On a Multiplicative Partition Function

Yifan Yang
Department of Mathematics
University of Illinois

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Abstract

Let $D(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ be the Dirichlet series generated by the infinite product $\prod_{k=2}^{\infty}(1-k^{-s})$. The value of a_m for squarefree integers m with n prime factors depends only on the number n, and we let $f(n)$ denote this value. We prove an asymptotic estimate for $f(n)$ which allows us to solve several problems raised in a recent paper by M. V. Subbarao and A. Verma.

1 Introduction and Statements of Results

Let $D(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ be the Dirichlet series generated by the infinite product $\prod_{k=2}^{\infty}(1 - k^{-s})$. The coefficients a_m denote the excess of the number of (unordered) representations of m as a product of an even number of distinct integers > 1 over the number of representation of m as a product of an odd number of distinct integers > 1. The Dirichlet series $D(s)$ is closely related to the generating Dirichlet series in the "Factorisatio Numerorum" problem of Oppenheim (see [6]). Indeed, if we let b_m denote the number of (unordered) representations of m as a product of integers > 1, not necessarily distinct, then we have $\sum_{m=1}^{\infty} b_m m^{-s} = D(s)^{-1}$. Thus, by the Möbius inversion formula, the numbers a_m and b_m are related by the identity $a_m = \sum_{d|m} \mu(d) b_{m/d}$. Oppenheim [6] showed that

$$\frac{1}{x}\sum_{m\le x} b_m \sim \frac{e^{\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}.$$

In [3], E. R. Canfield, P. Erdős and C. Pomerance proved that if m is an integer such that $b_n < b_m$ for all $n < m$, then

$$b_m = m \exp\left\{-(1+o(1)) \log m \log_3 m / \log_2 m\right\},$$

where $\log_k$ denotes the k-times iterated logarithm.

In this paper, we consider the more difficult problem of investigating the asymptotic behavior of the numbers a_m. This problem was raised by M. V. Subbarao, who observed that $a_m = 0, \pm 1$ for all positive integers m with at most four prime factors and asked whether this is true for all m. It is easy to see that for a positive integer $m > 1$ the coefficient a_m depends only on the exponents $r_1, \ldots, r_n$ in the canonical prime factorization $m = p_1^{r_1} \ldots p_n^{r_n}$. In particular, for squarefree $m = p_1 \ldots p_n$, the value of a_m is a function of the number n of prime factors of m. We will denote this function by $f(n)$.

The function $f(n)$ can be interpreted as a set-partition function. Indeed, by identifying factors of $m = p_1 \ldots p_n$ with subsets of $\{1, 2, \ldots, n\}$, we see that $f(n)$ is equal to the excess of the number of ways to partition a set S of n elements into an even number of non-empty subsets over the number of ways to partition S into an odd number of non-empty subsets. Therefore, $f(n)$ can also be written as

$$f(n) = \sum_{k=1}^{n} (-1)^k S_2(n, k), \tag{1}$$

where the numbers $S_2(n, k)$ are the Stirling numbers of the second kind, which denote the number of partitions of an n-element set into k non-empty subsets (see, e.g., [8, Section 3.6]).

A further motivation for studying the function $f(n)$ is the following observation of D. Bowman [2]. For each integer $n > 0$ there exist exactly one integer b_n and a polynomial $P_n(x, y)$ such that

$$\sum_{k=0}^{m} (k^{n-1} + b_n) k! = P_n(m!, m)$$

holds for all integers m. It turns out that this integer b_n is equal to $f(n)$. By a simple proof by induction, we have $\sum_{k=0}^{m} k \cdot k! = (m+1)! - 1$, and hence $f(2) = b_2 = 0$. The case $n = 2$ is the only known case with $b_n = 0$. H. S. Wilf raised the question whether $b_n = 0$ (or equivalently $f(n) = 0$) infinitely often.

By (1) we have the trivial upper bound

$$|f(n)| \le \sum_{k=1}^{n} S_2(n, k).$$

The numbers $B(n) = \sum_{k=1}^{n} S_2(n, k)$ are known as Bell numbers (see, e.g., [8, Section 1.6]). De Bruijn [4] gave a detailed asymptotic analysis of $B(n)$, using the saddle point method. In particular, de Bruijn [4, p. 108] showed that

$$\log B(n) = n \left(L - L_2 - 1 + \frac{L_2 + 1}{L} + \frac{L_2^2}{2L^2} + O\left(\frac{L_2^3}{L^3} \right) \right), \tag{2}$$

where $L = \log n$ and $L_2 = \log \log n$. Therefore we have the upper bound

$$\limsup_{n \to \infty} \frac{\log |f(n)|}{n \log n} \le 1. \tag{3}$$

In a recent paper Subbarao and A. Verma [7] showed that in fact

$$\limsup_{n \to \infty} \frac{\log |f(n)|}{n \log n} = 1.$$

Thus the coefficients a_m in the Dirichlet series $\sum_{m=1}^{\infty} a_m m^{-s} = \prod_{k=2}^{\infty}(1 - k^{-s})$ are not uniformly bounded. This answers the question of Subbarao mentioned earlier. (This result was also obtained by P. T. Bateman [1].)

In this paper we provide a detailed asymptotic analysis of $f(n)$, which allows us to answer some open problems mentioned in [7]. Our main result is the following theorem, which gives an asymptotic estimate for $f(n)$.

Theorem 1 *Let z_n be the solution to the equation $ze^z = -n-1$ with the smallest positive imaginary part. Let $\phi_n(z) = -e^z - (n+1)\log z$, and let w_n be the solution of $w_n^2 = -2/\phi_n''(z_n)$ with $\pi/2 < \arg w_n < \pi$. Then we have*

$$f(n) = \operatorname{Im} \Phi(n) + O\left(\frac{\log n}{n}|\Phi(n)|\right),$$

where

$$\Phi(n) = \frac{n!e}{\sqrt{\pi}} w_n \exp\{\phi_n(z_n)\}.$$

Using estimates for z_n and w_n (see Lemma 1 below), we obtain the following asymptotic upper bound for $\log|f(n)|$, which sharpens (3). We recall here the notations

$$L = \log n, \qquad L_2 = \log\log n \tag{4}$$

introduced earlier.

Corollary 1 *We have, for $n \geq 3$,*

$$\log|f(n)| \leq n\left(L - L_2 - 1 + \frac{L_2 + 1}{L} + \frac{L_2^2 - \pi^2}{2L^2} + O\left(\frac{L_2^3}{L^3}\right)\right).$$

Comparing this bound with the estimate (2) for the Bell numbers $B(n)$, we obtain the following corollary, which shows the cancellation effect occuring in the sum $f(n) = \sum_{k=1}^{n}(-1)^k S_2(n,k)$, when compared to $B(n) = \sum_{k=1}^{n} S_2(n,k)$.

Corollary 2 *We have, for $n \geq 3$,*

$$\log|f(n)| \leq \log B(n) - \frac{\pi^2 n}{2L^2} + O\left(\frac{nL_2^3}{L^3}\right).$$

By investigating the behavior of the argument of $\log \Phi(n)$, we can determine how often $f(n)$ changes signs. This is the content of the following two corollaries.

Corollary 3 *Let $\Phi(n)$ be defined as in Theorem 1. Then we have*

$$f(n) = |\Phi(n)|\left(\sin\theta(n) + O\left(\frac{\log n}{n}\right)\right),$$

where $\theta(t)$ *is a differentiable function defined on* $[3, \infty)$ *satisfying*

$$\theta(t) = -\frac{\pi t}{\log t} + O\left(\frac{t \log\log t}{(\log t)^2}\right), \tag{5}$$

$$\theta'(t) = -\frac{\pi}{\log t} + O\left(\frac{\log\log t}{(\log t)^2}\right), \tag{6}$$

and

$$\theta''(t) = \frac{\pi}{t(\log t)^2} + O\left(\frac{\log\log t}{t(\log t)^3}\right). \tag{7}$$

This result shows that $f(n)$ changes signs infinitely often and that $|f(n)|$ is not eventually monotone. This answers two questions raised by Subbarao and Verma [7].

The following result gives a precise estimate for the locations of the sign changes of $f(n)$.

Corollary 4 *Let* $n_1 < n_2 < \ldots$ *denote the sequence of integers at which* $f(n)$ *changes signs, i.e., at which* $f(n_k) \leq 0 < f(n_k + 1)$ *or* $f(n_k) \geq 0 > f(n_k + 1)$*. Then*

$$n_k = k \log k + O(k \log\log k) \tag{8}$$

and

$$n_{k+1} - n_k = \log k + O(\log\log k). \tag{9}$$

Corollary 4 implies that the density of zeros of $f(n)$ is zero. In particular, we have

$$|\{n \leq x : f(n) = 0\}| \ll \frac{x}{\log x}.$$

However, by a different approach, we can improve this bound.

Theorem 2 *We have*

$$|\{n \leq x : f(n) = 0\}| \ll x^{2/3}.$$

This result provides a partial answer to the question mentioned above whether $f(n) = 0$ infinitely often.

To prove Theorem 1, we adapt the approach used by de Bruijn [4] to study the behavior of $B(n)$. We then use exponential sum estimates to prove Theorem 2.

2 Proof of Theorem 1

In this section we continue to use the notations L, L_2 given in (4). We first deduce some useful estimates for the quantities z_n, w_n and $\phi_n(z_n)$ defined in the statement of Theorem 1.

Lemma 1 *Let z_n, w_n and $\phi_n(z)$ be defined as in the statement of Theorem 1. Then we have*

$$z_n = L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right), \tag{10}$$

$$w_n = \sqrt{\frac{2L}{n}}\left(-\frac{\pi}{2L} + i - \frac{iL_2}{2L} - \frac{i}{2L} + O\left(\frac{L_2^2}{L^2}\right)\right), \tag{11}$$

$$\phi_n(z_n) = n\left(-L_2 + \frac{L_2+1}{L} - \frac{\pi i}{L} + \frac{L_2^2 - \pi^2}{2L^2} - \frac{\pi i L_2}{L^2} + O\left(\frac{L_2^3}{L^3}\right)\right). \tag{12}$$

Proof. By the definition of z_n, we have $e^{z_n} = -(n+1)/z_n$. This implies $|z_n| \ll L$, and by iteration we obtain

$$\begin{aligned}
z_n &= \log(n+1) - \log z_n + \pi i \\
&= L + \pi i - \log\left(L - \log z_n + \pi i\right) + O\left(\frac{1}{n}\right) \\
&= L - L_2 + \pi i + \frac{\log z_n}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right) \\
&= L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right).
\end{aligned}$$

This proves estimate (10).

Similarly, since $\phi_n''(z) = -e^z + (n+1)/z^2$ and thus $\phi_n''(z_n) = (n+1)/z_n + (n+1)/z_n^2$, we have, by (10),

$$\begin{aligned}
w_n^2 &= -\frac{2}{\phi_n''(z_n)} = -\frac{2z_n}{n+1}\left(1 + \frac{1}{z_n}\right)^{-1} \\
&= -\frac{2L}{n}\left(1 - \frac{L_2}{L} + \frac{\pi i}{L} + O\left(\frac{L_2}{L^2}\right)\right)\left(1 - \frac{1}{L} + O\left(\frac{L_2}{L^2}\right)\right).
\end{aligned}$$

We then recall that, by the definition of w_n, $\pi/2 < \arg w_n < \pi$. Therefore

$$\begin{aligned}
w_n &= i\sqrt{\frac{2L}{n}}\left(1 - \frac{L_2}{2L} + \frac{\pi i}{2L} + O\left(\frac{L_2^2}{L^2}\right)\right)\left(1 - \frac{1}{2L} + O\left(\frac{L_2}{L^2}\right)\right) \\
&= \sqrt{\frac{2L}{n}}\left(-\frac{\pi}{2L} + i - \frac{iL_2}{2L} - \frac{i}{2L} + O\left(\frac{L_2^2}{L^2}\right)\right),
\end{aligned}$$

which is the claimed estimate (11). It remains to prove the estimate (12) for $\phi_n(z_n)$.

By (10) and the definitions of $\phi_n(z)$ and z_n, we have

$$\begin{aligned}
\phi_n(z_n) &= -e^{z_n} - (n+1)\log z_n = \frac{n+1}{z_n} - (n+1)\log z_n \\
&= \frac{n}{L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right)} \\
&\qquad - n\log\left(L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right)\right) + O(L_2) \\
&= n\left(\frac{1}{L} + \frac{L_2}{L^2} - \frac{\pi i}{L^2} + O\left(\frac{L_2^2}{L^3}\right)\right) \\
&\qquad - n\left(L_2 - \frac{L_2}{L} + \frac{\pi i}{L} + \frac{L_2}{L^2} - \frac{\pi i}{L^2} - \frac{1}{2L^2}(L_2 - \pi i)^2 + O\left(\frac{L_2^3}{L^3}\right)\right) \\
&= n\left(-L_2 + \frac{L_2+1}{L} - \frac{\pi i}{L} + \frac{L_2^2 - \pi^2}{2L^2} - \frac{\pi i L_2}{L^2} + O\left(\frac{L_2^3}{L^3}\right)\right).
\end{aligned}$$

This proves (12) and completes the proof of the lemma. □

Proof of Theorem 1. By the definition of $f(n)$, we have

$$f(n) = \sum_{\substack{0<n_1<\cdots<n_r \\ a_1,\ldots,a_r>0 \\ a_1n_1+\cdots+a_rn_r=n}} \frac{(-1)^{a_1+\cdots+a_r} n!}{a_1!\ldots a_r!(n_1!)^{a_1}\ldots(n_r!)^{a_r}}.$$

Thus the exponential generating function for $f(n)$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f(n)}{n!} z^n &= \sum_{\substack{n_1<\cdots<n_r \\ a_1,\ldots,a_r>0}} \frac{(-1)^{a_1+\cdots+a_r} z^{a_1n_1+\cdots+a_rn_r}}{a_1!\ldots a_r!(n_1!)^{a_1}\ldots(n_r!)^{a_r}} \\
&= \prod_{n=1}^{\infty} \left\{ \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \left(\frac{z^n}{n!}\right)^a \right\} \\
&= \exp(-z)\exp\left(-\frac{z^2}{2!}\right)\exp\left(-\frac{z^3}{3!}\right)\ldots \\
&= \exp\left\{-(e^z - 1)\right\}.
\end{aligned}$$

(For an alternative derivation of this identity see [7].) Using this generating function and Cauchy's formula, we obtain

$$\frac{f(n)}{n!e} = \frac{1}{2\pi i}\int_{\mathcal{C}} \exp(-e^z) z^{-n-1}\, dz,$$

where $\mathcal{C}$ is a simple closed curve encircling the origin. Since $\exp(-e^z)$ is uniformly bounded in any half-plane $\{z : \operatorname{Re} z \le \sigma\}$, the integration path $\mathcal{C}$ can be replaced by $\Gamma_1 \cup \Gamma_2$, where

$\Gamma_1 = \{z_n + w_n t : -\mathrm{Im}\, z_n/\mathrm{Im}\, w_n \le t < \infty\}$ and $\Gamma_2 = \{\bar{z}_n - \bar{w}_n t : -\infty < t < \mathrm{Im}\, z_n/\mathrm{Im}\, w_n\}$, i.e., Γ_1 is the straight line lying in the upper half-plane that passes through z_n in direction w_n, and the path Γ_2 is the reflection of Γ_1 with respect to the real axis, with direction $-\bar{w}_n$.

We now estimate the integral along Γ_1. Setting $z = z_n + w_n t$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_1} \exp(-e^z) z^{-n-1}\, dz \\
&= \frac{w_n}{2\pi i}\int_{-\mathrm{Im}\, z_n/\mathrm{Im}\, w_n}^{\infty} \exp\left\{\phi_n(z_n + w_n t)\right\}\, dt \\
&= \frac{w_n \exp\left\{\phi_n(z_n)\right\}}{2\pi i}\left\{\int_{-\mathrm{Im}\, z_n/\mathrm{Im}\, w_n}^{-1/|w_n|^{1/3}} + \int_{-1/|w_n|^{1/3}}^{1/|w_n|^{1/3}} + \int_{1/|w_n|^{1/3}}^{|z_n/w_n|} + \int_{|z_n/w_n|}^{\infty}\right\} \\
&\qquad\qquad \exp\left\{\phi_n(z_n + w_n t) - \phi_n(z_n)\right\}\, dt \\
&= \frac{w_n \exp\left\{\phi_n(z_n)\right\}}{2\pi i}\left\{I_1 + I_2 + I_3 + I_4\right\}.
\end{aligned}$$

By estimates (10) and (11) of Lemma 1, we have, for $t \ge |z_n/w_n|$,

$$\mathrm{Re}\, w_n t \le \left(-\frac{\pi}{2L} + O\left(\frac{L_2}{L^2}\right)\right)(L + O(L_2)) = -\frac{\pi}{2} + O\left(\frac{L_2}{L}\right),$$

and thus

$$\begin{aligned}
\mathrm{Re}\,(e^{z_n} - e^{z_n + w_n t}) &\le -\mathrm{Re}\left(\frac{n+1}{z_n}\right) + e^{\mathrm{Re}\,(z_n + w_n t)} \\
&\le -(1 - e^{-\pi/2})\frac{n}{L}\left(1 + O\left(\frac{L_2}{L}\right)\right).
\end{aligned} \tag{13}$$

Furthermore, since, by the same lemma,

$$\arg w_n - \arg z_n = \frac{\pi}{2} + O\left(\frac{L_2}{L}\right), \tag{14}$$

we have $|z_n + w_n t| \ge |w_n t|$ for sufficiently large n and $t \ge |z_n/w_n|$. Using (13), it follows that

$$\begin{aligned}
I_4 &\le \int_{|z_n/w_n|}^{\infty} \exp\left\{\mathrm{Re}\left(e^{z_n} - e^{z_n + w_n t} - (n+1)\log\left|\frac{z_n + w_n t}{z_n}\right|\right)\right\}\, dt \\
&\le \int_{|z_n/w_n|}^{\infty} \exp\left\{-\frac{c_1 n}{L} - n\log\left(\frac{|w_n|}{|z_n|}t\right)\right\}\, dt \\
&= \frac{|z_n|}{(n-1)|w_n|}\exp\left\{-\frac{c_1 n}{L}\right\} \ll \sqrt{\frac{n}{L^3}}\exp\left\{-\frac{c_1 n}{L}\right\}
\end{aligned} \tag{15}$$

for sufficiently large n, where c_1 is a suitable positive constant.

We next estimate I_3. We first show that $\mathrm{Re}\,(e^{z_n} - e^{z_n+w_nt}) \ll t\sqrt{n/L^3}$ uniformly for all $t > 0$ and sufficiently large n. By the definition of z_n and (10), we have

$$\mathrm{Re}\, e^{z_n} = -\mathrm{Re}\,\frac{n+1}{z_n} = -\frac{n}{L}\left(1 + O\left(\frac{L_2}{L}\right)\right)$$

and

$$\mathrm{Im}\, e^{z_n} = -\mathrm{Im}\,\frac{n+1}{z_n} = -\frac{\pi n}{L^2}\left(1 + O\left(\frac{L^2}{L}\right)\right).$$

Using the inequality $0 < \sqrt{x^2+y^2} - x \le y^2/(2x)$, which holds uniformly for all x and y with $0 < y \le x$, we obtain

$$\left|\mathrm{Re}\, e^{z_n} + |e^{z_n}|\right| \le \frac{1}{2}\left|\frac{(\mathrm{Im}\, e^{z_n})^2}{\mathrm{Re}\, e^{z_n}}\right| \le \frac{c_2 n}{L^3},$$

where c_2 is a positive constant. Therefore if t is a real number satisfying $\mathrm{Re}\, w_n t < -2c_2/L^2$, i.e., $t > (4c_2/\pi + o(1))/(|w_n|L)$, then we have by (11)

$$\begin{aligned}\mathrm{Re}\,(e^{z_n} - e^{z_n+w_nt}) &\le (|e^{z_n}| + \mathrm{Re}\, e^{z_n}) + \left(|e^{z_n}|e^{\mathrm{Re}\, w_n t} - |e^{z_n}|\right)\\ &\le \frac{c_2 n}{L^3} - \frac{2c_2}{L^2}|e^{z_n}| \le 0,\end{aligned}$$

On the other hand, if t is in the range $0 < t \le (4c_2/\pi + o(1))/(|w_n|L)$, then, by (10) and (11),

$$\begin{aligned}\mathrm{Re}\,(e^{z_n} - e^{z_n+w_nt}) &= \mathrm{Re}\,(-e^{z_n}w_nt) + O(|e^{z_n}w_n^2|t^2)\\ &= \mathrm{Re}\left\{\frac{n}{L}\left(1 + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2}{L^2}\right)\right)\right.\\ &\qquad \left.\sqrt{\frac{2L}{n}}\left(-\frac{\pi}{2L} + i + \frac{iL^2 - i}{L} + O\left(\frac{L_2^2}{L^2}\right)\right)t\right\} + O(t^2)\\ &= \left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\frac{n}{L^3}}t + O(t^2) \le c_3\sqrt{\frac{n}{L^3}}t\end{aligned}$$

for sufficiently large n, where c_3 is a positive constant. This proves the assertion that $\mathrm{Re}\, e^{z_n} - \mathrm{Re}\, e^{z_n+w_nt} \ll t\sqrt{n/L^3}$ uniformly for all $t > 0$ and sufficiently large n.

We now estimate I_3. For t in the interval $[1/|w_n|^{1/3}, |z_n|/|w_n|]$, the estimate (14) implies that

$$\log\left|1 + \frac{w_nt}{z_n}\right| \ge \frac{|w_n|}{4|z_n|}t$$

for sufficiently large n. It follows that, by Lemma 1,

$$\begin{aligned}I_3 &\le \int_{1/|w_n|^{1/3}}^{|z_n/w_n|} \exp\left\{c_3\sqrt{\frac{n}{L^3}}t - \frac{n}{4}\frac{|w_n|}{|z_n|}t\right\}\, dt\\ &\le \frac{|z_n|}{|w_n|}\exp\left\{-\left(\frac{1}{4} + o(1)\right)\sqrt{\frac{n}{L}}|w_n|^{-1/3}\right\} \ll \exp\left\{-\frac{c_4 n^{2/3}}{L^{2/3}}\right\}\end{aligned} \tag{16}$$

for some suitable positive constant c_4. The same bound holds for I_1. It remains to estimate I_2.

In the range $-1/|w_n|^{1/3} \le t \le 1/|w_n|^{1/3}$, we have, by Lemma 1,

$$\phi_n^{(3)}(z_n) = -e^{z_n} - \frac{2(n+1)}{z_n^3} = \frac{n+1}{z_n} + O\left(\frac{n}{L^3}\right) \ll \frac{n}{L},$$

$$\phi_n^{(4)}(z_n + w_n t) = -e^{z_n + w_n t} + \frac{6(n+1)}{(z_n + w_n)^3} \ll \frac{n}{L} e^{|w_n|^{2/3}} + \frac{n}{L^3} \ll \frac{n}{L},$$

and thus

$$\phi_n^{(3)}(z_n)(w_n t)^3 \ll \frac{n}{L}|w_n|^2 \ll 1,$$

$$\phi_n(z_n + w_n t) - \phi_n(z_n) - \phi_n'(z_n) w_n t - \frac{\phi_n''(z_n)}{2}(w_n t)^2 - \frac{\phi_n^{(3)}(z_n)}{6}(w_n t)^3 \ll \frac{n}{L}|w_n t|^4 \ll 1.$$

Since, by the definition of z_n and w_n, $\phi_n'(z_n) = 0$ and $\phi_n''(z_n) w_n^2/2 = -1$, it follows that

$$\begin{aligned} I_2 &= \int_{-1/|w_n|^{1/3}}^{1/|w_n|^{1/3}} \exp\left\{ -t^2 + \frac{\phi_n^{(3)}(z_n)}{6}(w_n t)^3 + O\left(\frac{n}{L}|w_n t|^4\right) \right\} dt \\ &= \int_{-1/|w_n|^{1/3}}^{1/|w_n|^{1/3}} e^{-t^2} \left(1 + \frac{\phi_n^{(3)}(z_n) w_n^3}{6} t^3 + O\left(|\phi_n^{(3)}(z_n)^2 w_n^6 |t^6\right) + O\left(\frac{n|w_n|^4}{L} t^4\right) \right) dt \\ &= \sqrt{\pi} + O\left(\exp\left\{-|w_n|^{-2/3}\right\}\right) + O\left(|\phi_n^{(3)}(z_n)^2 w_n^6|\right) + O\left(\frac{n|w_n|^4}{L}\right) \\ &= \sqrt{\pi} + O\left(\frac{L}{n}\right). \end{aligned}$$

Combining this estimate, (15) and (16), we obtain

$$\int_{\Gamma_1} \exp\{\phi_n(z)\}\, dz = w_n \exp\{\phi_n(z_n)\} \left(\sqrt{\pi} + O\left(\frac{L}{n}\right) \right).$$

Since $\int_{\Gamma_2} = -\overline{\int}_{\Gamma_1}$, it follows that

$$\begin{aligned} f(n) &= \frac{n!e}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_2} \right) = \operatorname{Im} \frac{n!e}{\sqrt{\pi}} w_n \exp\{\phi_n(z_n)\} + O\left(\frac{L}{n} n! |w_n \exp\{\phi_n(z_n)\}|\right) \\ &= \operatorname{Im} \Phi(n) + O\left(\frac{L}{n}|\Phi(n)|\right). \end{aligned}$$

This completes the proof of Theorem 1. □

3 Proofs of Corollaries

Throughout this section, L will denote $\log n$ or $\log t$, and L_2 will denote $\log\log n$ or $\log\log t$, depending on the context.

Proof of Corollary 1. By Theorem 1, we have

$$|f(n)| \le \frac{n!e}{\sqrt{\pi}} \left| w_n \exp\{\phi_n(z_n)\}\right| \left(1 + O\left(\frac{L}{n}\right)\right).$$

By Lemma 1 and the Stirling formula for $n!$, it follows that

$$\begin{aligned}\log|f(n)| &\le (n+1/2)\log n - n + \operatorname{Re}\phi_n(z_n) + O(1)\\ &= n\left(L - L_2 - 1 + \frac{L_2+1}{L} + \frac{L^2-\pi^2}{2L^2} + O\left(\frac{L_2^3}{L^3}\right)\right).\end{aligned}$$

This proves Corollary 1. Corollary 2 is an immediate consequence of Corollary 1. □

Proof of Corollary 3. We first note that the domains of the functions z_n, w_n, $\phi_n(z)$ and $\Phi(n)$ can be extended from the set of positive integers to the set of positive real numbers, and the asymptotic formulas in Lemma 1 remain valid with n replaced by a positive real number t. From Theorem 1 we deduce that

$$f(n) = |\Phi(n)| \left(\sin\theta(n) + O\left(\frac{L}{n}\right)\right),$$

where

$$\theta(t) = \operatorname{Im}\left(\phi_t(z_t) + \log w_t\right). \tag{17}$$

By Lemma 1, we have

$$\operatorname{Im}\log w_t = \frac{\pi}{2} + O\left(\frac{1}{L}\right)$$

and

$$\operatorname{Im}\phi_t(z_t) = -\frac{\pi t}{L} + O\left(\frac{tL_2}{L^2}\right).$$

The claimed estimate (5) for $\theta(t)$ follows by inserting these estimates into (17).

We now prove estimate (6). By the definition of z_t, we have $z_t e^{z_t} + (t+1) = 0$. Thus, the chain rule yields

$$\frac{dz_t}{dt} = -\frac{1}{e^{z_t}(z_t+1)} = \frac{z_t}{(t+1)(z_t+1)}. \tag{18}$$

Since

$$w_t^2 = -\frac{2}{\phi_t''(z_t)} = -\frac{2}{-e^{z_t} + (t+1)/z_t^2} = -\frac{2}{(t+1)/z_t + (t+1)/z_t^2},$$

by estimate (10) of Lemma 1 and (18), we have

$$\frac{1}{w_t}\frac{dw_t}{dt} = -\frac{1}{2}\frac{\dfrac{d}{dt}\left(\dfrac{t+1}{z_t}+\dfrac{t+1}{z_t^2}\right)}{\left(\dfrac{t+1}{z_t}+\dfrac{t+1}{z_t^2}\right)} = -\frac{1}{2(t+1)} + \frac{z_t^{-2}+2z_t^{-3}}{2(z_t^{-1}+z_t^{-2})}\frac{dz_t}{dt} \ll \frac{1}{t}. \tag{19}$$

Similarly, we have

$$\begin{aligned}\frac{d\phi_t(z_t)}{dt} &= \frac{d}{dt}\left(-e^{z_t}-(t+1)\log z_t\right) \\ &= -\log z_t - \left(e^{z_t}+\frac{t+1}{z_t}\right)\frac{dz_t}{dt} = -\log z_t\end{aligned} \tag{20}$$

and thus, by (10),

$$\begin{aligned}\operatorname{Im}\frac{d\phi_t(z_t)}{dt} &= -\operatorname{Im}\left\{\log\left(L-L_2+\pi i+\frac{L_2}{L}-\frac{\pi i}{L}+O\left(\frac{L_2^2}{L^2}\right)\right)\right\} \\ &= -\frac{\pi}{L}+O\left(\frac{L_2}{L^2}\right).\end{aligned} \tag{21}$$

Combining this estimate and (19), we obtain

$$\theta'(t) = \operatorname{Im}\frac{1}{w_t}\frac{dw_t}{dt} + \operatorname{Im}\frac{d}{dt}\phi_t(z_t) = -\frac{\pi}{L}+O\left(\frac{L_2}{L^2}\right).$$

This proves the estimate (6).

The proof of (7) is essentially the same as that of (6). By (18) and (20), we have

$$\frac{d^2}{dt^2}\phi_t(z_t) = -\frac{d}{dt}\log z_t = -\frac{1}{z_t}\frac{z_t}{(t+1)(z_t+1)} = -\frac{1}{t\left(L-L_2+\pi i+1+O(L_2/L)\right)}$$

and hence

$$\operatorname{Im}\frac{d^2}{dt^2}\phi_t(z_t) = \frac{\pi}{tL^2}+O\left(\frac{L_2}{tL^3}\right).$$

Similarly, we have, by (18) and (19),

$$\frac{d^2}{dt^2}\log w_t = \frac{1}{(t+1)^2}+\frac{1}{2}\frac{d}{dt}\left\{-\frac{z_t^{-2}+2z_t^{-3}}{z_t^{-1}+z_t^{-2}}\frac{z_t}{(t+1)(z_t+1)}\right\} \ll \frac{1}{t^2}.$$

Thus we conclude that

$$\theta''(t) = \operatorname{Im}\frac{d^2}{dt^2}\phi_t(z_t) + \operatorname{Im}\frac{d^2}{dt^2}\log w_t = \frac{\pi}{tL^2}+O\left(\frac{L_2}{tL^3}\right).$$

This completes the proof of Corollary 3. □

Proof of Corollary 4. Let $\theta(t)$ be defined as in Corollary 3. Let k be a positive integer, and let t_k be the solution of $\theta(t) = -k\pi$. By Corollary 3, t_k satisfies

$$k = \frac{t_k}{\log t_k} + O\left(\frac{t_k \log\log t_k}{(\log t_k)^2}\right).$$

Hence, we obtain

$$t_k = k \log t_k + O\left(\frac{t_k \log\log t_k}{\log t_k}\right) = k \log k + O(k \log\log k). \tag{22}$$

From this estimate we deduce that $n_k = t_k + O(1)$, and therefore estimate (8) holds.

To prove the second part of the corollary, we note that, by the mean value theorem, $\pi = \theta(t_k) - \theta(t_{k+1}) = (t_k - t_{k+1})\theta'(\xi_k)$, where ξ_k is a real number between t_k and t_{k+1}. The estimate (22) implies that

$$\xi_k = t_k + O(t_{k+1} - t_k) = t_k + O(k \log\log k).$$

Hence, by (6) in Corollary 3, we have

$$\begin{aligned} n_{k+1} - n_k = t_{k+1} - t_k + O(1) &= -\frac{\pi}{\theta'(\xi_k)} \\ &= \frac{\pi}{\dfrac{\pi}{\log t_k} + O\left(\dfrac{\log\log t_k}{(\log t_k)^2}\right)} = \log k + O(\log\log k), \end{aligned}$$

which is the claimed result. □

4 Proof of Theorem 2

We will use the following well-known exponential sum estimate (see, e.g., [5, p. 17]).

Lemma 2 *Let a and b be integers with $a < b$, and let g be twice differentiable on $[a, b]$ with $g''(x) \geq \rho > 0$ or $g''(x) \leq -\rho < 0$ for some positive real number ρ and all $x \in [a, b]$. Then*

$$\left|\sum_{n=a}^{b} e^{ig(n)}\right| \ll (|g'(b) - g'(a)| + 1)(\rho^{-1/2} + 1).$$

Proof of Theorem 2. It suffices to show that

$$\left|\left\{x^{1/2} < n \leq x : f(n) = 0\right\}\right| \ll x^{2/3}.$$

In light of Corollary 3, we have for sufficiently large x

$$\left|\left\{x^{1/2} < n \leq x : f(n) = 0\right\}\right| \leq \left|\left\{x^{1/2} < n \leq x : \|\theta(n)\| < \frac{c_1 \log n}{n}\right\}\right|$$

for some positive constant c_1, where $\theta(t)$ is the function occuring in the statement of Corollary 3 and $\|\theta(n)\|$ denotes the distance from $\theta(n)$ to the closest integer multiple of π. On the other hand, if $H = H(x)$ is an integer-valued function satisfying $H(x) \leq (\pi/(2c_1))x^{1/2}/\log x$, then, for $x^{1/2} < n \leq x$, the condition $\|\theta(n)\| < c_1 \log n/n$ implies that

$$\left(\frac{\sin((H+1)\theta(n))}{\sin\theta(n)}\right)^2 \geq \frac{(H+1)^2}{\pi^2}.$$

We therefore have

$$\left|\left\{x^{1/2} \leq n \leq x : f(n) = 0\right\}\right| \ll \frac{1}{H^2} \sum_{x^{1/2}<n\leq x} \left(\frac{\sin((H+1)\theta(n))}{\sin\theta(n)}\right)^2. \tag{23}$$

Using the identity

$$\left(\frac{\sin\left((H+1)t\right)}{\sin t}\right)^2 = \sum_{h=-H}^{H} (H+1-|h|)e^{2iht},$$

the right-hand side of (23) becomes

$$\frac{1}{H^2}\sum_{h=-H}^{H}(H+1-|h|)S_h \leq \frac{x(H+1)}{H^2} + \frac{2}{H^2}\sum_{h=1}^{H}(H+1-h)|S_h|, \tag{24}$$

where $S_h = \sum_{x^{1/2}<n\leq x} e^{2ih\theta(n)}$.

We now estimate the exponential sum S_h. By Corollary 3, we have

$$|h\theta'(\lfloor x\rfloor) - h\theta'(\lceil x^{1/2}\rceil)| \ll \frac{h}{\log x}$$

and

$$h\theta''(t) \gg \frac{h}{x(\log x)^2}$$

for $x^{1/2} < t \leq x$. Thus, applying Lemma 2 with $g(t) = h\theta(t)$, we obtain

$$|S_h| \ll \left(\frac{h}{\log x} + 1\right)\left((\log x)\sqrt{\frac{x}{h}} + 1\right).$$

Inserting this estimate in (24), we arrive at

$$\begin{aligned}\left|\left\{x^{1/2} \leq n \leq x : f(n) = 0\right\}\right| &\ll \frac{x}{H} + \frac{1}{H}\sum_{h=1}^{H}\left(\sqrt{hx} + (\log x)\sqrt{\frac{x}{h}} + \frac{h}{\log x} + 1\right) \\ &\ll \frac{x}{H} + \sqrt{Hx} + (\log x)\sqrt{\frac{x}{H}} + \frac{H}{\log x} + 1.\end{aligned}$$

Taking $H = \lfloor x^{1/3}\rfloor$, we finally obtain

$$\left|\left\{x^{1/2} \leq n \leq x : f(n) = 0\right\}\right| \ll x^{2/3}.$$

This completes the proof. □

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