Inverse Perron values and connectivity of a uniform hypergraph

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Abstract
In this paper, we show that a uniform hypergraph \( \mathcal{G} \) is connected if and only if one of its inverse Perron values is larger than 0. We give some bounds on the bipartition width, isoperimetric number and eccentricities of \( \mathcal{G} \) in terms of inverse Perron values. By using the inverse Perron values, we give an estimation of the edge connectivity of a 2-design, and determine the explicit edge connectivity of a symmetric design. Moreover, relations between the inverse Perron values and resistance distance of a connected graph are presented.

Mathematics Subject Classifications: 05C50, 05C65, 05C40, 05C12, 15A69

1 Introduction

Let \( V(\mathcal{G}) \) and \( E(\mathcal{G}) \) denote the vertex set and edge set of a hypergraph \( \mathcal{G} \), respectively. \( \mathcal{G} \) is \( k \)-uniform if \( |e| = k \) for each \( e \in E(\mathcal{G}) \). In particular, 2-uniform hypergraphs are usual graphs. For \( i \in V(\mathcal{G}) \), \( E_i(\mathcal{G}) \) denotes the set of edges containing \( i \), and \( d_i = |E_i(\mathcal{G})| \)

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denotes the degree of \( i \). The adjacency tensor \([8]\) of a \( k \)-uniform hypergraph \( \mathcal{G} \), denoted by \( \mathcal{A}_\mathcal{G} \), is an order \( k \) dimension \(|V(\mathcal{G})|\) tensor with entries
\[
a_{i_1i_2\cdots i_k} = \begin{cases} 1/(k-1)! & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}
\]

The Laplacian tensor \([27]\) of \( \mathcal{G} \) is \( \mathcal{L}_\mathcal{G} = \mathcal{D}_\mathcal{G} - \mathcal{A}_\mathcal{G} \), where \( \mathcal{D}_\mathcal{G} \) is the diagonal tensor of vertex degrees of \( \mathcal{G} \). Recently, the research on spectral hypergraph theory via tensors has attracted much attention \([7-10, 14, 19, 24]\). The spectral properties of the Laplacian tensor of hypergraphs are studied in \([13, 25, 27, 29, 35]\).

For an order \( k \) dimension \( n \) tensor \( \mathcal{T} = (t_{i_1i_2\cdots i_k}) \), let \( \mathcal{T}\mathbf{x}^k = \sum_{i_1,\ldots,i_k=1}^n t_{i_1i_2\cdots i_k}x_{i_1}\cdots x_{i_k} \).

The algebraic connectivity of a graph plays important roles in spectral graph theory \([11]\). Analogue to the algebraic connectivity of a graph, Qi \([27]\) defined the analytic connectivity \( \alpha(\mathcal{G}) \) of a \( k \)-uniform hypergraph \( \mathcal{G} \) as
\[
\alpha(\mathcal{G}) = \min_{j=1,\ldots,n} \min \left\{ \mathcal{L}_\mathcal{G}\mathbf{x}^k : \mathbf{x} \in \mathbb{R}_+^n, \sum_{i=1}^n x_i = 1, x_j = 0 \right\},
\]
where \( n = |V(\mathcal{G})| \), \( \mathbb{R}_+^n \) denotes the set of nonnegative vectors of dimension \( n \). Qi proved that \( \mathcal{G} \) is connected if and only if \( \alpha(\mathcal{G}) > 0 \). In \([20]\), some bounds on \( \alpha(\mathcal{G}) \) were presented in terms of degree, vertex connectivity, diameter and isoperimetric number. A feasible trust region algorithm of \( \alpha(\mathcal{G}) \) was given in \([9]\).

For any vertex \( j \) of a \( k \)-uniform hypergraph \( \mathcal{G} \), we define the inverse Perron value of \( j \) as
\[
\alpha_j(\mathcal{G}) = \min \left\{ \mathcal{L}_\mathcal{G}\mathbf{x}^k : \mathbf{x} \in \mathbb{R}_+^n, \sum_{i=1}^n x_i = 1, x_j = 0 \right\}.
\]
Clearly, the analytic connectivity \( \alpha(\mathcal{G}) = \min_{j \in V(\mathcal{G})} \alpha_j(\mathcal{G}) \) is the minimum inverse Perron value. For a connected graph \( \mathcal{G} \), \( \alpha_j(\mathcal{G}) \) is the minimum eigenvalue of \( \mathcal{L}_\mathcal{G}(j) \), where \( \mathcal{L}_\mathcal{G}(j) \) is the principal submatrix of \( \mathcal{L}_\mathcal{G} \) obtained by deleting the row and column corresponding to \( j \). \( \mathcal{L}_\mathcal{G}(j) \) is nonsingular and its inverse \( \mathcal{L}_\mathcal{G}(j)^{-1} \) is a nonnegative matrix \([16]\). It is easy to see that \( \alpha_j^{-1}(\mathcal{G}) \) is the spectral radius of \( \mathcal{L}_\mathcal{G}(j)^{-1} \), which is called the Perron value of \( \mathcal{G} \). All inverse Perron values of a tree \( T \) can determine the algebraic connectivity of \( T \) \([1, 15]\).

The resistance distance \([17, 34]\) is a distance function on graphs. For two vertices \( i, j \) in a connected graph \( \mathcal{G} \), the resistance distance between \( i \) and \( j \), denoted by \( r_{ij}(\mathcal{G}) \), is defined to be the effective resistance between them when unit resistors are placed on every edge of \( \mathcal{G} \). The Kirchhoff index \([17, 33]\) of \( \mathcal{G} \), denoted by \( Kf(\mathcal{G}) \), is defined as the sum of resistance distances between all pairs of vertices in \( \mathcal{G} \), i.e., \( Kf(\mathcal{G}) = \sum_{\{i,j\} \subseteq V(\mathcal{G})} r_{ij}(\mathcal{G}) \). \( Kf(\mathcal{G}) \) is a global robustness index. The resistance distance and Kirchhoff index in graphs have been investigated extensively in mathematical and chemical literatures \([3, 4, 6, 12, 23, 31, 36]\).
This paper is organized as follows. In Section 2, some auxiliary lemmas are introduced. In Section 3, we show that a uniform hypergraph $G$ is connected if and only if one of its inverse Perron values is larger than 0, and some inequalities among the inverse Perron values, bipartition width, isoperimetric number and eccentricities of $G$ are established. Partial results improve some bounds in [20, 27]. We also use the inverse Perron values to estimate the edge connectivity of 2-designs. In Section 4, some inequalities among the inverse Perron values, resistance distance and Kirchhoff index of a connected graph are presented.

2 Preliminaries

For a positive integer $n$, let $[n] = \{1, 2, \ldots, n\}$. An order $m$ dimension $n$ tensor $T = (t_{ij})$ consists of $n^m$ entries, where $i, j \in [n]$. When $m = 2$, $T$ is an $n \times n$ matrix. Let $\mathbb{R}^{[m,n]}$ denote the set of order $m$ dimension $n$ real tensors, and let $\mathbb{R}^n_+$ denote the cone of nonnegative vectors in $\mathbb{R}^n$. For $T = (t_{ij}) \in \mathbb{R}^{[m,n]}$ and $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, let $Tx^{m-1} \in \mathbb{R}^n$ denote the vector whose $i$-th component is

$$(Tx^{m-1})_i = \sum_{i_2, i_3, \ldots, i_m = 1}^n t_{i_2 \ldots i_m} x_{i_2} x_{i_3} \cdots x_{i_m},$$

and let $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$. In 2005, Qi [26] and Lim [21] proposed the concept of eigenvalues of tensors, independently. For $T = (t_{ij}) \in \mathbb{R}^{[m,n]}$, if there exist a number $\lambda \in \mathbb{R}$ and a nonzero vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ such that $Tx^{m-1} = \lambda x^{[m-1]}$, then $\lambda$ is called an $H$-eigenvalue of $T$, $x$ is called an $H$-eigenvector of $T$ corresponding to $\lambda$.

For a vertex $j$ of a $k$-uniform hypergraph $G$, let $L_G(j) \in \mathbb{R}^{[k,n-1]}$ denote the principal subtensor of $L_G \in \mathbb{R}^{[k,n]}$ with index set $V(G) \setminus \{j\}$. By Lemma 2.3 in [32], we know that $\alpha_j$ is the smallest $H$-eigenvalue of $L_G(j)$ for any $j \in V(G)$.

A path $P$ of a uniform hypergraph $G$ is an alternating sequence of vertices and edges $v_0e_1v_1e_2 \cdots v_{l-1}e_lv_l$, where $v_0, \ldots, v_l$ are distinct vertices of $G$, $e_1, \ldots, e_l$ are distinct edges of $G$ and $v_{i-1}, v_i \in e_i$, for $i = 1, \ldots, l$. The number of edges in $P$ is the length of $P$. For all $u, v \in V(G)$, if there exists a path starting at $u$ and terminating at $v$, then $G$ is said to be connected [5].

**Lemma 1.** [27] The uniform hypergraph $G$ is connected if and only if $\alpha(G) > 0$.

**Lemma 2.** [27] Let $G$ be a $k$-uniform hypergraph with $n$ vertices. Then

$$e(G) \geq \frac{n}{k} \alpha(G).$$
The $\{1\}$-inverse of a matrix $M$ is a matrix $X$ such that $MXM = M$. Let $M^{(1)}$ denote any $\{1\}$-inverse of $M$, and let $(M)_{ij}$ denote the $(i, j)$-entry of $M$.

**Lemma 3.** [2, 34] Let $G$ be a connected graph. Then

$$r_{ij}(G) = (\mathcal{L}_G^{(1)})_{ii} + (\mathcal{L}_G^{(1)})_{jj} - (\mathcal{L}_G^{(1)})_{ij} - (\mathcal{L}_G^{(1)})_{ji}.$$  

Let $\text{tr}(A)$ denote the trace of the square matrix $A$, and let $e$ denote an all-ones column vector.

**Lemma 4.** [30] Let $G$ be a connected graph of order $n$. Then

$$Kf(G) = n\text{tr}(\mathcal{L}_G^{(1)}) - e^T\mathcal{L}_G^{(1)} e.$$  

**Lemma 5.** [2] Let $G$ be a connected graph with $n$ vertices and $i \in [n]$. Let

$$\mathcal{L}_G = \begin{pmatrix} L_1 & x & L_2 \\ x^T & d_i & y \\ L_2^T & y^T & L_3 \end{pmatrix},$$

where $L_1 \in \mathbb{R}^{(i-1) \times (i-1)}$, $L_3 \in \mathbb{R}^{(n-i) \times (n-i)}$, $x \in \mathbb{R}^{i-1}$, $y^T \in \mathbb{R}^{n-i}$.

Suppose $\mathcal{L}_G^{(i)} = \begin{pmatrix} \widetilde{L}_1 & \widetilde{L}_2 \\ \widetilde{L}_2^T & \widetilde{L}_3 \end{pmatrix}$, where $\widetilde{L}_1 \in \mathbb{R}^{(i-1) \times (i-1)}$, $\widetilde{L}_3 \in \mathbb{R}^{(n-i) \times (n-i)}$. Then

$$\begin{pmatrix} \widetilde{L}_1 & 0 & \widetilde{L}_2 \\ 0 & 0 & 0 \\ \widetilde{L}_2^T & 0 & \widetilde{L}_3 \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of $\mathcal{L}_G$.

### 3 Inverse Perron values of uniform hypergraphs

In the following theorem, the relationship between inverse Perron values and connectivity of a hypergraph is presented.

**Theorem 6.** Let $G$ be a $k$-uniform hypergraph. Then the following statements are equivalent:

1. $G$ is connected.
2. $\alpha_j(G) > 0$ for all $j \in V(G)$.
3. $\alpha_j(G) > 0$ for some $j \in V(G)$.

**Proof.** (1)$\implies$(2). If $G$ is connected, then by Lemma 1, we know that $\alpha_j(G) > 0$ for all $j \in V(G)$.

(2)$\implies$(3). Obviously.

(3)$\implies$(1). Suppose that $G$ is disconnected. For any $j \in V(G)$, let $G_1$ be the component of $G$ such that $j \notin V(G_1)$. Let $x = (x_1, \ldots, x_{|V(G)|})^T$ be the vector satisfying

$$x_i = \begin{cases} |V(G_1)|^{-\frac{1}{2}}, & \text{if } i \in V(G_1), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, we have $\sum_{i=1}^{n} x_i^k = 1$. Then we have $0 \leq \alpha_j(G) \leq \mathcal{L}_G x^k = 0$ for any $j \in V(G)$, a contradiction to (3). Hence $G$ is connected if (3) holds.  

\[\Box\]
The bipartition width of a hypergraph \( \mathcal{G} \) is defined as \([18, 28]\)
\[
\text{bw}(\mathcal{G}) = \min \left\{ |E(S, \overline{S})| : S \subseteq V(\mathcal{G}), |S| = \left\lfloor \frac{n}{2} \right\rfloor \right\},
\]
where \( \left\lfloor \frac{n}{2} \right\rfloor \) denotes the maximum integer not larger than \( \frac{n}{2} \). The computation of \( \text{bw}(\mathcal{G}) \) is very difficult even for the graph case. In [22], Mohar and Poljak used the algebraic connectivity to obtain a lower bound on the bipartition width of a graph. In the following theorem, we use the inverse Perron values to obtain a lower bound on the bipartition width of a uniform hypergraph.

**Theorem 7.** Let \( \mathcal{G} \) be a \( k \)-uniform hypergraph with \( n \) vertices. Then
\[
\text{bw}(\mathcal{G}) \geq \frac{n + (-1)^n}{k(n + 1)} \sum_{j=1}^{n} \alpha_j(\mathcal{G}).
\]

**Proof.** Suppose that \( S_0 \subseteq V(\mathcal{G}) \) satisfying \( |S_0| = \left\lfloor \frac{n}{2} \right\rfloor \) and \( |E(S_0, \overline{S_0})| = \text{bw}(\mathcal{G}) \). Let \( x = (x_1, \ldots, x_n)^{T} \) be the vector satisfying
\[
 x_i = \begin{cases} 
 |S_0|^{-\frac{1}{k}}, & i \in S_0, \\
 0, & i \in \overline{S_0}. 
\end{cases}
\]

Then \( \sum_{i=1}^{n} x_i^k = 1 \). For \( j \in \overline{S_0} \), we get
\[
\alpha_j(\mathcal{G}) \leq \mathcal{L}_o x^k = \sum_{\{i_1, \ldots, i_k\} \in E(\mathcal{G})} (x_{i_1}^k + \cdots + x_{i_k}^k - k x_{i_1} \cdots x_{i_k})
\]
and for \( j \in S_0 \),
\[
\alpha_j(\mathcal{G}) \leq \sum_{\{i_1, \ldots, i_k\} \in E(S_0, \overline{S_0})} (x_{i_1}^k + \cdots + x_{i_k}^k - k x_{i_1} \cdots x_{i_k})
\]
\[
= \frac{1}{|S_0|} \sum_{e \in E(S_0, \overline{S_0})} |e \cap S_0| = \frac{t(S_0)\text{bw}(\mathcal{G})}{|S_0|},
\]
where \( t(S_0) = \frac{1}{|E(S_0, \overline{S_0})|} \sum_{e \in E(S_0, \overline{S_0})} |e \cap S_0| \).

Similarly, for \( j \in S_0 \), we obtain
\[
\alpha_j(\mathcal{G}) \leq \frac{(k - t(S_0))\text{bw}(\mathcal{G})}{|S_0|}.
\]
Combining (1) and (2), we get
\[
\sum_{j=1}^{n} \alpha_j(\mathcal{G}) = \sum_{j \in S_0} \alpha_j(\mathcal{G}) + \sum_{j \in \overline{S_0}} \alpha_j(\mathcal{G}) \leq \frac{|S_0|(k - t(S_0))\text{bw}(\mathcal{G})}{|S_0|} + \frac{|\overline{S_0}|t(S_0)\text{bw}(\mathcal{G})}{|S_0|}.
\]
If \( n \) is even, then \(|S_0| = |\overline{S_0}|\) and \( bw(G) \geq \frac{1}{k} \sum_{j=1}^{n} \alpha_j(G)\). If \( n \) is odd, then \(|S_0| = |\overline{S_0}| - 1 = \frac{n-1}{2}\) and
\[
\sum_{j=1}^{n} \alpha_j(G) \leq k|\overline{S_0}|bw(G) = k(n+1)bw(G) \quad \text{and} \quad bw(G) \geq \frac{n-1}{k(n+1)} \sum_{j=1}^{n} \alpha_j(G).
\]

The isoperimetric number of a \( k \)-uniform hypergraph \( G \) is defined as
\[
i(G) = \min\left\{ \frac{|E(S, \overline{S})|}{|S|} : S \subseteq V(G), 0 < |S| \leq \frac{|V(G)|}{2} \right\}.
\]

Let \( \beta(G) = \max_{j \in V(G)} \alpha_j(G) \) denote the maximum inverse Perron value of \( G \). In [20], it was shown that \( i(G) \geq \frac{2}{k} \alpha(G) \). We improve it as follows.

**Theorem 8.** Let \( G \) be a \( k \)-uniform hypergraph. Then
\[
i(G) \geq \frac{\alpha(G) + \beta(G)}{k}.
\]

**Proof.** Suppose that \( S_1 \subseteq V(G) \) satisfying \( 0 \leq |S_1| \leq \frac{|V(G)|}{2} \) and \( \frac{|E(S_1, \overline{S_1})|}{|S_1|} = i(G) \). Let \( x = (x_1, \ldots, x_n)^T \) be the vector satisfying
\[
x_i = \begin{cases} |S_1|^{-1/2}, & i \in S_1, \\ 0, & i \in \overline{S_1}. \end{cases}
\]

Then \( \sum_{i=1}^{n} x_i^k = 1 \). For \( j \in S_1 \), we obtain
\[
\alpha_j(G) \leq L_0 x^k = \frac{t(S_1)|E(S_1, \overline{S_1})|}{|S_1|} = t(S_1)i(G),
\]
where \( t(S_1) = \frac{1}{|E(S_1, \overline{S_1})|} \sum_{e \in E(S_1, \overline{S_1})} |e \cap S_1| \).

Similarly, for \( j \in S_1 \), we get
\[
\alpha_j(G) \leq \frac{(k-t(S_1))|E(S_1, \overline{S_1})|}{|S_1|} \leq (k-t(S_1))i(G). \tag{4}
\]

Let \( \alpha_s(G) = \beta(G) \). If \( s \in \overline{S_1} \), by (3), we get
\[
\beta(G) = \alpha_s(G) \leq t(S_1)i(G).
\]

From (4), we have
\[
\alpha(G) = \min_{j \in V(G)} \alpha_j(G) \leq \min_{j \in S_1} \alpha_j(G) \leq (k-t(S_1))i(G).
\]
Then 
\[ \alpha(G) + \beta(G) \leq t(S_1)i(G) + (k - t(S_1))i(G) = ki(G). \]
Similarly, if \( s \in S_1 \), we can also obtain \( \alpha(G) + \beta(G) \leq ki(G) \).

From the above discussion, we get \( i(G) \geq \frac{\alpha(G) + \beta(G)}{k} \).

The distance \( d(u, v) \) between two distinct vertices \( u \) and \( v \) of \( G \) is the length of the shortest path connecting them. The eccentricity of a vertex \( v \) is \( ecc(v) = \max\{d(u, v) : u \in V(G)\} \). The diameter and radius of \( G \) are defined as \( diam(G) = \max_{v \in V(G)} ecc(v) \) and \( rad(G) = \min_{v \in V(G)} ecc(v) \), respectively.

**Theorem 9.** Let \( G \) be a connected \( k \)-uniform hypergraph with \( n \) vertices. Then 
\[ ecc(j) \geq \frac{k}{2(k - 1)(n - 1)\alpha_j(G)}, \quad j \in V(G). \]

**Proof.** For \( j \in V(G) \), let \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}_+^n \) satisfying \( x_j = 0 \), \( \sum_{i=1}^n x_i = 1 \) and \( \alpha_j(G) = L_G x^k \). Then
\[ \alpha_j(G) = L_G x^k = \sum_{\{i_1, \ldots, i_k\} \in E(G)} (x_{i_1}^k + \cdots + x_{i_k}^k - kx_{i_1} \cdots x_{i_k}). \] 
From AM-GM inequality, it yields that 
\[ \sum_{1 \leq s < t < k} x_{i_s}^k x_{i_t}^k \geq \frac{k(k - 1)}{2} \left( \prod_{1 \leq s < t < k} x_{i_s}^k x_{i_t}^k \right)^{\frac{k}{k-1}} = \frac{k(k - 1)}{2} x_{i_1} \cdots x_{i_k}. \] 
By (5) and (6), we have 
\[ \alpha_j(G) \geq \sum_{\{i_1, \ldots, i_k\} \in E(G)} \left( x_{i_1}^k + \cdots + x_{i_k}^k - \frac{2}{k - 1} \sum_{1 \leq s < t < k} x_{i_s}^k x_{i_t}^k \right)^{\frac{1}{k - 1}} \]
\[ = \frac{1}{k - 1} \sum_{\{i_1, \ldots, i_k\} \in E(G)} \sum_{1 \leq s < t < k} \left( x_{i_s}^k - x_{i_t}^k \right)^2 \]
\[ = \frac{1}{k - 1} \sum_{e \in E(G)} \sum_{s, t \in e} \left( x_{i_s}^k - x_{i_t}^k \right)^2. \] 
Let \( v_0 \in \{i | x_i = \max_{p \in V(G)} x_p\} \). Let \( P = v_0 v_1 v_2 \cdots v_{l-1} v_l \) be the shortest path from vertex \( v_0 \) to vertex \( v_l = j \). Then \( x_{v_0}^k \geq \frac{1}{n-1}, \quad x_{v_l} = 0 \) and 
\[ \sum_{e \in E(G)} \sum_{s, t \in e} \left( x_{i_s}^k - x_{i_t}^k \right)^2 \geq \sum_{e \in E(P)} \sum_{s, t \in e} \left( x_{i_s}^k - x_{i_t}^k \right)^2 \]
\[ \begin{aligned}
&\sum_{i=1}^{l} \left( \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 + \sum_{u_j \in e_i \setminus \{v_{i-1}, v_{i}\}} \left( \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{u_j}^2} \right)^2 + \left( \frac{b}{x_{u_j}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 \right) \right) \\
&\geq \sum_{i=1}^{l} \left( \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 + \frac{1}{2} \sum_{u_j \in e_i \setminus \{v_{i-1}, v_{i}\}} \left( \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{u_j}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 \right) \right) \\
&= \sum_{i=1}^{l} \left( \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 + \frac{k-2}{2} \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 \right) \\
&= \frac{k}{2} \sum_{i=1}^{l} \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2.
\end{aligned} \]

By Cauchy-Schwarz inequality, we obtain

\[ \sum_{e \in E(G)} \sum_{s,t \in e} \left( \frac{b}{x_{s}^2} - \frac{b}{x_{t}^2} \right)^2 \geq \frac{k}{2l} \sum_{i=1}^{l} \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 \geq \frac{k}{2l} \left( \sum_{i=1}^{l} \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 \right) \geq \frac{k}{2l} \left( \sum_{i=1}^{l} \left( \frac{b}{x_{v_{i-1}}^2} - \frac{b}{x_{v_{i}}^2} \right)^2 \right). \quad (8) \]

From (7) and (8), it yields that

\[ \alpha_j(G) \geq \frac{k}{2(k-1)(n-1) \text{ecc}(j)} \quad \text{ecc}(j) \geq \frac{k}{2(k-1)(n-1) \alpha_j(G)}. \]

For a connected \( k \)-uniform hypergraph \( G \) with \( n \) vertices, [20] showed that

\[ \text{diam}(G) \geq \frac{4}{n^2(k-1) \alpha(G)}. \]

By Theorem 9, we obtain the following improved result.

**Corollary 10.** Let \( G \) be a connected \( k \)-uniform hypergraph with \( n \) vertices. Then

\[ \text{diam}(G) \geq \frac{k}{2(k-1)(n-1) \alpha(G)}, \quad \text{rad}(G) \geq \frac{k}{2(k-1)(n-1) \beta(G)}. \]

In [27], it was shown that \( \alpha(G) \leq \delta \), where \( \delta \) is the minimum degree of \( G \). We improve it as follows.

**Theorem 11.** Let \( G \) be a \( k \)-uniform hypergraph with \( n \) vertices. Then

\[ \alpha_j(G) \leq \frac{(k-1)d_j}{n-1}, \quad j \in V(G). \]
Proof. For \( j \in V(\mathcal{G}) \), let \( x = (x_1, \ldots, x_n)^T \) be the vector satisfying

\[
x_i = \begin{cases} 
(n - 1)^{-\frac{1}{k}}, & i \neq j, \\
0, & i = j.
\end{cases}
\]

Then \( \sum_{i=1}^{n} x_i^k = 1 \), and we get

\[
\alpha_j(\mathcal{G}) \leq \mathcal{L}_G x^k = \sum_{\{i_1, \ldots, i_k\} \in E(\mathcal{G})} (x_{i_1}^k + \cdots + x_{i_k}^k - k x_{i_1} \cdots x_{i_k})
\]

\[
= \sum_{\{i_1, \ldots, i_k\} \in E_j(\mathcal{G})} (x_{i_1}^k + \cdots + x_{i_k}^k) = \frac{(k - 1)d_j}{n - 1},
\]

where \( E_j(\mathcal{G}) \) denotes the set of edges containing \( j \). \qed

By Theorem 11, we obtain the following result.

**Corollary 12.** Let \( \mathcal{G} \) be a \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges. Then

\[
\sum_{j=1}^{n} \alpha_j(\mathcal{G}) \leq \frac{(k - 1)km}{n - 1}, \quad j \in V(\mathcal{G}).
\]

Let \( \mathcal{G} \) be a \( k \)-uniform hypergraph. For \( x, y \in V(\mathcal{G}) \), let \( c(x, y) = |\{e \in E(\mathcal{G}) : x, y \in e\}| \). A 2-\((n, b, k, r, \lambda)\) design can be regarded as a \( k \)-uniform \( r \)-regular hypergraph \( \mathcal{G} \) on \( n \) vertices, \( b \) edges, and \( c(x, y) = \lambda \) for any pair of distinct \( x, y \in V(\mathcal{G}) \). A 2-design satisfying \( n = b \) is called a symmetric design.

**Theorem 13.** Let \( \mathcal{G} \) be a connected \( k \)-uniform hypergraph with \( n \) vertices. Then \( \mathcal{G} \) is a 2-design if and only if \( \alpha_1(\mathcal{G}) = \cdots = \alpha_n(\mathcal{G}) = \frac{\Delta(k-1)}{n-1} \), where \( \Delta \) is the maximum degree of \( \mathcal{G} \).

Proof. We first prove the necessity. If \( \mathcal{G} \) is a 2-\((n, b, k, r, \lambda)\) design, then \( \lambda(n-1) = r(k-1) \) and \( \Delta = r = d_1 = \cdots = d_n \). For any \( j \in V(\mathcal{G}) \), by Theorem 11, we have

\[
\alpha_j(\mathcal{G}) \leq \frac{r(k-1)}{n-1} = \lambda.
\]

Let \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}_+^n \) satisfying \( x_j = 0, \sum_{i=1}^{n} x_i^k = 1 \) and \( \alpha_j(\mathcal{G}) = \mathcal{L}_G x_k^k \). Then we get

\[
\alpha_j(\mathcal{G}) = \mathcal{L}_G x_k^k \geq \sum_{\{i_1, \ldots, i_k\} \in E(\mathcal{G})} (x_{i_1}^k + \cdots + x_{i_k}^k - k x_{i_1} \cdots x_{i_k}) = \lambda \sum_{i \neq j} x_i^k = \lambda.
\]

Combining (9) and (10), we get

\[
\alpha_1(\mathcal{G}) = \cdots = \alpha_n(\mathcal{G}) = \lambda = \frac{r(k-1)}{n-1} = \frac{\Delta(k-1)}{n-1}.
\]
Moreover, if \( d_1 = \cdots = d_n = \Delta \). Let \( z = \left( (n-1)^{-\frac{1}{2}}, \ldots, (n-1)^{-\frac{1}{2}} \right)^T \in \mathbb{R}^{n-1} \).

For \( j \in V(G) \), let \( y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n \) be a vector such that \( y_i = 0 \) if \( i = j \) and \( y_i = (n-1)^{-\frac{1}{2}} \) otherwise. Then
\[
\mathcal{L}_G y^k = \sum_{\{i_1, \ldots, i_k\} \in E(G)} (y_{i_1}^k + \cdots + y_{i_k}^k - ky_{i_1} \cdots y_{i_k}) = \sum_{\{i_1, \ldots, i_k\} \in E_j(G)} (y_{i_1}^k + \cdots + y_{i_k}^k) = \frac{\Delta(k-1)}{n-1} = \alpha_j(G) = \alpha(G).
\]

We know that \( \alpha(G) = \alpha_j(G) \) is the smallest H-eigenvalue of \( \mathcal{L}_G(j) \). Since \( \mathcal{L}_G(j) z^k = \mathcal{L}_G y^k = \alpha(G), z \) is an H-eigenvector corresponding to \( \alpha(G) \), that is
\[
\alpha(G) z^{[k-1]} = \mathcal{L}_G(j) z^{k-1}.
\]

For all \( i \in V(G) \setminus \{j\} \), we have
\[
\alpha(G) = \frac{1}{z_i k^{-1}} (\mathcal{L}_G(j) z^{k-1})_i = \frac{1}{z_i k^{-1}} \sum_{i_2, \ldots, i_k \neq j} (\mathcal{L}_G(j))_{i_2 \cdots i_k} z_{i_2} \cdots z_{i_k}
\]
\[
= \sum_{i_2, \ldots, i_k \neq j} (\mathcal{L}_G)_{i_2 \cdots i_k} = c(i, j).
\]

So \( c(i, j) = \alpha(G) \) for any pair of distinct \( i, j \in V(G) \), which implies that \( G \) is a 2-design. \( \square \)

We give an estimation of the edge connectivity of a 2-design as follows.

**Theorem 14.** Let \( G \) be a 2-(\( n, b, k, r, \lambda \)) design. Then
\[
\frac{n \lambda}{k} \leq c(G) \leq \frac{(n-1) \lambda}{k-1}.
\]

Moreover, if \( G \) is a symmetric design, then \( c(G) = k = r \).

**Proof.** Since \( G \) is a 2-(\( n, b, k, r, \lambda \)) design, we have \( \lambda(n-1) = r(k-1) \). By Theorem 13, we have
\[
\alpha(G) = \frac{r(k-1)}{n-1} = \lambda.
\]

It follows from Lemma 2 that
\[
\frac{n \lambda}{k} = \frac{n}{k} \alpha(G) \leq c(G) \leq r = \frac{(n-1) \lambda}{k-1}.
\]

(11)

Moreover, if \( G \) is a symmetric design, then \( n = b \). Since \( nr = bk \), we have \( r = k \). From \( \lambda(n-1) = r(k-1) \) and (11), we have
\[
\frac{n(k-1)}{n-1} \leq c(G) \leq k.
\]

Since \( c(G) \) is a positive integer, we get \( c(G) = k = r \). \( \square \)
4 Inverse Perron values and resistance distance of graphs

For a vertex $i$ of a connected graph $G$, we define its resistance eccentricity as $r_i(G) = \max_{j \in V(G)} r_{ij}$.

**Theorem 15.** Let $G$ be a connected graph. For any $i \in V(G)$, we have

$$r_i(G) \leq \frac{1}{\alpha_i(G)}.$$

**Proof.** Without loss of generality, assume that $i$ is the vertex corresponding to the last row of the Laplacian matrix $L_G$. Since $\alpha_i(G)$ is the minimum eigenvalue of the principal submatrix $L_G(i)$, $\alpha_i^{-1}(G)$ is the spectral radius of the symmetric nonnegative matrix $L_G(i)^{-1}$. So $\alpha_i^{-1}(G) \geq \max_{j \neq i} (L_G(i)^{-1})_{jj}$.

By Lemmas 5 and 3, we get $r_{ij}(G) = (L_G(i)^{-1})_{jj}$ for any $j \neq i$. Hence

$$\alpha_i^{-1}(G) \geq \max_{j \neq i} (L_G(i)^{-1})_{jj} = r_i(G),$$

$$r_i(G) \leq \frac{1}{\alpha_i(G)}.$$ \hfill $\Box$

For a vertex $i$ of a connected graph $G$, its resistance centrality is defined as $Kf_i(G) = \sum_{j \in V(G)} r_{ij}(G)$. It is used to measure the centrality of a network [4]. Note that $Kf(G) = \sum_{\{i,j\} \subseteq V(G)} r_{ij}(G) = \frac{1}{2} \sum_{i \in V(G)} Kf_i(G)$.

**Theorem 16.** Let $G$ be a connected graph with $n$ vertices. For any $i \in V(G)$, we have

$$nKf_i(G) - Kf(G) \leq \frac{n-1}{\alpha_i(G)}.$$

**Proof.** Note that $\alpha_i^{-1}(G)$ is the maximum eigenvalue of the symmetric matrix $L_G(i)^{-1}$. Let $\mathbf{e}$ be the all-ones column vector, then

$$\alpha_i^{-1}(G) \geq \frac{\mathbf{e}^\top L_G(i)^{-1} \mathbf{e}}{\mathbf{e}^\top \mathbf{e}} = \frac{\mathbf{e}^\top L_G(i)^{-1} \mathbf{e}}{n-1}.$$ 

By Lemmas 5 and 4, we have

$$Kf(G) = n \text{tr}(L_G(i)^{-1}) - \mathbf{e}^\top L_G(i)^{-1} \mathbf{e}.$$ 

From Lemmas 5 and 3, we get $r_{ij}(G) = (L_G(i)^{-1})_{jj}$ for any $j \neq i$. Hence $\text{tr}(L_G(i)^{-1}) = Kf_i(G)$ and

$$Kf(G) = nKf_i(G) - \mathbf{e}^\top L_G(i)^{-1} \mathbf{e}.$$
By $\alpha^{-1}_i(G) \geq \frac{e^T L_G(i)^{-1} e}{n-1}$ we get

$$\alpha^{-1}_i(G) \geq \frac{e^T L_G(i)^{-1} e}{n-1} = \frac{nKf_i(G) - Kf(G)}{n-1} = \frac{n-1}{\alpha_i(G)}.$$ $\square$

**Corollary 17.** Let $G$ be a connected graph with $n$ vertices. Then

$$Kf(G) \leq \frac{n-1}{n} \sum_{i=1}^{n} \alpha^{-1}_i(G).$$

**Proof.** By Theorem 16, we have

$$\sum_{i=1}^{n} \frac{n-1}{\alpha_i(G)} \geq \sum_{i=1}^{n} (nKf_i(G) - Kf(G)) = nKf(G),$$

$$Kf(G) \leq \frac{n-1}{n} \sum_{i=1}^{n} \alpha^{-1}_i(G).$$ $\square$

**References**


