Covering a Graph with Cycles of Length at least 4

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Abstract
Let $G$ be a graph of order $n \geq 4k$, where $k$ is a positive integer. Suppose that the minimum degree of $G$ is at least $\lceil n/2 \rceil$. We show that $G$ contains $k$ vertex-disjoint cycles covering all the vertices of $G$ such that $k - 1$ of them are quadrilaterals.

Keywords: cycles; disjoint cycles; cycle coverings

1 Introduction
Let $G$ be a graph. A set of subgraphs of $G$ is said to be independent if no two of them have any common vertex in $G$. Corrádi and Hajnal [3] investigated the maximum number of independent cycles in a graph. They proved that if $G$ is a graph of order at least $3k$ with minimum degree at least $2k$, then $G$ contains $k$ independent cycles. In particular, when the order of $G$ is exactly $3k$, then $G$ contains $k$ independent triangles. A cycle of length 4 is called a quadrilateral. Erdős and Faudree [6] conjectured that if $G$ is a graph of order $4k$ with minimum degree at least $2k$, then $G$ contains $k$ independent quadrilaterals. Alon and Yuster [1] proved that for any $\epsilon > 0$, there exists $k_0$ such that if $G$ is a graph of order $4k$ and has minimum degree at least $(2 + \epsilon)k$ with $k \geq k_0$, then $G$ contains $k$ independent quadrilaterals. We proved this conjecture in [11], that is

**Theorem A** [11] If $G$ is a graph of order $4k$ and the minimum degree of $G$ is at least $2k$, then $G$ contains $k$ independent quadrilaterals.

In [9], we proved the following theorem.

**Theorem B** [9] Let $G$ be a graph of order $n$ with $4k + 1 \leq n \leq 4k + 4$, where $k$ is a positive integer. Suppose that the minimum degree of $G$ is at least $2k + 1$. Then $G$ contains $k$ independent quadrilaterals.

In [4], El-Zahar conjectured that if $G$ is a graph of order $n = n_1 + n_2 + \cdots + n_k$, where each $n_i$ is an integer at least 3, such that $\delta(G) \geq \lceil n_1/2 \rceil + \lceil n_2/2 \rceil + \cdots + \lceil n_k/2 \rceil$,
then \( G \) contains \( k \) independent cycles of lengths \( n_1, n_2, \ldots, n_k \), respectively. Clearly, this conjecture generalizes the above conjecture by Erdős and Faudree. In [8], we confirmed the El-Zahar’s conjecture for the case \( n_1 = \cdots = n_{k-1} = 3 \) and \( n_k \geq 3 \). In this paper, we will prove the following theorem:

**Theorem C** Let \( G \) be a graph of order \( n \geq 4k \), where \( k \) is a positive integer. Suppose that the minimum degree of \( G \) is at least \( \lceil n/2 \rceil \). Then \( G \) contains \( k \) independent cycles covering all the vertices of \( G \) such that \( k - 1 \) of them are quadrilaterals.

The minimum degree condition in the theorem is sharp. To see this, we just need to observe \( K_{(n-1)/2,(n+1)/2} \) when \( n \) is odd and \( K_{(n-2)/2,(n+2)/2} \) when \( n \) is even.

We discuss only finite simple graphs and use standard terminology and notation from [2] except as indicated. Let \( G \) be a graph. For a vertex \( u \in V(G) \) and a subgraph \( H \) of \( G \) or a subset \( H \) of \( V(G) \), \( N(u, H) \) is the set of neighbors of \( u \) contained in \( H \). We let \( d(u, H) = |N(u, H)| \). Thus \( d(u, G) \) is the degree of \( u \) in \( G \). For a subset \( U \) of \( V(G) \), \( G[U] \) denotes the subgraph of \( G \) induced by \( U \). For a subset \( X \) of \( V(G) \), we use \( G - X \) to denote \( G[V(G) - X] \). If \( u \in V(G) \), we also write \( G - \{u\} \) as \( G - u \).

If \( C = x_1x_2 \cdots x_mx_1 \) is a cycle, then the subscripts of \( x_i \)'s will be taken modulo by \( m \) in \( \{1, 2, \ldots, m\} \). A chord of a cycle \( C \) in \( G \) is an edge of \( G - E(C) \) that joins two vertices of \( C \). We use \( \tau(C) \) to denote the number of chords of \( C \) in \( G \).

## 2 Lemmas

In the following, \( G = (V, E) \) is a graph of order \( n \geq 3 \).

**Lemma 2.1.** Let \( P = x_1 \ldots x_k \) be a path and \( u \) a vertex in \( G \) such that \( u \not\in V(P) \) and \( d(u, P) + d(x_k, P) \geq k \). Then either \( G \) has a path \( P' \) from \( x_1 \) to \( u \) such that \( V(P') = V(P) \cup \{u\} \), or \( k \geq 2 \), \( x_1u \in E \) and \( d(x_k, P) + d(u, P) = k \).

**Proof.** Let \( I = \{x_{i+1} \mid x_i, x_k \in E, 1 \leq i \leq k \} \). Clearly, \( x_1 \not\in I \). If \( N(u, P) \cap I \neq \emptyset \), say \( x_{i+1} \in N(u, P) \cap I \), then \( x_1 \ldots x_ix_kx_{k-1} \ldots x_{i+1}u \) is the required path from \( x_1 \) to \( u \). If \( N(u, P) \cap I = \emptyset \), then \( N(u, P) \cup I = V(P) \) since \( d(x_k, P) + d(u, P) \geq k \) and \( |I| = d(x_k, P) \), and then the lemma follows. \( \square \)

**Lemma 2.2.** Let \( Q \) be a quadrilateral and let \( x \) and \( y \) be two distinct vertices of \( G \) not on \( Q \). Suppose \( d(x, Q) + d(y, Q) \geq 5 \), then \( G[V(Q) \cup \{x, y\}] \) contains a quadrilateral \( Q' \) and an edge \( e \) such that \( Q' \) and \( e \) are independent and \( e \) is incident with exactly one of \( x \) and \( y \).

**Proof.** The lemma is clearly true if \( d(x, Q) = 4 \) or \( d(y, Q) = 4 \). So we may assume w.l.o.g. that \( d(x, Q) = 3 \) and \( d(y, Q) \geq 2 \). Label \( Q = a_1a_2a_3a_4a_1 \) such that \( N(x, Q) = \{a_1, a_2, a_3\} \). Then we see that the lemma is true if either \( a_2y \in E \) or \( a_4y \in E \). If \( a_2y \not\in E \) and \( a_4y \not\in E \), then \( Q' = a_1a_4a_3ya_1 \) and \( e = a_2x \) satisfy the requirement. \( \square \)

**Lemma 2.3.** Let \( Q \) be a quadrilateral and let \( x \) and \( y \) be two distinct vertices of \( G \) not on \( Q \). Suppose that \( d(x, Q) + d(y, Q) \geq 5 \) and \( G[V(Q) \cup \{x, y\}] \) is not hamiltonian. Then
$G[V(Q) \cup \{x, y\}]$ contains a quadrilateral $Q'$ with $\tau(Q') \geq \tau(Q)$ and an edge $e$ such that $Q'$ and $e$ are independent and $e$ is incident with exactly one of $x$ and $y$.

**Proof.** Let $Q = a_1 a_2 a_3 a_4 a_1$. We may assume that $d(x, Q) \geq d(y, Q)$ and $\{a_1, a_2, a_3\} \subseteq N(x, Q)$. Clearly, the lemma is true if $ya_4 \in E$ or $d(x, Q) = 4$. Hence we may assume that $ya_4 \notin E$ and $d(x, Q) = 3$. Thus $d(y, Q) \geq 2$. As $G[V(Q) \cup \{x, y\}]$ is not hamiltonian, we see that $\{a_1, a_2\} \notin N(y)$ and $\{a_2, a_3\} \notin N(y)$. It follows that $N(y, Q) = \{a_1, a_3\}$, and therefore $a_2 a_4 \notin E$ for otherwise $G[V(Q) \cup \{x, y\}]$ is hamiltonian. Let $Q' = ya_1 a_3 y$ and $P' = xa_2$. Clearly, $\tau(Q') = \tau(Q)$, and so the lemma holds.

**Lemma 2.4.** Suppose that $n \geq 5$ and $d(x, G) + d(y, G) \geq n$ for every two nonadjacent vertices $x$ and $y$ of $G$. Then for each $x \in V(G)$, $G$ has a quadrilateral $Q$ such that $G - V(Q)$ has a hamiltonian path starting at $x$ unless that $n \leq 6$, and in addition, if $n = 5$ then $d(u, G) + d(v, G) = 5$ for some two nonadjacent vertices $u$ and $v$ of $G$, and if $n = 6$ then $G$ has an edge $e$ such that $G$ has a hamiltonian path from $u$ to $v$, $G - u - v$ has a quadrilateral and $d(u, G) + d(v, G) = 6$.

**Proof.** For the proof, we suppose that the lemma fails. Let $x_0$ be a vertex of $G$ such that $G$ does not have a quadrilateral $Q$ such that $G - V(Q)$ has a hamiltonian path starting at $x_0$.

First, suppose that $G - x_0$ does not have a quadrilateral. Let $x$ and $y$ be two arbitrary nonadjacent vertices of $G - x_0$. Then $|N(x, G - x_0) \cap N(y, G - x_0)| \leq 1$. As $d(x, G) + d(y, G) \geq n$, we see that $N(x, G) \cup N(y, G) = V(G) - \{x, y\}$, $x_0 \in N(x, G) \cap N(y, G)$ and $|N(x, G - x_0) \cap N(y, G - x_0)| = 1$. Say $N(x, G) \cap N(y, G) = \{x_0, z\}$. Assume w.l.o.g. $d(x) \geq d(y)$. Suppose $d(x, G - x_0) \geq 4$. Let $\{x_1, x_2\} \subseteq N(x, G - x_0 - z)$ with $x_1 \neq x_2$. Then either $x_1 z \notin E$ or $x_2 z \notin E$ for otherwise $G - x_0$ has a quadrilateral. Say $x_1 z \notin E$. For the same reason, $x_1 y \notin E$ and $x_2 y \notin E$. Similarly, we must have $N(x_1, G) \cup N(y, G) = V(G) - \{x_1, y\}$ and $|N(x_1, G) \cap N(y, G)| = 2$. In particular, we also have that $x_1 x_2 \in E$. Let $y_1 \in N(y, G - x_0 - z)$ be such that $x_1 y_1 \in E$. Clearly, $x_2 z \notin E$ and $x_2 y_1 \notin E$ for otherwise $G - x_0$ has a quadrilateral. Similarly, we have that $|N(x_2, G) \cap N(y, G)| = 2$ and $N(x_2, G) \cup N(y, G) = V(G) - \{x_2, y\}$. Let $y_2 \in N(y, G)$ be such that $x_2 y_2 \in E$. Similarly, we can show $y_1 y_2 \in E$, and thus $x_1 x_2 y_2 y_1 x_1$ is a quadrilateral in $G - x_0$, a contradiction. Therefore we must have $d(x, G) = 3$. Thus $n \leq 6$. If $n = 5$, we have that $d(x, G) + d(y, G) = 5$ and we are done. Hence we assume $n = 6$. Thus $d(x) = d(y) = 3$. Let $V(G) - \{x_0, x, y, z\} = \{x_1, y_1\}$ be such that $\{xx_1, yy_1\} \subseteq E$. As $x_1 y_1 \notin E$ and $x_1 x_2 \notin E$, we can show, as before, that $\{x_0 x_1, x_0 y_1\} \subseteq E$. If $x_1 y_1 \notin E$, then $z \in N(x_1) \cap N(y_1)$ as $d(x_1, G) + d(y_1, G) \geq 6$, and consequently, the second statement of the lemma holds with $\{u, v\} = \{y_1, y\}$. Thus we assume that $x_1 y_1 \in E$. Then $x_1 x_2 \notin E$ and $z y_1 \notin E$ for otherwise $G - x_0$ has a quadrilateral. Then $x_0 z \in E$ as $d(x_1, G) + d(z, G) \geq 6$. Again, we see that the second statement of the lemma holds with $\{u, v\} = \{x_1, y_1\}$.

Next, suppose that $G - x_0$ has a quadrilateral. We now choose a quadrilateral $Q$ from $G - x_0$ such that

The length of a longest path starting at $x_0$ in $G - V(Q)$ is maximum. \hfill (1)
Let $P$ be a longest path starting at $x_0$ in $G - V(Q)$. Subject to (1), we choose $Q$ and $P$ such that

$$\tau(Q) \text{ is maximum}. \quad (2)$$

Let $P = x_0x_1 \ldots x_t$ and $Q = a_1a_2a_3a_4$. We need to show that $t = n - 5$. On the contrary, suppose $t < n - 5$. Let $D = G - V(P \cup Q)$ and $r = |V(D)|$. Then $t = n - 5 - r$. Let $y_0 \in V(D)$. By Lemma 2.1, we have

$$d(y_0, P) + d(x_t, P) \leq t + 1. \quad (3)$$

Therefore

$$d(y_0, Q) + d(x_t, Q) \geq n - (t + 1) - (r - 1) = 5. \quad (4)$$

We claim the following:

Claim A. For each $i \in \{1, 2\}$, \{a_i, a_{i+2}\} $\notin N(y_0, Q)$.

Proof of Claim A. On the contrary, say w.l.o.g. \{a_1, a_3\} $\subseteq N(y_0, Q)$. By (1), we see that \{a_2, a_4\} $\cap N(x_t, Q) = \emptyset$. Hence $d(y_0, Q) \geq 3$ by (4). Say $a_2y_0 \in E$. As $\tau(y_0a_1a_2a_3a_4y_0) \leq \tau(Q)$ by (2), we must have $a_2a_4 \in E$. Thus $G[\{a_1, a_2, a_3, a_4, y_0\} - \{a_i\}]$ contains a quadrilateral for each $i \in \{1, 2, 3, 4\}$, and therefore $d(x_t, Q) = 0$ by (1), contradicting with (4). Hence the claim holds.

We now divide the proof into the following two cases.

Case 1. $d(y_0, Q) = 2$.

In this case, $d(x_t, Q) \geq 3$. By Claim A, we may assume w.l.o.g. $N(y_0, Q) = \{a_1, a_2\}$. We may also assume w.l.o.g. \{a_2, a_4\} $\subseteq N(x_t, Q)$ as $d(x_t, Q) \geq 3$. Then $a_1a_3 \notin E$ for otherwise $y_0a_1a_3a_2y_0$ is a quadrilateral and $P + x_ta_4$ is longer than $P$ in $G$. As $y_0a_3 \notin E$, $d(y_0, G) + d(a_3, G) \geq n$ and so $|N(y_0, G) \cap N(a_3, G)| \geq 2$. Then it is easy to see that $t \geq 1$. Set $Q_1 = x_ta_2a_3a_4x_t$. Then we see that $x_{t-1}y_0 \notin E$ and $x_{t-1}a_1 \notin E$ by (1). For the same reason, $d(y_0, P - x_t) + d(x_{t-1}, P - x_t) \leq t$ and $N(y_0, D) \cap N(x_{t-1}, D) = \emptyset$. It follows that $d(y_0, P \cup D) + d(x_{t-1}, P \cup D) \leq n - 5$, and therefore $d(y_0, Q) + d(x_{t-1}, Q) \geq 5$. Therefore $N(x_{t-1}, Q) = \{a_2, a_3, a_4\}$. Furthermore, we see that $d(x_{t-1}, P - x_t) + d(y_0, P - x_t) = t$. By Lemma 2.1, $t - 1 \geq 1$. Let $Q_2 = x_{t-1}x_ta_2a_3x_{t-1}$. Then we see that \{y_0, a_1, a_2\} $\cap N(x_{t-2}, G) = \emptyset$ and $N(y_0, D) \cap N(x_{t-2}, D) = \emptyset$. This implies that $d(y_0, P - x_{t-1} - x_{t-1}) + d(x_{t-2}, P - x_{t-1}) \geq n - 5 - (r - 1) = t + 1$. By Lemma 2.1, $G[V(P) \cup \{y_0\} \setminus \{x_t, x_{t-1}\}]$ has a hamiltonian path $P'$ from $x_0$ to $y_0$. Therefore $P'y_0a_1a_2$ is longer than $P$ and independent of $Q_2$, contradicting (1).

Case 2. $d(y_0, Q) = 1$.

We have that $d(x_t, Q) = 4$. Say $y_0a_1 \in E$. As $|N(y_0, G) \cap N(a_3, G)| \geq 2$, we see that $t \geq 1$. By (1), $a_1x_{t-1} \notin E$ and $y_0x_{t-1} \notin E$. We also have, by (1), that $N(y_0, D) \cap N(x_{t-1}, D) = \emptyset$. It follows that $d(y_0, P - x_t) + d(x_{t-1}, P - x_t) \geq n - 5 - (r - 1) = t + 1$. By Lemma 2.1, $G[V(P) \cup \{y_0\} \setminus \{x_t\}]$ has a hamiltonian path $P''$ from $x_0$ to $y_0$. Then $P''y_0a_1$ is longer than $P$ and independent of $x_ta_2a_3a_4x_t$, contradicting (1). This proves the lemma. \[\Box\]
Lemma 2.5. Suppose that \( d(x, G) + d(y, G) \geq n \) for every two nonadjacent vertices \( x \) and \( y \) of \( G \). Then for any two distinct vertices \( u \) and \( v \), \( G \) has a hamiltonian path from \( u \) to \( v \) unless either \( \{u, v\} \) is a vertex-cut of \( G \) or \( G \) has an independent set \( X \) with \( |X| \geq n/2 \) and \( \{u, v\} \subseteq V(G) - X \).

Proof. For the proof, suppose that there exist two distinct vertices \( u \) and \( v \) such that \( G \) does not have a hamiltonian path from \( u \) to \( v \) and \( \{u, v\} \) is not a vertex-cut of \( G \). Then we shall prove that \( G \) has an independent set \( X \) with \( |X| \geq n/2 \) and \( \{u, v\} \subseteq V(G) - X \).

Let \( P \) be a longest path of \( G \) starting at one of \( u \) and \( v \) but not passing through the other. Let \( \{u, v\} = \{x_0, x_1\} \) and \( P = x_1x_2\ldots x_t \). Set \( D = G - V(P) \) and \( r = |V(D)| \). If \( r = 1 \), then \( x_0, x_1 \notin E \) and so \( d(x_0, P) + d(x_1, P) \geq n \) by Lemma 2.1, \( G \) has a hamiltonian path from \( x_1 \) to \( x_0 \), a contradiction. Hence \( r \geq 2 \). As \( \{x_0, x_1\} \) is not a vertex-cut of \( G \), we let \( s \) be the smallest integer in \( \{2, 3, \ldots, t - 1\} \) such that \( d(x_s, D - x_0) \geq 1 \). Let \( y_0 \in V(D) - \{x_0\} \) such that \( y_0x_s \in E \). For each \( y \in V(D) - \{x_0\} \), we must have that \( d(y, P) + d(x_0, P) \leq t \) by Lemma 2.1, and therefore \( d(y, D) + d(x_1, D) \geq r \). Thus \( d(x_1, D) > 0 \). It follows that \( N(x_1, D) = \{x_0\} \) and \( D \) is a complete subgraph of \( G \). Furthermore, \( d(y, P) + d(x_1, P) = t \) for each \( y \in V(D) - \{x_0\} \). Let \( L_1 = x_1x_2\ldots x_s \) and \( L_2 = x_s+1x_s+2\ldots x_t \). Set \( I = \{x_{i-1}, x_i, y_0 \in E, s + 1 \leq i \leq t - 1\} \). Clearly, \( y_0x_{s+1} \notin E \) and \( N(x_t, L_2) \cap I = \emptyset \) for otherwise \( G[V(P) \cup \{y_0\}] \) has a hamiltonian path starting at \( x_t \). This implies that \( d(y_0, L_2) + d(x_t, L_2) \leq |V(L_2)| - 1 = t - s - 1 \), and thus \( d(y_0, L_1) + d(x_t, L_1) \geq s + 1 \). As \( x_0y_0x_sx_{s+1}x_{s+2}\ldots x_t \) is a path in \( G \) and by the maximality of \( P \), we must have \( s \geq 3 \). Clearly, \( x_{s-1}x_t \notin E \) for otherwise \( x_1x_2\ldots x_{s-1}x_tx_{t-1}\ldots x_y \) is a longer path than \( P \) in \( G \). Therefore \( N(x_t, L_1) = V(L_1) - \{x_{s-1}\} \) and \( y_0x_t \in E \). If \( s \geq 4 \), then \( x_0y_0x_sx_{s+1}\ldots x_1x_2x_3\ldots x_{s-1} \) is a longer path than \( P \) in \( G \), a contradiction. Hence \( s = 3 \). If \( r \geq 3 \) then \( x_0y'y_0x_3x_4\ldots x_1 \) is a longer path than \( P \) in \( G \) with \( y' \in V(D) - \{x_0, y_0\} \), a contradiction. Hence \( r = 2 \). Let \( P' = x_0y_0x_3x_4\ldots x_1 \). Then \( P' \) is a path in \( G \) starting at \( x_0 \) without passing through \( x_1 \). Furthermore, \( P' \) and \( P \) have the same length. Therefore we may assume w.l.o.g. that \( d(y_0, G) \geq n/2 \) as \( d(y_0, G) + d(x_2, G) \geq n \). Let \( X = \{x_i, y_0 \in E, 1 \leq i \leq t - 1\} \cup \{y_0\} \). We see that \( X \) is an independent set of \( G \) for otherwise \( G[V(P) \cup \{y_0\}] \) has a hamiltonian path starting at \( x_t \). Clearly, \( |X| \geq n/2 \) and \( \{x_0, x_1\} \subseteq V(G) - X \). This proves the lemma.

Lemma 2.6. [7] If \( P = x_1x_2\ldots x_m \) is a path of \( G \) with \( m \geq 3 \) such that \( d(x_1, P) + d(x_m, P) \geq m \), then \( G \) has a cycle \( C \) such that \( V(C) = V(P) \). Moreover, if \( d(x, G) + d(y, G) \geq n \) for any two nonadjacent vertices \( x \) and \( y \) of \( G \), then \( G \) is hamiltonian.

Lemma 2.7. Let \( t \) be a positive integer and let \( G \) be a graph of order \( n \geq 4t \). Suppose that \( d(x) \geq [n/2] \) for each \( x \in V(G) \). Then \( G \) has \( t \) independent quadrilaterals \( Q_1, Q_2, \ldots, Q_t \) such that \( G - V(\cup_{i=1}^t Q_i) \) has a hamiltonian path.

Proof. Let \( r = n - 4t \). We use induction on \( r \) to prove the lemma. When \( r \in \{0, 1, 4\} \), the lemma is true by Theorem A and Theorem B. Suppose \( r = 2 \). Then \( G \) has \( t \) independent quadrilaterals \( Q_1, \ldots, Q_t \) by Theorem B. If the two vertices of \( G - V(\cup_{i=1}^t Q_i) \), say \( x \) and \( y \), are not adjacent, then we would have that \( d(x, \cup_{i=1}^t Q_i) + d(y, \cup_{i=1}^t Q_i) \geq 4t + 2 \),
and therefore $d(x, Q_i) + d(y, Q_i) \geq 5$ for some $i \in \{1, 2, \ldots, t\}$. By Lemma 2.2, the lemma holds. Next, suppose $r = 3$. Using the above proof, we see that $G$ has $t$ independent quadrilaterals $Q_1, \ldots, Q_t$ such that $G - V(\cup_{i=1}^t Q_i)$ has at least one edge. Subject to this, we let $\sum_{i=1}^t \tau(Q_i)$ be as large as possible. Let $V(G) - V(\cup_{i=1}^t Q_i) = \{x_1, x_2, x_3\}$ be such that $x_1x_2 \in E$. If $x_3x_1 \in E$ or $x_3x_2 \in E$, we have nothing to prove. Hence we assume that $x_3x_1 \notin E$ and $x_3x_2 \notin E$. Then we see that there exists $Q_i$, say $Q_i = Q_1$, such that $d(x_1, Q_1) + d(x_3, Q_1) \geq 4$. By Lemma 2.2, $G[V(Q_1) \cup \{x_1, x_3\}]$ contains a quadrilateral $Q'_1$ and an edge $e$ such that $Q'_1$ and $e$ are independent and $e$ is incident with exactly one of $x_1$ and $x_3$. If $e$ is incident with $x_1$, then $e$ and $x_1x_2$ together contains a path of order 3 and we are done. Therefore we may assume that $e$ is incident with $x_3$. Say $y_1 \in V(Q_1)$ with $y_1x_3 \in E$. Let $Q_1 = a_1a_2a_3a_4$. Suppose that $d(x_3, Q_1) \geq 3$. Say $N(x_3, Q_1) \supseteq \{a_1, a_2, a_3\}$. Then $\tau(x_3a_1a_2a_3x_3) \geq \tau(Q_1)$ with equality only if $a_2a_4 \in E$. By our choice of $Q_i(1 \leq i \leq t)$, we must have that $a_2a_4 \in E$. Thus for each $i \in \{1, 2, 3, 4\}$, $G[\{a_1, a_2, a_3, a_4, x_3 \} - \{a_i\}]$ contains a quadrilateral. As $d(x_1, Q_1) \geq 1$, we see that the lemma holds. Hence we may assume that $d(x_3, Q_1) \leq 2$. Thus $d(x_1, Q_1) \geq 3$. Suppose that we also have that $d(x_2, Q_1) + d(x_3, Q_1) \geq 5$. Then $d(x_2, Q_1) \geq 3$. This implies that there exists $\{i, j\} \subseteq \{1, 2, 3, 4\}$ with $i \neq j$ such that $x_1a_ia_{i+1}x_2x_1$ and $x_1a_ja_{j+1}x_2x_1$ are two quadrilaterals in $G$. Thus the lemma holds if $x_3$ is adjacent to some vertex of $\{a_{i+2}, a_{i+3}, a_{j+2}, a_{j+3}\}$. Hence we may assume that $x_3$ is not adjacent to any vertex of this set, which implies that $d(x_3, Q_1) \leq 1$. It follows that $d(x_1, Q_1) = d(x_2, Q_1) = 4$ and $d(x_3, Q_1) = 1$, and clearly, the lemma holds in this situation, too. To finish the proof, we finally assume that $d(x_2, Q_1) + d(x_3, Q_1) \leq 4$. Then $d(x_2, \cup_{i=1}^t Q_i) + d(x_3, \cup_{i=1}^t Q_i) \geq 4t + 2 - 5 = 4(t - 1) + 1$. This implies that there exists $Q_i$ with $i \geq 2$, say $i = 2$, such that $d(x_2, Q_2) + d(x_3, Q_2) \geq 5$. As above, with $Q_2$ and $x_2$ in place of $Q_1$ and $x_1$, we may assume that $Q_2$ has a vertex $y_2$ such that $G[V(Q_2) \cup \{x_2\} - \{y_2\}]$ contains a quadrilateral $Q'_2$ and $x_3y_2 \in E$. Since $y_1x_3y_2$ is a path in $G$, we see the lemma holds. Therefore the lemma holds if $r \leq 4$. We now assume that the lemma is true if the value of $(n - 4t)$ is less than $r$ with $r \geq 5$. Say $n - 4t = r$. Then $n - 4(t + 1) = r - 4$. By the induction hypothesis, $G$ has $t + 1$ independent quadrilaterals $Q_1, \ldots, Q_{t+1}$ such that $G - V(\cup_{i=1}^{t+1} Q_i)$ has a path $P$ of order $r - 4$. Let $P = x_1x_2 \ldots x_{r-4}$. Then we may assume that for each $j \in \{1, 2, \ldots, t + 1\}$, $d(x_1, Q_j) = d(x_{r-4}, Q_j) = 0$ holds for otherwise $G[V(Q_j \cup P)]$ has a hamiltonian path and we are done. Thus $d(x_1, P) + d(x_{r-4}, P) \geq n$, and by Lemma 2.6, $G[V(P)]$ is hamiltonian. As $G$ is connected, there exists $Q_j$ such that $\sum_{i=1}^{t-1} d(x_i, Q_j) > 0$ and therefore $G[V(Q_j \cup P)]$ has a hamiltonian path. Thus the lemma is true for $n - 4t = r$. This proves the lemma.

Lemma 2.8. [5] Let $C = x_1x_2 \ldots x_mx_1$ be a cycle of $G$. Let $x_i, x_j \in V(C)$ with $i \neq j$. If $d(x_i, C) + d(x_j, C) \geq m + 1$, then $G$ has a path $P$ from $x_{i+1}$ to $x_{j+1}$ such that $V(P) = V(C)$.

Lemma 2.9. Suppose that $G$ has a hamiltonian path and that $d(x, G) + d(y, G) \geq n + s$ for any two endvertices $x$ and $y$ of a hamiltonian path of $G$, where $s$ is a fixed nonnegative integer. Then for any two distinct vertices $u$ and $v$ of $G$, $d(u, G) + d(v, G) \geq n + s$ holds.

Proof. By Lemma 2.6, $G$ is hamiltonian. Let $C = x_1x_2 \ldots x_nx_1$ be a hamiltonian cycle. Suppose, for a contradiction, that $d(x_i, G) + d(x_j, G) \leq n + s - 1$ for some $1 \leq i <
$j \leq n$. By the hypothesis, $d(x_{i-1}, G) + d(x_i, G) \geq n + s$ and $d(x_{j-1}, G) + d(x_j, G) \geq n + s$. Then we see that $d(x_{i-1}, G) + d(x_{j-1}, G) \geq n + s + 1$. By Lemma 2.8, $G$ has a hamiltonian path from $x_i$ to $x_j$, and by the hypothesis again, $d(x_i, G) + d(x_j, G) \geq n + s$, a contradiction. □

3 Proof of Theorem C

Let $k$ be a positive integer and $G$ a graph of order $n \geq 4k$. Assume $\delta(G) \geq \lceil n/2 \rceil$. Suppose, for a contradiction, that $G$ does not contain $k$ independent cycles covering all the vertices of $G$ such that $k - 1$ of them are quadrilaterals. By Theorem A, $n > 4k$.

Let $t = n - 4(k - 1)$. By Lemma 2.7, $G$ has $k$ independent quadrilaterals $Q_1, \ldots, Q_k$ such that $G - V(\bigcup_{i=1}^k Q_i)$ has a hamiltonian path $P$. If $t - 4 = 1$, then we readily see that $d(u, Q_i) \geq 3$ where $V(P) = \{u\}$ for some $i \in \{1, \ldots, k\}$ because $\delta(G) \geq \lceil n/2 \rceil$ and so $G[V(Q_i \cup P)]$ is hamiltonian, a contradiction. Hence we have $t - 4 \geq 2$. For convenience, let $r = t - 4$. As $G[V(Q_i \cup P)]$ is not hamiltonian, for any two endvertices $u$ and $v$ of a hamiltonian path of $G - V(\bigcup_{i=1}^k Q_i)$, we have

$$d(u, Q_i) + d(v, Q_i) \leq 4 \text{ for all } i \in \{1, \ldots, k\} \tag{5}$$

and therefore

$$d(u, G - V(\bigcup_{i=1}^k Q_i)) + d(v, G - V(\bigcup_{i=1}^k Q_i)) \geq r + \sigma \tag{6}$$

where $\sigma = 1$ if $r$ is odd and otherwise $\sigma = 0$. By Lemma 2.6, $G - V(\bigcup_{i=1}^k Q_i)$ is hamiltonian if $r \geq 3$. Let $H = \bigcup_{i=1}^k Q_i$ and $D = G - V(H)$. By (6) and Lemma 2.9, we have

$$d(x, D) + d(y, D) \geq r + \sigma \text{ for all } \{x, y\} \subseteq V(D) \text{ with } x \neq y. \tag{7}$$

We now divide the proof into the following two cases.

Case 1. $r \geq 5$.

In this case, By (7), $D$ is hamiltonian. We choose a hamiltonian path $P$ of $D$ as follows. If for each $x \in V(D)$, $D$ has a quadrilateral $Q$ such that $D - V(Q)$ has a hamiltonian path starting at $x$, let $P$ be a hamiltonian path with an endvertex $u$ such that $d(u, Q) \geq 1$ for some $i \in \{1, \ldots, k\}$. Such a path exists because $G$ is connected. We may assume $e(u, Q) \geq 1$ in this case. Otherwise by (7) and Lemma 2.4, we see that $\sigma = 0$ and $D$ has order 6. Furthermore, $D$ has an edge $uv$ such that $D$ has a hamiltonian path $P$ from $u$ to $v,$ $D - u - v$ has a quadrilateral and $d(u, D) + d(v, D) = 6.$ Then equality holds in (5) and (6) with respect to $\{u, v\}$, and therefore $d(u, Q_i) + d(v, Q_i) \geq 1$ for some $i \in \{1, \ldots, k\}$. In this case, we may assume $d(u, Q_1) \geq 1$. In the former case, let $Q'_1$ be a quadrilateral of $D$ such that $D - V(Q'_1)$ has a hamiltonian path starting at $u$. Subject to this, we further choose $Q'_1$ such that $D - V(Q'_1)$ does not contain a vertex-cut of cardinality 2 if there exists such a choice. In the latter, let $Q'_1$ be a quadrilateral of $D - u - v$ and $P = uv$. Set $D' = G[V(D \cup Q_1) - V(Q'_1)].$ As $d(u, Q_1) \geq 1,$ $D'$ has a hamiltonian path.

Replacing $Q_1'$ and $D'$ in the above proof of (5), (6) and (7), we see that $D'$ is hamiltonian, too. Let $L$ be a hamiltonian cycle of $D'$. Then the number of edges of $L$ in
between $Q_1$ and $D' - V(Q_1)$ must be even. This allows us to see that there exist two independent edges $x_1y_1$ and $x_2y_2$ between $Q_1$ and $D' - V(Q_1)$ with $\{y_1, y_2\} \subseteq V(Q_1)$ such that $G[V(Q_1)]$ has a hamiltonian path from $y_1$ to $y_2$. If $D$ has a hamiltonian path from $x_1$ to $x_2$, then $G[V(D \cup Q_1)]$ is hamiltonian and we are done. By Lemma 2.5, we see that either $\{x_1, x_2\}$ is a vertex-cut of $D$ or $D$ has an independent set $X$ with $|X| \geq r/2$ and $\{x_1, x_2\} \subseteq V(D) - X$. Let us first assume that $\{x_1, x_2\}$ is a vertex-cut of $D$. By (7), we see that $D - x_1 - x_2$ has exactly two components, say $D_1$ and $D_2$ such that $D_1 \cong D_2 \cong K_{(r-2)/2}$ and $d(x_1, D_1 \cup D_2) = d(x_2, D_1 \cup D_2) = r - 2$. As $D_1 \cup D_2 \supseteq Q'_1$, we see that $r \geq 10$. Let $Q''_i$ be a quadrilateral in $D_1 + x_1$ with $x_1 \in V(Q''_i)$. Let $z_1 \in V(D_1) - V(Q''_i)$ and $z_2 \in V(D_2)$. Clearly, $d(z_1, D) + d(z_2, D) = r$ and $D - V(Q''_i)$ has a hamiltonian path from $z_1$ to $z_2$. Then equality holds in (5) and (6) with respect to $\{z_1, z_2\}$. It follows that $d(z_j, Q_i) \geq 1$ for some $j \in \{1, 2\}$ and $i \in \{1, \ldots, k\}$. But $D - V(Q''_i)$ does not contain a vertex-cut of $D$ with cardinality 2. This contradicts the choice of $Q'_1$. Therefore $D$ has an independent set $X$ with $|X| \geq r/2$ and $\{x_1, x_2\} \subseteq V(D) - X$. By (7), we see that $|X| = r/2$ and $D$ contains a complete bipartite subgraph with $(X, V(D) - X)$ as its bipartition. As mentioned in the beginning of this paragraph, $D'$ is hamiltonian and so we readily see that $d(x, Q_1) > 0$ for some $x \in X$. It follows that $G[V(D \cup Q_1)]$ is hamiltonian, a contradiction.

Case 2. $2 \leq r \leq 4$.

In this case, we choose the $k$ independent quadrilaterals $Q_1, \ldots, Q_k$ and the path $P$ of order $r$ such that

$$\sum_{i=1}^{k} \tau(Q_i) \text{ is maximum.} \quad (8)$$

By (6), $D$ is hamiltonian if $r \geq 3$. We break into the following three cases.

Case 2.1. $r = 3$.

Then $D$ is a triangle, say $D = x_1x_2x_3x_1$. We have that $\sum_{i=1}^{3} d(x_i, H) \geq 3\lceil n/2 \rceil - 6 = 3(2k + 2) - 6 = 6k$. This implies that there exists $Q_i$ in $H$, say $Q_1 = Q_i$, such that $\sum_{i=1}^{3} d(x_i, Q_1) \geq 6$. This further implies that there exist two independent edges $e_1$ and $e_2$ between $D$ and $Q_1$. Say $Q_1 = a_1a_2a_3a_4a_1$. As $G[V(D \cup Q_1)]$ is not hamiltonian, we may assume that $e_1 = a_1x_1$ and $e_2 = a_3x_2$, and then we see that $d(a_2, D) = d(a_4, D) = 0$ and $a_2a_4 \notin E$. Consequently, $d(a_1, D) = d(a_3, D) = 3$. Let $Q'_1$ be a quadrilateral of $D + a_1$ and $P' = a_2a_3a_4$. Clearly, $\tau(Q'_1) > \tau(Q_1)$, contradicting (8).

Case 2.2. $r = 4$.

Since $G$ is hamiltonian, we have nothing to prove when $k = 1$. Therefore $k \geq 2$. Let $Q_0 = x_1x_2x_3x_4x_1$ be a quadrilateral of $D$. Clearly, $\sum_{i=1}^{4} d(x_i, H) \geq 4\lceil n/2 \rceil - 12 = 8(k + 1) - 12 = 6k + 2k - 4$. This implies that there exists $Q_i$ in $H$, say $Q_i = Q_1$, such that $\sum_{i=1}^{4} d(x_i, Q_1) \geq 6$. This further implies that there exist two independent edges, say $u_1w_1$ and $u_2w_2$ with $\{u_1, u_2\} \subseteq V(Q_0)$ and $\{w_1, w_2\} \subseteq V(Q_1)$, between $Q_0$ and $Q_1$ such that either $u_1u_2 \in E(Q_0)$ or $w_1w_2 \in E(Q_1)$. We may assume w.l.o.g. that $w_1w_2 \in E(Q_1)$. Therefore $u_1u_2 \notin E(Q_0)$ as $G[V(Q_0 \cup Q_1)]$ is not hamiltonian. Say w.l.o.g.
\{u_1, u_2\} = \{x_1, x_3\}. Then for the same reason, we see that \(d(x_2, Q_1) = d(x_4, Q_1) = 0\) and 
\(x_2x_4 \notin E\). Thus \(d(x_2, H - V(Q_1) + d(x_4, H - V(Q_1)) \geq 2[n/2] - 4 = 4(k - 1) + 4\). This implies that there exists \(Q_1\) in \(H - V(Q_1)\), say \(Q_1 = Q_2\), such that \(d(x_2, Q_2) + d(x_4, Q_2) \geq 5\).

As \(G[V(Q_0 \cup Q_2)]\) is not hamiltonian, we see that \(x_1x_3 \notin E\). Then equality holds in (5) and (6) with respect to \(\{x_j, x_{j+1}\}\) for each \(j \in \{1, 2, 3, 4\}\) and \(i \in \{1, 2, \ldots, k\}\), that is, \(d(x_j, Q_1) + d(x_{j+1}, Q_1) = 4\) for each \(j \in \{1, 2, 3, 4\}\) and \(j \in \{1, 2, \ldots, k\}\). Thus \(d(x_1, Q_1) = d(x_3, Q_1) = 4\). As \(d(x_2, Q_2) + d(x_4, Q_2) \geq 5\), it is also easy to see that \(d(x_1, Q_2) = d(x_3, Q_2) = 0\) for otherwise \(G[V(Q_0 \cup Q_2)]\) is hamiltonian. Thus \(d(x_2, Q_2) = d(x_4, Q_2) = d(x_1, Q_1) = d(x_3, Q_1) = 4\). Let \(y\) be an arbitrary vertex of \(Q_1\) and \(z\) an arbitrary vertex of \(Q_2\). Clearly, \(Q_1 - y + x_3\) and \(Q_2 - z + x_4\) are hamiltonian and \(yx_1x_2z\) is a path in \(G\). Similar to the proof of (6), we see that \(G[\{y, x_1, x_2, z\}]\) must be hamiltonian. Consequently, \(yz \in E\). This argument implies that \(d(w, Q_2) = 4\) for all \(w \in V(Q_1)\). It follows that \(G[V(Q_0 \cup Q_1 \cup Q_2) - \{y, x_1, x_2, z\}]\) is hamiltonian and we are done.

Case 2.3. \(r = 2\).

Let \(D = x_1x_2\). As \(\delta(D) \geq 2(k + 1)\) and by (5) and (6), we see that \(d(x_1, Q_i) + d(x_2, Q_i) = 4\) for all \(i \in \{1, 2, \ldots, k\}\). We claim that for each \(i \in \{1, 2, \ldots, k\}\), either \(d(x_1, Q_i) = 0\) or 
\(d(x_2, Q_i) = 0\). If this is not true, say \(d(x_1, Q_i) > 0\) and \(d(x_2, Q_i) > 0\). Let \(Q_i = a_1a_2a_3a_4a_1\) with \(x_1a_1 \in E\). As \(G[V(D \cup Q_i)]\) is not hamiltonian, we see that \(N(x_1, Q_i) = N(x_2, Q_i) = \{a_1, a_3\}\) and \(a_2a_4 \notin E\). Let \(Q_i' = x_1a_1a_3x_1a_3\). Clearly, \(\tau(Q_i') = \tau(Q_i) + 1\). We also have that \(d(a_2, H - V(Q_i)) + d(a_4, H - V(Q_i)) \geq 2(2k + 1) - 4 = 4(k - 1) + 2\). This implies that there exists \(Q_i\) in \(H - V(Q_1)\), say \(Q_i = Q_2\), such that \(d(a_2, Q_2) + d(a_4, Q_2) \geq 5\). As \(G[V(Q_2) \cup \{a_2, a_4\}]\) is not hamiltonian and by Lemma 2.3, \(G[V(Q_2) \cup \{a_2, a_4\}]\) has a quadrilateral \(Q_2'\) and a path \(P'\) of order 2 such that \(\tau(Q_2') \geq \tau(Q_2)\) and \(V(Q_2') \cap V(P') = \emptyset\). Replacing \(Q_1\) and \(Q_2\) by \(Q_i'\) and \(Q_2\), we see that (8) is violated. Therefore our claim holds.

Next, we claim that \(G[N(x_1, G - x_2)]\) is a complete subgraph of \(G\) and \(G[N(x_2, G - x_1)]\) is a complete subgraph of \(G\). For the proof, let \(u\) be an arbitrary vertex of \(N(x_1, G - x_2)\). We shall show that \(u\) is adjacent to every vertex of \(N(x_1, G - x_2 - u)\). Let \(Q_1 = a_1a_2a_3a_4a_1\). Say w.l.o.g. \(u = a_1\). Let \(G_1 = G[V(D \cup Q_1)]\) and \(H_1 = H - V(Q_1)\). Then \(d(x_2, H_1) + d(a_1, H_1) \geq 2(2k + 1) - 5 = 4(k - 1) + 1\). This implies that there exists \(Q_i\) in \(H_1\), say \(Q_i = Q_2\), such that \(d(x_2, Q_2) + d(a_1, Q_2) \geq 5\). Thus we must have \(d(x_2, Q_2) = 4\) and 
\(d(x_1, Q_2) = 0\). If \(d(a_1, Q_2) \geq 2\), then \(G(V(Q_1) \cup \{x_2, a_1\})\) is hamiltonian. As \(Q_1 - a_1 + x_1\) is hamiltonian, the theorem holds, a contradiction. Hence \(d(a_1, Q_2) = 1\). Let \(Q_2 = b_1b_2b_3b_4b_1\) with \(a_1b_1 \in E\). Similarly, we must have that \(d(b_1, Q_1) = 1\). Replacing \(Q_1\) and \(Q_2\) by \(Q_i' = x_1a_1a_3a_4x_1\) and \(Q_2 = x_2b_2b_3b_2x_2\), we see, by (8), that \(\{a_1a_3, b_1b_3\} \subseteq E\). Note that this argument implies that \(G(V(Q_i')\) is a complete graph of order 4 for all \(i \in \{1, \ldots, k\}\). With respect to the choice of \(\{a_1b_1, Q_i', Q_2, Q_3, \ldots, Q_k\}\), we can also show that for each \(i \in \{3, \ldots, k\}\), either \(d(a_1, Q_i) = 4\) or \(d(b_1, Q_i) = 4\). If there exists \(Q_i\) in \(\{Q_3, \ldots, Q_k\}\) such that \(d(x_1, Q_i) = d(b_1, Q_i) = 4\), then we would see that each of \(Q_i + x_1 + b_1\) and \(Q_2 - b_1 + x_2\) is hamiltonian and we are done. Hence we must have \(d(a_1, Q_i) = 4\) for each \(i \in \{3, \ldots, k\}\) with \(d(x_1, Q_i) = 4\). Therefore \(G[N(x_1, G - x_2)]\) is a complete subgraph of \(G\). Similarly, \(G[N(x_2, G - x_1)]\) is a complete subgraph of \(G\). The above argument also implies that \(d(w, G[N(x_1, G - x_j)]) = 1\) for each \(w \in N(x_j, G - x_i)\) with \(\{i, j\} = \{1, 2\}\), that is, there are 2\(k\) independent edges between \(N(x_1, G - x_2)\) and \(N(x_2, G - x_1)\). It is
easy to see that $G$ contains $k$ required independent cycles in this case. This completes the proof of the theorem.

References


