A characterization of Hermitian varieties as codewords

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Abstract

It is known that the Hermitian varieties are codewords in the code defined by the points and hyperplanes of the projective spaces PG(r, q^2). In finite geometry, also quasi-Hermitian varieties are defined. These are sets of points of PG(r, q^2) of the same size as a non-singular Hermitian variety of PG(r, q^2), having the same intersection sizes with the hyperplanes of PG(r, q^2). In the planar case, this reduces to the definition of a unital. A famous result of Blokhuis, Brouwer, and Wilbrink states that every unital in the code of the points and lines of PG(2, q^2) is a Hermitian curve. We prove a similar result for the quasi-Hermitian varieties in PG(3, q^2), q = p^h, as well as in PG(r, q^2), q = p prime, or q = p^2, p prime, and r \geq 4.

Keywords: Hermitian variety; incidence vector; codes of projective spaces; quasi-Hermitian variety
1 Introduction

Consider the non-singular Hermitian varieties $\mathcal{H}(r, q^2)$ in $\mathcal{H}(r, q^2)$. A non-singular Hermitian variety $\mathcal{H}(r, q^2)$ in $\mathcal{H}(r, q^2)$ is the set of absolute points of a Hermitian polarity of $\mathcal{H}(r, q^2)$. Many properties of a non-singular Hermitian variety $\mathcal{H}(r, q^2)$ in $\mathcal{H}(r, q^2)$ are known. In particular, its size is $(q^{r+1}+(-1)^r)(q^r-(-1)^r)/(q^2-1)$, and its intersection numbers with the hyperplanes of $\mathcal{H}(r, q^2)$ are equal to $(q^r+(-1)^r-1)(q^r-(-1)^r-1)/(q^2-1)$, in case the hyperplane is a non-tangent hyperplane to $\mathcal{H}(r, q^2)$, and equal to $1+q^2(q^r+(-1)^r)(q^r-2-(-1)^r)/(q^2-1)$ in case the hyperplane is a tangent hyperplane to $\mathcal{H}(r, q^2)$; see [16].

Quasi-Hermitian varieties $\mathcal{V}$ in $\mathcal{H}(r, q^2)$ are generalizations of the non-singular Hermitian variety $\mathcal{H}(r, q^2)$ so that $\mathcal{V}$ and $\mathcal{H}(r, q^2)$ have the same size and the same intersection numbers with hyperplanes.

Obviously, a Hermitian variety $\mathcal{H}(r, q^2)$ can be viewed as a trivial quasi–Hermitian variety and we call $\mathcal{H}(r, q^2)$ the classical quasi–Hermitian variety of $\mathcal{H}(r, q^2)$. In the 2-dimensional case, $\mathcal{H}(r, q^2)$ is also known as the classical example of a unital of the projective plane $\mathcal{H}(2, q^2)$.

As far as we know, the only known non-classical quasi-Hermitian varieties of $\mathcal{H}(r, q^2)$ were constructed in [1, 2, 8, 9, 14, 15].

In [6], it is shown that a unital in $\mathcal{H}(2, q^2)$ is a Hermitian curve if and only if it is in the $\mathbb{F}_p$-code spanned by the lines of $\mathcal{H}(2, q^2)$, with $q = p^h$, $p$ prime and $h \in \mathbb{N}$.

In this article, we prove the following result.

**Theorem 1.1.** A quasi-Hermitian variety $\mathcal{V}$ of $\mathcal{H}(r, q^2)$, with $r = 3$ and $q = p^h \neq 4$, $p$ prime, or $r \geq 4$, $q = p \geq 5$, or $r \geq 4$, $q = p^2$, $p \neq 2$ prime, is classical if and only if it is in the $\mathbb{F}_p$-code spanned by the hyperplanes of $\mathcal{H}(r, q^2)$.

Furthermore we consider singular quasi-Hermitian varieties, that is point sets having the same number of points as a singular Hermitian variety $\mathcal{S}$ and for which the intersection numbers with respect to hyperplanes are also the intersection numbers of $\mathcal{S}$ with respect to hyperplanes. We show that Theorem 1.1 also holds in the case in which $\mathcal{V}$ is assumed to be a singular quasi-Hermitian variety of $\mathcal{H}(r, q^2)$.

2 Preliminaries

A subset $\mathcal{K}$ of $\mathcal{H}(r, q^2)$ is a $k_{n,r,q^2}$ if $n$ is a fixed integer, with $1 \leq n \leq q^2$, such that:

(i) $|\mathcal{K}| = k$;

(ii) $|\ell \cap \mathcal{K}| = 1$, $n$, or $q^2 + 1$ for each line $\ell$;

(iii) $|\ell \cap \mathcal{K}| = n$ for some line $\ell$.

A point $P$ of $\mathcal{K}$ is singular if every line through $P$ is either a unisecant or a line of $\mathcal{K}$. The set $\mathcal{K}$ is called singular or non-singular according as it has singular points or not.

Furthermore, a subset $\mathcal{K}$ of $\mathcal{H}(r, q^2)$ is called regular if
(a) $K$ is a $k_{n,r,q^2}$;
(b) $3 \leq n \leq q^2 - 1$;
(c) no planar section of $K$ is the complement of a set of type $(0, q^2 + 1 - n)$.

Theorem 2.1. [10, Theorem 19.5.13] Let $K$ be a $k_{n,3,q^2}$ in $H(3,q^2)$, where $q$ is any prime power and $n \neq \frac{1}{2}q^2 + 1$. Suppose furthermore that every point in $K$ lies on at least one $n$-secant. Then $n = q + 1$ and $K$ is a non-singular Hermitian surface.

Theorem 2.2. [12, Theorem 23.5.19] If $K$ is a regular, non-singular $k_{n,r,q^2}$, with $r \geq 4$ and $q > 2$, then $K$ is a non-singular Hermitian variety.

Theorem 2.3. [12, Th. 23.5.1] If $K$ is a singular $k_{n,3,q^2}$ in $H(3,q^2)$, with $3 \leq n \leq q^2 - 1$, $q > 2$, then the following holds: $K$ is $n$ planes through a line or a cone with vertex a point and base $K'$ a plane section of type

I. a unital;
II. a subplane $H(2,q)$;
III. a set of type $(0, n - 1)$ plus an external line;
IV. the complement of a set of type $(0, q^2 + 1 - n)$.

Theorem 2.4. [12, Lemma 23.5.2 and Th. 25.5.3] If $K$ is a singular $k_{n,r,q}$ with $r \geq 4$, then the singular points of $K$ form a subspace $\Pi_d$ of dimension $d$ and one of the following possibilities holds:

1. $d = r - 1$ and $K$ is a hyperplane;
2. $d = r - 2$ and $K$ consists of $n > 1$ hyperplanes through $\Pi_d$;
3. $d \leq r - 3$ and $K$ is equal to a cone $\Pi_dK'$, with $\pi_d$ as vertex and with $K$ as base, where $K'$ is a non singular $k_{n,r-d-1,q}$.

A multiset in $H(r,q)$ is a set in which multiple instances of the elements are allowed.

Result 2.5. [17, Remark 2.4 and Lemma 2.5] Let $M$ be a multiset in $H(2,q)$, $q = p^h$, where $p$ is prime. Assume that the number of lines intersecting $M$ in not $k \pmod{p}$ points is $\delta$. Then, the number $s$ of non $k \pmod{p}$ secants through any point of $M$ satisfies $qs - s(s - 1) \leq \delta$. In particular, if $\delta < \frac{\delta}{16}(q + 1)^2$, then the number of non $k \pmod{p}$ secants through any point is at most $\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^2}$ or at least $q + 1 - \left(\frac{\delta}{q+1} + \frac{2\delta^2}{(q+1)^2}\right)$.

Property 2.6 ([17]). Let $M$ be a multiset in $H(2,q)$, $q = p^h$, where $p$ is prime. Assume that there are $\delta$ lines that intersect $M$ in not $k \pmod{p}$ points. If through a point there are more than $q/2$ lines intersecting $M$ in not $k \pmod{p}$ points, then there exists a value $r$ such that the intersection multiplicity of at least $2\frac{\delta}{q+1} + 5$ of these lines with $M$ is $r$. 

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Result 2.7 ([17]). Let \( \mathcal{M} \) be a multiset in \( \mathcal{H}(2,q) \), \( 17 < q \), \( q = p^h \), where \( p \) is prime. Assume that the number of lines intersecting \( \mathcal{M} \) in not \( k \) (mod \( p \)) points is \( \delta \), where
\[
\delta < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor).
\] Assume furthermore that Property 2.6 holds. Then there exists a multiset \( \mathcal{M}' \) with the property that it intersects every line in \( k \) (mod \( p \)) points and the number of different points in \( (\mathcal{M} \cup \mathcal{M}') \setminus (\mathcal{M} \cap \mathcal{M}') \) is exactly \( \left\lceil \frac{\delta}{q+1} \right\rceil \).

Result 2.8 ([17]). Let \( B \) be a proper point set in \( \mathcal{H}(2,q) \), \( 17 < q \). Suppose that \( B \) is a codeword of the lines of \( \mathcal{H}(2,q) \). Assume also that \( |B| < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor) \). Then \( B \) is a linear combination of at most \( \left\lceil \frac{|B|}{q+1} \right\rceil \) lines. \( \square \)

### 3 Proof of Theorem 1.1

Let \( V \) be the vector space of dimension \( q^{2r} + q^{2(r-1)} + \cdots + q^2 + 1 \) over the prime field \( \mathbb{F}_p \), where the coordinate positions for the vectors in \( V \) correspond to the points of \( \mathcal{H}(r,q^2) \) in some fixed order. If \( S \) is a subset of points in \( \mathcal{H}(r,q^2) \), then let \( v^S \) denote the vector in \( V \) with coordinate 1 in the positions corresponding to the points in \( S \) and with coordinate 0 in all other positions; that is \( v^S \) is the characteristic vector of \( S \). Let \( C_p \) denote the subspace of \( V \) spanned by the characteristic vectors of all the hyperplanes in \( \mathcal{H}(r,q^2) \). This code \( C_p \) is called the linear code of \( \mathcal{H}(r,q^2) \).

From [13, Theorem 1], we know that the characteristic vector \( v^V \) of a Hermitian variety \( V \in \mathcal{H}(r,q^2) \) is in \( C_p \). So from now on, we will assume that \( V \) is a quasi-Hermitian variety in \( \mathcal{H}(r,q^2) \) and \( v^V \in C_p \). In the remainder of this section, we will show that \( V \) is a classical Hermitian variety for specific values of \( q \).

The next lemmas hold for \( r \geq 3 \) and for any \( q = p^h \), \( p \) prime, \( h \geq 1 \).

**Lemma 3.1.** Every line of \( \mathcal{H}(r,q^2) \), \( q = p^h \), \( p \) prime, \( h \geq 1 \), meets \( V \) in 1 (mod \( p \)) points.

**Proof.** We may express
\[
v^V = v^{H_1} + \cdots + v^{H_t},
\]
where \( H_1, \ldots, H_t \) are (not necessarily distinct) hyperplanes of \( \mathcal{H}(r,q^2) \). Denote by \( \cdot \) the usual dot product. We get \( v^V \cdot v^V = |V| \equiv 1 \) (mod \( p \)). On the other hand,
\[
v^V \cdot v^V = v^V \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod{p},
\]
since every hyperplane of \( \mathcal{H}(r,q^2) \) meets \( V \) in 1 (mod \( p \)) points. Hence, we have \( t \equiv 1 \) (mod \( p \)). Finally, for a line \( \ell \) of \( \mathcal{H}(r,q^2) \),
\[
\ell \cdot v^V = \ell \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod{p},
\]
as every line of \( \mathcal{H}(r,q^2) \) meets a hyperplane in 1 or \( q^2 + 1 \) points. That is, \( |\ell \cap V| \equiv 1 \) (mod \( p \)) and in particular no lines of \( \mathcal{H}(r,q^2) \) are external to \( V \). \( \square \)
Remark 3.2. The preceding proof also shows that $V$ is a linear combination of $1 \pmod{p}$ (not necessarily distinct) hyperplanes, all having coefficient one.

Lemma 3.3. For every hyperplane $H$ of $H(r,q^2)$, $q = p^h$, $p$ prime, $h \geq 1$, the intersection $H \cap V$ is in the code of points and hyperplanes of $H$ itself.

Proof. Let $\Sigma$ denote the set of all hyperplanes of $H(r,q^2)$. By assumption, $v^V = \sum_{H \in \Sigma} \lambda_i v^{H_i}.$ \hspace{1cm} (1)

For every $H \in \Sigma$, let $\pi$ denote a hyperplane of $H$; then $\pi = H_{j_1} \cap \cdots \cap H_{j_{q^2+1}}$, where $H_{j_1}, \ldots, H_{j_{q^2+1}}$ are the hyperplanes of $H(r,q^2)$ through $\pi$. We assume $H = H_{j_{q^2+1}}$. For every hyperplane $\pi$ of $H$, we set \hspace{1cm} \lambda_{\pi} = \sum_{k=1}^{q^2+1} \lambda_{j_k},

where $\lambda_{j_k}$ is the coefficient in (1) of $v^{H_{j_k}}$ and $H_{j_k}$ is one of the $q^2 + 1$ hyperplanes through $\pi$.

Now, consider \hspace{1cm} T = \sum_{\pi \in \Sigma'} \lambda_{\pi} v^{\pi}, \hspace{1cm} (2)

where $\Sigma'$ is the set of all hyperplanes in $H$. We are going to show that $T = v^{V \cap H}$.

In fact, it is clear that at the positions belonging to the points outside of $H$ we see zeros. At a position belonging to a point in $H$, we see the original coefficients of $v^V$ plus $(|\Sigma'| - 1)\lambda_{j_{q^2+1}}$. Note that this last term is $0 \pmod{p}$, hence $T = v^{V \cap H}$. \qed

Corollary 3.4. For every subspace $S$ of $H(r,q^2)$, $q = p^h$, $p$ prime, $h \geq 1$, the intersection $S \cap V$ is in the code of points and hyperplanes of $S$ itself. \qed

Remark 3.5. Lemma 3.3 and Corollary 3.4 are valid for any set of points in $H(r,q^2)$ whose incidence vector belongs to the code of points and hyperplanes of $H(r,q^2)$. In particular, it follows that for every plane $\pi$ the intersection $\pi \cap V$ is a codeword of the points and lines of $\pi$, $\pi \cap V$ has size $1 \pmod{p}$ and so it is a linear combination of $1 \pmod{p}$ not necessarily distinct lines.

Lemma 3.6. Let $\ell$ be a line of $H(r,q^2)$. Then there exists at least one plane through $\ell$ meeting $V$ in $\delta$ points, with $\delta \leq q^3 + q^2 + q + 1$.

Proof. By way of contradiction, assume that all planes through $\ell$ meet $V$ in more than $q^3 + q^2 + q + 1$ points. Set $x = |\ell \cap V|$. We get \hspace{1cm} \frac{(q^{r+1} + (-1)^r)(q^r - (-1)^r)}{q^2 - 1} > m(q^3 + q^2 + q + 1 - x) + x, \hspace{1cm} (3)

where $m = q^{2(r-2)} + q^{2(r-3)} + \cdots + q^2 + 1$ is the number of planes in $H(r,q^2)$ through $\ell$. From (3), we obtain $x > q^2 + 1$, a contradiction. \qed
Lemma 3.7. For each line \( \ell \) of \( \mathcal{H}(r, q^2) \), \( q > 4 \) and \( q = p^h \), \( p \) odd prime, \( h \geq 1 \), either \( |\ell \cap \mathcal{V}| \leq q + 1 \) or \( |\ell \cap \mathcal{V}| \geq q^2 - q + 1 \).

Proof. Let \( \ell \) be a line of \( \mathcal{H}(r, q^2) \) and let \( \pi \) be a plane through \( \ell \) such that \( |\pi \cap \mathcal{V}| \leq q^3 + q^2 + q + 1 \); Lemma 3.6 shows that such a plane exists. Set \( B = \pi \cap \mathcal{V} \). By Corollary 3.4, \( B \) is a codeword of the code of the lines of \( \pi \), so we can write it as a linear combination of some lines of \( \pi \), that is \( \sum \lambda_i v_i^\ast \), where \( v_i^\ast \) are the characteristic vectors of the lines \( e_i \) in \( \pi \).

Let \( B^\ast \) be the multiset consisting of the lines \( e_i \), with multiplicity \( \lambda_i \), in the dual plane of \( \pi \). The weight of the codeword \( B \) is at most \( q^3 + q^2 + q + 1 \), hence in the dual plane this is the number of lines intersecting \( B^\ast \) in not 0 (mod \( p \)) points. Actually, as \( B \) is a proper set, we know that each non 0 (mod \( p \)) secant of \( B^\ast \) must be a 1 (mod \( p \)) secant. Using Result 2.5, with \( \delta = q^3 + q^2 + q + 1 \), in \( \mathcal{H}(2, q^2) \), the number of non 0 (mod \( p \)) secants through any point is at most

\[
\frac{\delta}{q^2 + 1} + 2 \frac{\delta^2}{(q^2 + 1)^3} = q + 1 + 2 \frac{(q + 1)^2}{q^2 + 1} < q + 4
\]

or at least

\[
q^2 + 1 - \left( \frac{\delta}{q^2 + 1} + 2 \frac{\delta^2}{(q^2 + 1)^3} \right) > q^2 - q - 3.
\]

In the original plane \( \pi \), this means that each line intersects \( B \) in either at most \( q + 3 \) or in at least \( q^2 - q - 2 \) points. Since such lines must be 1 (mod \( p \)) secants and \( p > 2 \), then each line intersects \( B \) in either at most \( q + 1 \) or in at least \( q^2 - q + 1 \) points. \( \square \)

Proposition 3.8. Assume that \( \pi \) is a plane of \( \mathcal{H}(r, q^2) \), \( q > 4 \), and \( q = p^h \), \( p \) odd prime, \( h \geq 1 \), such that \( |\pi \cap \mathcal{V}| \leq q^3 + 2q^2 \). Furthermore, suppose also that there exists a line \( \ell \) meeting \( \pi \cap \mathcal{V} \) in at least \( q^2 - q + 1 \) points, when \( q^3 + 1 \leq |\pi \cap \mathcal{V}| \). Then \( \pi \cap \mathcal{V} \) is a linear combination of at most \( q + 1 \) lines, each with weight 1.

Proof. Let \( B \) be the point set \( \pi \cap \mathcal{V} \). By Corollary 3.4, \( B \) is the corresponding point set of a codeword \( c \) of lines of \( \pi \), that is \( c = \sum \lambda_i v_i^\ast \), where lines of \( \pi \) are denoted by \( e_i \). Let \( C^\ast \) be the multiset in the dual plane containing the dual of each line \( e_i \) with multiplicity \( \lambda_i \). Clearly the number of lines intersecting \( C^\ast \) in not 0 (mod \( p \)) points is \( w(c) = |B| \). Note also, that every line that is not a 0 (mod \( p \)) secant is a 1 (mod \( p \)) secant, as \( B \) is a proper point set.

Our very first aim is to show that \( c \) is a linear combination of at most \( q + 3 \) different lines. When \( |B| < q^3 + 1 \), then, by Result 2.8, it is a linear combination of at most \( q \) different lines.

Next assume that \( |B| \geq q^3 + 1 \). From the assumption of the proposition, we know that there exists a line \( \ell \) meeting \( \pi \cap \mathcal{V} \) in at least \( q^2 - q + 1 \) points and from Lemma 3.7, we also know that each line intersects \( B \) in either at most \( q + 1 \) or in at least \( q^2 - q + 1 \) points. Hence, if we add the line \( \ell \) to \( c \) with multiplicity \(-1\), we reduce the weight by at least \( q^2 - q + 1 - q \) and by at most \( q^2 + 1 \). If \( w(c - v^\ell) < q^3 + 1 \), then from the above we know that \( c - v^\ell \) is a linear combination of \( \left\lceil \frac{w(c - v^\ell)}{q^2 + 1} \right\rceil \) lines. Hence, \( c \) is a linear
combination of at most \( q + 1 \) lines. If \( w(c - v^f) \geq q^3 + 1 \), then \( w(c) \geq q^3 + q^2 - 2q - 2 \) (see above) and so it follows that through any point of \( B \), there passes at least one line intersecting \( B \) in at least \( q^2 - q + 1 \) points. This means that we easily find three lines \( \ell_1 \), \( \ell_2 \), and \( \ell_3 \) intersecting \( B \) in at least \( q^2 - q + 1 \) points. Since \( w(c) \leq q^3 + 2q^2 \), we get that \( w(c - v^{f+1} - v^{f+2} - v^{f+3}) \leq q^3 + 2q^2 - 3 \cdot (q^2 - 2q - 2) < q^3 + 1 \). Hence, similarly as before, we get that \( c \) is a linear combination of at most \( q + 3 \) lines.

Next we show that each line in the linear combination (that constructs \( c \)) has weight 1. Take a line \( \ell \) which is in the linear combination with coefficient \( \lambda \neq 0 \). Then there are at least \( q^2 + 1 - (q + 2) \) positions, such that the corresponding point is in \( \ell \) and the value at that position is \( \lambda \). As \( B \) is a proper set, this yields that \( \lambda = 1 \). By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of \( c \) must be 1 (mod \( p \)); hence it can be at most \( q + 1 \).

\( \square \)

**Proposition 3.9.** Assume that \( \pi \) is a plane of \( \mathcal{H}(r,q^2) \), \( q > 4 \), and \( q = p^h \), \( p \) odd prime, \( h \geq 1 \), such that \( |\pi \cap V| \leq q^3 + 2q^2 \). Furthermore, suppose that every line meets \( \pi \cap V \) in at most \( q + 1 \) points. Then \( \pi \cap V \) is a classical unital.

**Proof.** Again let \( B = \pi \cap V \) and first assume that \( |B| < q^3 + 1 \). Proposition 3.8 shows that \( B \) is a linear combination of at most \( q + 1 \) lines, each with weight 1. But this yields that these lines intersect \( B \) in at least \( q^2 + 1 - q \) points. So this case cannot occur.

Hence, \( q^3 + 1 \leq |B| \leq q^3 + 2q^2 \). We are going to prove that there exists at least a tangent line to \( B \) in \( \pi \). Let \( t_i \) be the number of lines meeting \( B \) in \( i \) points. Set \( x = |B| \). Then double counting arguments give the following equations for the integers \( t_i \).

\[
\begin{align*}
\sum_{i=1}^{q+1} t_i &= q^4 + q^2 + 1 \\
\sum_{i=1}^{q+1} it_i &= x(q^2 + 1) \\
\sum_{i=1}^{q+1} i(i-1)t_i &= x(x - 1).
\end{align*}
\]

(4)

Consider \( f(x) = \sum_{i=1}^{q+1} (i - 2)(q + 1 - i)t_i \). From (4), we get

\[
f(x) = -x^2 + x[(q^2 + 1)(q + 2) + 1] - 2(q + 1)(q^4 + q^2 + 1).
\]

Since \( f(q^3/2) > 0 \), whereas \( f(q^3 + 1) < 0 \) and \( f(q^3 + 2q^2) < 0 \), it follows that if \( q^3 + 1 \leq x \leq q^3 + 2q^2 \), then \( f(x) < 0 \) and thus \( t_1 \) must be different from zero. Consider now the quantity \( \sum_{i=1}^{q+1} (i - 1)(q + 1 - i)t_i \). We have that

\[
\begin{align*}
\sum_{i=1}^{q+1} (i - 1)(q + 1 - i)t_i &= f(x) + \sum_{i=1}^{q+1} (q + 1 - i)t_i = f(x) + (q + 1) \sum_{i=1}^{q+1} t_i - \sum_{i=1}^{q+1} it_i \\
&= f(x) + (q + 1)(q^4 + q^2 + 1) - x(q^2 + 1) = -x^2 + x[(q^2 + 1)(q + 1) + 1] - (q + 1)(q^4 + q^2 + 1).
\end{align*}
\]

Since \( \sum_{i=1}^{q+1} (i - 1)(q + 1 - i)t_i \geq 0 \), we have that \( x \leq \frac{(q^2 + 1)(q + 1) + 1 + (q^4 - q^2 - q)}{2} = q^3 + 1 \). Therefore, \( x = q^3 + 1 \) and \( \sum_{i=1}^{q+1} (i - 1)(q + 1 - i)t_i = 0 \).
Since \((i - 1)(q + 1 - i) > 0\), for \(2 \leq i \leq q\), we obtain \(t_2 = t_3 = \cdots = t_q = 0\), that is, \(B\) is a set of \(q^3 + 1\) points such that each line is a 1-secant or a \((q + 1)\)-secant of \(B\). Namely, \(B\) is a unital and precisely a classical unital since \(B\) is a codeword of \(\pi\) [6].

The above two propositions and Lemma 3.7 imply the following corollary.

**Corollary 3.10.** Assume that \(\pi\) is a plane of \(H(r,q^2), q > 4\) and \(q = p^h, p\) odd prime, \(h \geq 1\), such that \(|\pi \cap V| \leq q^3 + 2q^2\). Then \(\pi \cap V\) is a linear combination of at most \(q + 1\) lines, each with weight 1, or it is a classical unital.

**Corollary 3.11.** Suppose that \(\pi\) is a plane of \(H(r,q^2), q > 4\) and \(q = p^h, p\) odd prime, \(h \geq 1\), containing exactly \(q^3 + 1\) points of \(V\). Then \(\pi \cap V\) is a classical unital.

**Proof.** Let \(B\) be the point set \(\pi \cap V\). We know that \(B\) is the support of a codeword of lines of \(\pi\). By Proposition 3.8, if there is a line intersecting \(B\) in at least \(q^2 - q + 1\) points, then \(B\) is a linear combination of at most \(q + 1\) lines, each with multiplicity 1. First of all note that a codeword that is a linear combination of \(q + 1\) lines has weight at least \((q^2 + 1)(q + 1) - 2\binom{q^2 + 1}{2}\), that is exactly \(q^3 + 1\). In fact, in a linear combination of \(q + 1\) lines the minimum number of points is obtained if there is a hole at the intersection of any two lines. There are \(\binom{q^2 + 1}{2}\) intersections and each intersection is counted twice, therefore we have to subtract \(2\binom{q^2 + 1}{2}\). To achieve this, we need that the intersection points of any two lines from such a linear combination are all different and the sum of the coefficients of any two lines is zero; which is clearly not the case (as all the coefficients are 1). From Remark 3.5, in this case \(B\) would be a linear combination of at most \(q + 1 - p\) lines and so its weight would be less than \(q^3 + 1\), a contradiction. Hence, there is no line intersecting \(B\) in at least \(q^2 - q + 1\) points, so Proposition 3.9 finishes the proof.

### 3.1 Case \(r = 3\)

In \(H(3,q^2)\), each plane intersects \(V\) in either \(q^3 + 1\) or \(q^3 + q^2 + 1\) points since these are the intersection numbers of a quasi-Hermitian variety with a plane of \(H(3,q^2)\).

#### 3.1.1 \(q = p\)

Let \(V\) be a quasi-Hermitian variety of \(H(3,p^2)\), \(p\) prime.

**Lemma 3.12.** Every plane \(\pi\) of \(H(3,p^2)\) sharing \(p^3 + 1\) points with \(V\) intersects \(V\) in a unital of \(\pi\).

**Proof.** Set \(U = \pi \cap V\). Let \(P\) be a point in \(U\). Assume that every line \(\ell\) in \(\pi\) through the point \(P\) meets \(U\) in at least \(p + 1\) points. We get \(|\pi \cap V| = p^3 + 1 \geq (p^2 + 1)p + 1 = p^3 + p + 1\), which is impossible.

Thus, \(P\) lies on at least one tangent line to \(U\) and this implies that \(U\) is a minimal blocking set in \(\pi\) of size \(p^3 + 1\). From a result obtained by Bruen and Thas, see [7], it follows that \(U\) is a unital of \(\pi\) and hence every line in \(\pi\) meets \(U\) in either 1 or \(p + 1\) points.
**Lemma 3.13.** Let \( \pi \) be a plane in \( \mathcal{H}(3, p^2) \) such that \( |\pi \cap \mathcal{V}| = p^3 + p^2 + 1 \), then every line in \( \pi \) meets \( \pi \cap \mathcal{V} \) in either 1 or \( p + 1 \) or \( p^2 + 1 \) points.

*Proof.* Set \( C = \pi \cap \mathcal{V} \) and let \( m \) be a line in \( \pi \) such that \( |m \cap C| = s \) with \( s \neq 1 \) and \( s \neq p + 1 \). Thus, from Lemma 3.12, every plane through \( m \) has to meet \( \mathcal{V} \) in \( p^3 + p^2 + 1 \) points and thus

\[
|\mathcal{V}| = (p^2 + 1)(p^3 + p^2 + 1 - s) + s,
\]

which gives \( s = p^2 + 1 \).

From Lemmas 3.12 and 3.13, it follows that every line in \( \mathcal{H}(3, p^2) \) meets \( \mathcal{V} \) in either 1 or \( p + 1 \) or \( p^2 + 1 \) points.

### 3.1.2 \( q = p^h, \ q \geq 5 \text{ odd} \)

Let \( \mathcal{V} \) be a quasi-Hermitian variety of \( \mathcal{H}(3, q^2) \), \( q \geq 5 \text{ odd} \).

**Lemma 3.14.** Let \( \pi \) be a plane in \( \mathcal{H}(3, q^2) \) such that \( |\pi \cap \mathcal{V}| = q^3 + q^2 + 1 \), then every line in \( \pi \) meets \( \pi \cap \mathcal{V} \) in either 1, \( q + 1 \) or \( q^2 + 1 \) points.

*Proof.* Set \( C = \pi \cap \mathcal{V} \) and let \( m \) be a line in \( \pi \) such that \( |m \cap C| = s \), with \( s \neq 1 \) and \( s \neq q + 1 \). Thus, from Corollary 3.11, every plane through \( m \) has to meet \( \mathcal{V} \) in \( q^3 + q^2 + 1 \) points and thus

\[
|\mathcal{V}| = (q^2 + 1)(q^3 + q^2 + 1 - s) + s,
\]

which gives \( s = q^2 + 1 \).

From Corollary 3.11 and Lemma 3.14, it follows that every line in \( \mathcal{H}(3, q^2) \) meets \( \mathcal{V} \) in either 1, \( q + 1 \), or \( q^2 + 1 \) points.

### 3.1.3 \( q = 2^h, \ h > 2 \)

Let \( \mathcal{V} \) be a quasi-Hermitian variety of \( \mathcal{H}(3, 2^{2h}) \), \( h > 2 \).

**Lemma 3.15.** For each line \( \ell \) of \( \mathcal{H}(3, 2^{2h}) \), \( h > 2 \), either \( |\ell \cap \mathcal{V}| \leq q + 1 \) or \( |\ell \cap \mathcal{V}| \geq q^2 - q - 1 \).

*Proof.* Let \( \ell \) be a line of \( \mathcal{H}(3, 2^{2h}) \). Since \( \ell \) is at least a tangent to \( \mathcal{V} \), there exists a plane through \( \ell \) meeting \( \mathcal{V} \) in \( q^3 + q^2 + 1 \) points. Let \( \pi \) be a plane through \( \ell \) such that \( |\pi \cap \mathcal{V}| = q^3 + q^2 + 1 \). Set \( B = \pi \cap \mathcal{V} \). As before, by Corollary 3.4, \( B \) is a codeword of the code of the lines of \( \pi \), so we can write it as a linear combination of some lines of \( \pi \), that is \( \sum \lambda_i v^{e_i} \), where \( v^{e_i} \) are the characteristic vectors of the lines \( e_i \) in \( \pi \).

Let \( B^* \) be the multiset consisting of the lines \( e_i \), with multiplicity \( \lambda_i \), in the dual plane of \( \pi \). The weight of the codeword \( B \) is \( q^3 + q^2 + 1 \), hence in the dual plane this is the number of lines intersecting \( B^* \) in not 0 \((\text{mod} \ p)\) points. Actually, as \( B \) is a proper set, we know that each non 0 \((\text{mod} \ p)\) secant of \( B^* \) must be a 1 \((\text{mod} \ p)\) secant. Using Result 2.5, with \( \delta = q^3 + q^2 + 1 \) in \( \mathcal{H}(2, 2^{2h}) \), the number \( s \) of non 0 \((\text{mod} \ p)\) secants through any...
point of $B$ satisfies the inequality $s^2 - (q^2 + 1)s - (q^3 + q^2 + 1)\geq 0$. Since the determinant, $(q^3 + 1)^2 + 4(q^3 + q^2 + 1) > ((q^2 + 1) - 2(q + 3))^2$, we get $s < q + 3$ or $s > q^2 - q - 2$

In the original plane $\pi$, this means that each line intersects $B$ in either at most $q + 2$ or in at least $q^2 - q - 1$ points. Since such lines must be $1$ (mod $p$) secants and $p = 2$, then each line intersects $B$ in either at most $q + 1$ or in at least $q^2 - q - 1$ points.

Let $\alpha$ be a plane meeting $\mathcal{V}$ in a point set $B'$ of size $q^3 + q^2 + 1$ points. We want to prove that $\alpha$ contains some $s$-secant, with $s$ at least $q^2 - q - 1$. Assume on the contrary that each line in $\alpha$ meets $\mathcal{V}$ in at most $q + 1$ points. Let $P$ be a point of $B'$ and consider the $q^2 + 1$ lines through $P$. We get $q^3 + q^2 + 1 \leq (q^2 + 1)q + 1$, a contradiction. Therefore, there exists a line $\ell$ in $\alpha$ meeting $B'$ in at least $q^2 - q - 1$ points. We are going to show that $B'$ is a linear combination of exactly $q + 1$ lines each with weight 1.

Again, by Corollary 3.4, $B'$ is the corresponding point set of a codeword $c'$ of lines of $\pi$, that is $c' = \sum_{i} \lambda_i v^{e_i}$, where lines of $\pi$ are denoted by $e_i$. Let $C''$ be the multiset in the dual plane containing the dual of each line $e_i$ with multiplicity $\lambda_i$. As before, the number of lines intersecting $C''$ in not 0 (mod $p$) points is $w(c') = |B'| = q^3 + q^2 + 1$ and every line that is not a 0 (mod $p$) secant is a 1 (mod $p$) secant, as $B'$ is a proper point set.

Hence, if we add the line $\ell$ to $c$ with multiplicity 1, we reduce the weight by at least $q^2 - q - 1 - q - 2 = q^2 - 2q - 3$ and at most by $q^2 + 1$. Now, through any point of $B'$, there passes at least one line intersecting $B'$ in at least $q^2 - q - 1$ points. Thus, we easily find two lines $\ell_1$ and $\ell_2$ intersecting $B'$ in at least $q^2 - q - 1$ points. We get that $w(c' - v^{e_1} - v^{e_2}) < q^3 + 1$. Hence, similarly as before, we get that $c' - v^{e_1} - v^{e_2}$ is a linear combination of $[w(c' - v^{e_1} - v^{e_2})]$ lines. Hence, $c'$ is a linear combination of at most $q + 2$ lines. By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of $c'$ must be 1 (mod $p$); hence, as $p = 2$ it can be at most $q + 1$. For $p = 2$, a codeword that is a linear combination of at most $q + 1$ lines each with weight 1, has weight at most $q^3 + q^2 + 1$ and this is achieved when the $q + 1$ lines are concurrent. This implies that each line in $\alpha$ is either a 1 or $q + 1$ or $q^2 + 1$-secant to $\mathcal{V}$. Now consider a line $m'$ that is an $s$-secant to $\mathcal{V}$ with different from 1, $q + 1$, and also different from $q^2 + 1$. Each plane through $m'$ has to meet $\mathcal{V}$ in $q^3 + 1$ points. From $|\mathcal{V}| = q^3 + q^2 + q^2 + 1 = (q^2 + 1)(q^3 + 1 - s) + s$ we get $s = 0$, a contradiction.

Thus each line of $\mathcal{H}(3, 2^h)$ meets $\mathcal{V}$ in either 1 or $q + 1$ or $q^2 + 1$ points.

Proof of Theorem 1.1 (case $r = 3$): From all previous lemmas of this section, it follows that every line in $\mathcal{H}(3, q^2)$, with $q = p^h \neq 4$ and $p$ any prime, meets $\mathcal{V}$ in either 1, $q + 1$, or $q^2 + 1$ points. Now, suppose on the contrary that there exists a singular point $P$ on $\mathcal{V}$; this means that all lines through $P$ are either tangents or $(q^2 + 1)$-secants to $\mathcal{V}$. Take a plane $\pi$ which does not contain $P$. Then $|\mathcal{V}| = q^2 |\pi \cap \mathcal{V}| + 1$ and since the two possible sizes of the planar sections are $q^3 + 1$ or $q^2 + q^2 + 1$, we get a contradiction. Thus, every point in $\mathcal{V}$ lies on at least one $(q + 1)$-secant and, from Theorem 2.1, we obtain that $\mathcal{V}$ is a Hermitian surface. □
3.2 Case \( r \geq 4 \) and \( q = p \geq 5 \)

We first prove the following result.

**Lemma 3.16.** If \( \pi \) is a plane of \( \mathcal{H}(r, p^2) \), which is not contained in \( \mathcal{V} \), then either

\[
|\pi \cap \mathcal{V}| = p^2 + 1 \text{ or } |\pi \cap \mathcal{V}| \geq p^3 + 1.
\]

**Proof.** Let \( \pi \) be a plane of \( \mathcal{H}(r, p^2) \) and set \( B = \pi \cap \mathcal{V} \). By Remark 3.5, \( B \) is a linear combination of \( 1 \) (mod \( p \)) not necessarily distinct lines.

If \( |B| < p^3 + 1 \), then by Result 2.8, \( B \) is a linear combination of at most \( p \) distinct lines. This and the previous observation yield that when \( |B| < p^3 + 1 \), then it is the scalar multiple of one line; hence \( |B| = p^2 + 1 \).

**Proposition 3.17.** Let \( \pi \) be a plane of \( \mathcal{H}(r, p^2) \), such that \( |\pi \cap \mathcal{V}| \leq p^3 + p^2 + p + 1 \). Then \( B = \pi \cap \mathcal{V} \) is either a classical unital or a linear combination of \( p + 1 \) concurrent lines or just one line, each with weight 1.

**Proof.** From Corollary 3.10, we have that \( B \) is either a linear combination of at most \( p + 1 \) lines or a classical unital. In the first case, since \( B \) intersects every line in \( 1 \) (mod \( p \)) points and \( B \) is a proper point set, the only possibilities are that \( B \) is a linear combination of \( p + 1 \) concurrent lines or just one line, each with weight 1.

**Proof of Theorem 1.1 (case \( r \geq 4 \), \( q = p \)):** Consider a line \( \ell \) of \( \mathcal{H}(r, p^2) \) which is not contained in \( \mathcal{V} \). By Lemma 3.6, there is a plane \( \pi \) through \( \ell \) such that \( |\pi \cap \mathcal{V}| \leq q^3 + q^2 + q + 1 \). From Proposition 3.17, we have that \( \ell \) is either a unisecant or a \((p + 1)\)-secant of \( \mathcal{V} \) and we also have that \( \mathcal{V} \) has no plane section of size \((p + 1)(p^2 + 1)\). Finally, it is easy to see like in the previous case \( r = 3 \), that \( \mathcal{V} \) has no singular points, thus \( \mathcal{V} \) turns out to be a Hermitian variety of \( \mathcal{H}(r, p^2) \) (Theorem 2.2).

3.3 Case \( r \geq 4 \) and \( q = p^2 \), \( p \) odd

Assume now that \( \mathcal{V} \) is a quasi-Hermitian variety of \( \mathcal{H}(r, p^4) \), with \( r \geq 4 \).

Lemma 3.7 states that every line contains at most \( p^2 + 1 \) points of \( \mathcal{V} \) or at least \( p^4 - p^2 + 1 \) points of \( \mathcal{V} \).

**Lemma 3.18.** If \( \ell \) is a line of \( \mathcal{H}(r, p^4) \), such that \( |\ell \cap \mathcal{V}| \geq p^4 - p^2 + 1 \), then \( |\ell \cap \mathcal{V}| \geq p^4 - p + 1 \).

**Proof.** Set \( |\ell \cap \mathcal{V}| = p^4 - x + 1 \), where \( x \leq p^2 \). It suffices to prove that \( x < p + 2 \). Let \( \pi \) be a plane through \( \ell \) and \( B = \pi \cap \mathcal{V} \). Choose \( \pi \) such that \( |B| = |\pi \cap \mathcal{V}| \leq p^6 + p^4 + p^2 + 1 \) (Lemma 3.6). Then, by Proposition 3.8, \( B \) is a linear combination of at most \( p^2 + 1 \) lines, each with weight 1. Let \( c \) be the codeword corresponding to \( B \). We observe that \( \ell \) must be one of the lines of \( c \), otherwise \( |B \cap \ell| \leq p^2 + 1 \), which is impossible. Thus if \( P \) is a point in \( \ell \setminus B \), then through \( P \) there pass at least \( p - 1 \) other lines of \( c \). If \( x \geq p + 2 \), then the number of lines necessary to define the codeword \( c \) would be at least \((p + 2)(p - 1) + 1\), a contradiction.
Lemma 3.19. For each plane $\pi$ of $H(r, p^4)$, either $|\pi \cap V| \leq p^6 + 2p^4 - p^2 - p + 1$ or $|\pi \cap V| \geq p^8 - p^5 + p^4 - p + 1$.

Proof. Let $B = \pi \cap V$, $x = |B|$, and let $t_i$ be the number of lines in $\pi$ meeting $B$ in $i$ points. Then, in this case, Equations (4) read

$$\begin{align*}
\sum_{i=1}^{p^4+1} t_i &= p^8 + p^4 + 1 \\
\sum_{i=1}^{p^4+1} it_i &= x(p^4 + 1) \\
\sum_{i=1}^{p^4+1} i(i-1)t_i &= x(x-1). 
\end{align*}$$

Set $f(x) = \sum_{i=1}^{p^4+1} (p^2 + 1 - i)(i - (p^4 - p + 1))t_i$. From (5) we obtain

$$f(x) = -x^2 + [(p^4 + 1)(p^4 + p^2 - p + 1) + 1]x - (p^8 + p^4 + 1)(p^2 + 1)(p^4 - p + 1).$$

Because of Lemma 3.18, we get $f(x) \leq 0$, while $f(p^6 + 2p^4 - p^2 + 1) > 0$, $f(p^8 - p^5 + p^4 - p) > 0$. This finishes the proof of the lemma.

Lemma 3.20. If $\pi$ is a plane of $H(r, p^4)$, such that $|\pi \cap V| \geq p^8 - p^5 + p^4 - p + 1$, then either $\pi$ is entirely contained in $V$ or $\pi \cap V$ consists of $p^8 - p^5 + p^4 + 1$ points and it only contains $i$-secants, with $i \in \{1, p^4 - p + 1, p^4 + 1\}$. 

Proof. Set $S = \pi \setminus V$. Suppose that there exists some point $P \in S$. We have the following two possibilities: either each line of the pencil with center at $P$ is a $(p^4 - p + 1)$-secant or only one line through $P$ is an $i$-secant, with $1 \leq i \leq p^2 + 1$, whereas the other $p^4$ lines through $P$ are $(p^4 - p + 1)$-secants. In the former case, when there are no $i$-secants, $1 \leq i \leq p^2 + 1$, each line $\ell$ in $\pi$ either is disjoint from $S$ or it meets $S$ in $p$ points since $\ell$ is a $(p^4 - p + 1)$-secant. This implies that $S$ is a maximal arc and this is impossible for $p \neq 2$ [4, 5].

In the latter case, we observe that the size of $\pi \cap V$ must be $p^8 - p^5 + p^4 + i$, where $1 \leq i \leq p^2 + 1$. Next, we denote by $t_s$ the number of $s$-secants in $\pi$, where $s \in \{i, p^4 - p + 1, p^4 + 1\}$. We have that

$$\begin{align*}
\sum_s t_s &= p^8 + p^4 + 1 \\
\sum_s st_s &= (p^4 + 1)(p^8 - p^5 + p^4 + i) \\
\sum_s s(s-1)t_s &= (p^8 - p^5 + p^4 + i)(p^8 - p^5 + p^4 + i - 1). 
\end{align*}$$

From (6) we get

$$t_i = \frac{p(p^4 - p - i + 1)(p^5 - i + 1)}{p(p^4 - p - i + 1)(p^4 - i + 1)} = \frac{p^5 - i + 1}{p^4 - i + 1},$$

and we can see that the only possibility for $t_i$ to be an integer is $ip - p - i + 1 = 0$, that is $i = 1$. For $i = 1$, we get $|B| = p^8 - p^5 + p^4 + 1$. 

\[\square\]
Lemma 3.21. If π is a plane of \(\mathcal{H}(r, p^4)\), not contained in \(\mathcal{V}\) and which does not contain any \((p^4 - p + 1)\)-secant, then \(\pi \cap \mathcal{V}\) is either a classical unital or the union of \(i\) concurrent lines, with \(1 \leq i \leq p^2 + 1\).

Proof. Because of Lemmas 3.18, 3.19 and 3.20, the plane \(\pi\) meets \(\mathcal{V}\) in at most \(p^6 + 2p^4 - p^2 - p + 1\) points. Furthermore, each line of \(\pi\) which is not contained in \(\mathcal{V}\) is an \(i\)-secant, with \(1 \leq i \leq p^2 + 1\) (Lemma 3.18 and the sentence preceding Lemma 3.18). Set \(B = \pi \cap \mathcal{V}\). If in \(\pi\) there are no \((p^4 + 1)\)-secants to \(B\), then \(|B| \leq p^6 + p^2 + 1\) and by Proposition 3.9 it follows that \(B\) is a classical unital.

If there is a \((p^4 + 1)\)-secant to \(B\) in \(\pi\), then arguing as in the proof of Proposition 3.8, we get that \(B\) is still a linear combination of \(m\) lines, with \(m \leq p^2 + 1\). Each of these \(m\) lines is a \((p^4 + 1)\)-secant to \(\mathcal{V}\). In fact if one of these lines, say \(v\), was an \(s\)-secant, with \(1 \leq s \leq p^2 + 1\), then through each point \(P \in v \setminus B\), there would pass at least \(p\) lines of the codeword corresponding to \(B\) and hence \(B\) would be a linear combination of at least \((p^4 + 1 - s)(p - 1) + 1 > p^2 + 1\) lines, which is impossible.

We are going to prove that these \(m\) lines, say \(\ell_1, \ldots, \ell_m\), are concurrent. Assume on the contrary that they are not. We can assume that through a point \(P \in \ell_m\), there pass at least \(p + 1\) lines of our codeword but there is a line \(\ell_j\) which does not pass through \(P\). Thus through at least \(p + 1\) points on \(\ell_j\), there are at least \(p + 1\) lines of our codeword and thus we find at least \((p + 1)p + 1 > m\) lines of \(B\), a contradiction. \(\square\)

Lemma 3.22. A plane \(\pi\) of \(\mathcal{H}(r, p^4)\) meeting \(\mathcal{V}\) in at most \(p^6 + 2p^4 - p^2 - p + 1\) points and containing a \((p^4 - p + 1)\)-secant to \(\mathcal{V}\) has at most \((p^2 + 1)(p^4 - p + 1)\) points of \(\mathcal{V}\).

Proof. Let \(\ell\) be a line of \(\pi\) which is a \((p^4 - p + 1)\)-secant to \(\mathcal{V}\). In this case, \(\pi \cap \mathcal{V}\) is a linear combination of at most \(p^2 + 1\) lines, each with weight 1 (Proposition 3.8). A line not in the codeword can contain at most \(p^2 + 1\) points. In particular, since \(\ell\) contains more than \(p^4 - p^2 + 1\) points of \(\mathcal{V}\), \(\ell\) is a line of the codeword and hence through each of the missing points of \(\ell\) there are at least \(p\) lines of the codeword corresponding to \(B\). On these \(p\) lines we can see at most \(p^4 - p + 1\) points of \(\mathcal{V}\).

So let \(\ell_1, \ell_2, \ldots, \ell_p\) be \(p\) lines of the codeword through a point of \(\ell \setminus \mathcal{V}\). Each of these lines contains at most \(p^4 - p + 1\) points of \(\mathcal{V}\). Thus these \(p\) lines contain together at most \(p(p^4 - p + 1)\) points of \(\mathcal{V}\). Now take any other line of the codeword, say \(e\). If \(e\) goes through the common point of the lines \(\ell_i\), then there is already one point missing from \(e\), so adding \(e\) to our set, we can add at most \(p^4 - p + 1\) points. If \(e\) does not go through the common point, then it intersects \(\ell_i\) in \(p\) different points. These points either do not belong to the set \(\pi \cap \mathcal{V}\) or they belong to the set \(\pi \cap \mathcal{V}\), but we have already counted them when we counted the points of \(\ell_i\), so again \(e\) can add at most \(p^4 - p + 1\) points to the set \(\pi \cap \mathcal{V}\). Thus adding the lines of the codewords one by one to \(\ell_i\) and counting the number of points, each time we add only at most \(p^4 - p + 1\) points to the set \(\pi \cap \mathcal{V}\). Hence, the plane \(\pi\) contains at most \((p^2 + 1)(p^4 - p + 1)\) points of \(\mathcal{V}\). \(\square\)

Lemma 3.23. Let \(\pi\) be a plane of \(\mathcal{H}(r, p^4)\), containing an \(i\)-secant, \(1 < i < p^2 + 1\), to \(\mathcal{V}\). Then \(\pi \cap \mathcal{V}\) is either the union of \(i\) concurrent lines or it is a linear combination of \(p^2 + 1\) lines (each with weight 1) so that they form a subplane of order \(p\), minus \(p\) concurrent lines.
Proof. By Lemmas 3.19, 3.20 and 3.21, $\pi$ meets $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points and must contain a $(p^4 - p + 1)$-secant to $\mathcal{V}$ or $\pi \cap \mathcal{V}$ is the union of $i$ concurrent lines. Hence, from now on, we assume that $\pi$ contains a $(p^4 - p + 1)$-secant. By Result 2.8, such a plane is a linear combination of at most $p^2 + 1$ lines. As before each line from the linear combination has weight 1. Note that the above two statements imply that a line of the linear combination will be either a $(p^4 + 1)$-secant or a $(p^4 - p + 1)$-secant.

As we have at most $p^2 + 1$ lines in the combination, a $(p^4 - p + 1)$-secant must be one of these lines. This also means that through each of the $p$ missing points of this line, there must pass at least $p - 1$ other lines from the linear combination. Hence, we already get $(p - 1)p + 1$ lines.

In the case in which the linear combination contains exactly $p^2 - p + 1$ lines, then from each of these lines there are exactly $p$ points missing and through each missing point there are exactly $p$ lines from the linear combination. Hence, the missing points and these lines form a projective plane of order $p - 1$, a contradiction as $p > 3$.

Therefore, as the number of the lines of the linear combination must be 1 (mod $p$) and at most $p^2 + 1$, we can assume that the linear combination contains $p^2 + 1$ lines. We are going to prove that through each point of the plane there pass either 0, 1, $p$ or $p + 1$ lines from the linear combination. From earlier arguments, we know that the number of lines through one point $P$ is 0 or 1 (mod $p$). Assume to the contrary that through $P$ there pass at least $p + 2$ of such lines. These $p^2 + 1$ lines forming the linear combination are not concurrent, so there is a line $\ell$ not through $P$. Through each of the intersection points of $\ell$ and a line through $P$, there pass at least $p - 1$ more other lines of the linear combination, so in total we get at least $(p - 1)(p + 2) + 1$ lines forming the linear combination, a contradiction.

Since there are $p^2 + 1$ lines forming the linear combination and through each point of the plane there pass either 0, 1, $p$ or $p + 1$ of these lines, we obtain that on a $(p^4 - p + 1)$-secant there is exactly one point, say $P$, through which there pass exactly $p + 1$ lines from the linear combination and $p$ points, not in the quasi Hermitian variety, through each of which there pass exactly $p$ lines.

If all the $p^2 + 1$ lines forming the linear combination, were $(p^4 - p + 1)$-secants then the number of points through which there pass exactly $p$ lines would be $(p^2 + 1)p/p$. On the other hand, through $P$ there pass $p + 1$ $(p^4 - p + 1)$-secants, hence we already get $(p + 1)p$ such points, a contradiction. Thus, there exists a line $m$ of the linear combination that is a $(p^4 + 1)$-secant. From the above arguments, on this line there are exactly $p$ points through each of which there pass exactly $p + 1$ lines, whereas through the rest of the points of the line $m$ there pass no other lines of the linear combination.

Assume that there is a line $m' \neq m$ of the linear combination that is also a $(p^4 + 1)$-secant. Then there is a point $Q$ on $m'$ but not on $m$ through which there pass $p + 1$ lines. This would mean that there are at least $p + 1$ points on $m$, through which there pass more than 2 lines of the linear combination, a contradiction.

Hence, there is exactly one line $m$ of the linear combination that is a $(p^4 + 1)$-secant and all the other lines of the linear combination are $(p^4 - p + 1)$-secants. It is easy to check that the points through which there are more than 2 lines plus the $(p^4 - p + 1)$-secants
form a dual affine plane. Hence our lemma follows. □

**Lemma 3.24.** There are no $i$-secants to $V$, with $1 < i < p^2 + 1$.

**Proof.** By Lemma 3.23, if a plane $\pi$ contains an $i$-secant, $1 < i < p^2 + 1$, then $\pi \cap V$ is a linear combination of either $i$ concurrent lines or lines of an embedded subplane of order $p$ minus $p$ concurrent lines. In the latter case, if $i > 1$ then an $i$-secant is at least a $(p^2 - p + 1)$-secant. In fact, since the $p^2 + 1$ lines of the linear combination do not intersect outside of $H(2, p)$, each of the $p^2 + 1$ lines of the linear combination is at least a $(p^4 - p + 1)$-secant, whereas the $p$ concurrent lines are 1-secants. Also, all the other lines of $H(2, p^4)$ intersect $H(2, p)$ in either 1 or zero points. If they intersect $H(2, p)$ in zero points they are $(p^2 + 1)$-secants to $V$. If they intersect $H(2, p)$ in a unique point $P$, then they are at least $(p^2 - p + 1)$-secants since $P$ lies on $p$ or $p + 1$ lines of the linear combination and each line intersects $V$ in 1 $(\mod p)$ points.

Hence, if there is an $i$-secant with $1 < i < p^2 - p + 1$, say $\ell$, we get that for each plane $\alpha$ through $\ell$, $\alpha \cap V$ is a linear combination of $i$ concurrent lines. Therefore

$$|V| = m(ip^4 + 1 - i) + i,$$

where $m = p^4(r - 2) + p^4(r - 3) + \ldots + p^4 + 1$ is the number of planes in $H(r, p^4)$ through $\ell$.

Setting $r = 2\sigma + \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$ according to $r$ is even or odd, we can write

$$|V| = 1 + p^4 + \ldots + p^{4(r - \sigma - 1)} + (p^{4(r - \sigma - \epsilon)} + p^{4((r - \sigma - \epsilon) + 1)} + \ldots + p^{4(r - 1)})p^2$$

Hence, (8) becomes

$$1 + p^4 + \ldots + p^{4(r - \sigma - 1)} + (p^{4(r - \sigma - \epsilon)} + p^{4((r - \sigma - \epsilon) + 1)} + \ldots + p^{4(r - 1)})p^2
-(p^{4(r - 2)} + p^{4(r - 3)} + \ldots + p^4 + 1) = ip^{4(r - 1)}.$$  (9)

Since $\sigma \geq 2$, we see that $p^{4(r - 1)}$ does not divide the left hand side of (9), a contradiction.

Thus, there can only be 1-, $(p^2 - p + 1)$-, $(p^2 + 1)$-, $(p^4 - p + 1)$- or $(p^4 + 1)$-secants to $V$. Now, suppose that $\ell$ is a $(p^2 - p + 1)$-secant to $V$. Again by Lemma 3.23, each plane through $\ell$ either has $x = (p^2 - p + 1)p^4 + 1$ or $y = p^2(p^4 - p) + p^4 + 1$ points of $V$. Next, denote by $t_j$ the number of $j$-secant planes through $\ell$ to $V$. We get

$$\begin{cases} 
t_x + t_y = m \\
t_x(x - p^2 + p - 1) + t_y(y - p^2 + p - 1) + p^2 - p + 1 = |V|. \end{cases}$$

(10)

Recover the value of $t_y$ from the first equation and substitute it in the second. We obtain

$$(m - t_y)(p^6 - p^5 + p^4 - p^2 + p) + t_y(p^6 + p^4 - p^3 - p^2 + p) + p^2 - p + 1 = |V|,$$

that is,

$$p^3(p^2 - 1)t_y = |V| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 - p + 1.$$

The case $t_y = 0$ must be excluded, since by direct computations $|V| \neq m(p^6 - p^5 + p^4 - p^2 + p) + p^2 - p + 1$. It is easy to check that $|V| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 + p - 1$ is not divisible by $p + 1$ and hence, $t_y$ turns out not to be an integer, which is impossible. □
Lemma 3.25. No plane meeting \( V \) in at most \( p^6 + 2p^4 - p^2 - p + 1 \) points contains a \((p^4 - p + 1)\)-secant.

Proof. Let \( \pi \) be a plane of \( \mathcal{H}(r, p^4) \) such that \(|\pi \cap V| \leq p^6 + 2p^4 - p^2 - p + 1\). It can contain only 1-, \((p^2 + 1)\)-, \((p^4 - p + 1)\)-, \((p^4 + 1)\)-secants (Lemma 3.18 and Lemma 3.24). If \( \pi \cap V \) contains a \((p^4 - p + 1)\)-secant, we know from Proposition 3.8 that it is a linear combination of at most \( p^2 + 1 \) lines, each with weight 1. Suppose that \( \pi \cap V \) contains a \((p^4 - p + 1)\)-secant. We know that there must be at least \( p - 1 \) other lines of the codeword through \( P \) and \( Q \). Let \( f \) and \( g \) be two such lines through \( Q \). The number of lines of the codeword through \( P \) is at most \( 2p \) and since the lines \( f \) and \( g \) are lines of the codeword through \( Q \), they are \((p^4 - p + 1)\)-secants. There exist at most \((p - 1)^2 \) lines through \( P \) distinct from \( e \) intersecting \( f \) or \( g \) in a point not in \( V \). Therefore we can find a line, say \( m \), of the plane through \( P \), that intersects \( f \) and \( g \) in a point of \( V \) and that is not a line of the codeword. Then \(|m \cap V| \geq 1 + p| \equiv 1 \pmod{p}|. \) Recall that \( m \) is not a line of the codeword and the total number of lines of the codeword is at most \( p^2 + 1 \). Also, \( m \) meets in \( P \) at least \( p \) lines of the codeword and therefore it can intersect all the remaining lines in at most \( p^2 - p + 1 \) pairwise distinct points. This means that the line \( m \) satisfies \( p + 1 \leq |m \cap V| \leq p^2 - p + 1 \), and this contradicts Lemma 3.24.

Lemma 3.26. There are no \((p^4 - p + 1)\)-secants to \( V \).

Proof. If there was a \((p^4 - p + 1)\)-secant to \( V \), say \( \ell \), then, by Lemma 3.25, all the planes through \( \ell \) would contain at least \( p^8 - p^5 + p^4 + 1 \) points of \( V \), and thus

\[
|V| \geq (p^{4(r-2)} + p^{4(r-3)} + \cdots + p^4 + 1)(p^8 - p^5 + p) + p^4 - p + 1, \tag{11}
\]
a contradiction.

Proof of Theorem 1.1 (case \( r \geq 4 \) and \( q = p^2 \)): Consider a line \( \ell \) which is not contained in \( V \). From the preceding lemmas we have that \( \ell \) is either a 1-secant or a \((p^2 + 1)\)-secant of \( V \). Furthermore, \( V \) has no plane section of size \((p^2 + 1)(p^4 + 1)\) because of Lemma 3.21. Finally, as in the case \( r = 3 \), it is easy to see that \( V \) has no singular points, thus, by Theorem 2.2, \( V \) turns out to be a Hermitian variety of \( \mathcal{H}(r, p^4) \).

4 Singular quasi-Hermitian varieties

In this section, we consider sets having the same behavior with respect to hyperplanes as singular Hermitian varieties.

Definition 4.1. A \( d \)-singular quasi-Hermitian variety is a subset of points of \( \mathcal{H}(r, q^2) \) having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension \( d \).

We prove the following result.
Theorem 4.2. Let $\mathcal{S}$ be a $d$-singular quasi-Hermitian variety in $\mathcal{H}(r, q^2)$. Suppose that either

- $r = 3$, $d = 0$, $q = p^h \neq 4$, $h \geq 1$, $p$ any prime, or
- $r \geq 4$, $d \leq r - 3$, $q = p \geq 5$, or
- $r \geq 4$, $d \leq r - 3$, $q = p^2$, $p$ odd prime.

Then $\mathcal{S}$ is a singular Hermitian variety with a singular space of dimension $d$ if and only if its incidence vector is in the $\mathbb{F}_p$-code spanned by the hyperplanes of $\mathcal{H}(r, q^2)$.

Proof. Let $\mathcal{S}$ be a singular Hermitian variety of $\mathcal{H}(r, q^2)$. The characteristic vector $v^\mathcal{S}$ of $\mathcal{S}$ is in $\mathbb{C}_p$ since [13, Theorem 1] also holds for singular Hermitian varieties. Now assume that $\mathcal{S}$ is a $d$-singular quasi-Hermitian variety. As in the non-singular case, by Lemma 3.1, each line of $\mathcal{H}(r, q^2)$ intersects $\mathcal{S}$ in $1 \pmod{p}$ points.

4.1 Case $r = 3$

Suppose that $r = 3$ and therefore $d = 0$. Then $\mathcal{S}$ has $q^3 + q^2 + 1$ points. Let $\pi$ be a plane of $\mathcal{H}(3, q^2)$. In this case, $\pi$ meets $\mathcal{S}$ in either $q^2 + 1$, or $q^3 + 1$ or $q^3 + q^2 + 1$ points. In particular, a planar section of $\mathcal{S}$ with $q^2 + 1$ points is a line since it is a blocking set with respect to lines of a projective plane.

In the case in which $q$ is a prime $p$, Lemma 3.12 and Lemma 3.13 are still valid in the singular case for $\mathcal{V} = \mathcal{S}$ and thus, the planar sections of $\mathcal{S}$ with $p^3 + 1$ or $p^3 + p^2 + 1$ points have to be unitals or pencils of $p + 1$ lines, respectively. Hence, each line of $\mathcal{H}(3, p^2)$ meets $\mathcal{S}$ in $1$, $p + 1$ or $p^2 + 1$ points.

When $q \geq 5$ is an odd prime power, Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10 are still valid in the singular case for $r = 3$.

Thus, if $|\pi \cap \mathcal{S}| = q^2 + 1$, then Proposition 3.8 implies that $\pi \cap \mathcal{S}$ is a line of $\pi$, whereas if $|\pi \cap \mathcal{S}| = q^3 + 1$, then Corollary 3.11 gives that $\pi \cap \mathcal{S}$ is a classical unital of $\pi$. Now suppose that $|\pi \cap \mathcal{S}| = q^3 + q^2 + 1$. Let $\ell$ be a line of $\pi$ such that $|\ell \cap \mathcal{S}| = s$ with $s \neq 1, q + 1, q^2 + 1$.

Each plane through $\ell$ must meet $\mathcal{S}$ in $q^3 + q^2 + 1$ points and this gives

$$(q^3 + q^2 + 1 - s)(q^2 + 1) + s = q^5 + q^2 + 1,$$

that is, $s = q^2 + q + 1$, which is impossible.

Thus in $\mathcal{H}(3, q^2)$, where $q$ is an odd prime power, each line intersects $\mathcal{S}$ in either 1, or $q + 1$ or $q^2 + 1$ points.

Next, assume $q = 2^h$, $h > 2$. Arguing as in the corresponding non-singular case, it turns out that a $(q^3 + q^2 + 1)$-plane meets $\mathcal{S}$ in a pencil of $q + 1$ lines. Now, assume that there is an $i$-secant line to $\mathcal{S}$, say $m$, with $2 < i < q$. Then, each plane through $m$ has to be a $(q^3 + 1)$-plane of $\mathcal{S}$. Counting the number of points of $\mathcal{S}$ by using all planes through $m$ we obtain $q^3 + q^2 + 1 = q^5 + q^3 + q^2 - iq^2 + 1$, hence $i = q$, a contradiction since every line contains $1 \pmod{2}$ points of $\mathcal{V}$. 

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Therefore, each line of $H(3,2^{2h})$, $h \neq 2$ meets $S$ in 1, $q + 1$ or $q^2 + 1$ points. Finally, $S$ is a $k_{q+1,3,q^2}$ for all $q \neq 4$. Also, $S$ cannot be non-singular by assumption. When $q \neq 2$, Theorem 2.3 applies and $S$ turns out to be a cone $\Pi_0S'$ with $S'$ of type I, II, III or IV as the possible intersection sizes with planes are $q^2 + 1, q^3 + 1, q^3 + q^2 + 1$.

Possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $S = \Pi_0 H$, where $H$ is a non-singular Hermitian curve.

For $q = 2$, there is just one point set in $H(3, 4)$ up to equivalence, meeting each line in 1, 3 or 5 points and each plane in 5, 9 or 13 points, that is the Hermitian cone, see [11, Theorem 19.6.8].

4.2 Case $r \geq 4$

Let $\ell$ be a line of $H(r, q^2)$ containing $x < q^2 + 1$ points of $S$. We are going to prove that there exists at least one plane through $\ell$ containing less than $q^3 + q^2 + q + 1$ points of $S$. If we suppose that all the planes through $\ell$ contain at least $q^3 + q^2 + q + 1$ points of $S$, then

$$q^{2(d+1)}(q^{-d} + (-1)^{r-d-1})(q^{r-d-1} - (-1)^{r-d-1}) \cdot q^2 - 1 + q^{2d} + q^{2(d-1)} + \cdots + q^2 + 1 \geq$$

$$m(q^3 + q^2 + q + 1 - x) + x,$$

where $m = q^{2(r-2)} + q^{2(r-3)} + \cdots + q^2 + 1$ is the number of planes through $\ell$ in $H(r, q^2)$. We obtain $x > q^2 + 1$, a contradiction.

Therefore, there exists at least one plane through $\ell$ having less than $q^3 + q^2 + q + 1$ points of $S$ and hence Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10, are still valid in this singular case for any odd $q > 4$.

Next, we are going to prove that $S$ is a $k_{q+1, r, q^2}$, with $q = p^h > 4$, $h = 1, 2$.

Case $q = p \geq 5$: Let $\ell$ be a line of $H(r, p^2)$. As we have seen, there is a plane $\pi$ through $\ell$ such that $|\pi \cap V| < p^3 + p^2 + p + 1$. Proposition 3.17 is still valid in this case and thus we have that $\ell$ is either a unisecant or a $(p + 1)$-secant of $S$. Furthermore, we also have that $S$ has no plane section of size $(p + 1)(p^2 + 1)$ and hence $S$ is a regular $k_{p+1, r, p^2}$.

Case $q = p^2$, $p$ odd: We first observe that (8) and (11) hold true in the case in which $V$ is assumed to be a singular quasi-Hermitian variety. This implies that all lemmas stated in the subparagraph 3.3 are valid in our case. Thus, we obtain that $S$ is a $k_{p^2+1, r, p^4}$ and it is straightforward to check that $S$ is also regular.

Finally, in both cases $q = p$ or $q = p^2$, we have that $S$ is a singular $k_{q+1, r, q^2}$ because if $S$ were a non-singular $k_{q+1, r, q^2}$, then, from Theorem 2.2, $S$ would be a non-singular Hermitian variety and this is not possible by our assumptions.

Therefore, by Theorem 2.4, the only possibility is that $S$ is a cone $\Pi_0S'$, with $S'$ a non-singular $k_{q+1, r-d-1, q^2}$. By Lemma 3.3, $S'$ belongs to the code of points and hyperplanes of $H(r - d - 1, q^2)$. Since $r - d - 1 \geq 2$, then, by [6] and Theorem 1.1, $S'$ is a non-singular Hermitian variety and, therefore, $S$ is a singular Hermitian variety with a vertex of dimension $d$. 

\[\Box\]
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