A DYNAMIC SURVEY OF GRAPH LABELING

JOSEPH A. GALLIAN
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MINNESOTA, DULUTH
DULUTH, MINNESOTA 55812
JGALLIAN@D.UMN.EDU
AMS SUBJECT CLASSIFICATION: 05C78.
SUBMITTED: SEPTEMBER 1, 1996; ACCEPTED: NOVEMBER 14, 1997;
THIS EDITION MAY 19, 1998

Abstract. A vertex *labeling* of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. The two best known labeling methods are called graceful and harmonious labelings. A function f is called a *graceful* labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A function f is called *harmonious* if it is an injection from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x)+f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. Over the past three decades many variations of graceful and harmonious labelings have evolved and about 300 papers have been on the subject of graph labeling. In this article we survey what about known the various methods.

1. Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [Ro1] in 1967, or one given by Graham and Sloane [GS] in 1980. Rosa [Ro1] called a function f a *β-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [Go] subsequently called such labelings *graceful* and this is now the popular term. Rosa introduced β-valuations as well as a number of other valuations as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [Ri] that K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. Although an unpublished result of Erdős says that most graphs are not graceful (cf. [GS]), most graphs that have some sort of regularity of structure are graceful. Sheppard [Sh] has shown that there are exactly $e!$ gracefully labeled graphs with e edges. Balakrishnan and Sampathkumar [BS] have shown that every graph is a subgraph of a graceful graph. Rosa [Ro2] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges", (2) G "has too many edges", and (3) G "has the wrong parity". An infinite class of graphs that are not graceful for the second reason is given in [BG]. As an example of the third condition Rosa [Ro1] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Harmonious graphs naturally arose in the study by Graham and Sloane [GS] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number q of edges and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [LZ2] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \ldots, d_p$ then $\gcd(d_1, d_2, \ldots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph.

Over the past three decades approximately 300 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes have been done by several authors. In this article, we survey what is known about numerious graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two methods, include [Be], [Bl], [KRT2], [Ga2], and [Ga4]. The extension of graceful labelings to directed graphs arose in the characterization of finite neofields by Hsu and Keedwell [HK1], [HK2]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings and neofields is discussed in [BH1], [BH2]. The connection between graceful labelings and perfect systems of difference sets is given in [BKT]. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design and communication network addressing–see [BG1] and [BG2] for details. Terms and notation not defined below follow that used in [CL] and [Ga2].

2. Graceful and Harmonious Labelings

2.1. **Trees.** The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Among the trees known to be graceful are: caterpillars [Ro1] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [HKR], [Zha] and [JMWG]; trees with at most 27 vertices [AlM]; trees with diameter at most 4 [HKR]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [BS]; and olive trees [PR] (a rooted tree consisting of k branches, where the ith branch is a path of length i). Stanton and Zarnke [SZ] and Koh, Rogers and Tan [KRT3] gave methods for combining graceful trees to yield larger graceful trees. Burzio and Ferrarese [BF] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. Aldred and McKay [AlM] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [GS]. In 1979 Bermond [Be] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a caterpillar). Special cases of this conjecture have been done by Ng [N1] and by Wang, Jin, Lu and Zhang [WJLZ]. Whether or not lobsters are harmonious seems to attracted no attention thus far. Chen, Lü and Yeh [CLY] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Some bananas trees have been shown to be graceful by Bhat-Nayak and Deshmukh [BD2]. Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. More specialized results about trees are contained in [Be], [Bl], [KRT2], [LL], [C4] and [JLLLLZ].

2.2. **Cycle-Related Graphs.** Cycle-related graphs have been the major focus of attention. Rosa [Ro1] showed that the n-cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4) and Graham and Sloane [GS] proved that C_n is harmonious if and only if $n \equiv 1$ or 3 (mod 4). Wheels $W_n = C_n + K_1$ are both graceful and harmonious – [F1], [HK] and [GS]. Notice that a subgraph of a graceful (harmonious) graph need not be graceful (harmonious). The *helm* H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle. Helms have been shown to be graceful [AF] and harmonious [Gn], [LiuY3], [LiuY4] (see also [LZ2], [SY1], [LiuB2] and [RP1]). Koh, et al. [KRTY] define a *web* graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. They ask whether such graphs are graceful. This was proved by Kang, Liang, Gao and Yang [KLGY]. Yang has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendant points to the new cycle. In his notation, $W(2, n)$ is the web graph whereas $W(t, n)$ is the generalized web with t n-cycles. Yang has shown that $W(3, n)$ and $W(4, n)$ are graceful (see [KLGY]). Gnanajothi [Gn] has shown that webs with odd cycles are harmonious. Seoud and Youssef [SY1] define a *closed helm* as the graph obtained from a helm by joining each pendant vertex to form a cycle and a *flower* as the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A *gear graph* is obtained from a wheel by adding a vertex between every pair of adjacent vertices of the n-cycle. Ma and Feng [MF2] have proved all gears are graceful. Liu [LiuY3] has shown that if two or more vertices are added between every pair of vertices of the n-cycle of a wheel, the resulting graph is graceful. Liu [LiuY1] has also proved that the graph obtain from a gear graph by attaching one or more pendant points to each vertex between the cycle vertices is graceful.

Delorme, et al. [DMTKT] and Ma and Feng [MF1] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher and Wegner

[BSW2], who proved various special cases. Koh and Yap [KY] generalized this by defining a *cycle with a P_k-chord* to be a cycle with the path P_k joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when $k = 3$ and conjectured that all cycles with a P_k-chord are graceful. This was proved for $k \geq 4$ by Punnim and Pabhapote in 1987 [PP]. Chen [CheZ2], [CheZ3] obtained the same result except for three cases which were then handled by Gao [Gu2]. Xu [X2] proved that all cycles with a chord are harmonious except for C_6 in the case where the distance in C_6 between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords have also been investigated. For $3 \leq p \leq n - r$, let $C_n(p, r)$ denote the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the r chords $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$ have been added. Koh and others, [KRTY] and [KP], have handled the cases $r = 2, 3$ and $n - 3$ where n is the length of the cycle. Since then, Ma [Ma] has shown that $C_n(p, n - p)$ is graceful when $p \equiv 0, 3 \pmod 4$. Ma, Liu and Liu [MLL] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph $C_n(3, n - 3)$ any number k_i of paths of length 2 from the vertex v_1 to the vertex v_i for $i = 2, \ldots, n$, the resulting graph is graceful. Chen [CheZ1] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. This generalizes the result that a cycle plus a chord is graceful. Liu [LiuR] has shown that the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords v_1v_k and v_1v_{k+2} $(2 \leq k \leq n - 3)$ are adjoined is graceful.

Truszczyński [T] studied unicyclic graphs (i.e., graphs with a unique cycle) and proved several classes of such graphs are graceful. Among these are what he calls dragons. A *dragon* is formed by joining the end point of a path to a cycle (Koh, et al. [KRTY] call these *tadpoles*). This work led Truszczyński to conjecture that all unicyclic graphs except C_n, where $n \equiv 1$ or $2 \pmod 4$, are graceful. Guo [Gu] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 (mod 4). In his Master's thesis, Doma [Do] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Because of the immense diversity of unicyclic graphs, a proof of Truszczyński's conjecture seems out of reach in the near future.

Cycles that share a vertex have received some attention. Let $C_n^{(t)}$ denote the one-point union of t cycles of length n. Bermond and others ([BBG] and [BKT]) proved that $C_3^{(t)}$ (that is, the friendship graph or Dutch t-windmill) is graceful if and only if $t \equiv 0$ or $1 \pmod 4$ while Graham and Sloane [GS] proved $C_3^{(t)}$ is harmonious if and only if $t \not\equiv 2 \pmod 4$. Koh, et al. [KRLT] conjecture that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0$ or $3 \pmod 4$. Qian [Q] verifies this conjecture for the case that $t = 2$ and n is even. Bodendiek, Schumacher and Wegner [BSW1] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [S2] has proved that $C_4^{(t)}$ is graceful for all t..

Another class of cycle-related graphs is that of triangular cacti. A *triangular cactus* is a connected graph all of whose blocks are triangles. A *triangular snake* is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n - 1$). Rosa [Ro2] conjectured that all triangular cacti with $t \equiv 0$ or $1 \pmod 4$ blocks are graceful (the cases where $t \equiv 2$ or 3 (mod 4) fail to be graceful because of the parity condition.) Moulton [Mo] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular cacti) seems hopelessly difficult. Liu and Zhang [LZ2] have shown triangular snakes with an odd number of triangles are harmonious while triangular snakes with $n \equiv 2 \pmod 4$ triangles are not harmonious. Xu [X2] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to 2 (mod 4).

Defining K_4-snakes analogous to triangular snakes, Grace [Gr3] showed that these are harmonious. Rosa [Ro2] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases. Gnanajothi [Gn, pp 25-31] has shown that quadrilateral snakes are graceful.

Several people have studied cycles with pendant edges attached. Frucht [F1] proved that any cycle with a pendant edge attached at each vertex (i.e., a "crown") is graceful. Bu, Zhang and He [BZH] have shown that any cycle with a fixed number of pendant edges adjoined to each vertex is graceful. Grace [Gr2] show that an odd cycle with one or more pendant edges at each vertex is harmonious and conjectured that an even cycle with one pendant edge attached at each vertex is harmonious. This has been proved by Liu and Zhang [LZ1], Liu [LiuY3] and [LiuY4], Huang [Hua] and Bu [Bu2]. For any $n \geq 3$ and any t with $1 \leq t \leq n$, let C_n^{+t} denote the class of graphs formed by adding a single pendant edge to t vertices of a cycle of length n. Ropp [Rop] proved that for every n and t the class C_n^{+t} contains a graceful graph. Gallian and Ropp [Ga2] conjecture that for all n and t, all members of C_n^{+t} are graceful. This was proved by Qian [Q] and by Kang, Liang, Gao and Yang [KLGY]. Of course, this is just a special case of the aforementioned conjecture of Truszczyński that all unicyclic graphs except C_n for $n \equiv 1$ or 2 (mod 4) are graceful.

2.3. **Product Related Graphs.** Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids, $P_m \times P_n$, are graceful was proved by Acharya and Gill [AG] in 1978 although the much simpler labeling scheme given by Maheo [Mah] in 1980 for $P_m \times P_2$ readily extends to all grids. In 1980 Graham and Sloane [GS] proved ladders, $P_m \times P_2$, are harmonious when $m > 2$ and in 1992 Jungreis and Reid [JR] showed that the grids $P_m \times P_n$ are harmonious when $(m, n) \neq (2, 2)$. A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [Kat1] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful [Kat2]. Acharya [A2] has shown that certain subgraphs of grid graphs are graceful. Lee [L1] defines a *Mongolian tent* as a graph obtained from $P_m \times P_n$, n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex. A *Mongolian village* is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A *Young tableau* is a subgraph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and K. C. Ng [LNK] have proved that all Young tableaus are graceful. Lee [L1] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful.

Prisms are graphs of the form $C_m \times P_n$. These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher and Wegner [BSW2] proved that $C_m \times P_2$ is graceful when $m \equiv 0 \pmod 4$ According to the survey by Bermond [Be], T. Gangopadhyay and S. P. Rao Hebbare did the case that m is even about the same time. In a 1979 paper, Frucht [F1] stated without proof that he had done all m. A complete proof of all cases and some related results were given by Frucht and Gallian [FG] in 1988. In 1992 Jungreis and Reid [JR] proved that all $C_m \times P_n$ are graceful when m and n are even or when $m \equiv 0 \pmod 4$. Yang and Wang [YW1] have shown that the prisms $C_{4n+2} \times P_{4m+3}$ are graceful. Singh [Sin1] proved that $C_3 \times P_n$ is graceful for all n. In their 1980 paper Graham and Sloane [GS] proved that $C_m \times P_n$ is harmonious when n is odd and they used a computer to show $C_4 \times P_2$, the cube, is not harmonious. In 1992 Gallian, Prout and Winters [GPW] proved that $C_m \times P_2$ is harmonious when $m \neq 4$. In 1992, Jungreis and Reid [JR] showed that $C_4 \times P_n$ is harmonious when $n \geq 3$. Huang and Skiena [HuS] have shown that $C_m \times P_n$ is graceful for all n when m is even and for all n with $3 \leq n \leq 12$ when m is odd.

Torus grids are graphs of the form $C_m \times C_n$ $(m > 2, n > 2)$. Very little success has been achieved with these graphs. The graceful parity condition is violated for $C_m \times C_n$ when m and n are odd and the harmonious parity condition [GS, Theorem 11] is violated for $C_m \times C_n$ when $m \equiv 1, 2, 3 \pmod 4$ and n is odd. The only result I'm aware of was done in 1992 by Jungreis and Reid [JR] who showed that $C_m \times C_n$ is graceful when $m \equiv 0 \pmod 4$ and n is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout and Winters [GPW] proved that all prisms $C_m \times P_2$ with a single vertex deleted or single edge deleted are graceful and harmonious. The *Möbius ladder* M_n is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n. In 1989 Gallian [Ga1] showed that all Möbius ladders are graceful and all but M_3 are harmonious. Ropp [Rop] has examined two classes of prisms with pendant edges attached. He proved that all $C_m \times P_2$ with a single pendant edge at each vertex are graceful and all $C_m \times P_2$ with a single pendant edge at each vertex of one of the cycles are graceful.

Another class of cartesian products that has been studied is that of books and "stacked" books. The *book* B_m is the graph $S_m \times P_2$ where S_m is the star with $m+1$ vertices. In 1980 Maheo [Mah] proved that the books of the form B_{2m} are graceful and conjectured that the books B_{4m+1} were also graceful. (The books B_{4m+3} do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [D] in 1980. Maheo [Mah] also proved that $L_n \times P_2$ and $B_{2m} \times P_2$ are graceful. Both Grace [Gr1] and Reid (see [GJ]) have given harmonious labelings for B_{2m}. The books B_{4m+3} do not satisfy the harmonious parity condition [GS, Theorem 11]. Gallian and Jungreis [GJ] conjectured that the books B_{4m+1} are harmonious. Gnanajothi [Gn] has verified this conjecture by showing B_{4m+1} has an even stronger form of labeling – see Section 4.1. Liang [Li] also proved the conjecture. In their 1988 paper Gallian and Jungreis [GJ] defined a *stacked book* as a graph of the form $S_m \times P_n$. They proved that the stacked books of the form $S_{2m} \times P_n$ are graceful and posed the case $S_{2m+1} \times P_n$ as an open question. The *n-cube* $K_2 \times K_2 \times \cdots \times K_2$ (n copies) was shown to be graceful by Kotzig [K1]—see also [Mah]. In 1986 Reid [Re] found a harmonious labeling for $K_4 \times P_n$.

The symmetric product $G_1 \oplus G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2)| \ x_1x_2 \in E(G_1) \text{ or } y_1y_2 \in E(G_2) \text{ but not both}\}$. The composition $G_1[G_2]$ is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2)| \ x_1x_2 \in E(G_1) \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(G_2)\}$. Seoud and Youssef [SY4] have proved that $P_n \oplus \overline{K_2}$ is graceful when $n > 1$ and $P_n[P_2]$ is harmonious for all n. They also observe that the graphs $C_m \oplus C_n$ and $C_m[C_n]$ violates parity conditions for graceful and harmonious graphs when m and n are odd.

2.4. **Complete Graphs.** The questions of the gracefulness and harmoniousness of the complete graphs K_n have been answered. In each case the answer is positive if and only if $n \leq 4$ ([Go], [Si], [GS]). Both Rosa [Ro1] and Golomb [Go] proved that the complete bipartite graphs $K_{m,n}$ are graceful while Graham and Sloane [GS] showed they are harmonious if and only if m or $n = 1$. Aravamudhan and Murugan [AM] have shown that the complete tripartite graph $K_{1,m,n}$ is both graceful and harmonious while Gnanajothi [Gn,pp.25-31] has shown that $K_{1,1,m,n}$ is both graceful and harmonious and $K_{2,m,n}$ is graceful. Some of the same results have been obtained by Seoud and Youssef [SY3]. They also observed that when m, n and p are congruent to 2 (mod 4), $K_{m,n,p}$ violates the parity conditions for harmonious graphs.

Define the *windmill* graphs $K_n^{(m)} (n > 3)$ to be the family of graphs consisting of m copies of K_n with a vertex in common. A necessary condition for $K_n^{(m)}$ to be graceful is that $n \leq 5$ – see [KRTY]. Bermond [Be] has conjectured that $K_4^{(m)}$ is graceful for all $m \geq 4$. This is known to be true for $m \leq 22$ [HuS]. Bermond, Kotzig and Turgeon [BKT] proved that $K_n^{(m)}$ is not graceful when $n = 4$ and $m = 2$ or 3 and when $m = 2$ and $n = 5$. In 1982 Hsu [Hs] proved that $K_4^{(m)}$ is harmonious for all m. Graham and Sloane [GS] conjectured that $K_n^{(2)}$ is harmonious if and only if $n = 4$. They verified this conjecture for the cases that n is odd or $n = 6$. Liu [LiuB2] has shown that $K_n^{(2)}$ is not harmonious if $n = 2^a p_1^{a_1} \cdots p_s^{a_s}$ where $a, a_1, \ldots a_s$ are positive integers and $p_1, \ldots, p_s$ are distinct odd primes and there is a j for which $p_j \equiv 3$ (mod 4) and a_j is odd. He also shows that $K_n^{(3)}$ is not harmonious when $n \equiv 0$ (mod 4) and $3n = 4^e(8k + 7)$ or $n \equiv 5$ (mod 8). Koh et al. [KRLT] and Rajasingh and Pushpam [RP2] have shown that $K_{m,n}^{(t)}$, the one-point union of t copies of $K_{m,n}$, is graceful.

Koh et al. [KRTY] introduced the notation $B(n, r, m)$ for the graph consisting of m copies of K_n with a K_r in common. (Guo [Gu2] has used the notation $B(n, r, m)$ to denote three independent paths of lengths n, r and m joining two vertices.) Bermond [Be] raised the question: "For which m, n and r is $B(n, r, m)$ graceful?" Of course, the case $r = 1$ is the same as $K_n^{(m)}$. For $r > 1$, $B(n, r, m)$ is graceful in the following cases: $n = 3, r = 2, m \geq 1$ [KRL]; $n = 4, r = 2, m \geq 1$ [D]; $n = 4, r = 3, m \geq 1$ (see [Be]), [KRL]. Seoud and Youssef [SY3] have proved $B(3, 2, m)$ and $B(4, 3, m)$ are harmonious. Liu [LiuB1] has shown that if there is a prime p such that $p \equiv 3$ (mod 4) and p divides both n and $n - 2$ and the highest power of p that divides n and $n - 2$ is odd, then $B(n, 2, 2)$ is not graceful. More generally, Bermond and Farhi [BF] have considered the class of graphs consisting of m copies of K_n having exactly a copies of K_r in common. They proved such graphs are not graceful for n sufficiently large compared to r.

2.5. **Disconnected Graphs.** There have been many papers dealing with graphs that are not connected. In 1975 Kotzig [K2] considered graphs that are the disjoint union of r cycles of length s, denoted by rC_s. When $rs \equiv 1$ or 2 (mod4), these graphs violate the parity condition and so are not graceful. Kotzig proved that when $r = 3$ and $s = 4k > 4$, then rC_s has a stronger form of graceful labeling called α-labeling (see Section 3.1) whereas when $r \geq 2$ and $s = 3$ or 5, rC_s is not graceful. In 1984 Kotzig [K4] once again investigated the gracefulness of rC_s as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of rC_s. He [He] has shown that graphs of the form $2C_{2m}$ and graphs obtained by connecting two copies of C_{2m} with an edge are graceful. Cahit [C7] has shown that rC_s is harmonious when r and s are odd and Seoud, Abdel el Maqsoud and Sheen [SAS1] noted that when r or s is even, rC_s is not harmonious. Seoud, Abdel el Maqsoud and Sheen [SAS1] proved that $C_n \cup C_{n+1}$ is harmonious if and only if $n \geq 4$. They conjecture that $C_3 \cup C_{2n}$ is harmonious when $k \geq 3$. In 1978 Kotzig and Turgeon [KT] proved that mK_n (i.e., the union of m disjoint copies of K_n) is graceful if and only if $m = 1$ and $n \leq 4$. Liu and Zhang [LZ2] have shown that mK_n is not harmonious for n odd and $m \equiv 2$ (mod 4) and is harmonious for $n = 3$ and m odd. They conjecture that mK_3 is not harmonious when $m \equiv 0$ (mod 4). Bu and Cao [BC1] give some sufficient conditions for the gracefulness of graphs of the form $K_{m,n} \cup G$ and that prove $K_{m,n} \cup P_t$ and the disjoint union of complete bipartite graphs are graceful under some conditions.

A *Skolem sequence* of order n is a sequence $s_1, s_2, \cdots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \cdots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. A Skolem sequence of order n exists if and only if $n \equiv 0$ or 1 (mod 4). Abrham [Ab2] has proved that any graceful 2-regular graph of order $n \equiv 0$ (mod 4) in which all the component cycles are even or of order $n \equiv 3$ (mod 4), with exactly one component an odd cycle, can be used to construct a Skolem sequence of order $n+1$. Also, he showed that certain special Skolem sequences of order n can be used to generate graceful labelings on certain 2-regular graphs.

In 1985 Frucht and Salinas [FS] conjectured that $C_s \cup P_n$ is graceful if and only if $s + n \geq 7$ and they proved the conjecture for the case that $s = 4$. Frucht [F3] did the case the $s = 3$ and the case that $s = 2n + 1$. Bhat-Nayak and Deshmukh [BD5] also did the case $s = 3$ and they have done the cases of the form $C_{2x+1} \cup P_{x-2\theta}$ where $1 \leq \theta \leq \lfloor (x-2)/2 \rfloor$ [BD1]. Choudum and Kishore [CK2] have done the cases where $s \geq 5$ and $n \geq (s+5)/2$ and Kishore [Kis] did the case $s = 5$. Gao and Liang [GL] have done the following cases: $s > 4, n = 2$ (see also [Gao]); $s = 4k, n = k+2, n = k+3, n = 2k+2$; $s = 4k+1, n = 2k, n = 3k-1, n = 4k-1$; $s = 4k+2, n = 3k, n = 3k+1, n = 4k+1$; $s = 4k+3, n = 2k+1, n = 3k, n = 4k$. Seoud, Abd el Maqsoud and Sheehan [SAS2] did the case that $s = 2k$ $(k \geq 3)$ and $n \geq k+1$. Seoud and Youssef [SY2] have shown that $K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q}$ $(m, n, p, q \geq 2), K_{m,n} \cup K_{p,q} \cup K_{r,s}$ $(m, n, p, q, r, s \geq 2,\ (p, q) \neq (2, 2))$, and $pK_{m,n}$ $(m, n \geq 2, (m, n) \neq (2, 2))$ are graceful. They also prove that $C_4 \cup K_{1,n}$ $(n \neq 2)$ is not graceful whereas Choudum and Kishore [CK4], [Kis] have proved that $C_s \cup K_{1,n}$ is graceful for every $s \geq 7$ and $n \geq 1$. Lee, Quach and Wang [LQW] established the gracefulness of $P_s \cup K_{1,n}$. Seoud and Wilson [SW] have shown that $C_3 \cup K_4, C_3 \cup C_3 \cup K_4$ and certain graphs of the form $C_3 \cup P_n$ and $C_3 \cup C_3 \cup P_n$ are not graceful. Abrham and Kotzig [AK4] proved that $C_p \cup C_q$ is graceful if and only if $p + q \equiv 0$ or 3 (mod 4). Zhou [Zh] proved that $K_m \cup K_n$ is graceful if and only if $\{m, n\} = \{4, 2\}$ or $\{5, 2\}$. Shee [S1] has shown that graphs of the form $P_2 \cup C_{2k+1}$ $(k > 1)$, $P_3 \cup C_{2k+1}$, $P_n \cup C_3$ and $S_n \cup C_{2k+1}$ all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [BD3] have shown that $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$ are graceful. Yang and Wang [YW2] proved that $S_m \cup S_n$ is graceful if and only if m or n is odd and that $S_m \cup S_n \cup S_k$ is graceful if and only if at least one of m, n or k is odd $(m > 1, n > 1, k > 1)$.

2.6. **Joins of Graphs.** A few classes of graphs that are the join of graphs have been shown to be graceful and harmonious. Among these are fans $P_n + K_1$ [GS] and double fans $P_n + \overline{K_2}$ [GS]. More generally, Reid [Re] proved that $P_n + \overline{K_t}$ is harmonious and Grace showed [Gr2] that if T is any graceful tree, then $T + \overline{K_t}$ is also graceful. Fu and Wu [FW] proved that if T is a graceful tree, then $T + S_k$ is graceful. Of course, wheels are of the form $C_n + K_1$ and are graceful and harmonious. Hebbare [H] showed that $S_m + K_1$ is graceful for all m. Shee [S] has proved $K_{m,n} + K_1$ is harmonious and observed that various cases of $K_{m,n} + K_t$ violate the parity condition in [GS]. Liu and Zhang [LZ2] have proved that $K_2 + K_2 + \cdots + K_2$ is harmonious. Yuan and Zhu [YZ] proved that $K_{m,n} + K_2$ is graceful. Gnanajothi [Ga, pp.80-127] obtained the following: $C_n + \overline{K_2}$ is harmonious when n is odd and not harmonious when $n \equiv 2, 4, 6$ (mod 8); $S_n + \overline{K_t}$ is harmonious; $P_n + \overline{K_t}$ is harmonious. Yuan and Zhu [YZ] proved that $K_{m,n} + K_2$ is harmonious. Balakrishnan and Kumar [BK2] have proved that the join of $\overline{K}_n$ and two disjoint copies of K_2 is harmonious if and only if n is even. Bu [Bu2] obtained partial results for the gracefulness of $K_n + \overline{K}_m$.

Seoud and Youssef [SY4] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and $C_n + \overline{K_p}$ is harmonious for every p when n is odd. They also prove that if any edge is added to $K_{m,n}$ the resulting graph is harmonious if m or n is at least 2. Deng [De] has shown certain cases of $C_n + \overline{K_p}$ are harmonious.

2.7. **Miscellaneous Results.** It is easy to see that P_n^2 is harmonious [Gr2] while a proof that P_n^2 is graceful has been given by Kang, Liang, Gao and Yang [KLGY]. (P_n^k, the kth power of P_n, is the graph obtained from P_n by adding edges that join all vertices u and v with $d(u, v) = k$.) This latter result proved a conjecture of Grace [Gr2]. Seoud, Abd el Maqsoud and Sheeham [SAS1] proved that P_n^3 is harmonious and conjecture that P_n^k is not harmonious when $k > 3$. However, Youssef [Yo] has observed that P_8^4 is harmonious. Gnanajothi [Gn, p.50] has shown that the graph that consists of n copies of C_6 that have exactly P_4 in common is graceful if and only if n is even. For a fixed n, let v_{i1}, v_{i2}, v_{i3} and v_{i4} $(1 \leq i \leq n)$ be consecutive vertices of a 4-cycle. Gnanajothi [Gn, p. 35] also proves: the graph obtained by joining each v_{i1} to $v_{i+1,3}$ is graceful for all n; the generalized Petersen graph $P(n, k)$ is harmonious in all cases (see also [LSS]). ($P(n, k)$, where $n \geq 5$ and $1 \leq k \leq n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i a_{i+1} \mid i = 0, 1, \cdots, n-1\} \cup \{a_i b_i \mid i = 0, 1, \ldots, n-1\} \cup \{b_{i+k} \mid i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo n.) The gracefulness of the generalized Petersen graphs appears to be an open problem.

Yuan and Zhu [YZ] define a generalization of P_n^2 as follows: $P_n(2k)$ is the graph obtained from the path P_n by adding edges that join all vertices x and y with $d(x,y) = 2k$. They proved that $P_n(2k)$ is harmonious when $1 \leq k \leq \frac{n-1}{2}$ and that $P_n(2k)$ has a stronger form of harmonious labeling (see Section 4.1) when $2k-1 \leq n \leq 4k-1$. Cahit [C7] defines a *p-star* as the graph obtained by joining p disjoint paths of length k to single vertex. He proves all such graphs are harmonious when p is odd and when $k = 2$ and p is even.

The total graph $T(P_n)$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n. Balakrishnan, Selvam and Yegnanarayanan [BSY1] have proved that $T(P_n)$ is harmonious.

For any graph G with vertices $v_1, \ldots, v_n$ and a vector $\mathbf{m} = (m_1, \ldots, m_n)$ of positive integers the corresponding *replicated graph*, $R_{\mathbf{m}}(G)$, of G is defined as follows. For each v_i form a stable set S_i consisting of m_i new vertices $i = 1, 2, \ldots, n$ (recall a *stable* set S consists of a set of vertices such that there is not an edge v_iv_j for all pairs v_i, v_j in S); two stable sets $S_i, S_j, i \neq j$, form a complete bipartite graph if each v_iv_j is an edge in G and otherwise there are no edges between S_i and S_j. Ramírez-Alfonsín [Ra] has proved that $R_{\mathbf{m}}(P_n)$ is graceful for all $\mathbf{m}$ and all $n > 1$ and that $R_{(2,2,\ldots,2)}(C_{2n})$ is graceful.

For any permutation f on $1, \ldots, n$, the *f-permutation* graph on a graph G, $P(G, f)$, consists of two disjoint copies of G, G_1 and G_2, each of which has vertices labeled $v_1, v_2, \ldots, v_n$ with n edges obtained by joining each v_i in G_1 to $v_{f(i)}$ in G_2. In 1983 Lee (see [LWK]) conjectured that for all $n > 1$ and all permutations on $1, 2, \ldots, n$, the permutation graph $P(P_n, f)$ is graceful. Lee, Wang and Kiang [LWK] proved that $P(P_{2k}, f)$ is graceful when $f = (12)(34)\cdots(k, k+1)\cdots(2k-1, 2k)$. They conjectured that if G is a graceful nonbipartite graph with n vertices then for any permutation f on $1, 2, \ldots, n$, the permutation graph $P(G, f)$ is graceful. Some families of graceful permutation graphs are given in [LLWK].

Gnanajothi [Gn, p.51] calls a graph G *bigraceful* if both G and its line graph are graceful. She shows the following are bigraceful: P_m; $P_m \times P_n$; C_n if and only if $n \equiv 0, 3 \pmod 4$; S_n; K_n if and only if $n \leq 3$; B_n if and only if $n \equiv 3 \pmod 4$. She also shows that $K_{m,n}$ is not bigraceful when $n \equiv 3 \pmod 4$. (Gangopadhyay and Hebbare [GH] used the term bigraceful to mean a bipartite graceful graph.)

Several well-known isolated graphs have been examined. Graceful labelings of the Petersen graph, the cube, the icosahedron and the dodecahedron can be found in [Go] and [Gar]. On the other hand, Graham and Sloane [GS] showed that all of these except the cube are harmonious. Winters [Wi] verified that the Grőtzsch graph (see [BM], p. 118), the Heawood graph (see [BM], p. 236) and the Herschel graph (see [BM], p. 53) are graceful.

2.8. **Summary.** The results and conjectures discussed above are summarized in the tables following. The letter G after a class of graphs indicates that the graphs in that class are known to be graceful; a question mark indicates that the gracefulness of the graphs in the class is an open problem; we put a "G" next to a question mark if the graphs have been conjectured to be graceful. The analogous notation with the letter H is used to indicate the status of the graphs with regard to being harmonious. The tables impart at a glimpse what has been done and what needs to be done to close out a particular class of graphs. Of course, there is an unlimited number of graphs one could consider. One wishes for some general results that would handle several broad classes at once but the experience of many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture alone has withstood the efforts of scores of people over the past three decades. Analogous sweeping conjectures are probably true but appear hopelessly difficult to prove.

Table 1. **Summary of Graceful Results**

Graph	*Graceful*
Trees	G if ≤ 27 vertices [AlM] G if diameter at most 4 [HKR] G if symmetrical [BS] G if at most 4 end-vertices [HKR] ?G Ringel-Kotzig
Cycles C_n	G iff $n \equiv 0, 3 \pmod 4$ [Ro1]
Wheels W_n	G [F1], [HK]
Helms (see §2.2)	G [AF]
Webs (see §2.2)	G [KLGY]
Gears (see §2.2)	G [MF2]
Cycles with P_k-chord (see §2.2)	G [DMTKT], [MF1], [KY], [PP]
C_n with k consecutive chords (see §2.2)	G if $k = 2, 3, n-3$ [KP], [KRTY]
Unicyclic graphs	?G iff G $\neq C_n, n \equiv 1, 2 \pmod 4$ [T]
$C_n^{(t)}$ (see §2.2)	$n = 3$ G iff $t \equiv 0, 1 \pmod 4$ [BBG], [BKT] ?G if $nt \equiv 0, 3 \pmod 4$ [KRLT] G if $n = 6, t$ even [KRLT] G if $n = 4, t > 1$ [S2] G if $t = 2$, n even [Q] G if $t = 2$, $n \equiv 0, 3 \pmod 4$ [BSW1] not G if $t = 2, n \equiv 1, 2 \pmod 4$ (parity conditions)
Triangular snakes (see §2.2)	G iff number of blocks $\equiv$ $0, 1 \pmod 4$ [Mo]
K_4-snakes (see §2.2)	?
Quadilateral snakes (see §2.2)	G [Gn], [Q]

TABLE 1. **continued**

Graph	*Graceful*
Crowns $C_n \odot K_1$	G [F1]
Grids $P_m \times P_n$	G [AG]
Prisms $C_m \times P_n$	G if $n = 2$ [FG] G if m even [HuS], G if m odd, $3 \leq n \leq 12$ [HuS] G if $m = 3$ [Sin1]
Torus grids $C_m \times C_n$	G if $m \equiv 0 \pmod 4$, n even [JR] not G if m, n odd (parity condition)
Vertex-deleted $C_m \times P_n$	G if $n = 2$ [GPW]
Edge-deleted $C_m \times P_n$	G if $n = 2$ [GP]
Möbius ladders M_n (see §2.3)	G [Ga1]
Stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, G iff $m \neq 3 \pmod 4$ [Mah], [D], [GJ] G if m even [GJ]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	G [K1]
$K_4 \times P_n$	?
K_n	G iff $n \leq 4$ [Go], [Si]
$K_{m,n}$	G [Ro1], [Go]
$K_{1,m,n}$	G [AM]
$K_{1,1,m,n}$	G [Gn]
Windmills $K_n^{(m)} (n > 3)$ (see §2.4)	G if $n = 4, m \leq 22$ [HuS] ?G if $n = 4, m \geq 4$ [Be] G if $n = 4, 4 \leq m \leq 22$ [HuS] not G if $n = 4, m = 2, 3$ [Be] not G if $(m, n) = (2, 5)$ [BKT] not G if $n > 5$ [KRTY]

TABLE 1. **continued**

Graph	*Graceful*
$B(n,r,m)$ $r>1$ (see §2.4)	G if $(n,r)=(3,2),(4,3)$ [KRL], (4,2) [D]
mK_n (see §2.5)	G iff $m=1, n\leq 4$ [KT]
$C_s \cup P_n$	? G iff $s+n\geq 7$ [FS] G if $s=3$ [F3], $s=4$ [FS], $s=5$ [Kis] G if $s>4, n=2$ [GL] G if $s=2n+1$ [F3] G if $s=2k, n\geq k+1$ [SAS2]
$C_p \cup C_q$	? G iff $p+q\equiv 0,3 \pmod 4$ [FS] G if $s=2n+1$ [F3], $s\geq 5$ and $n\geq (s+5)/2$ [CK2]
Fans $F_n = P_n + K_1$	G [GS]
Double fans $P_n + \overline{K_2}$	G [GS]
t-point suspension $P_n+\overline{K_t}$ of P_n	G [Gr2]
$S_m + K_1$	G [H]
Double cone $C_n + \overline{K_2}$	?
P_n^2 (see §2.7)	G [LK]
Petersen $P(n,k)$ (see §2.7)	?
Caterpillars	G [Ro1]
Lobsters	?G [Be]

Table 2. **Summary of Harmonious Results**

Graph	*Harmonious*
Trees	H if ≤ 26 vertices [AlM] ?H [GS]
Cycles C_n	H iff $n \equiv 1,3 \pmod 4$ [GS]
Wheels W_n	H [GS]
Helms (see §2.2)	H [Gn], [LiuY]
Webs (see §2.2)	H if cycle is odd
Gears (see §2.2)	?
Cycles with P_k-chord (see §2.2)	?
C_n with k consecutive chords (see §2.2)	?
Unicyclic graphs	?
$C_n^{(t)}$ (see §2.2)	$n = 3$ H iff $t \not\equiv 2 \pmod 4$ [GS] H if $n = 4,\ t > 1$ [S2]
Triangular snakes (see §2.2)	H if number of blocks is odd [LZ2] not H if number of blocks $\equiv 2$ (mod 4)[LZ2]
K_4-snakes (see §2.2)	H [Gr3]
Quadrilateral snakes (see §2.2)	?
Crowns $C_n \odot K_1$	H [Gr2], [LZ1]
Grids $P_m \times P_n$	H iff $(m,n) \neq (2,2)$ [JR]
Prisms $C_m \times P_n$	H if $n = 2, m \neq 4$ [GPW] H if n odd [GS] H if $m = 4$ and $n \geq 3$ [JR]

Table 2. **continued**

Graph	*Harmonious*
Torus grids $C_m \times C_n$,	H if $m = 4$, $n > 1$ [JR] not H if $m \equiv 1, 2, 3 \pmod 4$ and n odd [JR]
Vertex-deleted $C_m \times P_n$	H if $n = 2$ [GPW]
Edge-deleted $C_m \times P_n$	H if $n = 2$ [GPW]
Möbius ladders M_n (see §2.3)	H iff $n \neq 3$ [Ga]
Stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, H if m even [Gr1], [Re] not H $m \equiv 3 \pmod 4$, $n = 2$, (parity condition) H if $m \equiv 1 \pmod 4), n = 2$ [Gn]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	not H if $n = 2, 3$ [GS]
$K_4 \times P_n$	H [Re]
K_n	H iff $n \leq 4$ [GS]
$K_{m,n}$	H iff m or $n = 1$ [GS]
$K_{1,m,n}$	H [AM]
$K_{1,1,m,n}$	H [Gn]
Windmills $K_n^{(m)}$ $(n > 3)$ (see §2.4)	H if $n = 4$ [Hs] $m = 2$, ?H iff $n = 4$ [GS] not H if $m = 2, n$ odd or 6 [GS] not H for some cases $m = 3$ [LiuB2]
$B(n, r, m)$ $r > 1$ (see §2.4)	$(n, r) = (3, 2), (4, 3)$ [SY3]
mK_n (see §2.5)	H $n = 3$, m odd [LZ2] not H for n odd, $m \equiv 2 \pmod 4$ [LZ2]
$C_s \cup P_n$	?
Fans $F_n = P_n + K_1$	H [GS]

Table 2. **continued**

Graph	*Harmonious*
Double fans $P_n + \overline{K_2}$	H [GS]
t-point suspension $P_n + \overline{K_t}$ of P_n	H [Re]
$S_m + K_1$	H [Gn], [CHR]
Double cone $C_n + \overline{K_2}$	H if n odd [Re], [Gn] not H if $n \equiv 2, 4, 6 \pmod 8$ [Gn]
P_n^2 (see §2.7)	H [Gr2], [LZ1]
Petersen $P(n, k)$ (see §2.7)	H [Gn], [LSS]
Caterpillars	H [GS]
Lobsters	?

3. Variations of Graceful Labelings

3.1. α**-labelings.** In [Ro1] Rosa defined an *α-labeling* to be a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. (Other names for such labelings are *balanced* and *interlaced*.) It follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an α-labeling is necessarily bipartite and therefore can not contain a cycle of odd length.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an α-labeling is a typical approach. (See [CHR], [Gr2], [CLY] and [JR].) Moreover, Jungreis and Reid [JR] showed how sequential labelings of graphs (see Section 4.1) can often be obtained by modifying α-labelings of the graphs.

Graphs with α-labelings have proved useful in the development of the theory of graph decompositions. Rosa [Ro1], for instance, has shown that if G is a graph with q edges and has an α-labeling, then for every natural number p, the complete graph K_{2qp+1} can be decomposed into copies of G in such a way that the automorphism group of the decomposition itself contains Z_n, the cyclic group of order n. Although a proof of Ringel's conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have α-labelings are easy to construct (see [Ro1]).

As to which graphs have α-labelings, Rosa [Ro1] observed that the n-cycle has an α-labeling if and only if $n \equiv 0 \bmod 4$ while P_n always has an α-labeling. Other familiar graphs that have α-labelings include caterpillars [Ro1], the n-cube [Ro1],

B_{4n+1} (i.e., books with $4n+1$ pages) [GJ], $C_{2m} \cup C_{2m}$, and $C_{4m} \cup C_{4m} \cup C_{4m}$ for all $m > 1$ [K2], $P_n \times Q_n$ [Mah], $K_{1,2k} \times Q_n$ [Mah], $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$ [LV], $C_{4m} \cup C_{4n+2} \cup C_{4r+2}, C_{4m} \cup C_{4n} \cup C_{4r}$ when $m+n \leq r$ [AK4], $C_{4m} \cup C_{4n} \cup C_{4r} \cup C_{4s}$ when $m \geq n+r+s$ [ACE], $C_{4m} \cup C_{4n} \cup C_{4r+2} \cup C_{4s+2}$ when $m \geq n+r+s+1$ [ACE]. Abrham and Kotzig [AK4] have proved that $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m+n \equiv 0 \pmod 4$. Kotzig [K2] has also shown that $C_{2m+1} \cup C_{2m+1}$ and $C_4 \cup C_4 \cup C_4$ do not have α-labelings. He asked if $n = 3$ is the only integer such that the disjoint union of n copies of C_4 does not have an α-labeling. This was confirmed by Abrham and Kotzig in [AK3]. Eshghi [Es] proved that every 2-regular bipartite graph with 3 components has an α-labeling if and only if the number of edges is a multiple of four except for one special case.

Jungreis and Reid [JR] investigated the existence of α-labelings for graphs of the form $P_m \times P_n, C_m \times P_n$, and $C_m \times C_n$ (see also [Ga4]). Of course, the cases involving C_m with m odd are not bipartite, so there is no α-labeling. The only unresolved cases among these three families are $C_{4m+2} \times P_{2n+1}$ and $C_{4m+2} \times C_{4n+2}$. All other cases result in α-labelings. Balakrishman [Ba] uses the notation $Q_n(G)$ to denote the graph $P_2 \times P_2 \times \cdots \times P_2 \times G$ where P_2 occurs $n-1$ times. Snevily [Sn1] has shown that the graphs $Q_n(C_{4m})$ and the cycles C_{4m} and C_{4m+3} with the path P_n adjoined at each vertex have α-labelings. He also has shown [Sn2] that compositions of the form $G[\overline{K_n}]$ have an α-labeling whenever G does. Balakrishman and Kumar [BK1] have shown that all graphs of the form $Q_n(G)$ where G is $K_{3,3}, K_{4,4}$, or P_m have an α-labeling. Balakrishman [Ba] poses the following two problems. For which graphs G does $Q_n(G)$ have an α-labeling? For which graphs G does $Q_n(G)$ have a graceful labeling? Rosa [Ro1] has shown that $K_{m,n}$ has an α-labeling. Qian [Q] has proved that quadrilateral snakes have α-labelings. Fu and Wu [FW] showed that if T is a tree that has an α-labeling with partite sets V_1 and V_2 then the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to every vertex of V_1 has an α-labeling. Similarly, they prove that the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to the vertices of V_1 and new vertices $u_1, u_2, \ldots, u_t$ to every vertex of V_2 has an α-labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of V_1 or the vertices of both V_1 and V_2, the resulting graphs have α-labelings. Fu and Wu [FW] further show that if T is a tree with an α-labeling and the sizes of the two partite sets of T differ at by at most 1, then $T \times P_m$ has an α-labeling.

Given two bipartite graphs $G_1 = (H_1, L_1, E)$ and $G_2 = (H_2, L_2, E)$, Snevily [Sn1] defines their *weak tensor product* $G_1 \overline{\bigotimes} G_2$ as the bipartite graph with vertex set $(H_1 \times H_2, L_1 \times L_2)$ and with edge $(h_1, h_2)(l_1, l_2)$ if $h_1 l_1 \in E(G_1)$ and $h_2 l_2 \in E(G_2)$. He proves that if G_1 and G_2 have α-labelings then so does $G_1 \overline{\bigotimes} G_2$. This result considerably enlarges the class of graphs known to have α-labelings.

The sequential join of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_n$ by adding edges joining each vertex of G_i with each vertex of G_{i+1} for $1 \leq i \leq n-1$. Lee and Wang [LW1] have shown that for all $n \geq 2$ and any positive integers $a_1, a_2, \ldots, a_n$ the sequential join of the graphs $\overline{K}_{a_1}, \overline{K}_{a_2}, \ldots, \overline{K}_{a_n}$ has an α-labeling.

In [Ga2] Gallian and Ropp conjectured that every graph obtained by adding a single pendant edge to one or more vertices of a cycle is graceful. Qian [Q] has proved this conjecture and in the case that the cycle is even he showns the graphs have an

α-labeling. He further proves that for n even any graph obtained from an n-cycle by adding one or more pendant edges at some vertices has an α-labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

For any tree $T(V,E)$ whose vertices are properly 2-colored Rosa and Širáň [RS] define a *bipartite labeling* of T as a bijection $f: V \to \{0,1,2,\ldots,|E|\}$ for which there is a k such that whenever $f(w) \leq k \leq f(v)$, then u and v have different colors. They define the *α-size* of a tree T as the maximum number of distinct values of the induced edge labels $|f(u)-f(v)|, uv \in E$, taken over all bipartite labelings f of T. They prove that the α-size of any tree with n edges is at least $5(n+1)/7$ and that there exist trees whose α-size is at most$(5n+9)/6$. They conjectured that minimum of the α-sizes over all trees with n edges is asymptotically $5n/6$. This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [BoS]. Heinrich and Hell [HeH] defined the *gracesize* of a graph G with n vertices as the maximum, over all bijections $f\colon V(G) \to \{1,2,\ldots,n\}$, of the number of distinct values $|f(u)-f(v)|$ over all edges uv of G. So, from Rosa and Širáň's result, the gracesize of any tree with n edges is at least $5(n+1)/7$.

In [GPW] Gallian weakened the condition for an α-labeling somewhat by defining a *weakly α-labeling* as a graceful labeling for which there is an integer k so that for each edge xy either $f(x) \leq k \leq f(y)$ or $f(y) \leq k \leq f(x)$. This condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph. Namely, that the vertex with the label k must be on every odd cycle. Gallian, Prout and Winters [GPW] showed that the prisms $C_n \times P_2$ with a vertex deleted have α-labelings. The same paper reveals that $C_n \times P_2$ with an edge deleted from a cycle has an α-labeling when n is even and a weakly α-labeling when $n > 3$.

A special case of α-labeling called strongly graceful was introduced by Maheo [Mah] in 1980. A graceful labeling f of a graph G is called *strongly graceful* if G is bipartite with two partite sets A and B of the same order s, the number of edges is $2t+s$, there is an integer k with $t-s \leq k \leq t+s-1$ such that if $a \in A, f(a) \leq k$, and if $b \in B, f(b) > k$, and there is an involution π which is an automorphism of G such that: π exchanges A and B and the s edges $a\pi(a)$ where $a \in A$ have as labels the integers between $t+1$ and $t+s$. Maheo's main result is that if G is strongly graceful then so is $G \times Q_n$. In particular, she proved that $(P_n \times Q_n) \times K_2$, B_{2n} and $B_{2n} \times Q_n$ have strongly graceful labelings. El-Zanati and Vanden Eynden [EV] call a strongly graceful labeling a *strong α-valuation*. El-Zanati and Vanden Eynden proved that $K_{m,2} \times Q_n$ has a strong α-valuation and that $K_{m,2} \times P_n$ has an α-labeling for all n. They also prove that if G is a connected bipartite graph with partite sets of odd order and such that in each partite set each vertex has the same degree, then $G \times K_2$ does not have a strong α-valuation. As a corollary they have that $K_{m,n} \times K_2$ does not have a strong α-valuation when m and n are odd.

Another special case of α-labelings for trees was introduced by Ringel, Llado and Serra [RLS] in 1995 in an approach to proving their conjecture $K_{n,n}$ is edge-decomposable into n copies of any given tree with n edges. If T is a tree with n edges and partite sets A and B, they define a labeling f from the set of vertices to $\{1,2,\ldots,n\}$ to be a *bigraceful* labeling of T if f restricted to A is injective, f restricted to B is injective and the edge labels given by $f(y)-f(x)$ where yx is an edge with y in B and x in A is the set $\{0,1,2,\ldots,n-1\}$. (Notice that this terminology conflicts with that given in Section 2.7). Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars $S_{k,m}$ with k spokes of paths of length m, and complete d-ary trees for d odd. They also prove that if T is a tree then there is a vertex v and a nonnegative integer m such that the addition of m leaves to v results in a bigraceful tree. They conjecture that all trees are bigraceful.

3.2. **k-graceful Labelings.** A natural generalization of graceful graphs is the notion of k-graceful graphs introduced independently by Slater [Sl2] in 1982 and by Maheo and Thuillier [MT] in 1982. A graph G with q edges is *k-graceful* if there is labeling f from the vertices of G to $\{0, 1, 2, \ldots, q + k - 1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k+1, \ldots, q+k-1\}$. Obviously, 1-graceful is graceful and it is readily shown that any graph that has an α-labeling is k-graceful for all k. Graphs that are k-graceful for all k are sometimes called *arbitrarily graceful*. Ng [N2] has shown that there are graphs that are k-graceful for all k but do not have an α-labeling.

Results of Maheo and Thuillier [MT] together with those of Slater [Sl2] show that: C_n is k-graceful if and only if either $n \equiv 0$ or 1 (mod 4) with k even and $k \leq (n-1)/2$, or $n \equiv 3$ (mod 4) with k odd and $k \leq (n^2 - 1)/2$. Maheo and Thuillier [MT] also proved that the wheel W_{2k+1} is k-graceful and conjectured that W_{2k} is k-graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun and Xu [LSX]. Kang [Ka] proved that $P_m \times C_{4n}$ is k-graceful for all k. Lee and Wang [LW2] showed that all pyramids, lotuses and diamonds are k-graceful and Liang and Liu [LL] have shown that $K_{m,n}$ is k-graceful. Bu, Gao and Zhang [BGZ] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are k-graceful for all k. Acharya (see [A2]) has shown that a k-graceful Eulerian graph with q edges must satisfy one of the following conditions: $q \equiv 0$ (mod 4), $q \equiv 1$ (mod 4) if k is even, or $q \equiv 3$ (mod 4) if k is odd. Bu, Zhang and He [BZH] have shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is k-graceful.

Several authors have investigated the k-gracefulness of various classes of subgraphs of grid graphs. Acharya [A1] proved that all 2-dimensional polyminoes that are convex and Eulerian are k-graceful for all k; Lee [L1] showed that Mongolian tents and Mongolian villages are k-graceful for all k (see Section 2.3 for definitions); Lee and K. C. Ng [LNK] proved that all Young tableaus are k-graceful for all k (see Section 2.3 for definitions). Lee and H. K. Ng [LNH] subsequently generalized these results on Young tableaus to a wider class of planar graphs.

Let $c, m, p_1, p_2, \ldots, p_m$ be positive integers. For $i = 1, 2, \ldots, m$, let S_i be a set of $p_i + 1$ integers and let D_i be the set of positive differences of the pairs of elements of S_i. If all these differences are difference then the system $D_1, D_2, \ldots, D_m$ is called a *perfect system of difference sets starting at* c if the union of all the sets D_i is $c, c+1, \ldots, c - 1 + \sum_{i=1}^{m} \binom{p_i + 1}{2}$. There is a relationship between k-graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with c describes a c-graceful labeling of a graph which is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in [Ab1].

Acharya and Hegde [AH2] generalized k-graceful to *(k, d)-graceful labelings* by permitting the vertex labels to belong to $\{0, 1, 2, \ldots, k+(q-1)d\}$ and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be $\{k, k+d, k+2d, \ldots, k+(q-1)d\}$. They also introduce an analog of α-labelings in the obvious way. Notice that a (1,1)-graceful labeling is a graceful labeling and a $(k,1)$-graceful labeling is a k-graceful labeling. Bu and Zhang [BZ] have shown that $K_{m,n}$ is (k,d)-graceful for all k and d; for $n > 2, K_n$ is (k,d)-graceful if and only if $k = d$ and $n \leq 4$; if $m_i, n_i \geq 2$ and $\max\{m_i, n_i\} \geq 3$, then $K_{m_1,n_1} \cup K_{m_2,n_2} \cup \cdots \cup K_{m_r,n_r}$ is (k,d)-graceful for all k, d, and r; if G has an α-labeling, then G is (k,d)-graceful for all k and d; a k-graceful graph is a (kd,d)-graceful graph; a (kd,d)-graceful connected graph is k-graceful; and a (k,d)-graceful graph with q edges that is not bipartite has $k \leq (q-2)d$.

Slater [Sl5] has extended the definition of k-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices and the countably infinite Dutch windmill is k-graceful for all k.

More specialized results on k-graceful labelings can be found in [L1], [LNK], [LNH], [Sl2], [BF], [BH], [BGZ] and [CJ].

3.3. **Skolem-Graceful.** A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [LSh]) calls a graph G with p vertices and q edges *Skolem-graceful* if there is an injection from the set of vertices of G to $\{1, 2, \ldots, p\}$ such that the edge labels induced by $|f(x) - f(y)|$ for each edge xy are $1, 2, \ldots, q$. A necessary condition for a graph to be Skolem-graceful is that $p \geq q + 1$. Lee and Wui [LWu] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. They also prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has an even size. Denham, Leu and Liu [DLL] proved that the disjoint union of any four stars is Skolem-graceful. Choudum and Kishore [CK1] proved that all 5-stars are Skolem graceful. In [CK3] Choudum and Kishore show that the disjoint union of k copies of the star $K_{1,2p}$ is Skolem graceful if $k \leq 4p+1$ and the disjoint union of any number of copies of $K_{1,2}$ is Skolem graceful. Lee, Quach and Wang [LQW] showed that the disjoint union of the path P_n and the star of size m is Skolem-graceful if and only if $n = 2$ and m is even or $n \geq 3$ and $m \geq 1$. It follows from the work of Skolem [Sk] that nP_2, the disjoint union of n copies of P_2, is Skolem-graceful if and only if $n \equiv 0$ or 1 (mod 4). Harary and Hsu [HH] studied Skolem-graceful graphs under the name *node-graceful*. Frucht [F3] has shown that $P_m \cup P_n$ is Skolem-graceful when $m + n \geq 5$. Bhat-Nayak and Deshmukh [BD4] have shown that $P_{n_1} \cup P_{n_2} \cup P_{n_3}$ is Skolem-graceful when $n_1 < n_2 \leq n_3,\ n_2 = t(n_1+2)+1$ and n_1 is even and when $n_1 < n_2 \leq n_3,\ n_2 = t(n_1+3)+1$ and n_1 is odd. They also prove that the graphs of the form $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i}$ where $i \geq 4$ are Skolem-graceful under certain conditions. Kishore [Kis] has shown that a necessary condition for the disjoint union of graphs of the form $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_k}$ to be Skolem graceful is that some n_i is even or $k \equiv 0$ or 1 (mod 4). He conjectures that each one of these conditions is sufficient.

3.4. **Odd Graceful Labelings.** Gnanajothi [Gn, p. 182] defined a graph G with q edges to be *odd graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, \ldots, 2q-1\}$. She proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful. She also proved the following graphs are odd graceful: P_n; C_n if and only if n is even; $K_{m,n}$; combs $P_n \bigodot K_1$ (graphs obtained by joining a single pendant edge to each vertex of P_n); books; crowns $C_n \bigodot K_1$ (graphs obtained by joining a single pendant edge to each vertex of C_n); if and only if n is even; the disjoint union of m copies of C_4; the one-point union of m copies of C_4; $C_n \times K_2$ if and only if n is even; caterpillars; rooted trees of height 2; the graphs obtained from P_n $(n \geq 3)$ by adding exactly two leaves at each vertex of degree 2 of P_n; the graphs consisting of vertices $a_0, a_1, \ldots a_n, b_0, b_1, \cdots b_n$ with edges $a_i a_{i+1}, b_i b_{i+1}$ for $i = 0, \ldots, n-1$ and $a_i b_i$ for $i = 1, \ldots, n-1$; the graphs obtained from a star by adjoining to each end vertex the path P_3 or by adjoining to each end vertex the path P_4. She conjectures that all trees are odd graceful and proves the conjecture for all trees with order up to 10. Eldergill [E] generalized Gnanajothi's result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd graceful. He also proved that the one-point union of any number of copies of C_6 is odd graceful. Kathiresan [Kat3] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd graceful.

3.5. **Graceful-like Labelings.** As a means of attacking graph decomposition problems, Rosa [Ro2] invented another analogue of graceful labelings by permitting the vertices of a graph with q edges to assume labels from the set $\{0, 1, \ldots, q+1\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, \ldots, q-1, q\}$ or $\{1, 2, \ldots, q-1, q+1\}$. He calls these *nearly graceful labelings*, or *$\hat{\rho}$-labelings*. Frucht [F3] has shown that the following graphs have nearly graceful labelings with edge labels from $\{1, 2, \ldots, q-1, q+1\}$: $P_m \cup P_n$; $S_m \cup S_n$; $S_m \cup P_n$; $G \cup K_2$ where G is graceful; and $C_3 \cup K_2 \cup S_m$ where m is even or $m \equiv 3 \pmod{14}$.

Rosa [Ro2] has conjectured that triangular snakes with $t \equiv 0$ or $1 \pmod 4$ blocks are graceful and those with $t \equiv 2$ or $3 \pmod 4$ blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [Mo] proved Rosa's conjecture while introducing the slightly stronger concept of *almost graceful* by permitting the vertex labels to come from $\{0, 1, 2, \ldots, q-1, q+1\}$ while the edge labels are $\{1, 2, \ldots, q-1, q\}$, or $\{1, 2, \ldots, q-1, q+1\}$.

Yet another kind of labeling introduced by Rosa in his 1967 paper [Ro2] is a ρ-valuation. A *ρ-valuation* of a graph is an injection from the vertices of the graph with q edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, \ldots, a_q$, then $a_i = i$ or $a_i = 2q+1-i$. Rosa [Ro2] proved that a cyclic decomposition of the edge set of the complete graph K_{2q+1} into subsets inducing edge subgraphs isomorphic to a given graph G with q edges exists if and only if G has a ρ-valuation. (A decomposition of K_n into copies of G is called *cyclic* if the automorphism group of the decomposition itself contains Z_n.) Dufour [Du] and Eldergill [E] have some results on the decomposition of the complete graph using labeling methods. Balakrishnan and Sampathkumar [BS] showed that for each positive integer n the graph $\overline{K_n} + 2K_2$ admits a ρ-valuation. Balakrishnan [Ba] asks if it is true that $\overline{K_n} + mK_2$ admits a ρ-valuation for all n and m. Balakrishnan and Sampathkumar ask for which $m \geq 3$ is the graph $\overline{K_n} + mK_2$ graceful for all n. Bhat-Nayak and Gokhale [BG] have proved that $\overline{K_n} + 2K_2$ is not graceful.

For graphs with the property $p = q + 1$ (i.e., graphs that are trees or the disjoint union of a tree and unicyclic graphs), Frucht [F3] has introduced a stronger version of almost graceful graphs by permitting as vertex labels $\{0, 1, \ldots, q-1, q+1\}$ and as edge labels $\{1, 2, \ldots, q\}$. He calls such a labeling *pseudograceful*. Frucht proved that P_n $(n \geq 3)$, combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), $C_3 \cup P_n$ $(n \neq 3)$, and $C_4 \cup P_n$ $(n \neq 1)$ are pseudograceful while $K_{1,n}$ $(n \geq 3)$ is not. Kishore [Kis] proved that $C_s \cup P_n$ is pseudograceful when $s \geq 5$ and $n \geq (s+7)/2$ and that $C_s \cup S_n$ is pseudograceful when $s = 3, s = 4$, and $s \geq 7$. Seoud and Youssef [SY2] extended the definition of pseudograceful to all graphs with $p \leq q + 1$. They proved that $K_{m,n}$ $(n \geq 2)$ and $P_n + \overline{K_m}$ are pseudograceful. They also proved that if G is pseudograceful, then $G \cup K_{m,n}$ $(m \geq 2, n \geq 2)$ is graceful.

McTavish [Mc] has investigated labelings where the vertex and edge labels are from $\{0, \ldots, q, q+1\}$. She calls these *$\tilde{\rho}$-labelings*. Graphs that have $\tilde{\rho}$-labelings include cycles and the disjoint union of P_n or S_n with any graceful graph.

Frucht [F3] has made an observation about graceful labelings that yields nearly graceful analogs of α-labelings and weakly α-labelings in a natural way. Suppose $G(V, E)$ is a graceful graph with the vertex labeling f. For each edge xy in E, let $[f(x), f(y)]$ (where $f(x) \leq f(y)$) denote the interval of real numbers r with $f(x) \leq r \leq f(y)$. Then the intersection $\cap[f(x), f(y)]$ over all edges $xy \in E$ is a unit interval, a single point, or empty. Indeed, if f is an α-labeling of G then the intersection is a unit interval; if f is a weakly α-labeling, but not an α-labeling, then the intersection is a point; and, if f is graceful but not a weakly α-labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes.

3.6. **Cordial Labelings.** Cahit [C2] has introduced a variation of both graceful and harmonious labelings. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Call f a *cordial labeling* of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [C3] proved the following: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph $C_3^{(t)}$ is cordial if and only if $t \not\equiv 2 \pmod 4$; all fans are cordial; the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$; maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang and Kwong [KCK] determine all m and n for which mK_n is cordial. A *k-angular cactus* is a connected graph all of whose blocks are cycles with k vertices. In [C3] Cahit proved that a k-angular cactus with t cycles is cordial if and only if $kt \not\equiv 2 \pmod 4$. This was improved by Kirchherr [Ki1] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [Ki2] also gave a characterization of cordial graphs in terms of their adjacency matrices and conjectures that determining the set of cordial graphs is NP-complete. Ho, Lee and Shee [HLS2] proved: $P_n \times C_{4m}$ is cordial for all m and all odd n; the composition $G[H]$ is cordial if G is cordial and H is cordial and has odd order and even size; for $n \geq 4$ the composition $C_n[K_2]$ is cordial if and only if $n \not\equiv 2 \pmod 4$; the cartesian product of two cordial graphs of even sizes is cordial. The same authors [HLS1] showed that a unicyclic graph is cordial unless it is C_{4k+2} and that the generalized Petersen graph $P(n, k)$ is cordial if and only if $n \not\equiv 2 \pmod 4$.

Lee, Lee and Chang [LLC] prove the following graphs are cordial: the complete n-partite graph if and only if at most three of its partite sets have odd cardinality; the Cartesian product of an arbitrary number of paths; the Cartesian product of two cycles if and only if at least one of them is even; and the Cartesian product of an arbitrary number of cycles if at least one of them has length a multiple of 4 or at least two of them are even.

Shee and Ho [SH1] have investigated the cordiality of the one-point union of n copies of various graphs. For $C_m^{(n)}$, the one-point union of n copies of C_m, they proved:

(i) If $m \equiv 0 \pmod 4$, then $C_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 1$ or $3 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(iii) If $m \equiv 2 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if n is even.

For $K_m^{(n)}$, the one-point union of n copies of K_n, Shee and Ho [SH1] prove:

(i) If $m \equiv 0 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 3 \pmod 4$;

(ii) If $m \equiv 4 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 1 \pmod 4$;

(iii) If $m \equiv 5 \pmod 8$, then $K_m^{(n)}$ is not cordial for all odd n;

(iv) $K_4^{(n)}$ is cordial if and only if $n \not\equiv 1 \pmod 4$;

(v) $K_5^{(n)}$ is cordial if and only if n is even;

(vi) $K_6^{(n)}$ is cordial if and only if $n > 2$;

(vii) $K_7^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(viii) $K_n^{(2)}$ is cordial if and only if n has the form p^2 or $p^2 + 1$.

Benson and Lee [BL] have investigated the regular windmill graphs $K_m^{(n)}$ and determined precisely which ones are cordial for $m < 14$.

For $W_m^{(n)}$, the one-point union of n copies of the wheel W_m with the common vertex being the center, Shee and Ho [SH1] show:

(i) If $m \equiv 0$ or $2 \pmod 4$, then $W_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 3 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 1 \pmod 4$;

(iii) If $m \equiv 1 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 3 \pmod 4$.

For all n and all $m > 1$ Shee and Ho [SH1] prove $F_m^{(n)}$, the one-point union of n copies of the fan $F_m = P_m + K_1$ with the common point of the fans being the center, is cordial. The *flag* Fl_m is obtained by joining one vertex of C_m to an extra vertex called the *root*. Shee and Ho [SH1] show all $Fl_m^{(n)}$, the one-point union of n copies of Fl_m with the common point being the root, are cordial.

For graphs $G_1, , G_2, \ldots, G_n$ $(n \geq 2)$ which are all copies of a fixed graph, G Shee and Ho [SH2] call a graph obtained by adding an edge from G_i to G_{i+1} for $i = 1, \ldots, n-1$ a *path-union* of G (the resulting graph may depend on how the edges are chosen). Among their results they show the following graphs are cordial: path-unions of cycles; path-unions of n copies of K_m when $m = 4, 6$ or 7; path-unions of three or more copies of K_5; path-unions of two copies of K_m if and only if $m-2, m$ or $m+2$ is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees and various compositions.

Lee and Liu [LeL] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let H be a graph with an even number of edges and a cordial labeling such that the vertices of H can be divided into t parts $H_1, H_2, \ldots, H_t$ each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let G be any graph and $G_1, G_2, \ldots, G_t$ be any t subsets of the vertices of G. Let (G, H) be the graph which is the disjoint union of G and H augmented by edges joining every vertex in G_i to every vertex in H_i for all i. Then G is cordial if and only if (G, H) is. From this it follows that: all generalized fans $F_{m,n} = \overline{K_m} + P_n$ are cordial; the generalized bundle $B_{m,n}$ is cordial if and only if m is even or $n \not\equiv 2 \pmod 4$ ($B_{m,n}$ consists of $2n$ vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ with an edge from v_i to u_i and $2m$ vertices $x_1, x_2, \ldots x_m, y_1, y_2, \ldots, y_m$ with x_i joined to v_i and y_i joined to u_i); if m is odd a generalized wheel $W_{m,n} = \overline{K_m} + C_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$. If m is even, $W_{m,n}$ is cordial if and only if $n \not\equiv 2 \pmod 4$; a complete k-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Cahit [C8] calls a graph *H-cordial* if it is possible to label the edges with the numbers from the set $\{+1, -1\}$ in such a way that, for some K, at each vertex v the algebraic sum of the labels on the edges incident with v is either $+K$ or $-K$ and the inequalities $|v_f(+K) - v_f(-K)| \le 1$ and $|e_f(+1) - e_f(-1)| \le 1$ are also satisfied, where $v_f(i)$ and $e_f(j)$ are, respectively, the number of vertices labeled with i and the number of edges labeled with j. He proves: K_n is H-cordial if and only if $n \equiv 0 \pmod 4$; $K_{n,n}$ is H-cordial if and only if $n > 2$ and n is even; $K_{n,m}, n \ne m$, is H-cordial if and only if $n \equiv 0 \pmod 4$, m is even and $n > 2, m > 2$; W_n is H-cordial if and only if $n \equiv 1 \pmod 4$. By allowing 0 as the possible induced vertex label of an H-cordial labeling he studies semi-H-cordiality of trees. He also generalizes H-cordial labelings.

Hovey [Ho] has introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group A (under addition) and graph $G(V, E)$ he defines G to be *A-cordial* if there is a labeling of V with elements of A so that for all a and b in A when the edge ab is labeled with $f(a) + f(b)$, the number of vertices labeled with a and the number of vertices labeled b differ by at most one and the number of edges labeled with a and the number labeled with b differ by at most one. In the case where $A = Z_k$, the labeling is called *k-cordial.* With this definition we have: $G(V, E)$ is harmonious if and only if G is $|E|$-cordial; G is cordial if and only if G is 2-cordial.

Hovey has obtained the following: caterpillars are k-cordial for all k; all trees are k-cordial for $k = 3, 4$ and 5; odd cycles with pendant edges attached are k-cordial for all k; cycles are k-cordial for all odd k; for k even, C_{2mk+j} is k-cordial when $0 \le j \le \frac{k}{2} + 2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial; K_m is 3-cordial; and, for k even, K_{mk} is k-cordial if and only if $m = 1$.

Hovey advances the following conjectures: all trees are k-cordial for all k; all connected graphs are 3-cordial; and C_{2mk+j} is k-cordial if and only if $j \ne k$, where k and j are even and $0 \le j < 2k$. The last conjecture was verified by Tao, Tong and Wang [TTW]. This result combined with those of Hovey show that for all positive integers k the n-cycle is k-cordial with the exception that k is even and $n = 2mk + k$. Tao, Tong and Wang also proved that the crown with $2mk + j$ vertices is k-cordial unless $j = k$ is even and for $4 \le n \le k$ and the wheel W_n is k-cordial unless $k \equiv 5 \pmod 8$ and $n = (k + 1)/2$.

3.7. **k-equitable Labelings.** In 1990 Cahit [C4] proposed the idea of distributing the vertex and edge labels among $\{0, 1, \ldots, k - 1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V, E)$ and any positive integer k, assign vertex labels from $\{0, 1, \ldots, k-1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. Cahit has called a graph with such an assignment of labels *k-equitable*. Note that $G(V, E)$ is graceful if and only if it is $|E| + 1$-equitable and $G(V, E)$ is cordial if and only if it is 2-equitable. Cahit [C3] has shown the following: C_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; a triangular snake with n blocks is 3-equitable if and only if n is even; the friendship graph $C_3^{(n)}$ is 3-equitable if and only if n is even; W_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; an Eulerian graph with $q \equiv 3 \pmod 6$ edges is not 3-equitable; and all caterpillars are 3-equitable [C3]. He conjectures [C3] that a triangular cactus with n blocks is 3-equitable if and only if n is even. In [C4] Cahit proves that every tree with fewer than five end vertices has a 3-equitable labeling. He conjectures that all trees are k-equitable [C5].

Szaniszló [Sz] has proved the following: P_n is k-equitable for all k; K_n is 2-equitable if and only if $n = 1, 2$ or 3; K_n is not k-equitable for $3 \le k < n$; S_n is k-equitable for all k; $K_{2,n}$ is k-equitable if and only if $n \equiv k-1 \pmod k$, or $n \equiv 0, 1, 2, \ldots, \lfloor k/2 \rfloor - 1 \pmod k$, or $n = \lfloor k/2 \rfloor$ and k is odd. She also proves that C_n is k-equitable if and only if k meets all of the following conditions: $n \neq k$; if $k \equiv 2, 3 \pmod 4$, then $n \neq k-1$; if $k \equiv 2, 3 \pmod 4$ then $n \not\equiv k \pmod{2k}$.

Bloom has used the term k-equitable to describe another kind of labeling (see [W1], [W2] and [BR]). He calls a graph *k-equitable* if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label induced occurs exactly k times. A graph of order n is called *minimally k-equitable* if the vertex labels are 1, 2,. . ., n and it is k-equitable. Both Bloom and Wojciechowski [W1], [W2] proved that C_n is minimally k-equitable if and only if k is a proper divisor of n. Barrientos, Dejter and Hevia [BDH] have shown that forests of even size are 2-equitable (in the sense of Bloom). They also prove that for $k = 3$ or $k = 4$ a forest F of size kw is k-equitable if and only if the degree of F is at most $2w$ and that if 3 divides the size of the double star $S_{m,n}$ $(1 \le m \le n)$, then $S_{m,n}$ is 3-equitable if and only if $q/3 \le m \le \lfloor (q-1)/2 \rfloor$. ($S_{m,n}$ is K_2 with m pendant edges attached at one end and n pendant edges attached at the other end.) They discuss the k-equitability of forests for $k \ge 5$ and characterize all caterpillars of diameter 2 that are k-equitable for all possible values of k.

3.8. **Hamming-graceful Labelings.** Mollard, Payan and Shixin [MPS] introduced a generalization of graceful graphs called Hamming-graceful. A graph $G = (V, E)$ is called *Hamming-graceful* if there exists an injective labeling g from V to the set of binary $|E|$-tuples such that $\{d(g(v), g(u)) | uv \in E\} = \{1, 2, \ldots, |E|\}$. Shixin and Yu [ShY] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful; C_n is Hamming-graceful if and only if $n \equiv 0$ or $3 \pmod 4$; if K_n is Hamming-graceful, then n has the form k^2 or $k^2 + 2$; K_n is Hamming-graceful for $n = 2, 3, 4, 6, 9, 11, 16$, and 18. They conjecture that K_n is Hamming-graceful for n of the form k^2 and $k^2 + 2$ for $k \ge 5$.

4. Variations of Harmonious Labelings

4.1. **Sequential and Strongly c-harmonious Labelings.** Chang, Hsu and Rogers [CHR] and Grace [Gr1, Gr2] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling f of a graph G with q vertices to be *strongly c-harmonious* if the vertex labels are from $\{0, 1, \ldots, q-1\}$ and the edge labels induced by $f(x) + f(y)$ for each edge xy are $c, \ldots, c+q-1$. Grace called such a labeling *sequential*. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices while Grace allows the vertex labels to range from 0 to q with no vertex label used twice. By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [Gr2] proved that caterpillars, caterpillars with a pendant edge, cycles C_{2n+1} with zero or more pendant edges, trees with α-labelings, wheels W_{2n+1}, and P_n^2 (the square of P_n) are sequential. Liu and Zhang [LZ1] finished off the crowns $C_{2n} \odot K_1$. (The case $C_{2n+1} \odot K_1$ was a special case of Grace's results. Liu [LiuY3] proved crowns are harmonious.) Bu [Bu2] also proved that crowns are sequential as are all even cycles with m pendant edges attached at each vertex. Singh has proved the following: $C_n \odot K_2$ is sequential for all odd $n > 1$ [Sin2]; $C_n \odot P_3$ is sequential for all odd n [Sin4]; $K_2 \odot C_n$ (each vertex of the cycle is joined by edges to the end points of a copy of K_2) is sequential for all odd n [Sin4]; helms H_n are sequential when n is even [Sin4]; $K_{1,n} + K_2, K_{1,n} + \overline{K}_2$, and ladders are sequential [Sin5]. Both Grace [Gr1] and Reid (see [GJ]) have found sequential labelings for the books B_{2n}. Jungreis and Reid [JR] have shown the following graphs are sequential: $P_m \times P_n \ \ (m,n) \neq (2,2)$; $C_{4m} \times P_n \ \ (m,n) \neq (1,2)$; $C_{4m+2} \times P_{2n}$; $C_{2m+1} \times P_n$; and, $C_4 \times C_{2n} \ \ (n > 1)$. The graphs $C_{4m+2} \times C_{2n+1}$ and $C_{2m+1} \times C_{2n+1}$ fail to satisfy a necessary parity condition given by Graham and Sloane [GS]. The remaining cases of $C_m \times P_n$ and $C_m \times C_n$ are open. Gallian, Prout and Winters [GPW] proved that all graphs $C_n \times P_2$ with a vertex or edge deleted are sequential.

Gnanajothi [Gn, pp.68-78] has shown the following graphs are sequential: $K_{1,m,n}$; mC_n, the disjoint union of m copies of C_n, if and only if m and n are odd; books with triangular pages or pentagonal pages; and books of the form B_{4n+1}, thereby answering a question and proving a conjecture of Gallian and Jungreis [GJ]. Sun [Su] has also proved that B_n is sequential if and only if $n \not\equiv 3 \pmod 4$.

Yuan and Zhu [YZ] have shown that mC_n is sequential when m and n are odd. Although Graham and Sloane [GS] proved that the Möbius M_3 is not harmonious, Gallian [Ga1] established that all other Möbius ladders are sequential (see Section 2.3 for the definition). Chung, Hsu and Rogers [CHR] have shown that $K_{m,n} + K_1$, which includes $S_m + K_1$, is sequential. Hegde [Heg2] proved that every graph can be embedded as an induced subgraph of a sequential graph.

Among the strongly 1-harmonious (also called *strongly harmonious*) are: fans F_n with $n \geq 2$ [CHR]; wheels W_n with $n \not\equiv 2 \pmod 3$ [CHR]; $K_{m,n} + K_1$ [CHR]; French windmills $K_4^{(t)}$ [Hs], [KZ]; the friendship graphs $C_3^{(n)}$ if and only if $n \equiv 0$ or 1 (mod 4) [Hs], [KZ]; and $C_{4k}^{(t)}$ [SW]. Rajasingh and Pushpam [RP1] have shown that helms are strongly harmonious.

Acharya and Hegde [AH2] have generalized sequential labelings as follows. Let G be a graph with q edges and let k and d be positive integers. A labeling f of G is said to be (k, d)-*arithmetic* if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x)+f(y)$ for each edge xy are $k, k+d, k+2d, \ldots, k+(q-1)d$. They obtained a number of necessary conditions for various kinds of graphs to have a (k, d)-arithmetic labeling. The case where $k = 1$ and $d = 1$ was called *additively graceful* by Hegde [Heg1]. Hegde showed that K_n is additively graceful if and only if $n = 2, 3$ or 4; every additively graceful graph except K_2 or $K_{1,2}$ contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendant edge attached. Jinnah and Singh [JS] noted that P_n^2 is additively graceful. Bu and Shi [BS] proved that K_n is not (k, d)-arithmetic for $n \geq 5$ and that $K_{m,n}$ is (k, d)-arithmetic when k is not of the form id for $1 \leq i \leq n-1$. Yu [Yu] proved that a necessary condition for C_{4t+1} to be (k, d)-arithmetic is that $k = 2dt+r$ for some $r \geq 0$ and a necessary condition for C_{4t+3} to be (k, d)-arithmetic is that $k = (2t + 1)d + 2r$ for some $r \geq 0$. These conditions were conjectured by Acharya and Hegde [AH2].

A graph is called *arithmetic* if it is (k, d)-arithmetic for some k and d. Singh has proved that $P_m \times C_n$ is arithmetic for odd n [Sin5] and that both ladders and subdivisions of ladders are arithmetic [Sin3]. Jinnah and Singh [JS] ask if the disjoint union of two arithmetic graphs is arithmetic.

Acharya and Hegde [AH2] introduced a stronger form of sequential labeling by calling a (p, q) graph (V, E) *strongly k-indexable* if there is an injective function from V to $\{0, 1, 2, \ldots, p-1\}$ such that the set of edge labels induced by adding the vertex labels is $\{k, k + 1, k + 2, \ldots, k + q - 1\}$. Strongly 1-indexable graphs are simply called *strongly indexable*. Notice that for trees and unicyclic graphs the notions of sequential labelings and strongly k-indexable labelings coincide. Acharaya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are K_2, K_3 and $K_2 \times K_3$ and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [AH2] call a graph with p vertices *indexable* if there is an injective labeling of the vertices with labels from $\{0, 1, 2, \ldots, p-1\}$ such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [ArG] who also proved that all trees are indexable. Bu and Shi [BS2] also proved that all treees are indexable and that all uncyclic graphs with the cycle C_3 are indexable.

4.2. **Elegant Labelings.** An *elegant labeling* f of a graph G with q edges is an injective function from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $f(x) + f(y) \pmod{q + 1}$ the resulting edge labels are distinct and nonzero. This notion was introduced by Chang, Hsu and Rogers in 1981 [CHR]. Note that in contrast to the definition of a harmonious labeling, it is not necessary to make an exception for trees. While the cycle C_n is harmonious if and only if n is odd, Chang et al. [CHR] proved that C_n is elegant when $n \equiv 0$ or 3 (mod 4) and not elegant when $n \equiv 1$ (mod 4). Chang et al. further showed that all fans are elegant and the paths P_n are elegant for $n \not\equiv 0$ (mod 4). Cahit [C1] then showed that P_4 is the only path that is not elegant. Balakrishnan, Selvam and Yegnanarayanan [BSY2] have proved numerous graphs are elegant. Among them are: $K_{m,n}$ and the

mth-subdivision graph of $K_{1,2n}$. They prove that the bistar $B_{n,n}$ (K_2 with n pendant edges at each endpoint) is elegant if and only if n is even and that every simple graph is a subgraph of an elegant graph. They also prove several families of graphs are not elegant.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let G be a graph with q edges and H a finite Abelian group (under addition) of order q. Define G to be *H-harmonious* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x)+f(y)$ the resulting edge labels are distinct. When G is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley and Jungreis [BGHJ] have shown that if H is a finite Abelian group of order $n > 1$ then C_n is H-harmonious if and only if H has a non-cyclic or trivial Sylow 2-subgroup and H is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$. Thus, for example, C_{12} is not Z_{12}-harmonious but is $(Z_2 \times Z_2 \times Z_3)$-harmonious. Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let G be a graph with q edges and H a finite Abelian group (under addition) with $q+1$ elements. We say G is *H-elegant* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x)+f(y)$ the resulting set of edge labels is the non-identity elements of H. Beals et al. [BGHJ] proved that if H is a finite Abelian group of order n with $n \neq 1$ and $n \neq 3$, then C_{n-1} is H-elegant using only the non-identity elements of H as vertex labels if and only if H has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu and Rogers [CHR] by showing that C_n is elegant when $n \equiv 2 \pmod 4$. Mollard and Payan [MP] also proved that C_n is elegant when $n \equiv 2 \pmod 4$ and gave another proof that P_n is elegant when $n \neq 4$.

4.3. **Felicitous Labelings.** Another generalization of harmonious labelings are felicitous labelings. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, \ldots, q\}$ is called *felicitous* if the edge labels induced by $f(x)+f(y) \pmod q$ for each edge xy are distinct. This definition first appeared in a paper by Lee, Schmeichel and Shee in [LSS] and is attributed to E. Choo. Balakrishnan and Kumar [BK2] proved the conjecture of Lee, Schmeichel and Shee [LSS] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are: C_n except when $n \equiv 2 \pmod 4$ [LSS]; $K_{m,n}$ when $m, n > 1$ [LSS]; $P_2 \cup C_{2n+1}$ [LSS]; $P_3 \cup C_{2n+1}$ [LSS]; $S_m \cup C_{2n+1}$ [LSS]; the friendship graph $C_3^{(n)}$ for n odd [LSS]; $P_n \cup C_3$ [SL]; and the one-point union of an odd cycle and a caterpillar [SL]. Shee [S1] conjectured that $P_m \cup C_n$ is felicitous when $n > 2$ and $m > 3$. Lee, Schmeichel and Shee [SS] ask for which m and n is the one-point union of n copies of C_m felicitous. They showed that the case where mn is twice an odd integer is not felicitous. In contrast to the situation for felicitous labelings, we remark that C_{4k} and $K_{m,n}$ where $m, n > 1$ are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel and Shee [LSS] conjecture that the n-cube is felicitous. This is known to be true for $n = 2, 3$ and 4 ([LSS] and [BK2]).

Balakrishnan, Selvam and Yegnanarayanan [BSY1] obtained numerous results on felicitous labelings. The *wreath product*, $G * H$, of graphs G and H has vertex set $V(G) \times V(H)$ and (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and

$h_1h_2 \in E(H)$. They define $H_{n,n}$ as the graph with vertex set $\{u_1,\ldots,u_n;v_1,\ldots,v_n\}$ and edge set $\{u_iv_j|\ 1 \le i \le j \le n\}$. They let $\langle K_{1,n} : m\rangle$ denote the graph obtained by taking m disjoint copies of $K_{1,n}$, and joining a new vertex to the roots of the m copies of $K_{1,n}$. They prove the following are felicitous: $H_{n,n}; P_n * \overline{K_2}; \langle K_{1,m} : m\rangle; \langle K_{1,2} : m\rangle$ when $m \not\equiv 0 \pmod 3$ or $m \equiv 3 \pmod 6$ or $m \equiv 6 \pmod{12}$; $\langle K_{1,2n} : m\rangle$ for all m and $n \ge 2$; $\langle K_{1,2t+1} : 2n+1\rangle$ when $n \ge t$; P_n^k when $k = n-1$ and $n \not\equiv 2 \pmod 4$ or $k = 2t$ and $n \ge 3$ and $k < n-1$; the join of a star and $\overline{K_n}$; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [Y] conjectures that the graphs obtained from an even cycle by attaching n new vertices to each vertex of the cycle is felicitous.

Chang, Hsu and Rogers [CHR] have given a sequential counterpart to felecitous labelings. They call a graph *strongly c-elegant* if the vertex labels are from $\{0,1,\ldots,q\}$ and the edge labels induced by addition are $\{c, c+1,\cdots, c+q-1\}$. (A strongly 1-elegant labeling has also been called a *consecutive* labeling.) Notice that every strongly c-elegant graph is felicitous and that strongly c-elegant is the same as $(c,1)$-arithmetic in the case where the vertex labels are from $\{0,1,\ldots,q\}$. Results on strongly c-elegant graphs are meager. Chang et al. [CHR] have shown: K_n is strongly 1-elegant if and only if $n \equiv 2,3,4$; C_n is strongly 1-elegant if and only if $n \equiv 3$; and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [S2] has proved that $K_{m,n}$ is strongly c-elegant for a particular value of c and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

5. Total Labelings

In contrast to the labeling methods discussed thus far in which there is a function from the vertices of a graph to some set of labels, there are numerous methods that involve a function from the vertices and edges to some set of labels.

5.1. **k-sequential Labelings.** In 1981 Bange, Barkauskas and Slater [BBS1] defined a *k-sequential* labeling f of a graph $G(V,E)$ as one for which f is a bijection from $V \cup E$ to $\{k, k+1,\ldots,|V \cup E|+k-1\}$ such that for each edge xy in E, one has $f(xy) = |f(x) - f(y)|$. This generalized the notion of *simply sequential* where $k = 1$ introduced by Slater. Bange, Barkauskas and Slater showed that cycles are 1-sequential and if G is 1-sequential then the join of G and a point is graceful. In [Sl1], Slater proved: K_n is 1-sequential if and only if $n \le 3$; for $n \ge 2$, K_n is not k-sequential for all $k \ge 2$; and $K_{1,n}$ is k-sequential if and only if k divides n. Acharya and Hegde [AH1] proved: P_n is $\frac{n}{2}$-sequential if n is even; P_n is both $\frac{n-1}{2}$-sequential and $\frac{n+1}{2}$-sequential if n is odd; $K_{m,n}$ is k-sequential for $k = 1, m, n$; $K_{m,n,1}$ is 1-sequential; and the join of any caterpillar and $\overline{K_t}$ is 1-sequential. Acharya [A1] showed that if $G(E,V)$ is an odd graph with $|E|+|V| \equiv 1$ or $2 \pmod 4$ when k is odd or $|E|+|V| \equiv 2$ or $3 \pmod 4$ when k is even, then G is not k-sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig and Turgeon [BKT] we have: mK_4 is not k-sequential for any k when m is odd and mK_2 is not k-sequential for any odd k when $m \equiv 2$ or $3 \pmod 4$ or for any even k when $m \equiv 1$ or $2 \pmod 4$. He further noted that $K_{m,n}$ is not k-sequential when k is even and m and n are odd, while $K_{m,k}$ is k-sequential for all k. Acharya [A1] points out that the following result of Slater's [Sl2] for $k = 1$ linking k-graceful graphs and k-sequential graphs holds in general: A graph is k-sequential if and only if $G+v$ has a k-graceful labeling f with $f(v)=0$. Slater [Sl1] also proved the a k-sequential graph with p vertices and $q>0$ edges must satisfy $k \leq p-1$.

5.2. **Sequentially Additive Graphs.** Bange, Barkauskas and Slater [BBS2] defined a *k-sequentially additive labeling* f of a graph $G(V,E)$ to be a bijection from $V \cup E$ to $\{k, \ldots, k+|V \cup E|-1\}$ such that for each edge xy, $f(xy)=f(x)+f(y)$. They proved: K_n is 1-sequentially additive if and only if $n \leq 3$; C_{3n+1} is not k-sequentially additive for $k \equiv 0,2 \pmod 3$; C_{3n+2} is not k-sequentially additive for $k \equiv 1,2 \pmod 3$; C_n is 1-sequentially additive if and only if $n \equiv 0,1 \pmod 3$; and P_n is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive.

Acharya and Hegde [AH2] have generalized k-sequentially additive labelings by allowing the image of the bijection to be $\{k, k+d, \ldots, (k+|V \cup E|-1)d\}$. They call such a labeling *additively (k,d)-sequential.*

5.3. **Magic, Edge-magic and Antimagic Labelings.** Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedlácěk [Se] in 1963. Responding to a problem raised by Sedlácěk, Stewart [St1] and [St2] studied various ways to label the edges of a graph in the mid 60s. Stewart calls a connected graph *semi-magic* if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic* labeling. Stewart calls a magic labeling *supermagic* if the set of edge labels consists of consecutive integers. The classic concept of an $n \times n$ magic square in number theory corresponds to a supermagic labeling of $K_{n,n}$. Stewart [St1] proved the following: K_n is magic for $n=2$ and all $n \geq 5$; $K_{n,n}$ is magic for all $n \geq 3$; fans F_n are magic if and only if n is odd and $n \geq 3$; wheels W_n are magic for $n \geq 4$; W_n with one spoke deleted is magic for $n=4$ and for $n \geq 6$. Stewart [St1] also proved that $K_{m,n}$ is semi-magic with if and only if $m=n$. In [St2] Stewart proved that K_n is supermagic for $n \geq 5$ if and only if $n>5$ and $n \not\equiv 0 \bmod 4$. Sedláček [Se2] showed that Möbius ladders M_n are supermagic when $n \geq 3$ and n is odd and that $C_n \times P_2$ is magic, but not supermagic, when $n \geq 4$ and n is even. Sedláček defines a connected graph with at least two edges to be *pseudo-magic* if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when $n \geq 4$ and n is even, M_n is not pseudo-magic and when $m \geq 3$ and m is odd, $C_m \times P_2$ is not pseudo-magic. Sedláček also proves that graphs obtained from an odd cycle with at least 5 vertices in which every vertex v of the cycle has two chords joining v to the two vertices at greatest distance from v are magic. (He also calls these *Möbius ladders.*) Characterizations of regular magic graphs were given by Dood [Do] and necessary and sufficient conditions for a graph to be magic were given in [Je] and [JT]. Two classes of supermagic quartic graphs are given in [BHL].

In 1970 Kotzig and Rosa [KR1] defined a magic labeling of a graph $G(V,E)$ as a bijection f from $V \cup E$ to $\{1,2,\ldots,|V \cup E|\}$ such that for all edges xy, $f(x)+f(y)+f(xy)$ is constant. To distinguish between this usage and that of Stewart we will call this labeling an *edge-magic* labeling. Kotiz and Rosa proved: $K_{m,n}$ has a edge-magic labeling for all m and n; C_n has a edge-magic labeling for all $n \geq 3$; and the disjoint union of n copies of P_2 has a edge-magic labeling if and only if n is odd. They further state that K_m has a edge-magic labeling if and only if $n = 1, 2, 3, 4, 5$ or 6 (see [KR2] and [CT]) and ask whether all trees have edge-magic labelings. Balakrishnan and Kumar [BK2] proved that the join of $\overline{K_n}$ and two disjoint copies of K_2 is edge-magic if and only if $n = 3$. Ringel and Llado [RL] prove that a (p, q) graph is not edge-magic if q is even and $p + q \equiv 2 \pmod 4$ and each vertex has odd degree. They conjecture that trees are edge-magic. In 1993 Lee (see [LPC]) conjectured that a cubic graph with p edges is edge-magic if and only if $p \equiv 2 \pmod 4$. Figueroa-Centeno, Muntaner and Ichishima [FMI] have proved the following graphs are edge-magic: nK_2 if and only if n is odd; $P_4 \cup nK_2$ for n odd; $P_3 \cup nK_2; P_5 \cup nK_2; nP_i$ for n odd and $i = 3, 4, 5$; $2P_n; P_1 \cup P_2 \cup \cdots \cup P_n; mK_{1,n}; K_{1,n} \cup K_{1,n+1}; C_m \odot nK_1; K_1 \odot nK_2$ for n even; $W_{2n}; K_2 \times \overline{K}_n, nK_3$ for n odd; binary trees, generalized Petersen graphs, ladders, books, fans, and odd cycles with pendant edges attached to one vertex. Figueroa-Centeno et al. call an edge-magic labeling *strong edge-magic* if the set of vertex labels is $\{1, 2, \ldots, |V|\}$. They prove that a graph is strong edge-magic if and only if it is strongly 1-harmonious and that a strong edge-magic graph is cordial. They also prove that P_n^2 and $K_2 \times C_{2n+1}$ are strong edge-magic.

Hartsfield and Ringel [HR] introduced antimagic graphs in 1990. A graph with q edges is called *antimagic* if its edges can be labeled with $1, 2, \ldots, q$ so that the sums of the labels of the edges incident to each vertex are distinct. Among the antimagic graphs are [HR]: P_n $(n \geq 3)$, cycles, wheels, and K_n $(n \geq 3)$. Hartsfield and Ringel conjecture that every tree except P_2 is antimagic and, moreover, every connected graph except P_2 is antimagic.

The concept of an (a, d)-antimagic labelings was introduced by Wagner and Bodendiek [WB] in 1993. A connected graph $G = (V, E)$ is said to be (a, d)-*antimagic* if there exist positive integers a, d and a bijection $f \colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $g_f \colon V \to N$, defined by $g_f(v) = \sum\{f(u, v) \colon (u, v) \in E(G)\}$, is injective and $g_f(V) = \{a, a+d, \ldots, a+(|V|-1)d\}$. They prove ([BW1] and [BW2]) the Herschel graph is not (a, d)-antimagic and certain cases of graphs called *parachutes* $P_{g,b}$ are antimagic. ($P_{g,b}$ is the graph obtained from the wheel W_{g+p} by deleting p consecutive spokes.)

6. Miscellaneous Labelings

6.1. **Prime and Vertex Prime Labelings.** The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy and Howalla (see [LWY]). A graph with vertex set V is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge xy the labels assigned to x and y are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. So far, there has been little progress towards proving this conjecture. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with a one vertex of degree at least 3 and with all other vertices with degree at most 2) and all trees of order less than 16 (see [FH]). Other graphs with prime labelings include all cycles and the disjoint union of C_{2k} and C_n [DLM]. The complete graph K_n does not have a prime labeling for $n \geq 4$ and W_n is prime if and only if n is even (see [LWY]).

Given a collection of graphs $G_1, \ldots, G_n$ and some fixed vertex v_i from each G_i, Lee, Wui and Yeh [LWY] define $Amal\{(G_i, v_i)\}$, the almagamation of $\{(G_i, v_i)|i = 1, \ldots, n\}$, as the graph obtained by taking the union of the G_i and identifying $v_1, v_2, \ldots, v_n$. Lee et al. [LWY] have shown $Amal\{(G_i, v_i)\}$ has a prime labeling when G_i are paths and when G_i are cycles. They also showed that the almagamation of any number of copies of W_n, n odd, with a common vertex is not prime. They conjecture that for any tree T and v from T, the almagamation of two or more copies of T with v in common is prime. They further conjecture that the almagamation of two or more copies of W_n that share a common point is prime when n is even ($n \neq 4$).

A dual of prime labelings has been introduced by Deretsky, Lee and Mitchem [DLM]. They say a graph with edge set E has a *vertex prime labeling* if its edges can be labeled with distinct integers $1, \ldots, |E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs; $C_{2k} \cup C_n$; $C_{2m} \cup C_{2n} \cup C_{2k+1}$; $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$; and $5C_{2m}$. They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles.

6.2. **Edge-graceful Labelings.** In 1985, Lo [Lo] introduced the notion of edge-graceful graphs. A graph $G(V, E)$ is said to be *edge-graceful* if there exists a bijection f from E to $\{1, 2, \ldots, |E|\}$ so that the induced mapping f^+ from V to $\{0, 1, \ldots, |V|-1\}$ given by $f^+(x) = \sum\{f(xy)|xy \in E\} \pmod{|V|}$ is a bijection. Lee [L2] has conjectured that all trees of odd order are edge-graceful. Small [Sm] has proved that spiders of odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [KS] proved that all spiders of odd order and exactly three end vertices are edge-graceful. Cabaniss, Low and Mitchem [CLM] have shown that regular spiders of odd order are edge-graceful. Lee and Seah [LSe2] have shown that $K_{n,n,\ldots,n}$ is edge-graceful if and only if n is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [LSe1] have also proved that C_n^k (the kth power of C_n) is edge-graceful for $k < \lfloor n/2 \rfloor$ if and only if n is odd and for $k \geq \lfloor n/2 \rfloor$ if and only if n is a multiple of 4 or n is odd (see also [CLM]). Lee, Seah and Wang [LSW] gave a complete characterization of edge-graceful P_n^k graphs. Lee and Seah [LSe3] have investigated edge-gracefulness of multigraphs.

In 1997 Yilmaz and Cahit [YC] introduced a weaker version of edge-graceful called E-cordial. Let G be a graph with vertex set V and edge set E and let f a function from E to $\{0, 1\}$. Define f on V by $f(v) = \sum\{f(u, v)|uv \in E\} \pmod 2$. The function f is called an *E-cordial* labeling of G if the the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the the the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an E-cordial labeling is called *E-cordial*. Yilmaz and Cahit prove the following graphs are E-cordial: trees with n vertices if and only if $n \not\equiv 2 \pmod 4$; K_n if and only if $n \not\equiv 2 \pmod 4$; $K_{m,n}$ if and only if $m + n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; regular graphs of degree 1 on $2n$ vertices if and only if $n \not\equiv 1 \pmod 2$; friendship graphs $C_3^{(n)}$ for all n; fans F_n if and only if $n \not\equiv 1 \pmod 4$; and wheels W_n if and only if $n \neq 1 \pmod 4$. They observe that graphs with $n \equiv 2$ (mod 4) vertices can not be E-cordial. They generalize E-cordial labelings to E_k-cordial ($k > 1$) labelings by replacing $\{0,1\}$ by $\{0,1,2,\ldots,k-1\}$. Of course, E_2-cordial is the same as E-cordial.

6.3. **Line-graceful Labelings.** Gnanajothi [Gn] has defined a concept similar to edge-graceful. She calls a graph with n vertices *line-graceful* if it is possible to label its edges with $0,1,2,\ldots,n$ so that when each vertex is assigned the sum modulo n of all the edge labels incident with that vertex the resulting vertex labels are $0,1,\ldots,n-1$. A necessary condition for the line-gracefulness of a graph is that its order is not congruent to 2 (mod 4). Among line-graceful graphs are (see [Gn, pp. 132-181]) P_n if and only if $n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; $K_{1,n}$ if and only if $n \not\equiv 1 \pmod 4$; $P_n \bigodot K_1$ (combs) if and only if n is even; $(P_n \bigodot K_1) \bigodot K_1$ if and only if $n \not\equiv 2 \pmod 4$; (in general, if G has order n, $G \bigodot H$ is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H), mC_n when mn is odd; $C_n \bigodot K_1$ (crowns) if and only if n is even; mC_4 for all m; complete n-ary trees when n is even; $K_{1,n} \cup K_{1,n}$ if and only if n is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete n-ary tree when n is even; trees for which exactly one vertex has even degree. She conjectures that all trees with $p \not\equiv 2 \pmod 4$ vertices are line-graceful and proved this for $p \leq 9$.

Gnanajothi [Gn] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from $K_{1,4}$ by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has $p \not\equiv 2 \pmod 4$ vertices; the graph obtained from $K_{1,n}$ by replacing each spoke with a path of length m (counting the center vertex) is line-graceful in the following cases: $n = 2$; $n = 3$ and $m \not\equiv 3 \pmod 4$; m is even and $mn + 1 \equiv 0$ (mod 4).

Gnanajothi studied graphs obtained by joining disjoint graphs G and H with an edge. She proved such graphs are line-graceful in the following circumstances: $G = H$; $G = P_n, H = P_m$ and $m+n \not\equiv 0 \pmod 4$; $G = P_n \bigodot K_1$, $H = P_m \bigodot K_1$ and $m+n \not\equiv 0$ (mod 4).

6.4. **Sum Graphs.** In 1990, Harary [Ha1] introduced the notion of a sum graph. A graph $G(V,E)$ is called a *sum graph* if there is an bijective labeling f from V to a set of positive integers S such that $xy \in E$ if and only if $f(x) + f(y) \in S$. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. For a connected graph G, let $s(G)$, the *sum number of* G, denote the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph. Ellingham [El] proved the conjecture of Harary [Ha1] that $s(T) = 1$ for every tree $T \neq K_1$. Bergstand et al. [BHHJKW] proved that $s(K_n) = 2n - 3$. Hartsfield and Smyth [HaS] showed that $s(K_{m,n}) = \lceil 3m+n-3 \rceil/2$ when $n \geq m$ but Yan and Liu [YL1] found counterexamples to this result when $m \neq n$. Miller et al. [MRSS] proved that $s(W_n) = \frac{n}{2}+2$ for n even and $s(W_n) = n$ for $n \geq 5$ and n odd. Miller, Ryan and Smyth [MRS] prove that the complete n-partite graph on n sets of 2 nonadjacent vertices has sum number $4n - 5$ and obtain upper and lower bounds on the complete n-partite graph on n sets of m nonadjacent vertices. Gould and Rödl [GR] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [GBGGGJ] proved that $s(K_n - \text{edge}) \leq 2n - 4$. The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the *spum*. They proved spum of K_n is $4n - 6$ and the spum of C_n is at most $4n - 10$.

In 1994 Harary [Ha2] generalized sum graphs by permitting S to be any set of integers. He calls these graphs *integral sum graphs*. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [Sha] has shown that C_n and W_n are integral sum graphs for all $n \neq 4$. Chen [Che2] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. Chen also gives methods for constructing new connected integral sum graphs from given integral sum graphs by identification. The *integral sum number*, $\zeta(G)$, of G, is the minimum number of isolated vertices that must be added to G so that the resulting graph is an integral sum graph. Harary [Ha2] conjectured that for $n \geq 4$ the integral sum number $\zeta(K_n) = 2n - 3$. This conjecture was verified by Chen [Che] and by Sharary [Sha]. Yan and Liu proved: $\zeta(K_n - \text{edge}) = 2n - 4$ when $n \geq 5$ [YL1]; $\zeta(K_n - E(K_r)) = n - 1$ when $n \geq 6, n \equiv 0 \pmod 3$ and $r = 2n/3 - 1$ [YL2];$\zeta(K_{m.m}) = 2m - 1$ for $m \geq 2$ [YL2]; $\zeta(K_n - \text{edge}) = 2n - 4$ for $n \geq 5$ [YL2]; if $n \geq 5$ and $n - 3 \geq r$, then $\zeta(K_n - E(K_r)) \geq n - 1$ [YL2]; if $\lceil 2n/3 \rceil - 1 > r \geq 2$, then $\zeta(K_n - E(K_r)) \geq 2n - r - 2$ [YL2]; and if $2 \leq m < n$, and $n = (i + 1)(im - i + 2)/2$, then $s(K_{m,n}) = \zeta(K_{m,n}) = (m - 1)(i + 1) + 1$ while if $(i + 1)(im - i + 2)/2 < n < (i + 2)[(i + 1)m - i + 1]/2$, then $s(K_{m,n}) = \zeta(K_{m,n}) = \lceil ((m - 1)(i + 1)(i + 2) + 2n)/(2i + 2) \rceil$ [YL2].

Alon and Scheinermann [AS] generalized sum graphs by replacing the condition $f(x) + f(y) \in S$ with $g(f(x), f(y)) \in S$ where g is an arbitrary symmetric polynomial. They called a graph with this property a *g-graph* and proved that for a given symmetric polynomial g not all graphs are g-graphs. On the other hand, for every symmetric polynomial g and every graph G there is some vertex labeling so that G together with at most $|E(G)|$ isolated vertices is a g-graph.

Boland, Laskar, Turner, and Domke [BLTD] investigated a modular version of sum graphs. They call a graph $G(V, E)$ a *mod sum graph* (MSG) if there exists a positive integer n and an injective labeling from V to $\{1, 2, \ldots, n - 1\}$ such that $xy \in E$ if and only if $f(x) + f(y) \pmod n = f(z)$ for some vertex z. Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [BLTD] have shown the following graph are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and all $K_{2,n}$. They also proved that K_p $(p \geq 2)$ is not MSG and conjecture that W_p is MSG for $p \geq 4$.

Grimaldi [Gr] has investigated labeling the vertices of a graph $G(V, E)$ with n vertices with distinct elements of the ring Z_n so that $xy \in E$ whenever $(x + y)^{-1} \in Z_n$.

References

[Ab1] J. Abrham, Perfect systems of difference sets–A survey, *Ars Combin.* **17A** (1984) 5-36.

[Ab2] J. Abrham, Graceful 2-regular graphs and Skolem sequences, *Discrete Math.*, **93** (1991) 115–121.

[ACE] J. Abrham, M. Carter and K. Eshghi, personal communication.

[AK1] J. Abrham and A. Kotzig, Extensions of graceful valuations of 2-regular graphs consisting of 4-gons, *Ars Combin.,* **32**, (1991) 257-262.

[AK2] J. Abrham and A. Kotzig, Two sequences of 2-regular graceful graphs consisting of 4-gons, in J. Nesetril and M. Fiedler, eds., 4th Czechoslovakian Symp. on Combinatorics, Graphs, and Complexity (Elsevier, Amsterdam, (1992) 1-14.

[AK3] J. Abrham and A. Kotzig, All 2-regular graphs consisting of 4-cycles are graceful, *Discrete Math.*, **135** (1994) 1-14.

[AK4] J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.*, **150** (1996) 3-15.

[A1] B.D. Acharya, On d-sequential graphs, *J. Math. Phy. Sci.*, **17** (1983) 21-35.

[A2] B. D. Acharya, Are all polyminoes arbitrarily graceful?, *Proc. First Southeast Asian Graph Theory Colloquium*, Ed. K. M. Koh and H. P. Yap, Springer-Verlag, N. Y. 1984, 205-211.

[AG] B. D. Acharya and M. K. Gill, On the index of gracefulness of a graph and the gracefulness of two-dimensional square lattice graphs, *Indian J. Math.,* **23** (1981) 81-94.

[AH1] B.D. Acharya and S.M. Hegde, Further results on k-sequential graphs, *Nat. Acad. Sci. Lett.,* **8** (1985) 119-122.

[AH2] B.D. Acharya and S.M. Hegde, Arithmetic graphs, *J. Graph Theory*, **14** (1990) 275-299.

[AH3] B.D. Acharya and S.M. Hegde, Strongly indexable graphs, *Discrete Math.*, **93** (1991) 123-129.

[AlM] R.E. L. Aldred and B. D. McKay, Graceful and harmonious labellings of trees, preprint.

[AS] N. Alon and E. Scheinerman, Generalized sum graphs, *Graphs and Combin.,* **8** (1992) 23-29.

[AM] R. Aravamudhan and M. Murugan, Numbering of the vertices of $K_{a,1,b}$, preprint.

[ArG] S. Arumugam and K. Germina, On indexable graphs, *Discrete Math.*, **161** (1996) 285-289.

[AF] J. Ayel and O. Favaron, Helms are graceful, *Progress in Graph Theory* (Waterloo, Ont., 1982), Academic Press, Toronto, Ont. (1984) 89-92.

[BHL] B. Bača, I. Holländer and K.W. Lih, Two classes of super-magic quartic graphs, *JCMCC* **23** (1997) 113-120.

[Ba] R. Balakrishnan, Graph labelings, unpublished.

[BK1] R. Balakrishnan and R. Kumar, Decomposition of complete graphs into isomorphic bipartite subgraphs, *Graphs and Combin.,* **10** (1994) 19-25.

[BK2] R. Balakrishnan and R. Kumar, Existence and nonexistence of certain labellings for the graph $K_n^c \bigvee 2K_2$, *Utilitas Math.*, **46** (1994) 97-102.

[BS] R. Balakrishnan and R. Sampathkumar,, Decompositions of regular graphs into $K_n^c \bigvee 2K_2$, *Discrete Math.*, **156** (1996) 19-28.

[BSY1] R. Balakrishnan, A. Selvam and V. Yegnanarayanan, On felicitous labelings of graphs, *Proceed. National Workshop on Graph Theory and its Appl. Manonmaniam Sundaranar Univ., Tiruneli* (1996) 47-61.

[BSY2] R. Balakrishnan, A. Selvam and V. Yegnanarayanan, Some results on elegant graphs, *Indian J. Pure Appl. Math.*, **28** (1997) 905-916.

[BBS1] D. Bange, A. Barkauskas and P. Slater, Simply sequential and graceful graphs, *Proc. of the 10th S.E. Conf. on Combinat., Graph Theory, and Computing*, (Utilitas Math. Winnipeg, 1979) 155-162.

[BBS2] D. Bange, A. Barkauskas and P. Slater, Sequentially additive graphs, *Discrete Math.*, **44** (1983) 235-241.

[BHD] C. Barrientos, I. Dejter and H. Hevia, Equitable labelings of forests, *Combin. and Graph Th.*, **1** (1995) 1-26.

[BGHJ] R. Beals, J. Gallian, P. Headley and D. Jungreis, Harmonious groups, *J. Combin. Th., Series A*, **56** (1991) 223-238.

[BL] M. Benson and S.M. Lee, On cordialness of regular windmill graphs, *Congress. Numer.*, **68** (1989) 45-58.

[BHHJKW] D. Bergstrand, F. Harary, K. Hodges, G. Jennings, L. Kuklinski, and J. Wiener, The sum numbering of a complete graph, *Bull. Malaysian Math. Soc.,* **12** (1989) 25-28.

[Be] J.C. Bermond, Graceful graphs, radio antennae and French windmills, *Graph Theory and Combinatorics*, Pitman, London (1979) 13-37.

[BBG] J.C. Bermond, A. E. Brouwer and A. Germa, Systemes de triplets et differences associèes, *Problems Combinatories et Thèorie des Graphs*, Colloq. Intern. du Centre National de la Rech. Scient., 260, Editions du Centre Nationale de la Recherche Scientifique, Paris (1978) 35-38.

[BF] J. C. Bermond and G. Farhi, Sur un probleme combinatoire d'antennes en radioastronomie II, *Annals of Discrete. Math.,* **12** (1982) 49-53.

[BKT] J. C. Bermond, A. Kotzig and J. Turgeon, On a combinatorial problem of antennas in radioastronomy, in Combinatorics, A. Hajnal and V. T. Sós, eds., *Colloq. Math. Soc. János Bolyai*, **18**, 2 vols. North-Holland, Amsterdam (1978) 135-149.

[BS] J.C. Bermond and D. Sotteau, Graph decompositions and G-design, *Proc. 5th British Combin. Conf., 1975*, 53-72. (Second series), **12** (1989) 25-28.

[BD1] V. Bhat-Nayak and U. Deshmukh, Gracefulness of $C_{2x+1} \cup P_{x-2\theta}$, *Proc. International Conf. on Graph Theory and Number Theory, Trichy 1996,* to appear.

[BD2] V. Bhat-Nayak and U. Deshmukh, New families of graceful banana trees, *Proc. Indian Acad. Sci. Math. Sci.*, **106** (1996) 201-216.

[BD3] V. Bhat-Nayak and U. Deshmukh, Gracefulness of $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$, *J. Ramanujan Math. Soc.,* **11** (1996) 187-190.

[BD4] V. Bhat-Nayak and U. Deshmukh, Skolem-graceful labelings of unions of paths, personal commnication.

[BD5] V. Bhat-Nayak and U. Deshmukh, Gracefulness of $C_3 \cup P_n$, preprint.

[BG] V. N. Bhat-Nayak and S. K. Gokhale, Validity of Hebbare's conjecture, *Utilitas Math.*, **29** (1986) 49-59.

[Bl] G. S. Bloom, A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful, *Ann. N.Y. Acad. Sci.*, **326** (1979) 32-51.

[BG1] G. S. Bloom and S. W. Golomb, Applications of numbered undirected graphs, *Proc. IEEE,* **65** (1977) 562-570.

[BG2] G. S. Bloom and S. W. Golomb, Numbered complete graphs, unusual rulers, and assorted applications, in *Theory and Applications of Graphs, Lecture Notes in Math.*, 642, Springer-Verlag, New York (1978) 53-65.

[BH1] G. S. Bloom and D. F. Hsu, On graceful digraphs and a problem in network addressing, *Congress. Numer.,* **35** (1982) 91-103.

[BH2] G. S. Bloom and D. F. Hsu, On graceful directed graphs that are computational models of some algebraic systems, *Graph Theory with Applications to Algorithms and Computers*, Ed. Y. Alavi, Wiley, New York (1985).

[BH3] G. S. Bloom and D. F. Hsu, On graceful directed graphs, *SIAM J. Alg. Discrete Meth.,* **6** (1985) 519-536.

[BW1] R. Bodendiek and G. Walther, Aritmetisch antimagische Graphen, Graphentheorie III, K. Wagner and R. Bodendiek (eds), Mannheim, 1993.

[BW2] R. Bodendiek and G. Walther, (A, D) antimagic parachutes, *Ars Combin.*, **42** (1996), 129–149.

[BW3] R. Bodendiek and G. Walther, (a, d) antimagic parachutes II, *Ars Combin.*, **46** (1997), 33–63.

[BSW1] R. Bodendiek, H. Schumacher and H. Wegner, Űber eine spezielle Klasse groziőser Eulerscher Graphen, *Mitt. Math. Gesellsch. Hamburg*, **10** (1975) 241-248.

[BSW2] R. Bodendiek, H. Schumacher and H. Wegner, Űber graziőse Graphen, *Math.-Phys. Semesterberichte,* **24** (1977) 103-106.

[BLTD] J. Boland, R. Laskar, C. Turner and G. Domke, On mod sum graphs, *Congress. Numer.,* **70** (1990) 131-135.

[BM] J. Bondy and U. Murty, *Graph Theory with Applications*, North-Holland, New York (1976).

[Bu1] C. Bu, Gracefulness of graph $K_n + \overline{K}_m$, *J. Harbin Shipbuilding Eng. Inst.*, **15** (1994) 91-93.

[Bu2] C. Bu, Sequential labeling of the graph $C_n \bigodot \overline{K}_m$, preprint.

[BC1] C. Bu and C. Cao, The gracefulness for a class of disconnected graphs, *J. Natural Sci. Heilongjiang Univ.,* **12** (1995) 6-8.

[BC2] C. Bu and L. Chen, Some conclusions about graceful graphs, *J. Harbin Shipbuilding Eng. Ins.*, **14** (1993) 103-108.

[BF] C. Bu and W. Feng, Some composite theory about graceful graph, graph, *J. Harbin Eng. Univ.*, **16** (1995) 94-97.

[BGZ] C. Bu Z. Gao and D. Zhang, On k-gracefulness of $r - p_n \times p_2$, *J. Harbin Shipbuilding Eng. Ins.*, **15** (1994) 85-89.

[BF] C. Bu and B. He, The k-gracefulness of some graphs, *J. Harbin Eng. Univ.*, **14** (1993) 92-95.

[BS1] C. Bu and J. Shi, A class of (k, d)-arithmetic graphs, *J. Harbin Eng. Univ.*, **16** (1995) 79-81.

[BS2] C. Bu and J. Shi, Some conclusions about indexable graphs, *J. Harbin Eng. Univ.*, **16** (1995) 92-94.

[BZ] C. Bu and J. Zhang, (k, d)-graceful graph, preprint.

[BZH] C. Bu, D. Zhang and B. He, k-gracefulness of C_n^m *J. Harbin Shipbuilding Eng. Ins.*, **15** (1994) 95-99.

[BoS] C. P. Bonnington and J. Širàň, Bipartite labelings of trees with maximum degree three, preprint.

[CLM] S. Cabaniss, R. Low, Richard and J. Mitchem, On edge-graceful regular graphs and trees, *Ars Combin.*, **34** (1992) 129–142.

[C1] I. Cahit, Elegant valuation of the paths, *Ars Combin.*, **16** (1983) 223-227.

[C2] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars Combin.,* **23** (1987) 201-207.

[C3] I. Cahit, On cordial and 3-equitable labellings of graphs, *Utilitas Math.*, **37** (1990) 189-198.

[C4] I. Cahit, Status of graceful tree conjecture in 1989, in *Topics in Combinatorics and Graph Theory*, R. Bodendiek and R. Henn (eds), *Physica-Verlag*, Heidelberg 1990.

[C5] I. Cahit, Recent results and open problems on cordial graphs, *Contemporary Methods in Graph Theory*, R. Bodendiek (ed.), Wissenschaftsverlag, Mannheim, 1990, 209-230.

[C6] I. Cahit, Equitable tree labellings, *Ars Combin.,* **40**, (1995) 279-286.

[C7] I. Cahit, On harmonious tree labellings, *Ars Combin.,* **41**, (1995) 311-317.

[C8] I. Cahit, H-cordial graphs, *Bull. Inst. Combin. Appl.*, **18** (1996) 87-101.

[CHR] G. J. Chang, D. F. Hsu and D. G. Rogers, Additive variations on a graceful theme: some results on harmonious and other related graphs, *Congress. Numer.,* **32** (1981) 181-197.

[CL] G. Chartrand and L. Lesniak-Foster, *Graphs & Digraphs* 2nd. Wadsworth, Belmont, California (1986)

[CJ] D. L. Chen and C. J. Jiang, The K-gracefulness of the rhomb-ladder graph ∇_n^m, *Shandong Kuangye Xueyuan Xuebao*, **11** (1992) 196–199, 208.

[CLY] W.C. Chen, H.I. Lü and Y.N. Yeh, Operations of interlaced trees and graceful trees, *Southeast Asian Bull. Math.*, **21** (1997).

[Che1] Z. Chen, Two conjecture of Harary on sum graphs over all integers, *Discrete Math.*, to appear.

[Che2] Z. Chen, Integral sum graphs from identification, *Discrete Math.*, **181** (1998) 77-90.

[CheZ] Z.-Z. Chen, A generalization of the Bodendiek conjecture about graceful graphs, *Topics in Combinatorics and Graph Theory*, R. Bodendiek and R. Henn, eds., Physica-Verlag, Heidelberg, 1990, 737-746.

[CK1] S. A. Choudum and S. P. M. Kishore, All 5-stars are Skolem graceful, *Indian J. Pure and Appl. Math.*, **27** (1996) 1101-1105.

[CK2] S. A. Choudum and S. P. M. Kishore, Graceful labellings of the union of paths and cycles, preprint.

[CK3] S. A. Choudum and S. P. M. Kishore, On Skolem gracefulness of k-stars, *Ramanujan Mathematical Society Inter. Conf. on Discrete Math. and Number Theory*, 1996, to appear.

[CK4] S. A. Choudum and S. P. M. Kishore, Graceful labelling of the union of cycles and stars, preprint.

[CH] F. R. K. Chung and F. K. Hwang, Rotatable graceful graphs, *Ars Combin.,* **11** (1981) 239-250.

[CT] D. Craft and E. H. Tesar, On questions by Ringel and Erdős about edge-magic graphs, preprint.

[D] C. Delorme, Two sets of graceful graphs, *J. Graph Theory*, **4** (1980) 247-250.

[DMTKT] C. Delorme, M. Maheo, H. Thuillier, K. M. Koh and H. K. Teo, Cycles with a chord are graceful, *J. Graph Theory*, **4** (1980) 409-415.

[De] Y. X. Deng, Harmoniousness of the graphs $C_n + \overline{K}_t$, *J. Math. Res. Exposition*, **15** (1995), suppl., 79–81.

[DLL] G. Denham, M.G. Leu and A. Liu, All 4-stars are Skolem-graceful, *Ars Combin.,* **36** (1993) 183-191.

[DLM] T. Deretsky, S.M. Lee and J. Mitchem, On vertex prime labelings of graphs, in *Graph Theory, Combinatorics and Applications Vol. 1*, J.Alavi, G. Chartrand, O. Oellerman and A. Schwenk, eds., Proceedings 6th International Conference Theory and Applications of Graphs (Wiley, New York, 1991) 359-369.

[Do] J. Doma, Unicyclic Graceful Graphs, M. S. Thesis, McMaster Univ., 1991.

[Do] M. Doob, Characterizations of regular magic graphs, *J. Combin. Theory, Ser. B*, **25** (1978) 94-104.

[Du] M. Dufour, Sur la Decomposition d'un Graphe Complet en Arbres Isomorphes, Ph. D. Thesis, Universite de Montreal, 1995.

[E] P. Eldergill, Decomposition of the Complete Graph with an Even Number of Vertices, M. Sc. Thesis, McMaster University, 1997.

[El] M. N. Ellingham, Sum graphs from trees, *Ars Combin.*, **35** (1993) 335-349.

[EV] S. El-Zanati and C. Vanden Eynden, On graphs with strong α-valuations, preprint.

[Es] K. Eshghi, The Existence and Construction of α-valuations of 2-Regular Graphs with 3 Components, Ph. D. Thesis, Industrial Engineering Dept., University of Toronto, 1997.

[F1] R. Frucht, Graceful numbering of wheels and related graphs, *Ann. N.Y. Acad. of Sci.*, **319** (1979) 219-229.

[F2] R. Frucht, On mutually graceful and pseudograceful labelings of trees, *Scientia Series A,* **4** (1990/1991) 31-43.

[F3] R. Frucht, Nearly graceful labelings of graphs, *Scientia*, **5** (1992-1993) 47-59.

[FG] R. Frucht and J. A. Gallian, Labeling prisms, *Ars Combin.*, **26** (1988) 69-82.

[FS] R. W. Frucht and L. C. Salinas, Graceful numbering of snakes with constaints on the first label, *Ars Combin.*, **20** (1985), B, 143-157.

[FH] H. L. Fu and K. C. Huang, On prime labelling, *Discrete Math.,* **127** (1994) 181-186.

[FW] H. L. Fu and S. L. Wu, New results on graceful graphs, *J. Combin. Info. Sys. Sci.*, **15** (1990) 170-177.

[Ga1] J. A. Gallian, Labeling prisms and prism related graphs, *Congress. Numer.*, **59** (1989) 89-100.

[Ga2] J. A. Gallian, A survey: recent results, conjectures and open problems on labeling graphs, *J. Graph Theory*, **13** (1989) 491-504.

[Ga3] J. A. Gallian, Open problems in grid labeling, *Amer. Math. Monthly*, **97** (1990) 133-135.

[Ga4] J. A. Gallian, A guide to the graph labeling zoo, *Discrete Appl. Math.,* **49** (1994) 213-229.

[GJ] J. A. Gallian and D. S. Jungreis, Labeling books, *Scientia*, **1** (1988) 53-57.

[GPW] J. A. Gallian, J. Prout and S. Winters, Graceful and harmonious labelings of prisms and related graphs, *Ars Combin.*, **34** (1992) 213-222.

[GH] T. Gangopadhyay and S.P. Rao Hebbare, Bigraceful graphs-I, *Utilitas Math.*, **17** (1980) 271-275.

[Gao] Y. Z. Gao, On gracefulness of $C_m \cup P_2$, *J. Hebei Teachers College*, No. 3 (1993) 13 -21.

[GL] Y. Z. Gao and Z. H. Liang, private communication.

[Gar] M. Gardner, Mathematical games: the graceful graphs of Solomon Golomb, or how to number a graph parsimoniously, *Scientific American* 226, 3 (1972) 108-112; 226, 4 (1972) 104; 226, 6 (1972) 118.

[Gn] R.B. Gnanajothi, Topics in Graph Theory, Ph.D. Thesis, Madurai Kamaraj University, 1991.

[Go] S. W. Golomb, How to number a graph, in *Graph Theory and Computing*, R. C. Read, ed., Academic Press, New York (1972) 23-37.

[GBGGGJ] J. Goodell, A. Beveridge, M. Gallagher, D. Goodwin, J. Gyori, A. Joseph, Sum graphs, unpublished.

[GR] R.J. Gould and V. Rödl, Bounds on the number of isolated vertices in sum graphs, *Graph Th., Combin. and Appl.*, **1** (1991) 553-562.

[Gr1] T. Grace, Graceful, Harmonious, and Sequential Graphs, Ph.D. Thesis, University Illinois at Chicago Circle, 1982.

[Gr2] T. Grace, On sequential labelings of graphs, *J. Graph Theory*, **7** (1983) 195-201.

[Gr3] T. Grace, K_4 snakes are sequential, Technical Report, Department of Computer Science, Illinois Institute of Technology (1986).

[GS] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Meth.,* **1** (1980) 382-404.

[Gr] R.P. Grimaldi, Graphs from rings, *Congress. Numer.*, **71** (1990) 95-104.

[Gu1] W. F. Guo, Gracefulness of the graph $B(m, n)$, *J. Inner Mongolia Normal Univ.,* (1994) 24–29.

[Gu2] W. F. Guo, Gracefulness of the graph $B(m, n, p)$, *J. Math. (PRC)*, **15** (1995) 345-351.

[GL] A. Gyárfás and J. Lehel, A method to generate graceful trees, in *Colloque C.N.R.S. Problèmes Combinatories et Théorie des Graphes, Orsay, 1976* (1978) 207-209.

[Ha1] F. Harary, Sum graphs and difference graphs, *Congress. Numer.*, **72** (1990) 101-108.

[Ha2] F. Harary, Sum graphs over all the integers, *Discrete Math.,* **124** (1994) 99-105.

[HH] F. Harary and D. Hsu, Node-graceful graphs, *Comput. Math. Appl.*, **15** (1988) 291-298.

[HR] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, San Diego, 1990.

[HaS] N. Hartsfield and W.F. Smyth, The sum number of complete bipartite graphs, in *Graphs and Matrices* (ed. R. Rees), Marcel Dekker (1992) 205-211.

[HeH] K. Heinrich and P. Hell, On the problems of bandsize, *Graphs, Combinat.*, **3** (1987) 279-284.

[H] S. P. Rao Hebbare, Graceful cycles, *Utilitas Math.*, **10** (1976) 307-317.

[He] M. He, The gracefulness of the graph $2C_n$, *Neimenggu Daxue Xuebao Ziran Kexue*, **26** (1995) 247-251.

[Heg1] S. M. Hegde, Additively graceful graphs, *Nat. Acad. Sci. Lett.*, **12** (1989) 387-390.

[Heg2] S. M. Hegde, On K-sequential graphs, *Nat. Acad. Sci. Lett.*, **16** (1993) 299-301.

[HLS1] Y.S. Ho, S.M. Lee and S.C. Shee, Cordial labellings of unicyclic graphs and generalized Petersen graphs, *Congress. Numer.*, **68** (1989) 109-122.

[HLS2] Y.S. Ho, S.M. Lee and S.C. Shee, Cordial labellings of the cartesian product and composition of graphs, *Ars Combin.*, **29** (1990) 169-180.

[HK] C. Hoede and H. Kuiper, All wheels are graceful, *Utilitas Math.*, **14** (1987) 311.

[Ho] M. Hovey, A-cordial graphs, *Discrete Math.*, **93** (1991) 183-194.

[Hs] D. F. Hsu, Harmonious labelings of windmill graphs and related graphs, *J. Graph Theory*, **6** (1982) 85-87.

[HK1] D.F. Hsu and A.D. Keedwell, Generalized complete mappings, neofields, sequenceable groups and block designs, I, *Pacific J. Math.*, **111** (1984) 317-332.

[HK2] D.F. Hsu and A.D. Keedwell, Generalized complete mappings, neofields, sequenceable groups and block designs, II, *Pacific J. Math.*, **117** (1985) 291-312.

[Hua] Q. Huang, Harmonious labeling of crowns $C_n \odot K_1$, preprint.

[HKR] C. Huang, A. Kotzig and A. Rosa, Further results on tree labellings, *Utilitas Math.*, **21c** (1982) 31-48.

[HuS] J. Huang and S. Skiena, Gracefully labeling prisms, *Ars Combin.*, **38** (1994) 225-242.

[Je] R. H. Jeurissen, Magic graphs, a charterization, *Europ. J. Combinatorics*, **9** (1988) 363-368.

[JT] S. Jezný and M. Trenklér, Characterization of magic graphs, *Czechoslovak Math. J.*, **33** (1983) 435-438.

[JLLLLZ] D. J. Jin, S. Z. Liu, S. H. Lee, H. L. Liu, X. G. Lu and D. Zhang, The joint sum of graceful trees, *Comput. Math. Appl.*, **26** (1993) 83-87.

[JS] M. I. Jinnah and G. S. Singh, A note on arthmetic numberings of graphs, *Proc. Symposium on Graphs and Combinatorics, Kochi, Kerala, India* (1991) 83-87.

[JMWG] D. J. Jin, F. H. Meng and J. G. Wang, The gracefulness of trees with diameter 4, *Acta Sci. Natur. Univ. Jilin.*, 1993 17–22.

[J] R. P. Jones, Gracelessness, Proc. 10th S-E Conf. Combinatorics, Graph Theory, and Computing, Congress. Numer., **XXIII-XXIV**, *Utilitas Math.*, (1979) 547-552.

[JR] D. Jungreis and M. Reid, Labeling grids, *Ars Combin.*, **34** (1992) 167-182.

[Ka] Q.D. Kang, The k-gracefulness of the product graphs $P_m \times C_{4n}$, *J. Math. Res. Exposition*, **9** (1989) 623-627.

[KLGY] Q. D. Kang, Z.-H. Liang, Y.-Z. Gao and G.-H. Yang, On the labeling of some graphs, *J. Combin. Math. Combin. Comput.*, **22** (1996) 193-210.

[KZ] Q. D. Kang and X. Zhao, Strongly harmonious labelings of windmill graphs, *J. Hebei Normal College*, **2** (1992) 1-7.

[Kat1] K. Kathiresan, Subdivisions of ladders are graceful, *Indian J. Pure Appl. Math.*, **23** (1992) 21-23.

[Kat2] K. Kathiresan, Graceful labeling of ladders with pendant edges, preprint.

[Kat3] K. Kathiresan, Odd graceful graphs, preprint.

[KS] J. Keene and A. Simoson, Balanced strands for asymmetric, edge-graceful spiders, *Ars Combin.*, **42** (1996) 49-64.

[Ki1] W.W. Kirchherr, On the cordiality of some specific graphs, *Ars Combin.*, **31** (1991) 127-138.

[Ki2] W.W. Kirchherr, NEPS operations on cordial graphs, *Discrete Math.*, **115** (1993) 201-209.

[Kis] S. P. Kishore, Graceful Labellings of Certain Disconnected Graphs, Ph. D. Thesis, Indian Institute of Technology, Madras, 1996.

[KP] K. M. Koh and N. Punnim, On graceful graphs: cycles with 3-consecutive chords, *Bull. Malaysian Math. Soc.*, **5** (1982) 49-63.

[KRLT] K. M. Koh, D. G. Rogers, P. Y. Lee and C. W. Toh, On graceful graphs V: unions of graphs with one vertex in common, *Nanta Math.*, **12** (1979) 133-136.

[KRL] K. M. Koh, D. G. Rogers and C. K. Lim, On graceful graphs: sum of graphs, Research Report 78, College of Graduate Studies, Nanyang University (1979).

[KRT1] K. M. Koh, D. G. Rogers and T. Tan, On graceful trees, *Nanta Math.*, **10** (1977) 27-31.

[KRT2] K. M. Koh, D. G. Rogers and T. Tan, A graceful arboretum: a survey of graceful trees, in *Proceedings of Franco-Southeast Asian Conference*, Singapore, May 1979, **2** 278-287.

[KRT3] K. M. Koh, D. G. Rogers and T. Tan, Products of graceful trees, *Discrete Math.*, **31** (1980) 279-292.

[KRTY] K. M. Koh, D. G. Rogers, H. K. Teo and K. Y. Yap, Graceful graphs: some further results and problems, *Congress. Numer.*, **29** (1980) 559-571.

[KY] K. M. Koh and K. Y. Yap, Graceful numberings of cycles with a P_3-chord, *Bull. Inst. Math. Acad. Sinica,* **12** (1985) 41-48.

[K1] A. Kotzig, Decompositions of a complete graph into $4k$-gons (in Russian), *Matematický Casopis*, **15** (1965) 229-233.

[K2] A. Kotzig, β-valuations of quadratic graphs with isomorphic components, *Utilitas Math.*, **7** (1975) 263-279.

[K3] A. Kotzig, Decomposition of complete graphs into isomorphic cubes, *J. Comb. Theory, Series B,* **31** (1981) 292-296.

[K4] A. Kotzig, Recent results and open problems in graceful graphs, *Congress. Numer.,* **44** (1984) 197-219.

[KR1] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970) 451-461.

[KR2] A. Kotzig and A. Rosa, Magic valuations of complete graphs, Centre de Recherches Mathematiques, Universite de Montreal, (1972) CRM-175.

[KT] A. Kotzig and J. Turgeon, β-valuations of regular graphs with complete components, *Colloq. Math. Soc. János Bolyai 18, Combinatorics*, Keszthély, Hungary, 1976,

[KCK] D. Kuo, G. Chang, Y.-H. Kwong, Cordial labeling of mK_n, *Discrete Math.*, **169** (1997) 121-131.

[LV] D.R. Lashmi and S. Vangipuram, An α-valuation of quadratic graph $Q(4, 4k)$, *Proc. Nat. Acad. Sci. India* Sec. A, **57** (1987) 576-580.

[LLC] H. Y. Lee, H. M. Lee and G. J. Chang, Cordial labelings of graphs, *Chinese J. Math.,* **20** (1992) 263-273.

[L1] S. M. Lee, k-graceful labelling of Mongolian tents and related graphs, *Congress. Numer.*, **50** (1985) 85-96.

[L2] S. M. Lee, A conjecture on edge-graceful trees, *Scientia*, **3** (1989) 45-47.

[LLWK] S. M. Lee, K. Y. Lai, Y. S. Wang and M. K. Kiang, On the graceful permutation graphs conjecture, Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994), *Congr. Numer.*, **103** (1994), 193–201.

[LeL] S. M. Lee and A. Liu, A construction of cordial graphs from smaller cordial graphs, *Ars Combin.*, **32** (1991) 209-214.

[LNH] S. M. Lee and H. K. Ng, A class of k-graceful bipartite planar graphs, preprint.

[LNK] S. M. Lee and K. C. Ng, Every Young tableau graph is d-graceful, *Abstracts Amer. Math. Soc.,* **5** (1984) 84T-05-266.

[LPC] S. M. Lee, W. M. Pigg, and T. J. Cox, T. J., On edge-magic cubic graphs conjecture, Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994), *Congr. Numer.*, **105** (1994), 214–222.

[LQW] S.M. Lee, L. Quach and S. Wang, On Skolem-gracefulness of graphs which are disjoint union of paths and stars, *Congress. Numer.*, **61** (1988) 59-64.

[LSS] S.M. Lee, E. Schmeichel and S.C. Shee, On felicitous graphs, *Discrete Math.,* **93** (1991) 201-209.

[LSe1] S. M. Lee and E. Seah, On edge-gracefulness of kth power cycles, *Congress. Numer.*, **71** (1990) 237-242.

[LSe2] S. M. Lee and E. Seah, Edge-gracefulness labelings of regular complete K-partite graphs, *Congress. Numer.,* **75** (1990) 41-50.

[LSe3] S. M. Lee and E. Seah, On edge-gracefulness of composition of step graphs and null graphs, *Graph Theory, Combinatorics, Algorthms, and Applications (San Francisco, 1989)*, SIAM (1991) 325-330.

[LSh] S.M. Lee and S.C. Shee, On Skolem-graceful graphs, *Discrete Math.,* **93** (1991) 195-200.

[LSW] S. M. Lee, E. Seah, P.-C. Wang, On edge-gracefulness of kth power graphs, *Bull. Inst. Math. Acad. Sinica,* **18** (1990) 1-11.

[LT] S.M. Lee and S.K. Tan, A class of arbitrarily graceful planar bipartite graphs, *J. Combin. Math. Combin. Comput.,* **9** (1991) 119-127.

[LW1] S. M. Lee and P. Wang, On the k-gracefulness of the sequential join of null graphs, *Congress. Numer.*, **71** (1990) 243-254.

[LW2] S. M. Lee and G. Wang, All pyramids, lotuses and diamonds are d-graceful, unpublished.

[LWK] S. M. Lee, Y. S. Wong and M. K. Kiang, On graceful permutations graphs conjecture, preprint.

[LWu] S. M. Lee and I. Wui, On Skolem-gracefulness of 2-stars and 3-stars, *Bull. Malaysia Math. Soc.*, **10** (1987) 15-20.

[LWY] S. M. Lee, I. Wui and J. Yeh, On the amalgamation of prime graphs, *Bull. Malaysian Math. Soc. (Second Series),* **11** (1988) 59-67.

[KLGY] Q. Kang, Z. Liang, Y. Gao and G. Yang, On labeling of some graphs, *J. Combin. Math. Combin. Comput.*, **22** (1996) 193-210.

[LL] H. X. Liang and C. F. Liu, On k-gracefulness of graphs, *Dongbei Shida Xuebao*, **33** (1991) 41-44.

[Li] Z. Liang, The harmoniousness of book graph $St(4k+1) \times P_2$, *Southeast Asian Bull. Math.*, **21** (1997) 181-184.

[LSX] Z.H. Liang, D.Q. Sun and R.J. Xu, k-graceful labelings of the wheel graph W_{2k}, *J. Hebei Normal College*, **1** (1993) 33-44.

[LiuB1] B. Liu, Some new results on graceful graphs, *Math. Appl.,* **3** (1990) 108-110.

[LiuB2] B. Liu, Sums of squares and labels of graphs, *Math. Practice Theory*, (1994) 25-29.

[LiuR] R.Y. Liu, On Bodendiek's conjecture for graceful graphs, *Chinese Quart. J. Math.,* **4** (1989) 67-73.

[LiuY1] Y. Liu, The gracefulness of the star graph with top sides, *J. Sichuan Normal Univ.*, **18** (1995) 52-60.

[LiuY2] Y. Liu, Proof of a conjecture concerning gracefulness of graphs, *Huaihua Shizhuan Xuebao*, **15** (1996) 13-25.

[LiuY3] Y. Liu, Crowns graphs Q_{2n} are harmonious graphs, *Hunan Annals Math.*, **16** (1996) 125-128.

[LiuY4] Y. Liu, All crowns and helms are harmonious, preprint

[LiuZ] Z. S. Liu, A class of graceful graphs and two classes of graceful graphs derived from it, *Neimenggu Daxue Xuebao,* **16** (1985) 183-186.

[LZ1] B. Liu and X. Zhang, On a conjecture of harmonious graphs, *Systems Science and Math. Sciences*, **4** (1989) 325-328.

[LZ2] B. Liu and X. Zhang, On harmonious labelings of graphs, *Ars Combin.*, **36** (1993) 315-326.

[Lo] S. Lo, On edge-graceful labelings of graphs, *Congress. Numer.*, **50** (1985) 231-241.

[LL] X. Lu and X. F. Li, $P_1 \bigvee T_m$ graphs and a certifiation of its gracefulness, *Gongcheng Shuxue Xuebao*, **13** (1996) 109-113.

[Ma] X. Ma, A graceful numbering of a class of graphs, *J. Math. Res. and Exposition*, (1988) 215-216.

[MF1] K. J. Ma and C. J. Feng, About the Bodendiek's conjecture of graceful graph, *J. Math. Research and Exposition*, **4** (1984) 15-18.

[MF2] K. J. Ma and C. J. Feng, On the gracefulness of gear graphs, *Math. Practice Theory*, (1984) 72–73.

[MLL] X. Ma, Y. Liu and W. Liu, Graceful graphs: cycles with $(t-1)$ chords, *Math. Appl.*, **9** (1990), suppl., 6–8.

[Mah] M. Maheo, Strongly graceful graphs, *Discrete Math.,* **29** (1980) 39-46.

[MT] M. Maheo and H. Thuillier, On d-graceful graphs, *Ars Combin.,* **13** (1982) 181-192.

[MRSS] M. Miller, J. Ryan, Slamin and W. Smyth, Labelling wheels for the minimum sum number, preprint.

[MRS] M. Miller, J. Ryan and W. Smyth, The sum number of the cocktail party graph, preprint.

[Mc] K. McTavish, personal communication.

[MS] J. Mitchem and A. Simoson, On edge-graceful and super-edge-graceful graphs, *Ars Combin.*, **37** (1994) 97-111.

[MP] M. Mollard and C. Payan, Elegant labelings and edge-colorings: A proof of two conjectures of Hartman, and Chang, Hsu, Rogers, *Ars Combin.,* **36** (1993) 97-106.

[MPS] M. Mollard, C. Payan and S. Shixin, Graceful problems, *Seventh Hungarian Colloquium on Finite and Infinite Combinatorics*, Budapest July 1987.

[Mo] D. Moulton, Graceful labelings of triangular snakes, *Ars Combin.*, **28** (1989) 3-13.

[N1] H. K. Ng, Gracefulness of a class of lobsters, *Notices AMS,* **7** (1986) 825-05-294.

[N2] H. K. Ng, α-valuations and k-gracefulness, *Notices of AMS,* **7** (1986) 247.

[PR] A. M. Pastel and H. Raynaud, Les oliviers sont gracieux, in *Colloq. Grenoble*, Publications Université de Grenoble, 1978.

[PP] N. Punnim and N. Pabhapote, On graceful graphs: cycles with a P_k-chord, $k \geq 4$, *Ars Combin.,* **23A** (1987) 225-228.

[Q] J. Qian, On some conjectures and problems in graceful labelings graphs, preprint.

[RP1] I. Rajasingh and P. R. L. Pushpam, Strongly harmonious labeling of helms, preprint.

[RP2] I. Rajasingh and P. R. L. Pushpam, On graceful and harmonious labelings of t copies of $K_{m,n}$ and other special graphs, preprint.

[Ra] J. L. Ramírez-Alfonsín, Graceful labeling of certain graphs, preprint.

[Re] M. Reid, personal communication.

[Ri] G. Ringel, Problem 25, in Theory of Graphs and its Applications, *Proc. Symposium Smolenice 1963*, Prague (1964) 162.

[RL] G. Ringel and A. Llado, and O. Serra, Another tree conjecture, *Bull. Inst. Combin. Appl.*, **18** (1996) 83-85.

[RLS] G. Ringel, A. Llado and O. Serra, Decomposition of complete bipartite graphs into trees, DMAT Research Report 11/96, Univ. Politecnica de Catalunya.

[Rop] D. Ropp, Graceful labelings of cycles and prisms with pendant points, *Congress. Numer.,* **75** (1990) 218-234.

[Ro1] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966)*, Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.

[Ro2] A. Rosa, Cyclic Steiner triple systems and labelings of triangular cacti, *Scientia*, **1** (1988) 87-95.

[RS] A. Rosa and J. Širáň, Bipartite labelings of trees and the gracesize, *J. Graph Theory*, **19** (1995) 201-215.

[Se1] J. Sedláček, Problem 27, *in* Theory of Graphs and its Applications, *Proc. Symposium Smolenice,* June, (1963) 163-167.

[Se2] J. Sedláček, On magic graphs, *Math. Slov.,* **26** (1976) 329-335.

[SAS1] M. Seoud, A.E. I. Abd el Maqsoud and J. Sheehan, Harmonious graphs, *Utilitas Math.*, **47** (1995) 225-233.

[SAS2] M. Seoud, A.E. I. Abd el Maqsoud and J. Sheehan, Gracefulness of the union of cycles and graphs, preprint.

[SW] M. Seoud and R. J. Wilson, Some disgraceful graphs, *Internat. J. Math. Ed. Sci. Tech.,* **24** (1993) 435-441.

[SY1] M. Seoud and M. Youssef, Harmonious labellings of helms and related graphs, preprint.

[SY2] M. Seoud and M. Youssef, On gracefulness of disconnected graphs, preprint.

[SY3] M. Seoud and M. Youssef, On labelling complete tripartite graphs, *Int. J. Math. Ed. Sci. Tech.*, **28** (1997) 367-371.

[SY4] M. Seoud and M. Youssef, The effect of some operations on labelling of graphs, preprint.

[Sha] A. Sharary , Integal sum graphs from complete graphs, cycles and wheels, *Arab Gulf Sci. Res.,* **14-1** (1996) 1-14.

[S1] S. C. Shee, On harmonious and related graphs, *Ars Combin.,* **23** (1987) A, 237-247.

[S2] S.C. Shee, Some results on λ-valuation of graphs involving complete bipartite graphs, *Discrete Math.,* **28** (1991) 73-80.

[SH1] S.C. Shee and Y. S. Ho, The cordiality of one-point union of n-copies of a graph, *Discrete Math.,* **117** (1993) 225-243.

[SH2] S.C. Shee and Y. S. Ho, The cordiality of the path-union of n copies of a graph, *Discrete Math.,* **151** (1996) 221-229.

[SL] S. Shee and S. Lee, On harmonious and felicitious labeling of graphs, *Congress. Numer.,* **68** (1989) 155-170.

[Sh] D. A. Sheppard, The factorial representation of major balanced labelled graphs, *Discrete Math.,* **15** (1976) 379-388.

[SY] S. Shixin and S. Yu, On Hamming-graceful graphs, preprint.

[Si] G. J. Simmons, Synch-sets: a variant of difference sets, *Proc. 5th Southeastern Conference on Combinatorics, Graph Theory and Computing*, Utilitas Math. Pub. Co., Winnipeg (1974) 625-645.

[Sin1] G. S. Singh, A note on graceful prisms, *Nat. Acad. Sci. Lett.,* **15** (1992) 193-194.

[Sin2] G. S. Singh, A note on sequential crowns, *Nat. Acad. Sci. Lett.,* **16** (1993) 243-245.

[Sin3] G. S. Singh, Subdivisions of ladders are arithmetic, *Multidiscplinary Research Review,* **2** (1992) 23-25.

[Sin4] G. S. Singh, A note on labeling of graphs, *Graphs and Combin.*, (1997), to appear.

[Sin5] G. S. Singh, private communication.

[Sk] T. Skolem, On certain distribution of integers into pairs with given differences, *Math. Scand.*, **5** (1957) 57-68.

[Sl2] P. J. Slater, On k-graceful graphs, *Proc. of the 13th S.E. Conf. on Combinatorics, Graph Theory and Computing,* (1982) 53-57.

[Sl3] P. J. Slater, On k-graceful, locally finite graphs, *J. Comb. Theory, Series B,* **35** (1983) 319-322.

[Sl4] P. J. Slater, Problems in graph theory: graceful and sequential numbering of infinite graphs, *Southeast Asian Bull. Math.,* **9** (1985) 15-22.

[Sl5] P. J. Slater, On k-graceful, countably infinite graphs, *Discrete Math.,* **61** (1986) 293-303.

[Sn1] H. Snevily, Combinatorics of Finite Sets, Ph.D. Thesis, U. Illinois, 1991.

[Sn2] H. Snevily, New families of graphs that have α-labelings, *Discrete Math.*, **170** (1997) 185-194.

[SZ] R. Stanton and C. Zarnke, Labeling of balanced trees, *Proc. 4th Southeast Conf. Comb., Graph Theory, Computing* (1973) 479-495.

[St] B. M. Stewart, Magic graphs, *Canadian J. Math.,* **18** (1966) 1031-1059.

[St2] B. M. Stewart, Supermagic complete graphs, *Canadian J. Math.,* **19** (1967) 427-438.

[SW] D.Q. Sun and D.Y. Wang, private communication.

[Su] R. G. Sun, Harmonious and sequential labelings of the book graphs B_m, *Gaoxiao Yingyong Shuxue Xuebao Ser. A*, **9** (1994) 335–337.

[Sz] Z. Szaniszló, k-equitable labellings of cycles and some other graphs, *Ars Combin.,* **37** (1994) 49-63.

[TTW] R. Tao, J. Tong and C. Wang, On k-cordiality of cycles, crowns and wheels, preprint.

[T] M. Truszczyński, Graceful unicyclic graphs, *Demonstatio Mathematica,* **17** (1984) 377-387.

[WJLZ] J.-G. Wang, D. J. Jin, X.-G Lu, and D. Zhang, The gracefulness of a class of lobster trees, *Math. Comput. Modelling*, **20** (1994) 105-110.

[Wi] Steven Winters, personal communication.

[W1] J. Wojciechowski, Long Induced Cycles in the Hypercube and Colourings of Graphs, Ph.D. Thesis, Cambridge University, England, 1990.

[W2] J. Wojciechowski, Equitable labelings of cycles, *J. Graph Theory*, **17** (1993) 531-547.

[XL] L. T. Xie and G. Z. Liu, A survey of the problem of graceful trees, *Qufu Shiyuan Xuebao,* (1984) 8-15.

[X1] S. D. Xu, Harmonicity of triangular snakes, personal communication.

[X1] S. D. Xu, Cycles with a chord are harmonious, *Mathematica Applicata*, **8** (1995) 31–37. *J. Math. Res. Exposition*, **15** (1995) 475–476.

[YL1] W. Yan and B. Liu, Some results on integral sum graphs, preprint.

[YL2] W. Yan and B. Liu, The sum number and integral sum number of complete bipartite graphs, preprint.

[YW1] Y. C. Yang and X.G. Wang, On the gracefulness of the union of two stars and three stars, *Combinatorics, Graph Theory, Algorithms and Applications (Beijing, 1993)*, 417–424, World Sci. Publishing, River Edge, NJ, 1994 33

[YW2] Y. C. Yang and X.G. Wang, On the gracefulness of product graph $C_{4n+2} \times P_{4m+3}$, *Combinatorics, Graph Theory, Algorithms and Applications (Beijing, 1993)*, 425–431, World Sci. Publishing, River Edge, NJ, 1994.

[Y] V. Yegnanarayanan, On some additive analogues of graceful theme: cycle related graphs, *Southeast Asian Bulletin of Math.*, to appear.

[YC] R. Yilmaz and I. Cahit, E-cordial graphs, *Ars Combin.,* **46** (1997) 251-266.

[Yo] M. Z. Youssef, personal communication.

[Yu] P. Yu, A proof of a conjecture about arithmetic graphs, *J. Math. Res. Exposition,* **16** (1996) 594-598.

[YZ] J. Yuan and W. Zhu, Some results on harmonious labelings of graphs, preprint.

[Zha] S. L. Zhao, All trees of diameter four are graceful, *Graph Theory and its Applications: East and West (Jinan, 1986)*, 700–706, Ann. New York Acad. Sci., 576, New York Acad. Sci., New York, 1989.

[Zh] S. C. Zhou, Gracefulness of the graph $K_m \cup K_n$, *J. Lanzhou Railway Inst.*, **12** (1993) 70–72.