Using symbolic computation to prove nonexistence of distance-regular graphs

Janoš Vidali∗
Faculty of Mathematics and Physics
University of Ljubljana, 1000 Ljubljana, Slovenia
janos.vidali@fmf.uni-lj.si

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Abstract
A package for the Sage computer algebra system is developed for checking feasibility of a given intersection array for a distance-regular graph. We use this tool to show that there is no distance-regular graph with intersection array

\[(2r + 1)(4r + 1)(4t - 1), 8r(4rt - r + 2t), (r + t)(4r + 1); 1, (r + t)(4r + 1), 4r(2r + 1)(4t - 1) \] (r, t ≥ 1),

\{135,128,16; 1,16,120\}, \{234,165,12; 1,30,198\} or \{55,54,50,35,10; 1,5,20,45,55\}. In all cases, the proofs rely on equality in the Krein condition, from which triple intersection numbers are determined. Further combinatorial arguments are then used to derive nonexistence.

Mathematics Subject Classifications: 05E30

1 Introduction
Distance-regular graphs were introduced around 1970 by N. Biggs [1]. As they are intimately linked to many other combinatorial objects, such as finite simple groups, finite geometries, and codes, a natural goal is trying to classify them.

Many distance-regular graphs are known, however constructing new ones has proved to be a difficult task. A number of feasibility conditions for distance-regular graphs have been found, which allows us to compile a list of feasible intersection arrays for small distance-regular graphs (or related structures, such as Q-polynomial association schemes), see Brouwer et al. [2, 3, 4] and Williford [22]. However, feasibility is no guarantee for

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existence, so proofs of nonexistence of distance-regular graphs with feasible intersection arrays are also a contribution to the classification. In certain cases, single intersection arrays have been ruled out [12, 13], while other proofs may show nonexistence for a whole infinite family of feasible intersection arrays [6, 9, 19]. In this paper we give proofs of nonexistence for distance-regular graphs belonging to a two-parameter infinite family, as well as for graphs with intersection arrays
\[
\{135, 128, 16; 1, 16, 120\},
\{234, 165, 12; 1, 30, 198\},
\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}.
\]

We develop a package called **sage-drg** [21] for the Sage computer algebra system [18]. Sage is free open-source software written in the Python programming language [17], with many functionalities deriving from other free open-source software, such as Maxima [16], which Sage uses for symbolic computation. The **sage-drg** package is thus also free open-source software available under the MIT license, written in the Python programming language, making use of the Sage library. The package can be used to check for feasibility of a given intersection array against known feasibility conditions, see Van Dam, Koolen and Tanaka for an up-to-date survey [7]. Furthermore, using equality in the Krein condition (see Theorem 1), restrictions on triple intersection numbers can be derived. In this paper, we use them to derive some nonexistence results. The **sage-drg** package also includes Jupyter notebooks demonstrating its use to obtain these results, as well as the notebook `jupyter/Demo.ipynb` giving some general examples of use of the package. A more detailed description of the **sage-drg** package is given in the supplementary files\(^1\).

The results from Sections 3, 4 and 6 appeared in the author’s PhD thesis [20], where computation was done using a Mathematica [23] notebook originally developed by M. Urlep. Thus, the **sage-drg** package can be seen as a move from closed-source proprietary software to free open-source software, which allows one to check all code for correctness, thus making the results verifiable.

## 2 Preliminaries

In this section we review some basic definitions and concepts. See Brouwer, Cohen and Neumaier [3] for further details.

Let \( \Gamma \) be a connected graph with diameter \( d \) and \( n \) vertices, and let \( \partial(u, v) \) denote the distance between the vertices \( u \) and \( v \) of \( \Gamma \). The graph \( \Gamma \) is **distance-regular** if there exist constants \( p^h_{ij} \) \((0 \leq h, i, j \leq d)\), called the **intersection numbers**, such that for any pair of vertices \( u, v \) at distance \( h \), there are precisely \( p^h_{ij} \) vertices at distances \( i \) and \( j \) from \( u \) and \( v \), respectively. In fact, all intersection numbers can be computed given only the intersection numbers \( b_i = p^i_{1,i+1} \) and \( c_{i+1} = p^{i+1}_{i+1} \) \((0 \leq i \leq d - 1)\) [3, §4.1A]. These intersection numbers are usually gathered in the **intersection array** \( \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\} \). We also

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define the valency \( k = b_0 \) and \( a_i = k - b_i - c_i \) \((0 \leq i \leq d)\), where \( b_d = c_0 = 0\). A connected noncomplete strongly regular graph with parameters \((v, k, \lambda, \mu)\) is a distance-regular graph of diameter 2 with \( v \) vertices, valency \( k \) and intersection numbers \( a_1 = \lambda, \ c_2 = \mu\).

Let \( A_i \) \((0 \leq i \leq d)\) be a binary square matrix indexed with the vertices of a graph \( \Gamma \) of diameter \( d \), with entry corresponding to vertices \( u \) and \( v \) equal to 1 precisely when \( \partial(u, v) = i \). The matrix \( A = A_1 \) is the adjacency matrix of \( \Gamma \). The graph \( \Gamma \) is called primitive if all \( A_i \) \((1 \leq i \leq d)\) are adjacency matrices of connected graphs. A distance-regular graph of valency \( k \geq 3 \) that is not primitive is bipartite or antipodal (or both) \([3, \ Thm. \ 4.2.1]\). The spectrum of \( \Gamma \) is defined to be the spectrum of \( (\lambda, \ c_2 = \mu) \) matrices such that \( \lambda^2 = 0 \) is the empty adjacency matrix.

The ordering of eigenvalues is known as the natural ordering. We define the eigenmatrix \( P \) and dual eigenmatrix \( Q \) as \((d + 1) \times (d + 1)\) matrices such that \( A_i = \sum_{\ell=0}^d P_{ij} E_\ell \) and \( E_j = n^{-1} \sum_{i=0}^d Q_{ij} A_i \). The graph \( \Gamma \) is called formally self-dual \([3, \ p. \ 49]\) if \( P = Q \) holds for some ordering of eigenvalues. The graph \( \Gamma \) is called \( Q\)-polynomial \([3, \ §2.7]\) with respect to some ordering of eigenvalues if there exist real numbers \( z_0, \ldots, z_d \) and polynomials \( q_j \) of degree \( j \) such that \( Q_{ij} = q_j(z_i) \) \((0 \leq i, j \leq d)\). Finally, we define the Krein parameters \( q^h_{ij} \) \([3, \ §2.3]\) as such numbers that \( E_i \circ E_j = n^{-1} \sum_{b=0}^d q^h_{ij} E_b \), where \( \circ \) represents entrywise multiplication of matrices. A formally self-dual distance-regular graph is also \( Q\)-polynomial with respect to the corresponding ordering of eigenvalues and has \( p^h_{ij} = q^h_{ij} \) \((0 \leq i, j, h \leq d)\). In this paper, we will use the natural ordering for indexing, noting when a graph is \( Q\)-polynomial or formally self-dual for some other ordering.

For vertices \( u, v, w \) of the distance-regular graph \( \Gamma \) and integers \( i, j, h \) \((0 \leq i, j, h \leq d)\) we denote by \([u \ v \ w \ i \ j \ h]\) or simply \([i \ j \ h]\) when it is clear which triple \((u, v, w)\) we have in mind) the number of vertices of \( \Gamma \) that are at distances \( i, j, h \) from \( u, v, w \), respectively. We call these numbers triple intersection numbers. They have first been studied in the case of strongly regular graphs \([5]\), and later also for distance-regular graphs, see for example \([6, 8, 9, 10, 19]\). Unlike the intersection numbers, these numbers may depend on the particular choice of vertices \( u, v, w \) and not only on their pairwise distances. We may however write down a system of \( 3d^2 \) linear Diophantine equations with \( d^3 \) triple intersection numbers as variables, thus relating them to the intersection numbers, cf. \([9]\):

\[
\sum_{\ell=1}^d [\ell \ j \ h] = p^U_{j\ell} - [0 \ j \ h], \quad \sum_{\ell=1}^d [i \ \ell \ h] = p^V_{ih} - [i \ 0 \ h], \quad \sum_{\ell=1}^d [i \ j \ \ell] = p^W_{ij\ell} - [i \ j \ 0],
\]

(1)

where \( U = \partial(v, w), V = \partial(u, w), W = \partial(u, v) \), and

\[
[0 \ j \ h] = \delta_{jW} \delta_{hV}, \quad [i \ 0 \ h] = \delta_{iW} \delta_{hU}, \quad [i \ j \ 0] = \delta_{iW} \delta_{jU}.
\]

Furthermore, we can use the triangle inequality to conclude that certain triple intersection numbers must be zero. Moreover, the following theorem sometimes gives additional equations.

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Theorem 1. ([6, Theorem 3], cf. [3, Theorem 2.3.2]) Let $\Gamma$ be a distance-regular graph with diameter $d$, dual eigenmatrix $Q$ and Krein parameters $q^h_{ij}$ $(0 \leq i, j, h \leq d)$. Then,

\[
q^h_{ij} = 0 \iff \sum_{r,s,t=0}^d Q_{ri}Q_{sj}Q_{th}\begin{bmatrix} u & v & w \\ r & s & t \end{bmatrix} = 0 \quad \text{for all } u, v, w \in V\Gamma.
\]

Together with integrality and nonnegativity of triple intersection numbers, we can use all of the above to either derive that the system of equations has no solution, or arrive at a small number of solutions, which gives us new information on the structure of the graph and may lead to proving its nonexistence.

3 A two-parameter family of primitive graphs of diameter 3

In [9], graphs meeting necessary conditions for the existence of extremal codes were studied. One of the families of primitive graphs of diameter 3 for which these conditions were met was

\[
\{a(p + 1), (a + 1)p, c; 1, c, ap\},
\]

where $a = a_3$, $c = c_2$ and $p = p^3_{33}$. Graphs belonging to this family are $Q$-polynomial with respect to the natural ordering of eigenvalues precisely when the Krein parameter $q^3_{11}$ is zero, which is equivalent to

\[
c = \frac{1}{4} \left( (p + 1)^2 + \frac{2a(p + 1)}{p + 2} \right).
\]

Hence, $p + 2$ must divide $2a$ for $c$ to be integral. If $p = 2r - 1$, then $a = t(2r + 1)$ and $c = r(r + t)$ for some positive integers $r, t$, which gives us the two-parameter family

\[
\{2rt(2r + 1), (2r - 1)(2rt + t + 1), r(r + t); 1, r(r + t), t(4r^2 - 1)\}.
\]

In [9], nonexistence was shown for a feasible subfamily with $r = t \geq 2$. If, on the other hand, $p$ is even, integrality of the multiplicity of the second largest eigenvalue implies that we must have $p = 4r$, $a = (2r + 1)(4t - 1)$ and $c = (r + t)(4r + 1)$ for some positive integers $r, t$, giving the family

\[
\{(2r + 1)(4r + 1)(4t - 1), 8r(4rt - r + 2t), (r + t)(4r + 1); 1, (r + t)(4r + 1), 4r(2r + 1)(4t - 1)\}.
\]

We find two one-parameter infinite subfamilies of feasible intersection arrays by setting $t = 4r^2$ or $t = 4r^2 + 2r$:

\[
\{(2r + 1)(4r + 1)(16r^2 - 1), 8r^2(16r^2 + 8r - 1), r(4r + 1)^2; 1, r(4r + 1)^2, 4r(2r + 1)(16r^2 - 1)\},
\]

\[
\{(2r + 1)(4r + 1)(16r^2 + 8r - 1), 8r^2(4r + 1)(4r + 3), r(4r + 1)(4r + 3); 1, r(4r + 1)(4r + 3), 4r(2r + 1)(16r^2 + 8r - 1)\}.
\]
There are also other feasible cases – for instance, when \( r = 2 \), we have, besides the cases from the two subfamilies above, feasible examples when \( t \in \{ 4, 7, 196 \} \). The case with \( r = 1 \) and \( t = 4 \) belonging to the first subfamily above is also listed in the list of feasible parameter sets for 3-class \( Q \)-polynomial association schemes by J. S. Williford [22].

We now prove that a graph \( \Delta \) with intersection array (4) does not exist. The proof parallels that of [9, Lems. 1, 3] – in fact, a significant part of the proof may be extended to the entire family (3), as it has been done in [20]. The computation needed to obtain the results in this section is illustrated in the jupyter/DRG-d3-2param.ipynb notebook included in the sage-drg package [21].

**Lemma 2.** Let \( \Delta \) be a distance-regular graph with intersection array (4), and \( u', v, w \) be vertices of \( \Delta \) with \( \partial(u', v) = 1, \partial(u', w) = 2 \) and \( \partial(v, w) = 3 \). Then \( \begin{bmatrix} u' & v & w \\ 3 & 3 & 3 \end{bmatrix} = 1 \).

**Proof.** Let \( u \) be a vertex of \( \Delta \) at distance 3 from both \( v \) and \( w \) (such a vertex exists since \( p_{33} = 4r > 0 \)). We consider the triple intersection numbers \( [i \ j \ h] \) that correspond to \( (u, v, w) \). As \( q_{11}^3 = q_{13}^1 = q_{31}^1 = 0 \), Theorem 1 gives three additional equations to the system (1), allowing us to express its solution in terms of a single parameter \( \alpha = [3 \ 3 \ 3] \).

Let us express the counts of vertices at distance 1 or 2 from one of \( u, v, w \) and at distance 3 from the other two vertices:

\[
\begin{align*}
[3 \ 3 \ 1] &= [3 \ 1 \ 3] = [1 \ 3 \ 3] = \frac{(\alpha - 4r + 1)(4r + 1)}{4r - 1}, \\
[3 \ 3 \ 2] &= [3 \ 2 \ 3] = [2 \ 3 \ 3] = \frac{8r(4r - 1 - \alpha)}{4r - 1}.
\end{align*}
\]

For the values above to be nonnegative, we must have \( \alpha = 4r - 1 \), which means that they are all zero. As the choice of \( u, v, w \) was arbitrary, this implies that any pair of vertices at distance 3 induces a set of \( 4r + 2 \) vertices pairwise at distance 3 – in the terminology of [9], this is a maximal 1-code in \( \Delta \). Since we have \( a_3p_{13}^3 = 4r(2r + 1)(4t - 1) = c_3 \), it follows by [9, Prop. 2] that \( \begin{bmatrix} u' & v & w \\ 3 & 3 & 3 \end{bmatrix} = 1 \) holds. \( \square \)

**Theorem 3.** A distance-regular graph \( \Delta \) with intersection array (4) does not exist.

**Proof.** Let \( u', v, w \) be vertices of \( \Delta \) with \( \partial(u', v) = 1, \partial(u', w) = 2 \) and \( \partial(v, w) = 3 \) (such vertices exist, since we have \( p_{13}^2 = b_2 = (r + t)(4r + 1) > 0 \)). We consider the triple intersection numbers \( [i \ j \ h] \) that correspond to \( (u', v, w) \). By Lemma 2, we have \( [3 \ 3 \ 3] = 1 \). Using \( q_{11}^3 = 0 \), Theorem 1 gives an additional equation which allows us to obtain a unique solution to the system (1). However, we obtain \( [1 \ 1 \ 3] = 2t - 1/2 \), which is nonintegral for all integers \( t \). Therefore, the graph \( \Delta \) does not exist. \( \square \)

### 4 A primitive graph with diameter 3 and 1360 vertices

Let \( \Lambda \) be a distance-regular graph with intersection array

\[
\{135, 128, 16; 1, 16, 120\}. \tag{5}
\]
This intersection array can be obtained from (2) by setting \( a = 15, \ c = 16 \) and \( p = 8 \). The graph \( \Lambda \) has diameter 3 and 1360 vertices. It is not \( Q \)-polynomial, however its Krein parameter \( q_{33}^3 \) is zero. We show that such a graph does not exist. The computation needed to prove Theorem 4 is illustrated in the \textit{jupyter/DRG-135-128-16-1-16-120.ipynb} notebook included in the \texttt{sage-drg} package [21].

**Theorem 4.** A distance-regular graph \( \Lambda \) with intersection array (5) does not exist.

**Proof.** Let \( u, v, w \) be three pairwise adjacent vertices of \( \Lambda \) (such vertices exist, since we have \( p_{11}^1 = 6 > 0 \)). We consider triple intersection numbers \([i \ j \ h] \) that correspond to \((u, v, w)\). As \( q_{33}^3 = 0 \), Theorem 1 gives an additional equation to the system (1), allowing us to express its solution in terms of a single parameter \( \alpha = [1 \ 1 \ 1] \). In particular, we obtain
\[
[3 \ 3 \ 3] = 71 - 27\alpha.
\]
Clearly, \( \alpha \) must be a nonnegative integer. For \([3 \ 3 \ 3]\) to be nonnegative, we must have \( \alpha \in \{0, 1, 2\} \). However, \([3 \ 3 \ 3]\) is still nonintegral in these cases, showing that the graph \( \Lambda \) does not exist.

\(\square\)

5 A primitive graph with diameter 3 and 1600 vertices

Let \( \Xi \) be a distance-regular graph with intersection array
\[
\{234, 165, 12; 1, 30, 198\}.
\]

The graph \( \Xi \) has diameter 3 and 1600 vertices. The intersection array (6) has been found as an example of a feasible parameter set for a distance-regular graph which is formally self-dual for an ordering of eigenvalues distinct from the natural ordering – in fact, \( \Xi \) is \( Q \)-polynomial for the ordering 0, 2, 3, 1, so its Krein parameters \( q_{12}^1, q_{12}^2, q_{21}^2 \) are zero. The intersection array (6) is also listed in the list of feasible parameter sets for 3-class \( Q \)-polynomial association schemes by J. S. Williford [22]. We show that such a graph does not exist. The computation needed to prove Theorem 5 is illustrated in the \textit{jupyter/DRG-234-165-12-1-30-198.ipynb} notebook included in the \texttt{sage-drg} package [21].

**Theorem 5.** A distance-regular graph \( \Xi \) with intersection array (6) does not exist.

**Proof.** Let \( u, v, w \) be three vertices of \( \Xi \) that are pairwise at distance 3 (such vertices exist, since we have \( p_{33}^3 = 8 > 0 \)). We consider triple intersection numbers \([i \ j \ h] \) that correspond to \((u, v, w)\). As \( q_{22}^1 = q_{12}^2 = q_{21}^2 = 0 \), Theorem 1 gives three additional equations to the system (1), allowing us to express its solution in terms of a single parameter \( \alpha = [3 \ 3 \ 3] \). In particular, we obtain
\[
[3 \ 3 \ 2] = [3 \ 2 \ 3] = [2 \ 3 \ 3] = -17 - 4\alpha.
\]
Clearly, \( \alpha \) must be nonnegative, but then we have \([3 \ 3 \ 2] = [3 \ 2 \ 3] = [2 \ 3 \ 3] < 0 \), a contradiction. We conclude that the graph \( \Xi \) does not exist.

\(\square\)
Figure 1: The partition of vertices of $\Sigma$ by distance from a pair of vertices $u, v$ at distance 2. The part that is at distance $i$ from $u$ and distance $j$ from $v$ has size $p_{ij}^2$. As the graph is bipartite, the intersection number $p_{ij}^2$ is nonzero only when $i + j$ is even. Moreover, there are no edges within each part. It is natural to consider $[1 1 1]$ for $w$ at distance 2 from both $u$ and $v$, see Lemma 6.

6 A bipartite graph with diameter 5

Let $\Sigma$ be a distance-regular graph with intersection array

$$\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}. \quad (7)$$

This intersection array appears in the list of feasible intersection arrays for bipartite non-antipodal distance-regular graphs of diameter 5 by Brouwer et. al. [3, p. 418] as an open case. The existence of such a graph would give a counterexample to a conjecture by MacLean and Terwilliger [15], cf. Lang [14]. The computation needed to obtain the results in this section is illustrated in the jupyter/DRG-55-54-50-35-10-bipartite.ipynb notebook included in the sage-drg package [21].

The graph $\Sigma$ has diameter 5 and 3500 vertices. The partition of $\Sigma$ corresponding to two vertices at distance 2 is shown in Figure 1. The graph is $Q$-polynomial for the natural ordering of eigenvalues, see for example [3, p. 418]. Moreover, as the graph is bipartite, it is also $Q$-antipodal [3, Thm. 8.2.1]. Many Krein parameters are zero, in particular $q_{i1}^3$ and $q_{i1}^4$ due to the triangle inequality. We use this fact in the proof of the following statement.

Lemma 6. Let $\Sigma$ be a distance-regular graph with intersection array (7), and $u, v, w$ be vertices of $\Sigma$ that are pairwise at distance 2. Then $[u \ v \ w \ 1 \ 1 \ 1] \leq 1$.

Proof. We consider the triple intersection numbers $[i \ j \ h]$ that correspond to $(u, v, w)$. Since the graph $\Sigma$ is bipartite, we have $[i \ j \ h] = 0$ whenever any of the sums $i + j$, $j + h$, $h + i$ is odd. As $q_{i1}^3 = q_{i1}^4 = 0$, Theorem 1 gives us two additional equations to the system (1), thus allowing us to express the solution of the system in terms of a single parameter $\alpha = [1 \ 1 \ 1]$. In particular, we obtain

$$[5 \ 5 \ 5] = 20 - 12\alpha.$$

The integrality and nonnegativity of $[5 \ 5 \ 5]$ now gives $\alpha \leq [5/3] = 1$. \qed
Note. It can also be shown with a method similar to the one used in Lemma 6 that the graph $[\Sigma_5(u)]_2$ for a vertex $u \in V\Sigma$ (i.e., the graph of vertices at distance 5 from a vertex $u$, with adjacency corresponding to distance 2 in $\Sigma$) is strongly regular with parameters $(v, k, \lambda, \mu) = (210, 99, 48, 45)$. A strongly regular graph with such parameters has been constructed by M. Klin [11].

Theorem 7. A distance-regular graph $\Sigma$ with intersection array $(7)$ does not exist.

Proof. Let $u$ and $v$ be vertices of $\Sigma$ at distance 2, see Figure 1, and let $\{i\ j\}$ denote the set of vertices at distances $i$ and $j$ from $u$ and $v$, respectively. There are $p^2_{11}(k - 2) = 5 \cdot 53 = 265$ edges between the sets $\{1\ 1\}$ and $\{2\ 2\}$. However, the cardinality of the latter set is $p^2_{22} = 243 < 265$, so there is a vertex $w \in \{2\ 2\}$ that has at least two neighbours in $\{1\ 1\}$, i.e., $[u\ v\ w\ 1\ 1\ 1] \geq 2$, which is in contradiction with Lemma 6. Hence, the graph $\Sigma$ does not exist. □

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References


