# Line-transitive Automorphism Groups of Linear Spaces[1]

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**Abstract**

In this paper we prove the following theorem.

*Let $\mathcal{S}$ be a linear space. Assume that $\mathcal{S}$ has an automorphism group $G$ which is line-transitive and point-imprimitive with $k < 9$. Then $\mathcal{S}$ is one of the following:-*

**(a)** *A projective plane of order 4 or 7,*

**(a)** *One of 2 linear spaces with $v = 91$ and $k = 6$,*

**(b)** *One of 467 linear spaces with $v = 729$ and $k = 8$.*

*In all cases the full automorphism group Aut(S) is known.*

## 1 Introduction

A *linear space* $\mathcal{S}$ is a set of points, $\mathcal{P}$, together with a set of distinguished subsets, $\mathcal{L}$, called lines such that any two points lie on exactly one line. This paper will be concerned with linear spaces in which every line has the same number of points and we shall call such a space a *regular linear space*. Moreover, we shall also assume that $\mathcal{P}$ is finite and that $|\mathcal{L}| > 1$. The number of points will be denoted by $v$, the number of lines by $b$, the number of points on a line will be denoted by $k$ and the number of lines through a point by $r$. We shall assume that $k > 2$. Regular linear spaces are also called $2-(v, k, 1)$ block designs and sometimes Steiner Systems. The choice of notation was determined by the use of the language of linear spaces by a number of authors as well as the need to study the fixed points of automorphisms. Such subsets inherit the structure of the linear space but not of the block design.

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In this paper we investigate the properties of linear spaces which have an automorphism group which is transitive on lines. Clearly such a space is automatically a regular linear space.It follows from a result of Block [1] that a line-transitive automorphism group of a linear space is transitive on points. Recently Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck and Saxl [3] effectively classified all regular linear spaces with an automorphism group transitive on flags, that is on incident line-point pairs. (This classification is incomplete in that the so-called one-dimensional affine case remains open.) In a very interesting paper [9] it was shown that if a group of automorphisms was line-transitive but point-imprimitive then $v$ is small compared to $k$. This result makes the classification of line-transitive point-imprimitive linear spaces a possibility. This paper is a contribution to this problem.

The motivation for our work came from results in [3, 6, 9]. In this paper our main purpose is to prove the following theorem.

**Theorem 1 (The Main Theorem)** *Let $\mathcal{S}$ be a linear space. Assume that $\mathcal{S}$ has an automorphism group $G$ which is line-transitive and point-imprimitive with $k < 9$. Then $\mathcal{S}$ is one of the following:-*

**(a)** *A projective plane of order 4 or 7,*

**(a)** *One of 2 linear spaces with $v = 91$ and $k = 6$,*

**(b)** *One of 467 linear spaces with $v = 729$ and $k = 8$.*

*In all cases the full automorphism group Aut(S) is known.*

Before starting the body of the article we introduce some notation. Let $G$ act on a linear space $\mathcal{S}$ and let $l$ be a line of $\mathcal{S}$ . We use the following notation:-

- $G_l = \{g : lg = l\}$,
- $G_{(l)} = \{g : Pg = P \ \forall P \in l\}$,
- $G^l = G_l/G_{(l)}$,
- For any subset $H \subset G$, $\mathrm{Fix}(H) = \{P : Ph = P \ \forall h \in H\}$.

Thus $G^l$ denotes the action of the stabilizer of the line $l$ on the points of $l$.

This work is based on the thesis of the second author [15]. We would also like to express our thanks to Rachel Camina for her careful reading of the text and helpful comments. We would also like to express our gratitude to the referee.

## 2 Setting the scene

A key result, mentioned above, is the following, due to Anne Delandtsheer and Jean Doyen [9] is the following.

**Theorem 2** *Let $G$ act as a point-imprimitive, line-transitive automorphism group of a linear space $S$. Assume that $v = cd$ where $c$ is the size of a set of imprimitivity. Then there exist two positive integers $x$ and $y$ such that*

$$c = \frac{\binom{k}{2} - x}{y}$$

*and*

$$d = \frac{\binom{k}{2} - y}{x}.$$

The number $x$ can be interpreted as the number of pairs of points on a given line which are in the same set of imprimitivity, such pairs are called *inner pairs.* Thus for any given $k$ there are only a finite set of possible values for $v$.

We now list the possible values of the parameters for $k \leq 8$ recalling that $v \geq k^2 - k + 1$, (Fisher's inequality).

| k | x | y | c | d | v |
|---|---|---|---|---|---|
| 4 | 1 | 1 | 5 | 5 | 25 |
| 5 | 1 | 1 | 9 | 9 | 81 |
| 5 | 1 | 3 | 7 | 3 | 21 |
| 5 | 3 | 1 | 3 | 7 | 21 |
| 6 | 1 | 1 | 14 | 14 | 196 |
| 6 | 1 | 2 | 7 | 13 | 91 |
| 6 | 2 | 1 | 13 | 7 | 91 |
| 7 | 1 | 4 | 5 | 17 | 85 |
| 7 | 4 | 1 | 17 | 5 | 85 |
| 8 | 1 | 1 | 27 | 27 | 729 |
| 8 | 1 | 3 | 9 | 25 | 225 |
| 8 | 1 | 9 | 3 | 19 | 57 |
| 8 | 2 | 2 | 13 | 13 | 169 |
| 8 | 3 | 1 | 25 | 9 | 225 |
| 8 | 9 | 1 | 19 | 3 | 57 |

We will discuss what is known in the various cases above. When $x = y = 1$ there is a complete description of what happens, see [4, 17, 16]. This is described in the Theorem below.

**Theorem 3** *[17] Let $\mathcal{S}$ be a line-transitive, point-imprimitive linear space with $v = \left(\binom{k}{2} - 1\right)^2$. Then $v = 729$ and $k = 8$, the automorphism group is of the form $N.H$ where $H$ is cyclic of order 13 or the non-abelian group of order 39, and $N$ satisfies one of the following*

**(a)** $N = C_3^6$,

**(b)** $N = C_9^3$ *or*

**(c)** *$N$ is the relatively free, 3-generator, exponent 3, nilpotency class 2 group (of order 729)*

In [16] it is shown that, up to isomorphism, there are 467 such linear spaces. In conversation with C. E. Praeger we have been told that it is now known that $|H| = 13$.

The cases $k = 5$, $v = 21$ and $k = 8$, $v = 57$ both give rise to projective planes. There are unique projective planes of order 4 and 7, see [14, 2, 11]. These must be the projective planes over the appropriate fields. So in this situation there is a complete description see also [17], page 232. The situation when $k = 6$ and $v = 91$ is discussed in [5, 13]. It is shown that there are exactly two designs with these properties, both have soluble automorphism groups, one of order 273 and one of order 1092. Thus the following cases are left.

| k | x | y | c | d | v |
|---|---|---|---|---|---|
| 7 | 1 | 4 | 5 | 17 | 85 |
| 7 | 4 | 1 | 17 | 5 | 85 |
| 8 | 1 | 3 | 9 | 25 | 225 |
| 8 | 2 | 2 | 13 | 13 | 169 |
| 8 | 3 | 1 | 25 | 9 | 225 |

Section 5 of this paper deals with the situation when $k = 7$ and Section 6 deals with the situation when $k = 8$.

## 3 Some preliminary results

We begin this section with some simple lemmas concerning linear spaces with automorphism groups which satisfy the following hypothesis.

**Hypothesis 1** *Let $G$ be an automorphism group of a linear space $S$ which acts line-transitively but not flag-transitively.*

**Lemma 1** *Let $G$ satisfy Hypothesis 1. Let $s$ be an involution in $G$ and assume that there is a normal subgroup $N$ of $G$ with $[G : N] = 2$ such that $s \notin N$. Then $N$ also acts line-transitively.*

*Proof:* Since $s$ fixes at least one line, say $l$, we have $NG_l = G$ and the lemma follows.

**Lemma 2** *Let $G$ satisfy Hypothesis 1 so that it is minimal with respect to being line-transitive. Then any involution acts as an even permutation on both lines and points.*

*Proof:* This follows immediately from Lemma 1.

We now give a proof of a lemma to be found in the thesis of D. H. Davies [8].

**Lemma 3** *Let $g$ be a non-trivial automorphism of a regular linear space $S$. Let $g$ have prime order $p$. Then $g$ has at most $\max(r + k - p - 1, r)$ fixed points. Further if there is a point which does not lie on a line fixed by $g$ then $g$ has at most $r$ fixed points.*

*Proof:* Let $P$ be any point not fixed by $g$. Then there is at most one line through $P$ which can be fixed by $g$. A line not fixed by $g$ contains at most one fixed point. If $p \geq k$ then any line containing $P$ is of this form. If $p < k$ a line fixed by $g$ containing $P$ has at most $k - p$ fixed points and there is at most one of them. The lemma now follows.

**Lemma 4** *Let $G$ satisfy Hypothesis 1. Let $p$ be a prime such that $p$ divides $|G_{(l)}|$ but does not divide $|G^l|$. Let $H$ be a $p$-subgroup of $G_{(l)}$. Then the fixed point set of $H$ has the structure of a regular linear space with lines of size $k$. Hence $|\mathrm{Fix}(H)| \geq k^2 - k + 1$.*

*Proof:* From the conditions of the lemma it is clear that if $H$ fixes two points it has to fix all the points on the line joining the two points. Hence, either the fixed points of $H$ are just the points of the line $l$ or the conclusions of the lemma hold. If the fixed point set is just the points of $l$ then we can conclude from Lemma 3 of [7] that $G$ would act flag-transitively which is a contradiction.

**Lemma 5** *Let $G$ satisfy Hypothesis 1. Let $p$ be a prime .*

1. *If $p||G_{(l)}|$ and $k^2 - k + 1 > \max(r + k - p - 1, r)$ then $p\ ||G^l|$ for any line $l$,*
2. *If $p > k$ and $k^2 - k + 1 > r$ then $p|v$ or $p|(v - 1)$. Further if $T$ is a Sylow-p-subgroup of $G$ then $|\ T\ |$ divides $v$ or $v - 1$ respectively.*

*Proof:*

1. Let $H$ be the Sylow p-subgroup of $G_l$. Assume the conclusion is false and let $H$ have $d$ fixed points. By the preceeding Lemma we have $k^2 - k + 1 \leq d$ but by Lemma 3, $d \leq \max(r + k - p - 1, r)$ and the result follows.

2. Assume that $H$ is the Sylow $p$-subgroup of $G_l$ and that $H \neq 1$. Note that $p$ cannot divide $|G^l|$ in this case. We now get a contradiction since the fixed point set of $H$ cannot be a regular linear space with line size $k$. Hence we deduce that $H = 1$. Hence no $p$-subgroup of $G$ can fix more than 1 point so if $T$ is a Sylow $p$-subgroup of $G$ then $|T| \,\Big|\, v(v-1)$.

# 4 Imprimitivity

**Hypothesis 2** *Let $G$ be an automorphism group of a linear space $\mathcal{S}$ which acts transitively on lines but imprimitively on points. Let $X$ be a set of imprimitivity and let $\mid X \mid = c > 1$.*

We note that by [1] and [12] Hypothesis 2 implies Hypothesis 1. We now look at a simple consequence of Hypothesis 2.

**Lemma 6** *Let $G$ satisfy Hypothesis 2. For any line $l$ we have $\mid l \cap X \mid \leq [\frac{c+1}{2}]$, where $[n]$ denotes the greatest integer not greater than $n$.*

*Proof:* Let $a = |l \cap X| >$. Then $a > [\frac{c+1}{2}]$. Since each pair of points is on a unique line, $l$ is the unique line which intersects $X$ in $a$ points. Thus we get $G_l \supseteq G_X \supset G_P$, where $P \in X$. But $b \geq v$ and by transitivity

$$|G_l| \leq |G_P|.$$

This is a contradiction.

We now get a slightly more complex consequence of our hypothesis.

**Proposition 7** *Let $G$ satisfy Hypothesis 2. If $l$ is a line then each orbit of $G^l$ on the points of $l$ has order less than $k - 1$.*

*Proof:* If the orbit had length $k$ then $G$ would be flag-transitive and we know that implies point-primitivity, [12]. So we assume that $G^l$ has an orbit of size $k - 1$.

Let $\rho$ be the equivalence relation which comes from the system of imprimitivity given. We denote by $\rho(P)$ the equivalence class containing a point $P$. Let $P$ and $Q$ be two points on $l$. If $P$, $Q$ are in the same orbit of $G^l$ then $\exists g \in G_l$ so that $Pg = Q$ and so $\rho(P)g = \rho(Q)$. Hence we have $|\rho(P) \cap l| = |\rho(Q) \cap l|$. If there is an orbit $O$ of $G^l$ of size $k - 1$ we have that $|\rho(P) \cap l| = e$ for some integer $e$, $\forall P \in O$. Note that

$e > 1$. Also we have that $e|(k-1)$ and so there is an integer $f$ with $k-1 = ef$. So the number of internal pairs is given by $\binom{e}{2}f$. Now we can apply Theorem 2 to get:-

$$\begin{aligned} v &= \frac{\binom{k}{2} - x}{y} \times \frac{\binom{k}{2} - y}{x}, \\ &= \frac{\binom{ef+1}{2} - \binom{e}{2}f}{y} \times \frac{\binom{ef+1}{2} - y}{\binom{e}{2}f}, \\ &= \frac{ef(ef+1-(e-1))}{2y} \times \frac{ef(ef+1)-2y}{ef(e-1)}. \end{aligned} \tag{1}$$

Recall that $\frac{ef(ef+1)-2y}{ef(e-1)}$ is an integer. Thus we can deduce $ef|2y$ and so there is an integer, say a, so that $2y = aef$. Now substitute this in Equation 1 to get:-

$$\begin{aligned} v &= \frac{ef+1-(e-1)}{a} \times \frac{ef+1-a}{(e-1)}, \\ &= \frac{k-(e-1)}{a} \times \frac{k-a}{(e-1)}. \end{aligned}$$

Using the inequality $v \geq k^2 - k + 1$ gives

$$(k-(e-1))(k-a) \geq a(e-1)(k^2-k+1)$$

This is impossible given that $e > 1$, $a > 0$ and $k > 1$ and so the Proposition holds.

**Lemma 8** *Let $G$ satisfy Hypothesis 2. Assume that for some line $l$, $|l \cap X| = [\frac{c+1}{2}]$ then $c \leq 4$ and*

1. *if $c = 3$ then $\mathcal{S}$ is a projective plane.*
2. *if $c = 4$ then there is an integer $h$ so that $k = 8h+2$, $v = 4(24h^2+9h+1)$ and $G_X$ acts 2-transitively on the points of $X$.*

*Proof:* Let us begin by assuming that $c > 4$

Then assume that $c$ is even and let $c = 2m$, $m > 1$. Then our hypothesis implies that there exists a line $l$ such that $l \cap X = m$. We now count the number of lines say $a$ which can intersect $X$ in $m$ points. Each such intersection will contain $\frac{m(m-1)}{2}$ pairs. Thus we get

$$a\frac{m(m-1)}{2} \leq \frac{2m(2m-1)}{2}.$$

From this we deduce that

$$a \quad \leq \quad \frac{2(2m-1)}{m-1} \leq 6. \tag{2}$$

Equality can occur only in the above equation if $m = 2$.

We now assume that $c$ is odd and let $c = 2m+1$ and then $|l \cap X| = m+1$. Using a similar count we get

$$a\frac{m(m+1)}{2} \leq \frac{2m(2m+1)}{2}.$$

From this we deduce that

$$a \leq \frac{2(2m+1)}{m+1} \quad < \quad 4. \tag{3}$$

Thus in both cases we have that $a \leq 5$ if $c > 4$. If $P \in X$, then we get:-

$$|G_X| \quad \leq \quad 5|G_l|, \tag{4}$$

$$|G_X| \quad = \quad c|G_P|, \tag{5}$$

$$|G_l| \quad \leq \quad |G_P|. \tag{6}$$

Putting this altogether gives

$$c|G_l| \leq c|G_P| = |G_X| \leq 5|G_l|.$$

This can only happen if $c \leq 5$ but if $c = 5$ we can see from Equation 3 that this does not happen. So we have the first part of the lemma.

1. $c = 3$: we see that the number of lines which intersect $X$ in 2 points is 3 and from the above equations we deduce that $v = b$.

2. $c = 4$: in this situation there are 6 lines which can intersect $X$ in two points. The equations above can be strengthened by replacing 5 by the size of an orbit, say $n$ of $G_X$ on the lines which intersect in 2 points and obtaining the equation:-

$$|G_X| \quad = \quad n|G_l \cap G_X| \tag{7}$$

Combining this with above equations for $c = 4$ we conclude that $n = 6$ and $3v = 2b$. Given that $v$ has to be even we find that there is a parameter $h$ say so that

$$\begin{aligned} k &= 8h+2, \\ r &= 12h+3, \\ v &= 4(24h^2+9h+1) \text{ and} \\ b &= 6(24h^2+9h+1). \end{aligned}$$

Further since $n = 6$ we see that $G_X$ has to act 2-transitively on the 4 points of $X$.

In the above theorem it is easy to find examples where $k = 3$. Take a Desarguesian projective plane of order $q$ where $q \equiv 1 \pmod 3$, see also [17], page 232. The existence is established by considering the Singer cycle. However when $c = 4$ we have no idea how to proceed in general except to note that $G$ does not have a normal subgroup of order 4, see [6]. The referee has pointed out that the case when $h = 1$ is not possible.

## 5 The situation when $k = 7$

In this section we are going to consider groups and linear spaces satisfying the following:-

**Hypothesis 3** *Let $G$ satisfy Hypothesis 2 and let $k = 7$.*

¿From the results in Section 2 we need only consider the case $x = 1$ and $y = 4$ or $x = 4$ and $y = 1$. In this section we prove the following theorem.

**Theorem 4** *There is no $G$ satisfying Hypothesis 3.*

We will prove this theorem as a consequence of a series of lemmas proved under the assumption that $G$ satisfies Hypothesis 3.

**Lemma 9** *The only primes that can divide the order of $G$ are* 2, 3, 5 *and* 17.

*Proof:* We note that by the results of Lemma 5 we can exclude all the primes except 2, 3, 5, 17 and 7. Thus we need only consider the prime 7 to complete the proof of this lemma.

Let $T$ be a Sylow 7-subgroup of $G_l$ for some line $l$. By Lemma 5 we know that $7 \Big| |G^l|$. However since $k = 7$ we would conclude that $G^l$ acts transitively which contradicts Hypothesis 3. Thus we have that a Sylow 7-subgroup of $G$ does not fix a line. This is a contradiction since there are 170 lines.

**Lemma 10** *If $T$ is a non-trivial Sylow* 3*-subgroup of $G$ then $T$ fixes only one point.*

*Proof:* Assume that $|\text{Fix}(T)| = f \geq 2$. Then there is a line $l$ which $T$ fixes. By Lemma 2 of [7] we know $\text{Fix}(T)$ has the structure of a regular linear space with line size $k_0$, where $k_0$ is the number of fixed points of $T$ on $l$ or $\text{Fix}(T) \subset l$. Thus by the arguments of Lemma 5 we see that $k_0 = 4$. Firstly we consider the case when $\text{Fix}(T) \subset l$. Then $N_G(T) \subseteq G_l$ and so $[G_l : N_G(T)] \equiv 1 \pmod 3$ and $[G : N_G(T)] \equiv 1 \pmod 3$ but $[G : G_l] \equiv 2 \pmod 3$. This is a contradiction.

We now consider the alternative. Note that, again from Lemma2 of [7], $N_G(T)$ acts on $\text{Fix}(T)$ as a line transitive automorphism group. Thus from Lemma 3 we have that $|\text{Fix}(T)| = 13$ or 16. Since 13 does not divide the order $G$ we see that $|\mathrm{Fix}(T)| = 16$. Now again by Lemma 3 every point has to lie on a fixed line of $T$. But $T$ fixes only 20 lines so that altogether there are only $60 + 16$ points accounted for, recall $v = 85$. The lemma follows.

**Lemma 11** $(3, |G|) = 1$.

*Proof:* Since $G$ acts imprimitively, there are either 5 sets of imprimitivity of size 17 or 17 sets of imprimitivity of size 5. Since both 5 and 17 are congruent to 2 mod 3 we see that the Sylow 3-subgroup has to fix at least 2 sets of imprimitivity and two points on each such fixed set. Thus, a Sylow 3-subgroup of $G$ has to fix at least 4 points. But this contradicts the previous Lemma.

**Lemma 12** *$G$ is soluble.*

*Proof:* Since $G$ is not divisible by 3 we have that the only simple groups that can appear in $G$ are the Suzuki groups $\mathrm{Sz}(q)$ where $q = 2^{2n+1}$, see [10]. However $|\mathrm{Sz}(q)| = q^2(q^2+1)(q-1)$. It is easy to check that for no value of $q^n$ is $q^2(q^2+1)(q-1)$ divisible only by the primes 2, 5 and 17.

*Proof of Theorem 4:* The first observation is that by Lemma 5, $| G |= 17a$ where $(17, a) = 1$. We now have that $G$ is soluble and divisible by only the primes 2, 5 and 17. Let $F$ be the Fitting subgroup of $G$. We show first that $|F| = 85$. Assume that $G$ has a normal subgroup $N$ whose order is a power of 5. Then, by [[6], Theorem 1], $|N| = 5$. Since an element of order 17 has to centralise a group of order 5, $N$ cannot be the Fitting subgroup.

Now let $N$ be a normal subgroup of order 17. This time any element of order 5 has to centralize a group of order 17 and so $N$ cannot be the Fitting subgroup. So $|F| = 85$ and $F$ is a normal subgroup which is regular in its action on points.

Since there are 170 lines $G$ contains an involution, say $s$. Also since no involution can act fixed-point-freely, see [7] we see that $G/F$ has a unique involution and so $G$ has a unique class of involutions. Further $|G|$ divides $2^4.5.17$. Now $s$ has to fix either 5 or 17 points and the fixed points either lie on a line or have the structure of a regular linear space with line size either 3 or 5. Neither of the latter are possible so all the fixed points of an involution $s$ lie on a line, say $l$. But since all involutions are conjugate we would have $N_G(s) \subseteq G_l$ but $[G : G_l]$ is even. This contradiction completes the proof of Theorem 4.

## 6 The situation when $k = 8$

In order to complete the classification of line-transitive, point-imprimitive designs with $k < 9$ the only remaining case is when $k = 8$. With this aim we consider the following hypothesis.

**Hypothesis 4** *Let $G$ satisfy Hypothesis 2 and let $k = 8$.*

When we consider the results of Section 2 we see that the only possibilities we need to consider for $x$ and $y$ and $v$ are :-

**(a)** $x = 1$ and $y = 3$ or $x = 3$ and $y = 1$, $v = 225$ and

**(b)** $x = y = 2$, $v = 169$.

Before we look at each of these cases independently we prove two lemmas which apply in all cases.

**Lemma 13** *Let $G$ satisfy Hypothesis 4. Then we have that $|G|\Big|2^a3^b5^c\frac{v(v-1)}{56}$.*

*Proof:* By using Lemma 5 we can eliminate all the primes apart from 7. Now assume that $7\Big||G_l|$ for some line $l$. Then by Lemma 5 we get that $7\Big||G^l|$. Applying Proposition 7 gives a contradiction.

**Lemma 14** *Let $G$ satisfy Hypothesis 4. Assume that the Sylow 5-subgroup $T$ of $G_l$, for some line $l$, is non-trivial. Let $f = |\mathrm{Fix}(T)|$. Then*

1. *$N_G(T)$ acts line transitively on the fixed points of $T$ which have the structure of a Steiner triple system.*
2. *$6|f(f-1)$ and $\frac{f(f-1)}{6}||G|$.*
3. *$v \equiv f \pmod 5$.*
4. *$5\frac{f(f-1)}{6} + f \leq v$.*

*Proof:* The main point is that $T \not\subset G_{(l)}$ by Lemma 5. Thus $T$ fixes exactly 3 points on each line it fixes. But since $v \not\equiv 3 \pmod 5$ in all of the cases the first result follows from [7]. (2) follows immediately from (1) and (3) is straightforward. The last result follows by considering the non-fixed points which lie on fixed lines which are not fixed. Each such point is on a unique fixed line.

### 6.1 Case (a)

**Hypothesis 5** *$G$ is the group of smallest order satisfying Hypothesis 4 with $v = 225$.*

Our purpose in this section is to prove the following theorem.

**Theorem 5** *There is no $G$ satisfying Hypothesis 5.*

First we consider the Sylow 5-subgroup of $G$.

**Lemma 15** *Let $G$ satisfy Hypothesis 5. A Sylow 5-subgroup of $G$ has order $5^2$.*

*Proof:* Assume that the Sylow 5-subgroup $T$ of $G_l$, for some line $l$, is non-trivial. Let $f = |\mathrm{Fix}(T)|$. We know from Lemma 14 that $f = 15$ or 25. But simple calculations show that if $f = 15$ then 7 divides the order of $G$, this is false by Lemma 13. Further $f = 25$ contradicts the last assertion of Lemma 14. This completes the proof of the lemma since $|G| = 900|G_l|$.

In this situation $G$ can act imprimitively only on sets of size either 9 or 25. Hence when we examine the action of $G$ on the sets of imprimitivity we see that $G$ has to acts primitively on them. Fortunately we know about primitive group actions of degrees 9 and 25. Let $R$ denote the sets of imprimitivity.

**Lemma 16** *Let $G$ satisfy Hypothesis 5. Let $K$ be the kernel of the action of $G$ on the sets of imprimitivity. Then $|K| \neq 1$.*

*Proof:* Assume that $K = 1$. Then, since $G^R \cong G/K$, we have that $G^R = G$, where $G^R$ is primitive in its action on $R$. First we consider the case when $c = 9$. Thus $3^2$ has to divide $|G^R|$. However the only primitive groups of degree 25 divisible only by 2, 3 and 5 are $S_5$ wr $S_2$, $A_5$ wr $S_2$ and the primitive subgroups of $A\Gamma L(2,5)$ or $A\Gamma L(1,25)$. Since the order of the last two of these is not divisible by 9, it follows that $|K| \neq 1$ in these two cases.

However in the first two cases $G$ would have a normal subgroup of index 2 which does not contain all involutions. This contradicts Lemma 1, so we deduce that $|K| \neq 1$.

When $c = 25$ it is clear that $G^R$ cannot have order divisible by 25 so that $|K| \neq 1$.

**Proposition 17** *Let $G$ satisfy Hypothesis 5. Then $G$ has regular normal abelian subgroup $F$ of order 225.*

*Proof:* We will again consider the cases $c = 9$ and $c = 25$ separately. Let $c = 9$. We know that there is a normal subgroup $K$ which fixes each equivalence class. So $K$ is divisible only by the primes 2 and 3. Thus $K$ is soluble but then it has to have a normal Abelian subgroup $K_0$ with $|K_0| = 9$. If $K_0 = K$ then we use similar arguments to those in the previous lemma. $G/K$ again cannot be isomorphic to either $S_5$ wr $S_2$ or $A_5$ wr $S_2$. So $G/K$ is isomorphic to either $A\Gamma L(2,5)$ or $A\Gamma L(1,25)$. In both of these cases there is a normal subgroup $F$ of $G$ such that $F/K$ is of order 25. It is clear that $F$ has a subgroup with the required properties.

Now assume that $K_0 \neq K$. The only way this can occur is for $K$ to have an involution $s$ which inverts $K_0$ and fixes a unique point, say $\alpha$. This implies that $C_G(K) \cap K = 1$. In no case can 5 divide $|\mathrm{Aut}(K)|$ so that $C_G(K) \neq 1$ and since $C_G(s) \subseteq G_\alpha$ we have a contradiction.

We may now assume that $c = 25$. In this case $G/K$ acts as a primitive permutation group of degree 9. Since 7 does not divide the order of $G/K$ it follows that 5 does not divide the order of $G/K$ either for otherwise the action would be 2-transitive. Let $H$ be the stabilizer of an equivalence class. This means that $H$ is a primitive group of degree 25. One possibility is that $H \cong S_5$ wr $S_2$ with $K \cong A_5 \times A_5$ but then $C_G(K) \cap K = 1$ however $C_G(K) \neq 1$ as 9 divides $|G/K|$. Finally $G_G(K)$ has a normal subgroup of order 9 and we are in the first situation.

The other possibility is that $H$ has a normal subgroup of order 25. However it follows that $K$ will have a characteristic subgroup of order 25. So now we have a normal subgroup, say L, of $G$ so that $|L| = 25$ and $G/L$ is soluble. Thus $G$ is soluble. Let $F(G)$ be the Fitting subgroup of $G$. Then $L < F(G)$ since 9 does not divide $|C_G(L)|$. Thus the only possibility is that $|F(G)| = 225$.

*Proof of Theorem 5*

¿From Proposition 17, $G$ has a normal regular subgroup $F$ of order 225. Let $S$ and $T$ be the Sylow 5- and Sylow 3- subgroups of $F$ respectively. Now we consider the fixed point set of an involution $t$. Such a set can only have size 5, 9 or 25 where each of these corresponds to the order $C_F(t)$. We know that we have two distinct equivalence relations on the point set, one given by the cosets of $S$ and the other given by the cosets of $T$. Since the intersection of any line with any equivalence class contains at most one point, x=1, we see that $t$ can only fix two points on any line, since the fixed points are all in one equivalence class.

We consider each case individually.

- $|\text{Fix}(t)| = 5$. Then there are 10 lines of the design on which $t$ has fixed points. This will account for $5 + 10 \times 6 = 65$ points of the design. Since each point lies on a fixed line there are 80 more lines fixed by $t$ but then $t$ acts as an odd permutation on the lines which is a contradiction.
- $|\text{Fix}(t)| = 9$. Then there are 36 lines of the design on which $t$ has fixed points. This will account for $9 + 36 \times 6 = 225$ points of the design. Thus all the points are on lines of this type.
- $|\text{Fix}(t)| = 25$. Then there are 300 lines of the design on which $t$ has fixed points. This is too many.

We have now shown that each involution has to fix points on each line $l$ that it fixes. It is also known that the Sylow 2-subgroup of $G^l$ fixes no points on $l$. Thus $4||G^l|$ and so $16||G|$. When we consider $G/C_G(T)$ we see each involution acts fixed point freely. So the Sylow 2-subgroup of $G/C_G(T)$ has a unique involution. However no subgroup of $\text{GL}(2,5)$ with order 16 has this property and so we have completed the proof of Theorem 5.

## 6.2 Case (b)

In this section we discuss case (b) of this section - the design that arises when we choose $x = y = 2$ as a solution of the Delandtsheer-Doyen equation.

**Hypothesis 6** *G is the group of smallest order satisfying Hypothesis 4 with* $v = 169$.

Our purpose in this section is to prove the following theorem.

**Theorem 6** *There is no G satisfying Hypothesis 6.*

Before beginning the proof let $\rho$ be the equivalence relation which comes from the system of imprimitivity given. Denote the equivalence class containing the point $P$ by $\rho(P)$. Since the number of inner pairs is 2 we have that the intersections of a line with the set of imprimitivity have sizes 2, 2, 1, 1, 1 and 1.

**Lemma 18** *Let G satisfy Hypothesis 6. Then the order of G divides* $2^a 3^b 13^2$.

*Proof:* By Lemmas 5 and 13 the only prime we have to examine more closely is 5. Let $T$ be a Sylow 5-subgroup of $G$ and assume that $T \neq 1$. By Lemma 14, $|\text{Fix}(T)| = 9$. So $\text{Fix}(T)$ has the structure of a 2-dimensional affine geometry over $GF(3)$ with automorphism group $N_G(T)$. Given a line $l$ this implies that $G^l$ has two orbits on the points of $l$, one of length 5 and one of length 3. Now by Lemma 7 the orbit of length 5 has to be the union of points $P$ such that $\rho(P) \cap l$ is constant. Now, by the remark just preceeding the lemma we cannot have an orbit of length 5. So $T = 1$.

This leaves us with only the primes 2 and 3 to consider.

**Lemma 19** *Let G satisfy Hypothesis 6. Then the order of G divides* $2^a 13^2 3$

*Proof:* Let $T$ be a Sylow 3-subgroup of $G_l$ for some line $l$, let $g$ be an element of order 3 in $T^l$ and $(P, Q, R)$ be a three cycle of $g$. Using the comment before the preceding lemma we can assume that $|\rho(P) \cap l| = 2$. Then $|\rho(Q) \cap l| = 2$ and $|\rho(R) \cap l| = 2$. This implies that either $P, Q$ or $P, R$ is an inner pair but then all pairs would be inner, which is false. Thus $|\rho(P) \cap l| = 1$ and so $g$ fixes 5 points on $l$. It follows that $T$ fixes 5 points on $l$. Thus the set of fixed points of $T$ have the structure of a regular linear space with line size 5 since $169 \not\equiv 5 \pmod 3$ by Lemma 2 of [7]. The only possibility for this would be with 25 points but 5 does not divide $|G|$.

Now let us consider the action of G on the equivalence relation $\rho$ defined by the sets of imprimitivity. Let $K$ be the kernel of this action.

**Lemma 20** *K has a normal subgroup of order 13. Further* $G/K$ *is soluble.*

*Proof:* We see that $G/K$ is a transitive group of degree 13. So $13^2$ does not divide the order of $G/K$. Thus we have that 13 divides the order of $K$. However $K$ is a transitive group of degree 13. From the known list of 2-transitive groups the only 2-transitive non-soluble groups have orders divisible by 9. Thus the result follows.

Note that the same argument applies to $G/K$ and so the second statement holds.

**Lemma 21** *G has odd order.*

*Proof:* We see from the above proof that $G$ has to be soluble with a normal subgroup $N$ of order 169 which is regular on the points. Assume now that $G$ has even order. Since an involution cannot act fixed-point-freely each involution has to fix 13 points. Thus we know that $N$ is not cyclic. So $N$ is a direct product of two minimal normal subgroups of order 13. Each gives rise to a different system of imprimitivity and any line intersects any set of imprimitivity in at most two points. Since the fixed points of an involution all lie in a set of imprimitivity, we see that the an involution fixes at most two points on any line it fixes. So any involution fixes $13 \times 6 = 78$ lines each of which has two fixed points but these contain $6 \times 78$ distinct points, which is far too many.

*Proof of Theorem 6*
We now know that $G$ is soluble of order 507. The non-existence was completed by a computer search.

If we put all the results together we have completed the proof of the main theorem, Theorem 1.

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