Bijective Recurrences concerning Schröder paths

Robert A. Sulanke
Boise State University
Boise, Idaho, USA
sulanke@math.idbsu.edu

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Abstract

Consider lattice paths in $\mathbf{Z}^2$ with three step types: the *up diagonal* $(1,1)$, the *down diagonal* $(1,-1)$, and the *double horizontal* $(2,0)$. For $n \geq 1$, let S_n denote the set of such paths running from $(0,0)$ to $(2n,0)$ and remaining strictly above the x-axis except initially and terminally. It is well known that the cardinalities, $r_n = |S_n|$, are the large Schröder numbers. We use lattice paths to interpret bijectively the recurrence $(n+1)r_{n+1} = 3(2n-1)r_n - (n-2)r_{n-1}$, for $n \geq 2$, with $r_1 = 1$ and $r_2 = 2$.

We then use the bijective scheme to prove a result of Kreweras that the sum of the areas of the regions lying under the paths of S_n and above the x-axis, denoted by AS_n, satisfies $AS_{n+1} = 6AS_n - AS_{n-1}$, for $n \geq 2$, with $AS_1 = 1$, and $AS_2 = 7$. Hence $AS_n = 1, 7, 41, 239, 1393, \ldots$. The bijective scheme yields analogous recurrences for elevated Catalan paths.

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1 The paths and the recurrences

We will consider lattice paths in $\mathbf{Z}^2$ whose permitted step types are the *up diagonal* $(1,1)$ denoted by U, the *down diagonal* $(1,-1)$ denoted by D, and the *double horizontal* $(2,0)$ denoted by H. We will focus on paths that run from $(0,0)$ to $(2n,0)$, for $n \geq 1$, and that never touch or pass below the x-axis except initially and terminally. Let C_n denote the set of such paths when only U-steps and D-steps are allowed, and let S_n denote the set of such paths when all three types are allowed. It is well known that the cardinalities $c_n = |C_n|$ and $r_n = |S_n|$, for $n \geq 1$, are the Catalan numbers and the large Schröder numbers, respectively. (See Section 4, particularly Notes 2 and 4.) Hence, here one might view the elements of S_n as *elevated Schröder paths*. Let AC_n denote the sum of the areas of the regions lying under the paths of C_n and above the x-axis. Likewise, let AS_n denote the sum of the areas of the regions lying under the paths of S_n and above the x-axis.

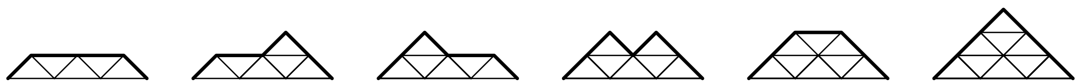


Figure 1: The 6 elevated Schröder paths of S_3 bound 41 triangles of unit area.

n	1	2	3	4	5	…
c_n	1	1	2	5	14	…
r_n	1	2	6	22	90	…
AC_n	1	4	16	64	256	…
AS_n	1	7	41	239	1393	…

The Catalan numbers and the Schröder numbers have been studied extensively; Section 4 references some studies related to lattice paths. In our notation their explicit formulas are, for $n \geq 1$,

$$c_n = \frac{1}{n}\binom{2n-2}{n-1} \quad \text{and} \quad r_n = \sum_{k=1}^{n} \frac{1}{k}\binom{n-2}{k-1}\binom{n-1}{k-1}2^k.$$

It is known that these sequences satisfy the recurrences

$$(n+1)c_{n+1} = 2(2n-1)c_n \tag{1}$$
$$(n+1)r_{n+1} = 3(2n-1)r_n - (n-2)r_{n-1} \tag{2}$$

for $n \geq 2$, with initial conditions $c_1 = 1$, $c_2 = 1$, $r_1 = 1$, and $r_2 = 2$.

We will give a bijective proof for (1) and (2) when the sequences c_n and r_n are defined in terms of the sets of lattice paths. We will then employ this bijective construction to obtain a combinatorial interpretation that the sequences for the total areas satisfy

$$AC_{n+1} = 4AC_n \tag{3}$$
$$AS_{n+1} = 6AS_n - AS_{n-1} \tag{4}$$

for $n \geq 2$ with initial conditions $AC_1 = 1$, $AC_2 = 4$, $AS_1 = 1$, and $AS_2 = 7$.

Using binary trees, Rémy [10] gave a combinatorial proof of recurrence (1). Recently, Foata and Zeilberger [3] showed bijectively, using well-weighted binary plane trees, that the small Schröder numbers satisfy (2) with initial conditions $r_1 = 1$ and $r_2 = 1$. (See Section 4 for "well-weighted" and "small".) Kreweras [4], using lattice paths equivalent to those of S_n showed $AS_n = \sum_{0\le k<n} 2^k\binom{2n-1}{2k}$ and derived recurrence (4). Following his results, Bonin, Shapiro, and Simion [2] proved (4) using generating functions and then wrote that "This recurrence cries out for a combinatorial interpretation." Section 3 comes to the rescue.

2 The proof of recurrences (1) and (2)

We will focus on recurrence (2) rearranged as $3(2n-1)r_n = (n+1)r_{n+1} + (n-2)r_{n-1}$. In S_n replicate each path, defined as a sequence of steps, $3(2n-1)$ times as follows: First repeatedly tag each path P by appending the symbol a, b, or c. Next, for each tagged path P, consecutively index its steps, as positioned in P, with the integers 1 through $2n-1$ so that each U-step and each non final D-step receives one integer and each H-step receives two consecutive integers. Then mark each path P by selecting an integer from $\{1,\ldots,2n-1\}$ and marking the corresponding step on P

– by the superscript x if the step is U or if the step is H with odd index, and

– by the superscript y if the step is D or if the step is H with even index.

We write the set of such replications as $\{a,b,c\}\times\{1,\ldots,2n-1\}\times S_n = \{a,b,c\}\times[2n-1]\times S_n$, where, in general, $[n]$ denotes $\{1,\ldots,n\}$. For instance, for $S_2 = \{UUDD, UHD\}$,

$$\begin{aligned}\{a,b,c\}\times[3]\times S_2 = \\ &\{U^xUDDa,\ UU^xDDa,\ UUD^yDa,\ U^xHDa,\ UH^yDa,\ UH^xDa,\\ &\ U^xUDDb,\ UU^xDDb,\ UUD^yDb,\ U^xHDb,\ UH^yDb,\ UH^xDb,\\ &\ U^xUDDc,\ UU^xDDc,\ UUD^yDc,\ U^xHDc,\ UH^yDc,\ UH^xDc\}.\end{aligned}$$

Next in S_{n+1} replicate each path $n+1$ times by sequentially marking one of its U-steps or H-steps by the symbol z. This replicated set is denoted as $[n+1]\times S_{n+1}$. Similarly, in S_{n-1} replicate each path $n-2$ times by sequentially marking one of its H-steps or non final D-steps by the symbol z. This replicated set is denoted as $[n-2]\times S_{n-1}$.

For $n\ge 2$, we now define the desired bijection,

$$f : \{a,b,c\}\times[2n-1]\times S_n \to [n+1]\times S_{n+1} \bigcup [n-2]\times S_{n-1}. \tag{5}$$

Suppose

$$P = p_1\cdots p_i\cdots p_j\cdots p_k\cdots p_m. \tag{6}$$

denotes a typical replicated path in $[2n-1]\times S_n$ for which the following four items hold.

– The positions i, j, and k satisfy $1\le i\le j<k\le m$.

– The step p_j is the step that is marked by x or y.

– The step p_i is the last U-step preceding p_{j+1} for which $\text{LEV}(p_i) = \text{LEV}(p_j)$. Here the *level* of arbitrary step p_ℓ, denoted $\text{LEV}(p_\ell)$, is the ordinate of its final point. When $p_j = U^x$, $i = j$.

– The step p_k is the first D-step after p_j for which $\text{LEV}(p_k) = \text{LEV}(p_j) - 1$.

Case 1a: If $p_j = U^x$, H^x, or D^y, $f(Pa) = p_1 \cdots p_i \cdots p_j U^z D p_{j+1} \cdots p_k \cdots p_m$.

(Here, $f(Pa)$ is obtained by inserting the pair $U^z D$ immediately after p_j. The tags x, y, and a are erased here; the tags b and c are erased in the following cases. If P appears in Fig. 2, $f(Pa)$ appears in Fig. 3. In the figures the dots pertain to an illustration for the proof of the next section.)

Case 1b: If $p_j = U^x$, H^x, or D^y, $f(Pb) = p_1 \cdots p_i^z \cdots p_j URD p_k \cdots p_m$, where $R = p_{j+1} \cdots p_{k-1}$.

(*Observation 1: In the path $Q = q_1 q_2 \cdots q_{m+2} = f(Pb)$ a D-step immediately precedes the D-step q_{k+2}, which is the translation of the step p_k.* The step q_{k+2} is the first step after $q_i^z = p_i^z$ for which $\text{LEV}(q_{k+2}) = \text{LEV}(q_i) - 1$; $q_j = p_j$ is now the last step before q_{k+2} such that $\text{LEV}(q_j) = \text{LEV}(q_{k+2}) - 1$. The path R may be empty.)

If P	$UUUDH^xHUDDD$	$UHUH^yUHDDD$	$UUUDH^yHUDDD$
then			
$f(Pa)$	$UUUDH\underline{U^zD}HUDDD$	$UHU\underline{U^z}H\underline{D}UHDDD$	$UUUD\underline{U^z}H\underline{D}UDDD$
$f(Pb)$	$UU^zUDH\underline{U}HUD\underline{D}DD$	$UHU^zHUHD\underline{H}DD$	$UU^zUD\underline{U}HUD\underline{D}UDDD$
$f(Pc)$	$UUUDH\underline{H^z}HUDDD$	$UHU^zUHDH\underline{H}DD$	$UUUD^zHUDDD$

Table 1: This table spells out the examples of the Figures 1 to 9 and 11 to 14. Underlining identifies inserted steps.

Case 1c: If $p_j = U^x$, H^x, or D^y, $f(Pc) = p_1 \cdots p_i \cdots p_j H^z p_{j+1} \cdots p_k \cdots p_m$.

Case 2a: If $p_j = H^y$, $f(Pa) = p_1 \cdots p_{j-1} U^z H D p_{j+1} \cdots p_m$.

Case 2b: If $p_j = H^y$, $f(Pb) = p_1 \cdots p_i^z \cdots p_{j-1} URDH p_k \cdots p_m$, where R is the subpath $p_{j+1} \cdots p_{k-1}$.

(Here a U-step and a D-step are inserted and the step $p_j = H$ is moved. *Observation 2: In the path $Q = f(Pb)$* exactly *one H-step immediately precedes q_{k+2}, the translation of p_k.* The step $q_{j-1} = p_{j-1}$ is now the last step before $q_k q_{k+1} q_{k+2} = DHp_k$ such that $\text{LEV}(q_{j-1}) = \text{LEV}(q_{k+2}) + 1$. R may be empty.)

Case 2c: If $p_{j-1}p_j = UH^y$, $f(Pc) = p_1 \cdots p_i^z \cdots p_{j-1} RHHp_k \cdots p_m$, where R is the subpath $p_{j+1} \cdots p_{k-1}$.

(*Observation 3: In the path $Q = f(Pc)$,* at least two *H-steps immediately precede the D-step q_{k+1}, the translation of p_k.*)

Case 3: If $p_{j-1}p_j = HH^y$ or DH^y, $f(Pc) = p_1 \cdots p_{j-1}^z p_{j+1} \cdots p_m$.

(Here $f(Pc) \in [n-2] \times S_{n-1}$ with the marked H-step being deleted.)

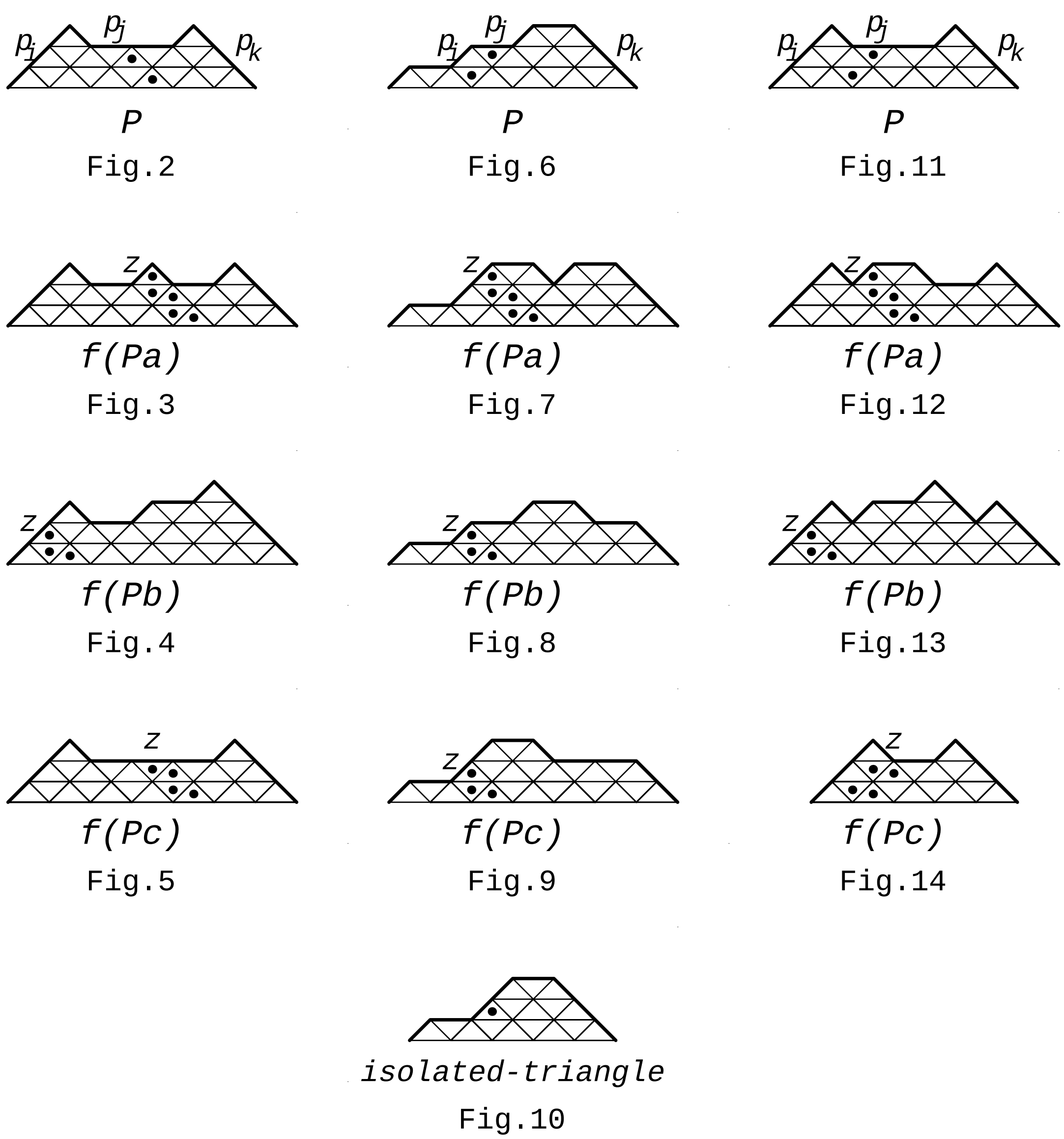
Table 1 and Figures 1 to 9 and 11 to 14 illustrate the map f. By giving special attention to the three *Observations* in the preceding, it is straight forward to check the necessary cases to show that f is a bijection. Assigning cardinalities to the sets in the bijection given in (5) yields the recurrence (2). To prove recurrence (1), simply remove all reference to the H-steps and to the tag c in the proof.

3 The proof of recurrences (3) and (4)

Retaining the previous notions, consider the recurrence (4). One can partition the region under a path and above the x-axis by copies of two isosceles right triangles whose hypotenuses have length 2 and are parallel to the x-axis. Figure 1 illustrates how these triangles of *unit area* uniquely partition the regions under the paths. A triangle is called an *up triangle* if its right angle is above its hypotenuse; otherwise, it is called a *down triangle*.

An *up-triangle-strip* (*down-triangle-strip*, respectively) under a path of S_n is a maximal array of up (down, respectively) triangles having the centers of their hypotenuses on a single line of slope -1 (slope 1, respectively). It is easily seen that each path in S_n has n up-triangle-strips and $n-1$ down-triangle-strips. The marked triangles in Figure 2 illustrate an up-triangle-strip; those in Figure 6 illustrate a down-triangle-strip. Each *marked* path $P \in [2n-1] \times S_n$ determines a unique strip under P as follows: If the step p_j is marked by x, then the corresponding strip is the up-triangle-strip whose line of centers of its triangles intersects the step p_j. If p_j is marked by y, then the corresponding strip is the down-triangle-strip whose line of centers of its triangles intersects the step p_j. In either case we designate by $6T_P$ six copies of the strip corresponding to the step p_j.

In the region under any path in S_{n+1} a *contiguous-strip* is a maximal array of up and down triangles having the centers of their hypotenuses on a single line of slope -1. Each marked path $P \in [n+1] \times S_{n+1}$ determines a unique strip under P, namely that contiguous-strip whose line of hypotenuse centers intersects the marked step of P. We designate this strip by TC_P. The marked triangles of Figure 3 indicate a contiguous-strip.



Figures 2 through 14.

In the region under any path in S_{n-1} each down triangle can always be paired with the contiguous up triangle on its right, but not visa-versa. A *diamond-strip* is a maximal array of such pairs of triangles whose common sides lie on a line of slope 1. The marked step of each path $P \in [n-2] \times S_{n-1}$ determines a unique diamond-strip under P, namely the diamond-strip whose line of common sides intersects the final point of the z-marked step of P. We designate this diamond-strip by TD_P. The marked triangles of Figure 14 indicate a diamond-strip.

Under a path $P \in [n-2] \times S_{n-1}$, any up triangle that is not contiguous along its left side with a down triangle is viewed as an *isolated-triangle*. Figure 10 illustrates an isolated-triangle. Since the left side of each isolated-triangle is a U-step of P and conversely, each U-step, say p_h, uniquely matches an isolated-triangle that we designate by $TI_{P,h}$. The disjoint collection of strips and isolated-triangles

$$\left(\bigcup_{P\in[n-2]\times S_{n-1}} \{TD_P\}\right) \bigcup \left(\bigcup_{P\in S_{n-1}} \bigcup_{h\in u(P)} \{TI_{P,h}\}\right)$$

partitions the total region under the paths in S_{n-1}, where $u(P)$ is the set of the positions of the U-steps on P.

To construct a function that yields a combinatorial proof of recurrence (4), consider defining

$$g : \bigcup_{P\in[2n-1]\times S_n} \{6T_P\} \to \mathcal{T} \bigcup \mathcal{Q}$$

where the elements of $\mathcal{T}$ and $\mathcal{Q}$ will be ordered triples and ordered quadruples, respectively, of mutually non overlapping strips partitioning the total region lying under the paths of S_{n+1} and S_{n-1}.

With P being an arbitrary path as in (6), the bijection f induces a function g so that

Case i: if $p_j = U^x$, H^x, or D^y, define

$$g(6T_P) = (TC_{f(Pa)}, TC_{f(Pb)}, TC_{f(Pc)}).$$

Case ii: if $p_{j-1}p_j = p_ip_j = UH^y$, define

$$g(6T_P) = (TC_{f(Pa)}, TC_{f(Pb)}, TC_{f(Pc)}, TI_{R,i}),$$

where $R = p_1 \cdots p_i p_{i+2} \cdots p_m$.

Case iii: if $p_{j-1}p_j = HH^y$ or DH^y, define

$$g(6T_P) = (TC_{f(Pa)}, TC_{f(Pb)}, TD_{f(Pc)}).$$

The mapping of six copies of the strip of Figure 2 to those of Figures 3 to 5 illustrates Case i. Likewise Figure 6 with Figures 7 to 10 illustrates Case ii, and Figure 11 with Figures 12 to 14 illustrates Case iii. Notice that each column of the array of figures shows the 6-fold transfer of area.

The function f being bijective implies g is bijective. The following three items can be routinely checked to show that g transfers area as claimed. Here $\mathrm{A}(T)$ denotes the area of an arbitrary strip T. For

Case i, $(\mathrm{A}(TC_{f(Pa)}), \mathrm{A}(TC_{f(Pb)}), \mathrm{A}(TC_{f(Pc)})) = (2\mathrm{A}(T_P)+1, 2\mathrm{A}(T_P)-1, 2\mathrm{A}(T_P))$;

Case ii, $(\mathrm{A}(TC_{f(Pa)}), \mathrm{A}(TC_{f(Pb)}), \mathrm{A}(TC_{f(Pc)}), \mathrm{A}(TI_{R,j})) = (2\mathrm{A}(T_P)+1, 2\mathrm{A}(T_P)-1, 2\mathrm{A}(T_P)-1, 1)$;

Case iii, $(\mathrm{A}(TC_{f(Pa)}), \mathrm{A}(TC_{f(Pb)}), \mathrm{A}(TD_{f(Pc)})) = (2\mathrm{A}(T_P)+1, 2\mathrm{A}(T_P)-1, 2\mathrm{A}(T_P))$;

Finally to prove recurrence (3) we remove all reference to H-steps and the tag c in this proof.

4 Notes

1. The following corollary of the construction of the function f originated as a fortuitous observation resulting in the definition for the crucial Case 3:

For $n \geq 2$, there are $(n-2)r_{n-1}$ step pairs of the form DH or HH on the totality of paths of S_n.

2. One of the more interesting of the many references to the Catalan numbers is Stanley's [15] collection of 66 combinatorial interpretations of these numbers. His book [15] lists other primary references in the vast literature for these numbers.

3. In lieu of the three step types employed in this paper, the step types $(0,1)$, $(1,0)$, and $(1,1)$ are the usual step types defining Schröder paths. For the latter three types, clearly the Schröder number r_n counts the paths running from $(0,0)$ to $(n-1,n-1)$ and never passing below the line $y=x$. In an early paper on paths with such step types Moser and Zayachkowski [7], realizing that the number of unrestricted paths from $(0,0)$ to (n,n) is a Legendre polynomial evaluated at 3, used a recurrence for these polynomials to derive essentially recurrence (2).

4. We use "r_n" for the large (or *double* as in [4]) Schröder numbers since $s_n = r_n/2$ for $n \geq 2$ with $s_1 = 1$ is reserved for the so-called *small Schröder numbers*: $1, 1, 3, 11, 45, \ldots$. Ernst Schröder formulated these numbers in the second problem of his 1870 paper [14]: In how many ways can one or more pairs of brackets be legally placed in $z_1, z_2 \cdots z_n$? For instance, when $n=3$, there are the three bracketings, $(z_1z_2z_3)$, $((z_1z_2)z_3)$, and $(z_1(z_2z_3))$. The problem of enumerating bracketings is equivalent both to the problem of enumerating dissections of convex polygons and to the problem of enumerating Schröder trees with a fixed number of leaves. (A Schröder tree is a plane trees whose internal nodes have at least two children.)

As noted in [16], David Hough discovered that the small Schröder numbers were apparently known to Hipparchus in the second century B.C. as counting certain logical propositions. The papers [2, 9, 12, 11, 13, 16, 17] form a selection of the studies concerning the Schröder numbers. Of particular interest is the result of Rogers and Shapiro, appearing implicitly in [12, 13], and later the result of Bonin, Shapiro, and

Simion [2], that give combinatorial maps relating the enumeration of bracketings to the enumeration of lattice paths of S_n.

5. A *well-weighted binary plane tree* is a binary tree where each node having a right internal child is labeled with a 1 or a 2. Foata and Zeilberger [3], after giving a rather simple bijection between Schröder trees and well-weighted binary plane trees, showed bijectively that the small Schröder numbers satisfy $(n+1)s_{n+1} = 3(2n-1)s_n - (n-2)s_{n-1}$ with the conditions $s_1 = 1$, and $s_2 = 1$. Their proof is not isomorphic to our proof of (2); this is not surprising since the bijections mentioned at the end of Note 3 between bracketings and S_n do not seem to be trivial.

6. In this and the next note define sequence AS_n purely as the one satisfying the formal recurrence (4), not specifically in terms of lattice paths. Barcucci, Brunetti, Del Lungo, and Del Rietoro [1] recently gave a combinatorial interpretation of formal recurrence (4) in terms of a regular language. In [5] the sequence AS_n is related to solutions of the diophantine equation, $x^2 + (x+1)^2 = y^2$, with $x = (AS_n - 1)/2$. Newman, Shanks, and Williams [8] found that the numbers AS_n correspond to the orders of certain simple groups.

7. The author [18] has considered the formal recurrences (1) to (4) bijectively in terms of parallelogram polyominoes. For $n \geq 2$, let $p_{\alpha,n}(w) = \sum_k p_{\alpha,n,k} w^k$, where $p_{0,n,k}$ denotes the number of parallelogram polyominoes with perimeter $2n$ and width k, and where $(n-1)p_{1,n,k}$ denotes the total area of such polyominoes. The paper [18] shows that the sequences $p_{0,n}(w)$ and $p_{1,n}(w)$ satisfy the recurrences

$$(n+1-\alpha)p_{\alpha,n+1}(w) = (2n-1-\alpha)(1+w)p_{\alpha,n}(w) - (n-2)(1-w)^2 p_{\alpha,n-1}(w),$$

with initial conditions $p_{\alpha,2}(w) = w$, $p_{\alpha,3}(w) = w + w^2$. The proof for the case $\alpha = 0$ in [18] is isomorphic to the proof in [3], but not to the proof of Section 2.

More specifically, the total area $(n-1)p_{1,n}(1)$ is 4^{n-2}; this result was recently derived by interesting generating-function argument by Woan, Shapiro, and Rogers [19]. The product $(n-1)p_{1,n}(2)$, corresponding to the sum of the areas of polyominoes having bi-colored columns, satisfies the recurrence $np_{1,n+1}(2) = 6(n-1)p_{1,n}(2) - (n-2)p_{1,n-1}(2)$ with early values $(n-1)p_{1,n}(2) = 1, 6, 35, 204, 1189, \ldots$, for $n = 2, 3, 4, 5, 6 \ldots$. These polynomial sequences, $p_{\alpha,n}(w)$, generalize other well-known sequences: e.g., $\{p_{0,n}(1)\}_{n\geq 2}$ are the Catalan numbers, $\{p_{0,n}(2)\}_{n\geq 2}$ are the large Schröder numbers, $\{((n-1)p_{1,n}(2)/2)^2\}_{n\geq 2}$ are the square-triangular numbers, and $\{p_{2,n}(1)\}$ are the central binomial coefficients.

8. Recently, Merlini, Sprugnoli, and Verri [6] used generating function methods in determining the sum of the areas bounded by constrained lattice paths belonging to sets that essentially generalize C_n and S_n. The paper [6] also contains additional relevant references to the literature.

References

[1] E. Barcucci, S. Brunetti, A. Del Lungo, and F. Del Rietoro, A combinatorial interpretation of the recurrence $f_{n+1} = 6f_n - f_{n-1}$, *Discrete Math.*, 190 (1998) 235-240.

[2] J. Bonin, L. Shapiro, and R. Simion, Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, *J. Statistical Planning and Inference* 34 (1993) 35-55.

[3] D. Foata and D. Zeilberger, A classic proof of a recurrence for a very classical sequence, *J. Comb. Theor., Ser. A.*, 80 (1997), 380-384.
Available at http://www.math.temple.edu/~zeilberg/papers1.html.

[4] G. Kreweras, Aires des chemins surdiagonaux a étapes obliques permises. *Cahier du B.U.R.O.* 24 (1976) 9-18.

[5] A. Martin, Diophantine analysis (Solutions of Problems), *Am. Math. Monthly*, 4 (1897) 24-25.

[6] D. Merlini, R. Sprugnoli, and M. C. Verri, The area determined by underdiagonal lattice paths, *Discrete Math.,* to appear.

[7] L. Moser and W. Zayachkowski, Lattice paths with diagonal steps, *Scripta Math.*, 26 (1963) 223-229.

[8] M. Newman, D. Shanks and H. C. Williams, Simple groups of square order and an interesting sequence of primes, *Acta Aritmetica* XXXVIII (1980) 129-140.

[9] E. Pergola and R. A. Sulanke, Schröder triangles, paths, and parallelogram polyominoes, *Journal of Integer Sequences*, Vol. 1 (1998), Available at http://www.research.att.com/~njas/sequences/JIS/

[10] J-L Rémy, Un procédé itératif de dénombrement d'arbres binaires et son application à leur génération aléatpoire, *RAIRO Inform. Théor.,* vol 19 (1985) 179 - 195.

[11] D. G. Rogers, A Schröder triangle, *Combinatorial Mathematics V: Proceedings of the Fifth Australian Conference. Lecture Notes in Mathematics,* vol 622, Springer-Verlag, Berlin (1977) 175-196.

[12] D. G. Rogers, The enumeration of a family of ladder graphs part II: Schröder and superconnective relations, *Quart. J. Math. Oxford* (2) 31 (1980) 491-506.

[13] D. G. Rogers and L. Shapiro, Some correspondences involving the Schröder numbers, *Combinatorial Mathematics: Proceedings of International Conference, Canberra, 1977. Lecture Notes in Math.* 686, Springer-Verlag (1978) 267-276.

[14] E. Schröder, Vier kombinatorische probleme, *Z. Math. Phys.* 15 (1870) 361 - 376.

[15] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999 (tentative)

[16] R. P. Stanley, Hipparchus, Plutarch, Schröder, Hough, *Am. Math. Monthly*, 104 (1997) 344 - 350.

[17] R. A. Sulanke, A recurrence restricted by a diagonal condition: generalized Catalan array, *Fibonacci Q.*, 27 (1989) 33 - 46.

[18] R. A. Sulanke, Three recurrences for parallelogram polyominoes, *J. of Difference Eq. and its Appl.*, (1998 tentative)
Available at http://diamond.idbsu.edu/~sulanke/recentpapindex.html

[19] W-J Woan, L. Shapiro, and D. G. Rogers, The Catalan numbers, the Lebesgue integral and 4^{n-2}, *Am. Math. Monthly*, 104 (1997) 926 - 931.