Perfect matchings in ϵ-regular graphs

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Abstract

A super (d,ϵ)-regular graph on $2n$ vertices is a bipartite graph on the classes of vertices V_1 and V_2, where $|V_1| = |V_2| = n$, in which the minimum degree and the maximum degree are between $(d-\epsilon)n$ and $(d+\epsilon)n$, and for every $U \subset V_1, W \subset V_2$ with $|U| \geq \epsilon n$, $|W| \geq \epsilon n$, $|\frac{e(U,W)}{|U||W|} - \frac{e(V_1,V_2)}{|V_1||V_2|}| < \epsilon$. We prove that for every $1 > d > 2\epsilon > 0$ and $n > n_0(\epsilon)$, the number of perfect matchings in any such graph is at least $(d-2\epsilon)^n n!$ and at most $(d+2\epsilon)^n n!$. The proof relies on the validity of two well known conjectures for permanents; the Minc conjecture, proved by Brégman, and the van der Waerden conjecture, proved by Falikman and Egorichev.

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An *ϵ-regular graph* on $2n$ vertices is a bipartite graph on the classes of vertices V_1 and V_2, where $|V_1| = |V_2| = n$, in which for every $U \subset V_1, W \subset V_2$ with $|U| \geq \epsilon n$, $|W| \geq \epsilon n$,

$$\left| \frac{e(U,W)}{|U||W|} - \frac{e(V_1,V_2)}{|V_1||V_2|} \right| < \epsilon, \tag{1}$$

where here $e(X,Y)$ denotes the number of edges between X and Y. The quantity $\frac{e(V_1,V_2)}{|V_1||V_2|}$ is called the *density* of the graph.

Such a graph is a *super (d,ϵ)-regular graph* if, in addition, its minimum degree δ and its maximum degree Δ satisfy

$$(d-\epsilon)n \leq \delta \leq \Delta \leq (d+\epsilon)n.$$

In this note we prove the following result

Theorem 1 *Let G be a super (d,ϵ)-regular graph on $2n$ vertices, where $d > 2\epsilon$ and $n > n_0(\epsilon)$. Then the number $M(G)$ of perfect matchings of G satisfies*

$$(d-2\epsilon)^n n! \leq M(G) \leq (d+2\epsilon)^n n!.$$

Thus, the number of perfect matchings in any super (d,ϵ)-regular graph on $2n$ vertices is close to the expected number of such matchings in a random bipartite graph with edge probability d (which is clearly $d^n n!$). This result is combined with some additional ideas in [7] to derive a new proof of the Blow-Up Lemma of Komlós, Sárközy and Szemerédi.

The upper bound in Theorem 1 is true for all bipartite graphs with maximum degree at most $(d+\epsilon)n$ on at least one side, and is an easy consequence of the Minc conjecture [6] for permanents, proved by Brégman [2] (c.f. also [1] for a probabilistic description of a proof of Schrijver). Indeed, the Minc conjecture states that the permanent of an n by n matrix A with $(0,1)$ entries satisfies

$$per(A) \leq \prod_{i=1}^{n} r_i!^{1/r_i} ,$$

where r_i is the sum of the entries of the i-th row of A. To derive the upper bound in Theorem 1 apply this estimate to the matrix $A = (a_{u,v})_{u \in V_1, v \in V_2}$ in which $a_{u,v} = 1$ if u,v are adjacent and $a_{u,v} = 0$ otherwise. Here $M(G) = per(A)$. Since the function $x!^{1/x}$ is increasing, $M(G) \leq (k!)^{n/k}$, where $k = \lfloor (d+\epsilon)n \rfloor$, and the upper bound follows by applying the Stirling approximation formula for factorials.

It is worth noting that since every ϵ-regular graph with density d and $2n$ vertices contains, in each color class, at most ϵn vertices of degree higher than $(d+\epsilon)n$, some version of the above upper bound is also true for any ϵ-regular graph of density d. Namely, one can show that for every $d > 0$, if ϵ is sufficiently small as a function of d, then for every ϵ-regular graph G on $2n$ vertices with density d we have

$$M(G) < (d+3\epsilon)^n n!$$

provided $n > n_0(\epsilon)$.

To prove the lower bound observe that by the van der Waerden conjecture, proved by Falikman [4] and Egorichev [3], the number of perfect matchings in a bipartite k-regular graph with n vertices in each color class is at least $(k/n)^n n!$. Thus it suffices to show that our graph contains a spanning k-regular subgraph (a k-factor), where $k = \lceil (d-2\epsilon)n \rceil$. This is proved in the next lemma.

Lemma 2 *Let G be a super (d,ϵ)-regular graph on $2n$ vertices, $d > 2\epsilon$. Then G contains a spanning k-factor, where $k = \lceil (d-2\epsilon)n \rceil$.*

In the proof of this lemma we will apply the following criterion for containing a k-factor, which can be found e.g. in [5], page 70, Thm. 2.4.2.

Theorem 3 *Let G be a bipartite graph on $2n$ vertices in the classes V_1 and V_2, where $|V_1| = |V_2| = n$. Then G has a k-factor if and only if for all $X \subseteq V_1$ and $Y \subseteq V_2$*

$$k|X| + k|Y| + e(V_1 - X, V_2 - Y) \geq kn \ .\Box \tag{2}$$

Proof of Lemma 2. We first assume, to simplify the notation and avoid using floor and ceiling signs when these are not crucial, that $(d-2\epsilon)n$ is an integer.

By Theorem 3, all we need is to prove inequality (2). If $|X|+|Y| \geq n$ then the left-hand side of (2) is at least nk, and we are done. Assume, thus, that $|X| + |Y| < n$. Without loss of generality we may and will assume that $|V_1 - X| \geq |V_2 - Y|$. If $|V_2 - Y| < \epsilon n$, then, since $|X| + |Y| < n$, it follows that $|X| < |V_2 - Y| < \epsilon n$ and thus every vertex of $V_2 - Y$ has at least $\delta - |X| > (d-2\epsilon)n = k$ neighbors in $V_1 - X$, implying that $e(V_1 - X, V_2 - Y) \geq (n - |Y|)k$, and showing that the left-hand side of (2) is at least $k|X| + k|Y| + k(n - |Y|) \geq kn$, as needed. Otherwise, $|V_1 - X| \geq |V_2 - Y| \geq \epsilon n$, and thus, by the ϵ-regularity assumption and the obvious fact that $e(V_1, V_2)/(|V_1||V_2|) \geq d-\epsilon$, it follows that $e(V_1-X, V_2-Y) > (d-2\epsilon)(n-|X|)(n-|Y|)$. Therefore, the left-hand side of (2) is at least

$$k|X| + k|Y| + e(V_1 - X, V_2 - Y) \geq (d-2\epsilon)(n|X| + n|Y| + (n-|X|)(n-|Y|))$$

$$= (d-2\epsilon)(n^2 + |X||Y|) \geq (d-2\epsilon)n^2 = kn.$$

This completes the proof. $\Box$

Remark: Note that in the last proof the assumption (1) may be relaxed, as we only used the fact that for every $U \subset V_1, W \subset V_2$, of cardinality at least ϵn each, $\frac{e(W,U)}{|W||U|} \geq \frac{e(V_1,V_2)}{|V_1||V_2|} - \epsilon$. For the lower bound in Theorem 1 the assumption about the maximum degree of G as well as the assumption that n is sufficiently large as a function of ϵ can also be omitted.

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