Rook placements and Jordan forms of upper-triangular nilpotent matrices

Martha Yip*
Department of Mathematics
University of Kentucky
Lexington, U.S.A.
martha.yip@uky.edu

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Abstract
The set of $n$ by $n$ upper-triangular nilpotent matrices with entries in a finite field $\mathbb{F}_q$ has Jordan canonical forms indexed by partitions $\lambda \vdash n$. We present a combinatorial formula for computing the number $F_\lambda(q)$ of matrices of Jordan type $\lambda$ as a weighted sum over standard Young tableaux. We construct a bijection between paths in a modified version of Young’s lattice and non-attacking rook placements, which leads to a refinement of the formula for $F_\lambda(q)$.

Keywords: nilpotent matrices, finite fields, Jordan form, rook placements, Young tableaux, set partitions.

1 Introduction
In the beautiful paper Variations on the Triangular Theme [7], Kirillov studied various structures on the set of triangular matrices. Let $G = G_n(\mathbb{F}_q)$ denote the group of $n$ by $n$ invertible upper-triangular matrices over the field $\mathbb{F}_q$ having $q$ elements, and let $\mathfrak{g} = \mathfrak{g}_n(\mathbb{F}_q) = \text{Lie}(G_n(\mathbb{F}_q))$ denote the corresponding Lie algebra of $n$ by $n$ upper-triangular nilpotent matrices over $\mathbb{F}_q$. The problem of determining the set $\mathcal{O}_n(\mathbb{F}_q)$ of adjoint $G$-orbits in $\mathfrak{g}$ remains challenging, and a more tractable task is to study a decomposition of $\mathcal{O}_n(\mathbb{F}_q)$ via the Jordan canonical form. Let $\lambda \vdash n$ be a partition of $n$ with $r$ positive parts.

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$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, and let

$$J_\lambda = J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots \oplus J_{\lambda_r},$$

where

$$J_i = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}_{i \times i}$$

is the $i$ by $i$ elementary Jordan matrix with all eigenvalues equal to zero. If $X \in g_n(\mathbb{F}_q)$ is similar to $J_\lambda$ under $GL_n(\mathbb{F}_q)$, then $X$ is said to have Jordan type $\lambda$. Each conjugacy class contains a unique Jordan matrix $J_\lambda$, so these classes are indexed by the partitions of $n$. Evidently, the Jordan type of $X$ depends only on its adjoint $G$-orbit.

Let $g_{n,\lambda}(\mathbb{F}_q) \subseteq g_n(\mathbb{F}_q)$ be the set of upper-triangular nilpotent matrices of fixed Jordan type $\lambda$, and let

$$F_\lambda(q) = \left| g_{n,\lambda}(\mathbb{F}_q) \right|.$$  

(1)

Springer showed that $g_{n,\lambda}(\mathbb{F}_q)$ is an algebraic manifold with $f_\lambda$ irreducible components, where $f_\lambda$ is the number of standard Young tableaux of shape $\lambda$, and each of which has dimension $\binom{n}{2} - n_\lambda$, where $n_\lambda$ is an integer defined in Equation 10. These quantities appear in the study of $F_\lambda(q)$.

In Section 2, we show that the numbers $F_\lambda(q)$ satisfy a simple recurrence equation, and that they are polynomials in $q$ with integer coefficients. As a consequence of the recurrence equation in Theorem 8, it follows that the coefficient of the highest degree term in $F_\lambda(q)$ is $f_\lambda$, and $\deg F_\lambda(q) = \binom{n}{2} - n_\lambda$. Equation (9) is a combinatorial formula for $F_\lambda(q)$ as a sum over standard Young tableaux of shape $\lambda$ that can be derived from the recurrence equation.

The cases $F_{(1^n)}(q) = 1$ and $F_{(n)}(q) = (q - 1)^{n-1}q^{\left(\binom{n-1}{2}\right)}$ are readily computed, since the matrix in $g_n(\mathbb{F}_q)$ of Jordan type $(1^n)$ is the matrix of zero rank, and the matrices in $g_n(\mathbb{F}_q)$ of Jordan type $(n)$ are the matrices of rank equal to $n - 1$. Section 2 concludes with explicit formulas for $F_\lambda(q)$ in several other special cases of $\lambda$, including hook shapes, two-rowed partitions and two-columned partitions.

In Section 3, we explore a connection of $F_\lambda(q)$ with rook placements. In their study of a formula of Frobenius, Garsia and Remmel [4] introduced the $q$-rook polynomial

$$R_{B,k}(q) = \sum_{c \in \mathcal{C}(B,k)} q^{\text{inv}(c)},$$

which is a sum over the set $\mathcal{C}(B,k)$ of non-attacking placements of $k$ rooks on a Ferrers board $B$, and $\text{inv}(c)$, defined in Equation (13), is the number of inversions of $c$. In the case when $B = B_n$ is the staircase-shaped board, Garsia and Remmel showed that $R_{B_n,k}(q) = S_{n,n-k}(q)$ is a $q$-Stirling number of the second kind. These numbers are defined by the recurrence equation

$$S_{n,k}(q) = q^{k-1}S_{n-1,k-1}(q) + [k]_q S_{n-1,k}(q) \quad \text{for} \quad 0 \leq k \leq n,$$
with initial conditions $S_{0,0}(q) = 1$, and $S_{n,k}(q) = 0$ for $k < 0$ or $k > n$.

It was shown by Solomon [12] that non-attacking placements of $k$ rooks on rectangular $m \times n$ boards are naturally associated to $m$ by $n$ matrices with rank $k$ over $\mathbb{F}_q$. By identifying a Ferrers board $B$ inside an $n$ by $n$ grid with the entries of an $n$ by $n$ matrix, Haglund [5] generalized Solomon’s result to the case of non-attacking placements of $k$ rooks on Ferrers boards, and obtained a formula for the number of $n$ by $n$ matrices with rank $k$ whose support is contained in the Ferrers board region. A special case of Haglund’s formula shows that the number of $n$ by $n$ nilpotent upper-triangular matrices of rank $k$ is

$$P_{B_n,k}(q) = (q - 1)^k q^{inom{n}{2} - k} R_{B_n,k}(q^{-1}).$$

(2)

Now, a matrix in $\mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ has rank $n - \ell(\lambda)$, where $\ell(\lambda)$ is the number of parts of $\lambda$, so the number of matrices in $\mathfrak{g}_n(\mathbb{F}_q)$ with rank $k$ is

$$F_{B_n,k}(q) = \sum_{\lambda \vdash n: \ell(\lambda) = n-k} F_{\lambda}(q).$$

(3)

Given Equations 2 and 3, it would be natural to ask whether it is possible to partition the placements $C(B_n,k)$ into disjoint subsets so that the sum over each subset of placements gives $F_{\lambda}(q)$. A central goal of this paper is to study the connection between upper-triangular nilpotent matrices over $\mathbb{F}_q$ and non-attacking rook placements on the staircase-shaped board $B_n$. Theorem 28 shows that there is a weight-preserving bijection $\Phi$ between rook placements on $B_n$ and paths in a graph $Z$ (see Figure 5), which is a multi-edged version of Young’s lattice. As a result, we obtain Corollary 30, which gives a formula for $F_{\lambda}(q)$ as a sum over certain rook placements that can be viewed as a generalization of Haglund’s formula in Equation (2).

There is a classically known bijection between rook placements in $C(B_n,k)$ and set partitions of $[n]$ with $n - k$ parts, so it is logical to next study the connection between $F_{\lambda}(q)$ and set partitions. We do this in Section 4. Theorem 34 describes the construction of a new (weight-preserving) bijection $\Psi$ between rook placements and set partitions. These bijections allow us to refine Equation (9) to a sum over set partitions (or rook placements). We also discuss the significance of the polynomials $F_{C}(q)$ indexed by rook placements in a special case.

This paper is the full version of the extended abstract [15].

2 Formulas for $F_{\lambda}(q)$

The recurrence equation for $F_{\lambda}(q)$ in Theorem 8 can be found in [1, Division Theorem], where Borodin considers the matrices as particles of a certain mass and studies the asymptotic behaviour of the formula. A preliminary version of the idea first appeared in [7]. In this section, we give an elementary proof of the formula, and investigate some of the combinatorial properties of $F_{\lambda}(q)$. 

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2.1 The recurrence equation for $F_\lambda(q)$

A partition $\lambda$ of a nonnegative integer $n$, denoted by $\lambda \vdash n$, is a non-increasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ with $|\lambda| = \sum_{i=1}^{n} \lambda_i = n$. If $\lambda$ has $r$ positive parts, write $\ell(\lambda) = r$. A partition $\lambda$ can be represented by its Ferrers diagram in the English notation, which is an array of $\lambda_i$ boxes in the $i$th row, with the boxes justified upwards and to the left. Let $\lambda'_j$ denote the size of the $j$th column of $\lambda$.

Young’s lattice $\mathcal{Y}$ is the lattice of partitions ordered by the inclusion of their Ferrers diagrams; that is, $\mu \subseteq \lambda$ if and only if $\mu_i \leq \lambda_i$ for every $i$. In particular, $\mu$ is covered by $\lambda$ in the Hasse diagram of $\mathcal{Y}$ and we write $\mu \lessdot \lambda$ if the Ferrers diagram of $\lambda$ can be obtained by adding a box to the Ferrers diagram of $\mu$. See Figure 1.

**Example 1.** The partition 

$$\lambda = (4, 2, 2, 1) \vdash 9$$

has diagram

and columns $\lambda'_1 = 4, \lambda'_2 = 3, \lambda'_3 = 1, \lambda'_4 = 1$.

**Lemma 2.** Let $\lambda \vdash n$ be a partition whose Ferrers diagram has $r$ rows and $c$ columns. The Jordan matrix $J_\lambda$ satisfies

$$\text{rank} (J^k_\lambda) = \begin{cases} 
\lambda'_{k+1} + \cdots + \lambda'_c, & \text{if } 0 \leq k < c, \\
0, & \text{if } k \geq c.
\end{cases}$$

**Proof.** The $i$ by $i$ elementary Jordan matrix $J_i$ has rank $(J^k_i) = i - k$ if $0 \leq k \leq i$, and its rank is zero otherwise, so the Jordan matrix $J_\lambda = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$ has

$$\text{rank} (J^k_\lambda) = \sum_{i=1}^{r} \text{rank} (J^k_{\lambda_i}) = \sum_{i: \lambda_i \geq k} \text{rank} (J^k_{\lambda_i}) = \sum_{j=k+1}^{c} \lambda'_j,$$

for $0 \leq k < c$, which is the number of boxes in the last $c - k$ columns of $\lambda$. \hfill $\square$

**Remark.** Matrices which are similar have the same rank, so if $X \in \mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$, then $\text{rank} (X^k) = \text{rank} (J^k_\lambda)$ for all $k \geq 0$. Conversely, let $\lambda, \nu \vdash n$. It follows from Lemma 2 that $\text{rank} (J^k_\lambda) = \text{rank} (J^k_\nu)$ for all $k \geq 0$ if and only if $\lambda = \nu$. Thus if $X \in \mathfrak{g}_n(\mathbb{F}_q)$ is a matrix such that $\text{rank} (X^k) = \text{rank} (J^k_\lambda)$ for all $k \geq 0$, then $X$ is similar to $J_\lambda$.

**Example 4.** If a matrix $X \in \mathfrak{g}_n(\mathbb{F}_q)$ has Jordan type $\lambda = (4, 2, 2, 1)$, then $\text{rank}(X) = 5$, $\text{rank}(X^2) = 2$, $\text{rank}(X^3) = 1$, and $\text{rank}(X^4) = 0$.

If $X \in \mathfrak{g}_n(\mathbb{F}_q)$ is a matrix of the form

$$X = \begin{bmatrix} J_\mu & \mathbf{v} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

then $\text{rank}(X) = \text{rank}(J_\mu) + \text{rank}(\mathbf{v})$. 

\[ \text{rank}(X^2) = \text{rank}(J_\mu^2) + \text{rank}(\mathbf{v}) \]

\[ \text{rank}(X^3) = \text{rank}(J_\mu^3) + \text{rank}(\mathbf{v}) \]

\[ \text{rank}(X^4) = \text{rank}(J_\mu^4) + \text{rank}(\mathbf{v}) \]
where $\mu \vdash n-1$, and $v = [v_1, \ldots, v_{n-1}]^T \in F_q^{n-1}$, then the first order leading principal submatrix of $X^k$ is $J^k_\mu$, and for $1 \leq k \leq n$, we define column vectors $v^k = [v^k_1, \ldots, v^k_{n-1}]^T \in F_q^{n-1}$ by

$$X^k = \begin{bmatrix} J^k_\mu & v^k \\ 0 & 0 \end{bmatrix}.$$ 

For $i \geq 1$, let $\alpha_i = \mu_1 + \cdots + \mu_i$ be the sum of the first $i$ parts of $\mu$. The $(i, j)$th entry of $J^k_\mu$ is nonzero if and only if $j = i + k$, and $i, i + k \leq \alpha_b$ for all $b \geq 1$. It follows from this that

$$v^k_i = \begin{cases} v_{i+k-1}, & \text{if } i, i + k - 1 \leq \alpha_b \text{ for all } b \geq 1, \\ 0, & \text{otherwise}. \end{cases} \quad (4)$$

There is a simple way to visualize the vectors $v^k$, which we illustrate with an example.

**Example 5.** Let $\mu = (4, 2, 1, 1)$, so that $\alpha_1 = 4, \alpha_2 = 6, \alpha_3 = 7$, and $\alpha_4 = 8$. Let

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & v_1 \\ 0 & 0 & 1 & 0 & v_2 \\ 0 & 0 & 0 & 1 & v_3 \\ 0 & 0 & 0 & 0 & v_4 \end{bmatrix} \quad \text{so that} \quad X^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & 0 & v_3 \\ 0 & 0 & 0 & 0 & v_4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

We may visualize the vectors $v$ and $v^2$ as fillings of the Ferrers diagram for $\mu$:

$$v = \begin{bmatrix} v_4 & v_3 & v_2 & v_1 \\ v_6 & v_5 \end{bmatrix} \quad \text{and} \quad v^2 = \begin{bmatrix} v_4 & v_3 & v_2 \\ v_6 \end{bmatrix}.$$ 

This way, a basis of $\ker X^k$ is the set of vectors filling the first $k$ columns of the diagram.

**Lemma 6.** If $X \in G_{n,\lambda}(F_q)$ and its first order leading principal submatrix $Y \in G_{n-1,\mu}(F_q)$, then $\lambda \triangleright \mu$.

**Proof.** We first consider the case $Y = J^k_\mu$. If $\mu$ has $s$ parts, let $\alpha_i = \mu_1 + \cdots + \mu_i$ for $1 \leq i \leq s$. Then

$$\text{rank}(X^k) = \text{rank}(J^k_\mu) = \begin{cases} 0, & \text{if } v_{\alpha_i} = 0 \text{ for all } i \text{ such that } \mu_i \geq k, \\ 1, & \text{otherwise}. \end{cases} \quad (5)$$

Let $c \leq n$ be the smallest positive integer for which $\text{rank}(X^c) - \text{rank}(J^c_\mu) = 0$. Then Equation (5) implies that

$$\text{rank}(X^k) = \text{rank}(J^k_\mu) = \begin{cases} 0, & \text{if } k \geq c, \\ 1, & \text{if } k < c. \end{cases}$$
Together with Lemma 2, we deduce that
\[ \lambda_k - \mu_k = (\text{rank}(X^{k-1}) - \text{rank}(X^k)) - (\text{rank}(J^{k-1}_\mu) - \text{rank}(J^k_\mu)) = \begin{cases} 1, & \text{if } k = c, \\ 0, & \text{if } k \neq c. \end{cases} \]

Therefore, \( \lambda \gg \mu \) in the case \( Y = J_\mu \).

In the general case where \( Y \) is any matrix of Jordan type \( \mu \), then \( \text{rank}(Y^k) = \text{rank}(J^k_\mu) \) for all \( k \geq 0 \), so the argument is the same.

Let \( \lambda \) be the partition whose diagram is obtained by adding a box to the \( i \)th row and \( j \)th column of the diagram of the partition \( \mu \). Define the coefficient
\[ c_{\mu,\lambda}(q) = \begin{cases} q^{|\mu| - \mu'_j}, & \text{if } j = 1, \\ q^{|\mu| - \mu'_{j-1}} (q^{\mu'_{j-1} - \mu'_j} - 1), & \text{if } j \geq 2. \end{cases} \]

Note that in the case \( j \geq 2 \), we have \( \mu'_{j-1} - \mu'_j \geq 1 \).

**Lemma 7.** Let \( Y \) be an upper-triangular nilpotent matrix of Jordan type \( \mu \vdash n - 1 \). If \( \mu \lessdot \lambda \), then there are \( c_{\mu,\lambda}(q) \) upper-triangular nilpotent matrices \( X \) of Jordan type \( \lambda \) whose first order leading principal submatrix is \( Y \).

**Proof.** By similarity, it suffices to consider the case \( Y = J_\mu = J_{\mu_1} \oplus \cdots \oplus J_{\mu_m} \), where \( \ell(\mu) = m \). Suppose \( X \) is a matrix of the form
\[ X = \begin{bmatrix} J_\mu & v \\ 0 & 0 \end{bmatrix} \]
of Jordan type \( \lambda \) such that \( \lambda \) is obtained by adding a box to \( \mu \) in the \( i \)th row and \( j \)th column.

First consider the case \( j \geq 2 \). Following the proof of Lemma 6, we know that \( j \) is the unique integer where \( \text{rank}(X^{j-1}) = \text{rank}(J^{j-1}_\mu) + 1 \), and \( \text{rank}(X^j) = \text{rank}(J^j_\mu) \). In order to satisfy the first condition, the entries in the vector \( v^{j-1} \) corresponding to the boxes in the \( (j - 1) \)th column and rows \( \geq i \) must not simultaneously be zero (refer to Equation (4) and Example 5), while in order to satisfy the second condition, the entries in the vector \( v^j \) corresponding to the boxes in the \( j \)th column of \( \mu \) must all be zero. The remaining \( n - 1 - \mu'_{j-1} \) entries of the vector \( v \) are free to be any element in \( \mathbb{F}_q \), so there are
\[ q^{n-1-\mu'_{j-1}} (q^{\mu'_{j-1} - \mu'_j} - 1) \]
possible matrices \( X \) whose leading principal submatrix is \( J_\mu \).

The case \( j = 1 \) is simpler. The necessary and sufficient condition that \( X \) and \( J_\mu \) must satisfy is that \( \text{rank}(X^k) = \text{rank}(J^k_\mu) \) for all \( k \geq 1 \), so the entries in the vector \( v \) corresponding to the boxes in the first column of the diagram for \( v^1 \) must all be zero, while the remaining \( n - 1 - \mu'_1 \) entries are free to be any element in \( \mathbb{F}_q \), so there are \( q^{n-1-\mu'_1} \) matrices \( X \) whose leading principal submatrix is \( J_\mu \) in this case. \( \square \)
The following recurrence equation for $F_{\lambda}(q)$ was first obtained by Borodin [1, Division Theorem]. Here, we provide an elementary proof before investigating some of the combinatorial properties of $F_{\lambda}(q)$.

**Theorem 8.** The number of $n$ by $n$ upper-triangular nilpotent matrices over $\mathbb{F}_q$ of Jordan type $\lambda \vdash n$ is

$$F_{\lambda}(q) = \sum_{\mu: \mu \preceq \lambda} c_{\mu, \lambda}(q) F_{\mu}(q),$$

with $F_{\emptyset}(q) = 1$.

**Proof.** Proceed by induction on $n$. For $n = 1$, the zero matrix is the only upper-triangular nilpotent matrix, and it has Jordan type $(1)$, agreeing with the formula $c_{\emptyset,(1)}(q) = 1$.

Suppose $\lambda \vdash n$. By Lemma 6, any matrix of Jordan type $\lambda$ has a leading principal submatrix of type $\mu \vdash n - 1$ for some $\mu \lessdot \lambda$. Furthermore, by Lemma 7, for each matrix $Y \in \mathfrak{g}_{n-1,\mu}(\mathbb{F}_q)$, there are $c_{\mu,\lambda}(q)$ matrices $X \in \mathfrak{g}_{n,\lambda}(\mathbb{F}_q)$ having $Y$ as its leading principal submatrix. Summing over all $\mu \lessdot \lambda$ gives the desired formula. \hfill \qed

**Remark 9** (Formulation in terms of standard Young tableaux). The formula for $F_{\lambda}(q)$ in Theorem 8 can be re-phrased as a sum over the set $\mathcal{P}_Y(\lambda)$ of paths in Young’s lattice $\mathcal{Y}$ from the empty partition $\emptyset$ to $\lambda$. If $\mu \lessdot \lambda$ in $\mathcal{Y}$, we assign the weight $c_{\mu,\lambda}(q)$ to the corresponding edge in $\mathcal{Y}$. Figure 1 shows Young’s lattice with weighted edges for partitions with up to four boxes. Let $P = (\emptyset = \pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)} = \lambda)$ denote a path in $\mathcal{Y}$ from $\emptyset$ to

Figure 1: Young’s lattice with edge weights $c_{\mu,\lambda}(q)$, up to $n = 4$. 

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**References**


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**Notes**

- **Theorem 8** provides a formula for the number of upper-triangular nilpotent matrices of a given Jordan type. The formula is given in terms of a sum over the partitions $\mu$ such that $\mu \preceq \lambda$.
- **Remark 9** allows the formula to be reformulated in terms of paths in Young’s lattice, assigning weights to the edges corresponding to the partitions.

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λ, where π\(^{(i)}\) is a partition of \(i\). To simplify notation, let \(\epsilon_i(q) = c_{\pi^{(i-1)},\pi^{(i)}}(q)\). Theorem 8 is equivalently re-phrased as

\[
F_{\lambda}(q) = \sum_{P \in \mathcal{P}_Y(\lambda)} F_P(q),
\]

where the weight of the path \(P\) is \(F_P(q) = \prod_{i=1}^{\ell} \epsilon_i(q)\).

The set of paths \(\mathcal{P}_Y(\lambda)\) is in bijection with the set \(\text{SYT}(\lambda)\) of standard Young tableaux of shape \(\lambda\), so we can also give an equation for \(F_{\lambda}(q)\) as a sum over standard Young tableaux.

A standard Young tableau \(T\) of shape \(\lambda\) is a filling of the Ferrers diagram of \(\lambda\) with \(1, \ldots, n\) such that the integers increase weakly along each row and strictly along each column. For \(1 \leq i \leq n\), let \(T^{(i)}\) denote the Young tableau of shape \(\lambda^{(i)}\) consisting of the boxes containing 1, \ldots, \(i\), and define weights

\[
T^{(i)}(q) = \begin{cases} q^{i-\ell(\lambda^{(i)})}, & \text{if the } i\text{th box is in the first column,} \\ q^{i-\lambda^{(i)}} - q^{i-1-\lambda^{(i-1)}}, & \text{if the } i\text{th box is in the } j\text{th column, } j \geq 2. \end{cases}
\]

Then

\[
F_{\lambda}(q) = \sum_{T \in \text{SYT}(\lambda)} F_T(q),
\]

where the weight of the standard Young tableau \(T\) is \(F_T(q) = \prod_{i=1}^{n} T^{(i)}(q)\).

### 2.2 Properties of \(F_{\lambda}(q)\)

Several properties of \(F_{\lambda}(q)\) follow readily from Theorem 8. For \(\lambda \vdash n\), let

\[
n_{\lambda} = \sum_{i \geq 1} (i - 1)\lambda_i = \sum_{b \in \lambda} \text{coleg}(b),
\]

where if a box \(b \in \lambda\) lies in the \(i\)th row of \(\lambda\), then \(\text{coleg}(b) = i - 1\).

**Corollary 10.** Let \(\lambda \vdash n\). As a polynomial in \(q\),

\[
\text{deg } F_{\lambda}(q) = \binom{n}{2} - n_{\lambda}.
\]

Moreover, the coefficient of the highest degree term in \(F_{\lambda}(q)\) is \(f^\lambda\), the number of standard Young tableaux of shape \(\lambda\).

**Proof.** Suppose \(P = (\emptyset = \pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)} = \lambda)\) is a path in \(\mathcal{Y}\) such that \(\pi^{(k)}\) is obtained by adding a box to the \(i\)th row and \(j\)th column of \(\pi^{(k-1)}\). Then \(\text{deg } c_{\pi^{(k-1)},\pi^{(k)}}(q) = k - i\), and therefore

\[
\text{deg } F_P(q) = \sum_{k=1}^{n} \text{deg } c_{\pi^{(k-1)},\pi^{(k)}}(q) = \sum_{k=1}^{n} k - \sum_{k \geq 1} k\lambda_k = \frac{n(n+1)}{2} - n_{\lambda}.
\]

In particular, every polynomial \(F_P(q)\) arising from a path \(P \in \mathcal{P}_Y(\lambda)\) has the same degree, so \(\text{deg } F_{\lambda}(q) = \frac{n(n+1)}{2} - n_{\lambda}\). Moreover, each \(F_P(q)\) is monic, so the coefficient of the highest degree term in \(F_{\lambda}(q)\) is the number of paths in \(\mathcal{Y}\) from \(\emptyset\) to \(\lambda\), which is \(f^\lambda\).
Corollary 11. Let $\lambda \vdash n$. The multiplicity of the factor $q - 1$ in $F_\lambda(q)$ is $n - \ell(\lambda)$.

Proof. The weight $c_{(k-1),2}(q)$ corresponding to the $k$th step in the path $P$ contributes a single factor of $q - 1$ to $F_P(q)$ if and only if the $k$th box added is not in the first column of $\lambda$. Therefore, the multiplicity of $q - 1$ in $F_P(q)$ is $n - \ell(\lambda)$, and it follows that the multiplicity of $q - 1$ in $F_\lambda(q)$ is $n - \ell(\lambda)$. \qed

Example 12. There are two partitions of 4 with two parts, namely $(3, 1)$ and $(2, 2)$.

There are three paths from $\emptyset$ to $(3, 1)$ in $\mathcal{Y}$, giving

$$F_{(3,1)}(q) = (q - 1) \cdot (q - 1)q \cdot q^2 + (q - 1) \cdot q \cdot (q - 1)q^2 + (q^2 - 1) \cdot (q - 1)q^2$$

$$= (q - 1)^2 (3q^3 + q^2),$$

and there are two paths from $\emptyset$ to $(2, 2)$ in $\mathcal{Y}$, giving

$$F_{(2,2)}(q) = (q - 1) \cdot q \cdot (q - 1)q + (q^2 - 1) \cdot (q - 1)q$$

$$= (q - 1)^2 (2q^2 + q).$$

Summing these gives a shift of the $q$-Stirling polynomial $(q - 1)^2 q \cdot S_{4,2}(q^{-1}) = (q - 1)^2 (3q^3 + 3q^2 + q)$.

2.3 Explicit formulas

In this section, we derive non-recursive formulas for some special cases of $\lambda$. Previously, we have noted the simple cases $F_{(1^n)} = 1$ and $F_{(n)} = q^{n-1}(q - 1)^{n-1}$.

Proposition 13 (Hook shapes). Let $n > k \geq 2$, and let $\lambda = (n - k + 1, 1^{k-1})$ be a hook-shaped partition of $n$ with $\ell(\lambda) = k$ parts. Then

$$F_\lambda(q) = (q - 1)^{n-k} \sum_{i=0}^{k-1} \binom{n - i - 1}{k - i - 1} q^{a-i}, \quad \text{where } a = \binom{n - 1}{2} - \binom{k - 1}{2}.$$ 

Proof. We make use of Equation (7). We enumerate paths from $\emptyset$ to $\lambda$ according to the first time a box is added to the second column, so for $0 \leq r \leq k - 1$, let $S_r$ be the set of paths in the sublattice $[\emptyset, \lambda]$ which contains the edge $((1,1^r), (2, 1^r))$. Such paths are formed by the concatenation of the unique path between $\emptyset$ and $\nu = (2, 1^r)$, which has weight $q^{r+1} - 1$, with any path in the sublattice $[\nu, \lambda]$. The sublattice $[\nu, \lambda]$ is the Cartesian product of a $(n - k)$-chain and a $(k - r - 1)$-chain, so it forms a rectangular grid, and therefore $|S_r| = \binom{n - r - 1}{k - r - 1}$. Notice that in any sublattice of the form

\[
\begin{array}{ccc}
(a, 1^{b+1}) & \overrightarrow{(q-1)^{a+b}} & (a+1, 1^{b+1}) \\
| & q^a | & |
(a, 1^b) & \overrightarrow{(q-1)^{a+b-1}} & (a+1, 1^b)
\end{array}
\]
the product of the edge weights is \((q - 1)q^{2a+b-1}\) no matter which path is taken from \((a, 1^b)\) to \((a + 1, 1^{b+1})\), so it follows that every path from \(\nu\) to \(\lambda\) has the same weight. By considering the path \((\nu, 21^{r+1}, \ldots, 21^{k-1}, 31^{k-1}, \ldots, \lambda)\), this weight is easily seen to be \((q - 1)^{n-k-1}q^{\alpha - r}\), for \(\alpha = \Big(\begin{array}{c} n-1 \\ 2 \end{array}\Big) - \Big(\begin{array}{c} k-1 \\ 2 \end{array}\Big)\). Altogether,

\[
F_{\lambda}(q) = (q - 1)^{n-k} \sum_{r=0}^{k-1} \binom{n-r-1}{k-r-1} \left( q^\alpha + q^{\alpha - 1} + \cdots + q^{\alpha - r} \right).
\]

For \(0 \leq i \leq k - 1\), the coefficient of \(q^{\alpha - i}\) in \(F_{\lambda}(q) / (q - 1)^{n-k}\) is

\[
\sum_{r=i}^{k-1} \binom{n-r-1}{k-r-1} = \sum_{u=0}^{i-1} \binom{n-k+u}{u} = \binom{n-i-1}{k-i-1},
\]

since \(\sum_{u=0}^{M} \binom{N+u}{u} = \binom{N+M+1}{M}\). Therefore,

\[
F_{\lambda}(q) = (q - 1)^{n-k} \sum_{i=0}^{k-1} \binom{n-i-1}{k-i-1} q^{\alpha - i},
\]

as claimed. \(\square\)

We next consider the case when \(\lambda\) is a partition with two parts. For \(n \geq k \geq 1\), let

\[
C_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1},
\]

and let \(C_{n,0} = 1\) for all \(n \geq 0\). These generalized Catalan numbers \(C_{n,k}\) (see OEIS [10, A009766]) enumerate lattice paths from \((0,0)\) to \((n,k)\), using the steps \((1,0)\) and \((0,1)\), which do not rise above the line \(y = x\). In the remainder of this section, we shall refer to these as Dyck paths.

The generalized Catalan numbers satisfy the simple recursive formula \(C_{n,k} = C_{n,k-1} + C_{n-1,k}\). Also, these are the usual Catalan numbers \(C_n = \frac{1}{n+1} \binom{2n}{n} = C_{n,n} = C_{n,n-1}\) when \(k = n\) or \(n - 1\). These facts will be used in the computations which follow.

**Proposition 14** (Partitions with two parts). If \(\lambda = (r,s)\vdash n\) such that \(r > s \geq 1\), then

\[
F_{(r,s)}(q) = (q - 1)^{r+s-2} q^{\binom{r+s-1}{2} - 2s+1} \sum_{i=0}^{s} C_{r+s-i,i} q^i.
\]

If \(r = s\), then

\[
F_{(r,r)}(q) = (q - 1)^{2r-2} q^{\binom{2r-2}{2}} \sum_{i=0}^{r-1} C_{2r-i,i} q^i.
\]

**Proof.** Proceed by induction on \(r + s\). The base cases are \(F_{(r)}(q) = (q - 1)^{r-1} q^{\binom{r-1}{2}}\) for \(r \geq 1\) and \(F_{(1,1)}(q) = 1\).
\[
\begin{array}{c|ccccccc}
  n \setminus k & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
  \hline
  0 & 1 & & & & & & \\
  1 & 1 & 1 & & & & & \\
  2 & 1 & 2 & 2 & & & & \\
  3 & 1 & 3 & 5 & 5 & & & \\
  4 & 1 & 4 & 9 & 14 & 14 & & \\
  5 & 1 & 5 & 14 & 28 & 42 & 42 & \\
  6 & 1 & 6 & 20 & 48 & 90 & 132 & 132 \\
\end{array}
\]

Figure 2: The Catalan triangle \( C_{n,k} \).

We first handle the case \( s = 1 \) separately. For \( r \geq 2 \),

\[
F_{(r,1)}(q) = q^{r-1} F_{(r)}(q) + (q - 1) q^{r-1} F_{(r-1,1)}(q)
\]

\[
= q^{r-1} \cdot (q - 1)^{r-1} q^\binom{r-1}{2} + (q - 1) q^{r-1} \cdot (q - 1)^{r-2} q^{r-2} (r^2)^{-1} ((r - 1) q + 1)
\]

\[
= (q - 1)^{r-1} q^\binom{r-2}{2} (rq + 1).
\]

Next, consider the case \( s = r \). For \( r \geq 2 \),

\[
F_{(r,r)}(q) = (q - 1) q^{2r-3} F_{(r,r-1)}(q)
\]

\[
= (q - 1)^{2r-2} q^\binom{2r-2}{2} \sum_{i=0}^{r-1} C_{2r-1-i,i} q^i.
\]

The case \( s = r - 1 \) is obtained as follows. For \( r \geq 3 \),

\[
F_{(r,r-1)}(q) = (q - 1) q^{2r-4} F_{(r,r-2)}(q) + (q^2 - 1) q^{2r-4} F_{(r-1,r-1)}(q)
\]

\[
= (q - 1)^{2r-3} q^{2r-4} q^\binom{2r-4}{2} \left( q \sum_{i=0}^{r-2} C_{2r-2-i,i} q^i + (q + 1) \sum_{i=0}^{r-2} C_{2r-3-i,i} q^i \right)
\]

\[
= (q - 1)^{2r-3} q^\binom{2r-3}{2} \left( (C_{r,r-2} + C_{r-1,r-2}) q^{r-1} \right.
\]

\[
+ \sum_{i=1}^{r-2} \left( C_{2r-1-i,i-1} + C_{2r-2-i,i-1} + C_{2r-3-i,i} \right) q^i + C_{2r-3,0} q^0 \bigg).
\]

Since \( C_{n,n-1} = C_{n,n} \), then \( C_{r,r-2} + C_{r-1,r-2} = C_{r,r-1} \). Similarly, we obtain \( C_{2r-1-i,i-1} + C_{2r-2-i,i-1} + C_{2r-3-i,i} = C_{2r-1-i,i} \) by applying the recurrence equation for the generalized Catalan numbers. Lastly, \( C_{n,0} = 1 \) for all \( n \geq 0 \), thus

\[
F_{(r,r-1)}(q) = (q - 1)^{2r-3} q^\binom{2r-3}{2} \sum_{i=0}^{r-1} C_{2r-1-i,i} q^i.
\]
which agrees with the formula for the case $s = r - 1$.

The last case to consider is the general case $r - s \geq 2$ where $s \geq 2$.

$$F(r,s) = (q - 1)q^{r+s-3}F(r,s-1) + (q - 1)q^{r+s-2}F(r-1,s)$$

$$= (q - 1)^{r+s-2}q^{r+s-3}q^{\binom{r+s-2}{2}-2s+1}\left(q^2\sum_{i=0}^{s-1}C_{r+s-1-i,i}q^i + q\sum_{i=0}^{s}C_{r-1+s-i,i}q^i\right)$$

$$= (q - 1)^{r+s-2}q^{\binom{r+s-1}{2}-2s+1}\left(C_{r+s,0}q^0 + \sum_{i=1}^{s} (C_{r+s-i,i-1} + C_{r-1+s-i,i}) q^i\right)$$

$$= (q - 1)^{r+s-2}q^{\binom{r+s-1}{2}-2s+1}\sum_{i=0}^{s}C_{r+s-i,i}q^i.$$  \[\square\]

The next equation is a formula for $F(r,r)(q)$ with a different factorization.

**Proposition 15** (Two equal parts). Let $\lambda = (r, r) \vdash n$, and $r \geq 1$. Then

$$F(r,r)(q) = (q - 1)^{2r-2}\sum_{i=0}^{r-1}C_{r-1,r-1-i}q^{2(r-1)^2-i}(q + 1)^i.$$  

**Proof.** The set of paths in the sublattice $[\emptyset, \lambda]$ are in bijection with the set of lattice paths
from \((0, 0)\) to \((r, r)\). In any sublattice of the form

\[
\begin{align*}
(a, b + 1) &\xrightarrow{(q-1)q^{a+b}} (a + 1, b + 1) \\
(a, b) &\xrightarrow{(q-1)q^{a+b-2}} (a + 1, b) \\
\end{align*}
\]

where \(b \leq a - 2\), the product of the edge weights is \((q - 1)^2 q^{2a + 2b - 2}\) no matter which path is taken from \((a, b)\) to \((a + 1, b + 1)\). As for sublattices of the form

\[
\begin{align*}
(a, a) &\xrightarrow{(q-1)q^{2a-2}} (a + 1, a) \\
(a, a - 1) &\xrightarrow{(q-1)q^{2a-2}} (a + 1, a - 1) \\
\end{align*}
\]

the product of the edge weights is \((q - 1)^2 q^{4a-4}\) via the lower horizontal edge, versus \((q - 1)^2 q^{4a-5}(q + 1)\) via the upper horizontal edge. It follows that if a path \(P\) from \(\emptyset\) to \(\lambda\) contains \(i\) partitions of the form \((a, a)\), then it has the weight

\[
F_p(q) = (q - 1)^{2r-2} q^{2(r-1)^2-i} (q + 1)^i.
\]

Dyck paths may be enumerated according to the points at which they touch the diagonal line \(y = x\), and the set of touch points are indexed by compositions \(\alpha = (\alpha_1, \ldots, \alpha_{i+1}) \models r\) where \(\alpha_j \geq 1\). The number of Dyck paths from \((0, 0)\) to \((r, r)\) which touch the diagonal exactly \(i\) times, not including the initial and the end points, is

\[
\sum_{\alpha \models r} \prod_{j=1}^{i+1} C_{\alpha_j-1}.
\]

On the other hand, the number \(C_{r-1,r-1-i}\) of Dyck paths from \((0, 0)\) to \((r-1, r-1-i)\) satisfies the same recurrence equation

\[
C_{r-1,r-1-i} = \sum_{\beta \models r-1-i} \prod_{j=1}^{i+1} C_{\beta_j},
\]

but the sum is over the set of weak compositions so that \(\beta_j \geq 0\). Under the appropriate shift in indices, it follows that the number of Dyck paths from \((0, 0)\) to \((r, r)\) which touch the diagonal exactly \(i\) times is \(C_{r-1,r-1-i}\). The result follows from this.

**Corollary 16.** For \(k \geq m \geq 0\),

\[
\sum_{j=m}^{k} \binom{j}{m} C_{k,j} = C_{2k+1-m,m}.
\]

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Proof. The two formulas for $F_{(k+1,k+1)}(q)$ yields the identity
\[
\sum_{i=0}^{k} C_{k,k-i} q^{2k^2-i}(q+1)^i = q^k \sum_{i=0}^{k} C_{2k-i,i} q^i.
\]
Exracting the coefficient of $q^{2k^2-n}$ in the above expressions yields the result. 

Remark 17. The formula for $F_{(r,r)}(q)$ provided in Proposition 15 can be viewed as a sum over Dyck paths, where each Dyck path $\pi$ contributes a term of the form $q^{s_1(\pi)}(q+1)^{s_2(\pi)}$ for some statistics $s_1$ and $s_2$ on the Dyck paths. This particular factorization for $F_{(r,r)}(q)$ is related to the work of Cai and Readdy on the $q$-Stirling numbers of the second kind, since the polynomials $F_\lambda(q)$ can be viewed as a refinement of $S_{n,k}(q)$, as explained in Section 3.

Cai and Readdy obtained a formula [2, Theorem 3.2] for $\widetilde{S}_{n,k}(q)$ (they use a different recursive formula to define the $q$-Stirling numbers, and the two are related by $S_{n,k}(q) = q^k \widetilde{S}_{n,k}(q))$ as a sum over allowable restricted-growth words, where each allowable word $w$ gives rise to a term of the form $q^{a(w)}(q+1)^{b(w)}$ for some statistics $a(w)$ and $b(w)$. They also showed that this enumerative result has an interesting extension to the study of the Stirling poset of the second kind, providing a decomposition of that poset into Boolean sublattices.

For example, if we define polynomials $G_\lambda(q)$ by letting $G_\lambda(q) = F_\lambda(q)/(q-1)^{-\ell(\lambda)}$ (see Equation (12)), then $G_{(3,1)}(q) + G_{(2,2)}(q) = q^3 \widetilde{S}_{4,2}(q^{-1})$. The formula of Cai and Readdy yields $q^3 \widetilde{S}_{4,2}(q^{-1}) = q(q+1)^2 + q^2(q+1) + q^3$, while our factorization yields $G_{(3,1)}(q) + G_{(2,2)}(q) = (q^3 + q^3 + q^2(q+1)) + (q^2 + q(q+1))$. So, the result of Proposition 15 gives a different expression for $S_{n,n-2}(q)$ as a sum with terms of the form $q^{s_1(\pi)}(q+1)^{s_2(\pi)}$, and it may be interesting to further investigate such factorizations of $F_\lambda(q)$.

Example 18. The first few $F_{(k,k)}$ are
\[
F_{(1,1)} = 1
\]
\[
F_{(2,2)} = (q-1)^2 (q^2 + q(q+1))
= (q-1)^2 (2q^2 + q)
\]
\[
F_{(3,3)} = (q-1)^4 (2q^8 + 2q^7(q+1) + q^6(q+1)^2)
= (q-1)^4 (5q^8 + 4q^7 + q^6)
\]
\[
F_{(4,4)} = (q-1)^6 (5q^{18} + 5q^{17}(q+1) + 3q^{16}(q+1)^2 + q^{15}(q+1)^3)
= (q-1)^6 (14q^{18} + 14q^{17} + 6q^{16} + q^{15})
\]
\[
F_{(5,5)} = (q-1)^8 (14q^{32} + 14q^{31}(q+1) + 9q^{30}(q+1)^2 + 4q^{29}(q+1)^3 + q^{28}(q+1)^4)
= (q-1)^8 (42q^{32} + 48q^{31} + 27q^{30} + 8q^{29} + q^{28})
\]

We end this section with one more closed formula for $F_\lambda(q)$ where $\lambda$ is a rectangular shape with two columns. Let $D(n,k)$ denote the set of Dyck paths from $(0,0)$ to $(n,k)$. The coarea of a Dyck path $\pi$ is the number of whole unit squares lying between the path
Proposition 19. (Partitions with two columns) Let $n \geq 1$, let $\rho_1(\pi)$ be one plus the number of unit squares lying between the path and the line $y = x + 1$ in the $i$th row. For example, the following Dyck path $\pi$ has $\text{coarea}(\pi) = 12$, and $(\rho_1(\pi), \rho_2(\pi), \rho_3(\pi), \rho_4(\pi)) = (2, 2, 1, 1)$.

For $n \geq 1$, let $\lfloor n \rfloor_q = 1 + q + \cdots + q^{n-1}$.

**Proposition 19.** (Partitions with two columns) Let $\lambda = (2^r, 1^s) \vdash n$ such that $r, s \geq 0$. Then

$$F_{(2^r, 1^s)}(q) = (q - 1)^r q^{\binom{s}{2}} \sum_{\pi \in D(r+s,r)} q^{\text{coarea}(\pi)} \prod_{i=1}^{r} [\rho_i(\pi)]_q.$$  

**Proof.** By Corollary 11, we know the multiplicity of the factor $q - 1$ in $F_{(2^r, 1^s)}(q)$ is $n - \ell(\lambda) = \lambda'_2 = r$, so we focus on computing $F_{(2^r, 1^s)}(q)/(q - 1)^r$. The paths in Young’s lattice from $\emptyset$ to $(2^r, 1^s)$ are in bijection with the Dyck paths $D(r+s, r)$, so we identify these paths; adding a box in the first column of a partition corresponds to a $(1, 0)$ step in the Dyck path, and adding a box in the second column of a partition corresponds to a $(0, 1)$ step in the Dyck path. As seen in Figure 4, a vertical step $(i,j)$ to $(i,j+1)$ has
weight \( q^{j[i]} \), while a horizontal step \((i, j)\) to \((i + 1, j)\) has weight \( q^j \). Thus the product of the edge weights of the \( r \) vertical steps of a given Dyck path \( \pi \) is \( q^{j[i]} \prod_{i=1}^{r} [\rho_i(\pi)]_q \), while the product of the edge weights of the \( r + s \) horizontal steps of a given Dyck path is \( q^{\text{coarea}(\pi)} \). The result follows.

**Example 20.** The first few \( F_{(2^n)} \) are

\[
\begin{align*}
F_{(2)} &= (q - 1)(q + 1) \\
F_{(2^2)} &= (q - 1)^3 (q + (q + 1)) \\
F_{(2^3)} &= (q - 1)^3 q^3 (q^3 + 2q^2(q + 1) + q(q + 1)^2 + (q^2 + q + 1)(q + 1)) \\
\end{align*}
\]

**Remark 21.** Kirillov and Melnkov [8] considered the number \( A_n(q) \) of \( n \times n \) upper-triangular matrices over \( \mathbb{F}_q \) satisfying \( X^2 = 0 \). In their first characterization of these polynomials, they considered the number \( A_r^n(q) \) of matrices of a given rank \( r \), so that \( A_n(q) = \sum_{r \geq 0} A_r^n(q) \), and observed that \( A_r^n(q) \) satisfies the recurrence equation

\[
A_r^n(q) = q^r A_r^{n-1}(q) + (q^{n-r} - q^r) A_r^n(q), \quad A_0^n(q) = 1 .
\]

We may think of \( A_n(q) \) as the sum of \( F_\lambda(q) \) over \( \lambda \vdash n \) with at most two columns, so Theorem 8 is a generalization of this recurrence equation.

It was also conjectured in [8] that the same sequence of polynomials arise in a number of different ways. Ekhad and Zeilberger [3] proved that one of the conjectured alternate definitions of \( A_n(q) \), namely

\[
C_n(q) = \sum_s c_{n+1,s} q^{\frac{n^2}{2} + \frac{1-s^2}{4}} ,
\]

is a sum over all \( s \in [-n - 1, n + 1] \) which satisfy \( s \equiv n + 1 \) mod 2 and \( s \equiv (-1)^n \) mod 3, and \( c_{n+1,s} \) are entries in the signed Catalan triangle, is indeed the same as \( A_n(q) \). It would be interesting to see what other combinatorics may arise from considering the sum of \( F_\lambda(q) \) over \( \lambda \vdash n \) with at most \( k \) columns for a fixed \( k \).

### 3 Jordan canonical forms and q-rook placements

In light of Corollary 11, we define polynomials \( G_\lambda(q) \in \mathbb{Z}[q] \) by

\[
F_\lambda(q) = (q - 1)^{n-\ell(\lambda)} G_\lambda(q) . \tag{12}
\]

In fact, we can deduce from Corollary 11 that \( G_\lambda(q) \in \mathbb{N}[q] \). In this section, we explore the connection between the nonnegative coefficients of \( G_\lambda(q) \) and rook placements.
3.1 Background on rook polynomials

A board $B$ is a subset of an $n$ by $n$ grid of squares. In this paper, we follow Haglund [5] and Solomon [12], and index the squares using the convention for the entries of a matrix. A Ferrers board is a board $B$ where if a square $s \in B$, then every square lying north and/or east of $s$ is also in $B$. Our Ferrers boards have squares justified upwards and to the right. Let $B_n$ denote the staircase-shaped board with $n$ columns of sizes $0, 1, \ldots, n - 1$. Let area($B$) be the number of squares in $B$, so that in particular, area($B_n$) = \binom{n}{2}.

A placement of $k$ rooks on a board $B$ is non-attacking if there is at most one rook in each row and each column of $B$. Let $C(B,k)$ denote the set of non-attacking placements of $k$ rooks on $B$. All rook placements considered in this article are non-attacking, so from this point forward, we drop the qualifier. For a placement $C \in C(B,k)$, let ne($C$) be the number of squares in $B$ lying directly north or directly east of a rook. The inversion of the placement is the number

$$\text{inv}(C) = \text{area}(B) - k - \text{ne}(C).$$

As noted in [4], the statistic inv($C$) is a generalization of the number of inversions of a permutation, since permutations can be identified with rook placements on a square-shaped board.

For $i = 1, \ldots, n$, the weight of the $i$th column $C_i$ of $C$ is

$$C_i(q) = (q - 1)^{\#\text{rooks in } C_i} q^{\text{ne}(C_i)},$$

and the weight of $C$ is defined by $FC(q) = \prod_{i=1}^{n} C_i(q)$. Alternatively, if $C \in C(B,k)$, then $F_C(q) = (q - 1)^k q^{\text{ne}(C)}$.

**Example 22.** We use $\times$ to mark a rook and use $\bullet$ to mark squares lying directly north or directly east of a rook (these squares shall be referred to as the north-east squares of the placement). The following illustration is a placement of four rooks on the staircase-shaped board $B_7$.

![Rook Placement Illustration](image)

This rook placement has ne($C$) = 11, inv($C$) = 6, and weight $F_C(q) = (q - 1)^4 q^{11}$.

For $k \geq 0$, the $q$-rook polynomial of a Ferrers board $B$ is defined by Garsia and Remmel [4, I.4] as

$$R_{B,k}(q) = \sum_{C \in C(B,k)} q^{\text{inv}(C)}.$$

The following result explains the role of rook polynomials in the enumeration of matrices of given rank. The support of a matrix $X$ is $\{(i,j) \mid x_{ij} \neq 0\}$. Given a Ferrers board $B$ with $n$ columns, we may identify the squares in $B$ with the entries in an $n$ by $n$ matrix.
Theorem 23 (Haglund). If $B$ is a Ferrers board, then the number $P_{B,k}(q)$ of $n$ by $n$ matrices of rank $k$ with support contained in $B$ is

$$P_{B,k}(q) = (q - 1)^k q^{\text{area}(B) - k} R_{B,k}(q^{-1}).$$

Looking ahead, it will be convenient to consider Theorem 23 in the following equivalent form:

$$P_{B,k}(q) = \sum_{C \in \mathcal{C}(B,k)} (q - 1)^k q^{\text{net}(C)} = \sum_{C \in \mathcal{C}(B,k)} F_C(q). \quad (16)$$

Example 24. We list the seven rook placements on $B_4$ with two rooks, along with their weights.

$$\begin{align*}
(q - 1)^2q^3 & \quad (q - 1)^2q^3 & \quad (q - 1)^2q^3 & \quad (q - 1)^2q^3 \\
(q - 1)^2q^2 & \quad (q - 1)^2q^2 & \quad (q - 1)^2q^2 & \quad (q - 1)^2q^2
\end{align*} \quad (17)$$

Thus $P_{B_4,2}(q) = (q - 1)^2(3q^3 + 3q^2 + q)$.

3.2 Rook placements and Jordan forms

The purpose of this section is to generalize Haglund’s formula (16) to a formula for $F_\lambda(q)$ (Corollary 30) as a sum over a set of rook placements. We achieve this by defining a multigraph $Z$ that is related to $Y$, and show that paths in $Z$ are equivalent to rook placements.

The multigraph $Z$ is constructed from $Y$ by replacing each edge of $Y$ by one or more edges as follows. If there is an edge from $\mu$ to $\lambda$ in $Y$ of weight $q^{\mu_1 - \mu'_1 - 1} \left( q^{\mu'_j - \mu_j - 1} - 1 \right)$, then this edge is replaced by $\mu_j - \mu'_j$ edges from $\mu$ to $\lambda$ with weights

$$(q - 1)q^{\mu_1 - \mu'_1 - 1}, \ldots, (q - 1)q^{\mu_1 - \mu'_1 - 1} \quad (19)$$

in $Z$. All other edges remain as before. See Figure 5.

Let $P_Z(\lambda)$ denote the set of paths in the graph $Z$ from the empty partition $\emptyset$ to $\lambda$. For a path $P = (\emptyset = \pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)} = \lambda)$ in $P_Z(\lambda)$, let $\epsilon_i(q)$ denote the weight of the $i$th edge, for $i = 1, \ldots, n$. Naturally, we define the weight of the path by $F_P(q) = \prod_{i=1}^n \epsilon_i(q)$, so that

$$F_\lambda(q) = \sum_{P \in P_Z(\lambda)} F_P(q). \quad (20)$$

Lemma 25. Let $\mu \vdash n - 1$ be a partition with $\ell(\mu) = \ell$ parts. Then there are $\ell + 1$ edges leaving $\mu$ in the graph $Z$, with weights

$$(q - 1)q^{\mu_1 - 1}, (q - 1)q^{\mu_2}, \ldots, (q - 1)q^{\mu_\ell}, and q^{\mu_1 - \ell}. \quad (21)$$
Proof. If a partition $\lambda \vdash n$ is obtained by adding a box to the first column of $\mu$, then there is a unique edge from $\mu$ to $\lambda$ in $\mathcal{Z}$ with weight $q^{\mu_i - \ell}$. Otherwise, if we consider the set of all partitions which can be obtained from $\mu$ by adding a box anywhere except in the first column, then there are a total of

$$\sum_{j \geq 2} (\mu_j' - \mu'_j) = \ell$$

edges from $\mu$ to some partition of $n$. Moreover, by Equation (19), these $\ell$ weights are $(q - 1)q^{\mu_i - i}$ for $i = 1, \ldots, \ell$.

A sequence of nonnegative integers is $\mathcal{P}_Z$-admissible if it is the degree sequence of a path $P = (\emptyset, \pi^{(1)}, \ldots, \pi^{(n)})$ in $\mathcal{Z}$. That is, $(d_1, \ldots, d_n) = (\deg \epsilon_1(q), \ldots, \deg \epsilon_n(q))$.

**Corollary 26.** A $\mathcal{P}_Z$-admissible sequence determines a unique path in $\mathcal{Z}$.

**Proof.** Induct on $n$. When $n = 1$, the only path is the from $\emptyset$ to $(1)$, and it has degree sequence $(0)$.

Given a $\mathcal{P}_Z$-admissible sequence $(d_1, \ldots, d_n)$, the subsequence $(d_1, \ldots, d_{n-1})$ determines a unique path $P' = (\emptyset, \pi^{(1)}, \ldots, \pi^{(n-1)})$. Suppose $\mu = \pi^{(n-1)}$ has $\ell$ parts. Then $|\mu| - \ell + 1 \leq d_n \leq |\mu|$, and by Lemma 25, there is a unique edge leaving $\mu$ with degree $d_n$. 

---

Figure 5: The multigraph $\mathcal{Z}$, up to $n = 4$.
3.3 The construction of $\Phi$

Let $P_Z(n, n-k)$ denote the set of paths in $Z$ from $\emptyset$ to a partition of $n$ with $n-k$ parts. In this section, we define a weight-preserving bijection $\Phi : C(B_n, k) \to P_Z(n, n-k)$.

**Proposition 27.** Let $n \geq 1$ and $k = 0, \ldots, n-1$. Let $C \in C(B_n, k)$ be a rook placement with columns $C_1, \ldots, C_n$. There exists a unique path $P \in P_Z(n, n-k)$ with edge weights $(\epsilon_1(q), \ldots, \epsilon_n(q)) = (C_1(q), \ldots, C_n(q))$.

**Proof.** Proceed by induction on $n+k$. When $n = 1$ and $k = 0$, there is a unique rook placement on the empty board $B_1$ with no rooks having weight one, corresponding to the unique path $P = (\emptyset, (1))$ in $Z$ with the same weight.

Assume the result holds for all rook placements in $C(B_{n-1}, k)$ and $C(B_{n-1}, k-1)$. Given a rook placement $C \in C(B_n, k)$, let $C'$ be the sub-placement consisting of the first $n-1$ columns of $C$. By induction, the sequence $(C_1(q), \ldots, C_{n-1}(q))$ determines a unique path $(\emptyset, \pi^{(1)}, \ldots, \pi^{(n-1-1)})$ in $Z$ such that $\epsilon_i(q) = C_i(q)$ for $i = 1, \ldots, n-1$.

There are now two cases two consider. The first case is if $C' \in C(B_{n-1}, k)$, so that $\ell(\pi^{(n-1)}) = n-k-1$. There are $k$ rooks in $C'$, so the $n$th column of $C$ does not contain any rooks, and $C_n(q) = q^k$. By Lemma 25, there exists a unique edge in the graph $Z$ originating at $\pi^{(n-1)}$ with weight $q^k$. Thus $C$ corresponds to the path $P = (\emptyset, \pi^{(1)}, \ldots, \pi^{(n-1)}, \pi^{(n)})$ where $\pi^{(n)}$ is obtained from $\pi^{(n-1)}$ by adding a box to the first column, and $\epsilon_n(q) = q^k$. Moreover, $\ell(\pi^{(n)}) = n-k$.

The second case is if $C' \in C(B_{n-1}, k-1)$, so that $\ell(\pi^{(n-1)}) = n-k$. There must be $k-1$ ‘northeast’ squares in the $n$th column of $C$, and there are $n-k$ remaining squares in that column where a rook may be placed. Label these available squares $a_0, a_1, \ldots, a_{n-k-1}$ from the top to the bottom. Observe that $C_n(q) = (q - 1)q^{k-1+i}$ if a rook is placed in the square $a_i$, for $0 \leq i \leq n-k-1$. Again by Lemma 25, there exists $n-k$ edges in the graph $Z$ originating at $\pi^{(n-1)}$ with the weights $(q-1)q^k$ for $k-1 \leq h \leq n-2$. Thus if the $k$th rook of $C$ is placed in the square $a_i$, then $C$ corresponds to the path $P = (\emptyset, \pi^{(1)}, \ldots, \pi^{(n-1)}, \pi^{(n)})$ with $\epsilon_n(q) = (q-1)q^{k-1+i}$, and $\ell(\pi^{(n)}) = n-k$. \(\square\)

Given a rook placement $C \in C(B_n, k)$, let $\Phi(C)$ be the path in $P_Z(n, n-k)$ with edge weights $(\epsilon_1(q), \ldots, \epsilon_n(q)) = (C_1(q), \ldots, C_n(q))$.

**Theorem 28.** The map $\Phi : C(B_n, k) \to P_Z(n, n-k)$ is a weight-preserving bijection.

**Proof.** Proposition 27 shows that the map $\Phi$ is an injective weight-preserving map, since each column of the rook placement determines each edge of the path $\Phi(C)$:

$$F_C(q) = \prod_{i=1}^{n} C_i(q) = \prod_{i=1}^{n} \epsilon_i(q) = F_{\Phi(C)}(q).$$

In fact, the proof of the Proposition also shows that $\Phi$ is surjective because the number of possible ways to add a column to an existing rook placement is equal to the number of possible ways to extend a path in $Z$ by one edge. Therefore, $\Phi$ is a weight-preserving bijection. \(\square\)
A sequence of nonnegative integers is \( C \)-admissible if it is the degree sequence of a rook placement. That is, \((d_1, \ldots, d_n) = (\deg C_1(q)), \ldots, \deg C_n(q))\) for \( C \in C(B_n, k)\). The next Corollary follows easily from Theorem 28.

**Corollary 29.** A \( C \)-admissible sequence determines a unique rook placement.

It follows from Theorem 28 that we may associate a partition type to each rook placement on \( B_n \). The partition type of a rook placement \( C \) is the partition at the endpoint of the path \( \Phi(C) \) in \( Z \). Let \( C(\lambda) = \Phi^{-1}(P_Z(\lambda)) \) denote the set of rook placements of partition type \( \lambda \).

**Corollary 30.** Let \( \lambda \vdash n \) be a partition with \( \ell(\lambda) = n - k \) parts. Then

\[
F(\lambda)(q) = \sum_{C \in C(\lambda)} F_C(q) = (q - 1)^{n - \ell(\lambda)} \sum_{C \in C(\lambda)} q^{\text{ne}(C)}.
\]

**Proof.** The result follows from Equation 20 and the bijection \( \Phi \).

**Remark 31.** The polynomial \( G(\lambda) \in \mathbb{N}[q] \) defined in Equation (12) is simply a sum over the rook placements of type \( \lambda \) involving the north-east statistic.

### 4 A connection with set partitions

The results of the previous section naturally leads to a decomposition of \( F_T(q) \), indexed by some tableau \( T \), into a sum of polynomials indexed by set partitions, which we explain below.

A set partition is a set \( S = \{s_1, \ldots, s_k\} \) of nonempty disjoint subsets of \([n]\) such that \( \bigcup_{i=1}^k s_i = [n] \). The \( s_i \)'s are the blocks of \( \sigma \). Let \( \ell(S) \) denote the number of blocks of \( S \), and let \( S(n, n - k) \) denote the set of set partitions of \([n]\) with \( n - k \) blocks. We adopt the convention of listing the blocks in order so that

\[
|s_1| \geq |s_2| \geq \cdots \geq |s_k|, \quad \text{and} \quad \min s_i < \min s_{i+1} \text{ if } |s_i| = |s_{i+1}|. \tag{21}
\]

This allows us to represent a set partition with a diagram similar to that of a standard Young tableau; the \( i \)th row of the diagram consists of the elements in the block \( s_i \) listed in increasing order, but there are no restrictions on the entries in each column of the diagram. A set partition \( S = (s_1, \ldots, s_m) \) has partition type \( \lambda \) if \( \lambda = (|s_1|, \ldots, |s_m|) \).

For \( i = 1, \ldots, n \), let \( S^{(i)} \) denote the sub-diagram of \( S \) consisting of the boxes containing \( 1, \ldots, i \), with rows ordered according to the convention set forth in Equation (21). If the box containing \( i \) is not in the first column of the diagram, let \( u \) be the least element in the same row as \( i \) in \( S^{(i)} \), and suppose \( u \) is in the \( r \)th row of \( S^{(i-1)} \) for some \( 1 \leq r \leq \ell(S^{(i-1)}) \). The weight arising from the \( i \)th box is

\[
S^{(i)}(q) = \begin{cases} 
q^{i-1-\ell(S^{(i-1)})}, & \text{if the } i \text{th box is in the first column,} \\
(q - 1)q^{i-1-r}, & \text{if the } i \text{th box is in the } j \text{th column, } j \geq 2.
\end{cases}
\tag{22}
\]
We define the weight of $S$ as $F_S(q) = \prod_{i=1}^{n} S_i(q)$.

A sequence of nonnegative integers is $S$-admissible if it is the degree sequence of a set partition. That is, $(d_1, \ldots, d_n) = (\deg S^{(1)}(q), \ldots, \deg S^{(n)}(q))$ for a $S \in S(n)$.

**Lemma 32.** An $S$-admissible sequence determines a unique set partition.

**Proof.** Induct on $n$. When $n = 1$, the only set partition is $\{\{1\}\}$, and its degree sequence is $(0)$.

Given an $S$-admissible sequence $(d_1, \ldots, d_n)$, the subsequence $(d_1, \ldots, d_{n-1})$ determines a unique set partition $S^{(n-1)} = (S_1^{(n-1)}, \ldots, S_m^{(n-1)})$. By Equation (22), $n-1-m \leq d_n \leq n-1$, and each of the $m+1$ choices for $d_n$ determines the block of $S^{(n-1)}$ into which $n$ should be inserted. \hfill $\square$

We have already constructed a weight-preserving bijection $\Phi$ between rook placements and paths in $Z$. We now construct a weight-preserving bijection $\Psi$ between rook placements and set partitions, effectively showing that paths in $Z$ are equivalent to set partitions, so that $F_Z(q) = F_C(q) = F_S(q)$ if $Z \leftrightarrow C \leftrightarrow S$ for $Z \in \mathcal{P}_Z(n, n-k)$, $C \in \mathcal{C}(B_n, k)$, and $S \in S(n, n-k)$.

**Remark 33.** There is a classically known bijection (see [14]) between the set of rook placements on the staircase board $B_n$ with $k$ rooks and the set of set partitions of $[n] = \{1, \ldots, n\}$ with $n-k$ blocks: the placement $C$ corresponds to the set partition where the integers $i$ and $j$ are in the same block if and only if there is a rook in the square $(i, j) \in C$. This bijection is different from the one described in Theorem 34. For example, the classical bijection associates the rook placement

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\times & \bullet & \bullet & \\
2 & & \bullet & \\
3 & & & \times \\
4 & & & \\
\end{array}
$$

to the set partition $\{(1, 2), \{3, 4\}\}$ and so has partition type $(2, 2)$, but as we shall see below, this placement is associated to the set partition $\{(1, 2, 4), \{3\}\}$ under the bijection in Theorem 34 and has partition type $(3, 1)$.

**4.1 The construction of $\Psi$**

Let $C \in \mathcal{C}(B_n, k)$ be a rook placement. The main idea is that the degree of $C_i(q)$ arising from the $i$th column of $C$ determines the block of the set partition in which we place $i$. In the construction of the set partition $\Psi(C)$, we will create a sequence of intermediate set partitions $S^{(i)}$ of $[i]$ for $i = 1, \ldots, n$.

The initial case is always $\deg(C_1(q)) = \deg(1) = 0$, so $S^{(1)} = \{\{1\}\}$. Assume that $S^{(i-1)} = \{S_1^{(i-1)}, \ldots, S_m^{(i-1)}\}$ is the set partition which corresponds to the first $i-1$ columns of $C$, so that $m = \ell(S^{(i-1)})$. Observe that there are $m+1$ possible blocks in which to insert $i$ to obtain $S^{(i)}$. By Corollary 26, we know that

$$i - 1 - \ell(S^{(i-1)}) \leq \deg(C_i(q)) \leq i - 1,$$
so we construct $S^{(i)}$ by placing $i$ in the $j$th block of $S^{(i-1)}$, where $j = i - \deg(C_i(q))$, and then rearranging the blocks to fit the convention in Equation (21) if necessary.

**Theorem 34.** The map $\Psi : C(n, k) \to S(n, n-k)$ is a weight-preserving bijection.

**Proof.** Let $S = \Psi(C)$. The map $\Psi$ is weight-preserving, as $C_i(q) = S^{(i)}(q)$ by construction, for each $i = 1, \ldots, n$. Now, since the degrees $\deg C_i(q) = \deg S^{(i)}(q)$, and by Corollary 29 and Lemma 32 the sequences of degrees completely determine $C$ and $S$ respectively, then $\Psi$ is injective. Finally, we note that $|C(n, k)| = |S(n, n-k)|$, so $\Psi$ is a bijection. \square

**Corollary 35.** Let $S(\lambda)$ denote the set of all set partitions of partition type $\lambda$. Then

$$F_\lambda(q) = \sum_{S \in S(\lambda)} F_S(q).$$

\square

**Example 36.** Let $C$ be the rook placement

The associated sequence of set partition diagrams associated to $C$ is

$$
\emptyset \to \epsilon_1 \to \epsilon_2 \to \epsilon_3 \to \epsilon_4 \to \epsilon_5 \to \epsilon_6 \to \epsilon_7 \to \epsilon_8 \to \epsilon_9,
$$

so the set partition associated to the rook placement $C$ is

$$S = \Psi(C) = (\{3, 8, 9\}, \{1, 5\}, \{6, 7\}, \{2\}, \{4\}).$$

**Remark 37.** An intriguing question is to ask for a geometric interpretation of the polynomials $F_C(q)$, indexed by rook placements (or set partitions or paths in $Z$).

The problem of determining the number of adjoint $G_n(F_q)$ orbits on $g_n(F_q)$ remains open. In the case $q = 2$, this number has been computed for $n \leq 16$ by Pak and Soffer [11, Appendix B]. Let $O_n(1)$ denote the orbits of rank $k$ matrices. When $k = 1$, it turns out that the polynomials $F_C(q)$ indexed by rook placements with exactly one rook gives the sizes of the $\binom{n}{2}$ orbits in $O_n(1)$. For $2 \leq i < j \leq n$, each orbit contains a unique matrix $E_{ij}$ whose $ij$th entry is $1$, and is zero everywhere else. The orbit containing $E_{ij}$ is associated to the rook placement $C(i, j)$ with a single rook in the $ij$th square, and the size of the associated orbit is $F_{C(i,j)}(q) = (q - 1)q^{n-1-(j-i)}$. 

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In particular, the formula in Proposition 13 applied to the partition \( \lambda = (2, 1^{n-2}) \) gives the generating function

\[
F_{(2, 1^{n-2})}(q) = (q - 1) \left( (n - 1)q^{n-2} + (n - 2)q^{n-3} + \cdots + 3q^2 + 2q + 1 \right)
\]

for rank one orbits of \( G_n(\mathbb{F}_q) \) on \( g_n(\mathbb{F}_q) \).

**Remark 38.** To close, we mention a related problem which may provide a geometric interpretation of \( F_C(q) \) for every rook placement \( C \). Let \( N \) be an \( n \times n \) nilpotent matrix with entries in an algebraically closed field \( k \) containing \( \mathbb{F}_q \), and suppose \( N \) has Jordan type \( \lambda \vdash n \). A complete flag \( f = (f_1, \ldots, f_n) \) is a sequence of subspaces in \( k^n \) such that \( f_1 \subset \cdots \subset f_n \) and \( \dim f_i = i \) for all \( i \). A flag is \( N\)-stable if \( N(f_i) \subseteq f_i \) for all \( i \). Spaltenstein [13] showed that the variety \( X_\lambda \) of \( N \)-stable flags is a disjoint union of \( f^\lambda \) smooth irreducible subvarieties \( X_T \) indexed by the standard Young tableaux of shape \( \lambda \). Moreover, the closures \( \overline{X_T} \) are the irreducible components of \( X_\lambda \), each of which has dimension \( n_\lambda \). The number of \( \mathbb{F}_q \)-rational points in \( X_\lambda \) is given by Green’s polynomials \( Q^\lambda_{(1^n)}(q) \) [9, III.7]. Evidently,

\[
\left( \prod_{i \geq 1} [m_i(\lambda)]_q \right)^{-1} Q^\lambda_{(1^n)}(q) = ((q - 1)^{n-\ell(\lambda)} q^m)^{-1} F_\lambda(q),
\]

with \( m = \min_{C \in C(\lambda)} \text{ne}(C) \). Based on some computations for small values of \( n \), we expect that \( F_C(q) \) plays a role in counting points in certain intersections of the irreducible components \( \overline{X_T} \).
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