Meet and Join within the Lattice of Set Partitions

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Abstract

We build on work of Boris Pittel [5] concerning the number of t-tuples of partitions whose meet (join) is the minimal (maximal) element in the lattice of set partitions.

1 Introduction

Recall that a *partition* of the set $[n] = \{1, 2, \ldots n\}$ is a collection of nonempty, pairwise disjoint subsets of $[n]$ whose union is $[n]$. The subsets are called *blocks*. One partition π_1 is said to *refine* another π_2, denoted $\pi_1 \le \pi_2$, provided every block of π_1 is contained in some block of π_2. The refinement relation is a partial ordering of the set Π_n of all partitions of $[n]$. Given two partitions π_1 and π_2, their *meet*, $\pi_1 \wedge \pi_2$, (respectively *join*, $\pi_1 \vee \pi_2$) is the largest (respectively smallest) partition which refines (respectively is refined by) both π_1 and π_2. The meet has as blocks all nonempty intersections of a block from π_1 with a block from π_2. The blocks of the join are the smallest subsets which are exactly a union of blocks from both π_1 and π_2. Under these operations, the poset Π_n is a lattice.

Recently Pittel has considered the number $M_n^{(t)}$ of t-tuples of partitions whose meet is the minimal partition $\{\{1\}, \{2\}, \ldots \{n\}\}$, and $J_n^{(t)}$ the number of t-tuples whose join is the maximal partition $\{\{1, 2, \ldots, n\}\}$. We shall prove

Theorem 1 *Let $M_t(x)$ and $J_t(x)$ be the exponential generating functions for the sequences $M_n^{(t)}$ and $J_n^{(t)}$. Then*

$$M_t(e^x - 1) \quad = \quad \sum_{n=0}^{\infty} (B_n)^t \frac{x^n}{n!} \quad = \quad \exp\{J_t(x) - 1\}.$$

where B_n is the n-th Bell number, the total number of partitions of the set $[n]$.

Remark. What about $n = 0$ and/or $t = 0$? The lattice Π_0 has exactly one element, and thus is isomorphic to Π_1. Generally, one takes the empty meet to be the maximal element and the empty join to be the minimal element. Thus, there is some logical justification to define

$$\begin{aligned} M_0^{(t)} &= J_0^{(t)} = 1 \text{ for all } t \\ M_n^{(0)} &= J_n^{(0)} = 1 \text{ for } n = 0, 1 \\ M_n^{(0)} &= J_n^{(0)} = 0 \text{ for all } n \geq 2 \end{aligned}$$

In particular, $M_0(x) = J_0(x) = 1 + x$, and $M_1(x) = J_1(x) = e^x$. These latter two when inserted in the theorem yield immediately recognized identities.

To prove the first equality of Theorem 1 we shall use the following known result:

Theorem 2 *Let E_n be the edge set of the complete graph K_n, G_S the graph with vertex set $[n]$ and edge set $S \subseteq E_n$, and $\mathrm{c}(G)$ the number of connected components in the graph G. Then,*

$$\sum_{S \subseteq E_n} (-1)^{|S|} X^{\mathrm{c}(G_S)} = X(X-1)\cdots(X-n+1).$$

A consequence of Theorem 2 is our later formula (2) which gives $M_n^{(t)}$ as a sum of products of Bell number powers with Stirling numbers of the first kind. With the second equality of Theorem 1 we can prove

Theorem 3

$$J_n^{(2)} = (B_n)^2 \times (1 - \frac{r^2}{n} - \frac{2r^3 + 2r^4 + 2r^5 + r^6}{(r+1)^2 n^2} + \mathrm{O}(r^7/n^3))$$

where r is the positive real solution of the equation $re^r = n$.

This improves on Pittel's estimate that $J_n^{(t)}$ is $(B_n)^t(1 + \mathrm{O}(r^{t+1}/n^{t-1})$. The method by which we prove Theorem 3 yields in principle a complete asymptotic expansion of $J_n^{(t)}$ in descending powers of n, although the later terms are quite complicated. In the final section of our paper, we present a generalization of the first equality in Theorem 1.

2 Discussion of Theorem 2

We shall not give a proof of this theorem, since many are available. Indeed, using the Principle of Inclusion-Exclusion, the left side can be interpreted as the number of ways to color properly the complete graph K_n with X colors, which agrees with the right side. More generally, we may replace the graph K_n on the left with an arbitrary graph G, and then on the right we replace the displayed polynomial with the chromatic polynomial of G. A good reference for this is [2].

Since the coefficients of $X(X-1)\cdots(X-n+1)$ are the (signed) Stirling numbers of the first kind, $s(n,k)$, Theorem 2 is equivalent to:

$$\sum_{\substack{S\subseteq E_n \\ \mathrm{c}(G_S)=k}} (-1)^{|S|} \quad = \quad s(n,k).$$

In this form the theorem states that among graphs of n vertices and k connected components, the excess of the number with an even number of edges over those with an odd number of edges is the signed Stirling number of the first kind $s(n,k)$. The case $k=1$ of this interesting interpretation appeared as a Monthly Problem a few years ago, and in the solution the generalization to larger k was noted, [3].

We close this section with a useful inclusion/exclusion enumeration formula based on Theorem 2.

Corollary. Let X be a set of combinatorial objects which may have properties corresponding to the pairs E_n, $n \geq 1$. Suppose that for $S \subseteq E_n$, the number of objects which have at least all the properties of S depends only on $\mathrm{c}(G_S)$, the number of connected components of the graph G_S determined by the pairs S. If this number is $f(\mathrm{c}(G_S))$, then,

$$\#\{x \in X : x \text{ has no property}\} \quad = \quad \sum_{k=1}^{n} s(n,k)\, f(k).$$

3 An Application

We shall now use the above inclusion/exclusion formula to give another proof of the beautiful formula found by Boris Pittel [5]. The formula is

$$M_n^{(t)} \quad = \quad e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{(i_1 \cdots i_t)_n}{i_1! \cdots i_t!}, \tag{1}$$

where, again, $M_n^{(t)}$ is the number of t-tuples of partitions satisfying

$$\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_t \quad = \quad \{\{1\},\{2\},\ldots,\{n\}\}.$$

A striking feature of Pittel's formula is its resemblance to Dobinski's formula (see [6])

$$B_n \quad = \quad e^{-1} \sum_{i=1}^{\infty} \frac{i^n}{i!},$$

or its t-th power:

$$(B_n)^t \quad = \quad e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{(i_1 \cdots i_t)^n}{i_1! \cdots i_t!}.$$

A collection of t partitions will have nontrivial meet precisely when there is at least one pair of integers i and j which belong to the same block in all t of the partitions. Let X be the set of all t-tuples of partitions, and let (i,j) be the property that when the meet of a t-tuple is formed, elements i and j are still in the same block. Then, by the Corollary of the previous section,

$$M_n^{(t)} \quad = \quad \sum_{k=1}^{n} s(n,k)(B_k)^t. \tag{2}$$

Herb Wilf pointed out that the previous identity is equivalent to, (1). Indeed,

$$\begin{aligned} e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{(i_1 \cdots i_t)_n}{i_1! \cdots i_t!} \quad &= \quad e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{\sum_{k=1}^{n} s(n,k)(i_1 \cdots i_t)^k}{i_1! \cdots i_t!} \\ &= \quad \sum_{k=1}^{n} s(n,k) \left(e^{-1} \sum_{i=1}^{\infty} \frac{i^k}{i!} \right)^t \\ &= \quad \sum_{k=1}^{n} s(n,k)(B_k)^t, \end{aligned}$$

proving the theorem.

4 Proof of the First Equality in Theorem 1

Since (see for example [1])

$$\sum_{n \geq 0} s(n,k) \frac{x^n}{n!} \quad = \quad \frac{(\log(1+x))^k}{k!},$$

equation (2) is equivalent to

$$[\frac{x^n}{n!}]\, M_t(x) \;\; = \;\; \sum_{k \geq 0} (B_k)^t \, [\frac{x^n}{n!}] \frac{(\log(1+x))^k}{k!}.$$

The linear operator $[\frac{x^n}{n!}]$, "take the coefficient of $\frac{x^n}{n!}$," can be moved outside the summation on the right. Then, we may drop the $[\frac{x^n}{n!}]$ from both sides, leaving an identity. The identity is exactly the first equality in Theorem 1, after substituting $e^x - 1$ for x.

5 Proof of the Second Equality in Theorem 1

There is a Basic Principle of Exponential Generating Functions which says that if $J(x)$ is the egf of certain labeled combinatorial objects, then $\exp\{J(x)-1\}$ is the egf for partitions of n with a J-object built on each block. A very good account of this exponential formula is given in [7], Chapter 3. It suffices, therefore, to establish a bijection

$$\underbrace{\Pi_n \times \Pi_n \times \cdots \times \Pi_n}_{t \text{ factors}} \quad \longleftrightarrow \quad Q_{n\,t} \tag{3}$$

where $Q_{n\,t}$ consists of all sets of t-tuples of partitions

$$\{ \; (x_{1\,1}, x_{1\,2}, \ldots x_{1\,t}), \quad \ldots \quad , \; (x_{\ell\,1}, x_{\ell\,2}, \ldots x_{\ell\,t}) \; \} \tag{4}$$

with the property that each join

$$x_{i\,1} \;\vee\; x_{i\,2} \;\vee\; \cdots \;\vee\; x_{i\,t}$$

is a one-block partition $\{S_i\}$, where $S_i \subseteq [n]$ and $\{S_i : 1 \le i \le \ell\}$ is a partition of $[n]$. To repeat for clarity, each member of $Q_{n\,t}$ is a nonempty set (whose size is denoted here $\ell \ge 1$), each element of which is a t-tuple $(x_{i\,1}, \ldots, x_{i\,t})$. The various $x_{i\,j}$ are themselves partitions of a set $S_i \subseteq [n]$; the join (over j) of the $x_{i\,j}$ equals $\{S_i\}$; and $\pi = \{S_i : 1 \le i \le \ell\}$ is a partition of $[n]$.

Once the definition of the set $Q_{n\,t}$ has been comprehended, the bijection (3) with the Cartesian product $(\Pi_n)^t$ is fairly natural. In the direction $\longrightarrow$, let a t-tuple of partitions $(\pi_1, \ldots, \pi_t)$, be given. Let $\pi = \{S_i : 1 \le i \le \ell\}$ be their join. The partitions $x_{i\,j}$, $(1 \le i \le \ell, 1 \le j \le t)$, are the nonempty intersections of the blocks of π_j with the set S_i.

In the other direction $\longleftarrow$, let T be a set of the form (4), consisting of t-tuples of partitions $x_{i\,j}$. We know that each join $\vee_{j=1}^{t} x_{i\,j}$ is a one-block partition $\{S_i\}$. Since $x_{i\,j}$ is a partition of S_i, and $\{S_i : 1 \le i \le \ell\}$ is itself a partition of $[n]$, it follows that

$$\pi_j \;=\; x_{1\,j} \;\cup\; x_{2\,j} \;\cup\; \cdots \;\cup\; x_{\ell\,j}$$

is a partition of $[n]$. The t-tuple $(\pi_1, \pi_2, \ldots \pi_t)$ so formed is the one to be associated by the bijection with the initially given set T.

6 Calculations

The equation (2) yields efficient calculation of $M_n^{(2)}$. By differentiating the second equality of Theorem 1, we obtain, by a familiar technique, the recursion

$$J_{n+1}^{(t)} \;=\; (B_{n+1})^t \;-\; \sum_{j=1}^{n} \binom{n}{j} (B_j)^t J_{n-j+1}^{(t)}, \quad n \ge 0, \tag{5}$$

and this permits efficient calculation of $J_n^{(t)}$. By these means we determine the following table for $t = 2$.

n	$M_n^{(2)}$	$J_n^{(2)}$
0	1	1
1	1	1
2	3	3
3	15	15
4	113	119
5	1153	1343
6	15125	19905
7	245829	369113
8	4815403	8285261
9	111308699	219627683
10	2985997351	6746244739

7 Proof of Theorem 3

To simplify and avoid proliferation of cases, we take $t = 2$ and accuracy n^{-2}; the method can be adapted for any fixed $t \geq 2$, and any desired accuracy. It is an iterative method, and we need an initial estimate. From [5] we know

$$J_n^{(2)} \quad = \quad (B_n)^2 \, (1 + \mathrm{O}(r^3/n)), \tag{6}$$

where r is the positive real solution of $re^r = n$. By the Moser-Wyman method [4] we have

$$\frac{B_{n+1}}{B_n} \quad = \quad \frac{n+1}{r} \, (1 + \mathrm{O}(n^{-1})), \tag{7}$$

and from the recursion (5),

$$\frac{J_{n+1}^{(2)}}{(B_{n+1})^2} \quad = \quad 1 \; - \; n \frac{J_n^{(2)}}{(B_{n+1})^2} \; - \; \frac{1}{(B_{n+1})^2} \sum_{j=2}^{n} \binom{n}{j} (B_j)^2 J_{n+1-j}^{(2)}.$$

We bound the summation above by replacing $J_{n+1-j}^{(2)}$ with $(B_{n+1-j})^2$. The resulting convolution can be further bounded as in the proof of Theorem 5 in [5]; namely, it is the terms at the extreme ends of the sum which dominate:

$$\frac{1}{(B_{n+1})^2} \sum_{j=2}^{n} \binom{n}{j} (B_j B_{n+1-j})^2 \quad = \quad \mathrm{O}(r^4/n^2).$$

With

$$\frac{J_n^{(2)}}{(B_{n+1})^2} \quad = \quad \frac{J_n^{(2)}}{(B_n)^2} \, (\frac{B_n}{B_{n+1}})^2,$$

the bound for the summation, (6), and (7) we have

$$\frac{J_n}{(B_n)^2} \quad = \quad 1 \; - \; \frac{r^2}{n} \; + \; \mathrm{O}(r^5/n^2).$$

(When we replace n by $n - 1$, we must replace r by $r + \mathrm{O}(n^{-1})$.) We now repeat the process. This time we substitute into

$$\begin{aligned} \frac{J_{n+1}}{(B_{n+1})^2} \quad &= \quad 1 \; - \; n \frac{J_n^{(2)}}{(B_{n+1})^2} \; - \; 4 \binom{n}{2} \frac{J_{n-1}^{(2)}}{(B_{n+1})^2} \; - \; J_1^{(2)} \, (\frac{B_n}{B_{n+1}})^2 \\ &\quad - \quad \frac{1}{(B_{n+1})^2} \sum_{j=3}^{n-1} \binom{n}{j} (B_j)^2 J_{n+1-j}^{(2)}, \end{aligned}$$

using in place of (7) the more accurate

$$\frac{B_{n+1}}{B_n} \quad = \quad \frac{n+1}{r} \, (1 - \frac{2 + 4r + r^2}{2(r+1)^2 n} + \mathrm{O}(r^2/n^2)),$$

and

$$\frac{1}{(B_{n+1})^2} \sum_{j=3}^{n-1} \binom{n}{j} (B_j B_{n+1-j})^2 \quad = \quad \mathrm{O}(r^6/n^3).$$

The result, after some algebra, including this time a replacement of n by $n - 1$ and of r by $r - r/(1 + r)n + \mathrm{O}(n^{-2})$, is the formula stated as Theorem 3.

8 A Generalization in Terms of Whitney Numbers

In the lattice Π_n, 0 is the finest partition $\{\{1\},\ldots,\{n\}\}$, and 1 is the coarsest $\{\{1,\ldots,n\}\}$. The intervals of Π_n have an interesting recursive structure. Consider first an interval of the form $[\pi, 1]$ Observe that the latter interval is isomorphic to Π_k, where π has k blocks. Now, if we take any t-tuple of partitions, and form their meet, we obtain some partition π. Thus, we can count all t-tuples according to their meet, as follows:

$$(B_n)^t = \sum_k S(n,k) M_k^{(t)}.$$

This provides, by inversion, another proof and further understanding of equation 2. We can formalize this as follows.

Theorem 4 *Let L_n be a sequence of lattices with rank$(1) = n$. Assume that each interval $[x, 1] \subseteq L_n$ is isomorphic to L_k if $x \in L_n$ and rank$(x) = n - k$. If ${M_L}^{(t)}_n$ equals the number of t-tuples of points in L_n whose meet is 0, then*

$$|L_n|^t = \sum_k W_{n-k} {M_L}^{(t)}_k,$$

where W_k are the Whitney numbers of the second kind, the number of elements of rank k.

As an example, consider the lattice B_n of subsets of $[n]$. Theorem 4 tells us

$$2^{nt} = \sum_k \binom{n}{k} {M_B}^{(t)}_k.$$

By inversion, we conclude there are

$$\sum_k (-1)^k \binom{n}{k} 2^{tk} = (2^t - 1)^n$$

t-tuples of subsets of $[n]$ whose intersection is empty.

A similar remark can be made for the join operation in Π_n. Namely, the interval $[0, \pi]$ is isomorphic to a Cartesian product of λ_1 copies Π_1 with λ_2 copies Π_2, etc., where the shape of partition π is $1^{\lambda_1}, \ldots n^{\lambda_n}$. Hence,

$$(B_n)^t = \sum_{\lambda \vdash n} \frac{n!}{\prod_i (i!)^{\lambda_i} \lambda_i!} \prod_i (J_i^{(t)})^{\lambda_i}.$$

In this equation, the fraction on the right is the well known [1] formula for the number of partitions of shape λ. This identity is equivalent to the second equality of Theorem 1. We will not formulate a generalization, since no examples other than Π_n come to mind!

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