An Asymptotic Expansion for the Number of Permutations with a Certain Number of Inversions

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Abstract

Let $b(n,k)$ denote the number of permutations of $\{1,\ldots,n\}$ with precisely k inversions. We represent $b(n,k)$ as a real trigonometric integral and then use the method of Laplace to give a complete asymptotic expansion of the integral. Among the consequences, we have a complete asymptotic expansion for $b(n,k)/n!$ for a range of k including the maximum of the $b(n,k)/n!$.

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A permutation $\sigma = (\sigma(1),\ldots,\sigma(n))$ of $[n] = \{1,\ldots,n\}$ has an **inversion at** (i,j), where $1 \leq i < j \leq n$, if and only if $\sigma(i) > \sigma(j)$. Let $b(n,k)$ denote the number of permutations of $[n]$ with precisely k inversions. Then $b(n,k) = b(n,\binom{n}{2} - k)$ for all integers k, while, $b(n,k) \neq 0$ if and only if $0 \leq k \leq \binom{n}{2}$. Bender [2; p. 110] showed that the $b(n,k)$ are log concave in k. Hence, the maximum $B(n)$ of the $b(n,k)$ occurs at $k = \lfloor \binom{n}{2}/2 \rfloor$, as well as $\lceil \binom{n}{2}/2 \rceil$ for odd $\binom{n}{2}$. See [3; pps. 236–240] for further results.

Random permutations show (see [3; pps. 282–283], for example) that the $b(n,k)$ satisfy a central limit theorem with $\mu_n = \binom{n}{2}/2$ and $\sigma_n^2 = n(n-1)(2n+5)/72$ (see [2; Theorem 1]). Bender [2; p. 109] remarks that "the theorems of Section 4 do not apply" to the $b(n,k)$. He then shows [2; p. 110] that the $b(n,k)$ are log concave in k "so that Lemma 2 applies." This will give a (first term) asymptotic formula for $b(n,k)/n!$ when $k = \lfloor \mu_n + x\sigma_n \rfloor$ where x is a fixed real number.

In this paper, we represent $b(n,k)$ as a real trigonometric integral. We then use the method of Laplace to give a complete asymptotic expansion of this integral in terms of the Bernoulli numbers and Hermite polynomials. Hence, we have the complete asymptotic expansion

$$\frac{b(n,k)}{n!} = 6(2\pi)^{-1/2}n^{-3/2}\mathrm{e}^{-x^2/2}\left\{1+\sum_{q=1}^{2m-2}(-2)^{-q}S_{2q}(n)H_{2q}(2^{-1/2}x)\right\}$$
$$+O\left(\frac{\ln^{2m^2+1} n}{n^{m+3/2}}\right) \quad \text{as } n\to\infty\,, \tag{1}$$

when $2k = \binom{n}{2} \pm xn^{3/2}/3$ where $x^2 = x^2(n) \le \ln n$ and m is a fixed integer at least 2. Here, H_{2q} are the Hermite polynomials defined before Theorem 1 and the S_{2q} are defined in Theorem 3. In particular, we have a complete asymptotic expansion for $B(n)/n!$ when $\binom{n}{2}$ is even. See Corollaries 2, 4 for other asymptotic expansions.

In what follows, k, ℓ and n are integers with $0 \le k \le \binom{n}{2}$ and $2 \le \ell \le n$. We denote the nonnegative integers by $\mathbb{N}$. All asymptotic formulas are for $n \to \infty$.

Muir [5] (see also [3; p. 239]) showed that $b(n,k)$ is the coefficient of z^k in $\prod_{\ell=2}^{n}(1+z+\cdots+z^{\ell-1})$. Then,

$$\begin{aligned} b(n,k) &= \frac{1}{2\pi i}\oint_C \frac{\prod_{\ell=2}^{n}(1+z+\cdots+z^{\ell-1})}{z^{k+1}}\,dz \\ &= \frac{1}{2\pi i}\oint_C z^{-k-1}\prod_{\ell=2}^{n}\left(\frac{z^\ell-1}{z-1}\right)\,dz, \end{aligned}$$

where C is the unit circle. Hence,

$$b(n,k) = \frac{2n!}{\pi}\int_0^{\pi/2}\prod_{\ell=2}^{n}\left(\frac{\sin \ell t}{\ell \sin t}\right)\cos\left(\left(\binom{n}{2}-2k\right)t\right)\,dt, \tag{2}$$

upon parameterizing C ($z = \mathrm{e}^{it}; t\in[0,2\pi]$) and using the symmetry of the integrand.

For an integer $n \ge 2$ and real numbers a, b and x, let

$$I(n,x,a,b) := \int_a^b \prod_{\ell=2}^{n}\left(\frac{\sin \ell t}{\ell \sin t}\right)\cos\left(\frac{xtn^{3/2}}{3}\right)\,dt$$

and

$$I(n,x) := I\left(n,x,0,\frac{\pi}{2}\right)$$

(where all discontinuities of the integrand have been removed). Then (2) gives

$$\frac{b(n,k)}{n!} = \frac{2}{\pi}I(n,x), \tag{3}$$

for all integers k,n where $0 \le k \le \binom{n}{2}$, $n \ge 2$ and $2k = \binom{n}{2} \pm xn^{3/2}/3$.

For a nonnegative integer q and real number x, let

$$F_q(x) := \int_0^\infty \exp(-u^2/2)u^q \cos(ux)\,du$$

denote the Fourier cosine transform of $\exp(-u^2/2)u^q$. Then $F_{2q}(x) = (-1)^q\pi^{1/2}2^{-q-1/2}\mathrm{e}^{-x^2/2}H_{2q}(2^{-1/2}x)$. Here $H_n(x)$ are the Hermite polynomials given by $H_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k n!(2x)^{n-2k}/k!(n-2k)!$ (see [4; pps. 60-64]).

We use the following Taylor series approximations which are valid for all real numbers t.

$$\sin t = t - \frac{t^3}{6} + a(t); \quad |a(t)| \le \frac{t^4}{24} \text{ for all real } t \text{ and } a(t) \ge 0 \text{ for } t \in [0,\pi]; \tag{4}$$

$$\cos t = 1 - \frac{t^2}{2} + b(t); \quad 0 \le b(t) \le t^3 \text{ for } t \in [0,\pi]; \tag{5}$$

and for an integer $m \ge 1$,

$$\mathrm{e}^{\,t} = 1 + t + \cdots + \frac{t^{m-1}}{(m-1)!} + c_m(t); \quad |c_m(t)| \le \mathrm{e}^{\,|t|}\,|t|^m\,. \tag{6}$$

Of course, our error terms a, b and c_m are all infinitely-differentiable functions over the reals. We also require the following inequality (integration by parts). For a real number $x > 0$,

$$\int_x^\infty \mathrm{e}^{\,-t^2/2}\,dt \le \frac{1}{x}\,\mathrm{e}^{\,-x^2/2}. \tag{7}$$

We now give our first result.

Theorem 1. *For $x^2 = x^2(n) \le \ln n$, we have the asymptotic expansion*

$$I(n,x) = 3\left(\frac{\pi}{2}\right)^{1/2} n^{-3/2} e^{-x^2/2}\Big\{1 - \frac{1}{100n}(9x^4 - 129x^2 + 102) + \frac{1}{980000n^2}\big(3969x^8 - 141282x^6 + 1340865x^4 - 4579480x^2 + 2259370\big)\Big\} + O\left(\frac{\ln^{19} n}{n^{9/2}}\right) \text{ as } n \to \infty.$$

Proof. We use the method of Laplace. For $0 < a \le 1$ and an integer $\ell \ge 2$, let $M_\ell(a) := \max\{|\sin \ell t/\sin t| : t \in [a,\pi/2]\}$ and $b := \cos a \in (0,1)$. For all integers $\ell \ge 2$, $M_\ell(a) \le b^{\ell-1} + b^{\ell-2} + \cdots + b + 1 \le \min\{\ell, (1-b)^{-1}\}$ by induction on ℓ, while $a^2/3 \le 1-b$. Here,

$$\prod_{\ell=2}^n \left|\frac{\sin \ell t}{\ell \sin t}\right| \le \frac{(1-b)^{-n}}{n!} \le \left(\frac{3\mathrm{e}}{a^2 n}\right)^n,$$

and, hence, for all $n \geq 9$ and all real numbers x,

$$|I(n, x, 3n^{-0.5}, \pi/2)| \leq 2\left(\frac{\mathrm{e}}{3}\right)^n. \tag{8}$$

For all integers ℓ and all real numbers t with $\sin t \neq 0$, (4) gives $\sin \ell t/\ell \sin t = 1 - (\ell^2-1)t^2/6 + d(\ell, t)$ where $|d(\ell, t)| \leq \ell^3 t^3/12$ for $t \in (0, 1]$ and $\ell \geq 2$. Hence,

$$0 < \frac{\sin \ell t}{\ell \sin t} \leq 1 - \frac{\ell^2 t^2}{24} \leq \exp\left(-\frac{\ell^2 t^2}{24}\right) \text{ for } \ell t \in [0, 1] \text{ and } \ell \geq 2. \tag{9}$$

(Naturally, we define $\sin \ell t/\ell \sin t = 1$ when $t = 0$ to remove that discontinuity.) For all $n \geq 144$ and all real numbers x, (9) gives

$$|I(n, x, n^{-0.7}, 3n^{-0.5})| \leq \int_{n^{-0.7}}^{3n^{-0.5}} \prod_{\ell=2}^{\lfloor n^{0.5}/3 \rfloor} \frac{\sin \ell t}{\ell \sin t}\, dt \leq \exp\left(\frac{-n^{0.1}}{4608}\right), \tag{10}$$

$$|I(n, x, n^{-1}, n^{-0.7})| \leq \int_{n^{-1}}^{n^{-0.7}} \prod_{\ell=2}^{\lfloor n^{0.7} \rfloor} \frac{\sin \ell t}{\ell \sin t}\, dt \leq \exp\left(\frac{-n^{0.1}}{576}\right), \tag{11}$$

and

$$|I(n, x, n^{-3/2} \ln n, n^{-1})| \leq \int_{n^{-3/2} \ln n}^{n^{-1}} \prod_{\ell=2}^{n} \frac{\sin \ell t}{\ell \sin t}\, dt \leq \exp\left(-\frac{\ln^2 n}{72}\right). \tag{12}$$

Recall that $\cot t = t^{-1} + \sum_{k=1}^{\infty} (-4)^k B_{2k} t^{2k-1}/(2k)!$, for real t with $0 < |t| < \pi$. Here B_n are the Bernoulli numbers defined by $z/(\mathrm{e}^z - 1) = \sum_{n=0}^{\infty} B_n z^n/n!$ for complex z with $|z| < 2\pi$ (see [3; pps. 48, 88]). Then, $\frac{d}{dt}\{\ln(\sin \ell t/\ell \sin t)\} = \ell \cot \ell t - \cot t = \sum_{k=1}^{\infty} (-4)^k B_{2k} (\ell^{2k} - 1) t^{2k-1}/(2k)!$ for $0 < |\ell t| < \pi$, hence,

$$\ln\left(\frac{\sin \ell t}{\ell \sin t}\right) = \sum_{k=1}^{\infty} (-4)^k B_{2k} (\ell^{2k} - 1) \frac{t^{2k}}{(2k)(2k)!} \text{ for } |\ell t| < \pi. \tag{13}$$

For a nonnegative integer m, $|\ell t| \leq 1$ and $\ell \geq 1$ (see [1; p. 805]),

$$\left| \sum_{k=m+1}^{\infty} (-4)^k B_{2k} (\ell^{2k} - 1) \frac{t^{2k}}{(2k)(2k)!} \right| \leq \ell^{2m+2} t^{2m+2}. \tag{14}$$

For $n \geq 2$ and $\theta_k(n) := \sum_{\ell=2}^{n} (\ell^k - 1)$ (see [3; p. 155]), (13), (14; $m = 3$) and (6; $m = 1$)

give

$$\begin{aligned}
&I(n,x,0,n^{-3/2}\ln n)\\
&=\int_0^{n^{-3/2}\ln n}\exp\left\{-\sum_{\ell=2}^{n}\left(\frac{\ell^2-1}{6}t^2+\frac{\ell^4-1}{180}t^4+\frac{\ell^6-1}{2835}t^6+O(n^8t^8)\right)\right\}\cos\left(\frac{xtn^{3/2}}{3}\right)dt\\
&=\int_0^{n^{-3/2}\ln n}\exp\left\{-\frac{\theta_2(n)t^2}{6}-\frac{\theta_4(n)t^4}{180}-\frac{\theta_6(n)t^6}{2835}+O(n^9t^8)\right\}\cos\left(\frac{xtn^{3/2}}{3}\right)dt\\
&=\int_0^{n^{-3/2}\ln n}\exp\left\{-\frac{\theta_2(n)t^2}{6}-\frac{\theta_4(n)t^4}{180}-\frac{\theta_6(n)t^6}{2835}\right\}\cos\left(\frac{xtn^{3/2}}{3}\right)dt+O\left(\frac{\ln^9 n}{n^{9/2}}\right)\\
&=\frac{3}{n^{3/2}}\int_0^{\ln n/3}\exp\left(-\frac{u^2}{2}\right)\exp\left\{R_2(n)u^2+R_4(n)u^4+R_6(n)u^6\right\}\cos(ux)\,du\\
&\qquad+O\left(\frac{\ln^9 n}{n^{9/2}}\right), \qquad (15)
\end{aligned}$$

upon setting $u=n^{3/2}t/3$, where $R_2(n)=-3/4n+5/4n^2$, $R_4(n)=-9/100n-9/40n^2-3/20n^3+93/200n^5$ and $R_6(n)=-9/245n^2-9/70n^3-9/70n^4+3/70n^6+123/490n^8$. It is readily seen that the error term in (15) is at most $e\,n^{-9/2}\ln^9 n$ for all $n\ge 2$ and all real numbers x. For $0\le u\le \ln n/3$, (6; $m=3$) gives

$$\begin{aligned}
&\exp\left\{R_2(n)u^2+R_4(n)u^4+R_6(n)u^6\right\}\\
&\qquad=1+S_2(n)u^2+S_4(n)u^4+S_6(n)u^6+S_8(n)u^8+O\left(\frac{\ln^{18} n}{n^3}\right), \qquad (16)
\end{aligned}$$

where $S_2(n)=-3/4n+5/4n^2$, $S_4(n)=-9/100n+9/160n^2$, $S_6(n)=603/19600n^2$ and $S_8(n)=81/20000n^2$. Hence, (15) and (16) give

$$\begin{aligned}
&I(n,x,0,n^{-3/2}\ln n)\\
&=\frac{3}{n^{3/2}}\int_0^{\ln n/3}\exp\left(-\frac{u^2}{2}\right)\left\{1+S_2(n)u^2+S_4(n)u^4+S_6(n)u^6+S_8(n)u^8\right.\\
&\qquad\left.+O\left(\frac{\ln^{18} n}{n^3}\right)\right\}\cos(ux)\,du+O\left(\frac{\ln^9 n}{n^{9/2}}\right)\\
&=\frac{3}{n^{3/2}}\int_0^{\ln n/3}\exp\left(-\frac{u^2}{2}\right)\left\{1+S_2(n)u^2+S_4(n)u^4+S_6(n)u^6+S_8(n)u^8\right\}\cos(ux)\,du\\
&\qquad+O\left(\frac{\ln^{19} n}{n^{9/2}}\right)\\
&=\frac{3}{n^{3/2}}\int_0^{\infty}\exp\left(-\frac{u^2}{2}\right)\left\{1+S_2(n)u^2+S_4(n)u^4+S_6(n)u^6+S_8(n)u^8\right\}\cos(ux)\,du\\
&\qquad+O\left(\int_{\ln n/3}^{\infty}\exp\left(-\frac{u^2}{4}\right)du\right)+O\left(\frac{\ln^{19} n}{n^{9/2}}\right)
\end{aligned}$$

$$= \frac{3}{n^{3/2}} \int_0^\infty \exp\left(-\frac{u^2}{2}\right) \left\{1 + S_2(n)u^2 + S_4(n)u^4 + S_6(n)u^6 + S_8(n)u^8\right\} \cos(ux)\, du + O\left(\frac{\ln^{19} n}{n^{9/2}}\right), \tag{17}$$

where the last equation follows from (7). The error term in the first equation holds uniformly for all real numbers x by the comments after (15) and, since $|\cos(ux)| \le 1$, the error term in the second equation holds uniformly for all real numbers x by (16) as does the error term in the third equation involving the integral. Then (8), (10–12) and (17) give

$$I(n,x) = \frac{3}{n^{3/2}} \left\{F_0(x) + S_2(n)F_2(x) + S_4(n)F_4(x) + S_6(n)F_6(x) + S_8(n)F_8(x)\right\} + O\left(\frac{\ln^{19} n}{n^{9/2}}\right), \tag{18}$$

where our error term holds uniformly for all real numbers x. Hence, after simplifying (18) we obtain

$$\begin{aligned} I(n,x) =& 3\left(\frac{\pi}{2}\right)^{1/2} n^{-3/2} \mathrm{e}^{-x^2/2} \Big\{1 - \frac{1}{100n}(9x^4 - 129x^2 + 102) \\ &+ \frac{1}{980000n^2}(3969x^8 - 141282x^6 + 1340865x^4 \\ &- 4579480x^2 + 2259370)\Big\} + O\left(\frac{\ln^{19} n}{n^{9/2}}\right), \end{aligned} \tag{19}$$

where our error term holds uniformly for all real numbers x. Our result follows since, apart from the error term, the smallest term in (19) has order of magnitude at least n^{-4} for $x^2 = x^2(n) \le \ln n$. ■

We note several consequences of Theorem 1.

Corollary 2. *For $x^2 = x^2(n) \le \ln n$, we have the asymptotic expansion*

$$\begin{aligned} \frac{b(n,k)}{n!} =& 6(2\pi)^{-1/2} n^{-3/2} e^{-x^2/2} \Big\{1 - \frac{1}{100n}(9x^4 - 129x^2 + 102) \\ &+ \frac{1}{980000n^2}(3969x^8 - 141282x^6 + 1340865x^4 \\ &- 4579480x^2 + 2259370)\Big\} + O\left(\frac{\ln^{19} n}{n^{9/2}}\right) \quad \text{as } n \to \infty, \end{aligned}$$

when $2k = \binom{n}{2} \pm xn^{3/2}/3$. We also have the asymptotic expansion

$$\frac{b(n,k)}{n!} = 6(2\pi)^{-1/2} n^{-3/2} \left(1 - \frac{51}{50n} + \frac{225937}{98000n^2}\right) + o\left(\frac{1}{n^{7/2}}\right) \quad \text{as } n \to \infty,$$

provided $2k = \binom{n}{2} + o(n^{1/2} \ln^{-3/2} n)$. In particular, $B(n)/n!$ has the same asymptotic expansion.

Proof. The asymptotic expansion for $b(n,k)/n!$ when $2k = \binom{n}{2} \pm xn^{3/2}/3$ where $x^2 = x^2(n) \le \ln n$ follows immediately from (3) and Theorem 1. For all $n \ge \mathrm{e}^{141}$ and all real numbers x, (8) and (10–12) give

$$\left| \int_{n^{-3/2}\ln n}^{\pi/2} \prod_{\ell=2}^{n} \left(\frac{\sin \ell t}{\ell \sin t} \right) \left\{ 1 - \cos\left(\frac{xtn^{3/2}}{3} \right) \right\} dt \right| \le 10 \exp\left(-\frac{\ln^2 n}{72} \right). \tag{20}$$

For an integer $\ell \ge 2$ and all $t \in [0, \pi/2\ell]$, $\sin \ell t / \ell \sin t \in [0,1]$ by induction on ℓ. Then, for all $n \ge 2$ and all $x \in [0, \ln^{-1} n]$, (5) gives

$$\begin{aligned} 0 &\le \int_0^{n^{-3/2}\ln n} \prod_{\ell=2}^{n} \left(\frac{\sin \ell t}{\ell \sin t} \right) \left\{ 1 - \cos\left(\frac{xtn^{3/2}}{3} \right) \right\} dt \\ &\le \int_0^{n^{-3/2}\ln n} \frac{x^2 t^2 n^3}{18}\, dt = \frac{x^2 \ln^3 n}{54 n^{3/2}}. \end{aligned} \tag{21}$$

Hence, for all $n \ge \mathrm{e}^{141}$ and all $x \in [0, \ln^{-1} n]$, (20) and (21) give

$$\bigl| I(n,0) - I(n,x) \bigr| \le \frac{x^2 \ln^3 n}{54 n^{3/2}} + 10 \exp\left(-\frac{\ln^2 n}{72} \right). \tag{22}$$

Assume $\binom{n}{2}$ is even (odd $\binom{n}{2}$ is similar) and $n \ge \mathrm{e}^{141}$. Let $\ell := \lfloor \binom{n}{2}/2 + n^{3/2}/6 \ln n \rfloor$ so that $2\ell = \binom{n}{2} + xn^{3/2}/3$ with $x \in [0, \ln^{-1} n]$. For $\binom{n}{2} \le 2k \le 2\ell$, log concavity of the $b(n,k)$ implies

$$b\left(n, \frac{\binom{n}{2}}{2} \right) \ge b(n,k) \ge b(n,\ell),$$

so that (3) and (22) give

$$\frac{2}{\pi} I(n,0) \ge \frac{b(n,k)}{n!} \ge \frac{2}{\pi} I(n,0) - \frac{x^2 \ln^3 n}{27 \pi n^{3/2}} - \frac{20}{\pi} \exp\left(-\frac{\ln^2 n}{72} \right).$$

Hence, Theorem 1 gives

$$\frac{b(n,k)}{n!} = 6(2\pi)^{-1/2} n^{-3/2} \left(1 - \frac{51}{50n} + \frac{225937}{98000 n^2} \right) + o\left(\frac{1}{n^{7/2}} \right),$$

for $2k = \binom{n}{2} + o\bigl(n^{1/2} \ln^{-3/2} n\bigr)$. ■

Remark. We can replace the $o(n^{-7/2})$ error term in the asymptotic expansion of $B(n)/n!$ with $O(n^{-9/2} \ln^{19} n)$.

The following extension of Theorem 1 (the case $m = 3$) giving a complete asymptotic expansion of $I(n,x)$ can be immediately read out of its proof.

Theorem 3. *Fix an integer $m \geq 2$. For $x^2 = x^2(n) \leq \ln n$, we have the asymptotic expansion*

$$I(n,x) = 3\left(\frac{\pi}{2}\right)^{1/2} n^{-3/2} e^{-x^2/2} \left\{1 + \sum_{q=1}^{2m-2} (-2)^{-q} S_{2q}(n) H_{2q}(2^{-1/2}x)\right\} + O\left(\frac{\ln^{2m^2+1} n}{n^{m+3/2}}\right) \text{ as } n \to \infty.$$

(The $S_{2q}(n)$ are defined in the proof.)

Proof. For $2 \leq \ell \leq n$ and $t \in [0, n^{-1}]$, (13) and (14) give

$$\ln\left(\frac{\sin \ell t}{\ell \sin t}\right) = \sum_{k=1}^{m} c_{2k}(\ell^{2k} - 1)t^{2k} + O(n^{2m+2}t^{2m+2}), \tag{23}$$

where $c_{2k} := (-4)^k B_{2k}/(2k)(2k)! < 0$, while,

$$0 \leq \theta_{2k}(n) = \sum_{\ell=2}^{n} (\ell^{2k} - 1) = \frac{1}{2k+1} \sum_{j=0}^{2k} B_j \binom{2k+1}{j} (n+1)^{2k+1-j} - n\,.$$

Hence, (23) and (6; $m = 1$) give

$$\begin{aligned} &I(n,x,0,n^{-3/2}\ln n) \\ &= \frac{3}{n^{3/2}} \int_0^{\ln n/3} \exp\left\{\sum_{k=1}^{m} 9^k c_{2k}\theta_{2k}(n)u^{2k}n^{-3k}\right\} \cos(ux)\,du + O\left(\frac{\ln^{2m+3} n}{n^{m+3/2}}\right) \\ &= \frac{3}{n^{3/2}} \int_0^{\ln n/3} \exp\left(-\frac{u^2}{2}\right) \exp\left\{R_2(n)u^2 + \cdots + R_{2m}(n)u^{2m}\right\} \cos(ux)\,du \\ &\qquad + O\left(\frac{\ln^{2m+3} n}{n^{m+3/2}}\right), \end{aligned} \tag{24}$$

where $R_2(n) = -3/4n + 5/4n^2$ and, for $2 \leq k \leq m$,

$$R_{2k}(n) := \frac{(-36)^k B_{2k}}{(2k)(2k+1)!} \sum_{j=0}^{2k} B_j \binom{2k+1}{j} n^{-3k}(n+1)^{2k+1-j} - \frac{(-36)^k B_{2k}}{(2k)(2k)!} n^{-3k+1}.$$

The error term in (24) holds uniformly for all real numbers x. For $2 \leq k \leq m \leq n-1$, crude estimates (see [1; p. 805]) give

$$|R_{2k}(n)| \leq 60(2k+1)!\, n^{-k+1}, \tag{25}$$

(in fact, $R_{2k}(n)$ involves n^{-k+1} and smaller integer powers of n). For all $n \geq m+1$ and all $0 \leq u \leq \ln n/3$, (25) gives

$$|R_2(n)u^2 + \cdots + R_{2m}(n)u^{2m}| \leq m(2m+1)! \left(\frac{\ln^{2m} n}{n}\right). \tag{26}$$

Hence, (6) and (26) give

$$\exp\left\{R_2(n)u^2 + \cdots + R_{2m}(n)u^{2m}\right\} = 1 + \sum_{q=1}^{2m-2} S_{2q}(n)u^{2q} + O\left(\frac{\ln^{2m^2} n}{n^m}\right), \tag{27}$$

where $S_{2q}(n)$ is that part of

$$\sum_{r=1}^{m-1} \sum_{\substack{(e_2,\ldots,e_{2m})\in\mathbb{N}^m \\ e_2+\cdots+e_{2m}=r \\ 2e_2+\cdots+2me_{2m}=2q}} \frac{R_2^{e_2}(n)\cdots R_{2m}^{e_{2m}}(n)}{e_2!\cdots e_{2m}!}$$

involving only $n^{-1}, \ldots, n^{-m+1}$ upon expansion. Here $R_2^{e_2}(n)\cdots R_{2m}^{e_{2m}}(n)$ involves $n^{-(e_2+\cdots+(m-1)e_{2m})} = n^{-(q-r+e_2)}$ and smaller integer powers of n while $q-r+e_2 \geq m$ if $q \geq 2m-1$. Then, (24) and (27) give

$$\begin{aligned} &I(n,x,0,n^{-3/2}\ln n) \\ &= \frac{3}{n^{3/2}} \int_0^{\ln n/3} \exp\left(-\frac{u^2}{2}\right)\left\{1 + \sum_{q=1}^{2m-2} S_{2q}(n)u^{2q}\right\}\cos(ux)\,du + O\left(\frac{\ln^{2m^2+1} n}{n^{m+3/2}}\right) \\ &= \frac{3}{n^{3/2}} \int_0^{\infty} \exp\left(-\frac{u^2}{2}\right)\left\{1 + \sum_{q=1}^{2m-2} S_{2q}(n)u^{2q}\right\}\cos(ux)\,du + O\left(\frac{\ln^{2m^2+1} n}{n^{m+3/2}}\right), \end{aligned} \tag{28}$$

where our error term holds uniformly for all real numbers x. Hence, after simplifying, (8), (10–12) and (28) give

$$\begin{aligned} I(n,x) = 3\left(\frac{\pi}{2}\right)^{1/2} n^{-3/2}\mathrm{e}^{-x^2/2}\left\{1 + \sum_{q=1}^{2m-2} (-2)^{-q} S_{2q}(n) H_{2q}(2^{-1/2}x)\right\} \\ + O\left(\frac{\ln^{2m^2+1} n}{n^{m+3/2}}\right), \end{aligned} \tag{29}$$

where our error term holds uniformly for all real numbers x. Our result follows since, apart from the error term, the smallest term in (29) has order of magnitude at least n^{-m-1} for $x^2 = x^2(n) \leq \ln n$. ■

As a consequence of Theorem 3, we have a complete asymptotic expansion for $b(n,k)/n!$ when $2k = \binom{n}{2} \pm xn^{3/2}/3$ where $x^2 = x^2(n) \leq \ln n$, as well as for $B(n)/n!$ when $\binom{n}{2}$ is even.

Corollary 4. *Fix an integer $m \geq 2$. For $x^2 = x^2(n) \leq \ln n$, we have the asymptotic expansion*

$$\frac{b(n,k)}{n!} = 6(2\pi)^{-1/2} n^{-3/2} e^{-x^2/2} \left\{ 1 + \sum_{q=1}^{2m-2} (-2)^{-q} S_{2q}(n) H_{2q}(2^{-1/2}x) \right\} + O\left(\frac{\ln^{2m^2+1} n}{n^{m+3/2}} \right) \quad \text{as } n \to \infty,$$

when $2k = \binom{n}{2} \pm xn^{3/2}/3$. In particular, we have the asymptotic expansion

$$\frac{B(n)}{n!} = 6(2\pi)^{-1/2} n^{-3/2} \left\{ 1 + \sum_{q=1}^{2m-2} 2^{-q} \frac{(2q)!}{q!} S_{2q}(n) \right\} + O\left(\frac{\ln^{2m^2+1} n}{n^{m+3/2}} \right) \quad \text{as } n \to \infty,$$

when $\binom{n}{2}$ is even. ■

In the following table we compare the exact value of $B(n)/n!$ (found by expanding the generating function for the $b(n,k)$) with the approximations (given by Corollary 4 for $m = 2, 3$) for $n = 40$ and 80.

	$B(40)/40!$	$B(80)/80!$
Exact Value	$0.009233258744992\cdots$	$0.003303747524408\cdots$
Approximation ($m = 2$)	$0.009220472410157\cdots$	$0.003302581000634\cdots$
Relative Error	0.138481%	0.035309%
Error as a function of n	$40^{-3.05435\cdots}$	$80^{-3.11761\cdots}$
Approximation ($m = 3$)	$0.009234106075478\cdots$	$0.003303786057784\cdots$
Relative Error	0.009176%	0.001166%
Error as a function of n	$40^{-3.79008\cdots}$	$80^{-3.89585\cdots}$

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