# Sequenceable Groups and Related Topics

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#### Abstract

In 1980, about 20 years after sequenceable groups were introduced by Gordon to construct row-complete latin squares, Keedwell published a survey of all the available results concerning sequencings. This was updated (jointly with Dénes) in 1991 and a short overview, including results about complete mappings and R-sequencings, was given in the CRC Handbook of Combinatorial Designs in 1995. In Sections 1 and 2 we give a survey of the current situation concerning sequencings, including details of the most important constructions. In Section 3 we consider some concepts closely related to sequenceable groups: R-sequencings, harmonious groups, supersequenceable groups (also known as super P-groups), terraces and the Gordon game. We also look at constructions for row-complete latin squares that do not use sequencings.

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# 1 Introduction

The problem of finding sequencings for groups was introduced by Gordon in 1961 [56], although similar ideas for cyclic groups go back at least as far as 1892 [74]. Our focus here is on the sequenceable group problem of Gordon; we give the necessary definitions and motivation in this section and look at the progress made in the next one. We consider some related problems: R-sequencings, harmonious groups, supersequenceable groups (also known as super P-groups), terraces and the Gordon game in Section 3 and also give the conclusion to the row-complete latin square question that was one of the motivators for the original problem.

Among topics not covered here are Fibonacci sequences in groups, the study of which appears to have initiated in [102] and sometimes includes the phrase "sequenceable group" [97]. Also not included are problems that naturally arise in looking for paths or cycles in Cayley graphs, where using a small number of generating elements for the Cayley graph is usually desired.

Unless explicitly stated, group theoretic terms may be found in [96]. Sequencings have been previously surveyed by Keedwell [68, 71] and Dénes and Keedwell [48, chapter 3].

A non-trivial finite group G of order n is said to be *sequenceable* if its elements can be arranged in a sequence  $(b_1, b_2, \ldots, b_n)$  in such a way that the partial products  $(a_1, a_2, \ldots, a_n)$ , where  $a_i = b_1 b_2 \cdots b_i$ , are distinct. The sequence  $(b_1, b_2, \ldots, b_n)$  is called a *sequencing* for G. If  $(b_1, b_2, \ldots, b_n)$  is a sequencing for G then  $b_1 = e$ , where e is the identity of G (if  $b_i = e$  for some  $i \neq 1$  then  $a_{i-1} = a_i$ ). The sequence  $(a_1, a_2, \ldots, a_n)$  is called a basic directed terrace (for any element  $g \in G$  the sequence  $(ga_1, ga_2, \ldots, ga_n)$  is called a directed terrace—observe that Theorem 1 still holds for this more general definition).

Note that a sequencing uniquely determines a basic directed terrace and that a basic directed terrace uniquely determines a sequencing. It is common practice in more recent papers to omit the initial identity element of the sequencing.

A latin square of order n is an  $n \times n$  array defined on a set X with n elements such that every element of X appears once in each row and once in each column. The notation  $L = (l_{ij})$  represents a latin square L with  $l_{ij}$  in the *i*th row and *j*th column. A latin square is said to be *based* on a group G if the latin square can be bordered with the elements of G to form the Cayley table of G.

An  $n \times n$  latin square is said to be row complete if every pair  $\{x, y\}$  of distinct elements of X occurs exactly once in each order in adjacent horizontal cells. A latin square is said to be column complete if every pair  $\{x, y\}$  of distinct elements of X occurs exactly once in each order in adjacent vertical cells. If a latin square is both row complete and column complete then it is said to be complete.

Another application is to graph theory. If there is a row-complete latin square of order n then the complete directed graph on n vertices can be decomposed into n disjoint Hamiltonian paths (a Hamiltonian path is a path which passes through each vertex exactly once; paths are disjoint if they have no edges in common). This is done by associating each symbol in the latin square with a vertex in the graph and taking a path to traverse the vertices in the order a row lists the symbols. As we are using a latin square the paths are Hamiltonian (since each symbol occurs exactly once in each row). As the latin square is row complete each ordered pair of symbols (x, y) occurs exactly once in adjacent horizontal cells, thus no edge is repeated and the paths are disjoint. Example 1 demonstrates this for n = 4. Observe that we have not used the property that each symbol occurs once in each column. If this property is removed from the definition of a row-complete latin square then we have a Tuscan square. A Tuscan square of order n is equivalent to a decomposition of the complete graph on n vertices into n Hamiltonian paths. See [55] for more details about Tuscan squares.

**Theorem 1** [56] Let G be a sequenceable group and  $(b_1, b_2, \ldots, b_n)$  be a sequencing with associated basic directed terrace  $(a_1, a_2, \ldots, a_n)$ . Then  $L = (l_{ij})$ , where  $l_{ij} = a_i^{-1}a_j$  for  $1 \leq i, j \leq n$ , is a complete latin square.

Proof: Suppose  $l_{ij} = l_{ik}$  for some  $1 \leq i, j, k \leq n$ . Then  $a_i^{-1}a_j = a_i^{-1}a_k$ , giving  $a_j = a_k$ . Therefore j = k and L has no repeated entries in any row. Similarly, L has no repeated entries in any column. Therefore L is a latin square.

To show that L is row complete we need  $a_i^{-1}a_j = x$  and  $a_i^{-1}a_{j+1} = y$  to have a unique solution for i and j given any ordered pair (x, y) of distinct elements of G.

Inverting both sides of the first equation and post-multiplying by the second gives  $a_j^{-1}a_{j+1} = x^{-1}y$ , that is  $b_{j+1} = x^{-1}y$ , uniquely determining j. Now  $a_i^{-1}a_j = x$  uniquely determines i, and L is row complete.

An analogous argument shows that L is also column complete. Therefore L is a complete latin square.  $\Box$ 

**Example 1** Let  $G = \mathbb{Z}_4$ , the additively written cyclic group of order 4. Then (0, 3, 2, 1) is a sequencing of G with basic directed terrace (0, 3, 1, 2). The corresponding complete latin square L is given in Figure 1. Figure 2 shows how this leads to a decomposition of the complete directed graph on 4 vertices into disjoint hamiltonian paths.

0	3	1	2
1	0	2	3
3	2	0	1
2	1	3	0

Figure 1: L



Figure 2: Decomposition of the complete directed graph with 4 vertices

Sequencings with special properties have been used to solve problems concerning bipartite tournaments balanced for carry-over effects [16, 37, 89], to give additional balance properties to latin squares [8, 10], to study 1-rotational Hamiltonian cycle systems of the complete graph [31, 39, 40], to solve some cases of the Oberwolfach problem [86, 88], to construct rainbow-difference paths [72], and to construct Hamiltonian double Latin squares [62, 84]. Bate and Jones [34] give a survey of the use of sequencings and similar ideas in the field of experimental design.

If a group is sequenceable then the Cayley table of the group has an almost transversal. This result follows because a sequencing gives rise to a near-complete mapping; see [71] for an explanation of this and other results concerning complete and near-complete mappings.

Vanden Eynden [101] extends the idea of a sequencing to groups of countably infinite order, showing that all such groups are sequenceable. Caulfield [43] shows that this notion corresponds to the ability to construct a quarter-plane complete infinite Latin square and uses similar ideas to show that full-plane complete infinite Latin squares also exist.

# 2 Classifying Sequenceable Groups

In his paper [56], which introduced the concept of a sequencing, Gordon also completely classified the sequenceable abelian groups: see Section 2.1. He also noted that the quaternion group,  $Q_8$ , of order 8 and  $D_6$  and  $D_8$ , the dihedral groups of order 6 and 8 respectively,

are not sequenceable. He did find, however, that  $D_{10}$  is sequenceable. In 1968 Dénes and Tőrők [49] confirmed these results and added  $D_{12}$ ,  $D_{14}$ ,  $D_{16}$  and the non-abelian group of order 21 (the smallest non-abelian group of odd order) to the list of known sequenceable groups. Also in 1968 Mendelsohn [75] published an independently obtained sequencing for the non-abelian group of order 21. In 1973 Keedwell [67] sequenced the non-abelian group of order 27 with exponent 9 and Wang [108] sequenced the non-abelian groups of orders 39, 55 and 57.

In 1976 more significant headway started to be made with the question of which dihedral groups are sequenceable: see Section 2.2. Also in 1976 the concept of a symmetric sequencing was introduced. This set in motion the now nearly complete classification of sequenceable binary groups: see Section 2.3. We define a *binary group* to be a group with a single element of order 2. This does not contradict Dickson's [50] use of the term and fits well as a generalisation of binary polyhedral groups: see [45]. In the literature on sequencings, binary groups are usually called  $\Lambda$ -groups. In 1981 Keedwell was the first to give sequencings of infinitely many non-abelian groups of odd order: see Section 2.4.

Throughout this section we give the constructions for sequencings of the groups in question but usually refer the reader to the relevant papers for proofs of their correctness.

## 2.1 Abelian Groups

In this section we shall write abelian groups additively. In [56] Gordon proved the following theorem, which shows exactly which abelian groups are sequenceable. Recall that a binary group is defined to be a group with a single element of order 2.

**Theorem 2** [56] A finite abelian group G is sequenceable if and only if G is a binary group.

Proof  $(\Rightarrow)$ : Suppose  $(b_1, \ldots, b_n)$  is a sequencing for G with associated basic directed terrace  $(a_1, \ldots, a_n)$ . Since G is abelian we have that  $a_n$  is the sum of the elements of G written in any order.

We first suppose that G has no elements of order 2. For each  $g \in G \setminus \{0\}$  we have  $g \neq -g$ , so the non-identity elements of the sequencing will cancel in pairs. This gives  $a_n = 0$ , contradicting  $a_1 = 0$ .

Now suppose that G has k elements,  $h_i$ , of order 2, where k > 1. These elements, along with 0, form a subgroup H of G of order  $k + 1 = 2^l$  for some l > 1. Then H has a basis  $\{u_1, \ldots, u_l\}$  for some  $u_1, \ldots, u_l \in H$ , thus each  $h_i$  is expressible in the form  $\epsilon_1 u_1 + \cdots + \epsilon_l u_l$  with each  $\epsilon_i \in \{0, 1\}$ . Each expression of this form represents one of the elements of order 2. Therefore  $2^{l-1}$  elements of H involve the generator  $u_i$  for each i. Since each element  $u_i$  occurs an even number of times in the expression for  $a_n$ , and  $2u_i = 0$  for all i, we again reach the contradiction  $a_n = 0$ .

Note that if G has exactly one element, h, of order 2 then  $a_n = h$ .

( $\Leftarrow$ ): Gordon gave a direct construction of sequencings in abelian binary groups. However, later results have made possible a simpler proof; we give this simpler proof in Section 3.4.  $\Box$  **Example 2** Consider the cyclic group of even order,  $\mathbb{Z}_{2n}$ . The following **b** is a sequencing and has corresponding basic directed terrace **a**.

$$\mathbf{b} = (0, 1, 2n - 2, 3, 2n - 4, 5, \dots, 4, 2n - 3, 2, 2n - 1)$$
$$\mathbf{a} = (0, 1, 2n - 1, 2, 2n - 2, 3, \dots, n + 2, n - 1, n + 1, n).$$

This sequencing was first given (implicitly) by Lucas (who gave credit to Walecki) in [74], where it was used to solve a problem concerning schoolchildren performing round-dances. It is often referred to as the Lucas-Walecki-Williams (directed) terrace/sequencing, with the addition of "Williams" due to his similar construction for a non-directed terrace, see Section 3.4. Further solutions to this problem which use sequencings and related ideas are given in [31].

Combining Example 2 and Theorem 1 gives a method for constructing a complete latin square of any even order.

## 2.2 Dihedral Groups

Let  $n \ge 3$ . We describe the dihedral group  $D_{2n}$ , of order 2n, as the set of ordered pairs  $(x, \epsilon)$  with  $x \in \mathbb{Z}_n$  and  $\epsilon \in \mathbb{Z}_2$  and multiplication defined by:

$$\begin{array}{lll} (x,0)(y,\delta) &=& (x+y,\delta) \\ (x,1)(y,\delta) &=& (x-y,1+\delta) \end{array}$$

In 1976 Anderson [2] showed that  $D_{2p}$  is sequenceable if p is a prime with a primitive root r such that  $3r \equiv -1 \pmod{p}$ . Also in 1976 Friedlander [53] showed that  $D_{2p}$  is sequenceable if p is prime and  $p \equiv 1 \pmod{4}$ . In 1981 Hoghton and Keedwell [63] added the groups  $D_{2p}$ , where p is a prime such that  $p \equiv 7 \pmod{8}$  and p has a primitive root rsuch that  $2r \equiv -1 \pmod{p}$ .

All of these results were obtained using quotient sequencings (see Section 2.4) and number theoretic arguments of varying intricacy.

In 1987 Anderson [3] used a computer search to show that all dihedral groups  $D_{2n}$  for  $5 \leq n \leq 50$  are sequenceable. In 1990 Isbell [65] produced a general argument which, when allied to Anderson's computer search, covered all of the infinite classes mentioned above and more:

**Theorem 3** [65] The dihedral groups  $D_{2n}$ , of order 2n, are sequenceable for all n, where  $n \neq 3$  ( $D_6$  is not sequenceable) and  $n \neq 4k$ .

Construction: We split the construction into five cases and then some anomalous small examples. For the first three cases we exhibit a sequencing of the form  $\mathbf{b} = (e, \alpha, \beta, \gamma)$  where e is the identity,  $\alpha$  and  $\gamma$  partition the remaining elements of the form (x, 0) and  $\beta$  consists of the elements of the form (x, 1).

Case 1; n = 4k + 1: We define  $\alpha$ ,  $\beta$  and  $\gamma$  as follows:

$$\begin{aligned} \alpha &= (2k-1,0), (2-2k,0), (2k-3,0), (4-2k,0), \dots, (3,0), (-2,0), \\ &(1,0), (2k,0) \end{aligned}$$
  
$$\beta &= (0,1), (1,1), (2,1), \dots, (2k-1,1), (4k,1), (2k,1), (2k+1,1), \dots, \\ &(4k-1,1) \end{aligned}$$
  
$$\gamma &= (-2k,0), (-1,0), (2,0), (-3,0), (4,0), \dots, (3-2k,0), (2k-2,0), \\ &(1-2k,0). \end{aligned}$$

Case 2; n = 8k + 7,  $(k \ge 1)$ : We produce  $\beta$  in the same manner as before:

$$\beta = (0,1), (1,1), \dots, (4k+2,1), (8k+6,1), (4k+3,1), \dots, (8k+5,1).$$

Now, working in  $\mathbb{Z}_{8k+7}$ , consider the following sequence:

$$\sigma = -(2k+1), (4k+2), -(4k+1), 4k, \dots, -(2k+3), (2k+2), -1, -2k, (2k-1), -(2k-2), \dots, 3, -2.$$

Define  $\alpha$  to be the sequence in  $D_{2n}$  with  $\sigma$  in the first co-ordinates and 0's in the second, followed by (-(4k+3), 0). Define  $\gamma$  to be (4k+3, 0) followed by the sequence with  $-\sigma$  in the first co-ordinates and 0's in the second. Now the sequence  $(e, \alpha, \beta, \gamma)$  lists all elements of  $D_{2n}$  and is the required sequencing.

Case 3; n = 8k+3,  $(k \neq 1, 2, 4)$ : Again we use the list  $(e, \alpha, \beta, \gamma)$  but here  $\beta$  is slightly more complicated:

$$\beta = (0,1), (1,1), \dots, (4k-1,1), (8k,1), (8k+1,1), (8k+2,1), (4k,1), (4k+1,1), \dots, (8k-1,1).$$

Similarly to the n = 8k + 7 case we look at sequences in  $\mathbb{Z}_{8k+3}$  first. Define

$$X_{(k)} = \{ x : -k \le x \le k - 1, x \neq 1, -1 \}.$$

We construct orderings  $X_k$  of  $X_{(k)}$  beginning with 2 and ending with -k such that the 2k-3 differences between consecutive elements contain exactly one of i and -i for  $2 \leq i \leq 2k-2$ . This condition is satisfied by the following three orderings:

$$X_3 = (2, -2, 0, -3)$$
  

$$X_5 = (2, 0, -4, 4, -3, 3, -2, -5)$$
  

$$X_7 = (2, -5, 4, 0, -2, 3, -3, 5, -6, 6, -4, -7).$$

We now extend inductively from k to k + 3. Note that the penultimate element in each case is -(k - 3); this condition will also be preserved by the induction.

To order  $X_{(k+3)}$  list  $X_k$  as far as the penultimate element -(k-3), then continue k+2, -(k+2), k+1, -(k+1), k, -k, -(k+3). This satisfies the conditions.

Consider the sets  $Y_{(k)} (\supseteq X_{(k)})$  of integers defined as follows:

$$Y_{(k)} = \{ x : -(2k-1) \le x \le 2k, x \ne -1 \}.$$

We define an ordering  $Y_k$  of  $Y_{(k)}$  beginning with 2, ending with 1 and having differences of consecutive elements exactly one of i and -i for  $1 \le i \le 4k - 2$ :

$$(\underbrace{2,\ldots,-k}_{X_k}, k, -(k+1), k+1, \ldots, -(2k-1), (2k-1), 2k, 1).$$

Let  $\tau_k$  be the sequence of differences of consecutive elements in this ordering  $Y_k$ : then the partial sums of  $\tau_k$  list the translate  $-2 + Y_{(k)}$  without repetition. Define  $\alpha$  to be the sequence with  $\tau_k$  in the first co-ordinates and 0's in the second, followed by (-(4k+1), 0), (4k-1, 0), (-4k, 0). Define  $\gamma$  to be (4k, 0) followed by the sequence with  $-\tau_k$  in the first co-ordinates and 0's in the second, finishing with (4k+1, 0), (-(4k-1), 0). We now have  $(e, \alpha, \beta, \gamma)$  listing  $D_{2n}$  without repetition and this is the required sequencing.

Case 4; n = 4k + 2, k even  $(k \ge 2)$ : For this we use the sequence  $(e, \beta, \alpha, \delta, \gamma)$  where e is the identity,  $\alpha$  and  $\gamma$  partition the remaining elements (x, 0) (here  $\alpha$  and  $\gamma$  are not of equal length),  $\delta$  is (4k + 1, 1) and  $\beta$  covers the other elements (x, 1). We construct  $\beta$  in the same manner as in case n = 4k + 1, that is

$$\beta = (0,1), (1,1), (2,1), \dots, (2k-1,1), (4k,1), (2k,1), (2k+1,1), \dots, (4k-1,1).$$

Consider the following two sequences in  $\mathbb{Z}_{4k+2}$ :

$$\sigma_{1} = -3, 5, -7, 9, \dots, 2k - 3, \underbrace{1 - 2k, 2k - 2}_{q_{2}, -(2k - 3), 2k - 5, \dots, -5, 3}$$
  
$$\sigma_{2} = -2, 4, -6, 8, \dots, 2k - 4, \underbrace{-(2k - 2), 1, 2k - 1, -1}_{2k - 6, \dots, -4, 2}, -(2k - 4),$$

Define  $\alpha$  to be (2k + 2, 0) followed by the sequence with  $\sigma_1$  in the first co-ordinates and 0's in the second, followed by (2k, 0), (2k + 1, 0). Define  $\gamma$  to be the sequence with  $\sigma_2$  as the first co-ordinates and 0's as the second. Now  $\alpha$  and  $\gamma$  cover all elements (x, 0)such that  $x \neq 0$ , and  $(e, \beta, \alpha, \delta, \gamma)$  is a sequencing of  $D_{2n}$ .

Case 5; n = 4k + 2, k odd  $(k \ge 3)$ : We define the sequencing  $(e, \beta, \alpha, \delta, \gamma)$  as in the previous case, but we need to modify  $\sigma_1$  and  $\sigma_2$  slightly as the length of the list each side of the braces is now odd, meaning that the sign alternation causes a problem. This problem is rectified by reversing the order of the terms in the braces, that is

$$\sigma_{1} = -3, 5, -7, 9, \dots, -(2k-3), \underbrace{2k-2, 1-2k}_{-5,3}, 2k-3, -(2k-5), \dots \\ -5, 3 \\ \sigma_{2} = -2, 4, -6, 8, \dots, -(2k-4), \underbrace{-1, 2k-1, 1, -(2k-2)}_{-(2k-6), \dots, -4, 2}, 2k-4,$$

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The construction now goes through as before.

The anomalous cases: The sequencings given here are those due to Anderson [3], though Isbell did produce sequencings for  $D_{14}$ ,  $D_{22}$ ,  $D_{38}$  and  $D_{70}$  similar in style to his sequencings of the infinite classes. For brevity, we identify (i, 0) with i + 1 and (i, 1) with n + i + 1 (exactly as in [100]). Recall that  $D_6$  is not sequenceable.

Below,  $S_{2n}$  is a sequencing for  $D_{2n}$ .

$$\begin{split} \mathcal{S}_{12} &: (1, 11, 2, 7, 3, 9, 12, 10, 6, 5, 8, 4) \\ \mathcal{S}_{14} &: (1, 8, 2, 10, 7, 6, 9, 5, 11, 4, 14, 13, 12, 3) \\ \mathcal{S}_{22} &: (1, 8, 18, 15, 16, 5, 10, 4, 6, 13, 11, 3, 19, 17, 7, 9, 22, 2, 14, 12, 20, 21) \\ \mathcal{S}_{38} &: (1, 32, 15, 24, 23, 8, 38, 14, 22, 19, 37, 34, 5, 33, 36, 26, 12, 25, 13, 6, 28, 21, 7, 29, 10, 4, 20, 11, 31, 18, 31, 35, 32, 16, 17, 27, 9) \\ \mathcal{S}_{70} &: (1, 3, 45, 10, 22, 33, 11, 16, 32, 54, 47, 61, 43, 62, 31, 12, 53, 20, 67, 35, 8, 46, 29, 21, 7, 60, 25, 39, 34, 57, 64, 59, 6, 55, 66, 4, 38, 63, 65, 51, 70, 2, 13, 68, 28, 37, 26, 50, 30, 24, 23, 58, 5, 40, 27, 69, 15, 48, 19, 42, 56, 9, 18, 36, 17, 41, 44, 49, 14, 52) \end{split}$$

In 1997 Li [73] completed the classification of sequenceable dihedral groups by sequencing  $D_{2n}$  where  $n \equiv 0 \pmod{4}$ ,  $n \neq 4$ . Recourse to Anderson's computer search [3] was again needed for some small cases.

**Theorem 4** [73] The dihedral groups  $D_{2n}$  are sequenceable when n = 4k, except when n = 4.

Construction: The construction varies slightly as k varies modulo 4. For each case the sequencing is  $((a), (b), \ldots, (s))$  from the appropriate table amongst Tables 1, 2, 3 and 4. Note that for some small values of k some of the components may be empty.

The anomalous cases: As in the proof of Theorem 3 we identify (i, 0) with i + 1and (i, 1) with n + i + 1. Recall that  $D_8$  is not sequenceable. Here we give Anderson's sequencing  $S_{2n}$  for  $D_{2n}$  [3]:

 $\begin{aligned} \mathcal{S}_{16} &: & (1, 13, 11, 16, 4, 14, 3, 5, 6, 15, 8, 7, 9, 12, 2, 10) \\ \mathcal{S}_{24} &: & (1, 17, 4, 11, 2, 8, 9, 13, 10, 3, 23, 24, 22, 15, 14, 6, 20, 18, 16, 7, 21, 19, 12, 5) \end{aligned}$ 

We have now covered all required values of n.  $\Box$ 

Secu	encing	No. of terms
- Sequ		10. 01 0011115
(a)	(0,0)	1
(b)	$(0,1),(1,1),(2,1),\ldots,(2k-2,1)$	2k - 1
(c)	(4k-2,1)	1
(d)	$(2k-1), (2k, 1), (2k+1, 1), \dots, (4k-3, 1)$	2k - 1
(e)	(2k,0)	1
(f)	$(4k-3,0), (5,0), (4k-7,0), (9,0), \dots, (2k-3,0)$	k-2
(g)	(2k+2,0)	1
(h)	$(2k-1,0), (2k+3,0), (2k-5,0), (2k+7,0), \dots, (3,0)$	k-1
(i)	(4k - 2, 0)	1
(j)	(4k - 1, 1)	1
(k)	$(2,0), (4k-4,0), (6,0), (4k-8,0), \dots, (k-2,0)$	k/2 - 1
(1)	(1,0)	1
(m)	$(3k-4,0), (k+6,0), (3k-8,0), (k+10,0), \dots, (2k-2,0)$	k/2 - 2
(n)	(3k,0)	1
(o)	(2k + 1, 0)	1
(p)	(k+2,0)	1
(q)	$(2k-4,0), (2k+6,0), (2k-8,0), (2k+10,0), \dots, (3k-2,0)$	k/2 - 2
(r)	(4k-1,0)	1
(s)	$(k,0), (3k+2,0), (k-4,0), (3k+6,0), \dots, (4,0)$	k/2 - 1

Table 1:  $k \equiv 0 \pmod{4}, k \ge 4$ 

Table 2:  $k \equiv 1 \pmod{4}, k \ge 5$ 

Sequ	No. of terms	
(a)	(0,0)	1
(b)	$(0,1), (1,1), (2,1), \dots, (2k-2,1)$	2k - 1
(c)	(4k-2,1)	1
(d)	$(2k-1), (2k,1), (2k+1,1), \dots, (4k-3,1)$	2k - 1
(e)	(2k,0)	1
(f)	$(4k-3,0), (5,0), (4k-7,0), (9,0), \dots, (2k-1,0)$	k-1
(g)	(2k+2,0)	1
(h)	$(2k-3,0), (2k+5,0), (2k-7,0), (2k+9,0), \dots, (3,0)$	k-2
(i)	(4k - 2, 0)	1
(j)	(4k - 1, 1)	1
(k)	$(2,0), (4k-4,0), (6,0), (4k-8,0), \dots, (k,0)$	(k-3)/2
(1)	(1,0)	1
(m)	$(3k-3,0), (k+5,0), (3k-7,0), (k+9,0), \dots, (2k-4,0)$	(k-5)/2
(n)	(3k + 1, 0)	1
(o)	(2k+1,0)	1
(p)	(k+1,0)	1
(q)	$(2k-2,0), (2k+4,0), (2k-6,0), (2k+8,0), \dots, (3k-1,0)$	(k-1)/2
(r)	(4k - 1, 0)	1
(s)	$(k-1,0), (3k+3,0), (k-5,0), (3k+7,0), \dots, (4,0)$	(k-3)/2

Secu								
Sequ	encing	No. of terms						
(a)	(0,0)	1						
(b)	$(0,1),(1,1),(2,1),\ldots,(2k-2,1)$	2k - 1						
(c)	(4k-2,1)	1						
(d)	$(2k-1), (2k, 1), (2k+1, 1), \dots, (4k-3, 1)$	2k - 1						
(e)	(2k,0)	1						
(f)	$(4k-3,0), (5,0), (4k-7,0), (9,0), \dots, (2k-3,0)$	k-2						
(g)	(2k+2,0)	1						
(h)	$(2k-1,0), (2k+3,0), (2k-5,0), (2k+7,0), \dots, (3,0)$	k-1						
(i)	(4k-2,0)	1						
(j)	(4k - 1, 1)	1						
(k)	$(2,0), (4k-4,0), (6,0), (4k-8,0), \dots, (k,0)$	k/2						
(1)	(4k - 1, 0)	1						
(m)	$(3k-2,0), (k+4,0), (3k-6,0), (k+8,0), \dots, (2k-2,0)$	k/2 - 1						
(n)	(3k,0)	1						
(o)	(2k+1,0)	1						
(p)	(k+2,0)	1						
(q)	$(2k-4,0), (2k+6,0), (2k-8,0), (2k+10,0), \dots, (3k-4,0)$	k/2 - 3						
(r)	(1,0)	1						
(s)	$(k-2,0), (3k+4,0), (k-6,0), (3k+8,0), \dots, (4,0)$	k/2-2						

Table 3:  $k \equiv 2 \pmod{4}, k \ge 6$ 

Table 4:  $k \equiv 3 \pmod{4}, k \ge 7$ 

Sequ	No. of terms	
(a)	(0,0)	1
(b)	$(0,1), (1,1), (2,1), \dots, (2k-2,1)$	2k - 1
(c)	(4k-2,1)	1
(d)	$(2k-1), (2k,1), (2k+1,1), \dots, (4k-3,1)$	2k - 1
(e)	(2k,0)	1
(f)	$(4k-3,0), (5,0), (4k-7,0), (9,0), \dots, (2k-1,0)$	k-1
(g)	(2k+2,0)	1
(h)	$(2k-3,0), (2k+5,0), (2k-7,0), (2k+9,0), \dots, (3,0)$	k-2
(i)	(4k-2,0)	1
(j)	(4k-1,1)	1
(k)	$(2,0), (4k-4,0), (6,0), (4k-8,0), \dots, (k-1,0)$	(k-1)/2
(1)	(4k-1,0)	1
(m)	$(3k-1,0), (k+3,0), (3k-5,0), (k+7,0), \dots, (2k-4,0)$	(k-3)/2
(n)	(3k + 1, 0)	1
(o)	(2k+1,0)	1
(p)	(k+1,0)	1
(q)	$(2k-2,0), (2k+4,0), (2k-6,0), (2k+8,0), \dots, (3k-3,0)$	(k-3)/2
(r)	(1,0)	1
(s)	$(k-3,0), (3k+5,0), (k-7,0), (3k+9,0), \dots, (4,0)$	(k-5)/2

## 2.3 Binary Groups

Recall that a binary group is defined to be a group with a unique element of order 2. If G is a binary group then we denote the unique subgroup of order 2 by  $\Lambda(G)$ . The subgroup  $\Lambda(G)$  is necessarily normal. Let G be a binary group of order 2n with z as its unique element of order 2. A sequencing **b** of G is said to be *symmetric* if it is of the form

$$\mathbf{b} = (e, b_2, b_3, \dots, b_n, z, b_n^{-1}, \dots, b_3^{-1}, b_2^{-1}).$$

Note that as z is the only element of order 2 we have immediately that  $b_i \neq b_i^{-1}$  for  $2 \leq i \leq n$ . Gordon's construction (Theorem 2) for sequencing abelian groups gives symmetric sequencings, as does our new proof of that theorem in Section 3.4.

The aim of this section is to find symmetric sequencings for binary groups. We begin by considering the structure of binary groups.

The class of binary groups has arisen in several different contexts. For example, a Frobenius complement of even order (in particular, the multiplicative group of a nearfield) is a binary group [93, chapter 3.18], as is the automorphism group of a switching class of tournaments [29]. We have already noted that the binary polyhedral groups are binary groups. Coxeter[45, p. 82] posed the problem of classifying the binary groups, which is now solved (as we outline below). It is unclear who first solved this problem. Babai and Cameron [29] give a classification due to Glauberman but report that "[t]his result is known to some group theorists, but we are not aware of a proof in the literature".

If G is a binary group, then so is any subgroup of even order; in particular, each Sylow 2-subgroup. Now, 2-groups with a unique involution are known [41, p. 132]: they are cyclic or generalised quaternion groups. Here, the generalised quaternion group  $Q_{2^n}$  is defined by

$$Q_{2^n} = \langle u, v \colon u^{2^{n-1}} = e, v^2 = u^{2^{n-2}}, vuv^{-1} = u^{-1} \rangle.$$

The Sylow 2-subgroups of  $G/\Lambda(G)$  have the form  $S/\Lambda(S)$  for Sylow 2-subgroups S of G. The quotient  $S/\Lambda(S)$  is cyclic or dihedral according as S is cyclic or generalised quaternion.

Conversely, a cohomological argument due to Glauberman, reported in [29], shows that, if H is a finite group with cyclic or dihedral Sylow 2-subgroups, then there is a unique binary group G with  $G/\Lambda(G) \cong H$ .

So the classification of binary groups reduces to that of groups with cyclic or dihedral Sylow 2-subgroups.

This classification is provided by Burnside's Transfer Theorem [41, p. 155] and the Gorenstein–Walter Theorem [57, 38]. The result is as follows. Recall that O(G) is the largest normal subgroup of G of odd order. Let H be a finite group with Sylow 2-subgroup T. Then

- if T is cyclic, then  $H/O(H) \cong T$ ;
- if T is dihedral, then H/O(H) is isomorphic to the alternating group  $A_7$ , or to a subgroup of  $P\Gamma L(2,q)$  containing PSL(2,q) (where q is an odd prime power), or to T.

In particular, if G is a soluble binary group, then  $G/O(G)\Lambda(G)$  is isomorphic to  $A_4$ ,  $S_4$ , V, or a cyclic or dihedral 2-group. (V denotes the elementary abelian 2-group of order 4).

It is not completely straightforward to describe the corresponding binary groups. Glauberman's argument gives a description in cohomological terms.

The search for symmetric sequencings was effectively initiated by Lucas [74], although he did not use this terminology. Following on from Gordon's construction, further results have been obtained by Bailey and Praeger [33], Nilrat and Praeger [80] and Anderson, working alone and with Ihrig and Leonard. Symmetric sequencings with special properties have been used to solve some cases of the Oberwolfach problem [86]. Theorem 5 is an early result which the rest of the work can be seen as generalising:

**Theorem 5** [2] If G is a sequenceable group of odd order n then  $G \times C_2$  has a symmetric sequencing.

Proof: Let z be the non-identity element of  $C_2$ . Observe first that  $G \times C_2$  is a binary group with (e, z) as its unique element of order two. Let  $(e, d_2, \ldots, d_n)$  be a sequencing of G. Since G is of odd order, every non-identity element is distinct from its inverse. Partition  $G \setminus \{e\}$  into (n - 1)/2 two-element subsets of the form  $\{g, g^{-1}\}$  and choose an element from each subset.

We now define a symmetric sequencing **b**, where  $\mathbf{b} = (b_1, b_2, \dots, b_{2n})$ , for  $G \times C_2$ :

$$(b_1, b_2, \dots, b_n) = ((e, e), (d_2, \epsilon_2), \dots, (d_n, \epsilon_n))$$

where

$$\epsilon_i = \begin{cases} z & \text{if } d_i \text{ is a chosen element} \\ e & \text{otherwise,} \end{cases}$$
$$b_{n+1} = (e, z)$$

and

$$(b_{n+2}, b_{n+3}, \dots, b_{2n}) = ((d_n^{-1}, \epsilon_n), (d_{n-1}^{-1}, \epsilon_{n-1}) \dots, (d_2^{-1}, \epsilon_2))$$

Now **b** lists  $G \times C_2$  without repetition and does so symmetrically (since  $(b_i, \epsilon_i)^{-1} = (b_i^{-1}, \epsilon_i)$ ).

Also, all of the elements in  $G \times C_2$  are in the sequence of partial products. The partial products move through the basic directed terrace for G, with associated e's and z's in the second co-ordinate. Then, from the (n + 1)th position, they move back through G's basic directed terrace with the e's and z's switched, finishing on (e, z).  $\Box$ 

A key concept on which the work relies is that of a 2-sequencing (or equivalently a basic terrace), introduced by Bailey [30]. A 2-sequencing of H, a group of order n, is a sequence of elements  $(e, d_2, d_3, \ldots, d_n)$ , not necessarily distinct, such that:

• the associated partial products  $e, ed_2, ed_2d_3, \ldots, ed_2 \cdots d_n$  are all distinct; this sequence is called a *basic terrace*,

• if  $h \in H$  and  $h \neq h^{-1}$  then

$$|\{i: 2 \leq i \leq n, d_i \in \{h, h^{-1}\}\}| = 2,$$

• if  $h \in H$  and  $h = h^{-1}$  then

$$|\{i: 1 \le i \le n, d_i = h\}| = 1.$$

Note that a sequencing for H is also a 2-sequencing for H. We look at the theory of terraces and 2-sequencings more in Section 3.4.

The following generalisation of Theorem 5 is pivotal:

**Theorem 6** [4] Let G be a binary group of order 2n. Then G has a symmetric sequencing if and only if  $G/\Lambda(G)$  has a 2-sequencing.

Proof: Let  $\pi : G \to G/\Lambda(G)$  be the natural projection. Then it is straightforward to check that if  $(e, b_2, \ldots, b_n, z, b_n^{-1}, \ldots, b_2^{-1})$  is a symmetric sequencing of G then  $(\pi(e), \pi(b_2), \ldots, \pi(b_n))$  is a 2-sequencing of  $G/\Lambda(G)$ .

Let z be the element of order 2 in G. If  $y \in G/\Lambda(G)$  then  $y = \{x, xz\}$  for some  $x \in G$ . Suppose we are given a 2-sequencing of  $G/\Lambda(G)$ . Lift this back to a sequence in G as follows:

(i): If  $y \in G/\Lambda(G)$ ,  $y \neq y^{-1}$  and y (equivalently  $y^{-1}$ ) occurs twice in our 2-sequencing, the two occurrences of  $y = \{x, xz\}$  can be lifted back to x and xz (in either order).

(ii): If  $y \in G/\Lambda(G)$ ,  $y \neq y^{-1}$  and both y and  $y^{-1}$  occur once in our 2-sequencing, say  $y = \{x, xz\}$  and  $y^{-1} = \{x^{-1}, x^{-1}z\}$ , we can either lift y to x and  $y^{-1}$  to  $x^{-1}z$  or y to xz and  $y^{-1}$  to  $x^{-1}$ .

(iii): If  $y \in G/\Lambda(G)$ ,  $y = y^{-1}$  and  $y \neq \{e, z\}$  then y must occur once in our 2-sequencing. Now  $y = \{x, x^{-1}\}$  and y may be lifted back to either x or  $x^{-1}$ .

(iv): If  $y = \{e, z\} \in G/\Lambda(G)$  then y must be lifted to e.

This process clearly gives a sequence of the form  $(e, b_2, \ldots, b_n)$  in G where  $b_i \neq b_j$  and  $b_i^{-1} \neq b_j$  for  $i, j \leq n$ , where  $i \neq j$ . Extend this to a sequence of all the elements of G:  $(e, b_2, \ldots, b_n, z, b_n^{-1}, \ldots, b_2^{-1})$ .

We claim that this is a symmetric sequencing of G.

The partial products are  $(e, a_2, \ldots, a_n, a_n z, a_{n-1} z, \ldots, a_2 z)$ : we therefore need that  $\{e, a_2, \ldots, a_n\}$  is a transversal of  $\{\{w, wz\} : w \in G\}$ . This follows because each  $a_i$  is either  $w_i$  or  $w_i z$  for some  $w_i \in G$  and the elements  $\{e, z\}, \{w_2, w_2 z\}, \ldots, \{w_n, w_n z\}$  of  $G/\Lambda(G)$  are distinct as we started from a 2-sequencing. The sequence is clearly symmetric, so the claim is verified and the proof is complete.  $\Box$ 

Note that the construction of Theorem 6 gives  $2^{(n+k-1)/2}$  different symmetric sequencings of G for each 2-sequencing of  $G/\Lambda(G)$ , where k is the number of elements of order 2 in  $G/\Lambda(G)$ .

Anderson and Ihrig extend this further to show:

**Theorem 7** [13] If G is a binary group and  $G/O(G)\Lambda(G)$  has a 2-sequencing then G has a symmetric sequencing.

The groups  $A_4$  and  $S_4$  have sequencings [3] and hence have 2-sequencings. We have also seen that cyclic and dihedral 2-groups have sequencings (Example 2 and Theorem 4). Prior to the proof of Theorem 4, Anderson had shown that all dihedral groups have 2-sequencings [4, 6, 7].

The case  $G/O(G)\Lambda(G) \cong V$  only arises when we consider binary groups with  $Q_8$ as their Sylow 2-subgroup. The group  $Q_8$  itself is not sequenceable, but Anderson and Leonard [15] show that the groups  $Q_8 \times B$ , where B is a non-trivial abelian group of odd order, have symmetric sequencings. These groups are Hamiltonian groups; that is, each is a non-abelian group every subgroup of which is normal. In fact, they are the only Hamiltonian binary groups. Anderson and Ihrig [13] show that all soluble binary groups G with  $G/O(G)\Lambda(G) \cong V$ , where  $G \neq Q_8$ , have symmetric sequencings. Theorem 7 now gives:

**Theorem 8** [13] All finite soluble binary groups, except  $Q_8$ , have symmetric sequencings.

In [14] Anderson and Ihrig consider the structure of insoluble binary groups. They show that to find sequencings of all insoluble binary groups it is sufficient to find 2sequencings of  $A_7$ , PSL(2, q) and PGL(2, q) for q an odd prime power greater than 3. They also show that there is no redundancy here; finding a 2-sequencing of each of these groups gives symmetric sequencings for an infinite set of insoluble binary groups and these sets are disjoint. The only result in this direction to date is the sequencing of  $PSL(2,5) \cong A_5$ [5], showing that such infinite sets of sequenceable insoluble binary groups do exist.

## 2.4 Groups of Odd Order

Keedwell [69] and Wang [105] have sequenced some non-abelian groups of odd order which have a cyclic normal subgroup with prime index. We do not give the full constructions here, merely introduce the two crucial concepts.

The first concept is the quotient sequencing (this concept was introduced by Friedlander [53]). Let G be a group of order pq with normal subgroup H of order q. A sequence Q of length pq containing elements of G/H is said to be a quotient sequencing of G/H if each element of G/H occurs q times in both Q and the partial product sequence (the basic quotient directed terrace) of Q. Note that the natural map  $G \to G/H$  maps a sequencing of G to a quotient sequencing of G/H; however, most quotient sequencings cannot be lifted to a sequencing of the parent group.

Suppose, with the above notation, that  $G/H \cong C_p$  for some odd prime p, where  $C_p = \langle u : u^p = e \rangle$ . Let  $\beta$  be a primitive root of p such that  $\beta/(\beta - 1)$  is also a primitive root of p (Wang [105] reports that such a  $\beta$  exists, using the results of [44]). Wang [105] gives a quotient sequencing for G/H. Here we just give the associated basic quotient directed terrace as that may be expressed more simply:

 $e, e, \ldots, e, x$  (q elements)

followed by q-2 copies of the sequence

$$x^{\beta^{p-2}}, x^{\beta^{p-3}}, \dots, x \quad (p-1 \text{ elements})$$

and finishing with

 $x^{\beta^{p-2}}, x^{\beta^{p-3}}, \dots, x^{\beta}, x^{\beta}, x^{\beta^2/(\beta-1)}, x^{\beta^3/(\beta-1)^2}, \dots, x^{\beta-1}, e \quad (2(p-1) \text{ elements}).$ 

Wang observes that this is a generalisation of the quotient directed terrace used by Keedwell [69].

The second important concept is the R-sequencing. An *R*-sequencing (sometimes called a *near-sequencing*) of a group G is a sequence  $(e, b_2, b_3, \ldots, b_n)$  of all the elements of G such that the partial products  $(e, eb_2, eb_2b_3, \ldots, eb_2b_3 \cdots b_{n-1})$  are distinct and  $eb_2b_3 \cdots b_n = e$ . Keedwell and Wang both consider groups G with a normal cyclic subgroup of order q and index a prime p. The method they use is to find an R-sequencing of  $C_q$  and use the first q-1 elements of this for the first q-1 elements of the sequencing, filling the rest of the sequencing in a way that is also compatible with the above quotient directed terrace.

Suppose that p and q are odd primes, with p < q. Then there is a non-abelian group of order pq if and only if q = 2ph + 1 for some positive integer h. This group has a cyclic normal subgroup of order q. Keedwell [69] found sequencings of groups of this type whenever 2 is a primitive root of p. Wang showed that it is sufficient to to find an R-sequencing of  $C_q$  in which  $x^{r-r^{1-\beta}}$  and  $x^{r-r^{1-\beta}-1}$  are adjacent for some r with  $r^p \equiv 1$ (mod q) and  $r \not\equiv 1 \pmod{q}$ . In [104] Wang gives some examples of such R-sequencings where 2 is not a primitive root of p.

Wang [105] also finds a sequencing which is compatible with the above quotient directed terrace for the unique non-abelian group of order  $p^m$  that has a cyclic normal subgroup of index p, where p is an odd prime and m > 3.

Let G be a group of odd order n. A sequencing  $(e, b_2, \ldots, b_n)$ , is said to be a startertranslate sequencing (Anderson [9] abbreviates this to st-sequencing) if both of the sets  $\{b_2, b_4, \ldots, b_{n-1}\}$  and  $\{b_3, b_5, \ldots, b_n\}$  contain precisely one of g and  $g^{-1}$  for each  $g \in G \setminus \{e\}$ . Anderson [9] shows that if G and H are groups with st-sequencings then  $G \times H$  also has an st-sequencing. He also shows that Keedwell's sequencing of the non-abelian group of order pq is starter-translate whenever both p and q are congruent to 3 modulo 4. This considerably extends the set of odd integers n for which a sequenceable group of order n is known to exist.

## 2.5 Summary

In addition to the results given already in this chapter, Anderson [3, 5] has used a hillclimbing algorithm to sequence all non-abelian groups of order n, where  $10 \le n \le 32$ , and  $A_5$  and  $S_5$ , the alternating and symmetric groups on 5 symbols. Therefore the following groups are known to be sequenceable.

- Dihedral groups of order at least 10
- Soluble (including abelian) binary groups, except  $Q_8$
- Insoluble binary groups G with  $A_5$  as their only non-abelian composition factor

- Some groups of order pq where p and q are odd primes
- Direct products of some of the groups of the previous type if both p and q are congruent to 3 modulo 4
- At least one of the non-abelian groups of order  $p^m$ , for p an odd prime and  $m \ge 3$
- Non-abelian groups of order n, where  $10 \le n \le 32$
- $A_5$  and  $S_5$

The only groups known to be non-sequenceable are abelian groups which do not have a unique element of order 2 and the non-abelian groups  $D_6$ ,  $D_8$  and  $Q_8$ .

**Conjecture 1** (Keedwell)  $D_6$ ,  $D_8$  and  $Q_8$  are the only non-abelian non-sequenceable groups.

A milder conjecture is

**Conjecture 2** (Anderson)  $Q_8$  is the only binary group which does not have a symmetric sequencing.

# 3 Related Concepts

In this section we look at some concepts related to sequencings: R-sequencings, harmonious groups, supersequenceable groups (also known as super P-groups), terraces and the Gordon Game. We also look at an alternative method for constructing row-complete latin squares.

## 3.1 R-sequencings

A pair of latin squares  $(l_{ij})$  and  $(l'_{ij})$  are said to be *orthogonal* if every ordered pair of symbols occurs exactly once among the  $n^2$  pairs  $(l_{ij}, l'_{ij})$ . It is shown in [92] that the existence of a group of order n having an R-sequencing (see page 16 for the definition) is a sufficient condition for there to exist a pair of orthogonal latin squares of order n. (More specifically, it is shown that having an R-sequencing is a sufficient condition for a group to have a complete mapping and having a complete mapping in a group of order n is sufficient to produce a pair of orthogonal latin squares of order n. See [71] for a summary of these and related topics.)

The study of orthogonal latin squares was originally motivated by a problem of Euler, set in 1779: "Thirty-six officers of six different ranks and taken from six different regiments, one of each rank in each regiment, are to be arranged, if possible, in a solid square formation of six by six, so that each row and each column contains one and only one officer of each rank and one and only one officer from each regiment". The solution of this problem is equivalent to the construction of a pair of orthogonal latin squares of order 6. In 1782 Euler conjectured that no such pairs of latin squares exist for orders n = 4k + 2; this was proved true for n = 6 by Tarry in 1900 and false for all n > 6 by Bose, Shrikhande and Parker in 1960. It is easily seen to be true for n = 2. See [47, chapter 5] for a thorough account of orthogonal latin squares and the history of Euler's conjecture.

Observe that for abelian groups the properties of being sequenceable and R-sequenceable are mutually exclusive as the final element of the partial product sequence is invariant.

**Theorem 9** The following types of abelian group are R-sequenceable:

(i)  $C_n$ , where n is odd, (ii) Abelian groups of odd order with (possibly trivial) cyclic Sylow 3-subgroups, (iii)  $C_3^n$ , where  $n \ge 2$ , (iv)  $C_3 \times C_{3n}$ , where  $n \ge 2$ , (v)  $C_2 \times C_{4n}$ , where  $n \ge 1$ , (vi) Abelian groups with Sylow 2-subgroups  $C_2^n$ , where n = 2 or  $n \ge 4$ , (vii) Abelian groups with Sylow 2-subgroups  $C_2 \times C_{2^n}$ , where n is odd.

Proof. Part (iv) is proved in [106]; all of the others are proved in [54].  $\Box$ 

Alternative constructions for R-sequencings of  $C_n$  are given in [1].

It is reported in [48, Chapter 3] that Ringel claims to have shown that  $C_2 \times C_{6n+2}$  is R-sequenceable for  $n \ge 1$ .

Since then Headley [58] has extended these results to include all abelian groups whose Sylow 2-subgroups are neither cyclic (including trivial) nor  $C_2 \times C_4$  nor  $C_2 \times C_2 \times C_2$ . Wang [106] has shown that  $C_3 \times C_{3m}$  is R-sequenceable for all  $m \ge 1$ .

The following theorem gives the position for non-abelian groups. The dicyclic group  $Q_{4n}$  is defined by

$$Q_{4n} = \langle a, b : a^{2n} = e, b^2 = a^n, ab = ba^{-1} \rangle.$$

If n is a power of 2 then  $Q_{4n}$  is a generalised quaternion group.

**Theorem 10** (i) The dihedral group  $D_{2n}$  of order 2n is R-sequenceable if and only if n is even.

(ii) The dicyclic group  $Q_{4n}$  is R-sequenceable if and only if n is an even integer greater than 2.

(iii) The non-abelian groups of order pq, where p and q are odd primes, are R-sequenceable.(iv) The two non-abelian groups of order 27 are R-sequenceable.

Proof: (i): See [70].

(ii): See [107].

(iii): The case when p < q and p has 2 as a primitive root was covered by Keedwell in [70]. Wang and Leonard subsequently removed the primitive root condition in [107].

(iv): See [36].  $\Box$ 

## 3.2 Harmonious Groups

Similarly to R-sequencings, a harmonious or #-harmonious sequence for a group of order n gives rise to a complete mapping of the group and hence a pair of orthogonal latin squares. They were introduced in [35].

Let G be a non-trivial group of order n and let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  be an arrangement of the elements of G. Let  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  be defined by  $b_i = a_i a_{i+1}$  for i < n and  $b_n = a_n a_1$ . If the elements of **b** also include all of the elements of G then **a** is a harmonious sequence and G is harmonious.

The situation for abelian groups is completely settled:

**Theorem 11** [35] An abelian group is harmonious if and only if it has non-cyclic or trivial Sylow 2-subgroups and is not an elementary abelian 2-group.

Here is the current state of knowledge for non-abelian groups:

**Theorem 12** The following non-abelian groups are harmonious:

(i) non-abelian groups of odd order, (ii)  $D_{8t}$  and  $D_{24t-12}$ , for  $t \ge 1$ ,

(iii)  $Q_{8t}$ , for  $t \ge 2$ .

Proof. Parts (i) and (ii) are proved in [35] and (iii) is proved in [107].  $\Box$ 

Groups of twice an odd order and  $Q_8$  and the dicyclic groups of four times an odd order are not harmonious [35, 107].

The notion of #-harmoniousness is similar, but with the identity of G removed. Let G be a non-trivial group of order n with identity e and let  $\mathbf{a} = (a_1, a_2, \ldots, a_{n-1})$  be an arrangement of the elements of  $G \setminus \{e\}$ . Let  $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1})$  be defined by  $b_i = a_i a_{i+1}$  for i < n-1 and  $b_{n-1} = a_{n-1}a_1$ . If the elements of  $\mathbf{b}$  also include all of the elements of  $G \setminus \{e\}$  then  $\mathbf{a}$  is a #-harmonious sequence and G is #-harmonious.

**Theorem 13** [35] An abelian group is #-harmonious if and only if it has non-cyclic or trivial Sylow 2-subgroups and is not  $\mathbb{Z}_3$ .

## 3.3 Supersequenceable Groups

Let G be a group of order n with derived group G'. As the elements of G commute modulo G', the products of all the elements of G lie in the same coset hG' of G', regardless of the order of multiplication. It is known [46, 95] that each element of this special coset may be expressed as the product of all of the group elements in some order (groups with this property were originally known as *P*-groups, but this name is now redundant).

In 1983 Keedwell [70] defined super P-groups, which we now call supersequenceable groups. A super sequenceable group is a finite group G in which each element g of the special coset is either

• the last element of some basic directed terrace, or

• the last element of the partial product sequence associated with some R-sequencing.

The second condition is used only when g = e; we have g = e only when hG' = G'. Keedwell [70] proved the following two theorems:

**Theorem 14** Let G be an abelian group. Then G is a supersequenceable group if and only if G is sequenceable or R-sequenceable.

Proof: Observe that  $G' = \{0\}$ , so the relevant coset of G' has just one element. This element is the identity if G is not a binary group, and is the unique element of order 2 if G is a binary group (see Theorem 2). Thus, if G is not a binary group then G is a supersequenceable group if and only if G is R-sequenceable. If G is a binary group then G is a supersequenceable group if and only if G is sequenceable.  $\Box$ 

**Theorem 15** The following groups are supersequenceable groups:

(i) Dihedral groups  $D_{2n}$  where  $n \ge 5$  is odd.

(ii) Dihedral groups  $D_{2n}$  where n is twice an odd prime.

(iii) Groups of order pq where p and q are primes, p < q and 2 is a primitive root of p.

Also, Bedford [36] has shown that both of the non-abelian groups of order 27 are supersequenceable.

## **3.4** Terraces

As we noted in Section 2.3, terraces and 2-sequencings are equivalent. We say that a group that has a terrace is *terraced*. Terraces were introduced by Bailey [30] to prove Theorem 16—an analogue of Theorem 1 for quasi-complete latin squares. An  $n \times n$  latin square is said to be *row quasi-complete* if each distinct pair of symbols  $\{x, y\}$  occurs in adjacent horizontal cells twice (in either order). It is said to be *column quasi-complete* if each pair of distinct symbols  $\{x, y\}$  occurs in adjacent vertical cells twice (in either order). A latin square that is both row quasi-complete and column quasi-complete is said to be *quasi-complete*.

Row-quasi-complete latin squares were used by Williams [109] for designing experiments where carry-over effects are thought to be present. He uses them in pairs, one containing the reverses of the other's rows, giving a design in which each pair of treatments occurs twice in each order as row-neighbours. He gives an example of such a design being used in practice to study the effect of diet on the milk yield of cows. An application where quasi-completeness is the natural requirement, rather than being used when completeness is unavailable, is given in [32]. The experiment described concerns five methods of controlling insects on spring beans. A quasi-complete latin square of order 5 is advocated because it was felt that there may be neighbour effects between adjacent plots from insects overspilling from a plot containing spring beans with a treatment that does little (or nothing) to repel them. It is pointed out that row neighbours should be kept distinct from column neighbours as plots in this type of experiment are rarely square—in this instance they measured  $1.2m \times 1m$ . A similar experiment is described in [98]. However, this experiment has six treatments and their quasi-complete latin square is also complete.

Quasi-complete latin squares have also been considered by Freeman [51, 52] and Campbell and Geller [42].

**Theorem 16** [30] Let G be a terraced group with terrace  $(a_1, a_2, \ldots, a_n)$ . Then the square  $(l_{ij}) = (a_i^{-1}a_j)$ , where  $1 \leq i, j \leq n$ , is a quasi-complete latin square.

Proof: Similar to the proof of Theorem 1.  $\Box$ 

We would therefore like to know which groups are terraced. The following result is originally due to Williams [109] and has been rediscovered, in various guises, by many authors.

**Theorem 17** [109] For all positive integers n, the cyclic group  $\mathbb{Z}_n$  is terraced.

Proof: A terrace for  $\mathbb{Z}_n$  is (0, 1, n - 1, 2, n - 2, ...), having (0, 1, n - 2, 3, n - 4, ...) as its 2-sequencing.  $\Box$ 

Note that when n is even the terrace given in Theorem 17 is directed. This is the one given in Example 2, explaining the Lucas-Walecki-Williams name given there.

Much work has been done on finding terraces for cyclic groups that have special properties. Sometimes the interest is in the Latin squares or other desgins that can be constructed from them [8, 10, 18, 81, 83, 86, 88, 99]; sometimes it is in properties of the terraces themselves [17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 90, 94].

Returning to the question of which groups are terraced, the following two results are due to Bailey.

**Theorem 18** [30]  $G = \mathbb{Z}_2^n$  is not 2-sequenceable for n > 1.

Proof: For each  $g \in G$ , we have g = -g. Thus G is 2-sequenceable if and only if G is sequenceable, but G is not sequenceable, by Theorem 2.  $\Box$ 

**Theorem 19** [30] Abelian groups of odd order are terraced.

Proof: Let  $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$  be an abelian group of odd order. We use induction on the number of summands. By Theorem 17,  $\mathbb{Z}_{n_1}$  is terraced.

Suppose that  $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$  is terraced with terrace  $\mathbf{a} = (a_1, a_2, \ldots, a_m)$  and let  $\mathbf{c} = (c_1, c_2, \ldots, c_{n_k})$  be the Williams terrace for  $\mathbb{Z}_{n_k}$ . For i > 0 let

$$\begin{array}{lll}
\alpha^{(0)} &=& (a_1,0), (a_2,0), \dots, (a_m,0) \\
\alpha^{(i)} &=& (a_m,c_i), (a_{m-1},c_{i+1}), (a_{m-2},c_i), (a_{m-3},c_{i+1}), \dots, (a_1,c_i) & \text{if } i \text{ is odd} \\
\alpha^{(i)} &=& (a_1,c_i), (a_2,c_{i-1}), (a_3,c_i), (a_4,c_{i-1}), \dots, (a_m,c_i) & \text{if } i \text{ is even} \\
\end{array}$$

We claim that  $(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n_k)})$  is a terrace for  $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ :

An allowable set of differences with zeros in the second co-ordinate occurs within the differences produced by  $\alpha^{(0)}$  as **a** is a terrace. An allowable set of differences with zeros in the first co-ordinate occurs within the differences produced by the juxtapositions of  $\alpha^{(i)}$  and  $\alpha^{(i+1)}$  as **c** is a terrace. These are the only occurrences of differences with zero in either co-ordinate.

Let x be a non-zero element of  $\mathbb{Z}_{n_k}$  such that  $c_{i+1} - c_i = x$  for some odd i (this covers exactly one of x and -x for each  $x \in \mathbb{Z}_{n_k}$ ). Then x or -x occurs in the second co-ordinate of the differences in  $\alpha^{(i)}$  and  $\alpha^{(i+1)}$  (and nowhere else). As **a** is a terrace, for each non-zero y, where  $y \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{k-1}}$ , we get one of the following combinations of differences among the differences produced by  $\alpha^{(i)}$  and  $\alpha^{(i+1)}$ :

- two occurrences each of (y, x) and (-y, x) (and no occurrences of (y, -x) or (-y, -x))
- one occurrence each of (y, x), (-y, x), (y, -x) and (-y, -x)
- two occurrences each of (y, -x) and (-y, -x) (and no occurrences of (y, x) or (-y, x)).

None of these combinations contravene the definition of a terrace, so  $(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n_k)})$  is a terrace as claimed. The result now follows by induction on k.  $\Box$ 

We can now complete the proof of Theorem 2; that is, abelian binary groups have directed terraces:

Proof of Theorem 2: Let A be an abelian binary group, say  $A = \mathbb{Z}_{2^m} \times B$  where m > 1and B is an abelian group of odd order. Then  $A/\mathbb{Z}_2 \cong \mathbb{Z}_{2^{m-1}} \times B$  and Theorem 6 says that if  $A/\mathbb{Z}_2$  is terraced then A has a directed terrace. The preceding theorem gives a terrace for B, so the result now follows by induction on m.  $\Box$ 

The question of which abelian groups are terraced is almost settled:

**Theorem 20** All abelian groups, except non-cyclic elementary abelian 2-groups and possibly those of order coprime to 15 with an elementary abelian Sylow 2-subgroup of order 8, are terraced.

Proof. Abelian groups of odd order were terraced by Bailey [30]. Groups of the form  $C_{2n} \times C_2$  were terraced by Anderson and Leonard [15]. The remaining groups were covered in a series of papers by Ollis and Willmott [82, 85, 90].  $\Box$ 

Here is the situation for non-abelian groups. The semidihedral group of order 8t, denoted  $SD_{8t}$  and another similarly structured group of the same order that we denote  $M_{8t}$ , are given by:

$$SD_{8t} = \langle u, v : u^{4t} = e = v^2, vu = u^{2t-1}v \rangle$$
  
$$M_{8t} = \langle u, v : u^{4t} = e = v^2, vu = u^{2t+1}v \rangle$$

Call a non-trivial abelian group a generalised Klein group if it as a composition series with all factors isomorphic to the Klein 4-group  $C_2^2$ .

**Theorem 21** The following non-abelian groups are terraced:

(i) all non-abelian sequenceable groups,

(ii) all non-abelian groups of odd order,

(iii) all non-abelian groups with a terraced normal subgroup of odd index and non-abelian groups with a odd-order normal subgroup with a terraced quotient group,

(iv)  $SD_{8t}$  and  $M_{8t}$ , for  $t \ge 2$ ,

(v) various groups with a central subgroup isomorphic to  $C_2^2$ , including those of the form  $A \times G$  where A is a generalised Klein group and G is  $D_{8t}$ ,  $SD_{8t}$ ,  $M_{8t}$ , for  $t \ge 2$ , or a non-abelian group of order 12, 16 or 20,

(vi) all non-abelian groups of order up to 86 (except possibly 64).

Proof: Part (i) follows immediately from the definitions; for (ii) see [12]; for (iii) see [12, 14]; for (iv) and (v) see [91]; and for (vi), see [11].  $\Box$ 

**Conjecture 3** (Bailey) All finite groups, except the elementary abelian 2-groups of order at least 4, are terraced.

In 1988 Morgan [76] generalised the concept of a terrace as follows. Let G be a group of order n and let **a** be a list  $(a_1, a_2, \ldots, a_p)$  of elements of G (repeats and omissions of elements permitted) where p = 1 + m(n-1)/2 for some integer m. Note that if n is even then m must also be even. For such an **a** let  $\mathbf{b} = (a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{p-1}^{-1}a_p)$ . We say that **a** is an m-terrace of G if each element of G occurs in **a** either  $\lfloor p/n \rfloor$  or  $\lfloor p/n \rfloor + 1$ times  $(\lfloor k \rfloor$  denotes the least integer greater than k-1) and if **b** consists of

- m/2 occurrences of each non-identity element g which satisfies  $g = g^{-1}$
- *m* total occurrences from the pair  $\{g, g^{-1}\}$  for each element *g* which does not satisfy  $g = g^{-1}$ .

Observe that a 2-terrace is a terrace as previously defined. The cyclic groups  $\mathbb{Z}_n$  have 2- and 4-terraces when n is even and 1-, 2-, 3- and 4-terraces when n is odd [76]. That all abelian groups of odd order have a 1-terrace follows immediately from the existence of a half-and-half terrace for such groups, proved in [87].

The purpose of this generalisation is for use in the construction of "polycross designs". We refer the reader to the papers [76, 77, 78, 79, 110] for more on this topic.

## 3.5 The Gordon Game

In 1992 Isbell [66] introduced the idea of competitive sequencing: the Gordon game  $\Gamma(G)$  for a given finite group G is played as follows.

A counter is placed on the identity, e, of G. White and Black then take turns (White first) to move the counter around the group subject to the condition that the (n + 1)st move (to  $x_{n+1}$ ) must satisfy

$$x_{n+1} \notin \{e, x_1, \dots, x_n\}$$

and

$$x_n^{-1}x_{n+1} \notin \{x_1, x_1^{-1}x_2, \dots, x_{n-1}^{-1}x_n\}.$$

That is, if a game contained as many moves as the group had non-identity elements then the sequence  $(e, x_1, \ldots, x_{|G|-1})$  would be a directed terrace for G. The first player unable to make a move loses.

Is bell investigated the Gordon game for groups of small order, finding the following results. Here W and B denote forced wins for White and Black respectively.

$C_2$ :	W	$C_3$ :	W	$C_4$ :	W
$C_2 \times C_2$ :	B	$C_5$ :	W	$C_6$ :	B
$D_6$ :	W	$C_7:$	B	$C_8:$	B
$C_2 \times C_4$ :	W	$C_2 \times C_2 \times C_2$ :	B	$D_8$ :	B
$Q_8$ :	W	$C_9$ :	B	$C_3 \times C_3$ :	B
$C_{10}:$	W	$C_{11}:$	B	$C_{13}:$	B
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Isbell tentatively suggests

#### **Conjecture 4** Black wins $\Gamma(C_p)$ for primes p > 5.

The reasoning behind this conjecture is (as Isbell freely admits) shaky. Fix  $h \in C_p \setminus \{e\}$ . White's first move is irrelevant as for each  $g \in C_p \setminus \{e\}$  there is an automorphism which maps g to h. However, this automorphism is unique and the argument for Black winning is that "in the unique game which Black faces after White's first move in  $\Gamma(C_p)$  the p-3possible opening moves are all different (i.e. inequivalent by automorphisms). For large p, it is very unlikely that all are losing moves".

As far as the author is aware, no further work has been done on this problem.

## **3.6** Row-Complete Latin Squares

We have already noted that sequencings were primarily investigated because they can be used to construct row-complete latin squares. In this section we outline another construction method.

Let q be an odd prime power. Let A be a  $q \times mq$  array of symbols from  $\mathbb{F}_q \times \mathbb{Z}_m$ , where  $\mathbb{F}_q$  is the field with q elements. Write  $A_{ij} = (x_{ij}, y_{ij})$  for  $1 \leq i \leq q, 1 \leq j \leq mq$ . Then A is a generating array if the following conditions hold:

- each symbol appears once in each row of A;
- if  $x_{ij} = x_{i'j}$  then i = i';
- if  $y_{i,j+1} y_{ij} = y_{i',j'+1} y_{i'j'}$  and  $(x_{ij}, x_{i,j+1}) = (x_{i'j'}, x_{i',j'+1})$  then (i, j) = (i', j').

Given a  $q \times mq$  generating array, A, define L to be the  $mq \times mq$  array (with symbols from  $\mathbb{F}_q \times \mathbb{Z}_m$ ) with

$$L_{kq+i,j} = (x_{ij}, y_{ij} + k)$$

where  $1 \leq i \leq q$ ,  $1 \leq j \leq mq$  and  $0 \leq k \leq m-1$ .

**Theorem 22** [28] L, as defined above, is an  $mq \times mq$  row-complete latin square.

Thus to construct an  $mq \times mq$  row-complete latin square we need only to construct a  $q \times mq$  generating array.

**Example 3** [28] Let n = 9,  $\mathbb{F}_3 = \{0, 1, 2\}$ ,  $\mathbb{Z}_3 = \{0, 1, 2\}$ . Then the array A given in Figure 3 is a generating array. The corresponding row-complete latin square L is given in Figure 4. It is obtained by using the map  $\phi : \mathbb{F}_3 \times \mathbb{Z}_3 \to \{1, 2, \dots, 9\}$ ,  $(x, y) \mapsto 3x + y + 1$  as integers.

Figure 3: A $3 \times 9$ generating array, $\Delta$	A.
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(0, 0)	(1, 0)	(2, 0)	(0,1)	(1, 2)	(2, 1)	(1, 1)	(2, 2)	(0, 2)
(1, 0)	(0, 1)	(0, 0)	(2, 1)	(2, 0)	(0, 2)	(2, 2)	(1, 1)	(1, 2)
(2, 0)	(2, 1)	(1, 2)	(1, 1)	(0, 1)	(1, 0)	(0, 2)	(0, 0)	(2, 2)

$L_{1}$ $L_{1$	Figure 4: 1	$A 9 \times$	9 row-com	plete latii	n square,	L
--	-------------	--------------	-----------	-------------	-----------	---

1	4	7	2	6	8	5	9	3
4	2	1	8	7	3	9	5	6
7	8	6	5	2	4	3	1	9
2	5	8	3	4	9	6	7	1
5	3	2	9	8	1	7	6	4
8	9	4	6	3	5	1	2	7
3	6	9	1	5	7	4	8	2
6	1	3	7	9	2	8	4	5
9	7	5	4	1	6	2	3	8

It seems that this method was used by Mertz and Sonneman to construct row-complete latin squares of orders 9 and 15 respectively, reported in [59] (they are given as columncomplete latin squares there; transposing gives row-complete squares). Archdeacon, Dinitz, Stinson and Tillson [28] first formally defined generating arrays; they also constructed row-complete latin squares of orders 9 and 15 and found examples of orders 21 and 27 not based on groups. In [48, chapter 3] it is reported that Owens, Dinitz and Stinson have found examples of orders 25 and 33. Until 1997 squares constructed from sequencings were the only other known row-complete latin squares of odd order. Higham [60] combined these squares to make larger ones, showing that if there is a sequenceable group of odd order m and a row-complete latin square of odd order n then there is a rowcomplete latin square of order mn. In 1998 he returned to the concept of a generating array to show

**Theorem 23** [61] Row-complete latin squares of all odd composite orders exist.

Proof: We will show how to construct the appropriate generating arrays. Detailed proofs of their correctness may be found in [61].

Let q be an odd prime power, q > 3. We construct a  $q \times mq$  generating array A, where  $A_{ij} = (x_{ij}, y_{ij})$ . Note that every odd composite number except 9 has a proper prime power divisor greater than 3 and we have already seen a  $9 \times 9$  row complete latin square.

Case 1, m = 4r + 1: We set the  $y_{ij}$ 's first. Choose  $y_{ij}$  to be constant within each column,  $y_{ij} = s_j$  say.

Now,

$$(0, 4r, 1, 4r - 1, \dots, r - 2, 3r + 2, r - 1, 3r + 1, r,$$

$$3r - 1, r + 1, 3r - 2, \dots, 2r - 2, 2r + 1, 2r - 1, 2r, 0$$

is the sequence of partial sums of an R-sequencing of  $\mathbb{Z}_m$  (see Section 3.1).

Let w be the sequence obtained from this by removing the final 0, adding r+1 to each term and cyclically shifting the new sequence forward 2r places. That is,

$$w = (2r + 1, 4r, 2r + 2, 4r - 1, \dots, 3r - 1, 3r + 2, 3r, 3r + 1,$$
$$r + 1, r, r + 2, r - 1, \dots, 2r - 1, 2, 2r, 1).$$

Let  $s_j$  be the *j*th element of the sequence that begins with q-1 0's, then has q-1 copies of w, then the reverse of w and finishes with a 0.

To allocate the  $x_{ij}$ 's we produce  $m q \times q$  component latin squares,  $C^{(k)}$ ,  $0 \leq k \leq m-1$ . The kth component square is then matched with the symbol k where it occurs in the second co-ordinate of the generating array. More specifically, put the *l*th column of  $C^{(k)}$  in the first co-ordinates of the *l*th column of A which consists of the symbol k in the second co-ordinate  $(1 \leq l \leq q)$ .

We now define these component squares. Let  $\sigma$  be a primitive element of  $\mathbb{F}_q$  such that  $\sigma \neq 2$ . The field  $\mathbb{F}_q$  has such a  $\sigma$  for q an odd prime power  $\geq 5$ . Let  $\mathbb{F}_q = \{f_1, f_2, \ldots f_q\}$ . Then

$$C^{(k)}{}_{ij} = \begin{cases} a_k f_i + b_k \sigma^j + c_k & \text{if } 1 \leq j < q \\ a_k f_i + c_k & \text{if } j = q \end{cases}$$

where  $a_0 = \cdots = a_{2r} = 1$ ,  $a_{2r+1} = \cdots = a_{4r} = -1$ ,  $b_0 = 1$ ,  $b_1 = 1 - \sigma$ ,  $b_2 = \cdots = b_r = 1$ ,  $b_{r+1} = \cdots = b_{2r} = -1$ ,  $b_{2r+1} = \cdots = b_{3r} = 1/2 - 1/\sigma$ ,  $b_{3r+1} = \cdots = b_{4r} = 1/2$ ,  $c_0 = -\sigma/2$ and  $c_1 = \cdots = c_{4r} = 0$ .

We now have a generating array for m = 4r + 1.

Case 2, m = 4r + 3: This differs only slightly from the previous case. Again choose  $y_{ij} = s_j$ 

Now,

 $(0, 4r+2, 1, 4r+1, \dots, r-2, 3r+4, r-1, 3r+3, r,$ 

$$3r + 1, r + 1, 3r, r + 2, 3r - 1, \dots, 2r - 1, 2r + 2, 2r, 2r + 1, 0)$$

is the sequence of partial sums of an R-sequencing in  $\mathbb{Z}_m$ .

Let w be the sequence obtained from this by removing the final 0, adding r + 1 to each term, reversing the sequence and cyclically shifting the new sequence forward 2r + 1places. That is,

$$w = (2r+1, 1, 2r, 2, \dots, r+3, r-1, r+2, r, r+1, 3r+2, 3r+1, 3r+3, 3r, 3r+4, \dots, 4r, 2r+3, 4r+1, 2r+2, 4r+2).$$

Again, let  $s_j$  be the *j*th element of the sequence that begins with q-1 0's then has q-1 copies of w, then the reverse of w and finishes with a 0.

We use component squares as before to allocate the  $x_{ij}$ 's. Let  $\sigma$  be a primitive element of  $\mathbb{F}_q$  such that  $\sigma \neq 2$  and  $3\sigma \neq 2$ . The field  $\mathbb{F}_q$  has such a  $\sigma$  for q an odd prime power power  $\geq 5$ . Let  $\mathbb{F}_q = \{f_1, f_2, \ldots, f_q\}$ . Then

$$C^{(k)}{}_{ij} = \begin{cases} a_k f_i + b_k \sigma^j + c_k & \text{if } 1 \leq j < q \\ a_k f_i + c_k & \text{if } j = q \end{cases}$$

where  $a_0 = \cdots = a_{2r} = 1$ ,  $a_{2r+1} = \cdots = a_{4r+1} = -1$ ,  $a_{4r+2} = 1$ ,  $b_0 = \cdots = b_r = 1$ ,  $b_{r+1} = \cdots = b_{2r} = -1$ ,  $b_{2r+1} = 1/2 - 1/\sigma$ ,  $b_{2r+2} = \cdots = b_{3r+1} = 1$ ,  $b_{3r+2} = \cdots = b_{4r+1} = -1$ ,  $c_0 = -\sigma/2$  and  $c_1 = \cdots = c_{4r} = 0$ .

We now have a generating array for m = 4r + 3 and hence we have a row complete latin square of every odd composite order.  $\Box$ 

It is known that there are no  $n \times n$  row-complete latin squares for n = 3, 5 or 7, but the question for other odd primes remains open.

# 4 Index Of Notation

$\mathbb{Z}_n$	:	The integers	modulo $n$	(considered	as	the	additively	written	cyclic
		group of orde	$(\operatorname{r} n)$						

- $C_n$ : The (multiplicatively written) cyclic group of order n
- $D_{2n}$  : The dihedral group of order 2n
- $SD_{2n}$  : The semidihedral group of order 2n
- $Q_{4n}$ : The dicyclic group of order 4n; this is a generalised quaternion group if n is a power of 2
  - $A_n$ : The alternating group on *n* symbols
  - $S_n$ : The symmetric group on *n* symbols
  - $\mathbb{F}_q$ : The field with q elements (q must be a prime power)
- PSL(2,q) : The projective special linear group of  $2 \times 2$  matrices over  $\mathbb{F}_q$
- PGL(2,q): The projective general linear group of  $2 \times 2$  matrices over  $\mathbb{F}_q$
- $P\Gamma L(2,q)$  : The automorphism group of PSL(2,q)
  - G' : The derived subgroup of a group G
    - O(G) : The largest normal subgroup of odd order of a group G
    - $\Lambda(G)$  : The normal subgroup of order 2 of a binary group G
      - |k| : The smallest integer greater than k-1

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