# Tight Upper Bounds for the Domination Numbers of Graphs with Given Order and Minimum Degree, II

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**Abstract**

Let $\gamma(n, \delta)$ denote the largest possible domination number for a graph of order $n$ and minimum degree $\delta$. This paper is concerned with the behavior of the right side of the sequence

$$n = \gamma(n, 0) \geq \gamma(n, 1) \geq \cdots \geq \gamma(n, n-1) = 1.$$

We set $\delta_k(n) = \max\{\delta \,|\, \gamma(n, \delta) \geq k\}$, $k \geq 1$. Our main result is that for any fixed $k \geq 2$ there is a constant $c_k$ such that for sufficiently large $n$,

$$n - c_k n^{(k-1)/k} \leq \delta_{k+1}(n) \leq n - n^{(k-1)/k}.$$

The lower bound is obtained by use of circulant graphs. We also show that for $n$ sufficiently large relative to $k$, $\gamma(n, \delta_k(n)) = k$. The case $k = 3$ is examined in further detail. The existence of circulant graphs with domination number greater than 2 is related to a kind of difference set in $\mathbb{Z}_n$.

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| $n/\delta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | | |
| 2 | 1 | | | | | | | | | | | | | | |
| 3 | 1 | 1 | | | | | | | | | | | | | |
| 4 | 2 | 2 | 1 | | | | | | | | | | | | |
| 5 | 2 | 2 | 1 | 1 | | | | | | | | | | | |
| 6 | 3 | 2 | 2 | 2 | 1 | | | | | | | | | | |
| 7 | 3 | 3 | 2 | 2 | 1 | 1 | | | | | | | | | |
| 8 | 4 | 4 | 3 | 2 | 2 | 2 | 1 | | | | | | | | |
| 9 | 4 | 4 | 3 | 3 | 2 | 2 | 1 | 1 | | | | | | | |
| 10 | 5 | 4 | 3 | 3 | 2 | 2 | 2 | 2 | 1 | | | | | | |
| 11 | 5 | 5 | 4 | 3 | 3 | 3 | 2 | 2 | 1 | 1 | | | | | |
| 12 | 6 | 6 | 4 | 4 | 3 | 3 | 2 | 2 | 2 | 2 | 1 | | | | |
| 13 | 6 | 6 | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 2 | 1 | 1 | | | |
| 14 | 7 | 6 | 5 | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 1 | | |
| 15 | 7 | 7 | 5 | 5 | 4 | *4 | 3 | 3 | †3 | 2 | 2 | 2 | 1 | 1 | |
| 16 | 8 | 8 | 6 | 5 | *5 | 4 | *4 | 3 | †3 | †3 | 2 | 2 | 2 | 2 | 1 |

Table 1: Values of $\gamma(n, \delta)$ for $1 \leq n \leq 16$. Entries marked with asterisks are unknown. For these cases the best known upper bounds for $\gamma(n, \delta)$ are given. Entries determined in Section 5 are marked by daggers.

# 1 Introduction

As in [2], we say that a (simple) graph $\Gamma$ with $n$ vertices and minimum degree $\delta$ is an $(n, \delta)$-graph and we define

$$\gamma(n, \delta) = \max\{\gamma(\Gamma) \mid \Gamma \text{ is an } (n, \delta)\text{-graph}\}$$

where $\gamma(\Gamma)$ denotes the domination number of $\Gamma$.

We are interested in the behavior of the right side of the sequence

$$n = \gamma(n, 0) \geq \gamma(n, 1) \geq \cdots \geq \gamma(n, n-1) = 1. \tag{1.1}$$

In [2] the values $\gamma(n, \delta)$ for $\delta = 0, 1, 2, 3$ were determined. Table 1 taken from [2] depicts the sequences (1.1) for small values of $n$. Actually there were six undecided entries in the table given in [2], three of which are decided in Section 5 of this paper. The remaining three unknown entries are marked by asterisks. The values given for these cases are the best known upper bounds.

One easily sees that $\gamma(n,\delta)$ is a non-increasing function in $\delta$. We are interested in determining the numbers $\delta_k(n)$ where

$$\delta_k(n) = \max\{\delta \,|\, \gamma(n,\delta) \geq k\}, \quad k \geq 1.$$

Since the domination number of an $(n,\delta)$-graph $G$ is 1 if and only if there is a vertex of degree $n-1$, it is not difficult to see that $\delta_1(n) = n-1$ and that for $n \geq 4$, $\delta_2(n) \geq n-2$ if $n$ is even while $\delta_2(n) \geq n-3$ if $n$ is odd. A little reflection shows that these are in fact the actual values of $\delta_2(n)$ because when $n$ is even, the graph whose complement is a perfect matching is an $(n, n-2)$-graph with domination number 2. When $n \geq 5$ is odd, the graph whose complement is a Hamilton cycle is an $(n, n-3)$-graph with domination number 2. Therefore, for $n \geq 4$,

$$\delta_2(n) = \begin{cases} n-2, & \text{if } n \text{ is even,} \\ n-3, & \text{if } n \text{ is odd.} \end{cases}$$

In this paper, we investigate for each fixed $k \geq 3$, the behavior of $\delta_k(n)$ for all sufficiently large $n$. We shall also consider the case $k = 3$ in more detail. There are various known upper bounds of $\gamma(n,\delta)$ (see for example [3]). The upper bound $\gamma^*(n,\delta)$ in Theorem 2 below differs only trivially from the upper bound $\gamma_6(n,\delta)$ in [3]. This bound actually gives the exact values of $\gamma(n,\delta)$ for most of the cases under our consideration (see Theorem 7).

**Theorem 1** ([3]) *Let $\Lambda = \delta + 1$ if $n\delta$ is odd, and let $\Lambda = \delta$, otherwise. Define the sequence $g_1, g_2, \ldots$ as follows:*

$$g_1 = n - \Lambda - 1 \quad \textit{and} \quad g_{t+1} = \left\lfloor g_t \left(1 - \frac{\delta+1}{n-t}\right) \right\rfloor, \qquad \textit{for } t \geq 1.$$

*Set $\gamma^*(n,\delta) = \min\{t \,|\, g_t = 0\}$. Then $\gamma(n,\delta) \leq \gamma^*(n,\delta)$.* ∎

**Theorem 2** *For $k \geq 2$, $\delta_{k+1}(n) < n - n^{(k-1)/k}$.*

**Proof.** Assume $\delta \geq n - n^{(k-1)/k}$. From the fact that

$$g_1 < n - \delta, \quad \text{and} \quad g_{t+1} < g_t \left(\frac{n-\delta}{n}\right), \quad t \geq 1,$$

we have

$$g_k < n \left(\frac{n-\delta}{n}\right)^k \leq n(n^{-1/k})^k = 1.$$

Hence $g_k = 0$ and $\gamma(n,\delta) \le \gamma^*(n,\delta) \le k$. The theorem therefore follows from the definition of $\delta_{k+1}(n)$ which is the maximum value of $\delta$ for which $\gamma(n,\delta) \ge k+1$. ■

We shall show that this upper bound is quite tight in the sense that for all sufficiently large $n$, there is a constant $c_k$ such that

$$\delta_{k+1}(n) \ge n - c_k n^{(k-1)/k}. \tag{1.2}$$

Such a lower bound can be established by showing that there exists a graph $G$ with appropriate minimum degree and domination number greater than $k$. Notice that this is not trivial as our lower bound for $\delta_k(n)$ is quite close to its upper bound in Theorem 2. We shall in fact construct a circulant graph with the required properties. This requires the construction of a suitably small subset $W$ of the additive group $\mathbb{Z}_n = \{0,1,2,\ldots,n-1\}$ of integers modulo $n$ with the following property:

$$\mathbb{Z}_n^{k-1} = \bigcup_{w \in W} (w - W)^{k-1},$$

where for $x_0 \in X \subseteq \mathbb{Z}_n$, $x_0 - X = \{x_0 - x \,|\, x \in X\}$ and the superscripts indicate Cartesian set products.

In Sections 4 and 5 we obtain more detailed results in the case of $\delta_3(n)$. For this it is useful to find circulant graphs of order n with large minimum degree and with domination number at least 3. This turns out to be related to the existence of what we call a *symmetric, pseudo difference set*, that is, a subset $T$ of $\mathbb{Z}_n$ such that $0 \notin T$, $T = -T$, and $\mathbb{Z}_n = T - T$. In Section 4 we prove that if $T$ is a symmetric, pseudo difference set of minimum size then

$$\sqrt{2}\sqrt{n} - 1 \le |T| \le 2\sqrt{n} + 3.$$

## 2 Circulant graphs with $\gamma > k$

We first review the definition of a circulant graph. Let

$$\mathbb{Z}_n = \{0,1,2,\ldots,n-1\}$$

denote the additive group of integers modulo $n$. For $X, Y \subseteq \mathbb{Z}_n$ we define

$$-X = \{-x \,|\, x \in X\} \text{ and } X \pm Y = \{x \pm y \,|\, x \in X, y \in Y\}.$$

If $S \subseteq \mathbb{Z}_n$ satisfies the two conditions

$$0 \notin S \text{ and } S = -S \tag{2.1}$$

the *circulant graph* with *connection set* $S$ is the graph $C(n, S)$ with vertex set $\mathbb{Z}_n$ and adjacency relation $\sim$ defined by

$$i \sim j \Longleftrightarrow j - i \in S.$$

See Alspach [1] for general results concerning isomorphism of circulant graphs. For each $S \subseteq \{\pm 1, \pm 2, \ldots, \pm 9\}$ Fisher and Spaulding [5] obtained a formula for the domination number of the circulant graph $C(S, n)$ as a function of $n$ and $S$, but results and techniques do not appear to be useful for our purposes.

Note that the closed neighborhood of a vertex $i$ of $C(n, S)$ is given by

$$N[\,i\,] = \{i\} \cup i + S = \{i\} \cup \{i + j \mid j \in S\}.$$

To illustrate our construction technique we first consider directed circulant graphs. Suppose $R \subseteq \mathbb{Z}_n$ and $0 \notin R$. Then the *circulant digraph* with *connection set* $R$ is the digraph $D(n, R)$ with vertex set $\mathbb{Z}_n$ and directed edges $(i, j)$ whenever $j - i \in R$. Let $W = \mathbb{Z}_n - (\{0\} \cup R)$. Notice that $i + W$ is the set of vertices not dominated by vertex $i$ in the digraph $D(n, R)$. Since both $C(n, S)$ and $D(n, R)$ are vertex transitive, we have the following result.

**Lemma 1** *If $R \subseteq \mathbb{Z}_n$, $0 \notin R$ and $W = \mathbb{Z}_n - (\{0\} \cup R)$, then $\gamma(D(n, R)) > k$ if and only if for all $x_1, x_2, \ldots, x_{k-1} \in \mathbb{Z}_n$, there exists $w_0, w_1, \ldots, w_{k-1} \in W$ such that $w_0 = x_i + w_i$, for $1 \leq i \leq k - 1$, that is,*

$$\mathbb{Z}_n^{k-1} = \bigcup_{w \in W} (w - W)^{k-1}. \tag{2.2}$$

*If also $W = -W$, then $R = -R$ and $\gamma(C(n, R)) > k$.*

**Proof.** Since $D(n, R)$ is vertex transitive, we have that $\gamma(D(n, R)) > k$ if and only if for any $x_1, x_2, \ldots, x_{k-1} \in \mathbb{Z}_n$, there is a vertex not dominated by any vertex in $\{0, x_1, x_2, \ldots, x_{k-1}\}$. This is equivalent to

$$W \cap (x_1 + W) \cap \ldots \cap (x_{k-1} + W) \neq \emptyset, \text{ for all } x_1, x_2, \ldots, x_{k-1} \in \mathbb{Z}_n,$$

which is equivalent to (2.2). ■

The following theorem gives the existence of suitably small sets $W$ satisfying (2.2) for all fixed $k \geq 2$ and all sufficiently large $n$.

**Theorem 3** *If $k \geq 2$ and let $A = a_1a_2\cdots a_{k-1}$ where $a_1, a_2, \ldots, a_{k-1}$ are pairwise relatively prime integers greater than 1 such that $kA < n$, there is a subset $W$ of $\mathbb{Z}_n - \{0\}$ which satisfies equation (2.2) and*

$$|W| \leq kA + \sum_{i=1}^{k-1} \lfloor (n-1)/a_i \rfloor$$

**Proof.** Write

$$W_0 = \{j \mid 1 \leq j \leq kA\},$$

and for $i = 1, 2, \ldots, k-1$,

$$W_i = \{ja_i \mid 1 \leq j \leq \lfloor (n-1)/a_i \rfloor\}.$$

Let

$$W = \bigcup_{i=0}^{k-1} W_i.$$

We shall show that $W$ satisfies condition (2.2). Let

$$x_1, x_2, \ldots, x_{k-1} \in \{0, 1, 2, \ldots, n-1\}.$$

Since there are $k$ intervals of the form

$$I_j = \{jA+1, jA+2, \ldots, (j+1)A\},$$

where $0 \leq j \leq k-1$, there is at least one value of $j$, say $\ell$, such that $x_i \notin I_\ell$, for $i = 1, \cdots, k-1$. For each $i$, define the indicator

$$b_i = \begin{cases} 0, & x_i < \ell A + 1, \\ 1, & x_i > (\ell+1)A. \end{cases}$$

Consider now the system of linear congruences with variable $x$:

$$x \equiv x_i - b_i n \pmod{a_i} \qquad 1 \leq i \leq k-1. \tag{2.3}$$

Let $w_0 \in I_\ell \subseteq W_0$ be a solution for $x$. From the Chinese Remainder Theorem, it follows that there exists a $w_0$ with the required properties. Thus there are integers $q_i$ such that

$$w_0 = x_i - b_i n + q_i a_i, \qquad 1 \leq i \leq k-1.$$

For $i = 1, 2, \ldots, k-1$, we define $w_i = q_i a_i$. We claim that $w_i \in W_i$. There are two cases. Suppose that $x_i < \ell A + 1$. Then

$$q_i a_i = w_0 - x_i, \quad \text{and} \quad 0 < w_0 - x_i < n,$$

which implies that $1 \leq q_i \leq \lfloor (n-1)/a_i \rfloor$. If $x_i > (\ell+1)A$, then

$$q_i a_i = w_0 - x_i + n, \quad \text{and} \quad 0 < n - (x_j - w_0) < n,$$

which implies again that $1 \leq q_i \leq \lfloor (n-1)/a_i \rfloor$. Therefore,

$$w_i = q_i a_i \in W_i,$$

and in $\mathbb{Z}_n$,

$$w_0 = x_i - b_i n + w_i = x_i + w_i.$$

We have therefore shown that (2.2) holds. Finally,

$$|W| \leq |W_0| + \sum_{i=1}^{k-1} |W_i| = kA + \sum_{i=1}^{k-1} \lfloor (n-1)/a_i \rfloor. \blacksquare$$

We next turn our attention to undirected circulant graphs. We could simply take $T = W \cup -W$ where $W$ is as in the above theorem. Then $S = \mathbb{Z}_n - \{0\} - T$ provides a connection set for a circulant graph $C(n, S)$ with domination number $> k$ with size at most twice that of $W$. However, with additional effort we obtain the following somewhat better result.

**Theorem 4** *Let $k \geq 2$ and let $A = a_1 a_2 \cdots a_{k-1}$ where $a_1, a_2, \ldots, a_{k-1}$ are pairwise relatively prime integers greater than 1 such that $\lceil k/2 \rceil A < n/2$. Then there is a subset $T$ of $\mathbb{Z}_n - \{0\}$ such that $T = -T$, (2.2) is satisfied and*

$$|T| \leq 2 \left\lceil \frac{k}{2} \right\rceil A + 2 \sum_{i=1}^{k-1} \left\lfloor \frac{n+2A}{2a_i} \right\rfloor. \tag{2.4}$$

**Proof.** Define the set

$$Q_0 = \{j \mid 1 \leq |j| \leq \lceil k/2 \rceil A\}.$$

And for $i = 1, 2, \ldots, k-1$, define the sets

$$Q_i = \{ja_i, \mid 1 \leq |j| \leq \lfloor (n+2A)/(2a_i) \rfloor\}.$$

Note that we consider the sets $Q_i$ to be subsets of $\mathbb{Z}_n$. Thus to show that an integer $u$ representing an element of $\mathbb{Z}_n$ is in $Q_i$ we need to show that $u \equiv v \pmod{n}$ where $v \in Q_i$. Let

$$T = Q_0 \cup Q_1 \cup \cdots \cup Q_{k-1},$$

and

$$x_1, x_2, \ldots, x_{k-1} \in \{0, 1, 2, \ldots, n-1\}.$$

We shall show that there are elements $t_0, t_1, \cdots, t_{k-1} \in T$ such that

$$x_i = t_0 - t_i, \quad i = 1, \cdots, k-1.$$

For $j = 0, 1, \ldots, \lceil k/2 \rceil - 1$, let $I_j$ be the interval

$$I_j = \{jA + 1, jA + 2, \ldots, (j+1)A\}.$$

Since the $2\lceil k/2 \rceil$ intervals $\pm I_j$, $0 \le j \le \lceil k/2 \rceil - 1$ form a partition of $Q_0$ there exists an interval that contains none of the $x_1, x_2, \ldots, x_{k-1}$. Since $T = -T$, we can assume $I_\ell$, for some $\ell \in \{0, 1, 2, \ldots, \lceil k/2 \rceil - 1\}$, is such an interval. Define the indicator

$$b_i = \begin{cases} 1, & -n < (\ell+1)A - x_i \le -n/2, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the system of linear congruences with variable $x$:

$$x \equiv x_i - b_i n \pmod{a_i}, \qquad 1 \le i \le k-1.$$

By the Chinese Remainder Theorem there is a solution $x = t_0$ in the interval $I_\ell$. Then for some integer $q_i$, we have

$$t_0 = x_i - b_i n + q_i a_i.$$

Clearly $t_0 \in Q_0$. We shall next find for each $i$, $t_i$ such that in $\mathbb{Z}_n$,

$$t_0 = x_i + t_i, \text{ and } t_i \in Q_i.$$

We consider the following four cases:

(I) $n/2 \le (\ell+1)A - x_i < n$. This is not possible since

$$(\ell+1)A \le \lceil k/2 \rceil A < n/2.$$

(II) $0 < (\ell+1)A - x_i < n/2$. Observe that $0 < t_0 - x_i < n/2$. Let $t_i = q_i a_i$. Then

$$t_0 = x_i + t_i = x_i + q_i a_i,$$

and

$$q_i = (t_0 - x_i)/a_i \in (0, n/(2a_i)),$$

which implies that $t_i = q_i a_i \in Q_i$.

(III) $-n/2 < (\ell+1)A - x_i < 0$. In this case we have

$$t_0 \geq \ell A + 1 \geq (\ell+1)A - A.$$

Then,

$$t_0 - x_i \geq (\ell+1)A - A - x_i > -n/2 - A.$$

Hence we have $-(n+2A)/2 < t_0 - x_i < 0$. Take $t_i = q_i a_i$. Then since $t_0 = x_i + q_i a_i$,

$$q_i = (t_0 - x_i)/a_i \in (-(n+2A)/(2a_i), 0),$$

which implies that $t_i \in Q_i$.

(IV) $-n < (\ell+1)A - x_i \leq -n/2$. Note that $b_i = 1$ in this case. We also have

$$t_0 - x_i \leq (\ell+1)A - x_i \leq -n/2$$

and $t_0 \geq 1$, so $-n < t_0 - x_i$. Hence

$$0 < n - (x_i - t_0) \leq n/2.$$

Take $t_i = -n + q_i a_i$. Then $t_0 = x_i - n + q_i a_i = x_i + t_i$, and

$$q_i = (n - (x_i - t_0))/a_i \in (0, n/(2a_i)],$$

which implies that $q_i a_i \in Q_i$. Since $t_i \equiv q_i a_i \pmod n$ it follows that $t_i \in Q_i$. Clearly (2.4) holds. To complete the proof, we note that

$$|T| \leq 2\left(|Q_0| + \sum_{i=1}^{k-1} |Q_i|\right) = 2\left\lceil \frac{k}{2} \right\rceil A + 2\sum_{i=1}^{k-1} \left\lfloor \frac{n+2A}{2a_i} \right\rfloor. \blacksquare$$

## 3 Upper and lower bounds for $\delta_{k+1}(n)$

We assume throughout that $k \geq 2$. From [4], Theorem 2, we may choose the pairwise relatively prime integers $a_1, a_2, \ldots, a_{k-1}$ in Theorem 4 so that for any small $\epsilon > 0$ and for all sufficiently large $n$,

$$(1-\epsilon)n^{1/k} \leq a_i \leq n^{1/k}, \quad 1 \leq i \leq k-1.$$

Also, if $A = a_1 a_2 \cdots a_{k-1}$ then $kA \leq kn^{(k-1)/k} < n$ for sufficiently large $n$. Then

$$\begin{aligned} |W| &\leq kA + \sum_{i=1}^{k-1} \lfloor (n-1)/a_i \rfloor \\ &\leq kn^{(k-1)/k} + (k-1)\frac{n}{(1-\epsilon)n^{1/k}} \\ &= (2k-1)n^{(k-1)/k} + \frac{(k-1)\epsilon}{1-\epsilon} n^{(k-1)/k}. \end{aligned}$$

Note that by [4], it is possible to have $\epsilon = Kn^{-1/k}$ for any increasing function $K$ of $n$. Also, From (2.2)

$$\mathbb{Z}_n^{k-1} = \bigcup_{w \in W} (w - W)^{k-1},$$

we have that

$$n^{k-1} \leq |W| \times |W|^{k-1},$$

from which we have the following lower bound of $|W|$:

$$|W| \geq n^{(k-1)/k}.$$

This means that the cardinality of set $W$ constructed above is of the correct order.

Now for the non-directed case, as above, from [4], Theorem 2, we may choose the pairwise relatively prime integers $a_1, a_2, \ldots, a_{k-1}$ in Theorem 4 so that for any small $\epsilon > 0$ and for all sufficiently large $n$,

$$(1-\epsilon)n^{1/k} \leq a_i \leq n^{1/k}, \quad 1 \leq i \leq k-1,$$

and

$$\lceil k/2 \rceil a_1 a_2 \cdots a_{k-1} < n/2.$$

Then from (2.4) we have

$$\begin{aligned} |T| &\leq 2\lceil k/2 \rceil n^{(k-1)/k} + 2(k-1)\frac{(n + 2n^{(k-1)/k})}{2(1-\epsilon)n^{1/k}} \\ &\leq (k+1)n^{(k-1)/k} + \frac{(k-1)(n^{(k-1)/k} + 2n^{(k-2)/k})}{1-\epsilon} \\ &= \left(2k + \frac{\epsilon(k-1)}{1-\epsilon}\right) n^{(k-1)/k} + \frac{2(k-1)}{1-\epsilon} n^{(k-2)/k}. \end{aligned}$$

Thus we have the following theorem.

**Theorem 5** *For any fixed $k \geq 2$, any $\epsilon > 0$ and all sufficiently large $n$, the following statements hold:*
*(I) There is a circulant digraph $H$ with $n$ vertices, outdegree at least*

$$n - 1 - (2k-1)n^{(k-1)/k} - \frac{(k-1)\epsilon}{1-\epsilon} n^{(k-1)/k}$$

*and $\gamma(H) > k$.*
*(II) There is a circulant graph $G$ with $n$ vertices, degree at least*

$$n - 1 - \left(2k + \frac{\epsilon(k-1)}{1-\epsilon}\right) n^{(k-1)/k} - \frac{2(k-1)}{1-\epsilon} n^{(k-2)/k}$$

*and $\gamma(G) > k$.* ■

Combining Theorem 2 and statement (II) in Theorem 5, we have the following estimates for $\delta_{k+1}(n)$.

**Theorem 6** *For any fixed $k \geq 2$ and all sufficiently large $n$,*

$$n - (2k+1)n^{(k-1)/k} \leq \delta_{k+1}(n) < n - n^{(k-1)/k}. \blacksquare$$

Note that our lower bound is of the form $\delta_{k+1}(n) \geq n - c_k n^{(k-1)/k}$ for some constant $c_k$ depending on $k$. It would be of interest to determine if $c_k$ can be replaced by $Lk^{\alpha}$, for some numbers $L$ and $\alpha < 1$. In the next section we give better estimates for $\delta_3(n)$ by dealing directly with the requirement that $S = -S$ (or $T = -T$).

We next consider the value of $\gamma(n, \delta_k(n))$. It is in general not true that $\gamma(n, \delta_k(n)) = k$. For example, in the sequence $\{\gamma(13, \delta)\}$, from Table 1, we have $\delta_5(13) = \delta_6(13) = 2$ and $\gamma(13, 2) = 6$. However, one might expect that for any fixed $k$ and for sufficiently large $n$, $\gamma(n, \delta_k(n)) = k$. This is in fact true, as we now show.

**Theorem 7** *For all fixed $k \geq 3$ and all sufficiently large $n$, $\gamma(n, \delta_k(n)) = k$.*

**Proof.** From Theorem 6, we have for all sufficiently large $n$,

$$\delta_k(n) \geq n - (2k+1)n^{(k-2)/(k-1)}.$$

Recall also that in our proof of Theorem 2, we see that if $\delta \geq n - n^{(k-1)/k}$ then $\gamma(n, \delta) \leq k$. Clearly for all sufficiently large $n$,

$$\delta_k(n) \geq n - (2k+1)n^{(k-2)/(k-1)} \geq n - n^{(k-1)/k},$$

and thus $\gamma(n, \delta_k(n)) \leq k$. But $\gamma(n, \delta_k(n)) \geq k$ by definition of $\delta_k(n)$. The theorem therefore follows. $\blacksquare$

In the next section we show that if $k = 3$ then the above theorem holds for $n \geq 6$. Note that the results in this section are stated for fixed $k$ and sufficiently large $n$. In fact, the same results hold if $k$ is a function of $n$ so long as $k$ does not grow too fast with $n$. For example, the reader can check that Theorems 5 and 6 remain true if $k \leq \ln n/(3 \ln \ln n)$.

The above results suggest the question: Given $n$ find the largest value $K(n)$ such that for $k \geq K(n)$, $\gamma(n, \delta_k(n)) = k$. From [2] $\gamma(n, 4) \geq \lfloor n/3 \rfloor$ and $\gamma(n, 3) = \lfloor 3n/8 \rfloor$. So for $n$ sufficiently large $K(n) \leq \lfloor n/3 \rfloor$. More generally we propose the following problem: Given $n$ find the spectrum of values

$$\mathcal{S}(n) = \{\gamma(n, \delta) \,|\, 0 \leq \delta \leq n-1\}.$$

From [2] we know the values of $\gamma(n, \delta)$ for $\delta = 0, 1, 2, 3$. From Theorem 7 we know that there is a function $K(n)$ such that $\mathcal{S}(n)$ contains

$$\{i \,|\, 1 \leq i \leq K(n)\}.$$

## 4 Circulant Graphs with $\gamma > 2$ and Pseudo Difference Sets

When $k = 2$ from Lemma 1 we obtain:

**Lemma 2** *Let $\mathbb{Z}_n = T \cup S \cup \{0\}$ be a partition of $\mathbb{Z}_n$. Then $S = -S$ and $\gamma(C(n, S)) > 2$ if and only if*

$$T = -T \quad \text{and} \quad \mathbb{Z}_n = T - T. \tag{4.1}$$

Let us say that a subset $T$ of $\mathbb{Z}_n$ is a *symmetric, pseudo difference set* if the following three conditions hold

(i) $0 \notin T$,

(ii) $\mathbb{Z}_n = T - T$, and

(iii) $T = -T$.

We note that in the presence of condition (iii), condition (ii) is equivalent to $\mathbb{Z}_n = T + T$. Such sets have been studied for general groups and are sometimes called *2-bases* for $\mathbb{Z}_n$ (see [7]). On the other hand, a $k$-subset $T$ of $\mathbb{Z}_n$ is called an $(n, k, \lambda)$-*difference set* if for each non-zero $i \in \mathbb{Z}_n$ there are exactly $\lambda$ ordered pairs $(u, v)$ such that $i = u - v$ (see, for example, [6].) As we will show, one can construct a small symmetric, pseudo difference set using an $(n, k, 1)$ difference set. In this case, $n = q^2 + q + 1$ where $q = k - 1$ and the corresponding block design is a projective plane of order $q$. The only known examples are when $q$ is a prime power. It is a famous open question whether or not there exist projective planes of non-prime power order. So we do not expect to be able to use $(n, k, 1)$ difference sets for very many values of $n$. However, the following lemma shows that we can do quite well for all $n$, even when an $(n, k, 1)$ difference set does not exist.

We are primarily interested in finding a symmetric, pseudo difference set in $\mathbb{Z}_n$ with the smallest size. From Lemma 2 this will give circulant graphs with large minimum degree and domination number greater than 2.

**Lemma 3** *There exists a symmetric, pseudo difference set $T \subset \mathbb{Z}_n$, $n \geq 4$, such that*

$$|T| \leq 2\left(\left\lfloor \frac{\sqrt{n}}{2} \right\rfloor + \left\lceil \frac{n}{4\lfloor \frac{\sqrt{n}}{2} \rfloor} \right\rceil\right) \leq 2\sqrt{n} + 3$$

**Proof.** Let $b$ be any positive integer satisfying $1 \leq b \leq \frac{n}{2}$. Define

$$T_1(b) = \{1, 2, \cdots, b\} \cup \{2ib \,|\, i = 1, 2, \cdots, \lceil n/(4b) \rceil\}.$$

It is easy to see that

$$\{0, 1, \cdots, \lfloor n/2 \rfloor\} \subseteq T_1(b) \pm T_1(b).$$

The critical case is when $x = (2i + 1)b + r \leq n/2$ where $i$ and $r$ are positive integers with $0 < r < b$. Then $x = 2(i + 1)b - (b - r)$. Hence

$$i + 1 \leq \frac{n}{4b} + \frac{1}{2} - \frac{r}{2b} < \frac{n}{4b} + \frac{1}{2}$$

and $i + 1 \leq \lceil n/(4b) \rceil$. Thus $x \in T_1(b) - T_1(b)$. It follows that the set

$$T(b) = T_1(b) \cup -T_1(b)$$

is a symmetric, pseudo difference set. Clearly,

$$|T(b)| \leq 2(b + \lceil n/(4b) \rceil).$$

Taking $b = \lfloor \sqrt{n}/2 \rfloor$ we obtain the desired symmetric, pseudo difference set. ■

In the next lemma we show how to construct symmetric, pseudo difference sets in $\mathbb{Z}_n$ of essentially the same size as that in Lemma 3 if there is an $(n, k, 1)$ difference set.

**Lemma 4** *If there esists an $(n,k,1)$ difference set, then there exists a symmetric, pseudo difference set $T$ satisfying*

$$|T| \leq 2\sqrt{n-3/4}+1.$$

**Proof** If $D$ is an $(n,k,1)$ difference set since $k(k-1)=n-1$ we have

$$|D| = \frac{1+\sqrt{4n-3}}{2}.$$

Since $D \neq \mathbb{Z}_n$ there is an element $a \in \mathbb{Z}_n$ such that $a \notin -D$. Then $B = a + D$ is also an $(n,k,1)$ difference set and $0 \notin B$. Hence $T = B \cup -B$ is a symmetric, pseudo difference set and

$$|T| \leq 2|D| = 1+\sqrt{4n-3}. \qquad \blacksquare$$

**Lemma 5** *If $T$ is a symmetric, pseudo difference set then*

$$|T| \geq \sqrt{2n-2} \geq \sqrt{2}\sqrt{n}-1.$$

**Proof** Assume that $T$ does not contain $n/2$. In this case, $T = X \cup -X$ where $X = \{a_1, a_2, \ldots, a_s\}$ and $|T| = 2s$. Then there are just the following seven types of elements in $T - T$:

1. $a_i - a_j$ where $i < j$,
2. $a_i - a_j$ where $j < i$,
3. $a_i + a_j$ where $i < j$.
4. $-a_i - a_j$ where $i < j$,
5. $a_i + a_i$,
6. $-a_i - a_i$,
7. 0

There are at most $\binom{s}{2}$ elements for each of the types (1), (2), (3) and (4). There are at most $s$ elements for each of the types (5) and (6). Since $\mathbb{Z}_n = T - T$, we must have

$$4\binom{s}{2} + 2s + 1 \geq n.$$

It follows that $s \geq \sqrt{(n-1)/2}$. Hence,

$$|T| = 2s \geq 2\sqrt{(n-1)/2} = \sqrt{2n-2}.$$

If $n$ happens to be even and $n/2 \in T$ then

$$T = X \cup -X \cup \{n/2\}$$

where as above $X$ has $s$ elements $a_1, a_2, \ldots, a_s$. In this case, in addition to elements of types (1)-(7) above, $T - T$ also contains elements of the form $n/2 \pm a_i$. There are at most 2s elements of this type. So we obtain the inequality

$$4\binom{s}{2} + 2s + 2s + 1 \geq n.$$

which gives

$$|T| = 2s + 1 \geq \sqrt{2n-1}.$$

Since this is larger that the previous bound, we obtain the desired lower bound for $|T|$. ■

The following theorem is immediate from Lemmas 3 and 5.

**Theorem 8** *If $T \subset \mathbb{Z}_n$, $n \geq 6$ is a symmetric, pseudo difference set of smallest size then*

$$\sqrt{2}\sqrt{n} - 1 \leq |T| \leq 2\sqrt{n} + 3. \quad ■$$

Since each such $T$ leads to a circulant graph $C(n, S)$, where $S = \mathbb{Z}_n - (\{0\} \cup T)$, with $\gamma(C(n, S)) > 2$ we have

**Corollary 1** *For every positive integer $n \geq 6$ there is a circulant graph of order $n$ with domination number at least 3 and minimum degree $\delta$ satisfying*

$$n - 2\sqrt{n} - 4 \leq \delta \leq n - \sqrt{2}\sqrt{n}. \quad ■$$

**Theorem 9** *For* $n \geq 4$,

$$n - 2\sqrt{n} - 4 \leq \delta_3(n) \leq n - 3/2 - \sqrt{n - 3/4}. \quad \blacksquare$$

**Proof.** From the definition of $g_k$ in Theorem 1 we see that

$$g_2 \leq \frac{(n-\delta-1)(n-\delta-2)}{n-1}. \tag{4.2}$$

From Theorem 2, $\gamma(n,\delta) \leq 2$ if $g_2 < 1$. Let $\delta = n - x$, then the right side (4.2) becomes

$$\frac{(x-1)(x-2)}{n-1} < 1, \tag{4.3}$$

which is equivalent to $x < (3 + \sqrt{4n-3})/2$. So if $\delta > n - \sqrt{n - 3/4} - 3/2$ then $\gamma(n,\delta) \leq 2$. This gives the desired upper bound for $\delta_3(n)$. The lower bound follows from Corollary 1. ■.

Using Theorem 9 we are able to establish the following result.

**Theorem 10** *If* $n \geq 6$ *then* $\gamma(n, \delta_3(n)) = 3$.

**Proof.** From the definition of $\delta_3(n)$ we only need to show that for each $n \geq 6$ there is some graph $\Gamma$ of order $n$ with domination number 3. For $6 \leq n \leq 16$ the result follows directly Table 1. For $16 \leq n \leq 150$ it is easy to show by straightforward computation that the small symmetric, pseudo difference sets $T$ constructed in Lemma 3 yield circulant graphs with domination number 3. For $n > 150$, from Theorem 9 it suffices to prove that if $\delta \geq n - 2\sqrt{n} - 4$ then $\gamma(n,\delta) \leq 3$. From Theorem 1 we only need to show for $\delta \geq n - 2\sqrt{n} - 4$ and $n > 150$ that $g_3 < 1$. Now

$$\begin{aligned} g_3 &\leq \frac{(n-\delta-1)(n-\delta-2)(n-\delta-3)}{(n-1)(n-2)} \\ &\leq \frac{(n-\delta-1)^3}{(n-2)^2} \\ &\leq \frac{(2\sqrt{n}+3)^3}{(n-2)^2}. \end{aligned}$$

It is easy to see that

$$\varphi(n) = \frac{(2\sqrt{n}+3)^3}{(n-2)^2}$$

is a decreasing function for $n > 2$ so it suffices to calculate

$$\varphi(150) = \frac{1}{39601}\left(2\sqrt{150}+3\right)^3 \approx .5248680758 < 1. \quad \blacksquare$$

# 5 Exact values of $\delta_3(n)$ for small $n$.

In this section we give exact values of $\delta_3(n)$ for $6 \le n \le 16$ and for $n = 19$. This entails showing that $\gamma(15,9) = 3$, $\gamma(16,10) = 3$, and $\gamma(19,12) = 3$. We also show that $\gamma(16,9) = 3$. This fills in three of the six unknown entries in the table of values of $\gamma(n,\delta)$ in [2]. The following table gives known values for $\delta_3(n)$ for $6 \le n \le 16$ and $n = 10$. (Note that for $n \le 5$, $\gamma(n,\delta) \le 2$.)

| $n$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_3(n)$ | 1 | 2 | 3 | 4 | 4 | 6 | 6 | 7 | 8 | 9 | 10 | 12 |

Exact values of $\delta_3(n)$ for $6 \le n \le 14$ are given in [2]. Since $\gamma(15,10) = 2$ and $\gamma(16,11) = 2$ by [2], to show that $\delta_3(15) = 9$ and $\delta_3(16) = 10$ it suffices to exhibit a $(15,9)$-graph with domination number 3 and a $(16,10)$-graph with domination number 3. The following is an adjacency list for a $(15,9)$-graph with domination number 3. The vertex set is $\{0,1,2,\ldots,14\}$. Note that all vertices have degree 9 except for vertex 14 which has degree 10.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 5 | 6 | 7 | 9 | 11 | 12 | 13 | 14 | |
| 1 | 0 | 2 | 6 | 8 | 9 | 10 | 11 | 13 | 14 | |
| 2 | 1 | 3 | 7 | 8 | 9 | 10 | 12 | 13 | 14 | |
| 3 | 2 | 4 | 6 | 7 | 9 | 10 | 11 | 12 | 14 | |
| 4 | 3 | 5 | 6 | 8 | 9 | 10 | 12 | 13 | 14 | |
| 5 | 0 | 4 | 7 | 8 | 9 | 10 | 11 | 13 | 14 | |
| 6 | 0 | 1 | 3 | 4 | 8 | 10 | 12 | 13 | 14 | |
| 7 | 0 | 2 | 3 | 5 | 8 | 10 | 11 | 13 | 14 | |
| 8 | 1 | 2 | 4 | 5 | 6 | 7 | 11 | 12 | 14 | |
| 9 | 0 | 1 | 2 | 3 | 4 | 5 | 11 | 12 | 14 | |
| 10 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 11 | 13 | |
| 11 | 0 | 1 | 3 | 5 | 7 | 8 | 9 | 10 | 12 | |
| 12 | 0 | 2 | 3 | 4 | 6 | 8 | 9 | 11 | 13 | |
| 13 | 0 | 1 | 2 | 4 | 5 | 6 | 7 | 10 | 12 | |
| 14 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

There is no $(16,10)$-circulant graph with domination number 3. However, we were able to find a Cayley graph on the semi-dihedral group of order 16 with $\delta = 10$ and $\gamma = 3$

showing that $\gamma(16, 10) = 3$ and $\delta_3(16) = 10$. Again, if we take the vertex set to be the integers $\{0, 1, 2, \dots, 15\}$, the adjacency list of the graph is:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 3 | 4 | 7 | 8 | 9 | 14 | 13 | 5 | 1 |
| 1 | 2 | 3 | 4 | 6 | 8 | 9 | 15 | 12 | 5 | 0 |
| 2 | 3 | 4 | 6 | 7 | 10 | 11 | 15 | 13 | 0 | 1 |
| 3 | 2 | 6 | 7 | 10 | 11 | 14 | 12 | 5 | 0 | 1 |
| 4 | 2 | 6 | 7 | 9 | 11 | 12 | 13 | 5 | 0 | 1 |
| 5 | 3 | 4 | 6 | 7 | 8 | 10 | 12 | 13 | 0 | 1 |
| 6 | 2 | 3 | 4 | 7 | 8 | 11 | 14 | 15 | 5 | 1 |
| 7 | 2 | 3 | 4 | 6 | 9 | 10 | 14 | 15 | 5 | 0 |
| 8 | 6 | 9 | 10 | 11 | 14 | 15 | 13 | 5 | 0 | 1 |
| 9 | 4 | 7 | 8 | 10 | 11 | 14 | 15 | 12 | 0 | 1 |
| 10 | 2 | 3 | 7 | 8 | 9 | 11 | 15 | 12 | 13 | 5 |
| 11 | 2 | 3 | 4 | 6 | 8 | 9 | 10 | 14 | 12 | 13 |
| 12 | 3 | 4 | 9 | 10 | 11 | 14 | 15 | 13 | 5 | 1 |
| 13 | 2 | 4 | 8 | 10 | 11 | 14 | 15 | 12 | 5 | 0 |
| 14 | 3 | 6 | 7 | 8 | 9 | 11 | 15 | 12 | 13 | 0 |
| 15 | 2 | 6 | 7 | 8 | 9 | 10 | 14 | 12 | 13 | 1 |

We have one additional exact value of $\delta_3(n)$, namely, $\delta_3(19) = 12$: A complete search of circulant graphs of order $n \leq 50$ finds the $(19, 12)$-graph $C(19, S_1)$ where

$$S_1 = \{1, 4, 6, 7, 8, 9, 10, 11, 12, 13, 15, 18\}$$

such that $\gamma(C(19, S_1)) = 3$, showing that $\delta_3(19) = 12$, since $\gamma^*(19, 12) \leq 3$ and $\gamma^*(19, 13) = 2$.

We also mention here the $(16, 9)$-graph $C(16, S_2)$ where

$$S_2 = \{1, 2, 3, 6, 8, 10, 13, 14, 15\}$$

which has domination number 3. This shows that $\gamma(16, 9) = 3$, thereby filling another missing entry in the table of values of $\gamma(n, \delta)$ in [2]

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