomial. In this section we review and prove some results about the multiplicities of irreducible $T$-modules (see also [4, Section 14]).

Fix a decomposition of the standard module $V$ into an orthogonal direct sum of irreducible $T$-modules. Let $W$ denote an irreducible $T$-module. Recall that the multiplicity of $W$ equals the number of irreducible $T$-modules in this sum which are isomorphic to $W$. As the isomorphism class of $W$ is determined by its endpoint and diameter, we introduce the following notation. For any integers $r, d$ ($0 \leq r, d \leq D$), we define $\text{mult}(r, d)$ to be the number of irreducible $T$-modules in this decomposition which have endpoint $r$ and diameter $d$. If no such modules exist, then we set $\text{mult}(r, d) = 0$. Note that if $W$ has endpoint $r$ and diameter $d$, then the multiplicity of $W$ equals $\text{mult}(r, d)$.
Definition 5.1 ([4, Definition 14.2]) With reference to Notation 4.1, assume that $\Gamma$ is $Q$-polynomial. Define a set $\Upsilon$ by

$$\Upsilon := \{(r,d) \in \mathbb{Z}^2 \mid 0 \leq d \leq D, D - d \text{ even}, \frac{D - d}{2} \leq r \leq D - d\}.$$ 

Observe that $\text{mult}(r,d) = 0$ for all integers $r,d$ such that $(r,d) \not\in \Upsilon$. We define a partial order $\preceq$ on $\Upsilon$ by

$$(r',d') \preceq (r,d) \quad \text{if and only if} \quad r' \leq r \text{ and } r + d \leq r' + d'.$$

To further describe $\text{mult}(r,d)$, we need a definition. Let $(r,d) \in \Upsilon$. Following [4, pp. 87-88] we define scalars $c_i(r,d)$ ($1 \leq i \leq d$) and $b_i(r,d)$ ($0 \leq i \leq d - 1$) by

$$c_i(r,d) = \frac{\theta_i(\theta^*_{r+i+1} - \theta^*_{r+i}) - \theta_{i+1}(\theta^*_{r+i+1} - \theta^*_{r})}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \quad (1 \leq i \leq d - 1),$$

$$b_i(r,d) = \frac{\theta_i(\theta^*_{r+i+1} - \theta^*_{r+i-1}) + \theta_{i+1}(\theta^*_{r+i+1} - \theta^*_{r})}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \quad (1 \leq i \leq d - 1),$$

where $t = (D - d)/2$. We also set $b_0(r,d) = c_d(r,d) = \theta_1$, $b_d(r,d) = c_0(r,d) = 0$. Assume now that $\Gamma$ is not $H(D,2)$ or $\overline{H}(2D,2)$. Then using Lemma 3.1(ii) we get that

$$c_i(r,d) = \frac{h(q^i - 1)(1 - s^*q^{2r+d+i+1})}{q^d+t(1 - s^*q^{2r+2i+1})} \quad (1 \leq i \leq d - 1),$$

$$b_i(r,d) = \frac{h(q^d - q^i)(1 - s^*q^{2r+i+1})}{q^d+t(1 - s^*q^{2r+2i+1})} \quad (1 \leq i \leq d - 1),$$

and $b_0(r,d) = c_d(r,d) = h(q^{-t} - q^{-D})$, $b_d(r,d) = c_0(r,d) = 0$.

Theorem 5.2 ([4, Theorem 14.7]) With reference to Definition 5.1, fix any $(r,d) \in \Upsilon$. Then

$$k_r \prod_{i=r}^{r+d-1} b_i c_{r+d-i} = \sum_{(r',d') \in \Upsilon \atop (r',d') \preceq (r,d)} \text{mult}(r',d') \prod_{i=r-r'} b_i (r',d') c_{i+1}(r',d').$$

Corollary 5.3 With reference to Definition 5.1, fix any $(r,d) \in \Upsilon$ such that $r + d = D (1 \leq r \leq D - 1)$. Then $r$ is even and

$$k_r \prod_{i=r}^{D-1} b_i c_{D-i} = \sum_{t=0}^{r/2} \text{mult}(2\ell,D-2\ell) \prod_{i=r-2\ell}^{D-2\ell-1} b_i(2\ell,D-2\ell) c_{i+1}(2\ell,D-2\ell).$$

Proof. Recall that $r = D - d$ is even by definition of the set $\Upsilon$. As $r + d = D$, it follows from the definition of $\preceq$ that the only pairs $(r',d') \in \Upsilon$ such that $(r',d') \preceq (r,d)$ are pairs with $r' \leq r$ and $r' + d' = D$. Note that $r' + d' = D$ implies $r'$ is even, and so the result follows from Theorem 5.2.
Lemma 5.4 With reference to Definition 5.1, fix any \((r, d) \in \mathcal{Y}\) such that \(r + d = D\) \((1 \leq r \leq D - 1)\). Assume that \(\Gamma\) is not \(H(D, 2)\) or \(\overline{H}(2D, 2)\). If there exists an irreducible \(T\)-module with endpoint \(r\) and diameter \(d\), then

\[
\prod_{i=0}^{D-r-1} b_i(r, D-r)c_{i+1}(r, D-r) \neq 0.
\]

Proof. Let \(q, s^*\) be as in Lemma 3.1. Recall that \(|q| > 1\) and \(s^* q^i \neq 1\) for \(2 \leq i \leq 2D + 1\). Using this, equations (7), (8) and \(b_0(r, d) = c_d(r, d) = h(q^{-t} - q^{t-D})\), we find that \(b_i(r, D-r) \neq 0\) and \(c_{i+1}(r, D-r) \neq 0\) for \(0 \leq i \leq D - r - 1\). The result follows.

Corollary 5.5 With reference to Definition 5.1, fix any \((r, d) \in \mathcal{Y}\) such that \(r + d = D\) \((1 \leq r \leq D - 1)\). Assume that \(\Gamma\) is not \(H(D, 2)\) or \(\overline{H}(2D, 2)\). Then \(\text{mult}(r, d)\) is equal to the quantity

\[
k_r \prod_{i=r}^{D-1} b_i c_{D-i} - \sum_{\ell=0}^{r/2-1} \text{mult}(2\ell, D - 2\ell) \prod_{i=r-2\ell}^{D-2\ell-1} b_i(2\ell, D - 2\ell)c_{i+1}(2\ell, D - 2\ell),
\]

divided by the quantity

\[
\prod_{i=0}^{D-r-1} b_i(r, D-r)c_{i+1}(r, D-r).
\]

Proof. Immediately from Corollary 5.3 and Lemma 5.4.

We now give explicit formulae for \(\text{mult}(r, d)\) for some specific values of \(r, d\). To do this we need the following definition. For \(a, b \in \mathbb{R}\) and for a non-negative integer \(n\) we set

\[
(a; b)_n = \prod_{i=1}^{n} (1 - ab^{-1}).
\]

Theorem 5.6 With reference to Definition 5.1 assume \(\Gamma\) is not \(H(D, 2)\) or \(\overline{H}(2D, 2)\). Let parameters \(q, s^*\) be as in Lemma 3.1. Then the following (i)–(iii) hold.

(i) \(\text{mult}(0, D) = 1\).

(ii) If \(r \in \{2, 4\}\), then

\[
\text{mult}(r, D-r) = \frac{(-1)^t q^{t(t-1)}(1 - s^* q^{2r})(q^{d+1}; q)_r(-s^* q^{D+1}; q)_t(s^* q^2; q^2)_t^{-1}}{(q^2; q^2)_t(s^* q^{D+t+1}; q)_t(s^* q^{D+d+2}; q^2)_t^{-1}},
\]

where \(d = D - r\) and \(t = (D - d)/2 = r/2\).

(iii) If \(r \in \{6, 8\}\) and \(D \in \{9, 10, 11\}\), then

\[
\text{mult}(r, D-r) = \frac{(-1)^t q^{t(t-1)}(1 - s^* q^{2r})(q^{d+1}; q)_r(-s^* q^{D+1}; q)_t(s^* q^2; q^2)_t^{-1}}{(q^2; q^2)_t(s^* q^{D+t+1}; q)_t(s^* q^{D+d+2}; q^2)_t^{-1}},
\]

where \(d = D - r\) and \(t = (D - d)/2 = r/2\).
Proof. (i), (ii) This is [4, Theorem 15.6 (i),(iii),(vii)].
(iii) The proof (although a bit tedious and lengthy) follows straightforward from Corollary 5.5 using (7), (8). We omit the details.

Remark 5.7 With reference to Definition 5.1 assume \( \Gamma \) is not \( H(D,2) \) or \( \overline{H}(2D,2) \). We conjecture that the formula for \( \text{mult}(r,D-r) \) given in Theorem 5.6 holds for any diameter \( D \) and for any even number \( r \leq D \). See also [4, Conjecture 15.8] for an extended conjecture about the multiplicities of irreducible \( T \)-modules of \( \Gamma \). However, for the purpose of this paper, Theorem 5.6 suffices.

6 Some results about parameter \( s^* \)

With reference to Notation 4.1 assume \( \Gamma \) is \( Q \)-polynomial, and let parameters \( q, s^* \) be as in Lemma 3.1. In this section we derive some restrictions on parameter \( s^* \). We first recall some results of Caughman.

Theorem 6.1 ([7, Theorem 4.1, Lemma 5.1, Lemma 6.6]) With reference to Notation 4.1 assume \( \Gamma \) is \( Q \)-polynomial and assume that \( \Gamma \) is not \( H(D,2) \) or \( \overline{H}(2D,2) \). Let parameters \( q, s^* \) be as in Lemma 3.1. Then the following (i)–(iii) hold.

(i) If \( D \geq 6 \), then \( q > 1 \).
(ii) If \( q > 1 \), then \(-q^{D-1} \leq s^* < q^{-2D-1} \).
(iii) If \( D \geq 7 \) and \(-q^{-13} \leq s^* \leq q^{-13} \), then \( s^* = 0 \).

Lemma 6.2 With reference to Notation 4.1 assume \( \Gamma \) is \( Q \)-polynomial and assume that \( \Gamma \) is not \( H(D,2) \) or \( \overline{H}(2D,2) \). Let parameters \( q, s^* \) be as in Lemma 3.1. Set \( \beta = q + 1/q \). If \( D \geq 5 \), then \( \beta \) is a rational number.

Proof. Assume that \( \Gamma \) is \( Q \)-polynomial with respect to \( E_0, E_1, \ldots, E_D \) and let \( \theta_1 \) denote the eigenvalue of \( \Gamma \) corresponding to \( E_1 \). By [6, Lemma 3.2] we have that \( \theta_1 \neq -1 \) and

\[
\beta = \frac{\theta_1^2 + c_2\theta_1 + b_2(k-2)}{b_2(\theta_1 + 1)}.
\]

If \( D \geq 5 \) then \( \theta_1 \) is integer by [9, Theorem 8.1.3], see also [6, Lemma 3.3(i)]. The result follows.

We note that it is easy to see that \( q^2 + q^{-2} = \beta^2 - 2 \), \( q^3 + q^{-3} = \beta^3 - 3\beta \) and \( q^4 + q^{-4} = \beta^4 - 4\beta^2 + 2 \). Also, if \( D \geq 6 \), then \( q > 1 \) by Theorem 6.1(i), and so \( \beta > 2 \) in this case.

The following result was proved by Lang.

Proposition 6.3 ([12, Lemma 9.3]) With reference to Notation 4.1 assume \( \Gamma \) is \( Q \)-polynomial and assume that \( \Gamma \) is not \( H(D,2) \) or \( \overline{H}(2D,2) \). Let parameters \( q, s^* \) be as in Lemma 3.1. If \( D \geq 5 \), then \( s^* \neq -q^{-D-3} \).
Proposition 6.4 With reference to Notation 4.1 assume $\Gamma$ is $Q$-polynomial and assume that $\Gamma$ is not $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters $q, s^*$ be as in Lemma 3.1. If $D \geq 9$, then $s^* \neq -q^{-D-2}$.

Proof. Assume on contrary that $D \geq 9$ and $s^* = -q^{-D-2}$. Observe that this implies $s^* < 0$ as $q > 1$ by Theorem 6.1(i). If $D \geq 11$, then $s^* = -q^{-D-2}$ implies $-q^{-13} \leq s^* \leq q^{-13}$. But then $s^* = 0$ by Theorem 6.1(iii), a contradiction.

If $D \in \{9,10\}$ then the proof is similar to the proof of Proposition 6.3 for the case $D = 9$. Let $\beta$ be as in Lemma 6.2. Assume first that $D = 10$. In this case we have

$$2c_2 = \beta^2 + 2\beta - 1 - \frac{2\beta^2 - \beta - 3}{\beta^3 - \beta^2 - 2\beta + 1},$$

which shows that $\beta$ is an algebraic integer. As $\beta$ is rational by Lemma 6.2, this implies that $\beta$ is an integer, and so $(2\beta^2 - \beta - 3)/(\beta^3 - \beta^2 - 2\beta + 1)$ is an integer. Observe that $2\beta^2 - \beta - 3$ and $\beta^3 - \beta^2 - 2\beta + 1$ are both positive for $\beta \geq 2$, so $2\beta^2 - \beta - 3 \geq \beta^3 - \beta^2 - 2\beta + 1$. But this implies $\beta = 2$, a contradiction (recall that $\beta > 2$).

Assume now $D = 9$. In this case we have

$$2c_2 = \beta^2 + 2\beta - 1 - \frac{2\beta^2 + \beta - 4}{\beta^3 - 3\beta^2},$$

which again shows that $\beta$ is an algebraic integer. As $\beta$ is rational by Lemma 6.2, this implies that $\beta$ is an integer, and so $(2\beta^2 + \beta - 4)/(\beta^3 - 3\beta^2)$ is an integer. Observe that $2\beta^2 + \beta - 4$ and $\beta^3 - 3\beta^2$ are both positive for $\beta \geq 2$, so $2\beta^2 + \beta - 4 \geq \beta^3 - 3\beta^2$. But this implies $\beta = 2$, a contradiction. This shows that $s^* \neq -q^{-D-2}$. 

Proposition 6.5 With reference to Notation 4.1 assume $\Gamma$ is $Q$-polynomial and assume that $\Gamma$ is not $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters $q, s^*$ be as in Lemma 3.1. If $D \geq 6$, then $s^* \neq -q^{-D-1}$.

Proof. We assume $s^* = -q^{-D-1}$ and derive a contradiction. By Theorem 5.6(ii) we find that $\text{mult}(2, D-2) = 0$. But now $\Gamma$ has, up to isomorphism, a unique irreducible $T$-module with endpoint 2, and this module has diameter $D - 4$. By [8, Theorem 3.12], $\Gamma$ is 2-homogeneous in the sense of Nomura [13]. However, as $D \geq 6$ we have that $\Gamma$ is $H(D,2)$ by [14, Theorem 1.2] (see also [8, Theorem 4.1]), a contradiction. This shows that $s^* \neq -q^{-D-1}$. 

7 Proof of the main theorem

In this section we prove our main theorem. To do this we first need the following result.

Theorem 7.1 With reference to Notation 4.1 assume $\Gamma$ is $Q$-polynomial and assume that $\Gamma$ is not $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters $q, s^*$ be as in Lemma 3.1. Assume further that $D \in \{9,10,11\}$. Then $s^* \geq -q^{-D-4}$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 25(1) (2018), #P1.52
**Proof.** The result obviously holds if $s^* \geq 0$ (recall that $q > 1$ by Theorem 6.1(i)), so assume that $s^* < 0$. Consider $\text{mult}(2, D - 2)$ and recall that this number is non-negative. It follows from Theorem 5.6(ii) that

$$(1 + s^* q^{D+1}) \geq 0.$$ 

As $s^* \neq -1/q^{D+1}$ by Proposition 6.5, we have that $s^* > -q^{-D-1}$.

Consider now $\text{mult}(4, D - 4)$ and recall that this number is non-negative. It follows from Theorem 5.6(ii) that

$$(1 + s^* q^{D+1})(1 + s^* q^{D+2}) \geq 0.$$ 

We have just proved that $s^* > -q^{-D-2}$ and so $1 + s^* q^{D+1} > 0$, implying that $1 + s^* q^{D+2} \geq 0$. As $s^* \neq -q^{-D-2}$ by Proposition 6.4, this shows that $s^* > -q^{-D-2}$.

Finally, consider $\text{mult}(6, D - 6)$ and recall that this number is non-negative. It follows from Theorem 5.6(iii) that

$$(1 + s^* q^{D+1})(1 + s^* q^{D+2})(1 + s^* q^{D+3}) \geq 0.$$ 

We have just proved that $s^* > -q^{-D-3}$ and so $(1 + s^* q^{D+1})(1 + s^* q^{D+2}) > 0$, implying that $1 + s^* q^{D+3} \geq 0$. As $s^* \neq -q^{-D-3}$ by Proposition 6.3, this shows that $s^* > -q^{-D-3}$.

Finally, consider $\text{mult}(8, D - 8)$ and recall that this number is non-negative. It follows from Theorem 5.6(iii) that

$$(1 + s^* q^{D+1})(1 + s^* q^{D+2})(1 + s^* q^{D+3})(1 + s^* q^{D+4}) \geq 0.$$ 

We have just proved that $s^* > -q^{-D-4}$ and so $(1 + s^* q^{D+1})(1 + s^* q^{D+2})(1 + s^* q^{D+3})(1 + s^* q^{D+4}) > 0$, implying that $1 + s^* q^{D+4} \geq 0$. This shows that $s^* \geq -q^{-D-4}$.

We are now ready to prove our main result.

**Theorem 7.2** Let $\Gamma$ denote a bipartite distance-regular graph with diameter $D \geq 9$. Then $\Gamma$ is $Q$-polynomial if and only if one of the following (i)–(iv) holds:

(i) $\Gamma$ is the ordinary $2D$-cycle.

(ii) $\Gamma$ is the Hamming cube $H(D, 2)$.

(iii) $\Gamma$ is the antipodal quotient of the Hamming cube $H(2D, 2)$.

(iv) The intersection numbers of $\Gamma$ satisfy

$$c_i = \frac{q^i - 1}{q - 1}, \quad b_i = \frac{q^D - q^i}{q - 1} \quad (0 \leq i \leq D),$$

where $q$ is an integer at least 2.
PROOF. If $D \geq 12$ then this is [7, Theorem 1.1], therefore we assume $D \in \{9,10,11\}$. Assume first that $\Gamma$ is $Q$-polynomial and that $\Gamma$ is not a $2D$-cycle, $H(D,2)$ or $\overline{H}(2D,2)$. Let parameters $q, s^*$ be as in Lemma 3.1. By Theorem 7.1 we have $s^* \geq -q^{-D-4}$. Together with Theorem 6.1(ii) this implies that $-q^{-13} \leq s^* \leq q^{-13}$, and so $s^* = 0$ by Theorem 6.1(iii). It follows from Lemma 3.1 that $c_i = \frac{q^i - 1}{q - 1}, \quad b_i = \frac{q^D - q^i}{q - 1} \quad (0 \leq i \leq D)$. But now $c_2 = q + 1$, and so $q$ is an integer. As $q > 1$ we have that $q \geq 2$.

Concerning the converse, assume that one of the cases (i)-(iv) holds. If (i) or (iii) holds, then $\Gamma$ is $Q$-polynomial by [2, Corollary 8.5.3(i),(iii)]. If (ii) or (iv) holds, then $\Gamma$ has classical parameters and it is $Q$-polynomial by [2, Corollary 8.4.2].

Acknowledgement

The author thanks John Caughman for giving this paper a close reading and offering valuable suggestions.

References


