Sequences containing no 3-term arithmetic progressions

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Abstract

A subsequence of the sequence $(1, 2, ..., n)$ is called a 3-AP-free sequence if it does not contain any three term arithmetic progression. By $r(n)$ we denote the length of the longest such 3-AP-free sequence. The exact values of the function $r(n)$ were known, for $n \leqslant 27$ and $41 \leqslant n \leqslant 43$. In the present paper we determine, with a use of computer, the exact values, for all $n \leqslant 123$. The value $r(122) = 32$ shows that the Szekeres' conjecture holds for $k = 5$.

1 Introduction

Let $\langle n \rangle$ denote the sequence $(1, 2, ..., n)$. A subsequence $(a_1, \ldots, a_k)$ of $\langle n \rangle$ is called a 3-AP-free sequence if it does not contain any three elements a_p, a_q, a_r such that $a_q - a_p = a_r - a_q$, or in other words if it does not contain any three term arithmetic progression. By $r(n)$ we denote the length of the longest 3-AP-free sequences in $\langle n \rangle$. Sometimes (see [1]) the problem is equivalently presented in the set version. A subset $S \subset \{1, 2, ..., n\}$ is called a 3-AP-free if $a + c \neq 2b$, for every distinct terms $a, b, c \in S$. The cardinality of the largest such subset S is equal to $r(n)$.

The study of the function $r(n)$ was initiated by Erdős and Turán [2]. They determined the values $r(n)$, for $n \leqslant 23$ and $n = 41$ (see Table 1), proved that $r(2N) \leqslant N$, for $n \geqslant 8$, and conjectured that $\lim_{n\to\infty} r(n)/n = 0$. This conjecture was proved in 1975 by Szemeredi [6]. Erdős and Turán also conjectured that $r(n) < n^{1-c}$, which was shown to be false by Salem and Spencer [4] who proved $r(n) > n^{1-c/\log\log n}$. The later result was improved by Behrend [1], who showed that $r(n) > n^{1-c/\sqrt{\log n}}$. The first non trivial upper bound was due by Roth [3] who proved $r(n) < \frac{cn}{\log\log n}$.

Recently Sharma [5] showed that Erdős and Turán gave the wrong value of $r(20)$ and determined the values of $r(n)$, for $n \leqslant 27$ and $41 \leqslant n \leqslant 43$ (see Tables 1 and 2). Erdős and Turán [2] noted that there is no three term arithmetic progression in the sequence of all numbers n, $0 \leqslant n \leqslant \frac{1}{2}(3^k - 1)$, which do not contain the digit 2 in the ternary scale.

Hence, if we add 1 to each of these numbers, we obtain the 3-AP-free sequence of length 2^k in $\langle \frac{1}{2}(3^k+1)\rangle$. Thus, we have

Lemma 1. *$r(\frac{1}{2}(3^k+1)) \geqslant 2^k$, for every $k \geqslant 1$.*

The Szekeres' conjecture (see [2]) says that $r(\frac{1}{2}(3^k+1)) = 2^k$. The values given in [2, 5] (see Tables 1 and 2) show that the conjecture is true for $k \leqslant 4$. The Szekeres' conjecture has several interesting implications described in [2, 6]. In the sequel we shall need the following lemmas from [2, 5].

Lemma 2. *If $(a_1, ..., a_k)$ is a 3-AP=free sequence in $\langle n\rangle$ and $j < a_1$, then also $(a_1 - j, ..., a_k - j)$ is a 3-AP-free sequence in $\langle n\rangle$.*

Lemma 3. *$r(n+m) \leqslant r(n) + r(m)$.*

Lemma 4. *$r(n) \leqslant r(n+1) \leqslant r(n) + 1$.*

If $r(n-1) < r(n)$ then we call n a jump node. Whenever $r(n-2) < r(n-1) < r(n)$ we call n a double jump.

Lemma 5. *Three consecutive numbers cannot be jump nodes.*

In this paper we determine, using computer, exact values of $r(n)$, for all $n \leqslant 123$. The value $r(122) = 32$ shows that the Szekeres' conjecture holds for $k = 5$. We also determined, for each jump node $n \leqslant 123$, $b(n)$ — the number of the longest 3-AP-free sequences in $\langle n\rangle$, and an example of the longest 3-AP-free sequence (see Tables 2 and 3).

2 Algorithm

In order to determine values of the function $r(n)$ we have designed a simple decision algorithm. This algorithm answers the question whether there is a 3-AP-free sequence of length k in $\langle n\rangle$. The algorithm uses the values $r(m)$, for $m < n$.

Algorithm 6. INPUT : $2 \leqslant k \leqslant n$ — natural numbers
OUTPUT : YES if there exists 3-AP-free sequence of length k in $\langle n\rangle$; NO otherwise.

```
a_1:=1; a_2:=2; l:=2;
APseq := (a_1, a_2)
repeat
   if APseq contains no 3-term arithmetic progression then
      APseq := (a_1, a_2, ..., a_l, a_l + 1)
      l := l + 1
   else
      while (l>1) and (r(n-a_l) < k-l+1)
         l := l - 1;
      APseq := (a_1, a_2, ..., a_{l-1}, a_l + 1)
until (a_1 ≠ 1) or 3-AP-free sequence of length k is found
if 3-AP-free sequence of length k is found, then return YES
if (a_1 ≠ 1), then return NO
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The algorithm looks for a 3-AP-free sequence of length k in $\langle n\rangle$. It starts (lines 1–2) with the sequence (1,2) of length $l = 2$. At the beginning of each round of the "repeat — until" loop (lines 3–11) the algorithm considers the sequence APseq $= (a_1, \dots, a_{l-1}, a_l)$ of length $l \geqslant 2$. First, it checks if the sequence contains a 3-term arithmetic progression. If not, then it adds the element $a_l + 1$ at the end of the sequence (lines 5–6). If APseq contains an arithmetic progression, then it looks for the next sequence to consider (lines 8–10). The first candidate is the sequence $(a_1, \dots, a_{l-1}, a_l + 1)$, but if $r(n - a_l) < k - l + 1$, it is not possible to find remaining $k - l + 1$ elements in the sequence $(a_l + 1, \dots, n)$ without arithmetic progression. In this case the algorithm sets $l := l - 1$ and checks if the sequence $(a_1, \dots, a_{l-1} + 1)$ will be good, and so on. The algorithm exits the main loop if it finds a 3-AP-free sequence of the length k, or if $a_1 > 1$. In the later case it returns the answer NO, because, by Lemma 2, it suffices to consider sequences which start with 1.

3 Results

We have used Algorithm 6 to determine the values of the function $r(n)$, for all $n \leqslant 120$. Observe that if we know the value $r(n)$, then we can obtain $r(n+1)$ after one application of the algorithm with the input $n+1$ and $k = r(n) + 1$. All computations took about 30h on personal computer. We have got the value $r(120) = 30$. By Lemma 1, $r(122) \geqslant 32$, thus by Lemma 4, $r(121) = 31$ and $r(122) = 32$. Finally, by Lemma 5, $r(123) = 32$.

Tables 1, 2 and 3 present: the value of $r(n)$, the value of $b(n)$ — the number of the longest 3-AP-free sequences, and an example of the longest 3-AP-free sequence in $\langle n\rangle$, for each jump node $n \leqslant 123$. Table 1 contains the results obtained by Erdős and Turán [2] and by Sharma [5]. The values for $n < 20$ were given by Erdős and Turán. They also gave

n	$r(n)$	$b(n)$	example of the longest 3-AP-free sequence
1	1	1	1
2	2	1	1 2
4	3	2	1 2 4
5	4	1	1 2 4 5
9	5	4	1 2 4 8 9
11	6	7	1 2 4 5 10 11
13	7	6	1 2 4 5 10 11 13
14	8	1	1 2 4 5 10 11 13 14
20	9	2	1 2 6 7 9 14 15 18 20
24	10	2	1 2 5 7 11 16 18 19 23 24
26	11	2	1 2 5 7 11 16 18 19 23 24 26

Table 1: Values of $r(n)$, $b(n)$, and examples of the longest 3-AP-free sequences, for jump nodes $n \leqslant 26$, obtained in [2, 5]. b(n) denotes the number of the longest 3-AP-free sequences in $\langle n\rangle$

the wrong value $r(20) = 8$ and based on this value proved $r(41) = 16$. Sharma gave the correct value of $r(20) = 9$ and a new proof of $r(41) = 16$. Our calculations confirm the values given by Erdos and Turan, for $n < 20$, and those given by Sharma, for $20 \leqslant n \leqslant 27$ and $41 \leqslant n \leqslant 43$ (including the correction of the value $r(20) = 9$). Tables 2 and 3 contain the results for $n \geqslant 30$. As we have mentioned above, the value r(41)=16 was given in [2,5], the others are new.

n	$r(n)$	$b(n)$	example of the longest 3-AP-free sequence
30	12	1	1 3 4 8 9 11 20 22 23 27 28 30
32	13	2	1 2 4 8 9 11 19 22 23 26 28 31 32
36	14	2	1 2 4 8 9 13 21 23 26 27 30 32 35 36
40	15	20	1 2 4 5 10 11 13 14 28 29 31 32 37 38 40
41	16	1	1 2 4 5 10 11 13 14 28 29 31 32 37 38 40 41
51	17	14	1 2 4 5 10 13 14 17 31 35 37 38 40 46 47 50 51
54	18	2	1 2 5 6 12 14 15 17 21 31 38 39 42 43 49 51 52 54
58	19	2	1 2 5 6 12 14 15 17 21 31 38 39 42 43 49 51 52 54 58
63	20	2	1 2 5 7 11 16 18 19 24 26 38 39 42 44 48 53 55 56 61 63
71	21	4	1 2 5 7 10 17 20 22 26 31 41 46 48 49 53 54 63 64 68 69 71
74	22	1	1 2 7 9 10 14 20 22 23 25 29 46 50 52 53 55 61 65 66 68 73 74
82	23	10	1 2 4 8 9 11 19 22 23 26 28 31 49 57 59 62 63 66 68 71 78 81 82

Table 2: Values of $r(n)$, $b(n)$, and examples of the longest 3-AP-free sequences, for jump nodes $27 \leqslant n \leqslant 82$. The value r(41)=16 was given in [2, 5].

4 Discussion and open problems

Sharma [5] asks if every $T_k = \frac{1}{2}(3^k + 1)$ is a jump node. Our calculations and previous results show that, for $k \leqslant 5$, every T_k is a double jump and there is only one longest 3-AP-free sequence in $\langle T_k \rangle$, just the one described before Lemma 1. We wonder if this is true for other k?

Using the values from Table 2 and 3 and a simple induction argument similar to the one described in [2] we can prove that $r(3n) \leqslant n$, for every $n \geqslant 16$. Indeed, we know that $r(3n) \leqslant n$, for every $16 \leqslant n \leqslant 31$, and $r(48) = 16$. For $n \geqslant 32$, we have $r(3n) = r(3(n - 16) + 48) \leqslant r(3(n - 16)) + r(48) \leqslant n - 16 + 16 = n$.

n	$r(n)$	$b(n)$	example of the longest 3-AP-free sequence
84	24	1	1 3 4 8 9 16 18 21 22 25 30 37 48 55 60 63 64 67 69 76 77 81 82 84
92	25	14	1 2 6 8 9 13 19 21 22 27 28 39 58 62 64 67 68 71 73 81 83 86 87 90 92
95	26	8	1 2 4 5 10 11 22 23 25 26 31 32 55 56 64 65 67 68 76 77 82 83 91 92 94 95
100	27	2	1 3 6 7 10 12 20 22 25 26 29 31 35 62 66 68 71 72 75 77 85 87 90 91 94 96 100
104	28	1	1 5 7 10 11 14 16 24 26 29 30 33 35 39 66 70 72 75 76 79 81 89 91 94 95 98 100 104
111	29	6	1 2 5 6 13 15 19 26 27 30 31 38 42 44 66 68 72 77 80 81 84 89 93 95 99 104 107 108 111
114	30	1	1 2 4 9 12 13 18 19 28 30 31 33 40 45 46 69 70 75 82 84 85 87 96 97 102 103 106 111 113 114
121	31	70	1 2 4 5 10 11 13 14 28 29 31 32 37 38 40 41 82 83 85 86 91 92 94 95 109 110 112 113 118 119 121
122	32	1	1 2 4 5 10 11 13 14 28 29 31 32 37 38 40 41 82 83 85 86 91 92 94 95 109 110 112 113 118 119 121 122

Table 3: Values of $r(n)$, $b(n)$, and examples of the longest 3-AP-free sequences, for jump nodes $83 \leqslant n \leqslant 123$

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