# An Analogue of Covering Space Theory for Ranked Posets

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**Abstract**

Suppose $P$ is a partially ordered set that is locally finite, has a least element, and admits a rank function. We call $P$ a weighted-relation poset if all the covering relations of $P$ are assigned a positive integer weight. We develop a theory of covering maps for weighted-relation posets, and in particular show that any weighted-relation poset $P$ has a universal cover $\widetilde{P} \to P$, unique up to isomorphism, so that

1. $\widetilde{P} \to P$ factors through any other covering map $P' \to P$;
2. every principal order ideal of $\widetilde{P}$ is a chain; and
3. the weight assigned to each covering relation of $\widetilde{P}$ is 1.

If $P$ is a poset of "natural" combinatorial objects, the elements of its universal cover $\widetilde{P}$ often have a simple description as well. For example, if $P$ is the poset of partitions ordered by inclusion of their Young diagrams, then the universal cover $\widetilde{P}$ is the poset of standard Young tableaux; if $P$ is the poset of rooted trees ordered by inclusion, then $\widetilde{P}$ consists of permutations. We discuss several other examples, including the posets of necklaces, bracket arrangements, and compositions.

## 1 Introduction

For topological spaces, the notion of a covering space is familiar (see, e.g., [9]): a covering map $p : X' \to X$ is a continuous surjection such that, for sufficiently small open sets $U \subset X$, $p^{-1}(U)$ is a disjoint union of open sets in $X'$ each of which $p$ maps homeomorphically onto $U$. For any space $X$ satisfying appropriate hypotheses (e.g., that $X$ is connected, locally arcwise connected and semilocally simply connected), there is a simply connected covering space $\pi : \widetilde{X} \to X$, which is universal in the sense that it "factors through" any other connected cover of $X$, i.e., if $p : X' \to X$ is any covering map with $X'$ conneced, then there is a covering map $f : \widetilde{X} \to X'$ so that $\pi = pf$. The universal covering space of $X$ is unique up to homeomorphism over $X$.

In this paper we develop a theory of covering maps for ranked posets. More precisely, we define covering maps of "weighted-relation" posets, which are locally finite ranked posets with least element that have a positive integer weight associated with each of their covering relations. We show that every such weighted-relation poset $P$ has a universal cover $\widetilde{P} \to P$, unique up to isomorphism in an appropriate category, which factors through any other cover $P' \to P$. The universal cover $\widetilde{P}$ is "simple" in the sense that its Hasse diagram is a tree and all its covering relations have weight 1.

In many cases where $P$ is a poset of familiar combinatorial objects, the elements of the universal cover $\widetilde{P}$ also have a simple description. For example, the poset of monomials in commuting variables $x_1, \ldots, x_k$ has a universal cover whose elements are monomials in $k$ noncommuting variables (Example 2 in §4 below); the poset of compositions (with an appropriate choice of weights) has as its universal cover the poset of Cayley permutations in the sense of [6] (Example 6). We discuss several other examples, including the posets of necklaces, bracket arrangements, partitions, and rooted trees.

## 2 Weighted-relation posets

Our terminology for posets follows [11]. Let $(P, \preceq)$ be a locally finite poset with least element $\hat{0}$ and rank function $|\cdot|$. By a weight system on the relations of $P$, we mean a function $n$ that assigns a nonnegative integer $n(x, y)$ to every pair $x, y \in P$ so that

1. $n(x, y) \neq 0$ if and only if $x \preceq y$;
2. for all elements $x \prec y$ and nonnegative integers $|x| \leq k \leq |y|$,

$$n(x, y) = \sum_{|z|=k} n(x, z)n(z, y).$$

(Note that the second condition implies $n(x, x) = 1$ for all $x \in P$.)

We call a poset $P$ together with a weight system on its relations a weighted-relation poset. By induction on $|y| - |x|$ it is easy to prove from the definition that for any $x \prec y$ in $P$

$$n(x, y) = \sum_{x=x_1 \prec x_2 \prec \cdots \prec x_k = y} n(x_1, x_2)n(x_2, x_3) \cdots n(x_{k-1}, x_k),$$

where the sum is over all saturated chains $x = x_1 \prec x_2 \prec \cdots \prec x_k = y$ from $x$ to $y$: thus, to define $n$ it suffices to give $n(x, y)$ when $y$ covers $x$. In particular, any ranked, locally finite poset with least element can be made a weighted-relation poset by assigning 1 to every covering relation.

The motivation for this definition comes from thinking of a covering relation $x \prec y$ of $P$ as indicating $y$ can be built from $x$ by some kind of elementary operation: $n(x, y)$ is the number of ways this can be done. Then in general $n(u, v)$ is the number of ways that $v$ can be built up from $u$ via a sequence of elementary operations. For examples see §4 below.

Let $\mathcal{W}$ be the category whose objects are weighted-relation posets, and whose morphisms are defined as follows. A morphism of weighted-relation posets $P$, $P'$ is a rank-preserving function $f : P \to P'$ such that, for any elements $t, s$ of $P$,

$$n(f(t), f(s)) \geq \sum_{s' \in f^{-1}(f(s))} n(t, s'). \tag{1}$$

In particular, any such function $f$ is order-preserving. Also, if $f$ has an inverse $f^{-1}$ that is also a morphism of weighted-relation posets, then $n(f(t), f(s)) = n(t, s)$ for all $t, s \in P$.

We call a weighted-relation poset $P$ simple if $n(x, y)$ is 1 or 0 for any $x, y \in P$. The following result is evident.

**Proposition 2.1.** *If $P$ is a weighted-relation poset, the following are equivalent:*
*(i) $P$ is simple;*
*(ii) the Hasse diagram of $P$ is a tree, and every covering relation has weight 1;*
*(iii) for every $x \in P$, $n(\hat{0}, x) = 1$.*

We also record the following fact, which is an immediate consequence of inequality (1).

**Proposition 2.2.** *If $f : P \to P'$ is a morphism of weighted-relation posets and $P'$ is simple, then $f$ is an injective function and $P$ is simple.*

# 3 Covering maps

Let $P'$ and $P$ be weighted-relation posets. We say that a rank-preserving function $\pi : P' \to P$ is a covering map if, whenever $s, r \in P$ with $\pi(s') = s$,

$$n(s, r) = \sum_{r' \in \pi^{-1}(r)} n(s', r'). \tag{2}$$

Note that equation (2) implies that $\pi$ is a morphism of weighted-relation posets, and taking $s = \hat{0}$, we see that $\pi$ is also surjective.

To prove that a given rank-preserving function is a covering map, it suffices to prove equation (2) for $|r| - |s| = 1$. For suppose (2) holds when $|r| - |s| = 1$, and suppose inductively it holds for $|r| - |s| < n$, $n > 1$. Let $r, s \in P$ with $|r| - |s| = n$, and let $\pi(s') = s$. Then

$$n(s, r) = \sum_{|t|=|s|+1} n(s, t)n(t, r) = \sum_{|t|=|s|+1} \sum_{t' \in \pi^{-1}(t)} \sum_{r' \in \pi^{-1}(r)} n(s', t')n(t', r'),$$

and since the sets $\pi^{-1}(t)$, as $t$ runs through the rank-$(|s| + 1)$ elements of $P$, partition the rank-$(|s| + 1)$ elements of $P'$,

$$n(s, r) = \sum_{|t'|=|s'|+1} \sum_{r' \in \pi^{-1}(r)} n(s', t')n(t', r') = \sum_{r' \in \pi^{-1}(r)} n(s', r').$$

If $P$ is a fixed weighted-relation poset, there is a category $\mathcal{W}/P$ of covers of $P$ whose objects are covering maps $\pi : P' \to P$. A morphism from $\pi_1 : P_1 \to P$ to $\pi_2 : P_2 \to P$ in $\mathcal{W}/P$ is a morphism $f : P_1 \to P_2$ in $\mathcal{W}$ such that $\pi_2 f = \pi_1$. In fact, all such functions $f$ are covering maps.

**Theorem 3.1.** *Suppose $\pi_i : P_i \to P$ is a covering map for $i = 1, 2$, and suppose $f : P_1 \to P_2$ is a morphism of weighted-relation posets such that $\pi_2 f = \pi_1$. Then $f$ is a covering map.*

*Proof.* We show $f$ satisfies equation (2) above. Let $s, r \in P_2$, $s' \in P_1$ with $f(s') = s$. Since $\pi_2$ is a covering map,

$$n(\pi_2(s), \pi_2(r)) = \sum_{i=1}^{k} n(s, r_i),$$

where $\pi_2^{-1}(\pi_2(r)) = \{r_1, \ldots, r_k\}$. For each $r_i$ in the image of $f$,

$$\sum_{r' \in f^{-1}(r_i)} n(s', r') \leq n(s, r_i). \tag{3}$$

Now $\bigcup_{i=1}^{k} f^{-1}(r_i) = \pi_1^{-1}(\pi_2(r))$, and since $\pi_1$ is a covering map we have

$$\sum_{r' \in \pi_1^{-1}(\pi_2(r))} n(s', r') = n(\pi_2(s), \pi_2(r)) = \sum_{i=1}^{k} n(s, r_i). \tag{4}$$

Comparing (3) and (4), we see there is a contradiction unless each of the sets $f^{-1}(r_i)$ is nonempty and (3) is an equality for all $i$. □

**Theorem 3.2.** *Suppose $\pi : P' \to P$ is a covering map and $f : Q \to P$ is a morphism of weighted-relation posets, with $Q$ simple. Then $f$ can be lifted to $P'$, i.e., there is a morphism of weighted-relation posets $f' : Q \to P'$ such that $\pi f' = f$.*

*Proof.* We define $f' : Q \to P'$ by induction on rank; there is no problem getting started since $f'$ must take $\hat{0} \in Q$ to $\hat{0} \in P'$. Suppose $f'$ has already been defined for rank $< n$. For a rank-$(n-1)$ element $z \in Q$ and a rank-$n$ element $x \in f(Q)$ with $x \succ f(z)$, let

$$C(x, z) = \{z' \in Q |\ z' \succ z,\ |z'| = n, \text{ and } f(z') = x\}.$$

Since the Hasse diagram of $Q$ is a tree, sets of the form $C(x, z)$ partition the rank-$n$ elements of $Q$. We shall extend $f'$ to $C(x, z)$. For $z' \in C(x, z)$,

$$n(f(z), x) \geq \sum_{z' \in C(x,z)} n(z, z') = \text{card } C(x, z).$$

Let $S = \{y \in P' |\ y \succ f'(z) \text{ and } \pi(y) = x\}$. For any $y \in S$,

$$n(f(z), x) = n(\pi f'(z), \pi(y)) = \sum_{y' \in S} n(f'(z), y')$$

and hence

$$\text{card } C(x,z) \leq \sum_{i=1}^{k} n(f'(z), y_i), \tag{5}$$

where $S = \{y_1, y_2, \ldots, y_k\}$. Choose a partition of $C(x,z)$ into disjoint subsets $S_1, \ldots, S_k$ (some possibly empty) so that $S_i$ has cardinality at most $n(f'(z), y_i)$: this is possible because of inequality (5). Extend $f'$ to $C(x,z)$ by setting $f'(z') = y_i$ for all $z' \in S_i$. Then for all $z' \in C(x,z)$,

$$n(f'(z), f'(z')) \geq \sum_{f'(z'')=f'(z')} n(z, z'').$$

Reasoning in the same way as in the paragraph following equation (2) above, we can conclude that $f'$ is extended as a morphism of weighted-relation posets; and by construction

$$\pi f'(z') = x = f(z')$$

for all $z' \in C(x,z)$. □

**Theorem 3.3.** *If $P$ is a weighted-relation poset, there is a poset $\widetilde{P}$ and a covering map $\pi : \widetilde{P} \to P$ so that $\widetilde{P}$ is a simple weighted-relation poset. Further, the fiber $\pi^{-1}(x)$ of each $x \in P$ contains $n(\hat{0}, x)$ elements.*

*Proof.* Again we proceed by induction on the rank. Let $P^{(n)}$ be the set of elements of $P$ of rank at most $n$. Suppose a covering $\pi : \widetilde{P}^{(n-1)} \to P^{(n-1)}$ with $\widetilde{P}^{(n-1)}$ simple has already been constructed, and let $x$ be a rank-$n$ element of $P$. Since $P$ is locally finite, the set $C(x)$ of elements covered by $x$ is finite: let $C(x) = \{x_1, \ldots, x_r\}$. Each fiber $\pi^{-1}(x_i)$ contains $n(\hat{0}, x_i)$ rank-$(n-1)$ elements of $\widetilde{P}$: call them $\widetilde{x}_{i1}, \widetilde{x}_{i2}, \ldots, \widetilde{x}_{im_i}$, where $m_i = n(\hat{0}, x_i)$. Let $K(x)$ be the set

$$\{(i,j,k) |\ 1 \leq i \leq \text{card } C(x),\ 1 \leq j \leq n(\hat{0}, x_i),\ 1 \leq k \leq n(x_i, x)\},$$

and define

$$\widetilde{P}^{(n)} = \widetilde{P}^{(n-1)} \cup \coprod_{x \in P, |x|=n} K(x).$$

Extend the weight system (and order) of $\widetilde{P}^{(n-1)}$ to $\widetilde{P}^{(n)}$ by putting

$$n(z, (i,j,k)) = \begin{cases} 1, & \text{if } z \preceq \widetilde{x}_{ij}, \\ 0, & \text{otherwise,} \end{cases}$$

for any $(i,j,k) \in K(x)$, $z \in \widetilde{P}^{(n-1)}$. Then $\widetilde{P}^{(n)}$ is simple: for any $(i,j,k) \in K(x)$ there is a unique chain to $\hat{0}$ passing through $\widetilde{x}_{ij}$, so

$$n(\hat{0}, (i,j,k)) = n(\hat{0}, \widetilde{x}_{ij}) n(\widetilde{x}_{ij}, (i,j,k)) = 1.$$

(The set $C(x) \cap C(x')$ may be nonempty for $x \neq x'$, so the same point of $\widetilde{P}^{(n-1)}$ may be labelled as both $\widetilde{x}_{ij}$ and $\widetilde{x}'_{pq}$, but this does not affect the conclusion since we are taking a disjoint union of the $K(x)$.)

Now extend $\pi$ to $\widetilde{P}^{(n)}$ by having $\pi$ send each element of $K(x)$ to $x$. Then $\pi^{-1}(x) = K(x)$ contains

$$\sum_{i=1}^{r} n(\hat{0}, x_i)n(x_i, x) = n(\hat{0}, x)$$

elements. Also, for any $z \in \widetilde{P}^{(n-1)}$ and rank-$n$ element $x$ of $P$, we have

$$n(\pi(z), x) = \sum_{i=1}^{r} n(\pi(z), x_i)n(x_i, x) =$$
$$\sum_{i=1}^{r}\sum_{j=1}^{m_i}\sum_{k=1}^{n(x,x_i)} n(z, \widetilde{x}_{ij})n(\widetilde{x}_{ij}, (i, j, k)) = \sum_{w\in\pi^{-1}(x)} n(z, w),$$

so $\pi$ is extended as a covering map. □

By a universal cover of $P$, we mean a cover $\widetilde{P} \to P$ so that, for any other cover $P' \to P$, there is a morphism of $\mathcal{W}/P$ from $\widetilde{P} \to P$ to $P' \to P$.

**Theorem 3.4.** *If $P$ is a weighted-relation poset, a cover $\widetilde{P} \to P$ is universal if and only if $\widetilde{P}$ is simple, and such a cover is unique up to isomorphism in $\mathcal{W}/P$.*

*Proof.* Suppose that $p : \widetilde{P} \to P$ is a cover with $\widetilde{P}$ simple, and let $\pi : P' \to P$ be another cover. By Theorem 3.2, $p$ can be lifted to a morphism $p' : \widetilde{P} \to P'$ of weighted-relation posets so that $\pi p' = p$: but this means $p : \widetilde{P} \to P$ is a universal cover. Thus, a simple cover is universal.

Now suppose $\pi' : P' \to P$ is a universal cover. By Theorem 3.3 there is a simple cover $\pi : \widetilde{P} \to P$, and by universality there is a morphism of $\mathcal{W}/P$ from $\pi'$ to $\pi$. Thus there is a morphism of weighted posets $f : P' \to \widetilde{P}$ which (by Theorem 3.1) is a covering map, hence surjective; and since $\widetilde{P}$ is simple, Proposition 2.2 says $f$ is injective and $P'$ is simple. It follows that $f$ is an isomorphism of $\mathcal{W}/P$. □

# 4 Examples

*Example* 1. Let $P$ be the poset of subsets of $\{1, 2, \ldots, n\}$, ordered by inclusion, with each covering relation given weight 1. Then the universal cover $\widetilde{P}$ can be identified with the set of linearly ordered subsets of $\{1, 2, \ldots, n\}$, with $A \preceq B$ in $\widetilde{P}$ if $A$ is an initial segment of $B$; and $\widetilde{P} \to P$ forgets the order. Evidently the fiber of any rank-$k$ element of $P$ has $k!$ elements, so there are a total of $k!\binom{n}{k}$ rank-$k$ elements in $\widetilde{P}$.

*Example* 2. Let $M$ be the poset of monomials in $k$ commuting variables $x_1, \ldots, x_k$, with $m \preceq m'$ in $M$ if there is a monomial $m''$ such that $m' = mm''$. The rank on $M$ is given by total degree, each of the $x_i$ having degree one; the least element of $M$ is the empty monomial 1; and the covering relations are all given weight 1. Then the universal cover

$\widetilde{M}$ is isomorphic to the poset of monomials in $k$ noncommuting variables $X_1, \ldots, X_k$, with weights given by

$$n(w, w') = \begin{cases} 1, & \text{if } w' = wX_i \text{ for some } i, \\ 0, & \text{otherwise,} \end{cases}$$

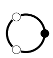
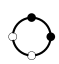
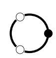
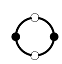
for $|w'| - |w| = 1$. Clearly $\widetilde{M}$ is simple. The function $\pi : \widetilde{M} \to M$ that sends $X_i$ to $x_i$ (so, e.g., $\pi^{-1}(x_1^2 x_2) = \{X_1^2 X_2, X_1 X_2 X_1, X_2 X_1^2\}$) is a covering map. The cardinality of the fiber of any monomial is given by

$$n(1, x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}) = \binom{i_1 + \cdots + i_k}{i_1 \; i_2 \; \cdots \; i_k},$$

and the total number of rank-$n$ elements of $\widetilde{M}$ is

$$\sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1 \; \cdots \; i_k} = k^n.$$

*Example* 3. Let $\mathcal{N}$ be the set of circular necklaces made of beads of $k$ colors: a rank-$m$ element of $\mathcal{N}$ is a necklace with $m$ beads, and the least element is the empty necklace $\emptyset$. For a rank-$(m-1)$ necklace $p$ and a rank-$m$ necklace $q$, $p \prec q$ if $q$ can be obtained from $p$ by insertion of a bead of any color, and $n(p, q)$ is the number of ways to insert a bead into $p$ to get $q$. For example, in the case $k = 2$,

$n($  ,  $) = 2$ and $n($  ,  $) = 1.$

The universal cover $\widetilde{\mathcal{N}}$ can be described as the poset of necklaces with labelled beads, i.e., the beads of a rank-$m$ necklace are labelled $1, 2 \ldots, m$, with $\widetilde{\mathcal{N}} \to \mathcal{N}$ the function that forgets the labels. It is clear that $\widetilde{\mathcal{N}}$ is simple, since there is a unique chain from any labelled necklace to $\emptyset$ via the operation of removing the highest-label bead. A rank-$m$ element of $\widetilde{\mathcal{N}}$ can be thought of as a "$k$-colored permutation" mod rotation, so there are $k^m(m-1)!$ such elements. Also, the fiber of a given necklace $p \in \mathcal{N}$ with $m$ beads has $n(\emptyset, p) = m!/N(p)$ elements, where $N(p)$ is the number of rotations that take $p$ to itself (necessarily a divisor of $m$): $p$ is called primitive if $N(p) = 1$. Evidently a necklace $p$ with $N(p) = d$ has a primitive "quotient necklace" of size $\frac{m}{d}$. Thus, if $P(m)$ is the number of primitive necklaces of size $m$, we have

$$\sum_{d|m} P(\frac{m}{d}) \frac{m!}{d} = \sum_{|p|=m} n(\emptyset, p) = k^m (m-1)!,$$

or $\sum_{d|m} P(d) d = k^m$. By Möbius inversion we obtain the classical result

$$P(m) = \frac{1}{m} \sum_{d|m} \mu(d) k^{\frac{m}{d}}.$$

Cf. [7, Theorem 7.1].

*Example* 4. Let $\mathcal{B}$ be the set of balanced bracket arrangements: a rank-$n$ element of $\mathcal{B}$ is a sequence of $n$ left brackets and $n$ right brackets so that, reading left to right, the number of right brackets never exceeds the number of left brackets. For $b, b' \in \mathcal{B}$ with $|b'| - |b| = 1$, let $n(b, b')$ be the number of ways to insert a balanced pair $\langle\rangle$ into $b$ to obtain $b'$, e.g., $n(\langle\rangle\langle\rangle, \langle\langle\rangle\rangle\langle\rangle) = 1$ and $n(\langle\rangle\langle\rangle, \langle\rangle\langle\rangle\langle\rangle) = 3$. The least element is the empty arrangement $\emptyset$. Then $\mathcal{B}$ is a weighted-relation poset.

The universal cover $\widetilde{\mathcal{B}}$ has rank-$n$ elements that are permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that, if $a_i > a_j$ and $i < j$, then there is some $k < j$, $k \neq i$, with $a_k = a_i$. In particular, if $s$ is a rank-$n$ element of $\widetilde{\mathcal{B}}$, then the two occurrences of $n$ in $s$ must be adjacent. We define a partial order on $\widetilde{\mathcal{B}}$ by declaring that the rank-$n$ element $a_1 a_2 \ldots a_{2n}$ covers the rank-$(n-1)$ element $a_1 \cdots a_{i-1} a_{i+2} \cdots a_{2n}$, where $a_i = a_{i+1} = n$, and define the weight of all covering relations to be 1. Then $\widetilde{\mathcal{B}}$ is evidently simple.

Define $\pi : \widetilde{\mathcal{B}} \to \mathcal{B}$ by sending $s \in \widetilde{\mathcal{B}}$ to the bracket arrangement obtained by replacing the first occurrence of each positive integer in $s$ by $\langle$, and the second occurrence of each positive integer by $\rangle$. Let $s$ be a rank-$(n-1)$ element of $\mathcal{B}$, with $\pi(s') = s$. Then a rank-$n$ element $r' \succ s'$ is obtained by inserting $nn$ into $s'$, corresponding to inserting $\langle\rangle$ into $s$. Thus, for any $r \succ s$ in $\mathcal{B}$ with $|r| - |s| = 1$,

$$\begin{aligned} n(s, r) &= \text{number of ways to insert } \langle\rangle \text{ into } s \text{ to get } r \\ &= \sum_{r' \in \pi^{-1}(r)} n(s', r'), \end{aligned}$$

so $\pi$ is a covering map.

It is well known that there are $C_n$ rank-$n$ elements of $\mathcal{B}$, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the $n$th Catalan number. The number of rank-$n$ elements of $\widetilde{\mathcal{B}}$ can be seen to be

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3 \cdot 1$$

as follows. If $s = a_1 a_2 \cdots a_{2n} \in \widetilde{\mathcal{B}}$, there are $2n-1$ possible choices of $i$ so that $a_i$ is the first occurrence of $n$ in $s$. Once $i$ is chosen, then $a_{i+1} = n$, so $s$ covers the rank-$(n-1)$ element $a_1 \cdots a_{i-1} a_{i+2} \cdots a_{2n}$ of $\widetilde{\mathcal{B}}$, which by induction can be chosen in $(2n-3)!!$ ways. The phenomenon that labelling elements of a set enumerated by Catalan numbers gives a set enumerated by double factorials was noted in [3].

*Example* 5. Let $\mathcal{F}$ be the set of partitions of nonnegative integers, ordered by inclusion of their Young diagrams. Thus, a partition $\lambda$ of $n$ covers a partition $\mu$ of $n-1$ if $\lambda$ can be obtained from $\mu$ by increasing one part of $\mu$ by 1, or by adding a new part of size 1 to $\mu$: and we assign weight 1 to every covering relation. Then a rank-$n$ element of the universal cover $\widetilde{\mathcal{F}}$ is a Young diagram with boxes labelled $1, 2 \ldots, n$ so that the labels increase from left to right and from top to bottom, i.e., a standard Young tableau. The ordering on $\widetilde{\mathcal{F}}$ is by inclusion, and $\widetilde{\mathcal{F}} \to \mathcal{F}$ is the obvious function. The cardinality $n(\emptyset, \lambda)$ of the fiber of a partition $\lambda$ is given by the hook-length formula (see [12, Cor. 7.21.6]). More generally, when $\mu \prec \lambda$ the number $n(\mu,\lambda)$ counts standard Young tableaux of skew shape $\lambda/\mu$ (see [12, Cor. 7.16.3] for a formula). There is also an algebraic interpretation of the numbers $n(\mu,\lambda)$: if we let $s_\lambda$ be the Schur symmetric function corresponding to the partition $\lambda$, then

$$s_1^k s_\mu = \sum_{|\lambda|=|\mu|+k} n(\mu,\lambda) s_\lambda$$

(see [12, Sect. 7.15]).

*Example* 6. Let $\mathcal{C}$ be the poset of compositions, i.e., finite sequences of integers, with rank given by the sum, and least element $\emptyset$. For compositions $I, J$ with $|J|-|I|=1$, we define

$$n(I,J) = \begin{cases} 1, & \text{if } J \text{ is obtained from } I \text{ by increasing one part;} \\ m, & \text{if there are } m \text{ ways to insert 1 into } I \text{ to get } J; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, e.g., $n(121,122)=1$, $n(121,1121)=2$, and $n(121,212)=0$. This defines a weight system of $\mathcal{C}$, so $\mathcal{C}$ is a weighted-relation poset.

A rank-$n$ element of the universal cover $\widetilde{\mathcal{C}}$ is a Cayley permutation of length $n$ as defined in [6], i.e., a length-$n$ sequence $s$ of positive integers such that any positive integer $i<j$ appears in $s$ whenever $j$ does. The partial order on $\widetilde{\mathcal{C}}$ is defined as follows. If $s=a_1\cdots a_n$ is a Cayley permutation, let $m(s)=\max\{a_1,\dots,a_n\}$. Then $s$ covers $a_1\cdots a_{n-1}$ if the latter is a Cayley permutation: otherwise, $s$ covers $p(a_1)\cdots p(a_{n-1})$, where $p$ is the order-preserving bijection from $\{a_1,\dots,a_{n-1}\}$ to $\{1,2,\dots,m(s)-1\}$. For example, the order ideal generated by 41332 is

$$41332 \succ 3122 \succ 312 \succ 21 \succ 1 \succ \emptyset.$$

If we give each covering relation weight 1, then $\widetilde{\mathcal{C}}$ is evidently simple.

Let $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ be the function that sends a sequence $s$ to the sequence $i_1 i_2 \cdots i_k$, where $i_j$ is the number of times $j$ occurs in $s$; e.g., $\pi^{-1}(13)=\{1222,2122,2212,2221\}$. To see that $\pi$ is a covering map, consider compositions $I, J$ with $|J|=|I|+1$. Let $I=i_1\cdots i_k$ and $s=a_1\cdots a_n\in\widetilde{\mathcal{C}}$ with $\pi(s)=I$. Suppose first that $J$ is obtained from $I$ by increasing the size of one part, so $J=i_1\cdots i_{r-1}(i_r+1)i_{r+1}\cdots i_k$. Then $n(I,J)=1$ and there is only one $t\succ s$ with $\pi(t)=J$, namely $t=a_1\cdots a_n r$. Now suppose $J$ is obtained from $I$ by inserting 1, i.e., $J=i_1\cdots i_r 1 i_{r+1}\cdots i_k$; without loss of generality we can assume $i_r\neq 1$. Then $J$ contains a string of 1's of length $n(I,J)$ after $i_r$. The possible elements $t\succ s$ in $\widetilde{\mathcal{C}}$ with $\pi(t)=J$ are of the form $t=q(a_1)q(a_2)\cdots q(a_n)(r+i)$, where $i$ runs from 1 to $n(I,J)$ and $q$ is the order-preserving bijection from $\{a_1,\dots,a_n\}=\{1,\dots,k\}$ to $\{1,\dots,r+i-1,r+i+1,\dots,k+1\}$. Finally, if $n(I,J)=0$, we must have $n(s,t)=0$ for any $t\in\widetilde{\mathcal{C}}$ with $\pi(t)=J$ since the previous two cases have exhausted all the possibilities for $t$ to cover $s$. So in any case,

$$n(I,J) = \sum_{t\in\pi^{-1}(J)} n(s,t)$$

when $|J|-|I|=1$ and $\pi(s)=I$.

The cardinality of $n(\emptyset, I)$ of the fiber of a composition $I=i_1\cdots i_k$ is evidently the multinomial coefficient

$$\binom{|I|}{i_1\ \cdots\ i_k}.$$

There is an algebraic interpretation of the numbers $n(I,J)$ analogous to that of the preceding example: if $M_I$ is the monomial quasisymmetric function corresponding to the composition $I$ (see [7, Sect. 9.4], or [12, Sect. 7.19] for definitions), then

$$M_1^k M_I = \sum_{|J|=|I|+k} n(I,J) M_J.$$

In particular, the multinomial coefficients $n(\emptyset, J)$ appear in the expansion of $M_1^k$.

*Example* 7. Let $\mathcal{T}$ be the poset of rooted trees ordered by inclusion, i.e., $t' \succ t$ if $t'$ can be obtained from $t$ by adding new edges and vertices. The rank function is given by

$$|t| = \text{number of vertices of } t - 1,$$

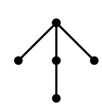
and the least element is the tree $\bullet$ consisting of the root vertex. The weight system is defined as follows: if $|t'|-|t|=1$, let $n(t,t')$ be the number of vertices of $t$ to which a new edge and terminal vertex may be added to obtain $t'$.

Rank-$n$ elements of $\widetilde{\mathcal{T}}$ are permutations of $\{1,2,\dots,n\}$. A permutation $\sigma = s_1 s_2\cdots s_n$ of $\{1,\dots,n\}$ with $s_i=n$ covers the permutation

$$\tau = s_1\cdots s_{i-1}s_{i+1}\cdots s_n$$

of $\{1,\dots,n-1\}$ (and no other). The least element is the empty permutation $\emptyset$. Then $\widetilde{\mathcal{T}}$ is clearly simple if we give each covering relation weight 1.

Now we define the covering map $\pi : \widetilde{\mathcal{T}} \to \mathcal{T}$. Let $\pi(\emptyset)=\bullet$, and given a nonempty permutation $\sigma = s_1 s_2 \cdots s_n$ define a rooted tree with vertices labelled $0,1,\dots,n$ as follows. Label the root 0, and attach the vertex labelled $i$ to the vertex labelled $j<i$ if $j$ is the last element of the sequence $s_1 s_2 \dots s_{k-1}$ that is smaller than $i$, where $s_k=i$; attach $i$ to the root if no such $j$ exists. This associates a labelled rooted tree with each $\sigma\in\widetilde{\mathcal{T}}$, and $\pi(\sigma)$ is just the rooted tree obtained by forgetting the labels. Thus, e.g.,

$\pi(4231) =$  .

To see that $\pi$ is a covering map, note first that terminal vertices of $\pi(\sigma)$ correspond either to descents of $\sigma$ (i.e., terms $s_i$ with $s_i > s_{i+1}$), or to the final term. Now a permutation $\sigma$ with $|\sigma|=n$ covers $\tau$ exactly when $\sigma$ is obtained by inserting $n$ into $\tau$, e.g., $2413 \succ 213$. This always introduces a new descent (or new final term) into $\tau$, and corresponds to adding a new edge and terminal vertex to $\pi(\tau)$; moreover, the $n$ possible places to insert $n$ in a rank-$(n-1)$ permutation $\tau$ correspond to the $n$ vertices of $\pi(\tau)$ where a new edge and vertex can be attached. Thus, for trees $r, s$ with $|r|-|s|=1$ and $s=\pi(\tau)$,

$$\begin{aligned} n(s,r) &= \text{number of permutations } \sigma \succ \tau \text{ with } \pi(\sigma)=r \\ &= \sum_{\sigma\in\pi^{-1}(r)} n(\tau,\sigma). \end{aligned}$$

The cardinality $n(\bullet,t)$ of the fiber of a rank-$n$ rooted tree $t$ is the number of distinct labelled rooted trees (with labels coming from $\{0,1,\ldots,n\}$ and strictly increasing as one moves away from the root) that are isomorphic to $t$ when the labels are removed. Since there are $n!$ rank-$n$ elements of $\widetilde{\mathfrak{T}}$,

$$\sum_{|t|=n} n(\bullet,t)=n!.$$

In fact, if we let $e(t)$ be the number of terminal vertices of the tree $t$ we have

$$\sum_{|t|=n,e(t)=k} n(\bullet,t)=\left\langle {n \atop k-1} \right\rangle,$$

where $\left\langle {p \atop q} \right\rangle$ is the number of permutations of $\{1,\ldots,p\}$ with $q$ descents (Eulerian number), because of the correspondence between descents of $\sigma$ and terminal vertices of $\pi(\sigma)$. Cf. [11, Prop. 1.3.16].

The numbers $n(\bullet,t)$ appear in connection with the "growth operator" on the Hopf algebra of rooted trees studied by Connes and Kreimer [2]. In [1, 5], $n(\bullet,t)$ is called the "Connes-Moscovici weight" of $t$, and some results about it are obtained. To describe them requires a few definitions. Given a tree $t$, let $V(t)$ be the set of vertices of $t$, and for $v\in V(t)$ let $t_v$ be the subtree consisting of $v$ and all its descendents (with $v$ as root): thus $t_r=t$ if $r$ is the root of $t$, and $t_v=\bullet$ if $v$ is a terminal vertex. Define the "tree factorial" of $t$ by

$$t! = \prod_{v\in V(t)} (|t_v|+1).$$

Also, if $v_1,\ldots,v_k$ are the children of the root of $t$, define the symmetry group $SG(t)$ to be the group of permutations $\sigma$ of $\{1,\ldots,k\}$ such that $t_{v_i}$ and $t_{v_j}$ are isomorphic rooted trees when $\sigma(i)=j$. Define the symmetry degree of $t$ to be

$$S_t=\prod_{v\in V(t)} \operatorname{card} SG(t_v).$$

For example, the tree $t=\pi(4231)$ above has $t!=10$ and $S_t=2$. In [1] it is shown that

$$n(\bullet,t)=\frac{(|t|+1)!}{t!S_t}. \tag{6}$$

Equation (6) is actually a variant of the generalized hook-length formula for rooted trees that appears in [10, Sect. 22], [4, Ex. 5.1.4-20], and [8]. To see this, note that if $T$ is a realization of the rooted tree $t$ as a planar directed graph (with arrows coming out from the root), then the number of ways to attach the labels $\{0, 1, \ldots, |t|\}$ to the vertices of $T$ so that the labels strictly increase outward from the root is $n(\bullet, t)S_t$. Also of interest is the result of [5] that

$$\sum_{|t|=n} \frac{n(\bullet, t)}{t!} = \frac{n!}{2^n}.$$

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