Bipartite Ramsey numbers for bipartite graphs of small bandwidth

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Submitted: Sep 17, 2017; Accepted: Apr 4, 2018; Published: Apr 27, 2018
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Abstract

A graph $H = (W, E_H)$ is said to have bandwidth at most $b$ if there exists a labeling of $W$ as $w_1, w_2, \ldots, w_n$ such that $|i - j| \leq b$ for every edge $w_iw_j \in E_H$, and a bipartite balanced $(\beta, \Delta)$-graph $H$ is a bipartite graph with bandwidth at most $\beta|W|$ and maximum degree at most $\Delta$, and furthermore it has a proper 2-coloring $\chi : W \to [2]$ such that $||\chi^{-1}(1)| - |\chi^{-1}(2)|| \leq \beta|\chi^{-1}(2)|$. We prove that for any fixed $0 < \gamma < 1$ and integer $\Delta \geq 1$, there exist a constant $\beta = \beta(\gamma, \Delta) > 0$ and a natural number $n_0$ such that for every balanced $(\beta, \Delta)$-graph $H$ on $n \geq n_0$ vertices the bipartite Ramsey number $br(H, H)$ is at most $(1 + \gamma)n$. In particular, $br(C_{2n}, C_{2n}) = (2 + o(1))n$.

Mathematics Subject Classifications: 05D10

1 Introduction

For graphs $G$, $H_1$ and $H_2$, denote $G \to (H_1, H_2)$ by that any red/blue edge coloring of $G$ containing either a red copy of $H_1$ or a blue copy of $H_2$. In Ramsey Theory, a well-known problem is to determine the Ramsey number $r(H_1, H_2)$, i.e. the minimum integer $N$ such that $K_N \to (H_1, H_2)$. We refer the readers to the book by Graham, Rothschild and Spencer [?] for an overview and a survey by Conlon, Fox and Sudakov [?] for many recent developments.

A natural generalization of the above problem is to determine the bipartite Ramsey number $br(H_1, H_2)$, the minimum integer $N$ such that $K_{N,N} \to (H_1, H_2)$, where $H_1$ and

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∗Supported by NSFC(11671088) and NSFFP(2016J01017).
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‡Supported by NSFC(11301086).
$H_2$ are bipartite graphs. Since an edge coloring of $K_{2N}$ induces an edge coloring of $K_{N,N}$, we have

$$r(H_1, H_2) \leq 2br(H_1, H_2).$$

The bipartite Ramsey numbers involving complete bipartite graphs have attracted most attention. Beineke and Schwenk [?] conjectured that $br(K_{t,n}, K_{t,n}) = 2^t(n - 1) + 1$ for any $n \geq t \geq 1$, which is trivially true if $t = 1$. In particular, the authors in [?] proved that $br(K_{2,n}, K_{2,n}) \leq 4n - 3$, and $br(K_{2,n}, K_{2,n}) = 4n - 3$ if there is a Hadamard matrix of order $2(n - 1)$ for odd $n$, and $br(K_{3,n}, K_{3,n}) \geq 8n - 7$ if there is a Hadamard matrix of order $4(n - 1)$. Irving [?] proved $br(K_{3,n}, K_{3,n}) \leq 8n - 7$, which together with the lower bound obtained in [?] gave $br(K_{3,n}, K_{3,n}) = 8n - 7$ for infinitely many values of $n$. However, for the diagonal case, Irving disproved the general conjecture by showing that $br(K_{n,n}, K_{n,n}) < 2^{n-1}(n - 1)$ for $n \geq 21$. Thomason [?] proved that $br(K_{t,n}, K_{t,n}) \leq 2^t(n - 1) + 1$ for any $n \geq t \geq 1$. For fixed $t \geq 2$, Li, Tang and Zang [?] proved that $br(K_{t,n}, K_{t,n}) = (1 + o(1))2^n n$ as $n \to \infty$. It was shown that

$$(1 + o(1))\frac{\sqrt{2}}{e}n^{2n/2} \leq br(K_{n,n}, K_{n,n}) \leq (1 + o(1))2^{n+1}\log n,$$

where the upper bound by Conlon [?] improved that by Irving [?], while the lower bound by Hattingh and Henning [?] is similar to Spencer’s lower bound [?] for ordinary Ramsey numbers by using Lovász Local Lemma, and here and henceforth logarithmic function has natural base 2. For fixed $t \geq 2$, Caro and Rousseau [?] proved that

$$c_1\left(\frac{n}{\log n}\right)^{(t+1)/2} < br(K_{t,t}, K_{n,n}) < c_2\left(\frac{n}{\log n}\right)^t,$$

where $c_i = c_i(t) > 0$ are constants. Recently, Lin and Li [?] proved that the order of magnitude of $br(K_{t,t}, K_{n,n})$ is $n^{t+1}/(\log n)^t$, but the orders of magnitude of $br(K_{n,n}, K_{n,n})$ and $br(K_{t,t}, K_{n,n})$, that are more interesting, seems to be very hard to obtain. For more bipartite Ramsey numbers involving small complete bipartite graphs, see e.g. [?, ?, ?, ?].

For the bipartite Ramsey numbers involving non-complete bipartite graphs, there are not too many references. Let $P_n$ be the path of order $n$, and let $C_n$ be the cycle of order $n$. A celebrated result by Faudree and Schelp [?] determined the “Ramsey bipartite number pair” $B(P_n, P_m)$ for all $m, n \geq 1$, which implied the corresponding bipartite Ramsey number $br(P_n, P_m)$. However, there are not too many references on the bipartite Ramsey numbers $br(C_{2n}, C_{2m})$. In [?, ?], Zhang, Sun and Wu obtained that $br(C_{2n}, C_4) = n + 1$, and $br(C_{2n}, C_6) = n + 2$ for $n \geq 4$.

In this paper, we mainly consider the bipartite balanced $(\beta, \Delta)$-graphs.

**Theorem 1.** For each fixed $0 < \gamma < 1$ and every natural number $\Delta \geq 1$, there exist a constant $\beta = \beta(\gamma, \Delta) > 0$ and a natural number $n_0$ such that for every bipartite balanced $(\beta, \Delta)$-graph $H$ on $n \geq n_0$ vertices we have

$$br(H, H) \leq (1 + \gamma)n.$$
From Theorem ??, the following corollary follows easily and the lower bound construction will be presented below.

**Corollary 2.** Let $C_{2n}$ be the cycle of length $2n$. Then

$$br(C_{2n}, C_{2n}) = (2 + o(1))n.$$  

**Proof.** Indeed, note that $C_{2n}$ defined on $[2n] = \{1, 2, \ldots, 2n\}$ is a balanced $(1/n, 2)$-graph if we label the vertices and denote the cycle as

$$1, 3, 5, \ldots, 2n - 1, 2n, 2n - 2, \ldots, 4, 2, 1,$$

so Theorem ?? implies that $br(C_{2n}, C_{2n}) \leq (2 + o(1))n$ for sufficiently large $n$. On the other hand, consider the bipartite graph $F$ with bipartition $U = \{u_1, u_2, \ldots, u_{2n-2}\}$ and $V = \{v_1, v_2, \ldots, v_{2n-2}\}$, in which $u_i v_j$ is an edge in $F$ if and only if $u_i \in U$ and $1 \leq j \leq n - 1$. Clearly, $F$ does not contain $C_{2n}$, as a subgraph, and its complement $\overline{F}$ (which is restricted in $K_{2n-2, 2n-2}$) also contains no copy of $C_{2n}$. This gives that $br(C_{2n}, C_{2n}) \geq 2n - 1$. Therefore, we have the above corollary as desired. 

\[ \qed \]

2 Preliminary results

Let $A$ be a set of positive integers and $A_n = A \cap [n]$. In the 1930s, Erdős and Turán conjectured that if $\lim_{n \to \infty} \frac{|A_n|}{n} > 0$, then $A$ contains arbitrarily long arithmetic progressions. The conjecture for the arithmetic progression of length 3 was proved by Roth [?, ?]. The full conjecture was proved by Szemerédi [?] with a deep and complicated combinatorial argument. In the proof, Szemerédi used a result, which is now called the bipartite regularity lemma, and then he proved the general Regularity Lemma, see [?]. The lemma has become a powerful tool in extremal graph theory. For many applications, we refer the readers to the survey of Komlós and Simonovits [?] and related references.

2.1 The regularity method

Let $G = (V, E)$ be a graph on $n$ vertices. The density of $G$ is given by $d_G = |E|/(n \choose 2)$. For disjoint vertex sets $A, B \subseteq V$, let $E_G(A, B)$ denote the number of edges of $G$ with one endpoint in $A$ and the other in $B$, and the density of $(A, B)$ is

$$d_G(A, B) = \frac{E_G(A, B)}{|A||B|}.$$

For $\epsilon > 0, d \leq 1$, a pair $(A, B)$ is $\epsilon$-regular if for all $X \subseteq A, Y \subseteq B$ with $|X| > \epsilon |A|$ and $|Y| > \epsilon |B|$ we have $|d_G(X, Y) - d_G(A, B)| < \epsilon$. Then, $(A, B)$ is called $(\epsilon, d)$-regular if it is $\epsilon$-regular and $d_G(A, B) \geq d$. Moreover, $(A, B)$ is called $(\epsilon, d)$-super-regular if it is $\epsilon$-regular and $deg_G(u) > d|B|$ for all $u \in A$ and $deg_G(v) > d|A|$ for all $v \in B$, where $deg_G(u)$ is the degree of a vertex $u$ in $G$.

The following property says that subgraphs of a regular pair are regular.
Claim 3. If \((A, B)\) is \((\epsilon, d)\)-regular, and let \(X \subseteq A\) and \(Y \subseteq B\) with \(|X| \geq \alpha|A|\) and \(|Y| \geq \alpha|B|\) for some \(\alpha > \epsilon\). Then \((X, Y)\) is \(\epsilon'\)-regular such that \(|d(A, B) - d(X, Y)| < \epsilon\), where \(\epsilon' = \max\{\frac{\epsilon}{2}, 2\epsilon\}\).

Let us have another property that any regular pair has a large subgraph which is super-regular, and we include a proof for completeness.

Claim 4. For \(0 < \epsilon < 1/2\) and \(d \leq 1\), if \((A, B)\) is \((\epsilon, d)\)-regular with \(|A| = |B| = m\) then there exist \(A_1 \subseteq A\) and \(B_1 \subseteq B\) with \(|A_1| = |B_1| = (1 - \epsilon)m\) such that \((A_1, B_1)\) is \((2\epsilon, d - 2\epsilon)\)-super-regular.

Proof. Let \(X \subseteq A\) consists of vertices with at most \((d - \epsilon)|A|\) neighbors in \(B\). Since \(e(X, B) \leq |X|(d - \epsilon)|B|\) we have \(|d(X, B) - d| \geq \epsilon\). According to the definition of \(\epsilon\)-regular, we know that \(|X| < \epsilon m\).

Similarly, let \(Y \subseteq B\) consists of vertices with at most \((d - \epsilon)|A|\) neighbors in \(A\), we know that \(|Y| < \epsilon m\).

Take \(A_1 \subseteq A\ \backslash X\) and \(B_1 \subseteq B\ \backslash Y\) with \(|A_1| = |B_1| = (1 - \epsilon)m\). Clearly, each vertex of \(A_1\) has at least \((d - \epsilon)m - \epsilon m = (d - 2\epsilon)m\) neighbors in \(B_1\), and similarly each vertex of \(B_1\) has at least \((d - 2\epsilon)m\) neighbors in \(A_1\). On the other hand, for any subset \(S \subseteq A_1\) and \(T \subseteq B_1\), if \(|S| \geq 2\epsilon|A_1|\) and \(|T| \geq 2\epsilon|B_1|\), then clearly \(|S| \geq \epsilon m\) and \(|T| \geq \epsilon m\).

From the fact that \((A, B)\) is \((\epsilon, d)\)-regular, we have

\[|d(S, T) - d(A_1, B_1)| \leq |d(S, T) - d(A, B)| + |d(A_1, B_1) - d(A, B)| < 2\epsilon.\]

This completes the proof. \(\square\)

In this paper, we shall use the following bipartite form of the regularity lemma. In [?], Böttcher, Heinig and Taraz give a proof (sketch), one can also find a detailed proof in Lin and Li [?, Lemma 5]. For a bipartite graph \(G\) with bipartition \((V^{(1)}, V^{(2)})\), a partition \(\{V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)}\}\) for each \(V^{(s)}\) \((s = 1, 2)\) is said to be \((\epsilon, d)\)-regular on \(R = R([k], [k]; E_R)\) if \(ij \in E_R\) implies that \((V_i^{(1)}, V_j^{(2)})\) is \((\epsilon, d)\)-regular in \(G\) for \(i, j \in [k]\).

If such a partition exists, we also say that \(R\) is an \((\epsilon, d)\)-reduced graph of \(G\).

Lemma 5. For any \(\epsilon > 0\) and integer \(k_0 \geq 1\), there exists \(K_0 = K_0(\epsilon, k_0)\) such that every bipartite graph \(G\) with bipartition \((V^{(1)}, V^{(2)})\) satisfying \(|V^{(1)}| = |V^{(2)}| \geq K_0\) has a partition \(\{V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)}\}\) for each \(V^{(s)}\) for \(s = 1, 2\), where \(k\) is the same for each part \(V^{(s)}\) with \(k_0 \leq k \leq K_0\), such that

1. \(|V_i^{(s)}| = \cdots = |V_k^{(s)}|\) and \(|V_0^{(s)}| \leq \epsilon|V^{(s)}|\) for \(s = 1, 2\);

2. All but at most \(\epsilon k^2\) pairs \((V_i^{(1)}, V_j^{(2)})\), \(1 \leq i, j \leq k\), are \(\epsilon\)-regular.

It is known that the Blow-up Lemma guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently large super-regular pairs. In order to prove Theorem 1.1, we need to find a copy of \(H\) in a monochromatic graph \(G\) of any red/blue edge coloring of \(K_{N,N}\). Similar to the Blow-up Lemma, the following Embedding Lemma allows us to embed \(H\) into \(G\) if \(H\) and \(G\) have “compatible” partitions.

Let \(H = (W, E_H)\) be a graph, for a vertex \(w \in W\), denote \(N_H(w)\) by the neighborhood of \(w\) in \(H\). For \(S \subseteq W\), denote \(N_H(S) = \cup_{v \in S} N_H(v)\) \(\backslash S\).
Definition 6. Let \( H = (W,E_H) \) and \( R = ([k],E_R) \) be graphs and let \( R' = ([k],E_{R'}) \) be a subgraph of \( R \). We say that a vertex partition \( W = (W_i)_{i \in [k]} \) of \( H \) is \( \varepsilon \)-compatible with a vertex partition \( V = (V_i)_{i \in [k]} \) of a graph \( G = (V,E_G) \) if the following holds.

For \( i,j \in [k] \) with \( i \neq j \), let \( S_i \subseteq W_i \) that has some neighbours in \( W_j \) for \( ij \in E_R \). Set \( S = \bigcup S_i \) and \( T_i = N_H(S) \cap (W_i \setminus S) \). Then

1. \( |W_i| \leq |V_i| \) for \( i \in [k] \).
2. \( xy \in E_H \) for \( x \in W_i, y \in W_j \) implies \( ij \in E_R \).
3. \( |S_i| \leq \varepsilon |V_i| \) for \( i \in [k] \).
4. \( |T_i| \leq \varepsilon \min\{|V_j| : ij \in E_{R'}\} \).

That is to say, from this definition, we particularly require that the edges of \( H \) run only between classes that correspond to a dense regular pair in \( G \) (condition (2)). The vertices that have neighbors not belong to the super-regular pairs are assumed to be few (condition (3)), and the neighbours of these vertices are assumed to be few too (condition (4)).

The following Embedding Lemma by Böttcher, Heinig and Taraz [?, ?] is crucial for us to prove our main result.

Lemma 7 (Embedding Lemma). For all \( d > 0 \) and integers \( \Delta, r \geq 1 \), there is a constant \( \varepsilon = \varepsilon(d,\Delta,r) > 0 \) such that the following holds. Let \( G = (V,E) \) be an \( N \)-vertex graph that has a partition \( (V_i)_{i \in [k]} \) with \( (\varepsilon,d) \)-reduced graph \( R \) on \( [k] \), and \( R' \subseteq R \) is \( (\varepsilon,d) \)-super-regular with connected components having at most \( r \) vertices each. Suppose \( H = (W,E_H) \) is a graph of order \( n \leq N \) with maximum degree \( \Delta(H) \leq \Delta \) that has a partition \( (W_i)_{i \in [k]} \) which is \( \varepsilon \)-compatible with \( (V_i)_{i \in [k]} \). Then \( H \subseteq G \).

In the above lemma, if \( R' = M \) is a matching of \( R \), then \( r = 2 \), and so \( \varepsilon = \varepsilon(d,\Delta) \) depends only on \( d \) and \( \Delta \).

2.2 Long path in bipartite reduced graph

In our proof, for any red/blue edge coloring of \( K_{N,N} \), we will embed the bipartite graph \( H \) into one of the monochromatic subgraph \( G \) of \( K_{N,N} \). From the regularity lemma, the reduced graph \( R \) corresponding to \( G \) has enough \( \varepsilon \)-regular pairs, and we need to find a long path of this bipartite graph \( R \) such that we can embed \( H \) into the subgraph induced by all edges corresponding to the regular pairs of this path. This can be achieved from the result by Gyárfás, Rousseau and Schelp[?, Theorem 1].

Lemma 8. Let \( H \) be a bipartite graph with bipartition \( (U,V) \), where \( |U| = k_1 \) and \( |V| = k_2 \) \( (k_1 \leq k_2) \). If \( c \leq k_1/2 \) and \( H \) contains no path \( P_{2t} \) for \( t > c \), then \( \varepsilon(H) \leq (k_1 + k_2 - 2c)c \).

From the above lemma, we have the following result immediately.

Corollary 9. Let \( 0 < \varepsilon < 1/4 \) and let \( H \) be a bipartite graph with bipartition \( (U,V) \), where \( |U| = k = |V| \). If \( \varepsilon(H) > (1/2 - \varepsilon)k^2 \), then \( H \) contains a path \( P_{2t} \) with \( t > \lfloor (1/2 - \sqrt{\varepsilon/2})k \rfloor \).
Proof. Take \( c = \lceil (\frac{1}{2} - \sqrt{\epsilon/2})k \rceil = \frac{1}{2} - \sqrt{\epsilon/2})k - \eta \) for some \( \eta \geq 0 \). From Lemma ??, if \( H \) contains no path \( P_{2t} \) with \( t > c \), then
\[
e(H) \leq (k + k - 2c)c = \left(2k - 2 \left( \frac{1}{2} - \sqrt{\frac{\epsilon}{2}} \right) k - \eta \right) \left( \frac{1}{2} - \sqrt{\frac{\epsilon}{2}} \right) k - \eta \]
\[
= \frac{1}{2} \left( k^2 - \left(2 \sqrt{\frac{\epsilon}{2}} k + 2\eta \right)^2 \right) \leq \left( \frac{1}{2} - \epsilon \right) k^2,
\]
a contradiction. This completes the proof. \( \square \)

In the following, for a red-blue edge coloring of \( K_{N,N} \), let us denote \( G_r \) and \( G_b \) by the graphs induced by all red edges and blue edges, respectively.

Lemma 10. For any \( 0 < \epsilon < 1/4 \) and any integer \( k_0 \geq 1 \), there exists \( K_0 \) such that any red-blue edge coloring of \( K_{N,N} \) with bipartition \( (V^{(1)}, V^{(2)}) \) and \( N \geq K_0 \) has an \( \epsilon \)-regular partition \( \{ V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)} \} \) for each \( V^{(s)} \) \( (s = 1, 2) \) with \( k_0 \leq k \leq K_0 \) satisfying the following property. The reduced graph contains a monochromatic path \( P_{2t} \) with \( t \geq (\frac{1}{2} - \sqrt{\epsilon/2})k \) such that all regular pairs corresponding to the edges of \( P_{2t} \) are \( (\epsilon, 1/2) \)-regular on \( G_r \) or all of these corresponding regular pairs are \( (\epsilon, 1/2) \)-regular on \( G_b \).

Proof. From Lemma ??, there is a partition \( \{ V_0^{(s)}, V_1^{(s)}, \ldots, V_k^{(s)} \} \) for each \( V^{(s)} \) for \( s = 1, 2 \), where \( k \) is the same for each part \( V^{(s)} \) and \( k_0 \leq k \leq K_0 \), such that (1) \( |V_1^{(s)}| = \cdots = |V_k^{(s)}| \) and \( |V_0^{(s)}| \leq \epsilon |V^{(s)}| \) for \( s = 1, 2 \); (2) All but at most \( \epsilon k^2 \) pairs \( (V_1^{(1)}, V_2^{(2)}) \), \( 1 \leq i, j \leq k \), are \( \epsilon \)-regular.

Let \( R \) be the reduced subgraph of \( K_{k,k} \) with bipartition \( ([k], [k]) \), in which a pair \( ij \) for \( i, j \in [k] \) is adjacent in \( R \) if and only if \( (V_i^{(1)}, V_j^{(2)}) \) is \( \epsilon \)-regular. Thus \( e(R) \geq (1 - \epsilon)k^2 \).

Color an edge \( ij \) of \( R \) green if \( d_{G_r}(V_i^{(1)}, V_j^{(2)}) \geq 1/2 \), or white if \( d_{G_r}(V_i^{(1)}, V_j^{(2)}) < 1/2 \). Denote by \( R_g \) and \( R_w \) the subgraphs spanned by green edges and white edges, respectively. Without loss of generality, we may assume that
\[
e(R_g) \geq \frac{(1 - \epsilon)k^2}{2} = \left( \frac{1}{2} - \frac{\epsilon}{2} \right) k^2.
\]
From Corollary ??, there exists a path \( P_{2t} \) in \( R_g \) with \( t > \lceil (\frac{1}{2} - \sqrt{\epsilon/2})k \rceil \), and all regular pairs corresponding to the edge of \( P_{2t} \) are \( (\epsilon, 1/2) \)-regular on \( G_r \), as desired. This completes the proof. \( \square \)

2.3 Locally balanced intervals

Given \( \beta > 0 \) and integer \( \Delta \geq 1 \), we have known that a balanced \((\beta, \Delta)\)-graph \( H \) has a proper 2-coloring \( \chi : V(H) \to [2] \) such that the sizes of the color classes are almost equal. This definition focuses on vertices of different colors as a whole. In fact, we can see that the two colors also have approximately the same number of vertices locally.

For a graph \( H = (W, E) \) with \( W = \{w_1, w_2, \ldots, w_n\} \), where \( w_i \) is a labeling of the vertices, let \( \chi : W \to [2] \) be a 2-coloring. For \( W' \subseteq W \), denote \( C_i(W') = |\chi^{-1}(i) \cap W'| \) for
We know that \( \chi \) is a \( \beta \)-balanced coloring of \( W \) if \( 1 - \beta \leq \frac{C_1(W)}{C_2(W)} \leq 1 + \beta \). A set \( I \subseteq W \) is called interval if there exists \( p < q \) such that \( I = \{w_p, w_{p+1}, \ldots, w_q\} \). Finally, let \( \sigma : [\ell] \to [\ell] \) be a permutation, and for a partition \( \Gamma = \{I_1, I_2, \ldots, I_\ell\} \) of \( W \), where each \( I_i \) is an interval, let \( C_\tau(\Gamma, \sigma, a, b) = \sum_{j=a}^{b} C_\tau(I_{\sigma(j)}) \) for \( \tau = 1, 2 \), and we always write \( C_\tau(\sigma, a, b) = C_\tau(\Gamma, \sigma, a, b) \) for simplicity. The following result by Mota, Sárközy, Schacht and Taraz [7, Lemma 2.11] means that every balanced bipartite graph is also balanced in local.

**Lemma 11.** For every \( \xi > 0 \) and integer \( \ell \geq 1 \) there exists \( n_0 \) such that if \( H = (W, E) \) is a graph on \( W = \{w_1, w_2, \ldots, w_n\} \) with \( n \geq n_0 \), then for every \( \beta \)-balanced 2-coloring \( \chi \) of \( W \) with \( \beta \leq 2/\ell \), and every partition of \( W \) into intervals \( I_1, I_2, \ldots, I_\ell \) with \( |I_1| \leq |I_2| \leq \ldots \leq |I_\ell| \leq |I_1| + 1 \) there exists a permutation \( \sigma : [\ell] \to [\ell] \) such that for every pair of integers \( 1 \leq a < b \leq \ell \) with \( b - a \geq 7/\xi \),

\[
|C_1(\sigma, a, b) - C_2(\sigma, a, b)| \leq \xi C_2(\sigma, a, b).
\]

## 3 Proof of Theorem 1.1

For every \( 0 < \gamma < 1 \) and integer \( \Delta \geq 1 \), we want to prove that there exists a constant \( \beta = \beta(\gamma, \Delta) > 0 \) and a natural number \( n_0 \) such that if \( H \) is a balanced \((\beta, \Delta)\)-graph on \( n \) vertices for \( n \geq n_0 \) then any red-blue edge coloring of \( K_{N,N} \) for \( N = (1 + \gamma/3)n \) contains a monochromatic copy of \( H \).

The main idea is as follows. First, we shall apply Lemma ?? to find a monochromatic subgraph, say red graph \( G_r \), of any red/blue edge coloring of \( K_{N,N} \) with sufficiently long path \( P \) of the reduced graph, and then use Claim ?? to get a subgraph \( G_P \) by deleting some vertices from \( G_r \) such that \( G_P \) contains sufficiently many dense super-regular pairs covering \((1 + o(1))n \) vertices. Second, we partition the vertices of \( H \), and apply Lemma ?? to show that the partition is \( 2c \)-compatible with the partition of \( G_P \). Finally, we shall find a copy of \( H \) in \( G_P \) by using the Embedding Lemma (Lemma ??). The details are as follows.

For \( 0 < \gamma < 1 \), \( \Delta \geq 1 \) be given and \( d = 1/3 \), we have \( \varepsilon_0 = \varepsilon_0(d, \Delta) \) by Lemma ??. Set

\[
\epsilon = \min\left\{ \frac{\varepsilon_0}{2}, \frac{\gamma^2}{25} \right\}.
\]

For such \( \epsilon > 0 \) defined above and integer \( k_0 \geq 1 \), \( K_0 \) is determined by \( k_0 \) and \( \epsilon \) from Lemma ???. Fix

\[
\xi = \frac{\gamma}{60},
\]

and let \( n_0 \) be obtained from Lemma ?? dependent on \( \xi \) and \( K_0 \). Set

\[
\beta = \frac{\epsilon \xi (1 + 2\xi)}{36 \Delta^2 K_0^2}.
\]

For sufficiently small \( \gamma > 0 \), we have

\[
\beta \ll \epsilon \ll \xi < \gamma.
\]
Let 
\[ c = c(\gamma) = 1 + \frac{\gamma}{3}, \]
and let \( H = (W, E_H) \) be a balanced \((\beta, \Delta)\)-graph on \( n \) vertices with \( n \leq N \), where \( N = cn \geq \max\{n_0, K_0\} \).

### 3.1 Preparing the host graph \( G_P \)

From Lemma ???, any red-blue edge coloring of \( K_{N,N} \) with bipartition \((V^{(1)}, V^{(2)})\) and \( N \geq K_0 \) has an \( \varepsilon \)-regular partition \( \{V_i^{(s)}, V_i^{(s)} : \ldots, V_i^{(s)}\} \) for each \( V^{(s)} \) \((s = 1, 2)\) with \( k_0 \leq k \leq K_0 \) satisfying the following property. The reduced graph contains a monochromatic path \( P_{2t} \) with \( t \geq (\frac{1}{2} - \frac{\sqrt{c}}{2})k \), and all regular pairs corresponding to the edges of \( P_{2t} \) are \((\varepsilon, 1/2)\)-regular on \( G_r \) or \( G_b \), say \( G_r \).

Without loss of generality, we may relabel those \( V_i^{(s)} \)'s for \( s = 1, 2 \) such that \( P_{2t} \) defined on \([-\lceil t \rceil, \lceil t \rceil] \) that corresopnding to \( V_i^{(1)} \) and \( V_i^{(2)} \) for \( i \in [t] \), where \( ii \) (corresponds to the regular pair \( V_i^{(1)}V_i^{(2)} \)) for \( 1 \leq i \leq t \) and \((i+1)i \) (corresponds to the regular pair \( V_i^{(1)}V_i^{(2)} \)) for \( 1 \leq i \leq t-1 \) are those edges of \( P_{2t} \), and suppose that all edges of type \( ii \) consist of the edges of the matching \( M \) of \( P_{2t} \).

Applying Claim ?? to the regular pair \((V_i^{(1)}, V_i^{(2)})\) for \( 1 \leq i \leq t \), we have \( A_i \subseteq V_i^{(1)} \) and \( B_i \subseteq V_i^{(2)} \) with \( |A_i| = |B_i| \geq (1-\varepsilon)m \) such that
\[ (A_i, B_i) \text{ is } (2\varepsilon, 1/2 - 2\varepsilon)-super regular. \]

Note that \( \gamma > 0, \varepsilon \leq \gamma^2/25, c = 1 + \gamma/3 \) and \( t \geq (1/2 - \sqrt{\varepsilon}/2)k \), so we have
\[ |A_i| = |B_i| \geq (1-\varepsilon)m \geq (1-\varepsilon) \frac{(1-\varepsilon)N}{k} \geq \left( 1 + \frac{\gamma}{30} \right) \frac{n}{2t}. \tag{1} \]

Slightly abusing the notations, we also denote \( P_{2t} \) by the path defined on \([-\lceil t \rceil, \lceil t \rceil] \) corresponding to the pairs \((A_i, B_i)\) for \( 1 \leq i \leq t \) and \((A_{i+1}, B_i)\) for \( 1 \leq i \leq t-1 \), and denote \( M \) by the matching corresponding to the pairs \((A_i, B_i)\) for \( 1 \leq i \leq t \) that are \((2\varepsilon, 1/2 - 2\varepsilon)-super regular. \)

Note that all the other pairs \((A_{i+1}, B_i)\) corresponding to edges of \( P_{2t} \setminus M \) are \((2\varepsilon, 1/2 - 2\varepsilon)-regular since the pairs \((V_i^{(1)}V_i^{(2)})\) are \((\varepsilon, 1/2)\)-regular for \( 1 \leq i \leq t-1 \). Let \( G_P \) be the subgraph induced by all of the red edges from \((\cup_{i=1}^t A_i) \cup (\cup_{i=1}^t B_i) \) of \( G_r \).

### 3.2 Preparing \( H \)

Since \( H = (W, E_H) \) is a balanced \((\beta, \Delta)\)-graph, there is a proper 2-coloring \( \chi : V(H) \to [2] \) such that \( ||\chi^{-1}(1)| - |\chi^{-1}(2)| || \leq \beta|\chi^{-1}(2)| \). Label the vertices of \( W \) as \( w_1, w_2, \ldots, w_n \) such that \( |g-h| \leq \beta n \) for every edge \( w_gw_h \in E_H \), and let \( \ell \) be the smallest integer divisible by \( t \) with \( \ell \geq 7(K_0/\xi) + t \geq t(7/\xi + 1) \). Moreover, we may choose \( n \) suitably such that \( n \) is divisible by \( \ell \), and let \( \Gamma = \{I_1, I_2, \ldots, I_\ell \} \) be the partition of \( W \) with \( |I_1| = |I_2| = \cdots = |I_\ell| = \frac{n}{\ell} \) in order, i.e., for \( 1 \leq i \leq \ell \),
\[ I_i = \{w_{(i-1)\frac{n}{\ell}+1}, w_{(i-1)\frac{n}{\ell}+2}, \ldots, w_{i\frac{n}{\ell}}\}. \]
Let $a_i = (i - 1)\frac{\ell}{t} + 1$ and $b_i = i\frac{\ell}{t}$. Then $b_i - a_i = \frac{\ell}{t} - 1 \geqslant 7/\xi$. Since $\beta = \frac{\xi(1+2\xi)}{36\Delta^2 K_0} < 1/\ell$, Lemma ?? implies that there exists a permutation $\sigma : [\ell] \to [\ell]$ such that

$$|C_1(\sigma, a_i, b_i) - C_2(\sigma, a_i, b_i)| \leqslant \xi C_2(\sigma, a_i, b_i),$$

where $C_\tau(\sigma, a_i, b_i) = \sum_{j=a_i}^{b_i} C_\tau(I_{\sigma(j)})$ for $\tau = 1, 2$. Denote $J_i = I_{\sigma(a_i)} \cup I_{\sigma(a_i+1)} \cup \cdots \cup I_{\sigma(b_i)}$, $C_\tau(J_i) = C_\tau(\sigma, a_i, b_i)$ for $\tau = 1, 2$. Clearly $|J_i| = (b_i - a_i + 1)n = \frac{n}{\ell}$, and

$$(1 - \xi)\frac{n}{2\ell} \leqslant C_1(J_i), C_2(J_i) \leqslant (1 + \xi)\frac{n}{2\ell}.$$ 

Note that

$$W = \bigcup_{i=1}^{\ell} I_i = \bigcup_{i=1}^{\ell} I_{\sigma(i)} = \bigcup_{i=1}^{t} J_i,$$

and we will partition $W$ into disjoint subsets $X_1, Y_1, X_2, Y_2, \ldots, X_t, Y_t$ as follows.

Noticing that the edges of $H$ can only belong to two successive intervals $I_i$ and $I_{i+1}$ since $H$ is a graph with bandwidth at most $\beta|W| < |W|/\ell$ from the definition of $\ell$.

For $i = \ell$, if $I_\ell \in J_j$ for some $1 \leqslant j \leqslant t$, then put $I_\ell \cap \chi^{-1}(1)$ into $X_j$ and put $I_\ell \cap \chi^{-1}(2)$ into $Y_j$. For $1 \leqslant i \leqslant \ell - 1$,

(i) if $I_i$ and $I_{i+1}$ belong to the same $J_j$ for some $1 \leqslant j \leqslant t$, then put $I_i \cap \chi^{-1}(1)$ into $X_j$ and put $I_i \cap \chi^{-1}(2)$ into $Y_j$.

(ii) if $I_i$ and $I_{i+1}$ belong to different $J_j$ and $J_{j'}$ for $j < j'$, then we divide $I_i$ into two disjoint subsets $L_i$ (we call it a link) and $K_i$, where $L_i \subseteq I_i$ consists all of the last $[2(j' - j) + 1] \beta n$ vertices of $I_i$, and $K_i = I_i \setminus L_i$. For $K_i$, put $K_i \cap \chi^{-1}(1)$ into $X_j$ and put $K_i \cap \chi^{-1}(2)$ into $Y_j$.

For $L_i$, let $p = 2(j' - j)$, and denote

$$L_i = \bigcup_{q=1}^{p+1} L_i(q),$$

where

$$L_i(q) = \left\{ w_{[i-(p+2-q)\beta \ell]} \frac{\beta \ell}{2} + 1, w_{[i-(p+2-q)\beta \ell]} \frac{\beta \ell}{2} + 2, \ldots, w_{[i-(p+1-q)\beta \ell]} \frac{\beta \ell}{2} \right\}$$

with $|L_i(q)| = \beta n$ for $1 \leqslant q \leqslant p + 1$.

For $q = 1$, put $L_i(1) \cap \chi^{-1}(2)$ into $Y_j$. For odd $q$, put $(L_i(q) \cup L_i(q+1)) \cap \chi^{-1}(1)$ into $X_{j+\frac{p+1}{2}}$. For even $q$, put $(L_i(q) \cup L_i(q+1)) \cap \chi^{-1}(2)$ into $Y_{j+\frac{p+1}{2}}$. And for $q = p + 1$, put $L_i(p+1) \cap \chi^{-1}(1)$ into $X_{j+\frac{p+1}{2}}$. See the figure below.

Now, we have put the vertices of $I_i$ for $1 \leqslant i \leqslant \ell$ into subsets $X_1, Y_1, X_2, Y_2, \ldots, X_t, Y_t$ that form a vertex partition of $W$. From the above construction, we can see that each
$\chi(H) = 2$ \hspace{1cm} $\chi(H) = 2$ \hspace{1cm} $\chi(H) = 1$

$X_i$ induces an independent set that consists of most vertices from $C_1(J_i)$ together with at most two pieces of a fixed link, and each $Y_i$ induces an independent set that consists of most vertices from $C_2(J_i)$ together with at most two pieces of a fixed link. Hence, by noting that $\xi = \gamma/60$, $\beta = \epsilon \xi (1 + 2 \xi)/(36 \Delta^2 K_0^2)$, $t \leq K_0$ and $\ell \leq (7 K_0 + 2 K_0 \xi)/\xi$. We have

$$|X_i| \leq C_1(J_i) + 2 \ell \cdot \beta n \leq (1 + \xi) \frac{n}{2t} + 2 \ell \beta n = (1 + \xi + 4 t \ell \beta) \frac{n}{2t} < |A_i|,$$

and similarly $|Y_i| < |B_i|$.

### 3.3 Embedding $H$ into $G_P$

Now, let $G_P$ be the graph with the vertex partition $\{A_1, B_1, \ldots, A_t, B_t\}$, and $H$ be the balanced $(\beta, \Delta)$-vertex of order $n \leq |V(G_P)|$ with the vertex partition $\{X_1, Y_1, \ldots, X_t, Y_t\}$ as above. Note that the matching $M$ whose edges correspond to $(A_i, B_i)$ for $1 \leq i \leq t$ are $(2 \epsilon, 1/2 - 2 \epsilon)$-super regular, and $(A_{i+1}, B_i)$ are $(2 \epsilon, 1/2 - 2 \epsilon)$-regular. We will apply the Embedding Lemma, i.e. Lemma ??, to the host graph $G_P$ with reduced graph $P_{2t}$. It suffices to prove that the partition $\{X_1, Y_1, \ldots, X_t, Y_t\}$ of $H$ is $2\epsilon$-compatible with the partition $\{A_1, B_1, \ldots, A_t, B_t\}$ of $G_P$. To this end, we shall check all of the four conditions of Definition ?? as follows.

Note that from the partition of $V(H)$, we have

1. $|X_i| \leq |A_i|$ and $|Y_i| \leq |B_i|$ for all $1 \leq i \leq t$.

2. $xy \in E_H$ for $x \in X_i, y \in Y_j$ ($j = i$ or $j = i - 1$) implies that $(i, j) \in E_{P_{2t}}$.

3. Note that all edges of $E_{P_{2t}} \setminus E_M$ if of type $(i, i - 1)$ for $2 \leq i \leq t$. From the vertex partition of $W$, the vertices of $X_i$ can be adjacent to at most two pieces of each link. Moreover, $Y_{i-1}$ contains at most $\ell$ links. Denote $S_i \subseteq X_i$ by the vertex set that has some neighbours in $Y_{i-1}$, i.e. $S_i = N_H(Y_{i-1}) \cap X_i$. Therefore, by noting that each piece of a link is of size at most $\beta n$, we have

$$|S_i| \leq \Delta \cdot (2 \ell \beta n) \leq \frac{\epsilon}{\Delta} (1 + \gamma/30) \frac{n}{2t} \leq \frac{\epsilon}{\Delta} |A_i|.$$

Similarly, denote $S'_{i-1} \subseteq Y_{i-1}$ by the vertex set that has some neighbours in $X_i$, we have $|S'_{i-1}| \leq \frac{\epsilon}{\Delta} |B_{i-1}|$. 

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(4) Set $S = \bigcup_{i=1}^{t}(S_i \cup S_i')$. Let $T_i = N_H(S) \cap (X_i \setminus S)$ and $T_i' = N_H(S) \cap (Y_i \setminus S)$. For $T_i = N_H(S) \cap (X_i \setminus S)$, noticing that the neighbors of vertices of $X_i$ can only locate in $Y_{i-1}$ and $Y_i$, hence $N_H(S) \cap (X_i \setminus S)$ consists of vertices of $N_H(S \cap Y_i) \cap (X_i \setminus S)$ and $N_H(S \cap Y_{i-1}) \cap (X_i \setminus S)$. Since $N_H(Y_{i-1}) \cap X_i = S_i$, we have $N_H(S \cap Y_{i-1}) \cap (X_i \setminus S) = S_i \cap (X_i \setminus S) = \emptyset$, and hence

$$|T_i| = |N_H(S \cap Y_i) \cap (X_i \setminus S)| < \Delta|S \cap Y_i| = \Delta|S_i'| \leq \epsilon|B_i|.$$ 

Similarly, for $T_i' = N_H(S) \cap (Y_i \setminus S)$, we have $|T_i'| < \epsilon|A_i|$.

Therefore, the partition $\{X_1,Y_1,\ldots,X_t,Y_t\}$ of $H$ is indeed $2\epsilon$-compatible with the partition $\{A_1,B_1,\ldots,A_t,B_t\}$ of $G_P$. This completes the proof of Theorem 1.2. □

Acknowledgments

The authors are grateful to the anonymous referees for their careful reading of the manuscript and invaluable comments which have improved the manuscript greatly.

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