Piercing axis-parallel boxes

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Abstract

Let \( \mathcal{F} \) be a finite family of axis-parallel boxes in \( \mathbb{R}^d \) such that \( \mathcal{F} \) contains no \( k + 1 \) pairwise disjoint boxes. We prove that if \( \mathcal{F} \) contains a subfamily \( \mathcal{M} \) of \( k \) pairwise disjoint boxes with the property that for every \( F \in \mathcal{F} \) and \( M \in \mathcal{M} \) with \( F \cap M \neq \emptyset \), either \( F \) contains a corner of \( M \) or \( M \) contains \( 2^{d-1} \) corners of \( F \), then \( \mathcal{F} \) can be pierced by \( O(k) \) points. One consequence of this result is that if \( d = 2 \) and the ratio between any of the side lengths of any box is bounded by a constant, then \( \mathcal{F} \) can be pierced by \( O(k) \) points. We further show that if for each two intersecting boxes in \( \mathcal{F} \) a corner of one is contained in the other, then \( \mathcal{F} \) can be pierced by at most \( O(k \log \log(k)) \) points, and in the special case where \( \mathcal{F} \) contains only cubes this bound improves to \( O(k) \).

1 Introduction

A matching in a hypergraph \( H = (V, E) \) on vertex set \( V \) and edge set \( E \) is a subset of disjoint edges in \( E \), and a cover of \( H \) is a subset of \( V \) that intersects all edges in \( E \). The matching number \( \nu(H) \) of \( H \) is the maximal size of a matching in \( H \), and the covering number \( \tau(H) \) of \( H \) is the minimal size of a cover. The fractional relaxations of these numbers are denoted as usual by \( \nu^*(H) \) and \( \tau^*(H) \). By LP duality we have that \( \nu^*(H) = \tau^*(H) \).

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Let $\mathcal{F}$ be a finite family of axis-parallel boxes in $\mathbb{R}^d$. We identify $\mathcal{F}$ with the hypergraph with vertex set $\mathbb{R}^d$ and edge set $\mathcal{F}$. Thus a matching in $\mathcal{F}$ is a subfamily of pairwise disjoint boxes (also called an independent set in the literature) and a cover in $\mathcal{F}$ is a set of points in $\mathbb{R}^d$ intersecting every box in $\mathcal{F}$ (also called a hitting set).

An old result due to Gallai is the following (see e.g. [8]):

**Theorem 1** (Gallai). If $\mathcal{F}$ is a family of intervals in $\mathbb{R}$ (i.e., a family of boxes in $\mathbb{R}$) then $\tau(\mathcal{F}) = \nu(\mathcal{F})$.

For a family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{F}) = 1$, Helly’s theorem [9] implies that $\tau(\mathcal{F}) = 1$.

**Observation 2** (Helly [9]). Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{F}) = 1$. Then $\tau(\mathcal{F}) = 1$.

A rectangle is an axis-parallel box in $\mathbb{R}^2$. In 1965, Wegner [14] conjectured that in a hypergraph of axis-parallel rectangles in $\mathbb{R}^2$, the ratio $\tau/\nu$ is bounded by 2. Gáyárfás and Lehel conjectured in [7] that the same ratio is bounded by a constant. The best known lower bound, $\tau = [5\nu/3]$, is attained by a construction due to Fon-Der-Flaass and Kostochka in [6]. Károlyi [10] proved that in families of axis-parallel boxes in $\mathbb{R}^d$ we have $\tau(\mathcal{F}) \leq \nu(\mathcal{F}) (1 + \log (\nu(\mathcal{F})))^{d-1}$, where $\log = \log_2$. Here is a short proof of Károlyi’s bound.

**Theorem 3** (Károlyi [10]). If $\mathcal{F}$ is a finite family of axis-parallel boxes in $\mathbb{R}^d$, then $\tau(\mathcal{F}) \leq \nu(\mathcal{F}) (1 + \log (\nu(\mathcal{F})))^{d-1}$.

**Proof.** We proceed by induction on $d$ and $\nu(\mathcal{F})$. Note that if $\nu(\mathcal{F}) \in \{0, 1\}$ then the result holds for all $d$ by Helly’s theorem [9]. Now let $d, n \in \mathbb{N}$. Let $F_d : \mathbb{R} \to \mathbb{R}$ be a function for which $\tau(\mathcal{T}) \leq F_d(\nu(\mathcal{T}))$ for every family $\mathcal{T}$ of axis-parallel boxes in $\mathbb{R}^d$ with $d' < d$, or with $d = d'$ and $\nu(\mathcal{T}) < n$.

Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{F}) = n$. For $a \in \mathbb{R}$, let $H_a$ be the hyperplane $\{x = (x_1, \ldots, x_d) : x_1 = a\}$. Write $L_a = \{x = (x_1, \ldots, x_d) : x_1 \leq a\}$, and let $\mathcal{F}_a = \{F \in \mathcal{F} : F \subseteq L_a\}$. Define $a^* = \min \{a : \nu(F_a) \geq \lfloor \nu/2 \rfloor\}$. The hyperplane $H_{a^*}$ gives rise to a partition $\mathcal{F} = \bigcup_{i=1}^{\lfloor \nu/2 \rfloor} \mathcal{F}_i$, where $\mathcal{F}_1 = \{F \in \mathcal{F} : F \subseteq L_{a^*} \setminus H_{a^*}\}$, $\mathcal{F}_2 = \{F \in \mathcal{F} : F \cap H_{a^*} \neq \emptyset\}$, and $\mathcal{F}_3 = \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. It follows from the choice of $a^*$ that $\nu(\mathcal{F}_1) \leq [\nu(\mathcal{F})/2] - 1$, $\nu(\mathcal{F}_2) \leq \nu(\mathcal{F})$, and $\nu(\mathcal{F}_3) \leq [\nu(\mathcal{F})/2]$.

Therefore,

\[
F_d (\nu(\mathcal{F})) \leq \tau(\mathcal{F}_1) + \tau(\mathcal{F}_3) + \tau(\{F \cap H_{a^*} : F \in \mathcal{F}_2\})
\]

\[
\leq F_d (\nu(\mathcal{F}_1)) + F_d (\nu(\mathcal{F}_3)) + F_{d-1} (\nu(\mathcal{F}_2))
\]

\[
\leq F_d \left( \left\lfloor \frac{\nu(\mathcal{F})}{2} \right\rfloor - 1 \right) + F_d \left( \left\lfloor \frac{\nu(\mathcal{F})}{2} \right\rfloor \right) + F_{d-1} (\nu(\mathcal{F}))
\]

\[
\leq 2 \frac{\nu(\mathcal{F})}{2} \left( 1 + \log \left( \frac{\nu(\mathcal{F})}{2} \right) \right)^{d-1} + \nu(\mathcal{F}) (1 + \log (\nu(\mathcal{F})))^{d-2}
\]

\[
\leq \nu(\mathcal{F}) (1 + \log (\nu(\mathcal{F})))^{d-1},
\]

implying the result. \qed
Note that for $\nu(\mathcal{F}) = 2$, we have that $\mathcal{F}_1 = \emptyset$, $\nu(\mathcal{F}_2) = 1$ and so $\tau(\mathcal{F}) \leq F_{d-1}(2) + 1$. Therefore, we have the following, which was also proved in [6].

**Observation 4** (Fon-der-Flaass and Kostochka [6]). Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$ with $\nu(\mathcal{F}) = 2$. Then $\tau(\mathcal{F}) \leq d + 1$.

The bound from Theorem 3 was improved by Akopyan [2] to $\tau(\mathcal{F}) \leq (1.5 \log_3 2 + o(1))\nu(\mathcal{F}) (\log_2 (\nu(\mathcal{F})))^{d-1}$.

A corner of a box $F$ in $\mathbb{R}^d$ is a zero-dimensional face of $F$. We say that two boxes in $\mathbb{R}^d$ intersect at a corner if one of them contains a corner of the other.

A family $\mathcal{F}$ of connected subsets of $\mathbb{R}^2$ is a family of pseudo-disks, if for every pair of distinct subsets in $\mathcal{F}$, their boundaries intersect in at most two points. In [4], Chan and Har-Peled proved that families of pseudo-disks in $\mathbb{R}^2$ satisfy $\tau = O(\nu)$. It is easy to check that if $\mathcal{F}$ is a family of axis-parallel rectangles in $\mathbb{R}^2$ in which every two intersecting rectangles intersect at a corner, then $\mathcal{F}$ is a family of pseudo-disks. Thus we have:

**Theorem 5** (Chan and Har-Peled [4]). There exists a constant $c$ such that for every family $\mathcal{F}$ of axis-parallel rectangles in $\mathbb{R}^2$ in which every two intersecting rectangles intersect at a corner, we have that $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$.

Here we prove a few different generalizations of this theorem. In Theorem 6 we prove the bound $\tau(\mathcal{F}) \leq c\nu(\mathcal{F}) \log \log(\nu(\mathcal{F}))$ for families $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ in which every two intersecting boxes intersect at a corner, and in Theorem 7 we prove $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$ for families $\mathcal{F}$ of axis-parallel cubes in $\mathbb{R}^d$, where in both cases $c$ is a constant depending only on the dimension $d$. We further prove in Theorem 8 that in families $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ satisfying certain assumptions on their pairwise intersections, the bound on the covering number improves to $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$. For $d = 2$, these assumptions are equivalent to the assumption that there is a maximum matching $\mathcal{M}$ in $\mathcal{F}$ such that every intersection between a box in $\mathcal{M}$ and a box in $\mathcal{F}\setminus\mathcal{M}$ occurs at a corner. We use this result to prove our Theorem 10, asserting that for every $r$, if $\mathcal{F}$ is a family of axis-parallel rectangles in $\mathbb{R}^2$ with the property that the ratio between the side lengths of every rectangle in $\mathcal{F}$ is bounded by $r$, then $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$ for some constant $c$ depending only on $r$.

Let us now describe our results in more detail. First, for general dimension $d$ we have the following.

**Theorem 6.** There exists a constant $c$ depending only on $d$, such that for every family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ in which every two intersecting boxes intersect at a corner we have $\tau(\mathcal{F}) \leq c\nu(\mathcal{F}) \log \log(\nu(\mathcal{F}))$.

For the proof, we first prove the bound $\tau^*(\mathcal{F}) \leq 2^d \nu(\mathcal{F})$ on the fractional covering number of $\mathcal{F}$, and then use Theorem 11 below for the bound $\tau(\mathcal{F}) = O(\tau^*(\mathcal{F}) \log \log(\tau^*(\mathcal{F})))$.

An axis-parallel box is a **cube** if all its side lengths are equal. Note that if $\mathcal{F}$ consists of axis-parallel cubes in $\mathbb{R}^d$, then every intersection in $\mathcal{F}$ occurs at a corner. Moreover, for axis-parallel cubes we have $\tau(\mathcal{F}) = O(\tau^*(\mathcal{F}))$ by Theorem 11, and thus we conclude the following.
Theorem 7. If $F$ is a family of axis-parallel cubes in $\mathbb{R}^d$, then $\tau(F) \leq c\nu(F)$ for some constant $c$ depending only on $d$.

To get a constant bound on the ratio $\tau/\nu$ in families of axis-parallel boxes in $\mathbb{R}^d$ which are not necessarily cubes, we make a more restrictive assumption on the intersections in $F$.

Theorem 8. Let $F$ be a family of axis-parallel boxes in $\mathbb{R}^d$. Suppose that there exists a maximum matching $M$ in $F$ such that for every $F \in F$ and $M \in M$, at least one of the following holds:

1. $F$ contains a corner of $M$;
2. $F \cap M = \emptyset$; or
3. $M$ contains $2^{d-1}$ corners of $F$.

Then $\tau(F) \leq (2^d + (4 + d)d)\nu(F)$.

For $d = 2$, this theorem implies the following corollary.

Corollary 9. Let $F$ be a family of axis-parallel rectangles in $\mathbb{R}^2$. Suppose that there exists a maximum matching $M$ in $F$ such that for every $F \in F$ and $M \in M$, if $F$ and $M$ intersect then they intersect at a corner. Then $\tau(F) \leq 16\nu(F)$.

Note that Corollary 9 is slightly stronger than Theorem 5. Here we only need that the intersections with rectangles in some fixed maximum matching $M$ occur at corners, but we do not restrict the intersections of two rectangles $F, F' \notin M$.

Given a constant $r > 0$, we say that a family $F$ of axis-parallel boxes in $\mathbb{R}^d$ has an $r$-bounded aspect ratio if every box $F \in F$ has $l_i(F)/l_j(F) \leq r$ for all $i, j \in \{1, \ldots, d\}$, where $l_i(F)$ is the length of the orthogonal projection of $F$ onto the $i$th coordinate.

For families of rectangles with bounded aspect ratio we prove the following.

Theorem 10. Let $F$ be a family of axis-parallel rectangles in $\mathbb{R}^2$ that has an $r$-bounded aspect ratio. Then $\tau(F) \leq (14 + 2r^2)\nu(F)$.

A result similar to Theorem 10 was announced in [1], but to the best of our knowledge the proof was not published.

An application of Theorem 10 is the existence of weak $\varepsilon$-nets of size $O\left(\frac{1}{\varepsilon}\right)$ for axis-parallel rectangles in $\mathbb{R}^2$ with bounded aspect ratio. More precisely, let $P$ be a set of $n$ points in $\mathbb{R}^d$ and let $F$ be a family of sets in $\mathbb{R}^d$, each containing at least $\varepsilon n$ points of $P$. A weak $\varepsilon$-net for $F$ is a cover of $F$, and a strong $\varepsilon$-net for $F$ is a cover of $F$ with points of $P$. The existence of weak $\varepsilon$-nets of size $O\left(\frac{1}{\varepsilon}\right)$ for pseudo-disks in $\mathbb{R}^2$ was proved by Pyrga and Ray in [12]. Aronov, Ezra and Sharir in [3] showed the existence of strong $\varepsilon$-nets of size $O\left(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}\right)$ for axis-parallel boxes in $\mathbb{R}^2$ and $\mathbb{R}^3$, and the existence of weak $\varepsilon$-nets of size $O\left(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}\right)$ for all $d$ was then proved by Ezra in [5]. Ezra also showed that for axis-parallel cubes in $\mathbb{R}^d$ there exists an $\varepsilon$-net of size $O\left(\frac{1}{\varepsilon^2}\right)$. These results imply the following.
Theorem 11 (Aronov, Ezra and Sharir [3]; Ezra [5]). If $\mathcal{F}$ is a family of axis-parallel boxes in $\mathbb{R}^d$ then $\tau(\mathcal{F}) \leq cr^*\mathcal{F} \log \log (r^*\mathcal{F})$ for some constant $c$ depending only on $d$. If $\mathcal{F}$ consists of cubes, then this bound improves to $\tau(\mathcal{F}) \leq cr^*\mathcal{F}$.

An example where the smallest strong $\varepsilon$-net for axis-parallel rectangles in $\mathbb{R}^2$ is of size $\Omega \left( \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon} \right)$ was constructed by Pach and Tardos in [11]. The question of whether weak $\varepsilon$-nets of size $O\left( \frac{1}{\varepsilon} \right)$ for axis-parallel rectangles in $\mathbb{R}^2$ exist was raised both in [3] and in [11].

Theorem 10 implies a positive answer for the family of axis-parallel rectangles in $\mathbb{R}^2$ satisfying the $r$-bounded aspect ratio property:

Corollary 12. For every fixed constant $r$, there exists a weak $\varepsilon$-net of size $O(\frac{1}{\varepsilon})$ for the family $\mathcal{F}$ of axis-parallel rectangles in $\mathbb{R}^2$ with aspect ratio bounded by $r$.

Proof. Given a set $P$ of $n$ points, there cannot be $\frac{1}{\varepsilon} + 1$ pairwise disjoint rectangles in $\mathcal{F}$, each containing at least $\varepsilon n$ points of $P$. Therefore $\nu(\mathcal{F}) \leq \frac{1}{\varepsilon}$. Theorem 10 implies that there is a cover of $\mathcal{F}$ of size $O(\frac{1}{\varepsilon})$.

This paper is organized as follows. In Section 2 we prove Theorem 6. Section 3 contains definitions and tools. Theorem 8 is then proved in Section 4 and Theorem 10 is proved in Section 5.

2 Proofs of Theorems 6 and 7

Let $\mathcal{F}$ be a finite family of axis-parallel boxes in $\mathbb{R}^d$, such that every intersection in $\mathcal{F}$ occurs at a corner. By performing small perturbations on the boxes, we may assume that no two corners of boxes of $\mathcal{F}$ coincide.

Proposition 13. We have $\tau^*\mathcal{F} \leq 2^d \nu(\mathcal{F})$.

Proof. We let $\nu(\mathcal{F}) = k$. Since an optimal fractional matching is an optimum solution to a linear program with integer coefficients, and by [13, Theorem 10.1], there exists an optimum fractional matching $g : \mathcal{F} \rightarrow \mathbb{Q}^+$ for $\mathcal{F}$. By choosing a common denominator $r$, we may assume that $g(F) = \frac{k_F}{r}$ for some $k_F \in \mathbb{N}$ for all $F \in \mathcal{F}$. We now let $\mathcal{F}'$ be the family of boxes that contains $k_F$ copies of each box $F \in \mathcal{F}$. Let $n$ be the number of boxes in $\mathcal{F}'$. It follows that $\tau^*(\mathcal{F}) = \nu(\mathcal{F}) = \frac{n}{r}$, and thus our aim is to show that $\frac{n}{r} \leq 2^d k$.

For $x \in \mathbb{R}^d$, we let $\mathcal{F}_x$ be the set of $F \in \mathcal{F}$ containing $x$. Since $g$ is a fractional matching, it follows that $\sum_{F \in \mathcal{F}_x} g(F) \leq 1$. Thus, the number of boxes in $\mathcal{F}'$ that intersect $x$ is at most $\sum_{F \in \mathcal{F}_x} k_F \leq r$.

Since a matching of $\mathcal{F}'$ cannot contain two copies of the same box in $\mathcal{F}$, it follows that $\nu(\mathcal{F}') \leq \nu(\mathcal{F})$. Since $\nu(\mathcal{F}') \leq k$, it follows from Turán’s theorem that there are at least $(n-k)/(2k)$ unordered intersecting pairs of boxes $\mathcal{F}'$. Each such unordered pair contributes at least two pairs of the form $(x, F)$, where $x$ is a corner of a box $F' \in \mathcal{F}'$, $F$ is box in $\mathcal{F}'$ different from $F'$, and $x$ pierces $F$. Therefore, since there are altogether $2^d n$ corners of boxes in $\mathcal{F}'$, there must exist a corner $x$ of a box $F \in \mathcal{F}'$ that pierces at least $(n-k)/(2k)$ boxes in $\mathcal{F}'$, all different from $F$. Together with $F$, $x$ intersects at least $n/2^d k$ boxes of $\mathcal{F}'$, implying that $n/2^d k \leq r$. Thus $\frac{n}{r} \leq 2^d k$, as desired. 

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Combining this bound with Theorem 11, we obtain the proofs of Theorems 6 and 7.

3 Definitions and tools

Let $R$ be an axis-parallel box in $\mathbb{R}^d$ with $R = [x_1, y_1] \times \cdots \times [x_d, y_d]$. For $i \in \{1, \ldots, d\}$, let $p_i(R) = [x_i, y_i]$ denote the orthogonal projection of $R$ onto the $i$-th coordinate. Two intervals $[a, b], [c, d] \subseteq \mathbb{R}$, are incomparable if $[a, b] \not\subseteq [c, d]$ and $[c, d] \not\subseteq [a, b]$. We say that $[a, b] < [c, d]$ if $b < c$. For two axis-parallel boxes $Q$ and $R$ we say that $Q \prec_i R$ if $p_i(Q) < p_i(R)$.

**Observation 14.** Let $Q, R$ be disjoint axis-parallel boxes in $\mathbb{R}^d$. Then there exists $i \in \{1, \ldots, d\}$ such that $Q \prec_i R$ or $R \prec_i Q$.

**Lemma 15.** Let $Q, R$ be axis-parallel boxes in $\mathbb{R}^d$ such that $Q$ contains a corner of $R$ but $R$ does not contain a corner of $Q$. Then, for all $i \in \{1, \ldots, d\}$, either $p_i(R)$ and $p_i(Q)$ are incomparable, or $p_i(R) \subseteq p_i(Q)$, and there exists $i \in \{1, \ldots, d\}$ such that $p_i(R) \not\subseteq p_i(Q)$.

Moreover, if $R \not\subseteq Q$, then there exists $j \in \{1, \ldots, d\} \setminus \{i\}$ such that $p_i(R)$ and $p_j(Q)$ are incomparable.

**Proof.** Let $x = (x_1, \ldots, x_d)$ be a corner of $R$ contained in $Q$. By symmetry, we may assume that $x_i = \max(p_i(R))$ for all $i \in \{1, \ldots, d\}$. Since $x_i \in p_i(Q)$ for all $i \in \{1, \ldots, d\}$, it follows that $\max(p_i(Q)) \geq \max(p_i(R))$ for all $i \in \{1, \ldots, d\}$. If $\min(p_i(Q)) < \min(p_i(R))$, then $p_i(R) \subseteq p_i(Q)$; otherwise, $p_i(Q)$ and $p_i(R)$ are incomparable. If $p_i(Q)$ and $p_i(R)$ are incomparable for all $i \in \{1, \ldots, d\}$, then $y = (y_1, \ldots, y_d)$ with $y_i = \min(p_i(Q))$ is a corner of $Q$ and since $\min(p_i(Q)) > \min(p_i(R))$, it follows that $y \in R$, a contradiction. It follows that there exists an $i \in \{1, \ldots, d\}$ such that $p_i(R) \not\subseteq p_i(Q)$.

If $p_i(R) \not\subseteq p_i(Q)$ for all $i \in \{1, \ldots, d\}$, then $R \subseteq Q$; this implies the result. $\square$

**Observation 16.** Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$. Let $\mathcal{F}'$ arise from $\mathcal{F}$ by removing every box in $\mathcal{F}$ that contains another box in $\mathcal{F}$. Then $\nu(\mathcal{F}) = \nu(\mathcal{F}')$ and $\tau(\mathcal{F}) = \tau(\mathcal{F}')$.

**Proof.** Since $\mathcal{F}' \subseteq \mathcal{F}$, it follows that $\nu(\mathcal{F}') \leq \nu(\mathcal{F})$ and $\tau(\mathcal{F}') \leq \tau(\mathcal{F})$. Let $\mathcal{M}$ be a matching in $\mathcal{F}$ of size $\nu(\mathcal{F})$. Let $\mathcal{M}'$ arise from $\mathcal{M}$ by replacing each box $R$ in $\mathcal{M} \setminus \mathcal{F}'$ with a box in $\mathcal{F}'$ contained in $R$. Then $\mathcal{M}'$ is a matching in $\mathcal{F}'$, and so $\nu(\mathcal{F}') = \nu(\mathcal{F})$. Moreover, let $P$ be a cover of $\mathcal{F}'$. Since every box in $\mathcal{F}$ contains a box in $\mathcal{F}'$ (possibly itself) which, in turn, contains a point in $P$, we deduce that $P$ is a cover of $\mathcal{F}$. It follows that $\tau(\mathcal{F}') = \tau(\mathcal{F})$. $\square$

A family $\mathcal{F}$ of axis-parallel boxes is *clean* if no box in $\mathcal{F}$ contains another box in $\mathcal{F}$. By Observation 16, we may restrict ourselves to clean families of boxes.
4 Proof of Theorem 8

Throughout this section, let $\mathcal{F}$ be a clean family of axis-parallel boxes in $\mathbb{R}^d$, and let $\mathcal{M}$ be a matching of maximum size in $\mathcal{F}$. We let $\mathcal{F}(\mathcal{M})$ denote the subfamily of $\mathcal{F}$ consisting of those boxes $R$ in $\mathcal{F}$ for which for every $M \in \mathcal{M}$, either $M$ is disjoint from $R$ or $M$ contains at least $2^{d-1}$ corners of $R$. Our goal is to bound $\tau(\mathcal{F}(\mathcal{M}))$.

**Lemma 17.** Let $R \in \mathcal{F}(\mathcal{M})$. Then $R$ intersects at least one and at most two boxes in $\mathcal{M}$. If $R$ intersects two boxes $M_1, M_2 \in \mathcal{M}$, then there exists $j \in \{1, \ldots, d\}$ such that $M_1 \prec_j M_2$ or $M_2 \prec_j M_1$, and for all $i \in \{1, \ldots, d\} \setminus \{j\}$, we have that $p_i(R) \subseteq p_i(M_1)$ and $p_i(R) \subseteq p_i(M_2)$.

**Proof.** If $R$ is disjoint from every box in $\mathcal{M}$, then $\mathcal{M} \cup \{R\}$ is a larger matching, a contradiction. So $R$ intersects at least one box in $\mathcal{M}$. Let $M_1$ be in $\mathcal{M}$ such that $R \cap M_1 \neq \emptyset$. We claim that there exists $j \in \{1, \ldots, d\}$ such that $M_1$ contains precisely the set of corners of $R$ with the same $j$th coordinate.

By Lemma 15, there exists $j \in \{1, \ldots, d\}$ such that $p_j(R) = [a, b]$ and $p_j(M_1)$ are incomparable. By symmetry, we may assume that $a \in p_j(M_1)$, $b \notin p_j(M_1)$. This proves that $M_1$ contains all $2^{d-1}$ corners of $R$ with $a$ as their $j$th coordinate, and our claim follows.

Consequently, $p_i(R) \subseteq p_i(M_1)$ for all $i \in \{1, \ldots, d\} \setminus \{j\}$. Since $R$ has exactly $2^d$ corners, and members of $\mathcal{M}$ are disjoint, it follows that there exist at most two boxes in $\mathcal{M}$ that intersect $R$. If $M_1$ is the only one such box, then the result follows. Let $M_2 \in \mathcal{M} \setminus \{M_1\}$ such that $R \cap M_1 \neq \emptyset$. By our claim, it follows that $M_2$ contains $2^{d-1}$ corners of $R$; and since $M_1$ is disjoint from $M_2$, it follows that $M_2$ contains precisely those corners of $R$ with $j$th coordinate equal to $b$. Therefore, $p_i(R) \subseteq p_i(M_2)$ for all $i \in \{1, \ldots, d\} \setminus \{j\}$. We conclude that $p_i(M_2)$ is not disjoint from $p_i(M_1)$ for all $i \in \{1, \ldots, d\} \setminus \{j\}$, and since $M_1, M_2$ are disjoint, it follows from Observation 14 that either $M_1 \prec_j M_2$ or $M_2 \prec_j M_1$. \qed

For $i \in \{1, \ldots, d\}$, we define a directed graph $G_i$ as follows. We let $V(G_i) = \mathcal{M}$, and for $M_1, M_2 \in \mathcal{M}$ we let $M_1 M_2 \in E(G_i)$ if and only if $M_1 \prec_i M_2$ and there exists $R \in \mathcal{F}(\mathcal{M})$ such that $R \cap M_1 \neq \emptyset$ and $R \cap M_2 \neq \emptyset$. In this case, we say that $R$ witnesses the edge $M_1 M_2$. For $i \in \{1, \ldots, d\}$, we say that $R$ is $i$-pendant at $M_1 \in \mathcal{M}$ if $M_1$ is the only box of $\mathcal{M}$ intersecting $R$ and $p_i(R)$ and $p_i(M_1)$ are incomparable. Note that by Lemma 17, every box $R$ in $\mathcal{F}(\mathcal{M})$ satisfies exactly one of the following: $R$ witnesses an edge in exactly one of the graphs $G_i$, $i \in \{1, \ldots, d\}$; or $R$ is $i$-pendant for exactly one $i \in \{1, \ldots, d\}$.

**Lemma 18.** Let $i \in \{1, \ldots, d\}$. Let $Q, R \in \mathcal{F}(\mathcal{M})$ be such that $Q$ witnesses an edge $M_1 M_2$ in $G_i$, and $R$ witnesses an edge $M_3 M_4$ in $G_i$. If $Q$ and $R$ intersect, then either $M_1 = M_4$, or $M_2 = M_3$, or $M_1 M_2 = M_3 M_4$.

**Proof.** By symmetry, we may assume that $i = 1$. Let $p_1(M_1) = [x_1, y_1]$ and $p_1(M_2) = [x_2, y_2]$. It follows that $p_1(Q) \subseteq [x_1, y_2]$. Let $a = (a_1, a_2, \ldots, a_d) \in Q \cap R$. It follows that $a_j \in p_j(Q) \subseteq p_j(M_1) \cap p_j(M_2)$ and $a_j \in p_j(R) \subseteq p_j(M_3) \cap p_j(M_4)$ for all $j \in \{2, \ldots, d\}$.
If $M_1 \in \{M_3, M_4\}$ and $M_2 \in \{M_3, M_4\}$, then $M_1 M_2 = M_3 M_4$, and the result follows. Therefore, we may assume that this does not happen. By symmetry, we may assume that $M_1$ is distinct from $M_3$ and $M_4$. (If $M_2$ is distinct from $M_3$ and $M_4$, and $M_1$ is not, then we reflect the family of boxes along the origin; this switches the roles of $M_1$ and $M_2$, and of $M_3$ and $M_4$.)

It follows that $a \notin M_1$, for otherwise $R$ intersects three distinct members of $\mathcal{M}$, contrary to Lemma 17. Since $R$ is disjoint from $M_1$, it follows that either $M_1 \prec_1 R$ or $R \prec_1 M_1$. But $p_1(Q) \subseteq [x_1, y_2]$, and since $Q \cap R \neq \emptyset$, it follows that $M_1 \prec_1 R$ (see Figure 1).
Since $M_3 \neq M_1$ and $p_j(M_3) \cap p_j(M_1) \ni a_j$ for all $j \in \{2, \ldots, d\}$, it follows that either $M_1 \prec_1 M_3$ or $M_1 \prec_1 M_3$. Since $M_1 \prec_1 R$ and $R \cap M_3 \neq \emptyset$, it follows that $M_1 \prec_1 M_3$.

Suppose that $a \in M_3$. Then $Q \cap M_3 \neq \emptyset$, and since $M_1 \prec_1 M_3$, we have that $M_3 = M_2$ as desired.

Therefore, we may assume that $a \notin M_3$, and thus $p_1(M_1) \prec p_1(M_3) \prec [a_1, a_1]$. Since $[y_1, a_1] \subseteq p_1(Q)$, it follows that $p_1(M_3) \cap p_1(Q) \neq \emptyset$. But $p_j(M_3) \cap p_j(Q) \ni a_j$ for all $j \in \{2, \ldots, d\}$, and hence $Q \cap M_3 \neq \emptyset$. But then $M_3 \in \{M_1, M_2\}$, and thus $M_3 = M_2$.

This concludes the proof.

The following is a well-known fact about directed graphs; we include a proof for completeness.

**Lemma 19.** Let $G$ be a directed graph. Then there exists an edge set $E \subseteq E(G)$ with $|E| \geq |E(G)|/4$ such that for every vertex $v \in V(G)$, either $E$ contains no incoming edge at $v$, or $E$ contains no outgoing edge at $v$.

**Proof.** For $A, B \subseteq V(G)$, let $E(A, B)$ denote the set of edges of $G$ with head in $A$ and tail in $B$.

Let $X_0 = Y_0 = \emptyset$, $V(G) = \{v_1, \ldots, v_n\}$. For $i = 1, \ldots, n$ we will construct $X_i, Y_i$ such that $X_i \cup Y_i = \{v_1, \ldots, v_i\}$, $X_i \cap Y_i = \emptyset$ and $|E(X_i, Y_i)| + |E(Y_i, X_i)| \geq |E(G|(X_i \cup Y_i))|/2$, where $G|(X_i \cup Y_i)$ denotes the induced subgraph of $G$ on vertex set $X_i \cup Y_i$. This holds for $X_0, Y_0$. Suppose that we have constructed $X_{i-1}, Y_{i-1}$ for some $i \in \{1, \ldots, n\}$. If $|E(X_{i-1}, \{v_i\})| + |E(\{v_i\}, X_{i-1})| \geq |E(Y_{i-1}, \{v_i\})| + |E(\{v_i\}, Y_{i-1})|$, we let $X_i = X_{i-1}, Y_i = Y_{i-1} \cup \{v_i\}$; otherwise, let $X_i = X_{i-1} \cup \{v_i\}, Y_i = Y_{i-1}$. It follows that $X_i, Y_i$ still have the desired properties. Thus, $|E(X_n, Y_n)| + |E(Y_n, X_n)| \geq |E(G)|/2$. By symmetry, we may assume that $|E(X_n, Y_n)| \geq |E(G)|/4$. But then $E(X_n, Y_n)$ is the desired set $E$; it contains only incoming edges at vertices in $X_n$, and only outgoing edges at vertices in $Y_n$. This concludes the proof.

**Theorem 20.** For $i \in \{1, \ldots, d\}$, $|E(G_i)| \leq 4\nu(F)$.

**Proof.** Let $E \subseteq E(G_i)$ as in Lemma 19. For each edge in $E$, we pick one box witnessing this edge; let $F'$ denote the family of these boxes. We claim that $F'$ is a matching. Indeed, suppose not, and let $Q, R \in F'$ be distinct and intersecting. Let $Q$ witness $M_1M_2$ and $R$ witness $M_3M_4$. By Lemma 18, it follows that either $M_1M_2 = M_3M_4$ (impossible since we picked exactly one witness per edge) or $M_1 = M_4$ (impossible because $E$ does not contain both an incoming and an outgoing edge at $M_1 = M_4$) or $M_2 = M_3$ (impossible because $E$ does not contain both an incoming and an outgoing edge at $M_2 = M_3$). This is a contradiction, and our claim follows. Now we have $\nu(F) \geq |F'| = |E| \geq |E(G_i)|/4$, which implies the result.

A matching $M$ of a clean family $F$ of boxes is extremal if for every $M \in M$ and $R \in F \setminus M$, either $(M \setminus \{M\}) \cup \{R\}$ is not a matching or there exists an $i \in \{1, \ldots, d\}$ such that $\max(p_i(R)) \geq \max(p_i(M))$. Every family $F$ of axis parallel boxes has an extremal maximum matching. For example, the maximum matching $M$ minimizing $\sum_{M \in M} \sum_{i=1}^d \max(p_i(M))$ is extremal.
Theorem 21. For \( i \in \{1, \ldots, d\} \), let \( F_i \) denote the set of boxes in \( F(M) \) that either are i-pendant or witness an edge in \( G_1 \). Then \( \tau(F_i) \leq (4 + d)\nu(F) \). If \( M \) is extremal, then \( \tau(F_i) \leq (3 + d)\nu(F) \).

Proof. By symmetry, it is enough to prove the theorem for \( i = 1 \). For \( M \in M \), let \( F_M \) denote the set of boxes in \( F_i \) that either are 1-pendant at \( M \), or witness an edge \( MM' \) of \( G_1 \). It follows that \( \bigcup_{M \in M} F_M = F_i \). For \( M \in M \), let \( d^+(M) \) denote the out-degree of \( M \) in \( G_1 \). We will prove that \( \tau(F_M) \leq d^+(M) + d \) for all \( M \in M \).

We fix a box \( M \in M \). Let \( A \) denote the set of boxes that are 1-pendant at \( M \). Suppose that \( A \) contains two disjoint boxes \( M_1, M_2 \). Then \( (M \setminus \{M\}) \cup \{M_1, M_2\} \) is a larger matching than \( M \), a contradiction. So every two boxes in \( A \) pairwise intersect. By Observation 2, it follows that \( \tau(A) = 1 \).

Let \( B = F_M \setminus A \), i.e. \( B \) is the set of boxes in \( F_i \) that witness an outgoing edge \( MM' \) at \( M \). For every edge \( MM' \in E(G_1) \), we let \( B(M') \) denote the set of boxes in \( F_i \) that witness the edge \( MM' \).

Suppose that there is an edge \( MM' \in E(G_1) \) such that the set \( B(M') \) satisfies \( \nu(B(M')) \geq 3 \). Then \( M \) is not a maximum matching, since removing \( M \) and \( M' \) from \( M \) and adding \( \nu(B(M')) \) disjoint rectangles in \( B(M') \) yields a larger matching. Moreover, for distinct \( M', M'' \in M \), every box in \( B(M') \) is disjoint from every box in \( B(M'') \) by Lemma 18. Thus, if there exist \( M', M'' \) such that \( \nu(B(M')) = \nu(B(M'')) = 2 \) and \( M' \neq M'' \), then removing \( M, M' \) and \( M'' \) and adding two disjoint rectangles from each of \( B(M') \) and \( B(M'') \) yields a bigger matching, a contradiction.

Let \( p_1(M) = [a, b] \). Two boxes in \( B(M') \) intersect if and only if their intersections with the hyperplane \( H = \{(x_1, \ldots, x_d) : x_1 = b\} \) intersect. If \( \nu(B(M')) = 1 \), then \( \tau(B(M')) = 1 \) by Observation 2. If \( \nu(B(M')) = 2 \), then \( \nu(\{F \cap H : F \in B(M')\}) = 2 \) and so

\[
\tau(B(M')) = \tau(\{F \cap H : F \in B(M')\}) \leq d
\]

by Observation 4.

Therefore,

\[
\tau(B) \leq \sum_{M' : MM' \in E(G_1)} \tau(B(M')) \leq d^+(M) - 1 + d,
\]

and since \( \tau(A) \leq 1 \), it follows that \( \tau(F_M) \leq d^+(M) + d \) as claimed (see Figure 2).

Summing over all rectangles in \( M \), we obtain

\[
\tau(F_i) \leq \sum_{M \in M} \tau(F_M) \leq \sum_{M \in M} (d^+(M) + d) = d|V(G_1)| + |E(G_1)| \leq d|M| + 4|M| = (4 + d)\nu(F),
\]

where we used Theorem 20 for the inequality \( |E(G_1)| \leq 4|M| \).

If \( M \) is extremal, then every 1-pendant box at \( M \) also intersects \( H \). Let \( M' \) be such that \( \nu(B(M')) \) is maximum. It follows that \( \nu(A \cup B(M')) \leq 2 \) and thus \( \tau(A \cup B(M')) \leq d \), implying \( \tau(F_M) \leq d^+(M) + d - 1 \). This concludes the proof of the second part of the theorem.

\[ \square \]

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Figure 2: Proof that $\tau(F_M) \leq d^+(M) = d$ for $d = 2$; here $d^+(M) = 3$. The red boxes in $A$ satisfy $\tau(A) = \nu(A) = 1$, since $M$ is the only box in $M$ they intersect. There is only one $M'$, namely $M' = M'_1$, such that $\nu(B(M')) > 1$; since all those boxes intersect the line $x = b$, $\tau(B(M')) \leq d = 2$. For all of the $d^+(M) - 1$ boxes $M'$ such that $M' \neq M'_1$, $\tau(B(M')) = \nu(B(M')) = 1$. So $\tau(F_M) \leq 5$, as shown.

Theorem 22. Let $F' \subseteq F$ be the set of boxes $R \in F$ such that for each $M \in M$, either $M \cap R = \emptyset$, or $M$ contains $2^{d-1}$ corners of $R$, or $R$ contains a corner of $M$. Then $\tau(F') \leq (2^d + (4 + d)d)\nu(F)$. If $M$ is extremal, then $\tau(F') \leq (2^d + (3 + d)d)\nu(F)$.

Proof. We proved in Theorem 21 that $\tau(F_i) \leq (4 + d)\nu(F)$ for $i = 1, \ldots, d$. Let $F'' = F' \setminus F(M)$. Then $F''$ consists of boxes $R$ such that $R$ contains a corner of some box
Let $P$ be the set of all corners of boxes in $\mathcal{M}$. It follows that $P$ covers $\mathcal{F}''$, and so $\tau(\mathcal{F}'') \leq 2^d \nu(\mathcal{F})$. Since $\mathcal{F}' = \mathcal{F}'' \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_d$, it follows that $\tau(\mathcal{F}') \leq (2^d + (4 + d)d) \nu(\mathcal{F})$. If $\mathcal{M}$ is extremal, the same argument yields that $\tau(\mathcal{F}') \leq (2^d + (3 + d)d) \nu(\mathcal{F})$, since $\tau(\mathcal{F}_i) \leq (3 + d) \nu(\mathcal{F})$ for $i = 1, \ldots, d$ by Theorem 21.

We are now ready to prove our main theorems.

Proof of Theorem 8. Let $\mathcal{F}$ be a family of axis-parallel boxes in $\mathbb{R}^d$, and let $\mathcal{M}$ be a maximum matching in $\mathcal{F}$ such that for every $F \in \mathcal{F}$ and $M \in \mathcal{M}$, either $F \cap M = \emptyset$, or $F$ contains a corner of $M$, or $M$ contains $2^{d-1}$ corners of $F$. It follows that $\mathcal{F} = \mathcal{F}'$ in Theorem 22, and therefore, $\tau(\mathcal{F}) \leq (2^d + (4 + d)d) \nu(\mathcal{F})$.

5 Proof of Theorem 10

Let $\mathcal{M}$ be a maximum matching in $\mathcal{F}$, and let $\mathcal{M}$ be extremal. Observe that each rectangle $R \in \mathcal{F}$ satisfies one of the following:

- $R$ contains a corner of some $M \in \mathcal{M}$;
- some $M \in \mathcal{M}$ contains two corners of $R$; or
- there exists $M \in \mathcal{M}$ such that $M \cap R \neq \emptyset$, and $p_i(R) \supseteq p_i(M)$ for some $i \in \{1, 2\}$.

By Theorem 22, $14 \nu(\mathcal{F})$ points suffice to cover every rectangle satisfying at least one of the first two conditions. Now, due to the $r$-bounded aspect ratio, for each $M \in \mathcal{M}$ and for each $i \in \{1, 2\}$, at most $r^2$ disjoint rectangles $R \in \mathcal{F}$ can satisfy the third condition for $M$ and $i$. Thus the family of projections of the rectangles satisfying the third condition for $M$ and $i$ onto the $(3-i)$th coordinate have a matching number at most $r^2$. Since all these rectangles intersect the boundary of $M$ twice, by Theorem 1, we need at most $r^2$ additional points to cover them for each $i \in \{1, 2\}$. We conclude that $\tau(\mathcal{F}) \leq (14 + 2r^2) \nu(\mathcal{F})$.

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References


