TWO CONGRUENCES INVOLVING 4–CORES

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Submitted: February 25, 1995; Accepted: June 30, 1995

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Abstract. The goal of this paper is to prove two new congruences involving 4–cores using elementary techniques; namely, if $a_4(n)$ denotes the number of 4–cores of n, then $a_4(9n+2) \equiv 0 \pmod 2$ and $a_4(9n+8) \equiv 0 \pmod 4$.

1991 *Mathematics Subject Classification.* 05A17, 11P83.

Key words and phrases. Congruence, partitions, t–cores.

*This work was done while the first author was on leave at Université Louis Pasteur, Strasbourg, France.

Section 1. Introduction.

Several papers have recently been published concerning t–cores and congruence properties that they satisfy. For examples of these, see Garvan [1], Kolitsch [4], and Garvan, Kim, and Stanton [2]. Most of the congruence results developed in the above works deal with prime t. In contrast, the goal of this paper is to focus on two congruences satisfied by 4–cores.

Before looking at the congruences themselves, a brief word is in order concerning the definition of a t–core. Given the partition π of the integer n, we say that π is a t–core if its Ferrers graph does not contain a hook of length t. See James and Kerber [3] for a fuller discussion of the definition of t–cores.

We will focus all attention in this paper on the case $t = 4$. The generating function for the number of 4–cores of n, denoted by $a_4(n)$, is given by

$$\sum_{n\geq 0} a_4(n)\, q^n = \frac{(q^4;q^4)_\infty^4}{(q\,;q)_\infty}$$

where

$$(a\,;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)\dots$$

Section 2. Congruences Modulo 2.

In this section, our goal is to prove the following:

Theorem 1. *For all $n \geq 0$,*

$$a_4(9n+2) \equiv 0 \pmod 2, \tag{1}$$

and

$$a_4(9n+8) \equiv 0 \pmod 2. \tag{2}$$

Proof. Note the following:

$$\begin{aligned}
\sum_{n\geq 0} a_4(n)\, q^n &= \frac{(q^4;q^4)_\infty^4}{(q\,;q)_\infty} \\
&= \frac{(q^4;q^4)_\infty (q^4;q^4)_\infty^3}{(q\,;q)_\infty} \\
&= (q\,;q)_\infty^3 (q^4;q^4)_\infty^3 \cdot \frac{(q^4;q^4)_\infty}{(q\,;q)_\infty^4} \\
&\equiv (q\,;q)_\infty^3 (q^4;q^4)_\infty^3 \pmod 2 \\
&= \left(\sum_{n\geq 0} (-1)^n (2n+1) q^{\binom{n+1}{2}}\right)\left(\sum_{m\geq 0} (-1)^m (2m+1) q^{4\binom{m+1}{2}}\right) \\
&\equiv \sum_{m,n\geq 0} q^{\binom{n+1}{2} + 4\binom{m+1}{2}} \pmod 2.
\end{aligned}$$

Here we have used Jacobi's famous result:

$$(q\,;q)_\infty^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{\binom{n+1}{2}}$$

Our theorem is now proven provided we show that all coefficients of the terms of the form q^{9n+2} and q^{9n+8} in the double sum above are divisible by 2.

In order to get a contribution to the term q^{9n+2}, we must have

$$\binom{n+1}{2} + 4\binom{m+1}{2} \equiv 2 \pmod 9.$$

However, note that $\binom{n+1}{2} \equiv 0, 1, 3$ or $6 \pmod 9$ for every $n \in \mathbb{Z}$. Hence, $\binom{n+1}{2} + 4\binom{m+1}{2} \equiv 0, 1, 3, 4, 5, 6,$ or $7 \pmod 9$, but never 2. Thus, there is no contribution to q^{9n+2}. This proves (1). Exactly the same approach proves (2); the fact needed to prove (2) is that 8 is not in the above list. ■

Section 3. A Congruence Modulo 4.

Congruence (1) is "best possible" in the sense that there are some values of $a_4(9n+2)$ which are even but not divisible by 4. The earliest example is $a_4(2)$, which equals 2. However, (2) is not best possible. In fact, the goal of this section is to prove the following strengthening of (2):

Theorem 2. *For all $n \geq 0$,*

$$a_4(9n+8) \equiv 0 \pmod 4. \tag{3}$$

Proof. We have

$$\begin{aligned}
\sum_{n\geq 0} a_4(n) q^{8n+5} &= q^5 \frac{(q^{32};q^{32})_\infty^4}{(q^8;q^8)_\infty} \\
&= q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}\left(q^2\frac{(q^{32};q^{32})_\infty^2}{(q^{16};q^{16})_\infty}\right)^2 \\
&= \sum_{\substack{n \text{ odd,} \\ \text{positive}}} q^{n^2} \left(\sum_{\substack{n \text{ odd,} \\ \text{positive}}} q^{2n^2} \right)^2 \\
&= \sum_{\substack{k,l,m \text{ odd,} \\ \text{positive}}} q^{k^2+2l^2+2m^2}
\end{aligned}$$

So we see that $a_4(n)$ equals the number of solutions of the equation $8n+5 = k^2+2l^2+2m^2$ with k, l, m odd and positive. This was noted by Ono [5], who used it to show that $a_4(n)$ is positive for all n. In particular, $a_4(9n+8)$ is the number of solutions of

$$72n + 69 = k^2 + 2l^2 + 2m^2. \tag{4}$$

We want to show that the number of solutions of (4) is divisible by 4.

As a quick example, note that if $n = 0$ in (4), we see that

$$\begin{aligned}
69 &= 1^2 + 2\times 3^2 + 2\times 5^2 = 1^2 + 2\times 5^2 + 2\times 3^2 \\
&= 7^2 + 2\times 3^2 + 2\times 1^2 = 7^2 + 2\times 1^2 + 2\times 3^2
\end{aligned}$$

and the number of solutions is 4.

Now, consider equation (4). Modulo 6 this becomes

$$k^2 + 2l^2 + 2m^2 \equiv 3 \pmod 6.$$

An odd square is 1 or 3 $\pmod 6$. From the tables

$k^2 \equiv 1 \pmod 6$:

l^2 :	1	1	3	3
m^2 :	1	3	1	3
$k^2+2l^2+2m^2$:	5	3	3	1

$k^2 \equiv 3 \pmod 6$:

l^2 :	1	1	3	3
m^2 :	1	3	1	3
$k^2+2l^2+2m^2$:	1	5	5	3

we see that the only solutions of $k^2+2l^2+2m^2 \equiv 3 \pmod 6$ are $(k^2, l^2, m^2) \equiv (1,1,3), (1,3,1)$ or $(3,3,3) \pmod 6$, that is, $(k,l,m) = (\pm 1, \pm 1, 3), (\pm 1, 3, \pm 1)$, or $(3,3,3) \pmod 6$.

However, if $(k,l,m) \equiv (3,3,3) \pmod 6$, then $(k^2, l^2, m^2) \equiv (9,9,9) \pmod{72}$ [since $(6k+3)^2 = 72\binom{k+1}{2} + 9$], and then $k^2+2l^2+2m^2 \equiv 45 \not\equiv 69 \pmod{72}$.

So, in (4), just one of l and m is $3 \pmod 6$. Suppose it is m. We have $m \equiv 3 \pmod 6$, $m^2 \equiv 9 \pmod{72}$, and

$$k^2 + 2l^2 = 72n + 69 - 2m^2 \equiv 51 \pmod{72}.$$

We will show that for each $m \equiv 3 \pmod 6$, the number of solutions is even. By the symmetry between l and m in (4), the total number of solutions will be divisible by 4.

We want to show that the number of solutions of

$$k^2 + 2l^2 = 72n + 51 \tag{5}$$

with k, l odd and positive is even. Allowing k, l to be positive or negative, we want to show that the number of solutions is divisible by 8. For example, if $n = 0$ in (5) above, we have

$$\begin{aligned} 51 &= (\pm 1)^2 + 2 \times (\pm 5)^2 \\ &= (\pm 7)^2 + 2 \times (\pm 1)^2, \end{aligned}$$

yielding 8 solutions.

Each solution of (5) with k, l positive gives rise to a family of four solutions $(\pm k, \pm l)$. Modulo 18, (5) becomes

$$k^2 + 2l^2 \equiv 15 \pmod{18}.$$

Note the following table, which gives $k^2 + 2l^2 \pmod{18}$:

l \ k	±1	±3	±5	±7	9
±1	3	11	9	15	11
±3	1	9	7	13	9
±5	15	5	3	9	5
±7	9	17	15	3	17
9	1	9	7	13	9

¿From this table, we see that $(k,l) \equiv (\pm1,\pm5), (\pm7,\pm1)$, or $(\pm5,\pm7) \pmod{18}$. Observe that each family includes just one solution with $(k,l) \equiv (1,1) \pmod 6$. We call that solution the representative of the family. Thus, the representative solution has $(k,l) \equiv (1,-5), (7,1)$, or $(-5,7) \pmod{18}$. For example, in

$$\begin{aligned} 51 &= (\pm1)^2 + 2\times(\pm5)^2 \\ &= (\pm7)^2 + 2\times(\pm1)^2, \end{aligned}$$

the family $(k,l) = (\pm1,\pm5)$ is represented by $(1,-5)$ and the family $(k,l) = (\pm7,\pm1)$ by $(7,1)$.

We now show that the 'families of four' come in pairs. Suppose $k^2+2l^2 = 72n+51$ and that $(k,l) \equiv (1,-5), (7,1)$ or $(-5,7) \pmod{18}$. Set

$$k' = \frac{k-4l}{3},\ l' = -\frac{2k+l}{3}.$$

Then, $(k',l') \equiv (1,1) \pmod 6$. That is, k' and l' are odd integers and

$$(k')^2 + 2(l')^2 = \left(\frac{k-4l}{3}\right)^2 + 2\left(-\frac{2k+l}{3}\right)^2 = k^2+2l^2 = 72n+51.$$

Note that (k',l') belongs to a different family from (k,l). First, $k' \neq -k$ since $k' \equiv k \equiv 1 \pmod 6$. Also, $k' \neq k$ since, if $k'=k$, $k=-2l$, and then

$$72n+51 = k^2+2l^2 = 6l^2 \equiv 0 \pmod 6,$$

which is false. Moreover, (k',l') is the representative of that family.

Therefore, since the transformation

$$\begin{pmatrix} k \\ l \end{pmatrix} \to \begin{pmatrix} k' \\ l' \end{pmatrix} = \begin{pmatrix} 1/3 & -4/3 \\ -2/3 & -1/3 \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}$$

is its own inverse, we know the families come in pairs. This completes the proof of (3). ■

Section 4. Concluding Remarks.

The referee has made the following observation. We showed in the proof of Theorem 1 that $a_4(n)$ is congruent modulo 2 to the number of representations of n in the form

$$n = \binom{l+1}{2} + 4\binom{m+1}{2}$$

or,

$$8n + 5 = (2l+1)^2 + (4m+2)^2$$

with $l, m \geq 0$.
It follows that

$$a_4(n) \equiv \frac{r_2(8n+5)}{8} \pmod 2$$

where $r_2(n)$ is the number of representations of n as a sum of two squares.
As is well-known, $r_2(n)$ is given for $n > 0$ by

$$r_2(n) = 4(d_1(n) - d_3(n))$$

where $d_i(n)$ is the number of divisors of n which are congruent to i modulo 4.
Thus $a_4(n)$ is even precisely when $d_1(8n+5) - d_3(8n+5)$ is divisible by 4, which gives the following

Theorem 3. $a_4(n)$ is even if and only if at least one of the following holds:

(α) $8n+5$ has a prime divisor $p \equiv 3 (\text{mod } 4)$ with $ord_p(8n+5)$ odd,

(β) $8n+5$ has a prime divisor $p \equiv 1 (\text{mod } 4)$ with $ord_p(8n+5) \equiv 3 (\text{mod } 4)$,

(γ) $8n+5$ has two prime divisors $p_1, p_2 \equiv 1 (\text{mod } 4)$ with $ord_{p_1}(8n+5)$ and $ord_{p_2}(8n+5)$ both odd.

(Here, $ord_p(n)$ is the highest power of p which divides n.)

In particular, if $n \equiv 2$ or $8 \pmod 9$ then $ord_3(8n+5) = 1$ is odd, and $a_4(n)$ is even.

We believe that $a_4(n) \equiv 0 \pmod 4$ for n in other arithmetic progressions, but we do not yet have proofs of these congruences.

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