Hypertopes with tetrahedral diagram

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Abstract

In this paper we construct an infinite family of hypertopes of rank four having the
complete graph \( K_4 \) as diagram. Their group of rotational symmetries is isomorphic
to \( PSL(2,q) \). It turns out some elements of this family are regular hypertopes and
some are chiral. Moreover, we show that the chiral ones have both improper and
proper correlations simultaneously.

Mathematics Subject Classifications: 51E24, 52B11, 20F05

1 Introduction

Abstract polytopes generalize (the face lattice of) convex polytopes to combinatorial
structures. The main interest of the theory of abstract polytopes has been the study of
their symmetries. Hence, highly symmetric polytopes (in particular those regular and
chiral), together with their automorphism groups, are the most studied ones. A polytope
can be regarded as a thin residually connected geometry with linear diagram. The concept
of hypertope was introduced in [2] and generalizes the concept of a polytope by dropping the linear condition on the diagram. This generalization was made in such a way that the concept of chirality can be extended to hypertopes. As it is the case with maps and polytopes, we are interested in understanding chiral hypertopes, and finding examples of them can be the first step to achieve this goal.

In the 1970’s Branko Grünbaum considered rank 4 polytopes that are locally toroidal, meaning that all their facets and vertex figures are either spherical or toroidal, and not all of them are spherical. In [5] an almost complete answer to Grünbaum’s problem, the classification of rank 4 regular locally toroidal polytopes, is given. In this paper we follow [2] and generalize the concept of a polytope to that of a hypertope, and study some interesting locally toroidal ones.

In [3] some examples of chiral hypertopes of rank 4 with certain diagrams are given, satisfying that their residues of rank 3 are either spherical or toroidal. In this paper we continue with the study of locally toroidal 4-hypertopes, in the sense that their residues of rank 3 are toroidal. Using $PSL(2, q)$ as their rotational subgroups, we found an infinite family of hypertopes that contains both regular and chiral hypertopes. Moreover, the chiral ones have the very interesting property that they admit both proper and improper correlations, a feature that is known to be impossible for polytopes (see [4, Lemma 3.1]).

We construct these hypertopes as coset geometries $\Gamma = (G; \{G_0, G_1, G_2, G_3\})$ where $G$ is $PSL(2, q)$ with $q = p$ or $q = p^2$ for certain primes $p$. In these geometries two maximal parabolic subgroups are alternating groups $A_4$ and the other two are $E_q : C_3$. Therefore 3 must divide $q - 1$ for our construction to work, and there is a third root of unity $e$ in the field of order $q$. In the case $q = p^2$ we require that $e$ is not in the subfield of order $p$, which is equivalent to requiring that $p \equiv 2 \mod 3$. We get an infinite family of regular hypertopes when $q = p^2$ and an infinite family of chiral hypertopes when $q = p$. More precisely in the case $q = p^2$ two residues are hypermaps of type $(3, 3, 3)_{(2, 0)}$ and the other two are hypermaps of type $(3, 3, 3)_{(p, 0)}$; in the case $q = p$ two residues are hypermaps $(3, 3, 3)_{(2, 0)}$ and the other two are both chiral hypermaps of types $(3, 3, 3)_{(1, e)}$ and $(3, 3, 3)_{(e, 1)}$, respectively.

The paper is organised as follows. In Section 2, we give the definitions and notation needed to understand this paper. In Section 3, we construct $PSL(2, q)$, with $q \in \{p, p^2\}$ and $p$ a prime, as a $C^+$-group. In Section 4, we show that the $C^+$-groups obtained in the previous section give hypertopes. In Section 5 we show that the hypertopes obtained are either regular or chiral. In Section 6, we give a geometric description of the hypertopes we constructed, in terms of objects of the projective line $PG(1, q)$.

## 2 Preliminaries

### 2.1 Incidence geometries

Following [1], an incidence system $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- $X$ is a set whose elements are called the elements of $\Gamma$;
• $I$ is a set whose elements are called the types of $\Gamma$;

• $t : X \to I$ is a type function, associating to each element $x \in X$ a type $t(x) \in I$;

• $*$ is a binary relation on $X$ called incidence, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The incidence graph of $\Gamma$ is the graph whose vertex set is $X$ and where two vertices are joined provided the corresponding elements of $\Gamma$ are incident. A flag is a clique of the incidence graph of $\Gamma$. The type of a flag $F$ is $\{t(x) : x \in F\}$. A chamber is a flag of type $I$.

An element $x$ is incident to a flag $F$ and we write $x * F$ for that, when $x$ is incident to all elements of $F$. An incidence system $\Gamma$ is a geometry or incidence geometry if every flag of $\Gamma$ is contained in a chamber. The rank of $\Gamma$ is the number of types of $\Gamma$, namely the cardinality of $I$.

Let $\Gamma := (X, *, t, I)$ be an incidence geometry and $F$ a flag of $\Gamma$. The residue of $F$ in $\Gamma$ is the incidence geometry $\Gamma_F := (X_F, *_F, t_F, I_F)$ where

- $X_F := \{x \in X : x * F, x \not\in F\};$
- $I_F := I \setminus t(F);$
- $t_F$ and $*_F$ are the restrictions of $t$ and $*$ to $X_F$ and $I_F$.

An incidence system $\Gamma$ is connected if its incidence graph is connected; $\Gamma$ is residually connected when each residue of rank at least two of $\Gamma$ (including itself) has a connected incidence graph; $\Gamma$ is called thin (resp. firm) when every residue of rank one of $\Gamma$ contains exactly two (resp. at least two) elements. As in [2], we say that a hypertope is a thin incidence geometry which is residually connected.

Let $\Gamma := (X, *, t, I)$ be an incidence system. An automorphism of $\Gamma$ is a mapping $\alpha : (X, I) \to (X, I) : (x, t(x)) \mapsto (\alpha(x), t(\alpha(x)))$ where

- $\alpha$ is a bijection on $X$;
- for each $x, y \in X$, $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- for each $x, y \in X$, $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

Note that $\alpha$ induces a bijection on $I$. When $t(x) = i$ we say that $x$ is an element of type $i$, or equivalently, that $x$ is an $i$-element. The set of automorphisms of $\Gamma$ is a group denoted by $\Aut(\Gamma)$.

An automorphism $\alpha$ of $\Gamma$ is called type preserving when for each $x \in X$, $t(\alpha(x)) = t(x)$ (i.e. $\alpha$ maps each element on an element of the same type). The set of type-preserving automorphisms of $\Gamma$ is a group denoted by $\Aut_I(\Gamma)$ and obviously $\Aut_I(\Gamma) \leq \Aut(\Gamma)$.

A correlation is a non-type-preserving automorphism, that is an element of $\Aut(\Gamma) \setminus \Aut_I(\Gamma)$. A duality is correlation that induces an involutory permutation on $I$.

An incidence geometry $\Gamma$ is chamber-transitive if $\Aut_I(\Gamma)$ is transitive on all chambers of $\Gamma$. Finally, an incidence geometry $\Gamma$ is regular if $\Aut_I(\Gamma)$ acts regularly on the chambers, that is, the action is semi-regular (free) and transitive.
Observe that chamber-transitivity implies flag-transitivity, that is, for each $J \subseteq I$, there is a unique orbits on the flags of type $J$ under the action of $\text{Aut}(\Gamma)$.

Sometimes, when $\text{Aut}(\Gamma)$ is not transitive on the chambers of $\Gamma$, it has two orbits. We are also interested in those hypertopes having two orbits with an extra condition. Two chambers $C$ and $C'$ of a thin incidence geometry of rank $r$ are called $i$-adjacent if $C$ and $C'$ differ only in their $i$-elements. We then denote $C'$ by $C^i$. Let $\Gamma(X, *, t, I)$ be a thin incidence geometry. We say that $\Gamma$ is chiral if $\text{Aut}_I(\Gamma)$ has two orbits on the chambers of $\Gamma$ such that any two adjacent chambers lie in distinct orbits. Moreover, if $\Gamma$ is residually connected, we call $\Gamma$ a chiral hypertope.

When $\Gamma$ is a chiral hypertope, if $\text{Aut}(\Gamma) \neq \text{Aut}_I(\Gamma)$, correlations may either interchange the two orbits or preserve them. A correlation that interchanges the two orbits is said to be improper and a correlation that preserves them is said to be proper.

The following proposition shows how to construct an incident geometry starting from a group.

**Proposition 1.** (Tits Algorithm, 1956) [7] Let $n$ be a positive integer and $I := \{1, \ldots, n\}$. Let $G$ be a group together with a family of subgroups $(G_i)_{i \in I}$, $X$ the set consisting of all cosets $G_i g$ with $g \in G$ and $i \in I$, and $t : X \to I$ defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by:

$$G_i g_1 \ast G_j g_2 \iff G_i g_1 \cap G_j g_2 \neq \emptyset.$$ 

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence system having a chamber. Moreover, the group $G$ acts by right multiplication as an automorphism group on $\Gamma$. Finally, the group $G$ is transitive on the flags of rank less than 3.

When a geometry $\Gamma$ is constructed using the proposition above, we denote it by $\Gamma(G; (G_i)_{i \in I})$ and call it a coset geometry. The subgroups $(G_i)_{i \in I}$ are called the maximal parabolic subgroups.

### 2.2 Coset geometries from $C^+$-groups

As in [2], consider a pair $(G^+, R)$ with $G^+$ being a group and $R := \{\alpha_1, \ldots, \alpha_{r-1}\}$ a set of generators of $G^+$. Define $\alpha_0 := 1_{G^+}$ and $\alpha_{ij} := \alpha_i^{-1} \alpha_j$ for all $i, j \in I := \{0, \ldots, r-1\}$. Let $G^+_J := \langle \alpha_{ij} | i, j \in J \rangle$ for $J \subseteq I$.

If the pair $(G^+, R)$ satisfies the following condition called the intersection property IP$^+$, we say that $(G^+, R)$ is a $C^+$-group.

$$G^+_J \cap G^+_K = G^+_{J \cap K},$$

for all $J, K \subseteq I$, with $|J|, |K| \geq 2$. The following construction produces an incidence system from a $C^+$-group.

**Construction 2.1.** [2] Let $R := \{\alpha_1, \ldots, \alpha_{r-1}\}$ be an independent generating set of the group $G^+$. Define $G_i := \langle \alpha_j | j \neq i \rangle$ for $i = 1, \ldots, r-1$ and $G_0 := \langle \alpha_1^{-1} \alpha_j | j \geq 2 \rangle$. The coset geometry $\Gamma(G^+, R) := \Gamma(G^+; (G_i)_{i \in \{0, \ldots, r-1\}})$ constructed using Tits’ algorithm (see Proposition 1) is the incidence system associated to the pair $(G^+, R)$. 

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We denote $\Gamma(G^+, R)$ simply by $\Gamma$ whenever $G^+$ and $R$ are clear from the context. We set $I := \{0, \ldots, r - 1\}$, where $r := |R| + 1$.

It is convenient to represent $(G^+, R)$ by a graph $\mathcal{B}$ with $r$ vertices which we call the $B$-diagram of $(G^+, R)$. The vertex set of $\mathcal{B}$ is the set $\{\alpha_0, \ldots, \alpha_{r-1}\}$. The edges $\{\alpha_i, \alpha_j\}$ of this graph are labelled by $o(\alpha_i^{-1}\alpha_j) = o(\alpha_j^{-1}\alpha_i)$. We take the convention of dropping an edge if its label is 2 and of not writing the label if it is 3. Vertices of $\mathcal{B}$ are represented by small circles. Finally, we sometimes attach to each vertex $\alpha_i$ the corresponding subgroup $G_i$ defined in Construction 2.1.

Observe that, thanks to Proposition 1 that ensures that $G^+$ is transitive on the flags of rank less than 3, we not only know the number of elements of type $i$ for every $i$, that is the index of $G_i$ in $G^+$, but also the number of elements of type $j$ incident to a given element of type $i$, that is the index of $G_i \cap G_j$ in $G_i$.

The residue of the element $G_i$ is the coset geometry $\Gamma(G_i, (G_i \cap G_j)_{j \in I \setminus \{i\}})$. Its diagram is obtained by removing the vertex $\alpha_i$ from the diagram of $\Gamma(G^+; (G_i)_{i \in I})$.

The coset geometry $\Gamma(G^+, R)$ gives an incidence system using Proposition 1. In what follows we prove that any such coset geometry has a connected incidence graph if its rank is at least 3.

**Proposition 2.** If $|R| \geq 2$, then $\Gamma(G^+, R)$ has a connected incidence graph.

**Proof.** As for any $g \in G^+$, $\{G_i g | i \in I\}$ is a set of mutually incident elements of $\Gamma$, it is sufficient to prove that $G_i$ and $G_i g$ are in the same connected component of the incidence graph (of $\Gamma$) for every $g \in G^+$ and every $i \in I$. As $G^+ = \langle \alpha_i | i \in I \rangle$, we can assume $g \in \{\alpha_i : i \in I\}$. If $i \neq 0$, then $G_i \cap G_i \alpha_j = G_i$ for any $j \in I \setminus \{i\}$. Moreover, $G_i G_k$ and $G_k G_i \alpha_i$ for any $k \in I \setminus \{0, i\}$, which is a non-empty set since $|I| - 2 = |R| - 1 \geq 1$. If $i = 0$, then $G_0 G_k$ and $G_k G_0 \alpha_j$ for any $k \in I \setminus \{1, j\}$. \hfill $\square$

Although the incidence geometry $\Gamma$ has a connected incidence graph, it need not be residually connected. Moreover, $\Gamma$ might not be a thin geometry, and hence $\Gamma$ need not be a hypertope. Furthermore, in general $\Gamma$ might not be transitive on flags of rank 3, and it might have many orbits of chambers. In the construction we shall give in the next section, the geometry that we obtain will in fact be a hypertope and have only two orbits of chambers. The following theorem will help us to decide if a hypertope is regular or chiral.

**Theorem 3.** [2] Let $(G^+, R)$ be a $C^+$-group. Let $\Gamma := \Gamma(G^+, R)$ be the coset geometry associated to $(G^+, R)$ using Construction 2.1. If $\Gamma$ is a hypertope and $G^+$ has two orbits on the set of chambers of $\Gamma$, then $\Gamma$ is chiral if and only if there is no automorphism of $G^+$ that inverts all the elements of $R$. Otherwise, there exists an automorphism $\sigma \in \text{Aut}(G^+)$ that inverts all the elements of $R$ and the group $G^+$ extended by $\sigma$ is regular on $\Gamma$.

### 2.3 Toroidal hypermaps of type $(3,3,3)_{(a,b)}$

The hypertopes we shall construct in Section 4 have residues of rank 3 with a known structure, namely, they are toroidal hypermaps of type $(3,3,3)$. 

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If $\Gamma$ is a hypertope of rank three, then it is also a hypermap. A toroidal hypertope (or toroidal hypermap) is either a map or a hypermap embeddable on the torus. The toroidal (regular or chiral) hypertopes of rank 3 are divided into the following families: the toroidal maps $\{3, 6\}_{(a,b)}$, $\{6, 3\}_{(a,b)}$, $\{4, 4\}_{(a,b)}$, and the hypermaps $\{3, 3, 3\}_{(a,b)}$ with $(a,b) \neq (1,1)$. Note that the hypermap $\{3, 3, 3\}_{(a,b)}$ is obtained from the toroidal map $\{6, 3\}_{(a,b)}$ by doubling the fundamental region. Indeed as $\{6, 3\}_{(a,b)}$ is bipartite it is possible to take one monochromatic set of vertices to be the hyperedges of the hypermap $\{3, 3, 3\}_{(a,b)}$ (see [8]).

The rotation subgroup $G$ of the automorphism group of a rank three toroidal hypermap is as follows for some integers $a$ and $b$.

$$\langle x, y \mid x^3, y^3, (x^{-1}y)^3, (xy^{-1}x)^a(xy)^b \rangle$$

This hypermap is denoted by $(3, 3, 3)_{(a,b)}$. The above presentation readily shows that such a hypermap has a B-diagram that is a triangle with no numbers on the edges.

2.4 $\text{PSL}(2,q)$ acting on the projective line

For a prime power $q = p^n$ let $GF(q)^*$ be the multiplicative group of the Galois field $GF(q)$ on $q$ elements.

A primitive element of $GF(q)$ is a generator of the multiplicative group $GF(q)^*$.

For any positive integer $k < q$, a solution of $x^k = 1$ which is not a solution of $x^j = 1$ for any $j \in \{1, \ldots, k - 1\}$ is called a $k$th primitive root of unity. If $i$ is a $k$th primitive root of unity, then $k$ divides $q - 1$ and $\sum_{j=0}^{k-1} i^j = 0$.

Let $V$ be the 2-dimensional vector space $GF(q)^2$ over $GF(q)$. Consider the relation $\sim$ in $V \setminus \{(0,0)\}$ defined as follows.

$$(x_0, x_1) \sim (y_0, y_1) \text{ if and only if } (y_0, y_1) = \lambda(x_0, x_1), \text{ for some } \lambda \in GF(q)^*.$$ 

The projective line $\text{PG}(1,q)$ is the set of equivalence classes $V \setminus \{(0,0)\} / \sim$. The elements of the projective line $[x_0, x_1]$ can be identified with their non-homogeneous coordinates by the following bijection where $i$ is a $(q - 1)$th root of unity.

$$\begin{align*}
\text{PG}(1,q) &\rightarrow \{0, 1, i, i^2, \ldots, i^{q-2}, \infty\}, \\
[x_0, x_1] &\rightarrow \begin{cases} 
\frac{x_0}{x_1} & \text{if } x_1 \neq 0, \\
\infty & \text{if } x_1 = 0.
\end{cases}
\end{align*}$$

Consider the special linear group $\text{SL}(2,q) = \{A \in \text{GL}(2,q) \mid \det(A) = 1\}$ and denote by $Id$ its identity matrix. As $\text{PSL}(2,q) = \text{SL}(2,q)/Z$ where $Z = \{\pm Id\}$, the elements of $\text{PSL}(2,q)$ can be seen as unordered pairs $\pm A$ with $A \in \text{SL}(2,q)$. Observe that in characteristic 2, $|Z| = 1$. For convenience, we shall denote $\text{PSL}(2,q)$ by $G^+$. An element of $G^+$ will be given by one of its two representative elements in $\text{SL}(2,q)$. Consequently
equalities are to be taken modulo $Z = \{\pm \text{Id}\}$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,q)$ and $z \in \text{PG}(1,q)$, consider the correspondence

$$\varphi : G^+ \times \text{PG}(1,q) \to \text{PG}(1,q), \ (\pm A, z) \mapsto Az$$

defined as follows.

$$Az = \frac{az + b}{cz + d}.$$ 

As $-Az = Az$, $\varphi$ is well-defined and gives an action of $G^+$ on $\text{PG}(1,q)$.

**Lemma 4.** [6, Lemma 5.3] Let $q$ be a power of a prime. Then $\text{SL}(2,q)$ is generated by the elementary matrices of the form

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ with } b, c \in \text{GF}(q).$$

### 3 A C$^+$-group for $\text{PSL}(2,q)$

Let $q \in \{p, p^2\}$ for a prime number $p$ with $3$ being a divisor of $q - 1$, and let $e \in \text{GF}(q)$ be a third primitive root of unity. In the case $q = p^2$ we also assume that $3$ is not a divisor of $p - 1$ so that

$$\text{GF}(p^2) \cong \{a + eb : a, b \in \text{GF}(p)\}.$$ 

In this section we prove that $\text{PSL}(2,q)$ is a $\text{C}^+$-group with the B-diagram of Figure 1 (where the labels of the corners are the groups corresponding to each rank 3 residue).

In order to construct this $\text{C}^+$-group we consider the following elements of $\text{PSL}(2,q)$:

$$\alpha_1 := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \ \alpha_2 := \begin{pmatrix} e & e^2 \\ 0 & e^2 \end{pmatrix} \text{ and } \alpha_3 := \begin{pmatrix} e^2 & e \\ 0 & e \end{pmatrix},$$

and we let $G^+ = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, $G_i := \langle \alpha_j | j \neq i \rangle$ for $i = 1, 2, 3$ and $G_0 := \langle \alpha_1^{-1} \alpha_j | j \geq 2 \rangle$. In what follows we show that $G_2 := \langle \alpha_1, \alpha_3 \rangle$ and $G_3 := \langle \alpha_1, \alpha_2 \rangle$ are both isomorphic to $A_4$, while $G_0 := \langle \alpha_1^{-1} \alpha_2, \alpha_1^{-1} \alpha_3 \rangle$ and $G_1 := \langle \alpha_2, \alpha_3 \rangle$ are both isomorphic to $E_q : C_3$.

**Lemma 5.** The groups $G_2$ and $G_3$ are both isomorphic to $A_4$ and their intersection is $\langle \alpha_1 \rangle$. Moreover $(G_2, \{\alpha_1, \alpha_3\})$ and $(G_3, \{\alpha_1, \alpha_2\})$ are $\text{C}^+$-groups with triangular B-diagram of type $(3,3,3)$. 

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**Figure 1:** Tetrahedral B-diagram for $\text{PSL}(2,q)$. 

$E_q : C_3 \quad \begin{array}{c} \text{4} \\ \text{3} \end{array} \quad A_4 \\
E_q : C_3 \quad \begin{array}{c} \text{4} \\ \text{3} \end{array} \quad A_4$
Proof. We first observe that, for each \( i \in \{2, 3\} \), \( \{\alpha_1, \alpha_i\} \) is an independent set, which is sufficient to guarantee that it gives a set of generators of a \( C^+ \)-group. Now let us compute the order of the elements that give the B-diagram. It is straightforward to see that \( \alpha_1 \) has order 3. To compute the order of the other elements we use the fact that \( e \) is a third root of unity and hence \( e^2 + e + 1 = 0 \). Therefore the following equalities hold for any \( c \in GF(q) \).

\[
\begin{pmatrix}
    e & 0 \\
    c & e^2
\end{pmatrix}^2 = \begin{pmatrix}
    e^2 & 0 \\
    -c & e
\end{pmatrix}, \quad \begin{pmatrix}
    e & 0 \\
    c & e^2
\end{pmatrix}^3 = Id;
\]

\[
\begin{pmatrix}
    e & c \\
    0 & e^2
\end{pmatrix}^2 = \begin{pmatrix}
    e^2 & -c \\
    0 & e
\end{pmatrix}, \quad \begin{pmatrix}
    e & c \\
    0 & e^2
\end{pmatrix}^3 = Id.
\]

In particular, this shows that both \( \alpha_2 \) and \( \alpha_3 \) have order 3. In addition,

\[
\alpha_1^{-1}\alpha_2 = \begin{pmatrix}
    e & 0 \\
    e & e^2
\end{pmatrix} \quad \text{and} \quad \alpha_1^{-1}\alpha_3 = \begin{pmatrix}
    e^2 & 0 \\
    e & e
\end{pmatrix},
\]

hence these elements have also order 3.

To prove that the groups are isomorphic to \( A_4 \) we first observe that for \( i \in \{2, 3\} \)

\[
\alpha_3^3 = \alpha_i^3 = (\alpha_1^{-1}\alpha_i)^3 = (\alpha_1\alpha_i^{-1}\alpha_1)^2 = Id.
\]

As \( Id, \alpha_1, \alpha_i, \alpha_i^2, \alpha_i\alpha_1, \alpha_1\alpha_i, \alpha_i^2, \alpha_1\alpha_i^2, \alpha_i\alpha_1^2, \alpha_1\alpha_i^2, \alpha_i^2\alpha_1, \alpha_1\alpha_i^2, \alpha_1^2\alpha_1 \) are distinct elements of \( \langle \alpha_1, \alpha_i \rangle \), from 1 we conclude that \( G_i \) is isomorphic to \( A_4 \), the rotational subgroup of the hypermap of type \( (3,3,3)_{(2,0)} \). From the given enumeration of the elements of these two groups we conclude that their intersection is \( \langle \alpha_1 \rangle \).

Figures 2 and 3 show that \( G_2 \) is in the stabilizer of \( \{0, 1, \infty, -e^2\} \), while \( G_3 \) is in the stabilizer of \( \{0, 1, \infty, -e\} \) (and both groups are isomorphic to \( A_4 \)).
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Moreover

\[ G := \text{groups with triangular } B\text{-diagram of type } (3) \]

When \( q = 0 \) are \( \mathbb{C} \)-groups. Now let us compute the order of the elements that give the \( G \) and \( (1,0,\infty,-e) \)

Figure 3: Action of \( G_3 \) on the 4-tuple \((0, 1, \infty, -e)\).

Lemma 6. The groups \( G_0 \) and \( G_1 \) are both isomorphic to \( E_2 : C_3 \), more precisely, \( G_0 = H_0 : \langle \alpha_2^{-1} \alpha_3 \rangle \) and \( G_1 = H_1 : \langle \alpha_2^{-1} \alpha_3 \rangle \) where \( H_0 \) and \( H_1 \) are the following elementary abelian \( p \)-groups of order \( q \).

\[
H_0 = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in GF(q) \right\} \quad \text{and} \quad H_1 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in GF(q) \right\}.
\]

Moreover \( G_0 \cap G_1 = \langle \alpha_2^{-1} \alpha_3 \rangle \). Finally, \((G_0, \{\alpha_i^{-1} \alpha_2, \alpha_1^{-1} \alpha_3\})\) and \((G_1, \{\alpha_2, \alpha_3\})\) are \( C^+ \)-groups with triangular \( B\)-diagram of type \((3,3,3)\).

Proof. First observe that the sets \( \{\alpha_2, \alpha_3\} \) and \( \{\alpha_1^{-1} \alpha_2, \alpha_1^{-1} \alpha_3\} \) are independent, thus \( G_0 \) and \( G_1 \) are \( C^+ \)-groups. Now let us compute the order of the elements that give the \( B\)-diagram. As noted above, the elements \( \alpha_2, \alpha_3, \alpha_1^{-1} \alpha_2 \) and \( \alpha_1^{-1} \alpha_3 \) have order 3. Now \( E := \alpha_2^{-1} \alpha_3 = \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix} \) has also order 3 and belongs to \( G_0 \cap G_1 \).

Consider the following subgroups of \( PSL(2, q) \):

\[
H_0 = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in GF(q) \right\} \quad \text{and} \quad H_1 = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in GF(q) \right\}
\]

When \( q = p \), \( H_0 \) and \( H_1 \) are cyclic groups of order \( p \). More precisely,

\[
H_0 = \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle \quad \text{and} \quad H_1 = \langle \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \rangle.
\]

When \( q = p^2 \), \( H_0 \) and \( H_1 \) are elementary abelian groups of order \( p^2 \). More precisely,

\[
H_0 = \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \rangle \quad \text{and} \quad H_1 = \langle \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & e \\ 1 & 0 \end{pmatrix} \rangle.
\]
since we assumed that 3 does not divide $p - 1$ and therefore $GF(q) = \{a + eb : a, b \in GF(p)\}$.

We claim that $G_0 \cong H_0 : \langle E \rangle$ and $G_1 \cong H_1 : \langle E \rangle$. The following equalities prove that $H_0$ and $H_1$ are subgroups of $G_0$ and $G_1$, respectively.

$$
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} = (\alpha_1^{-1}\alpha_3)(\alpha_1^{-1}\alpha_2)^{-1}(\alpha_1^{-1}\alpha_3) \quad \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} = \alpha_2\alpha_3^{-1}\alpha_2
$$

(2)

On the other hand, as

$$
\alpha_1^{-1}\alpha_2 = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
e & 0 \\
e^2 & 0
\end{pmatrix} \quad \alpha_1^{-1}\alpha_3 = \begin{pmatrix}
1 & 0 \\
e^2 & 1
\end{pmatrix} \begin{pmatrix}
0 & e \\
0 & e
\end{pmatrix}
$$

we have that $\alpha_1^{-1}\alpha_2, \alpha_1^{-1}\alpha_3 \in H_0 : \langle E \rangle$, while

$$
\alpha_2 = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
e & 0 \\
e^2 & 0
\end{pmatrix} \quad \alpha_3 = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & e \\
0 & e
\end{pmatrix}
$$

show that $\alpha_2, \alpha_3 \in H_1 : \langle E \rangle$.

With this we have that the groups $G_1$ and $G_0$ are both isomorphic to $E_i : \langle E \rangle$.

Since an element of $G_0$ is equal to $\begin{pmatrix} e^i & 0 \\
c & e^{2i} \end{pmatrix}$ for some $c \in GF(q)$ and $i \in \{0, 1, 2\}$, while an element of $G_1$ is equal to $\begin{pmatrix} e^j & b \\
0 & e^{2j} \end{pmatrix}$ for some $b \in GF(q)$ and $j \in \{0, 1, 2\}$, we have that $G_0 \cap G_1 = \langle E \rangle$. \hfill \Box

**Lemma 7.** $(G^+, \{\alpha_1, \alpha_2, \alpha_3\})$ is a $C^+$-group with tetrahedral B-diagram as in Figure 1.

**Proof.** By Lemmas 5 and 6 we have that $G_i$, $i = 0, 1, 2, 3$ are $C^+$-groups, $G_2 \cap G_3 = \langle \alpha_1 \rangle$ and $G_1 \cap G_0 = \langle \alpha_2^{-1}\alpha_3 \rangle$. We have that $G_0$ is a subgroup of the stabilizer of 0 in $G^+$ and $G_1$ is a subgroup of the stabilizer of $\infty$ in $G^+$. Figure 2 shows that the only elements of $G_2$ fixing 0 are the identity, $\alpha_2^2\alpha_1$ and $\alpha_2^3\alpha_3$, hence $G_0 \cap G_2 = \langle \alpha_1^{-1}\alpha_3 \rangle$. The same figure shows that the only elements of $G_2$ fixing $\infty$ are the identity, $\alpha_3$ and $\alpha_3^2$, thus $G_1 \cap G_2 = \langle \alpha_3 \rangle$. Figure 3 shows that the only elements of $G_3$ fixing 0 are the identity, $\alpha_3^2\alpha_1$ and $\alpha_3^2\alpha_2$, hence $G_0 \cap G_3 = \langle \alpha_1^{-1}\alpha_2 \rangle$. The same figure shows that the only elements of $G_3$ fixing $\infty$ are the identity, $\alpha_2$ and $\alpha_2^2$, thus $G_1 \cap G_3 = \langle \alpha_2 \rangle$. From this we conclude that $(G^+, \{\alpha_1, \alpha_2, \alpha_3\})$ is a $C^+$-group. Moreover, as $G_i \cap G_j$ are cyclic groups of order three, for $i, j \in \{0, 1, 2, 3\}$ with $(i \neq j)$, the B-diagram is a complete graph $K_4$ with edges labelled 3. \hfill \Box

**Lemma 8.** The group $G^+$ is isomorphic to $PSL(2, q)$.

**Proof.** This follows from the description of the subgroups $H_0$ and $H_1$ in Lemma 6 and Lemma 4. \hfill \Box
4 A hypertope for $PSL(2, q)$

As pointed out in [2], not every coset geometry is a hypertope. In this section we shall see that the coset geometry $\Gamma := \Gamma(G^+, R) := \Gamma(G^+; (G_i)_{i \in I})$ where $I = \{0, 1, 2, 3\}$, $G^+$ is isomorphic to $PSL(2, q)$ and the $G_i$'s are the groups defined in Section 3, is a hypertope. Since $\Gamma$ is a coset geometry, then we only have to show that it is thin and residually connected.

We observe that $\Gamma$ has a lot of symmetry, and we shall use such symmetry to show that it is a hypertope. Note that since $G^+$ is a $C^+$-group, then $G_1 \cap G_2 \cap G_3$ is trivial. This implies that the action of $G^+$ on the flags by right multiplication is free (semi-regular) on the chambers. Moreover, as $\Gamma$ is a coset geometry, it is transitive on the flags of rank less than 3. We shall show that in fact $\Gamma$ is transitive on the flags of rank 3. To show this, in the following two lemmas we analyze some incidences among elements of $\Gamma$, and some incidences among elements and flags of $\Gamma$.

From the B-diagram of $G^+$, since $G_i$ with $i \in \{2, 3\}$ is isomorphic to $A_4$ and $G_i \cap G_j \cong C_3$, we know that each element of type $i \in \{2, 3\}$ is incident to exactly four elements of type $j \neq i$. Similarly, since $G_i$ with $i \in \{0, 1\}$ is isomorphic to $E_q : C_3$, we have that each element of type $i \in \{0, 1\}$ is incident to exactly $q$ elements of type $j \neq i$. Remark also that, if $G_i g * G_j$ then $G_i^g = G_i h$ for some $h \in G_j$. Moreover, if $T$ is a transversal for $G_i \cap G_j$ in $G_j$, then $G_i g = G_i h = G_i t$ for some $t \in T$. More precisely we have the following lemma.

**Lemma 9.** Let $i, j \in I$ with $i \neq j$ and let $g \in G^+$. Then $G_i g * G_j \iff \left\{ \begin{array}{ll} G_g \in \{G_i h \mid h \in H_j\} & \text{if } j \in \{0, 1\}; \\ G_g \in \{G_i, G_j, G_i \alpha_i, G_i \alpha_i \alpha_k, G_i \alpha_i \alpha_k^{-1}\} & \text{if } j \in \{2, 3\} \text{ and } \{i, j, k\} = \{1, 2, 3\}; \\ G_g \in \{G_0, G_0 \alpha_1, G_0 \alpha_1^{-1}, G_0 \alpha_k\} & \text{if } j \in \{2, 3\} \text{ and } \{i, j, k\} = \{0, 2, 3\}. \end{array} \right.$

**Proof.** It can be easily checked that all of the cosets in the set on the right side of the equivalences are incident to the respective $G_j$.

Suppose that $j \in \{0, 1\}$. By Lemma 6, $H_j$ is a subgroup of $G_j$ of order $q$. Thus $g \in H_j$ implies that $G_i g \cap G_j \neq \emptyset$. As $G_i \cap G_j$ has order 3, if $G_i g * G_j$, then $|G_i g \cap G_j| = 3$. It remains to prove that the cardinality of $\{G_i g \mid g \in H_j\}$ is $q$. Suppose that $g, h \in H_j$ and $G_i g = G_i h$. Then $gh^{-1} \in G_i \cap H_j = (G_i \cap G_j) \cap H_j$. In all the cases $G_i \cap G_j$ is generated.

**Figure 4:** The intersections $G_i \cap G_j$. 

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by an element of order 3 which does not belong to $H_j$, therefore all the intersections $(G_i \cap G_j) \cap H_j$ are trivial, and so $g = h$.

For the cases when $j \in \{2, 3\}$, by Lemma 7 (see also Figure 4 we have the following:

- If $i \neq 0$, then $G_i \cap G_j = \langle \alpha_k \rangle$ where $\{i, j, k\} = \{1, 2, 3\}$. Hence
  \[ T = \{ \text{Id}, \alpha_i, \alpha_i \alpha_k, \alpha_i \alpha_k^{-1} \} \]
  is a transversal for $G_i \cap G_j$ in $G_j$ and $G_i g = G_i t$ for some $t \in T$.
- If $i = 0$, then $G_0 \cap G_j = \langle \alpha_1^{-1} \alpha_k \rangle$ where $\{j, k\} = \{2, 3\}$. Hence
  \[ T = \{ \text{Id}, \alpha_1, \alpha_1^{-1}, \alpha_k \} \]
  is a transversal for $G_0 \cap G_j$ in $G_j$ and $G_0 g = G_0 t$ for some $t \in T$. \[\square\]

**Proposition 10.** Let $i, j, k \in I$ be distinct and let $g \in G^+$. Then

\[ G_i g \ast \{G_j, G_k\} \iff G_i g = G_i h \text{ for some } h \in G_j \cap G_k. \]

**Proof.** Obviously the right hand side of the equivalence implies the left one. The proof of the converse will be divided in cases covering all the possibilities for the set $\{j, k\}$.

Assume that $G_i g \ast \{G_j, G_k\}$.

**Case 1.** If $\{j, k\} = \{0, 1\}$, then $i \in \{2, 3\}$ and

\[ G_i g \in \{G_i h_0 \mid h_0 \in H_0\} \cap \{G_i h_1 \mid h_1 \in H_1\} \]

by means of Lemma 9. Looking at the 12 elements of $G_i$, we see that for $h_0 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H_0$ and $h_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H_1$,

\[ h_0 h_1^{-1} = \begin{pmatrix} 1 & -b \\ c & 1 - bc \end{pmatrix} \in G_i \Leftrightarrow (b, c) \in \{(0, 0), (1, 1), (-e^{1-i}, -e^{-1-i})\}. \]

As before, we set $E := \alpha_3^{-1} \alpha_3$. We have to prove that $G_i g = G_i h$ for some $h \in \langle E \rangle = G_0 \cap G_1$ (Figure 4). If $(b, c) = (0, 0)$, then $G_i g = G_i h_0 = G_i h_1 = G_i$. If $(b, c) = (1, 1)$, then $h_1 = \alpha_2 \alpha_3^{-1} \alpha_2$ by 2. Hence

\[ G_i g = G_i h_0 = G_i h_1 = G_i \alpha_2 \alpha_3^{-1} \alpha_2 = G_i E^{i-1} \]

since $G_2 \alpha_2 \alpha_3^{-1} \alpha_2 = G_2 (\alpha_3^{-1} \alpha_2)^2 = G_2 E^{-2} = G_2 E$ and $G_3 \alpha_2 \alpha_3^{-1} \alpha_2 = G_3 \alpha_3^{-1} \alpha_2 = G_2 E^{-1} = G_3 E^2$.

If $(b, c) = (-e^{1-i}, -e^{-1-i})$, then

\[ G_i g = G_i h_0 = G_i h_1 = G_i h_1 E^{i-1} E^{1-i} = G_i E^{1-i} \]
since $h_1E^{i-1} = \left( \begin{array}{cc} e^{1-i} & e^{l-1} \\ 0 & e^{l-1} \end{array} \right)^{-1} \in G_i$. Thus $G_1g = G_1h$ for some $h \in \langle E \rangle = G_0 \cap G_1$.

**Case 2.** If $\{j, k\} = \{0, 2\}$ or $\{0, 3\}$, then $i \neq 0$ and we can assume $k = 0$, $j \in \{2, 3\}$. By means of Lemma 9 there is $h_0 \in H_0$ such that

$$G_1g = G_1h_0 \in \{G_i, G_i\alpha_i, G_i\alpha_i\alpha_l, G_i\alpha_i\alpha_l^{-1}\}$$

where $l \in I$ is such that $\{i, j, k, l\} = I$. In other words, $G_1g$ must belong to the set

$$S := \{G_1t \mid t \in \{Id, \alpha_i, \alpha_i\alpha_l, \alpha_i\alpha_l^{-1}\}, H_0 \cap G_1t \neq \emptyset \}.$$ 

Obviously $G_i \in S$. To see that $G_i\alpha_i \in S$, consider the following elements:

$$\beta_1 := \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \in G_1, \quad \beta_2 := \alpha_1\alpha_3^{-1}\alpha_1 \in G_2 \quad \text{and} \quad \beta_3 := \alpha_1\alpha_2^{-1}\alpha_1 \in G_3.$$ 

Then $\beta_i\alpha_i \in H_0 \cap G_i\alpha_i$ for any $i \in \{1, 2, 3\}$. To show that $G_i\alpha_i\alpha_l \not\in S$ we remark that

$$\alpha_i\alpha_l(0) = \alpha_i(1) = \begin{cases} \infty & \text{if } i = 1, \\ -e & \text{if } i = 2, \\ -e^2 & \text{if } i = 3, \end{cases}$$

from which it follows $H_0 \cap G_i\alpha_i\alpha_l = \emptyset$, since there is no element in $G_i$ sending $\alpha_i(1)$ to 0. Finally, considering the following elements

$$\gamma_1 := E^{j-1} \in G_1, \quad \gamma_2 := \alpha_1\alpha_3 \in G_2 \quad \text{and} \quad \gamma_3 := \alpha_1\alpha_2 \in G_3,$$

we have that $\gamma_i\alpha_i\alpha_l^{-1} \in H_0$ for any $i \in \{1, 2, 3\}$. Hence $G_i\alpha_i\alpha_l^{-1} \in S$ and so

$$S = \{G_i, G_i\alpha_i, G_i\alpha_i\alpha_l^{-1}\}.$$ 

It is now straightforward to check that $S = \{G_ih \mid h \in \langle \alpha_l^{-1}\alpha_l \rangle \}$, from which we conclude that $G_1g = G_1h$ for some $h \in \langle \alpha_l^{-1}\alpha_l \rangle = G_0 \cap G_j$ (Figure 4).

**Case 3.** If $\{j, k\} = \{1, 2\}$ or $\{1, 3\}$ we assume $k = 1$ and $j \in \{2, 3\}$. Let $l \in I$ such that $\{j, l\} = \{2, 3\}$. According to Lemma 9 we have the following:

- If $i = 0$ then $G_1g = G_0g$ belongs to

$$\{G_0t \mid t \in \{Id, \alpha_1, \alpha_1^{-1}, \alpha_l\}, H_1 \cap G_0t \neq \emptyset \}$$

which is equal to $\{G_0, G_0\alpha_1^{-1}, G_0\alpha_l\}$. Namely, $E_1, E_2$ and $\left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right)$ are elements of $G_0$ such that

$$E_2\alpha_2, E_3, \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) \alpha_1^{-1} \in H_1,$$

while $\alpha_1(\infty) = 0$ gives $H_1 \cap G_0\alpha_1 = \emptyset$. As $G_0\alpha_1^{-1} = G_0\alpha_l^{-1}$, we conclude that $G_0g = G_0h$ for some $h \in \langle \alpha_l \rangle = G_j \cap G_1 = G_j \cap G_k$ (Figure 4).
• If \( i \neq 0 \) then \( G_i g \) belongs to

\[
\{ G_i t \mid t \in \{ Id, \alpha_i, \alpha_i \alpha_1, \alpha_i \alpha_1^{-1} \}, \ H_1 \cap G_i t \neq \emptyset \}
\]

which is equal to \( \{ G_i, G_i \alpha_i, G_i \alpha_1 \} \). Namely, \( \alpha_j \alpha_i, \alpha_i \alpha_j \in H_1, G_i \alpha_i \alpha_1 = G_i \alpha_i^{-1} \alpha_1^{-1} = G_1 \alpha_1^{-1}, \) while \( \alpha_i \alpha_1^{-1}(\infty) = \alpha_i(1) = -e^{i-1} \) gives \( H_1 \cap G_i \alpha_1^{-1} = \emptyset \). We conclude so that \( G_i g = G_i h \) for some \( h \in \langle \alpha_i \rangle = G_j \cap G_1 = G_j \cap G_k \) (Figure 4).

**Case 4.** If \( \{ j, k \} = \{ 2, 3 \} \), then by Lemma 9 we have the following:

• If \( i = 0 \), then \( G_i g = G_0 g \) belongs to

\[
\{ G_0, G_0 \alpha_1, G_0 \alpha_1^{-1}, G_0 \alpha_2 \} \cap \{ G_0, G_0 \alpha_1, G_0 \alpha_1^{-1}, G_0 \alpha_3 \}
\]

which is equal to \( \{ G_0, G_0 \alpha_1, G_0 \alpha_1^{-1} \} \) since \( \alpha_2 \alpha_3^{-1} \notin G_0 \).

• If \( i = 1 \), then \( G_i g = G_1 g \) belongs to

\[
\{ G_1, G_1 \alpha_1, G_1 \alpha_1 \alpha_2, G_1 \alpha_1 \alpha_2^{-1} \} \cap \{ G_1, G_1 \alpha_1, G_1 \alpha_1 \alpha_3, G_1 \alpha_1 \alpha_3^{-1} \}.
\]

Remark that \( G_1 \alpha_1 \alpha_k = G_1 \alpha_1^{-1} \alpha_k^{-1} = G_1 \alpha_1^{-1} \) by means of the fact that \( \alpha_1 \alpha_k \) is an involution for \( k = 2, 3 \). Moreover \( (\alpha_1 \alpha_2^{-1})(\alpha_1 \alpha_3^{-1})^{-1} \notin G_1 \) because it does not fix \( \infty \). Hence \( G_1 g \) belongs to \( \{ G_1, G_1 \alpha_1, G_1 \alpha_1^{-1} \} \). \( \square \)

We are now ready to show that \( G^+ \) is transitive on the flags of rank 3.

**Proposition 11.** Let \( J = \{ i, j, k \} \subset \{ 0, 1, 2, 3 \} \) with \( i, j \) and \( k \) all distinct. The action of \( G^+ \) on the flags of type \( J \) is transitive.

**Proof.** Let \( F = \{ G_i, G_j, G_k \} \) and \( F' = \{ G_i a_i, G_j a_j, G_k a_k \} \). By Proposition 1, \( G^+ \) is transitive on the flags of rank 2. Hence, there exists a flag \( F'' \) and \( a \in G^+ \) such that \( F' = F'' a \) with \( F'' = \{ G_i g, G_j, G_k \} \). Without loss of generality we may assume that \( (i, j, k) \in \{ (0, 2, 3), (1, 2, 3), (0, 1, 3), (0, 1, 2) \} \). Then by Proposition 10, there exists \( h \in G_i g \cap G_j \cap G_k \). Hence \( F'' = Fh \) and therefore \( F' = Fha \). \( \square \)

To show that \( \Gamma \) is a thin geometry, we need to show that every residue of rank 1 has exactly two elements. A residue of rank 1 is the set of elements of \( \Gamma \) that are incident to a flag of type \( J \subset \{ 0, 1, 2, 3 \} \), with \( |J| = 3 \). By Proposition 11, all residues of rank 1 corresponding to flags of a given type \( J \) are isomorphic. Hence, it is enough to show that, for \( \{ i, j, k, l \} = \{ 0, 1, 2, 3 \} \), the sets \( \{ G_j g \mid G_j g \ast \{ G_i, G_k, G_l \} \} \) all have cardinality two.

**Proposition 12.** The coset geometry \( \Gamma \) is a thin geometry.

**Proof.** Following the above discussion, we need to show that four sets have cardinality two. To do this, we shall show that:

\[
\begin{align*}
G_0 g \ast \{ G_1, G_2, G_3 \} &\iff G_0 g \in \{ G_0, G_0 \alpha_1^{-1} \}; \\
G_1 g \ast \{ G_0, G_2, G_3 \} &\iff G_1 g \in \{ G_1, G_1 \alpha_1 \}; \\
G_2 g \ast \{ G_0, G_1, G_3 \} &\iff G_2 g \in \{ G_2, G_2 \alpha_2 \}; \\
G_3 g \ast \{ G_0, G_1, G_2 \} &\iff G_3 g \in \{ G_3, G_3 \alpha_3 \}.
\end{align*}
\]

\( \square \)
It is straightforward to see that the elements on the sets of the right are incident to the flags on the left.

We start by assuming that $G_i g \in \{G_j, G_2, G_3\}$ where $\{i, j\} = \{0, 1\}$. As $G_i g \in \{G_2, G_3\}$, by Proposition 10, $G_i g \in \{G_i, G_i \alpha_1, G_i \alpha_1^{-1}\}$. As we also have $G_i g \ast G_j$ we observe that

\[
\left(\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array}\right) \alpha_1^{-1} \in G_0 \alpha_1^{-1} \cap G_1
\]

while $G_0 \alpha_1 \cap G_1 = \emptyset$ since any element of $G_0 \alpha_1$ sends $\infty$ to 0 and so it does not belong to $G_1$ which is in the stabilizer of $\infty$. As $G_0 g \ast G_1 \Leftrightarrow G_0 \ast G_1 g^{-1}$, this proves the first two equivalences in 3.

Now let $G_i g \in \{G_0, G_1, G_j\}$ where $\{i, j\} = \{2, 3\}$. As $G_i g \ast G_0$ and $G_i g \ast G_1$, there exists an element of $G_i g$ fixing 0 and an element fixing $\infty$. In addition, as $G_i g \ast G_j$, by Lemma 9, $G_i g \in \{G_i, G_i \alpha_i, G_i \alpha_i \alpha_1, G_i \alpha_i \alpha_1^{-1}\}$.

Let $i = 2$. Looking at Figure 2 and Figure 3, we have that $\alpha_2 \alpha_1(0) = -e$ and $\alpha_2 \alpha_1^{-1}(\infty) = -e$. As $G_2$ stabilizes $\{0, 1, \infty, -e^2\}$, there is no element in $G_2 \alpha_2 \alpha_1$ fixing 0 and there is no element in $G_2 \alpha_2 \alpha_1^{-1}$ fixing $\infty$. Thus $G_2 g \in \{G_2, G_2 \alpha_3\}$, as $\alpha_3^{-1} \alpha_2 \in G_2 \alpha_2$ fixes both 0 and $\infty$.

Now consider $i = 3$. We have that $\alpha_3 \alpha_1(0) = -e^2$ and $\alpha_3 \alpha_1^{-1}(\infty) = -e^2$. As $G_2$ stabilizes $\{0, 1, \infty, -e\}$, there is no element in $G_3 \alpha_3 \alpha_1$ fixing 0 and there is no element in $G_2 \alpha_3 \alpha_1^{-1}$ fixing $\infty$. Thus $G_3 g \in \{G_3, G_3 \alpha_3\}$, as $\alpha_2^{-1} \alpha_3 \in G_3 \alpha_3$ fixes both 0 and $\infty$.

Therefore all the equivalences in (3) are satisfied and so $\Gamma$ is a thin geometry.

In order to show that the geometry $\Gamma$ is a hypertope, we need to show that it is residually connected. That is, we need to show that each residue of rank at least two has a connected incidence graph. We already showed that all the residues of rank 3 are hypermaps of type $(3, 3, 3)$ (Lemmas 5 and 6). Since we know that all such hypermaps are residually connected, as observed in [2], that will conclude the proof. More precisely, we have the following theorem.

**Theorem 13.** $\Gamma$ is a hypertope.

*Proof.* As pointed out before, $\Gamma$ is a geometry, and by Proposition 12, it is thin. By Lemmas 5 and 6, its rank 3 residues $\Gamma_i$ are toroidal hypertopes isomorphic to the coset geometries $\Gamma(G_i, (G_i \cap G_j)_{j \in I \setminus \{i\}})$. This, together with Proposition 2, implies that $\Gamma$ is residually connected. Therefore $\Gamma$ is a hypertope. $\square$

### 4.1 The residues of rank 3

The residues of rank three of $\Gamma$ are hypermaps of type $(3, 3, 3)_{(a, b)}$ for some $a$ and $b$ that depend on the type of the residue. To determine the vector $(a, b)$ of the rank 3 residues of $\Gamma$ we now fix an order for the generators of each $G_i$. Let the first generator of $G_i$ be the one with minimal label when $i \neq 0$ and let the first generator of $G_0$ be $\alpha_1^{-1} \alpha_2$.

**Lemma 14.** The residues $\Gamma_i$ with $i \in \{2, 3\}$ are regular hypermaps of type $(3, 3, 3)_{(0, 2)}$. When $q = p^2$ the residues $\Gamma_i$ with $i \in \{0, 1\}$ are regular hypermaps of type $(3, 3, 3)_{(0, p)}$. 

*Proof.* As noted in the previous section, $\Gamma$ is a hypertope, and by Proposition 12, it is thin. By Lemmas 5 and 6, its rank 3 residues $\Gamma_i$ are toroidal hypertopes isomorphic to the coset geometries $\Gamma(G_i, (G_i \cap G_j)_{j \in I \setminus \{i\}})$. This, together with Proposition 2, implies that $\Gamma$ is residually connected. Therefore $\Gamma$ is a hypertope. $\square$
When \( q = p \) the residue \( \Gamma_0 \) is the chiral hypermap of type \( (3,3,3)_{(1,e)} \) while the residue \( \Gamma_1 \) is the chiral hypermap of type \( (3,3,3)_{(e,1)} \), where \( e \) is a third root of unity (which is an integer, as \( q = p \)).

**Proof.** For the following see 1. As \( (\alpha_1 \alpha_2)^2 = (\alpha_1 \alpha_3)^2 = 1_{G^+} \), the residues \( \Gamma_2 \) and \( \Gamma_3 \) are regular hypermaps of type \( (3,3,3)_{(0,2)} \).

Consider first the case \( q = p^2 \). In this case the order of \( (\alpha_1^{-1} \alpha_2)(\alpha_1^{-1} \alpha_3) \) is \( p \). Hence \( \Gamma_0 \) and \( \Gamma_1 \) are regular hypermaps of type \( (3,3,3)_{(0,p)} \).

Now let \( q = p \). We have that \( \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) which gives the following relation in \( G^+ \),

\[
(\alpha_2 \alpha_3^{-1} \alpha_2)^e (\alpha_2 \alpha_3) = 1_{G^+}.
\]

Hence \( \Gamma_1 \) is a chiral hypermap of type \( (3,3,3)_{(e,1)} \). We also have that \( (\alpha_1^{-1} \alpha_2)(\alpha_1^{-1} \alpha_3) = \begin{pmatrix} 1 & e \\ 1 & 1 \end{pmatrix} \) and \( (\alpha_1^{-1} \alpha_2)(\alpha_1^{-1} \alpha_3)^{-1}(\alpha_1^{-1} \alpha_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \). As \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{e} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \) is the inverse of \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \),

\[
(\alpha_1^{-1} \alpha_2)(\alpha_1^{-1} \alpha_3)^{-1}(\alpha_1^{-1} \alpha_2)[(\alpha_1^{-1} \alpha_2)(\alpha_1^{-1} \alpha_3)]^e = 1_{G^+}.
\]

Hence \( \Gamma_0 \) is a chiral hypermap of type \( (3,3,3)_{(1,e)} \).

\( \square \)

5 The symmetries of \( \Gamma \)

As pointed out in the introduction, the hypertopes we have constructed in this paper are highly symmetric. Recall that we have two families: a family of hypertopes \( \Gamma^{(p)} = \Gamma(G^+, R) \) with \( G^+ = PSL(2, p) \) for any prime number \( p \) satisfying \( p \equiv 1 \mod 3 \), and a family of hypertopes \( \Gamma^{(p^2)} = \Gamma(G^+, R) \) with \( G^+ = PSL(2, p^2) \) for any prime \( p \) satisfying \( p \equiv 2 \mod 3 \), where \( R = \{\alpha_1, \alpha_2, \alpha_3\} \) for both families. In this section we shall show that all hypertopes \( \Gamma^{(p)} \) are chiral, while all hypertopes \( \Gamma^{(p^2)} \) are regular. Furthermore, we study the correlations of the hypertopes and find that all the chiral hypertopes \( \Gamma^{(p)} \) have both proper and improper correlations. For any \( \Gamma \) in one of these two families we have the following lemma.

**Lemma 15.** \( G^+ \) acts with two orbits on chambers of \( \Gamma \) with adjacent chambers in different orbits.

**Proof.** This is a consequence of Propositions 11 and 12.

\( \square \)

**Lemma 16.** Any hypertope \( \Gamma^{(p)} \) is chiral while any hypertope \( \Gamma^{(p^2)} \) is regular.

**Proof.** By Lemma 14, one residue of \( \Gamma^{(p)} \) is chiral. Hence \( \Gamma^{(p)} \) is itself chiral, indeed by Theorem 3 the residues of a regular hypertope must all be regular. For \( \Gamma^{(p^2)} \) we have
that $G^+ = PSL(2, p^2)$. Then, the Frobenius automorphism of $GF(p^2)$ gives rise to the involutory automorphism

$$\varphi : G^+ \to G^+, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$$

which fixes $\alpha_1$ and interchanges $\alpha_2$ and $\alpha_3$ (since $p \equiv 2 \mod 3$ and therefore $e^p = e^2$ in that case). Let $\psi$ be the automorphism of $G^+$ given by conjugation with

$$A := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2, p^2).$$

Then $\psi\varphi$ is an automorphism of $G^+$ inverting $\alpha_1$, $\alpha_2$ and $\alpha_3$. Hence, by Theorem 3 $\Gamma(p^2)$ is a regular hypertope.

For the chiral hypertopes $\Gamma(p)$ we will now exhibit proper and improper correlations. In order to give a proper correlation of $\Gamma(p)$, we remark the following general fact. Given a coset geometry $\Gamma := \Gamma(G_i, (G_i)_i \in I)$, any $\varphi \in Aut(G)$ satisfying $\{\varphi(G_i) \mid i \in I\} = \{G_i \mid i \in I\}$ gives rise to an automorphism of $\Gamma$ mapping $G_i g$ to $\varphi(G_i) \varphi(g)$. In particular, we have the following lemma.

**Lemma 17.** The automorphism $\eta$ of $G^+ = PSL(2, p)$ defined by conjugation with the involution

$$B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, p)$$

gives rise to a proper duality of $\Gamma(p)$.

**Proof.** As $\eta$ is an involution satisfying

$$\eta(\alpha_1) = \alpha_1^{-1}, \quad \eta(\alpha_2) = \alpha_2^{-1} \alpha_3, \quad \eta(\alpha_3) = \alpha_2^{-1} \alpha_1,$$

it interchanges $G_0$ with $G_1$ and $G_2$ with $G_3$. 

To give an improper correlation of $\Gamma(p)$, we extend the assignment

$$\mu : G_0 \mapsto G_0 \alpha_1^{-1}, \quad G_1 \mapsto G_1, \quad G_2 \mapsto G_3, \quad G_3 \mapsto G_2$$

to a bijection of the set of elements of $\Gamma(p)$ by setting

$$\mu(G_i g) := \mu(G_i) g A, \quad \text{where} \quad A := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2, p).$$

Note that $A$ is an involution and that $\alpha_1^A = \alpha_1^{-1}$, $\alpha_2^A = \alpha_3^{-1}$, $\alpha_3^A = \alpha_2^{-1}$. Therefore $\mu(G_0)^A = \alpha_1 G_0$ and $\mu(G_i)^A = G_i$ for any $i \in \{1, 2, 3\}$. Then, for any $j \in I$ we have

$$\mu(G_j g) = \mu(G_j h) \iff \mu(G_j) g A = \mu(G_j) h A \iff \mu(G_j)^A g = \mu(G_j)^A h \iff G_j g = G_j h,$$

showing that $\mu$ is well-defined and a bijection. To prove that $\mu$ is a correlation we need the following lemma.
Lemma 18. \( \{ G_i g^A \alpha | g \in H_0 \} = \{ G_i h | h \in H_0 \} \) for any \( i \in \{1, 2, 3\} \).

Proof. Let \( g = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H_0 \). Then
\[
g^A \alpha_1 = \begin{pmatrix} -c & 1 \\ c-1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \in \{ G_i h | h \in H_0 \} = \alpha_1 \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \in \{ G_2 h | h \in H_0 \} \cap \{ G_3 h | h \in H_0 \}.
\]

Hence \( G_i g^A \alpha_1 \in \{ G_i h | h \in H_0 \} \).

\( \square \)

Lemma 19. The bijection \( \mu \) is an improper duality of \( \Gamma^{(p)} \).

Proof. By definition \( \mu \) sends the chamber \( \{ G_0, G_1, G_2, G_3 \} \) to \( \{ G_0 \alpha_1^{-1}, G_1, G_2, G_3 \} \), the adjacent chamber. Hence we have only to prove that \( \mu \) is a correlation. It is enough to prove that for any \( g \in G^+ \) and any \( i, j \in I \) with \( i > j \)
\[
\mu(G_i g) \ast \mu(G_j) \Leftrightarrow G_i g \ast G_j.
\]

According to the definition of \( \mu \) and Lemma 9, we have
\[
\mu(G_i g) \ast \mu(G_j) \Leftrightarrow \mu(G_i) g^A \ast \mu(G_j)
\]
\[
\Leftrightarrow \begin{cases} 
\mu(G_i) g^A \alpha_1 \in \{ \mu(G_i) h | h \in H_0 \} & \text{if } j = 0, \\
\mu(G_i) g^A \alpha_1 \in \{ \mu(G_i) h | h \in H_1 \} & \text{if } j = 1, \\
G_3 g^A \alpha_1 \in \{ G_3, G_3 \alpha_3, G_3 \alpha_3 \alpha_1, G_3 \alpha_3 \alpha_1^{-1} \} & \text{if } j = 2 (\Rightarrow i = 3).
\end{cases}
\]
\[
\Leftrightarrow \begin{cases} 
G_i g \alpha_1^{-1} \in \{ G_i h^A | h \in H_0 \} & \text{if } j = 0, \\
G_i g \in \{ G_i h^A | h \in H_1 \} & \text{if } j = 1, \\
G_3 g \in \{ G_3, G_3 \alpha_3^{-1}, G_3 \alpha_3^{-1} \alpha_1^{-1}, G_3 \alpha_3^{-1} \alpha_1 \} & \text{if } j = 2.
\end{cases}
\]
\[
\Leftrightarrow \begin{cases} 
G_i g \in \{ G_i h^A \alpha_1 | h \in H_0 \} & \text{if } j = 0, \\
G_i g \in \{ G_i h^A | h \in H_1 \} & \text{if } j = 1, \\
G_3 g \in \{ G_3, G_3 \alpha_3^{-1}, G_3 \alpha_3^{-1} \alpha_1^{-1}, G_3 \alpha_3^{-1} \alpha_1 \} & \text{if } j = 2.
\end{cases}
\]

Using Lemma 18 \((j = 0)\), checking that \( A \) normalizes \( H_1 \) \((j = 1)\) and verifying that \[
\{ G_3, G_3 \alpha_3^{-1}, G_3 \alpha_3^{-1} \alpha_1^{-1}, G_3 \alpha_3^{-1} \alpha_1 \} = \{ G_3, G_3 \alpha_3 \alpha_1, G_3 \alpha_3, G_3 \alpha_3 \alpha_1^{-1} \} \quad (j = 2),
\]
we conclude that \( \mu(G_i g) \ast \mu(G_j) \Leftrightarrow G_i g \ast G_j \) for any \( i, j \in I \) with \( i > j \) using again Lemma 9.

\( \square \)

As it has been shown in Lemmas 17 and 19, the chiral hypertopes \( \Gamma^{(p)} \) have both proper and improper correlations simultaneously. We therefore have the following theorem.

Theorem 20. There exist chiral hypertopes of rank 4 having both proper and improper correlations simultaneously.
We finish this section with the following two remarks.

1. All residues of rank 3 of the regular hypertope \( \Gamma^{(4)} \) are regular hypermaps of type \((3,3,3)_0\) (Lemma 14). Consequently a correspondence fixing a generator of \( G^+ \) and interchanging the other two, can be extended to an automorphism of \( G^+ \). Thus \( \Gamma^{(4)} \) has all possible kinds of correlations. In other words, \( \text{Aut}(\Gamma^{(4)}) \cong S_5 \times S_4 \cong \text{PSL}(2, 4) \times S_4 \).

2. We observe that the tetrahedral groups \( G^+ \) arise naturally as subgroups of index 4 in the symmetry group \([6, 3, 6] = \langle r_0, r_1, r_2, r_3 \rangle\) of the hyperbolic tessellation \{6,3,6\} given in [6, Section 11] (generated by \( r_1, r_2, (r_1)^0 \) and \((r_2)\).

6 A geometric construction of \( \Gamma \)

We now give a geometric description of the rank 4 hypertopes \( \Gamma = (X, *, t, I) \) constructed in the previous sections. The set \( I = \{0, 1, 2, 3\} \) of types gives a partition of the set of elements of \( \Gamma \) into four sets \( X_0, X_1, X_2 \) and \( X_3 \), where \( x \in X_j \) if and only if \( t(x) = j \). The elements of \( X_0, X_1, X_2 \) and \( X_3 \) can be described from a certain action in the projective line \( PG(1, q) \), where \( q = p \) if \( \Gamma = \Gamma^{(p)} \) and \( q = p^2 \) if \( \Gamma = \Gamma^{(p^2)} \). In what follows we define each of the \( X_i \) and the incidence between them.

Let \( G^+ \) be the group \( PSL(2, q) \) acting on the projective line \( PG(1, q) = GF(q) \cup \{\infty\} \) as follows.

\[
PG(1, q) \times G^+ \to PG(1, q), \ (z, g) \mapsto z^g := g^{-1}(z),
\]

where \( g^{-1}(z) = Az \) if \( g^{-1} = \pm A \) with \( A \in SL(2, q) \), as in Section 2.4.

Consider a transversal \( U \) for the stabilizer \( G^+_{\infty} \) of \( \infty \). The index of \( G^+_{\infty} \) in \( G^+ \) is \( q + 1 \), since \( G^+_{\infty} \) is the semi-direct product of the elementary abelian \( p \)-group \( H_1 \) of order \( q \) (given in Lemma 6) with the cyclic group \( D \) of diagonal matrices in \( G^+ \), which has order \( \frac{q-1}{d} \) with \( d = \gcd(2, q - 1) \). As \( q + 1 = |PG(1, q)| \), we can identify any \( u \in U \) with the element \( z := \infty^u \) of \( PG(1, q) \) and write \( u = uz \).

By Lemma 6 the group \( G_1 = H_1 : (G_0 \cap G_1) \) is a subgroup of \( G^+_{\infty} = H_1 : D \). Consider a transversal \( T \) for \( G_1 \) in \( G^+_{\infty} \). Let \( X_0 \) and \( X_1 \) be two disjoint copies of \( T \times PG(1, q) \). For \( x_0 = (t, z) \in X_0 \) and \( x_1 = (s, w) \in X_1 \) we say that

\[
x_0 \ast x_1 \text{ whenever } t = s \text{ and } z \neq w.
\]

It is straightforward to see that an element \( x_0 \in X_0 \) is incident with exactly \( q \) elements of \( X_1 \) and vice-versa.

Let \( X_2 \) and \( X_3 \) be the orbits of the sets \( \{1, e, e^2, \infty\} \) and \( \{1, e, e^2, 0\} \) under the action of \( G^+ \), respectively. Remarking that all possible cross ratio of four distinct points in \( \{1, e, e^2, \infty\} \) or in \( \{1, e, e^2, 0\} \) takes only two values, we conclude that both sets have a stabilizer isomorphic to \( A_4 \). Hence \( |X_2| = |X_3| = \frac{(q+1)(q-1)}{12d} \).

Given \( x_2 \in X_2 \) and \( x_3 \in X_3 \), we shall say that

\[
x_2 \ast x_3 \text{ whenever } |x_2 \cap x_3| = 3.
\]
Then, each element \( x_2 \in X_2 \) is incident with exactly 4 elements of \( X_3 \) and vice-versa.

Before defining the incidences between any other two elements, we shall first define the incidence between \( \{1, e, e^2, \infty\} \in X_2 \) and an element \( (t, z) \) of \( X_0 \) or \( X_1 \). We say that

\[
\{1, e, e^2, \infty\} * (t, z) \text{ if and only if } t = 1_{G^n} \text{ and } z \in \{1, e, e^2, \infty\}.
\]

Having 6 in mind, the transversal \( U \) can be choosen in such a way that \( u_z \in G^+_{\{1, e, e^2, \infty\}} \) for \( z \in \{1, e, e^2, \infty\} \). This is possible and motivated by the following. If the coset \( G_1tu_z \) intersects \( G^+_{\{1, e, e^2, \infty\}} \), then \( z \in \{1, e, e^2, \infty\} \) and therefore \( u_z \in G^+_{\{1, e, e^2, \infty\}} \) by our choice of \( U \). It follows then that \( G_1t \) intersects \( G^+_{\{1, e, e^2, \infty\}} \). As \( G_1t \) is in \( G^\infty \) and \( G^\infty \cap G^+_{\{1, e, e^2, \infty\}} < G_1 \), we have that \( G_1t \not\supseteq G_1 \not\supseteq \emptyset \), that is \( t = 1_{G^n} \). Now we can say that an element \( \{1, e, e^2, \infty\}^g \in X_2 \) will be incident to those \( (t, z) \) (in \( X_0 \) or \( X_1 \)) such that \( G^+_{\{1, e, e^2, \infty\}} \cap G_1tu_zg^{-1} \not\supseteq \emptyset \). Note that, if \( g \in G^+_{\{1, e, e^2, \infty\}} \), then \( \{1, e, e^2, \infty\}^g = \{1, e, e^2, \infty\} \) will be incident to those \( (t, z) \) such that \( G^+_{\{1, e, e^2, \infty\}} \) is incident with \( G_1tu_zg^{-1} = G_1tu_zg^{-1} \) (by the choice of \( U \)). As we saw, these are the pairs \( (t, z) \) with \( t = 1_{G^n} \) and \( z \in \{1, e, e^2, \infty\} \). Thus incidence is well-defined.

Finally, an element \( \{1, e, e^2, 0\}^g \in X_3 \) will be incident with an element \( (t, z) \) in \( X_0 \) or \( X_1 \) if the corresponding element \( \{1, e, e^2, \infty\}^g \in X_2 \) is incident with \( (t, z) \).

By definition, \( \Gamma = (X, * , t, I) \) is an incidence system, where \( X = \cup_{j=0}^{3} X_j, I = \{0, 1, 2, 3\}, t : X \to I \) is such that \( t(x) = j \) whenever \( x \in X_j \) and the incidence \( * \) is given by (4), (5), (6) and the last paragraph above. Furthermore, it is straightforward to see that every flag is in a chamber, implying that \( \Gamma \) is in fact an incidence geometry. By construction, \( \Gamma \) is isomorphic to the hypertope given in section 4.

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