\(\lambda\)-Euler’s difference table for colored permutations

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Abstract
Motivated by the \(\lambda\)-Euler’s difference table of Eriksen et al. and colored Euler’s difference table of Faliharimalala and Zeng, we study the \(\lambda\)-analogue of colored Euler’s difference table and generalize their results. We generalize the number of permutations with \(k\)-excedances studied by Liese and Remmel in colored permutations. We also extend Wang et al.’s recent results about \(r\)-derangements by relating with the sequences arising from the difference table.

Mathematics Subject Classifications: 05A18, 05A15

1 Introduction
Euler [4] studied the difference table \((g^m_n)_{0 \leq m \leq n}\), where the coefficients are defined by
\[
g^m_n = n! \quad \text{and} \quad g^m_n = g^{m+1}_n - g^m_{n-1},
\]
for \(0 \leq m \leq n-1\). Dumont and Randrianarivony [4] studied the combinatorial interpretation of \(g^m_n\) in the symmetric group \(S_n\), which consists of permutations of \([n] = \{1, \ldots, n\}\). In particular, they showed that the sequence \(\{g^0_n\}_{n \geq 0}\) is the number of derangements, i.e., the fixed point free permutations in \(S_n\). Then Rakotondrajao [11] developed further combinatorial interpretations. The reader is referred to [4, 11, 12, 7, 3, 5, 10, 2], where several generalizations of Euler’s difference table with combinatorial meanings were studied.

Definition 1. For fixed integer \(\ell \geq 1\), we define \(\lambda\)-Euler’s difference table \((g^m_n(\lambda))_{0 \leq m \leq n}\) for \(C_\ell \wr S_n\), where the coefficients are defined by
\[
\begin{align*}
g^m_n(\lambda) &= \ell^m n! & (m = n); \\
g^m_n(\lambda) &= g^{m+1}_n(\lambda) + (\lambda - 1) g^m_{n-1}(\lambda) & (0 \leq m \leq n - 1).
\end{align*}
\]

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From the above definition, it is easy to see the coefficients $g_{\ell,n}^m(\lambda)$ are polynomials in $\lambda$. Faliharimalala and Zeng [7] studied the combinatorial interpretation of $g_{1,n}^m(0)$ in terms of $k$-circular successions in $C_{k} \wr S_n$. Eriksen et al. [5] gave a combinatorial interpretation for the coefficients $g_{1,n}^m(\lambda)$ by assuming that $\lambda$ is a non-negative integer. They showed that $g_{1,n}^m(\lambda)$ count the number of permutations of $\lambda$ colors. Liese and Remmel [10] interpreted the coefficients of polynomial $g_{1,n}^m(\lambda)$ by counting certain rook placements in the $[n] \times [n]$ board.

It is not hard to see that the coefficient $g_{\ell,n}^m(\lambda)$ is divisible by $\ell^m m!$. This prompted us to introduce $d_{\ell,n}^m(\lambda) = g_{\ell,n}^m(\lambda) / \ell^m m!$. Then we derive the following allied array $(d_{\ell,n}^m(\lambda))_{0 \leq m \leq n}$ from (1.2).

**Definition 2.** For a fixed integer $\ell \geq 1$, the coefficients of the $\lambda$-difference table

$$(d_{\ell,n}^m(\lambda))_{0 \leq m \leq n}$$

are defined by

$$
\begin{aligned}
    d_{\ell,n}^m(\lambda) &= 1 & (m = n); \\
    d_{\ell,n}^m(\lambda) &= \ell(m + 1)d_{\ell,n}^{m+1}(\lambda) + (\lambda - 1)d_{\ell,n}^{m-1}(\lambda) & (0 \leq m \leq n - 1).
\end{aligned}
$$

The first terms of these coefficients for $\ell = 1, 2$ are given in Tables 1 and 2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
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<td></td>
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<td></td>
<td></td>
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<tr>
<td>1</td>
<td></td>
<td>$\lambda$</td>
<td></td>
<td>1</td>
<td></td>
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</tr>
<tr>
<td>2</td>
<td></td>
<td>$\lambda^2 + 1$</td>
<td>$\lambda + 1$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$\lambda^3 + 3\lambda + 2$</td>
<td>$\lambda^2 + 2\lambda + 3$</td>
<td>$\lambda + 2$</td>
<td>1</td>
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</tr>
<tr>
<td>4</td>
<td></td>
<td>$\lambda^4 + 6\lambda^2 + 8\lambda + 9$</td>
<td>$\lambda^3 + 3\lambda^2 + 9\lambda + 11$</td>
<td>$\lambda^2 + 4\lambda + 7$</td>
<td>$\lambda + 3$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Values of $d_{\ell,n}^m(\lambda)$ for $0 \leq m \leq n \leq 4$ and $\ell = 1$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>$\lambda + 1$</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$\lambda^2 + 2\lambda + 5$</td>
<td>$\lambda + 3$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$\lambda^3 + 3\lambda^2 + 15\lambda + 29$</td>
<td>$\lambda^2 + 6\lambda + 17$</td>
<td>$\lambda + 5$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$\lambda^4 + 4\lambda^3 + 30\lambda^2 + 116\lambda + 233$</td>
<td>$\lambda^3 + 9\lambda^2 + 51\lambda + 131$</td>
<td>$\lambda^2 + 10\lambda + 37$</td>
<td>$\lambda + 7$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Values of $d_{\ell,n}^m(\lambda)$ for $0 \leq m \leq n \leq 4$ and $\ell = 2$.

Two combinatorial interpretations of $d_{\ell,n}^m(0)$ were given in [7]. When $\lambda$ is a non-negative integer, Eriksen et al. [5] gave a combinatorial interpretation for the coefficients
Motivated by [7, 5, 10, 14], we study the combinatorial interpretation of \(G_{\ell,n}(\lambda)\) and \(d_{\ell,n}(\lambda)\) in the colored symmetric group \(G_{\ell,n}\), i.e., the wreath product of a cyclic group and a symmetric group. The paper is organized as follows. In Sections 3 and 4, we interpret the polynomial \(G_{\ell,n}(\lambda)\) and the coefficients in \(G_{\ell,n}(\lambda)\), respectively. In Sections 5 and 6, we prove the linear combinatorial interpretation and cyclic combinatorial interpretation of \(d_{\ell,n}(\lambda)\), respectively. In Section 7, we obtain the generating functions and recurrence relations of \(d_{\ell,n}(\lambda)\). In Section 8, we generalize \(r\)-derangement number by relating with \(d_{\ell,n}(\lambda)\). In Section 9, we give a combinatorial proof of recurrence relation of \(d_{\ell,n}(\lambda)\).

2 Definitions and main results

For positive integers \(\ell, n \geq 1\), the group of colored permutations of \(n\) elements with \(\ell\) colors is the wreath product \(G_{\ell,n} := C_\ell \rtimes S_n\), where \(C_\ell\) is the \(\ell\)-cyclic group generated by \(\zeta = e^{2\pi i/\ell}(i^2 = -1)\). From definition, it is obvious to see the elements in \(G_{\ell,n}\) are pairs \((\epsilon, \sigma)\) \(\in C_\ell \times S_n\).

And \(G_{\ell,n}\) can also be seen as a permutation group on the colored set:

\[
\Sigma_{\ell,n} := C_\ell \times [n] = \{\zeta^j i | i \in [n], 0 \leq j \leq \ell - 1\}.
\]

Clearly there are \(\ell^n n!\) signed permutations in the group \(G_{\ell,n}\). For more details, see [6].

A signed permutation \(\pi \in G_{\ell,n}\) can be written in two-line form. For example, if \(\pi = (\epsilon, \sigma) \in G_{4,11}\), where \(\epsilon = (1, \zeta^3, 1, \zeta, 1, \zeta^2, \zeta, 1, \zeta, 1)\) and

\[
\sigma = 7 \ 5 \ 3 \ 1 \ 2 \ 6 \ 8 \ 9 \ 4 \ 10 \ 11,
\]

we write

\[
\pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\zeta^2 7 & 5 & 3 & 1 & \zeta^3 2 & 6 & \zeta 8 & 9 & \zeta 4 & \zeta 10 & 11
\end{pmatrix}.
\]

To be convenient, we write \(j\) bars over \(i\) instead of \(\zeta^j i\). Thus, we rewrite the above permutation in linear form as \(\pi = \bar{7} \ 5 \ 3 \ 1 \ 2 \ 6 \ \bar{8} \ 9 \ 4 \ \bar{10} \ 11\), or in disjoint cyclic form as

\[
\pi = (1, \bar{7}, \bar{8}, 9, \bar{4}) (\bar{2}, 5) (3) (6) (\bar{10}) (11).
\]

That is, when using disjoint cyclic notation to determine the image of a number, we ignore the sign on that number and only consider the sign on the number to which it is mapped. Thus, in the above example, we ignore the sign \(\zeta^2\) on the 7 and 7 maps to \(\zeta 8\) since the sign on 8 is \(\zeta\). Moreover, let \([m+1, n]\) denote the interval \(\{m+1, \ldots, n\}\), and we give the following conventions:

i) If \(\pi = (\epsilon, \sigma) \in G_{\ell,n}\), we define \(|\pi| = \sigma\) and \(\text{sign}_\epsilon(i) = \epsilon_i\) for \(i \in [n]\). For example, if \(\pi = 1 \bar{4} 3 \bar{2}\) then \(\epsilon = (1, \zeta^2, 1, \zeta)\) and \(\text{sign}_\epsilon(4) = \zeta\).
ii) For $i \in [n]$ and $j \in \{0, 1, \ldots, \ell - 1\}$, we define $\zeta^j i + k = \zeta^j(i + k)$ for $0 \leq k \leq n - i$, and $\zeta^j i - k = \zeta^j(i - k)$ for $0 \leq k \leq i$. For example, $\bar{2} + 1 = \bar{3}$ in $G_{4,11}$.

iii) We define the total ordering on $\Sigma_{\ell,n}$ as follows. For $i, j \in \{0, 1, \ldots, \ell - 1\}$ and $a, b \in [n]$, $\zeta^j a < \zeta^j b \iff i > j$ or $i = j$ and $a < b$.

In $G_{\ell,n}$, Faliharimalala and Zeng [7] introduced the $k$-successions as follows.

**Definition 3.** Given a permutation $\pi \in G_{\ell,n}$ and an integer $0 \leq k \leq n - 1$, $\pi(i)$ is a $k$-succession at position $i \in [n - k]$ if $\pi(i) = i + k$. In particular, the 0-succession is also called fixed point.

Note that the above $k$-succession $\pi(i)$ needs to be uncolored, that is, $\text{sign}_n(\pi(i)) = 1$.

To obtain the combinatorial interpretation of $g^m_{\ell,n}(\lambda)$, we introduce the following definition.

**Definition 4.** For any integer $0 \leq k \leq n - 1$, let $SUC_k(\pi)$ denote the set of $k$-successions in $\pi \in G_{\ell,n}$, i.e.,

$$SUC_k(\pi) = \{\pi(i)|\pi(i) = i + k, i \in [n - k], \pi \in G_{\ell,n}\}.$$

For an integer $0 \leq m \leq n$, we define the statistic $\text{suc}^{(k)}_{>m}(\pi)$ as the number of $k$-successions included in $[m + 1, n]$ for $\pi \in G_{\ell,n}$, i.e.,

$$\text{suc}^{(k)}_{>m}(\pi) = \#\{\pi(i) \in [m + 1, n]|\pi(i) \in SUC_k(\pi)\}.$$

In particular, for $\pi \in G_{\ell,n}$, by taking $k = 0$ and $k = m$, $\text{suc}^{(k)}_{>m}$ is the number of fixed points and $m$-successions concerning $\pi \in G_{\ell,n}$, respectively, which are included in $[m + 1, n]$.

For example, when $\pi \in G_{4,11}$, if

$$\pi = 531\bar{2}6\bar{9}4\bar{10}11\bar{7}$$

and

$$\pi' = 31\bar{2}6\bar{9}4\bar{10}11\bar{7}5,$$

we have $SUC_1(\pi) = SUC_2(\pi') = \{3, 6, 11\}$ and $\text{suc}^{(1)}_{>4}(\pi) = \text{suc}^{(2)}_{>4}(\pi') = 2$.

**Theorem 5.** For fixed integers $\ell$, $k$, $m$ and $n$, let $\ell \geq 1$ and $0 \leq k \leq m \leq n$, we have

$$g^m_{\ell,n}(\lambda) = \sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}^{(k)}_{>m}(\pi)}.$$

**Remark 6.** We recover Faliharimalala and Zeng’s result [7, Theorem 3] about the combinatorial interpretation of $g^m_{\ell,n}(0)$ in $G_{\ell,n}$. And we prove Theorem 5 in Section 3.
We give an example to illustrate the above theorem. For $\ell = 2$, $n = 2$ and $m = 1$, the permutations in $G_{2,2}$ are

$$1 \ 2, \ \bar{1} \ 2, \ 1 \ \bar{2}, \ 1 \ 2, \ 2 \ 1, \ \bar{2} \ 1, \ 2 \ \bar{1}, \ \bar{2} \ \bar{1}.$$ 

For $k = 0$, $\sum_{\pi \in G_{2,2}} \lambda^{\text{suc}(0)}(\pi) = 2\lambda + 6$. For $k = 1$, $\sum_{\pi \in G_{2,2}} \lambda^{\text{suc}(1)}(\pi) = 2\lambda + 6$.

For $n, m, s \geq 0$, Rakotondrajao [12] also studied the number of permutations in $S_n$ having exactly $s \ m$-successions. Similarly, we define that $c_{\ell,n,s}^m$ is the number of permutations $\pi \in G_{\ell,n}$ having $s \ m$-successions. In other words,

$$c_{\ell,n,s}^m = |\{\pi \in G_{\ell,n} \ | \ |SUC_m(\pi)| = s\}|, \ \text{for } n, s, m \geq 0.$$ 

With Theorem 5 and above definition, we state an expression of $g_{\ell,n}^m(\lambda)$ as follows.

**Corollary 7.** For $\ell \geq 1$, $0 \leq m \leq n$ and $0 \leq s \leq n - m$, we have

$$g_{\ell,n}^m(\lambda) = \sum_{s \geq 0} c_{\ell,n,s}^m \lambda^s.$$ 

(5)

**Remark 8.** With the equations (2) and (5), we obtain that

$$c_{\ell,n,s}^{m+1} = c_{\ell,n,s}^m + c_{\ell,n-1,s}^m - c_{\ell,n-1,s-1}^m,$$

which is the result of [7, Theorem 4].

To show the combinatorial interpretations and recursions of $c_{\ell,n,s}^m$, we review the generalized rook theory model in [1].

Let $B_n^{\ell}$ be the $n \times \ell n$ array of squares, we label the $n$ columns from left to right by $1, 2, \ldots, n$ and the $\ell n$ rows from bottom to top by

$$1, \zeta_1, \ldots, \zeta_{\ell-1} 1, 2, \zeta_2, \ldots, \zeta_{\ell-1} 2, \ldots, n, \zeta n, \ldots, \zeta_{\ell-1} n,$$

respectively. For instance, the board $B_n^{2}$ is pictured in Figure 1. The square in the column labeled with $i$ and the row labeled with $\zeta^r j$ is denoted by $(i, \zeta^r j)$. Each such square is called a cell and the rows labeled by $j, \zeta j, \ldots, \zeta_{\ell-1} j$ are called level $j$.

Given a board $B \subseteq B_n^{\ell}$, we let $R_{k,n}^\ell(B)$ denote the set of $k$ element subsets $P$ of $B$ such that no two elements lie in the same level or column for non-negative integers $k$. We call the subset $P$ a placement of non-attacking $\ell$-rooks in $B$. Since the cells in the placement are considered to contain $\ell$-rooks, we define the $k$th $\ell$-rook number of $B$ by $r_{k,n}^\ell(B) = |R_{k,n}^\ell(B)|$.

Given a permutation $\pi \in G_{\ell,n}$, we can identify $\pi$ with a placement $P_{\pi}$ of $n \ \ell$-rooks in $B_n^{\ell}$. In other word, $P_{\pi} = \{(i, \zeta^r j) : \pi(i) = \zeta^r j \ \text{for } 1 \leq i \leq n\}$, then we define the $k$th $\ell$-hit number of $B$ denoted by $h_{k,n}^\ell(B)$, which is the number of $\pi \in G_{\ell,n}$ such that the placement $P_{\pi}$ intersects the board $B$ in exactly $k$ cells, i.e.,

$$h_{k,n}^\ell(B) = |\{|P_{\pi} | \pi \in G_{\ell,n} \ \text{and} |P_{\pi} \cap B| = k\}|.$$ 

Briggs and Remmel [1, Theorem 1] found the following relationship between the $\ell$-hit numbers and the $\ell$-rook numbers.
Theorem 9 (Briggs-Remmel). Let $B$ be a board contained in $B_n^\ell$. Then
\[ \sum_{k=0}^{n} h_{k,n}^\ell(B)x^k = \sum_{k=0}^{n} r_{k,n}^\ell(B)\ell^{n-k}(n-k)!/(x-1)^k. \]

By interpreting $c_{\ell,n,s}^m$ in terms of $\ell$-hit numbers for a certain board, we obtain the following formula.

Theorem 10. For $\ell, n \geq 1$, $0 \leq m \leq n$ and $s \geq 0$, we have
\[ c_{\ell,n,s}^m = \sum_{t=s}^{n-m} (-1)^{t-s} \ell^{n-t}(n-t)! \binom{t}{s} \binom{n-m}{t}. \quad (6) \]

Remark 11. When $\ell = 1$, (6) reduce to the result of [10, Theorem 2.2]. And we prove Theorem 10 in Section 4.

To give the linear interpretation of $d_{\ell,n}(\lambda)$, we give the following definition.
Definition 12. For \(0 \leq m \leq n\), a permutation \(\pi = (\epsilon, \sigma) \in G_{\ell,n}\) is called an \(m\)-decreasing permutation if satisfies the following conditions:

i) \(\text{sign}_{\pi}(\pi(i)) = 1 (i \in [m]);\)

ii) \(\pi(1) > \pi(2) > \cdots > \pi(m)\).

Let \(L^m_{\ell,n}\) be the set of \(m\)-decreasing permutations in \(G_{\ell,n}\). For example, when \(\ell = 2, n = 3\) and \(m = 2\),

\[L^2_{2,3} = \{213, 21\bar{3}, 312, 31\bar{2}, 321, 32\bar{1}\}, \text{ and } \sum_{\pi \in L^2_{2,3}} \lambda^{\text{fix}>2(\pi)} = \lambda + 5.\]

Theorem 13. For \(0 \leq m \leq n\), we have

\[d^m_{\ell,n}(\lambda) = \sum_{\pi \in L^m_{\ell,n}} \lambda^{\text{fix}>m(\pi)}.\]

Remark 14. When \(\lambda = 0\), Theorem 13 reduce to the result of [7, Theorem 10], we prove above theorem in Section 5.

To give the cyclic interpretation of \(d^m_{\ell,n}(\lambda)\), we give the following definition.

Definition 15. For \(0 \leq m \leq n\), a permutation \(\pi = (\epsilon, \sigma) \in G_{\ell,n}\) is called \(m\)-separated permutation if satisfies the following conditions:

i) \(\text{sign}_{\pi}(i) = 1 (i \in [m]);\)

ii) the first \(m\) elements belong into distinct cycles.

Let \(C^m_{\ell,n}\) be the set of \(m\)-separated permutations in \(G_{\ell,n}\). For example, when \(\ell = 2, n = 3\) and \(m = 2\),

\[C^2_{2,3} = \{(13)(2), (1\bar{3})(2), (1)(23), (1)(2\bar{3}), (1)(2)(3), (1)(2)(\bar{3})\}, \text{ and } \sum_{\pi \in C^2_{2,3}} \lambda^{\text{fix}>2(\pi)} = \lambda + 5.\]

Theorem 16. For \(0 \leq m \leq n\), we have

\[d^m_{\ell,n}(\lambda) = \sum_{\pi \in C^m_{\ell,n}} \lambda^{\text{fix}>m(\pi)}.\]

Remark 17. When \(\lambda = 0\), Theorem 16 reduce to the result of [7, Theorem 12], we prove above theorem in Section 6.

To generalize the definition of \(r\)-derangement number, we give the following definition.

Definition 18. For \(0 \leq m \leq n\), a permutation \(\pi \in G_{\ell,n}\) is called \(m\)-fixed point-free colored permutation if satisfies the following conditions:
i) For \( i \in [m] \), let \( \pi(i) \in [m + 1, n] \) and \( \text{sign}_\pi(i) = \text{sign}_\pi(\pi(i)) = 1 \);

ii) no two elements of \([m]\) are in the same cycle.

Let \( F_{\ell,n+m}^m \) be the set of \( m \)-fixed point-free colored permutations in \( G_{\ell,n+2m} \), we define

\[
f_{\ell,n+m}^m(\lambda) = \sum_{\pi \in F_{\ell,n+m}^m} \lambda^{\text{fix}_m(\pi)}. \tag{7}
\]

For example, when \( \ell = 2, n = 1 \) and \( m = 1 \),

\[
F_{2,2}^1 = \{(12)(3), (12)(\bar{3}), (13)(2), (13)(\bar{2}), (123), (12\bar{3}), (132), (13\bar{2})\}
\]

and \( f_{2,2}^1(\lambda) = 2\lambda + 6 \).

Remark 19. When \((\ell, \lambda) = (1,0)\), the equation (7) reduce to the sum over \( \{\pi \in F_{\ell,n+m}^m | \text{fix}_m(\pi) = 0\} \),

then the polynomial \( f_{\ell,n+m}^m(\lambda) \) reduces to the \( r \)-derangement number, see [14, Definition 1].

By the above definition, we generalize the generating functions and recurrence relations of Wang et al. [14].

By observing the above definitions, we prove the following combinatorial relation between the \( f_{\ell,n+m}^m(\lambda) \) and \( d_{\ell,n+m}^m(\lambda) \) in Section 8.

**Theorem 20.** For \( \ell \geq 1 \) and \( m, n \geq 0 \), we have

\[
f_{\ell,n+m}^m(\lambda) = \frac{(n+m)!}{n!} d_{\ell,n+m}^m(\lambda). \tag{8}
\]

3 Proof of Theorem 5

In the section, to prove Theorem 5, we prove the following equations,

\[
\begin{align*}
\sum_{\pi \in G_{\ell,n}} \lambda^{\text{SUC}_{\lambda}^m(\pi)} &= \ell^n n! \quad (m = n); \\
\sum_{\pi \in G_{\ell,n}} \lambda^{\text{SUC}_{\lambda}^m(\pi)} &= \sum_{\pi \in G_{\ell,n}} \lambda^{\text{SUC}_{\lambda}^m(\pi)} + (\lambda - 1) \sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{SUC}_{\lambda}^m(\pi)} \quad (0 \leq m \leq n - 1).
\end{align*}
\]

\[
(9)
\]

**Lemma 21.** For any integer \( k \) such that \( 0 \leq k \leq m \) and \( 0 \leq m \leq n \), there holds

\[
\sum_{\pi \in G_{\ell,n}} \lambda^{\text{SUC}_{\lambda}^m(\pi)} = \lambda \sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{SUC}_{\lambda}^m(\pi)}. \tag{10}
\]
Proof. Let us define the bijection $\psi: G_{\ell,n} \mapsto G_{\ell,n-1}$. For $\pi \in G_{\ell,n}$, we delete the $m+1$ at position $m+1-k$ and define the $\psi(\pi) = \overset{\pi_1}{\pi_2} \ldots \overset{\pi_{m-k}}{\pi_{m-k+2}} \ldots \overset{\pi_n}{\pi_{n-1}} \in G_{\ell,n-1}$ where

$$\pi_i = \begin{cases} 
\overset{\pi_i}{\pi_i}, & \text{if } |\pi_i| < m+1; \\
\overset{\pi_i-1}{\pi_i}, & \text{if } |\pi_i| > m+1.
\end{cases}$$

Conversely, starting from $\psi(\pi) = \overset{\pi_1}{\pi_2} \ldots \overset{\pi_{m-k}}{\pi_{m-k+2}} \ldots \overset{\pi_n}{\pi_{n-1}} \in G_{\ell,n-1}$, we define $\pi = \overset{\pi_1}{\pi_2} \ldots \overset{\pi_n}{\pi_{n-1}} \in G_{\ell,n}$ where

$$\pi_i = \begin{cases} 
\overset{\pi_i}{\pi_i}, & \text{if } |\pi_i| < m+1; \\
\overset{\pi_i+1}{\pi_i}, & \text{if } |\pi_i| > m+1.
\end{cases}$$

Then we put $m+1$ at the position $m+1-k$, from the map, we can easily see $\text{suc}^k_{>m}(\pi) = \text{suc}^k_{>m}(\psi(\pi)) + 1$. □

For example $\ell = 4, n = 9, m = 4, k = 1$, $\pi = \overset{7}{3} 4 5 \overset{9}{2} \overset{6}{1} 8 9 \overset{6}{5}$, $\psi(\pi) = \overset{6}{3} 4 2 \overset{1}{1} 7 8 \overset{5}{5}$, and $\text{suc}^1_{>4}(\pi) = \text{suc}^1_{>4}(\psi(\pi)) + 1$.

Lemma 22. For $0 \leq m \leq n$, there holds

$$\sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{suc}^k_{>m}(\pi)} = \sum_{\pi \in G_{\ell,n} \setminus \text{SUC}_k(\pi)} \lambda^{\text{suc}^k_{>m+1}(\pi)}. \tag{11}$$

Proof. It follows similar arguments as in the proof of Lemma 21. □

Proof of Theorem 5. First we check the initial condition in (9), when $m = n$, $\text{suc}^k_{>n}(\pi) = 0$, $\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}^k_{>n}(\pi)} = \ell^n n!$.

We start to prove the recurrence in (9). Then, by considering the following equation,

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}^k_{>m}(\pi)} = \sum_{\pi \in G_{\ell,n} \setminus \text{SUC}_k(\pi)} \lambda^{\text{suc}^k_{>m}(\pi)} + \sum_{\pi \in G_{\ell,n} \setminus \text{SUC}_k(\pi)} \lambda^{\text{suc}^k_{>m+1}(\pi)}. \tag{12}$$

Because for $\pi \in G_{\ell,n}$ with $m+1 \notin \text{SUC}_k(\pi)$, we have $\text{suc}^k_{>m}(\pi) = \text{suc}^k_{>m+1}(\pi)$, then (12) is equivalent to

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}^k_{>m}(\pi)} = \sum_{\pi \in G_{\ell,n} \setminus \text{SUC}_k(\pi)} \lambda^{\text{suc}^k_{>m+1}(\pi)} + \sum_{\pi \in G_{\ell,n} \setminus \text{SUC}_k(\pi)} \lambda^{\text{suc}^k_{>m+1}(\pi)}. \tag{13}$$

By equations (10) and (13), we obtain that

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}^k_{>m}(\pi)} = \sum_{\pi \in G_{\ell,n} \setminus \text{SUC}_k(\pi)} \lambda^{\text{suc}^k_{>m+1}(\pi)} + \lambda \sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{suc}^k_{>m}(\pi)}. \tag{14}$$
Figure 2: The board $B^2_{4,1}$ corresponds to the shaded cells.

By combining the equations (11) and (14), we obtain

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{> m}(k)}(\pi) = (\lambda - 1) \sum_{\pi \in G_{\ell,n-1}} \lambda^{\text{suc}_{> m}(k)}(\pi) + \sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{> m+1}(k)}(\pi) + \sum_{\pi \in G_{\ell,n}} \lambda^{\text{suc}_{> m+1}(k)}(\pi).$$

With (12), it is easy to see that the above equation is equivalent to the recurrence relation in (9), this completes the proof of Theorem 5.

Remark 23. Since $\gamma_{\ell,n}^m(\lambda)$ is independent from $k$ ($0 \leq k \leq m$) in the above proof is not mentioned, we state an argument as follows. By considering the bijection $d$ which transforms $\pi = \pi_1\pi_2\pi_3 \cdots \pi_n$ into $d(\pi) = \pi' = \pi_2\pi_3 \cdots \pi_n\pi_1$. It is easy to see that the $k$-successions of $\pi$ are in $[m+1, n]$ if and only if the $(k+1)$-successions of $\pi'$ are in $[m+1, n]$. Hence, let the composition of $j$ times of $d$ is denoted by $d^j$, the application of $d^{k_2-k_1}$ permits to transfer the $k_1$-successions to $k_2$-successions if $k_1 < k_2$. In particular if we apply $d^m$ to a permutation whose fixed points are in $[m+1, n]$, then we obtain a permutation whose $m$-succession are in $[m+1, n]$ and vice versa.

4 Proof of Theorem 10

In this section, first we prove the following expressions of $c_{m}^{\ell,n,s}$ in Theorem 10,

$$c_{\ell,n,s}^{m} = \sum_{t=s}^{n-m} (-1)^{t-s} \ell^n (n-t)! \binom{t}{s} \binom{n-m}{t}.$$ \hfill (15)

Then we derive several recurrence relations of $c_{\ell,n,s}^{m}$.

Proof of Theorem 10. First, we give the combinatorial interpretation of $c_{\ell,n,s}^{m}$ as follows. Let $B_{n,m}^{\ell}$ be the board contained in $B_{n}^{\ell}$ consisting of the cells $(1, 1+m), (2, 2+m), (3, 3+m), \ldots, (n-m, n)$. For example, the board $B_{2,1}^{2}$ is pictured in Figure 2. Then the number of $\pi \in G_{\ell,n}$ with $s$ $m$-successions is the $s$-th $\ell$-hit number of $B_{n,m}^{\ell}$, i.e.,

$$c_{\ell,n,s}^{m} = h_{s,n}^{\ell}(B_{n,m}^{\ell}).$$ \hfill (16)
With the definitions of $B_{n,m}^\ell$, we have $r_{s,n}^\ell(B_{n,m}) = \binom{n-m}{s}$. By Theorem 9,

$$g_{\ell,n}^m(\lambda) = \sum_{s=0}^n c_{\ell,n,s}^m \lambda^s = \sum_{s=0}^n h_{s,n}(B_{n,m})^\ell \lambda^s$$

$$= \sum_{s=0}^n r_{s,n}^\ell(B_{n,m})^\ell \binom{n-s}{s} (n-s)! (\lambda-1)^s$$

$$= \sum_{s=0}^n \binom{n-m}{s} \ell^{n-s}(n-s)! (\lambda-1)^s. \tag{17}$$

Equating the coefficients of $\lambda^s$ yields (15) immediately.

Remark 24. We also obtain the above expression (17) of $g_{\ell,n}^m(\lambda)$ by generating function, see Proposition 41.

Let $s = n - m$ in (15), we obtain the following corollary.

**Corollary 25.** For all $\ell \geq 1$ and $n \geq m \geq 0$,

$$c_{\ell,n,n-m}^m = \ell^m m!$$

Next we show the recurrence relations of $c_{\ell,n,s}^m$ in colored symmetric group.

**Proposition 26.** For all $\ell \geq 1, n \geq 2, 0 \leq m < n$, and $s \geq 1$,

$$c_{\ell,n,s}^m = (\ell(n-s-1) + (\ell-1))c_{\ell,n-1,s}^m + \ell(s+1)c_{\ell,n-1,s+1}^m + c_{\ell,n-1,s-1}^m. \tag{18}$$

**Proof.** Let us consider the map from $\pi = \pi_1 \ldots \pi_{n-1} \in G_{\ell,n-1}$ to $\bar{\pi} \in G_{\ell,n}$ such that $\bar{\pi}$ has $s$ $m$-successions, we consider the following three cases.

1. **If $\pi \in G_{\ell,n-1}$ has $s$ $m$-successions.**

   - Let $\bar{\pi} = \pi_1 \pi_{n-m-1}(\zeta^j n) \pi_{n-m+1} \ldots \pi_{n-1} \pi_{n-m}$ and $1 \leq j \leq \ell - 1$, the number of $m$-successions of $\bar{\pi}$ and $\pi$ are the same, so there are $(\ell-1)c_{\ell,n-1,s}^m$ permutations in this case.

   - Let $\bar{\pi} = \pi_1 \pi_{i-1}(\zeta^j n) \pi_{i+1} \ldots \pi_{n-1} \pi_i$, where $i \neq n-m$ and $i$ is a position without $m$-successions, the number of $m$-successions of $\bar{\pi}$ and $\pi$ are the same. Since we have $n-s-1$ choices for position $i$ and $0 \leq j \leq \ell - 1$, there are $\ell(n-s-1)c_{\ell,n-1,s}^m$ permutations in this case.

2. **If $\pi \in G_{\ell,n-1}$ has $s + 1$ $m$-successions.** Let $\bar{\pi} = \pi_1 \pi_{i-1}(\zeta^j n) \pi_{i+1} \ldots \pi_{n-1} \pi_i$, where $i$ is a position with $m$-succession, the number of $m$-successions of $\bar{\pi}$ is the number of $m$-successions of $\pi$ minus one. Since we have $s + 1$ choices for position $i$ and $0 \leq j \leq \ell - 1$, there are $\ell(s+1)c_{\ell,n-1,s+1}^m$ permutations in this case.
• If $\pi \in G_{\ell,n-1}$ has $s - 1$ $m$-successions. Let $\bar{\pi} = \pi_1 \pi_{n-m-1} (\zeta^j n) \pi_{n-m+1} \ldots \pi_{n-1} \pi_{n-m}$ and $j = 0$, the number of $m$-successions of $\bar{\pi}$ is the number of $m$-successions of $\pi$ plus one, so there are $c_{\ell,n-1,s-1}^m$ permutations in this case.

**Proposition 27.** For all $\ell, n \geq 1$, $0 \leq m < n$, and $s \geq 0$,

$$c_{\ell,n,s}^m = \binom{n-m}{s} c_{\ell,n-s,0}^m. \quad (19)$$

**Proof.** Note that in (16), $c_{\ell,n,s}^m$ is the number of placements of $n$ non-attacking $\ell$-rooks in $B_{n,m}^{\ell}$ that intersect $B_{n,m}^{\ell}$ in exactly $s$ squares. By removing the level $i + m$ and column $i$ of these $\ell$-rooks which lie in the cell $(i, i + m)(1 \leq i \leq n - m)$, we obtain these placements of $n - s$ non-attacking $\ell$-rooks in $B_{n-s,m}^{\ell}$ that intersect $B_{n-s,m}^{\ell}$ in exactly 0 squares, which is counted by $c_{\ell,n-s,0}^m$. The process is pictured in Figure 3.

**Remark 28.** Faliharimalala and Zeng [7, Lemma 14] proved the above (19) directly by interpreting $c_{\ell,n,s}^m$ as the number of permutation in $G_{\ell,n}$ with $s$ $m$-successions. However, we give a trivial proof by interpreting $c_{\ell,n,s}^m$ as the number of placements of $n$ non-attacking $\ell$-rooks in $B_{n}^{\ell}$ that intersect $B_{n,m}^{\ell}$ in exactly $s$ squares.

**Proposition 29.** For all $\ell \geq 1$, $n \geq 2$ and $0 \leq m < n$,

$$c_{\ell,n,0}^m = (\ell n - 1) c_{\ell,n-1,0}^m + \ell (n - m - 1) c_{\ell,n-2,0}^m. \quad (20)$$

**Proof.** Let us consider the map from $\pi = \pi_1 \ldots \pi_n \in G_{\ell,n}$ to $\pi' \in G_{\ell,n-1}$, starting from $\pi$ without $m$-successions, we define

$$\pi' = \begin{cases} 
\pi_1 \ldots \pi_{i-1} \pi_{i+1} \ldots \pi_n, & \text{if } \pi_i = \zeta^j n (0 \leq j \leq \ell - 1) \text{ for } 1 \leq i < n; \\
\pi_1 \ldots \pi_{n-1}, & \text{if } \pi_n = \zeta^j n (0 \leq j \leq \ell - 1). 
\end{cases}$$
1. $\pi' \in G_{\ell,n-1}$ has no $m$-successions.

Either $\pi_n = \zeta^j n(0 \leq j \leq \ell - 1)$ or $\pi_i = \zeta^j n(1 \leq i < n, 0 \leq j \leq \ell - 1)$ and $\pi_n \neq i + m$, $\pi'$ has no $m$-successions. Conversely, for $\pi' \in G_{\ell,n-1}$ without $m$-successions, by inserting $\zeta^j n(0 \leq j \leq \ell - 1)$ into $\pi'$ in every position except putting $n$ in to position $n - m$, we obtain the permutation in $G_{\ell,n}$ without $m$-successions. Since $\zeta^j n(0 \leq j \leq \ell - 1)$ can be in any position except $\pi_{n-m} = n$, there are $(\ell n - 1)c_{n-1,0}^m$ permutations.

2. $\pi' \in G_{\ell,n-1}$ has 1 $m$-succession.

When $\pi_i = \zeta^j n(1 \leq i \leq n - 1 - m, 0 \leq j \leq \ell - 1)$ and $\pi_n = i + m$, $\pi'$ has 1 $m$-succession, then the 1 $m$-succession of $\pi'$ corresponds to the $\ell$-rook $(i,i+m)$ of the rook placement in the board $B_{n-1}^\ell$. For the rook placement corresponds to $\pi'$ in $B_{n-1}^\ell$, removing the column $i$ and level $i + m$ from the board $B_{n-1}^\ell$, we obtain the rook placement in $B_{n-2}^\ell$ without intersecting $B_{n-2,m}^\ell$, which corresponding to the permutation denoted by $\tilde{\pi} \in G_{\ell,n-2}$, and $\tilde{\pi}$ has no $m$-successions.

Conversely, let $\tilde{\pi}$ be a permutation in $G_{\ell,n-2}$ without $m$-successions, we obtain $\pi \in G_{\ell,n}$ in two steps.

Step 1. For $1 \leq i \leq n - 1 - m$, by adding the column $i$ and level $i + m$ to the boards $B_{n-2}^\ell$, we choose $(i,i + m)$ as the new $\ell$-rook and take the same rook placement corresponds to $\tilde{\pi}$, then we obtain the rook placement in $B_{n-1}^\ell$ corresponding to $\pi' \in G_{\ell,n-1}$ with 1 $m$-succession.

Step 2. Adding the column $n$ and level $n$ in the board $B_{n-1}^\ell$, by taking away the $\ell$-rook $(i,i+m)$ and putting $\ell$-rooks at $(i,\zeta^j n)(0 \leq j \leq \ell - 1)$ and $(n,i+m)$, we obtain the rook placement without intersecting $B_{n,m}^\ell$, which corresponds to the permutation $\pi \in G_{\ell,n}$ without $m$-successions. Since $1 \leq i \leq n - 1 - m$ and $0 \leq j \leq \ell - 1$, there are $(\ell(n - 1 - m)c_{n-2,0}^m)$ permutations.

Remark 30. When $m = 0$, we define that

$$D_n^\ell := c_{\ell,n,0}^0,$$  \hspace{1cm} (21)

which counts the number of derangements in $G_{\ell,n}$. It is easy to see (20) reduce to

$$D_n^\ell = (\ell n - 1)D_{n-1}^\ell + \ell(n - 1)D_{n-2}^\ell.$$ 

Proposition 31. For all $\ell \geq 1$, $n \geq 2$, and $1 \leq m < n$,

$$c_{\ell,n,0}^m = \ell mc_{\ell,n-1,0}^{m-1} + \ell(n-m)c_{\ell,n-1,0}^m.$$ \hspace{1cm} (22)

Proof. We prove the above equation by considering level 1 in the rook placement corresponding to the permutation $\pi \in G_{\ell,n}$ without $m$-successions.
1. When the $\ell$-rook of level 1 is in column $i$ ($1 \leq i \leq n - m$), if the $\ell$-rook of level $i + m$ lies at the position $(k, \zeta(i + m))(k \neq i$ and $0 \leq j \leq \ell - 1)$, by adding a $\ell$-rook at the position $(k, \zeta(j))$ in the level 1, we obtain a placement of $n + 1$ $\ell$-rooks without intersecting $B^\ell_{n,m}$. Then removing column $i$ and level $i + m$ will result in a placement of $n - 1$ non-attacking $\ell$-rooks without intersecting $B^\ell_{n-1,m}$. Since the rook placement corresponding to $\pi$ has $\ell$ different positions in level 1 and column $1 \leq i \leq n - m$, thus there are $\ell(n - m)c^m_{\ell,n-1,0}$ placements. The process is illustrated in top of the Figure 4.

2. When the $\ell$-rook of level 1 is in column $i(n - m < i \leq n)$, by removing column $i$ and level 1, we obtain a placement of $n - 1$ non-attacking $\ell$-rooks without intersecting $B^\ell_{n-1,m-1}$. Thus there are $\ell mc^m_{\ell,n-1,0}$ placements. The process is illustrated in the bottom of Figure 4. \qed

**Proposition 32.** For all $\ell, n \geq 1$ and $0 \leq m < n$,

$$c^m_{\ell,n,0} = \ell^m m! \sum_{r=0}^{m} \binom{m}{r} \binom{n - m}{m - r} c^m_{\ell,n-m,0}. \quad (23)$$

**Proof.** To obtain a placement of $n$ non-attacking $\ell$-rooks without intersecting $B^\ell_{n,m}$, starting from the lightly shaded cells in the lower right corner of the board in Figure 5, we see that 0 to $m$ $\ell$-rooks can be placed in this area. Suppose that we choose $r$ levels in
this area, there are \( \binom{m}{r} \) ways. Since there should be \( m \) \( \ell \)-rooks in the last \( m \) columns, we choose \( m - r \) \( \ell \)-rooks above the lower right corner of the board. Thus we choose \( m - r \) levels from the \( n - m \) levels, there are \( \binom{n-m}{m-r} \) ways.

After picking the \( m \) levels that contain the \( \ell \)-rooks in the last \( m \) columns, there are \( \ell^m m! \) ways to place the \( \ell \)-rooks in the last \( m \) columns.

Let \( P \) denote the non-attacking rook placement in the last \( m \) columns with \( r \) \( \ell \)-rooks falling in the lightly shaded area, we extend \( P \) to a non-attacking rook placement \( Q \) with \( n \) \( \ell \)-rooks, where there is no intersection with \( B_{n-m,m}^{\ell} \). By removing the levels and columns of rook placement \( P \), we obtain the non-attacking rook placement without intersecting \( B_{n-m,m-r}^{\ell} \), which is counted by \( c_{\ell,n-m,0}^{m-r} \). Summing over all possible values of \( r \) yields the desired result. \( \square \)

**Proposition 33.** For all \( \ell \geq 1 \), \( n \geq 2 \) and \( 0 \leq m < n \),

\[
c_{\ell,n,0}^{m} = \ell c_{\ell,n,1}^{m+1} + (\ell m + \ell - 1)c_{\ell,n-1,0}^{m}.
\] (24)

**Proof.** Let us consider the rook position of level \( n \) in the rook placement which corresponding to the permutation \( \pi \in G_{\ell,n} \) without \( m \)-successions.

1. When the \( \ell \)-rook of level \( n \) is in column \( i(1 \leq i \leq n - m - 1) \). If the \( \ell \)-rook of column \( i \) is in row \( n \), we keep it unchanged. If the \( \ell \)-rook of column \( i \) is in row \( \zeta(j) n(1 \leq j \leq \ell - 1) \), we exchange the row \( \zeta(j) n \) with row \( n \). Then we move the level \( n \) to the bottom level of the board, which is denoted by level \( 1' \), other levels are increased by one such as level \( 2' \), ... , level \( n' \). By exchanging the level \( 1' \) and level \( (i + m + 1)' \), we obtain a non-attacking rook placement that intersect \( B_{n,m+1}^{\ell} \) one rook \((i,i + m + 1)\). Since the \( \ell \)-rook can be in the row \( \zeta(j) n(0 \leq j \leq \ell - 1) \), there are \( \ell^m c_{n,1}^{m+1} \) permutations in this case. This process is shown in top of Figure 6.

2. When the \( \ell \)-rook of level \( n \) is in column \( i(n - m \leq i \leq n) \), the \( \ell \)-rook can be in the position \((i,\zeta(j)n)(n - m \leq i \leq n,0 \leq j \leq \ell - 1) \), since \( \pi \) has no \( m \)-successions, the \( \ell \)-rook can not be in the square \((n - m,n)\), so there are \( \ell m + \ell - 1 \) choices in level \( n \).
Proposition 34. For $\ell \geq 1, n \geq 2$ and $1 \leq m < n$,

$$c_{\ell,n,0}^m = c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1}. \quad (25)$$

Proof. Let us consider the non-attacking rook placement corresponding to the permutation $\pi \in G_{\ell,n}$ without $m$-successions. We move the bottom level to the top level and all other levels reduced by one, which is shown in Figure 7.

1. When the bottom $\ell$-rook is not in the position $(n-m+1,1)$, the process is shown in the top of Figure 7. After the movement of $\ell$-rooks in the board, we obtain the non-attacking rook placement without intersecting $B_{n-1,m}^\ell$. Thus there are $c_{\ell,n,0}^{m-1}$ permutations in this case.

2. When the bottom $\ell$-rook is in the position $(n-m+1,1)$, the process is shown in the bottom of Figure 7. The resulting rook placement intersect $B_{n-1,m+1}^\ell$ in the position $(n-m+1,n)$. By removing the column $n-m+1$ and level $n$, we get the non-attacking rook placement without intersecting $B_{n-1,m-1}^\ell$. Thus there are $c_{\ell,n-1,0}^{m-1}$ permutations in this case. \[\square\]
Figure 7: Moving the bottom level to top level in $B_{4,2}^2$.

**Proposition 35.** For $\ell, n \geq 1$ and $0 \leq m < n$,

$$c_{\ell,n,0}^m = \sum_{r=0}^{m} \binom{m}{r} D_{n-m+r}^\ell.$$  \hfill (26)

**Proof.** We prove this theorem by inductions on $m$. If $m = 0$, we have $c_{\ell,n,0}^0 = D_n^\ell$ by equation (21). Suppose that $c_{\ell,n,0}^i = \sum_{r=0}^{i} \binom{i}{r} D_{n-i+r}^\ell$ is satisfied for $i \leq m - 1$, then

$$c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1} = \sum_{r=0}^{m-1} \binom{m-1}{r} D_{n-m+1+r}^\ell + \sum_{r=0}^{m-1} \binom{m-1}{r} D_{n-m+r}^\ell.$$  \hfill (27)

By separating out the $m - 1$ term of the first sum and the 0 term of the second sum in (27), which is equivalent to

$$\binom{m-1}{m-1} D_n^\ell + \sum_{r=0}^{m-2} \binom{m-1}{r} D_{n-m+1+r}^\ell + \binom{m-1}{0} D_{n-m}^\ell + \sum_{r=1}^{m-1} \binom{m-1}{r} D_{n-m+r}^\ell.$$  

By transforming $r$ to $r - 1$ in the first sum and using $\binom{m-1}{r-1} + \binom{m-1}{r} = \binom{m}{r}$, we have

$$c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1} = \sum_{r=0}^{m} \binom{m}{r} D_{n-m+r}^\ell.$$
With the recurrence (25),
\[ c_{\ell,n,0}^m = c_{\ell,n,0}^{m-1} + c_{\ell,n-1,0}^{m-1}, \]
we obtain
\[ c_{\ell,n,0}^m = \sum_{r=0}^{m} \binom{m}{r} D_{n-m+r}^\ell. \]
The proof is thus completed. \( \square \)

With equations (19) and (26), we give the relation between \( c_{\ell,n,s}^m \) and \( D_n^\ell \) directly.

**Corollary 36.** For \( \ell, n \geq 1, 0 \leq m < n \) and \( s \geq 0 \),
\[ c_{\ell,n,s}^m = \binom{n-m}{s} \sum_{r=0}^{m} \binom{m}{r} D_{n-s-m+r}^\ell. \]

By observing the coefficients of polynomial \( g_{\ell,n}^m(\lambda) \), we find \( c_{\ell,n,s}^m \) decreases as \( s \) increases.

**Proposition 37.** For \( \ell, n \geq 1, 1 \leq m < n \) and \( s \geq 1 \),
\[ c_{\ell,n,s-1}^m \geq c_{\ell,n,s}^m. \]  \( (28) \)

**Proof.** With recursion (19) and (22), we have
\[
\begin{align*}
&\quad \quad c_{\ell,n,s-1}^m - c_{\ell,n,s}^m \\
&= \binom{n-m}{s-1} c_{\ell,n-s+1,0}^m - \binom{n-m}{s} c_{\ell,n-s,0}^m \\
&= (\ell(n-m-s+1)\binom{n-m}{s-1} - (n-m)\binom{n-m}{s}) c_{\ell,n-s,0}^m + \ell m \binom{n-m}{s-1} c_{\ell,n-s,0}^{m-1}. \\
\end{align*}
\]
Since
\[
\ell(n-m-s+1)\binom{n-m}{s-1} - (n-m)\binom{n-m}{s} = \frac{(n-m)!}{(n-m-s)!s!(\ell s - 1)},
\]
we obtain (28) immediately. \( \square \)

**Remark 38.** In particular, when \( m = 0 \), we have the similar result for \( c_{\ell,n,s}^0 \). By using \( D_n^\ell = \ell n D_{n-1}^\ell + (-1)^n [7, \text{equation (2.8)}] \) and similar arguments above, we have \( c_{\ell,n,s-1}^0 \geq c_{\ell,n,s}^0 \) for \( 2 \leq s \leq n-2 \) and \( n \geq 3 \).

**Remark 39.** When \( \ell = 1 \), the above expressions and relations of \( c_{\ell,n,s}^m \) in this section reduce to Liese and Remmel’s results [10, Sections 2 and 3].
5 Proof of Theorem 13

In this section, to prove Theorem 13, we prove the following equations,

\[ \sum_{\pi \in L_{\ell,n}^m} \lambda_{\text{fix} > m}(\pi) = \sum_{\pi \in G_{\ell,n}} \lambda_{\text{fix} > m}(\pi) / \ell^m m!, \quad \text{for } 0 \leq m \leq n. \tag{29} \]

**Proof of Theorem 13.** For \( 1 \leq k \leq n \), if \( \pi = \pi(1)\pi(2)\ldots\pi(k - 1)\pi(k)\ldots\pi(n) \in G_{\ell,n} \), let \( T(\pi) \) be the vector that record the numbers of the last \( n - k \) positions in \( \pi \), i.e., \( T(\pi) = (\pi(k + 1), \pi(k + 2), \ldots, \pi(n)) \). For example, if \( n = 12 \), \( k = 4 \), \( \pi = \bar{9} \ 5 \ 4 \ 3 \ 2 \ 6 \ 7 \ 10 \ \bar{12} \ \bar{1} \ \bar{11} \in G_{4,12} \), then \( T(\pi) = (3, \bar{8}, \ldots, \bar{11}) \). We define the relation \( \sim \) on \( G_{\ell,n} \) by

\[ \pi \sim \pi' \iff T(\pi) = T(\pi'), \]

it is easy to see this is an equivalence relation. Let us consider the map \( \delta : (\eta, \pi) \rightarrow \delta(\eta, \pi) \) from \( G_{\ell,m} \times G_{\ell,n} \) to \( G_{\ell,n} \), where \( G_{\ell,m} \) can be seen as a permutation group of colored set \( C_{\ell} \times \{|\pi|(1), |\pi|(2), \ldots, |\pi|(m)|\} \). Define the permutation \( \delta(\eta, \pi) \) such that \( \delta(\eta, \pi)(i) = \eta(i)(i \leq m) \), and \( \delta(\eta, \pi)(i) = \pi(i)(i > m) \). For example, if \( \pi = \bar{9} \ 5 \ 4 \ 1 \ 3 \ 8 \ 2 \ 6 \ 7 \ 10 \ \bar{12} \ 11 \in G_{4,12} \), and \( \eta = 5 \ 4 \ 1 \bar{9} \in G_{4,4} \), then

\[ \delta(\eta, \pi) = 5 \ 4 \ 1 \ 3 \ 8 \ 2 \ 6 \ 7 \ 10 \ \bar{12} \ 11. \]

So the equivalence class of \( \pi \in G_{\ell,n} \) is \( \{\delta(\eta, \pi)|\eta \in G_{\ell,m}\} \), it’s easy to see the cardinality of each equivalence class is \( \ell^m m! \), choosing the representative of the equivalence class \( \delta(\iota, \pi) \) such that

\[ \text{sign}(|\iota|(i)) = 1 \quad \text{and} \quad \iota(1) > \iota(2) \cdots > \iota(m). \]

Since the fix points of \( \pi \) and \( \delta(\iota, \pi) \) on \([m + 1, n] \) keep unaltered. By Theorem 5, we obtain that the number of equivalence class is \( g_{\ell,n}^m(\lambda) / \ell^m m! \), which yields the equation (29). \( \Box \)

**Remark 40.** As in the proof of Theorem 5, we can also prove the recurrence relations of (3) by constructing bijections directly, the proof is left to the interested reader.

6 Proof of Theorem 16

In this section, we give two proofs of Theorem 16. In the first proof, we give a bijection from \( C_{\ell,n}^m \) to \( L_{\ell,n}^m \), that is,

\[ \sum_{\pi \in L_{\ell,n}^m} \lambda_{\text{fix} > m}(\pi) = \sum_{\pi \in C_{\ell,n}^m} \lambda_{\text{fix} > m}(\pi) \text{ for } 0 \leq m \leq n. \tag{30} \]

In the second proof, we prove this cyclic result by constructing a equivalence relation on \( G_{\ell,n} \), that is,

\[ \sum_{\pi \in C_{\ell,n}^m} \lambda_{\text{fix} > m}(\pi) = \sum_{\pi \in G_{\ell,n}} \lambda_{\text{fix} > m}(\pi) / \ell^m m!, \text{ for } 0 \leq m \leq n. \tag{31} \]
6.1 First Proof

we will give a bijection \( \rho : \pi \to \pi' \) from \( C'_{\ell,n} \) to \( L'_{\ell,n} \) such that \( \text{fix}_{>m}(\pi) = \text{fix}_{>m}(\pi') \). First we give the map \( |\pi| \to |\pi'| \) and then construct the sign transformation.

- Let \( |\pi'| = |\pi'|(1), \ldots, |\pi'|(|\pi|), \ldots, |\pi'|(|\pi|) \), where \( |\pi'|(1), \ldots, |\pi'|(|\pi|) \) is decreasing rearrangement of \( |\pi|(1), \ldots, |\pi|(|\pi|) \) and \( |\pi'|(|\pi| + i) = |\pi|(|\pi| + i)(1 \leq i \leq n - m) \).

Conversely, we give the reverse map by \( |\pi'| \to |\pi| \). For \( \pi' \in L'_{\ell,n} \), we define \( P := \{|\pi'|(i) : i \in [m]\} \).

\[
(|\pi'|^{-s}(i), \ldots, |\pi'|^{-2}(i), |\pi'|^{-1}(i), i)
\]

be the cycle of \( |\pi| \) containing \( i(i \in [m]) \), where \( s \) is the least non-negative number such that \( |\pi'|^{-s}(i) \in P \) and if \( |\pi'|(j) = i(j \in [n]) \), then \( |\pi'|^{-1}(i) = j \). And setting \( |\pi'|^0(i) = i \), that is, if \( i \in P \cap [m] \), then \( s = 0 \), and \( i \) is a fixed point of \( |\pi| \). The other cycles keep in accordance with \( |\pi'| \).

- We define the sign transformation as follows. Since each element \( i \in [m] \) in \( \pi \) and \( \pi(i)(i \in [m]) \) in \( \pi' \) are uncolored, we exchange the sign of \( |\pi|(i) \in [m] \) in \( \pi \) and \( i \in [m] \) in \( \pi' \). In other words,

\[
\text{sign}_\pi(i) = \text{sign}_{\pi'}(|\pi|(i)) = 1 \text{ and } \text{sign}_{\pi}(i) = \text{sign}_{\pi'}(|\pi|(i)), \; i \in [m].
\]

The signs of other elements remain unchanged, i.e.,

\[
\text{sign}_{\pi'}(i) = \text{sign}_{\pi}(i), \quad i \notin [m] \cup \{|\pi|(i) : i \in [m]\}.
\]

For example: For \( \ell = 4, n = 12, m = 4 \), \( \pi = (\bar{1} \bar{9} \bar{7} \bar{2} \bar{5} \bar{3} \bar{4}) \bar{6} \bar{5} \bar{8} \bar{9} \bar{10} \bar{12}) \in C_{4,12}^4 \),

\[
\text{sign}_{\pi}(1) = \text{sign}_{\pi'}(|\pi|(1)) = \zeta^2, \text{sign}_{\pi'}(2) = \text{sign}_{\pi'}(|\pi|(2)) = \zeta,
\]

we have

\[
\pi' = 9 7 5 4 3 8 2 6 \bar{1} \bar{10} \bar{12} 11 \in L_{4,12}^4 \text{ and } \text{fix}_{>4}(\pi) = \text{fix}_{>4}(\pi') = 1.
\]

6.2 Second Proof

We decompose a permutation \( \pi \in G_{\ell,n} \) as a product of disjoint cycles. For each \( i \in [m] \), we define \( \omega_s(i) = \pi(i)\pi^2(i) \ldots \pi^{s-1}(i) \) where \( s \geq 1 \) is the least integer such that \( |\pi|^s(i) \in [m] \). Obviously \( \omega_s(i) = \emptyset \) if \( s = 1 \). Let \( \Omega(\pi) \) be the product of cycles of \( \pi \) which have no common elements with \( \{\zeta^j i : i \in [m], 0 \leq j \leq \ell - 1\} \), let \( \pi_m \in G_{\ell,m} \) be the permutation obtained from \( \pi \) by deleting elements in \( \omega_s(i) \) and the cycles in \( \Omega(\pi) \) for \( i \in [m] \).

For example, if \( \ell = 4, n = 12, m = 4 \) and \( \pi = (\bar{1} \bar{9} \bar{7} \bar{2} \bar{5} \bar{3} \bar{4}) \bar{6} \bar{8} \bar{10} \bar{12}) \), then \( \pi_4 = (1234) \) and

\[
\omega_1(1) = \bar{9} \bar{7}, \quad \omega_2(2) = 5, \quad \omega_3(3) = \emptyset, \quad \omega_4(4) = \emptyset, \quad \text{and } \Omega(\pi) = (6 \bar{8} \bar{10} \bar{12}).
\]
Setting $E(\pi) = (\omega_1(1), \omega_2(2), \cdots, \omega_n(k), \Omega(\pi))$, we define the relation $\sim$ on $G_{\ell,n}$ by

$$\pi_1 \sim \pi_2 \iff E(\pi_1) = E(\pi_2),$$

it is easy to see that this is an equivalence relation. Then we define the mapping $\theta : (\tau, \pi) \mapsto \theta(\tau, \pi)$ from $G_{\ell,m} \times G_{\ell,n}$ to $G_{\ell,n}$. We obtain the permutation $\theta(\tau, \pi)$ by inserting the elements $\omega_i(\pi)$ after the elements $\zeta_j(i \in [m], 0 \leq j \leq \ell - 1)$ of $\tau$ and adding the cycles of $\Omega(\pi)$.

For example, if $\pi = (1 \ 9 \ 7 \ 2 \ 5 \ 3 \ 4) \ (6 \ 8) \ (10) \ (11 \ \underline{12})$ and $\tau = (1 \ 2) \ (3) \ (4)$ then

$$\theta(\tau, \pi) = (1 \ 9 \ 7 \ 2 \ 5 \ 3 \ 4) \ (6 \ 8) \ (10) \ (11 \ \underline{12}).$$

Obviously $\{\theta(\tau, \pi) | \tau \in G_{\ell,m}\}$ is the equivalence class of $\pi \in G_{\ell,n}$. From the construction of $\theta(\tau, \pi)$, for $\tau \in G_{\ell,m}$ and $\pi \in G_{\ell,n}^m$, we have $\theta(\tau, \pi) \sim \pi$. Conversely, if $\pi' \sim \pi$, then $\pi' = \theta(\pi, \pi)$, and if $\theta(\tau, \pi) = \theta(\tau', \pi)$ = $\pi'$ for $\tau, \tau' \in G_{\ell,m}$, then $\tau = \tau' = \pi'_m$. Hence the cardinality of each equivalence class is $\ell^m m!$. Let $\eta$ be the identity permutation of $G_{\ell,m}$, then we choose $\theta(\eta, \pi)$ as the representative of each equivalence class $\{\theta(\tau, \pi) | \tau \in G_{\ell,m}\}$, that is, $\theta(\eta, \pi)$ represents the the permutation $\pi \in G_{\ell,n}$ where $\text{sign}_\pi(i) = 1 (i \in [m])$ with the first $m$ elements belong into distinct cycles. It is obvious to see $\text{fix}_{>m}(\pi) = \text{fix}_{>m}(\theta(\eta, \pi))$. By Theorem 5, the number of equivalence classes is $g_{\ell,m}^m(\lambda)/\ell^m m!$, as desired.

### 7 Generating functions and further recurrence relations

In this section, by using the recurrence relation (2), we obtain the generating functions and further recurrence relations of $g_{\ell,n}^m(\lambda)$ and $d_{\ell,n}^m(\lambda)$.

**Proposition 41.** For $m \geq 0$ we have the following identities:

\[
g_{\ell,n}^m(\lambda) = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right) \ell^{m+i}(m+i)!, \tag{32}
\]

\[
\sum_{n \geq 0} g_{\ell,n}^m(\lambda) \frac{u^n}{n!} = \frac{\ell^m m! \exp((\lambda - 1)u)}{(1 - \ell u)^{m+1}}, \tag{33}
\]

\[
\sum_{m,n \geq 0} g_{\ell,n}^m(\lambda) \frac{u^m}{m!} \frac{w^n}{n!} = \exp((\lambda - 1)u) \frac{1}{1 - \ell x - \ell u}. \tag{34}
\]

**Proof.** For any function $f(k)(k \geq 0) : \mathbb{Z}[\lambda] \to \mathbb{C}[\lambda]$, we define the operator $\Delta f(n)(\lambda) = f(n)(\lambda) + (\lambda - 1)f(n-1)(\lambda)$. By inductions on $N \geq 0$, we have

\[
\Delta^N f(n)(\lambda) = \sum_{i=0}^{N} (\lambda - 1)^i \left(\begin{array}{c} N \\ i \end{array}\right) f(n-i)(\lambda) = \sum_{i=0}^{N} (\lambda - 1)^{N-i} \left(\begin{array}{c} N \\ i \end{array}\right) f(n-N+i)(\lambda). \tag{35}
\]
If $f(n)(\lambda) = g_{\ell,n}^n(\lambda)$, thus $g_{\ell,n+m-i}^{n+m-i}(\lambda) = \Delta^i f(n+m)(\lambda)$ for $i \geq 0$. From (35), we obtain

$$g_{\ell,n+m}(\lambda) = \Delta^n f(n+m)(\lambda) = \sum_{i=0}^{n} (\lambda-1)^{n-i} \binom{n}{i} \ell^{m+i}(m+i)!. \quad (36)$$

For the above identity, multiplying both sides by $u^n/n!$ and summing over $n \geq 0$, we obtain

$$\sum_{n \geq 0} g_{\ell,n+m}(\lambda) \frac{u^n}{n!} = \ell^m m! \sum_{n,i \geq 0} (\lambda-1)^{n-i} \binom{m+i}{i} \frac{\ell^i u^n}{(n-i)!}.$$ 

By shifting $n$ to $n + i$, we have

$$\sum_{n \geq 0} g_{\ell,n+m}(\lambda) \frac{u^n}{n!} = \ell^m m! \left( \sum_{n \geq 0} (\lambda-1)^{n} \frac{u^n}{n!} \right) \cdot \left( \sum_{i \geq 0} \binom{m+i}{i} (\ell u)^i \right).$$

Clearly the above equation implies (33) immediately. Finally multiplying both sides of (33) by $x^m/m!$ and summing over $m \geq 0$ yields (34).

\[ \square \]

Remark 42. Setting $m = 0$ in (32), we obtain

$$d_{\ell,n}^{0}(\lambda) = g_{\ell,n}^{0}(\lambda) = n! \sum_{i=0}^{n} \frac{(\lambda-1)^i}{i!} \ell^{n-i}, \quad (37)$$

which implies immediately the following recurrence relation,

$$d_{\ell,n}^{0}(\lambda) = \ell n d_{\ell,n-1}^{0}(\lambda) + (\lambda-1)^n \quad (n \geq 1). \quad (38)$$

Proposition 43. For $\ell \geq 1$ and $0 \leq m \leq n-2$ we have

$$g_{\ell,n}^{m}(\lambda) = (\ell n + \lambda - 1) g_{\ell,n-1}^{m-1}(\lambda) - \ell (n - m - 1) (\lambda - 1) g_{\ell,n-2}^{m-1}(\lambda) \quad (n \geq 2); \quad (39)$$

$$g_{\ell,n}^{m}(\lambda) = \ell (n - m) g_{\ell,n-1}^{m-1}(\lambda) + \ell m g_{\ell,n-1}^{m-1}(\lambda) \quad (m \geq 1, n \geq 1); \quad (40)$$

$$g_{\ell,n}^{m}(\lambda) = \ell n g_{\ell,n-1}^{m-1}(\lambda) + \ell m (\lambda - 1) g_{\ell,n-2}^{m-1}(\lambda) \quad (m \geq 1, n \geq 2), \quad (41)$$

where $g_{\ell,0}^{0}(\lambda) = 1$, $g_{\ell,1}^{0}(\lambda) = \lambda + \ell - 1$ and $g_{\ell,1}^{0}(\lambda) = \ell$.

Proof. Let $F(u)$ denote the left-hand side of (33). By using the differentiation of $F(u)$ and (33), we obtain

$$(1 - \ell u) F'(u) = [\ell (m+1) + (\lambda-1)(1-\ell u)] F(u). \quad (42)$$

By equating the coefficients of $u^n/n!$ in (42), we have

$$g_{\ell,n+m+1}(\lambda) = [\ell (m+n+1) + \lambda - 1] g_{\ell,n+m}(\lambda) - \ell n (\lambda - 1) g_{\ell,n+m-1}(\lambda),$$

shifting $n + m + 1$ to $n$ yields (39) immediately.
Then multiplying both sides of (33) by $1 - \ell u$, we have

$$(1 - \ell u) \sum_{n \geq 0} g_{\ell,n+m}^m(\lambda) \frac{u^n}{n!} = \ell^m m! \exp((\lambda - 1)u) = \ell m \sum_{n \geq 0} g_{\ell,n+1,m-1}^m(\lambda) \frac{u^n}{n!}. \quad (43)$$

By equating the coefficients of $u^n/n!$, we have

$$g_{\ell,n+m}^m(\lambda) - \ell n g_{\ell,n+m-1}^m(\lambda) = \ell m g_{\ell,n-1}^{m-1}(\lambda), \quad (44)$$

shifting $n + m$ to $n$ yields (40).

Finally, from (40) and (2), we have

$$g_{\ell,n}^m(\lambda) = \ell n g_{\ell,n-1}^{m-1}(\lambda) - \ell m (g_{\ell,n-1}^m(\lambda) - g_{\ell,n-1}^{m-1}(\lambda)) = \ell n g_{\ell,n-1}^{m-1}(\lambda) + \ell m (\lambda - 1) g_{\ell,n-2}^{m-1}(\lambda),$$

which yields (41), the proof is completed.

With the above Proposition 43, we derive the following propositions immediately.

**Proposition 44.** For $\ell \geq 1$ and $0 \leq m \leq n - 2$ we have

$$d_{\ell,n}^m(\lambda) = (\ell n + \lambda - 1)d_{\ell,n-1}^m(\lambda) - \ell (n - m - 1)(\lambda - 1)d_{\ell,n-2}^m(\lambda) \quad (n \geq 2); \quad (45)$$

$$d_{\ell,n}^m(\lambda) = \ell (n - m)d_{\ell,n-1}^m(\lambda) + d_{\ell,n-1}^{m-1}(\lambda) \quad (m \geq 1, n \geq 1); \quad (46)$$

$$d_{\ell,n}^m(\lambda) - (\lambda - 1)d_{\ell,n-1}^{m-1}(\lambda) = \ell n d_{\ell,n-2}^m(\lambda) \quad (m \geq 1, n \geq 2), \quad (47)$$

where $d_{\ell,0}^0(\lambda) = 1$, $d_{\ell,1}^0(\lambda) = \lambda + \ell - 1$ and $d_{\ell,1}^1(\lambda) = 1$.

**Proof.** With Proposition 43, we can get these equations (45), (46) and (47) directly.

**Remark 45.** Setting $\ell = 1$, (3), (45), and (47) reduce to the result of Eriksen et al. [5, Propositions 8.1, 8.3 and 8.2]. In this case, (33) and (34) recover the result of Rakotondrajaoo [11, Theorem 6.7 and Theorem 6.8]. Setting $\lambda = 0$, Propositions 41, 43 and 44 reduce to the result of Faliharimalala and Zeng [7, Propositions 17, 18, and 19].

### 8 Proof of Theorem 20

In this section, to prove Theorem 20, we prove the following equation,

$$\sum_{\pi \in F_{\ell,n}^{\fix,>m}(\pi)} = \frac{(n + m)!}{n!} \sum_{\pi \in C_{\ell,n}^{\fix,>m}(\pi)} \lambda^{\fix,>m}(\pi) \quad \text{for } m, n \geq 0. \quad (48)$$

And with the generating functions of $d_{\ell,n}^m(\lambda)$, we obtain the generating functions and recurrence relations of $f_{\ell,n}^m(\lambda)$.
With Theorem 20, we construct such a permutation \( \pi \in F^m_{\ell,n+m} \) in following way, see Definition 18.

Starting from the set \([n+2m]\), we take \(m\) elements from the set \([m+1, 2m]\) as the image of \([1, m]\), which is labeled as \(\pi(i')(i \in [m])\). Clearly there are \((n+m)!m!\) ways to choose. Let \(i'(i \in [m])\) represent the two element set \(\{i, \pi(i)\}\), and let \(i'(i \in [m+1, n+m])\) denote the remaining element \([n+2m] \setminus \{i, \pi(i)\}\). Let \(\pi'\) denote the permutation on the colored set \(C_\ell \times \{1', 2', 3', \ldots, (m+n)\}\) such that \(\text{sign}_{\pi'}(i') = 1\) \((i' \in [m])\) and \(i'(i' \in [m])\) belong into distinct cycles, by transforming the \(i'\) into \(\{i, \pi(i)\}\), we obtain the desired permutation in \(F^m_{\ell,n+m}\) and vice versa. From this construction, we have \(\text{fix}_{\geq m}(\pi) = \text{fix}_{\geq m}(\pi')\). This completes the proof. \(\square\)

**Theorem 46.** For \(\ell \geq 1\) and \(0 \leq m \leq n\), we have

\[
\sum_{n \geq 0} f^m_{\ell,n}(\lambda) \frac{u^n}{n!} = \frac{u^m \exp(\lambda - 1)u}{(1 - \ell u)^{m+1}}. \tag{49}
\]

**Proof.** According to the generating function (33) of \(g^m_{\ell,n}(\lambda)\), it is clear to see that

\[
\sum_{n \geq 0} d^m_{\ell,n+m}(\lambda) \frac{u^n}{n!} = \frac{\exp(\lambda - 1)u}{(1 - \ell u)^{m+1}}.
\]

For the above identity, multiplying both sides by \(u^m\), we obtain

\[
\sum_{n \geq 0} \frac{(n+m)!}{n!} d^m_{\ell,n+m}(\lambda) \frac{u^{n+m}}{(n+m)!} = \frac{u^m \exp(\lambda - 1)u}{(1 - \ell u)^{m+1}}.
\]

With Theorem 20,

\[
\sum_{n \geq 0} f^m_{\ell,n+m} \frac{u^{n+m}}{(n+m)!} = \frac{u^m \exp(\lambda - 1)u}{(1 - \ell u)^{m+1}},
\]

which is (49) by shifting \(n+m\) to \(n\). \(\square\)

With (3), (45), (46), (47) and Theorem 20, we obtain the following corollary.

**Corollary 47.** For \(\ell \geq 1, 1 \leq m \leq n-2\), we have

\[
(n-m+1)f^m_{\ell,n-1}(\lambda) = \ell mf^m_{\ell,n}(\lambda) + (\lambda - 1)n f^m_{\ell,n-1}(\lambda); \tag{50}
\]

\[
(n-m)f^m_{\ell,n}(\lambda) = n((n-1+\lambda)f^m_{\ell,n-1}(\lambda) - \ell(\lambda - 1)n(n-1)f^m_{\ell,n-2}(\lambda); \tag{51}
\]

\[
f^m_{\ell,n}(\lambda) = \ell nf^m_{\ell,n-1}(\lambda) + nf^m_{\ell,n-1}(\lambda); \tag{52}
\]

\[
(n-m)f^m_{\ell,n}(\lambda) = (\lambda - 1)n(n-1)f^m_{\ell,n-2}(\lambda) + \ell n^2 f^m_{\ell,n-1}(\lambda); \tag{53}
\]

where \(f^0_{\ell,0}(\lambda) = 1, f^0_{\ell,1}(\lambda) = \lambda + \ell - 1\) and \(f^1_{\ell,1}(\lambda) = 1\).

With (51) and (52), we have the following corollary.

**Corollary 48.** For \(\ell \geq 1\) and \(1 \leq m \leq n-2\), we have

\[
f^m_{\ell,n}(\lambda) = mf^m_{\ell,n-1}(\lambda) - \ell(\lambda - 1)(n-1)f^m_{\ell,n-2}(\lambda) + (\ell m + \ell n - 1 + \lambda)f^m_{\ell,n-1}(\lambda). \tag{54}
\]

**Remark 49.** When \((\ell, \lambda) = (1, 0)\), (49) and (54) reduce to the results of [14, Theorem 3 and Theorem 2].
9 Combinatorial proof of recurrence relation (46)

In this section, we give the combinatorial proof of recurrence (46), that is,

\[
\sum_{\pi \in C_{m}^{\ell,n}} \lambda_{\text{fix}>m}(\pi) = \ell(n-m) \sum_{\pi \in C_{m-1}^{\ell,n}} \lambda_{\text{fix}>m}(\pi) + \sum_{\pi \in C_{m-1}^{\ell,n}} \lambda_{\text{fix}>m-1}(\pi),
\]

(55)

other recurrences (45) and (47) can be proved in similar ways.

Lemma 50. For \(0 \leq m \leq n\),

\[
\sum_{\pi \in C_{m}^{\ell,n}} \lambda_{\text{fix}>m}(\pi) = \sum_{\pi \in C_{m-1}^{\ell,n}} \lambda_{\text{fix}>m-1}(\pi).
\]

(56)

Proof. It follows similar arguments as in the proof of Lemma 21.

Lemma 51. For \(0 \leq m \leq n\),

\[
\sum_{\pi \in C_{m}^{\ell,n}} \lambda_{\text{fix}>m}(\pi) = \ell(n-m) \sum_{\pi \in C_{m-1}^{\ell,n}} \lambda_{\text{fix}>m}(\pi).
\]

(57)

Proof. Let us consider the map \(\chi : \pi \to (\epsilon, \beta, \pi')\) from \(C_{m}^{\ell,n} \cap \{\pi \in G_{\ell,n} | m \notin \text{FIX}(\pi)\}\) to \(C_{\ell} \times [n-m] \times C_{\ell,n-1}^{m}\) such that \(\text{fix}_{>m}(\pi) = \text{fix}_{>m}(\pi')\).

For \(\pi \in C_{m}^{\ell,n} \cap \{\pi \in G_{\ell,n} | m \notin \text{FIX}(\pi)\}\), we decompose \(\pi\) as the product of disjoint cycles. Let \(\pi(m) = \beta\), it is easy to see \(|\beta| \in [m+1, n]\) and \(\text{sign}_\pi(\beta) = \epsilon\).

For the element \(i \in \pi\), we delete the element \(\beta\) and define the element \(i' \in [n-1]\) in \(\pi'\) by

\[
i' = \begin{cases} i, & \text{if } |i| < |\beta|; \\ i - 1, & \text{if } |i| > |\beta|. \end{cases}
\]

Conversely, starting from \((\epsilon, \beta, \pi') \in C_{\ell} \times [n-k] \times C_{\ell,n-1}^{m}\), for the element \(i' \in \pi'\), we define the element \(i \in [n]\) in \(\pi\) by

\[
i = \begin{cases} i', & \text{if } |i'| < |\beta|; \\ i' + 1, & \text{if } |i'| \geq |\beta|. \end{cases}
\]

and let \(\pi(m) = (\epsilon, \beta)\).

For example, let \(\ell = 4, n = 9, k = 4\), if \(\pi = (1 \ 7) (2 \ 5) (3 \ 8) (4 \ 9) (6) \in C_{4,9}^{4}\),

\[\epsilon = \zeta, \beta = 9, \quad \pi' = (1 \ 7) (2 \ 5) (3 \ 8) (4 \ 6) \in C_{4,8}^{4}, \quad \text{and} \quad \text{fix}_{>4}(\pi) = \text{fix}_{>4}(\pi').\]

for \(\pi = (1 \ 7) (2) (3 \ 8) (4 \ 5) (6) (9) \in C_{4,9}^{4}\),

\[\epsilon = \zeta^2, \beta = 5, \quad \pi' = (1 \ 6) (2) (3 \ 7) (4 \ 5) (8) \in C_{4,8}^{4}, \quad \text{and} \quad \text{fix}_{>4}(\pi) = \text{fix}_{>4}(\pi').\]
Proof of equation (55). By considering the following equation,
\[
\sum_{\pi \in C_{m}^{\ell, n}} \lambda_{\text{fix}>m}(\pi) = \sum_{\pi \in C_{m}^{\ell, n}} \lambda_{\text{fix}>m}(\pi) + \sum_{\pi \in C_{m}^{\ell, n}} \lambda_{\text{fix}>m}(\pi),
\]
by Lemma 50, the (58) is equivalent to
\[
\sum_{\pi \in C_{m}^{\ell, n}} \lambda_{\text{fix}>m}(\pi) = \sum_{\pi \in C_{m}^{\ell, n}} \lambda_{\text{fix}>m}(\pi) + \sum_{\pi \in C_{m-1}^{\ell, n-1}} \lambda_{\text{fix}>m-1}(\pi).
\]
By Lemma 51, we obtain (55) immediately. This completes the proof.

10 Final remarks

Faliharimalala and Zeng [8, eq. (1.2)] studied the wreath product analogue of Euler’s \(q\)-difference table \(\{g_{\ell,n}^{m}(q)\}_{0 \leq m \leq n}\) as follows.

**Definition 52** (Faliharimalala-Zeng). For fixed integer \(\ell \geq 1\), the coefficients of Euler’s \(q\)-difference table \((g_{\ell,n}^{m}(q))_{0 \leq m \leq n}\) for \(C_{\ell} \wr S_{n}\) are defined by
\[
\begin{aligned}
g_{n}^{n}(q) &= [\ell]_{q} [2\ell]_{q} \cdots [n\ell]_{q}, \\
g_{m}^{m}(q) &= g_{m+1}^{m}(q) - q^{(n-m-1)} g_{m-1}^{m}(q) \quad (0 \leq m \leq n-1). 
\end{aligned}
\]

Faliharimalala and Zeng found a combinatorial interpretation of \((g_{\ell,n}^{m}(q))_{0 \leq m \leq n}\) by introducing a new Mahonian statistic \(fmaf\) on the wreath products. So the natural question is to find a \(q\)-\(\lambda\)-Euler’s difference table for \(\lambda\)-Euler’s difference table in Definition 1, it seems the statistic \(fmaf\) cannot help directly.

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References


