A Dynamic Survey of Graph Labeling

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Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the late 1960s. In the intervening years dozens of graph labelings techniques have been studied in over 1700 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

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1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [1209] in 1967, or one given by Graham and Sloane [586] in 1980. Rosa [1209] called a function f a *β-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [576] subsequently called such labelings *graceful* and this is now the popular term. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [1197] that K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. Although an unpublished result of Erdős says that most graphs are not graceful (see [586]), most graphs that have some sort of regularity of structure are graceful. Sheppard [1332] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Rosa [1209] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges," (2) G "has too many edges," and (3) G "has the wrong parity." The disjoint union of trees is a case where there are too many vertices for the number of edges. An infinite class of graphs that are not graceful for the second reason is given in [278]. As an example of the third condition Rosa [1209] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Acharya [12] proved that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph. Acharya, Rao, and Arumugam [30] proved: every triangle-free graph can be embedded as an induced subgraph of a triangle-free graceful graph; every planar graph can be embedded as an induced subgraph of a planar graceful graph; and every tree can be embedded as an induced subgraph of a graceful tree. These results demonstrate that there is no forbidden subgraph characterization of these particular kinds of graceful graphs.

Harmonious graphs naturally arose in the study by Graham and Sloane [586] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. They proved that almost all graphs are not harmonious. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [970] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \ldots, d_p$ then $\gcd(d_1, d_2, \ldots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph. More generally, Sethuraman and Elumalai [1302] have shown that any given set of graphs $G_1, G_2, \ldots, G_t$ can be embedded in a graceful or harmonious graph. Determining whether a graph has a harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001 (see [833]).

In the early 1980s Bloom and Hsu [287], [288],[268] extended graceful labelings to directed graphs by defining a graceful labeling on a directed graph $D(V, E)$ as a one-to-one map θ from V to $\{0, 1, 2, \ldots, |E|\}$ such that $\theta(y) - \theta(x) \bmod (|E| + 1)$ is distinct for every edge xy in E.

Graceful labelings of directed graphs also arose in the characterization of finite neofields by Hsu and Keedwell [667], [668]. Graceful labelings of directed graphs was the subject of Marr's 2007 Ph.D. dissertation [1027]. In [1027] and [1028] Marr presents results of graceful labelings of directed paths, stars, wheels, and umbrellas.

Over the past four decades in excess of 1700 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes of graphs have been done by several authors and in some cases the same terminology is used for different concepts. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two labeling methods, include [262], [283], [799], [530], and [532]. The book edited by Acharya, Arumugam, and Rosa [17] includes a variety of labeling methods that we do not discuss in this survey. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings, and neofields is discussed in [287] and [288]. The connection between graceful labelings and perfect systems of difference sets is given in [265]. Bloom and Hsu [289] extended the notion of graceful labeling to directed graphs. (See also [340]). Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, secret sharing schemes,and models for constraint programming over finite domains–see [284], [285], [1470], [1167], [1405], [1404], [99], [1392] and [1045] for details. Terms and notation not defined below follow that used in [363] and [530].

2 Graceful and Harmonious Labelings

2.1 Trees

The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Kotzig [671] has called the effort to prove it a "disease." Among the trees known to be graceful are: caterpillars [1209] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [671], [1720] and [745]; trees with diameter at most 5 [1720] and [664]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [266], [1141]; rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity [339]; rooted trees with diameter D where every vertex has even degree except for one root and the leaves in level $\lfloor D/2 \rfloor$ [211]; rooted trees with diameter D where every vertex has even degree except for one root and the leaves, which are in level $\lfloor D/2 \rfloor$ [211]; rooted trees with diameter D where every vertex has even degree except for one root, the vertices in level $\lfloor D/2 \rfloor - 1$, and the leaves which are in level $\lfloor D/2 \rfloor$ [211]; the graph obtained by identifying the endpoints any number of paths of a fixed length except for the case that the length has the form $4r+1$, $r > 1$ and the number of paths is of the form $4m$ with $m > r$ [1249]; regular bamboo trees [1249] (a rooted tree consisting of branches of equal length the endpoints of which are identified with end points of stars of equal size); and olive trees [1125], [3] (a rooted tree consisting of k branches, where the ith branch is a path of length i); Bahls, Lake, and Wertheim [201] proved that spiders for which the lengths of every path from the center to a leaf differ by at most one are graceful. (A *spider* is a tree that has at most one vertex (called the *center*) of degree greater than 2.) Motivated by Horton's work [662], in 2010 Fang [484] used a deterministic back-tracking algorithm to prove that all trees with at most 35 vertices are graceful. In 2011 Fang [485] used a hybrid algorithm that involved probabilistic backtracking, tabu searching, and constraint programming satisfaction to verify that every tree with at most 31 vertices is harmonious.

Aldred, Širáň and Širáň [67] have proved that the number of graceful labelings of P_n grows at least as fast as $(5/3)^n$. They mention that this fact has an application to topological graph theory. One such application was provided by Goddyn, Richter, and and Širáň [574] who used graceful labelings of paths on $2s+1$ vertices ($s \geq 2$) to obtain 2^{2s} cyclic oriented triangular embeddings of the complete graph on $12s+7$ vertices. The Aldred, Širáň and Širáň bound was improved by Adamaszek [36] to $(2.37)^n$ with the aid of a computer. Cattell [350] has shown that when finding a graceful labeling of a path one has almost complete freedom to choose a particular label i for any given vertex v. In particular, he shows that the only cases of P_n when this cannot be done are when $n \equiv 3 \pmod 4$ or $n \equiv 1 \pmod{12}$, v is in the smaller of the two partite sets of vertices, and $i = (n-1)/2$. Pradhan and Kumar [1165] proved that all combs $P_n \odot K_1$ with perfect matching are graceful.

In [475] and [476] Eshghi and Azimi [475] discuss a programming model for finding graceful labelings of large graphs. The computational results show that the models can easily solve the graceful labeling problems for large graphs. They used this method to verify that all trees with 30, 35, or 40 vertices are graceful. Stanton and Zarnke [1435] and Koh, Rogers, and Tan [800], [801], [804] gave methods for combining graceful trees to yield larger graceful trees. Rogers in [1206] and Koh, Tan, and Rogers in [803] provide recursive constructions to create graceful trees. Burzio and Ferrarese [326] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. In 1979 Bermond [262] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a caterpillar). Morgan [1070] has shown that all lobsters with perfect matchings are graceful. Ghosh [569] used adjacency matrices to prove that three classes of lobsters are graceful.

A *Skolem sequence* of order n is a sequence $s_1, s_2, \ldots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. A Skolem sequence of order n exists if and only if $n \equiv 0$ or 1 (mod 4). Morgan [1071] has used Skolem sequences to construct classes of graceful trees. Morgan and Rees [1072] used Skolem and Hooked-Skolem sequences to generate classes of graceful lobsters.

Mishra and Panigrahi [1063] and [1117] found classes of graceful lobsters of diameter at least five. They show other classes of lobsters are graceful in [1064] and [1065]. In [1305] Sethuraman and Jesintha [1305] explores how one can generate graceful lobsters from a graceful caterpillar while in [1309] and [1310] (see also [711]) they show how to generate graceful trees from a graceful star. More special cases of Bermond's conjecture have been done by Ng [1093], by Wang, Jin, Lu, and Zhang [1607], Abhyanker [2], and by Mishra and Panigrahi [1064]. Renuka, Balaganesan, Selvaraju [1196] proved spider trees with n legs of even length t and odd $n \geq 3$ and lobsters for which each vertex of the spine is adjacent to a path of length two are harmonious.

Barrientos [230] defines a *y-tree* as a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point. He proves that graphs obtained from a y-tree T by replacing every edge e_i of T by a copy of K_{2,n_i} in such a way that the ends of e_i are merged with the two independent vertices of K_{2,n_i} after removing the edge e_i from T are graceful.

Sethuraman and Jesintha [1306], [1307] and [1308] (see also [711]) proved that rooted trees obtained by identifying one of the end vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. They also proved that rooted trees obtained by identifying either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. In [1306], [1307], and [1308] (see also [711] and [712]) Sethuraman and Jesintha prove that all rooted trees in which every level contains pendent vertices and the degrees of the internal vertices in the same level are equal are graceful. Kanetkar and Sane [761] show that trees formed by identifying one end vertex of each of six or fewer paths whose lengths determine an arithmetic progression are graceful.

Chen, Lü, and Yeh [371] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Before Sethuraman and Jesintha [1312] and [1311] (see also [711]) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful, various kinds of bananas trees had been shown to be graceful by Bhat-Nayak and Deshmukh [273], by Murugan and Arumugam [1086], [1084] and by Vilfred [1584].

Consider a set of caterpillars, having equal diameter, in which one of the penultimate vertices has arbitrary degree and all the other internal vertices including the other penultimate vertex is of fixed even degree. Jesintha and Sethuraman [714] call the rooted tree obtained by merging an end-vertex adjacent to the penultimate vertex of fixed even degree of each caterpillar a *arbitrarily fixed generalized banana tree*. They prove that such trees are graceful. From this it follows that all banana trees are graceful and all generalized banana trees are graceful.

Zhenbin [1722] has shown that graphs obtained by starting with any number of identical stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. He also shows that graphs obtained by starting with any two stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. In [713] Jesintha and Sethuraman use a method of Hrnciar and Havier [664] to generate graceful trees from a graceful star with n edges.

Aldred and McKay [65] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [586]. In a paper published in 2004 Krishnaa [830] claims to proved that all trees have both graceful and harmonious labelings. However, her proofs were flawed.

Vietri [1578] utilized a counting technique that generalizes Rosa's graceful parity condition and provides contraints on possible graceful labelings of certain classes of trees. He expresses doubts about the validity of the graceful tree conjecture.

Using a variant of the Matrix Tree Theorem, Whitty [1643] specifies an $n \times n$ matrix of indeterminates whose determinant is a multivariate polynomial that enumerates the gracefully labelled $(n+1)$-vertex trees. Whitty also gives a bijection between gracefully labelled graphs and rook placements on a chessboard on the Möbius strip.

In [308] Brankovic and Wanless describe applications of graceful and graceful-like labelings of trees to several well known combinatorial problems including complete graph decompositions, the Oberwolfach problem, and coloring. They also discuss the connection between α-labeling of paths and near transversals in Latin squares and show how spectral graph theory might be used to further the progress on the graceful tree conjecture.

Graceful labelings of trees have been used in multi protocol label switching (MPLS) routing platforms in IP networks [99].

Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. More specialized results about trees are contained in [262], [283], [799], [1006], [333], [744], and [1210]. In [451] Edwards and Howard provide a lengthy survey paper on graceful trees. Alfalayleh, Brankovic, and Giggins [66] survey results related to the graceful tree conjecture as of 2004 and conclude with five open problems.

2.2 Cycle-Related Graphs

Cycle-related graphs have been a major focus of attention. Rosa [1209] showed that the n-cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4) and Graham and Sloane [586] proved that C_n is harmonious if and only if n is odd. Wheels $W_n = C_n + K_1$ are both graceful and harmonious – [515], [660] and [586]. As a consequence we have that a subgraph of a graceful (harmonious) graph need not be graceful (harmonious). The *n-cone* (also called the *n-point suspension* of C_m) $C_m + \overline{K_n}$ has been shown to be graceful when $m \equiv 0$ or 3 (mod 12) by Bhat-Nayak and Selvam [279]. When n is even and m is 2, 6 or 10 (mod 12) $C_m + \overline{K_n}$ violates the parity condition for a graceful graph. Bhat-Nayak and Selvam [279] also prove that the following cones are graceful: $C_4 + \overline{K_n}, C_5 + \overline{K_2}, C_7 + \overline{K_n}, C_9 + \overline{K_2}, C_{11} + \overline{K_n}$ and $C_{19} + \overline{K_n}$. The *helm* H_n is the graph obtained from a wheel by attaching a pendent edge at each vertex of the n-cycle. Helms have been shown to be graceful [108] and harmonious [572], [981], [982] (see also [970], [1294], [968], [418] and [1181]). Koh, Rogers, Teo, and Yap, [802] define a *web* graph as one obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. They asked whether such graphs are graceful. This was proved by Kang, Liang, Gao, and Yang [764]. Yang has extended the notion of a web by iterating the process of adding pendent points and joining them to form a cycle and then adding pendent points to the new cycle. In his notation, $W(2,n)$ is the web graph whereas $W(t,n)$ is the *generalized web* with t n-cycles. Yang has shown that $W(3,n)$ and $W(4,n)$ are graceful (see [764]), Abhyanker and Bhat-Nayak [4] have done $W(5,n)$ and Abhyanker [2] has done $W(t,5)$ for $5 \leq t \leq 13$. Gnanajothi [572] has shown that webs with odd cycles are harmonious. Seoud and Youssef [1294] define a *closed helm* as the graph obtained from a helm by joining each pendent vertex to form a cycle and a *flower* as the graph obtained from a helm by joining each pendent vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A *gear graph* is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n-cycle. In 1984 Ma and Feng [1009] proved all gears are graceful while in a Master's thesis in 2006 Chen [372] proved all gears are harmonious. Liu [981] has shown that if two or more vertices are inserted between every pair of vertices of the n-cycle of the wheel W_n, the resulting graph is graceful. Liu [979] has also proved that the graph obtained from a gear graph by attaching one or more pendent edges to each vertex between the vertices of the n-cycle is graceful. Pradhan and Kumar [1165] proved that graphs obtained by adding a pendent edge to each pendent vertex of hairy cycle $C_n \odot K_1$ are graceful if $n \equiv 0(\text{mod} 4m)$. They further provide a rule for determining the missing numbers in the graceful labeling of $C_n \odot K_1$ and of the graph obtained by adding pendent edges to each pendent vertex of $C_n \odot K_1$.

Abhyanker [2] has investigated various unicyclic (that is, graphs with exactly one cycle) graphs. He proved that the unicyclic graphs obtained by identifying one vertex of C_4 with the root of the olive tree with $2n$ branches and identifying an adjacent vertex on C_4 with the end point of the path P_{2n-2} are graceful. He showed that if one attaches any number of pendent edges to these unicyclic graphs at the vertex of C_4 that is adjacent to the root of the olive tree but not adjacent to the end vertex of the attached path, the resulting graphs are graceful. Likewise, Abhyanker proved that the graph obtained by deleting the branch of length 1 from an olive tree with $2n$ branches and identifying the root of the edge deleted tree with a vertex of a cycle of the form C_{2n+3} is graceful. He also has a number of results similar to these.

Delorme, Maheo, Thuillier, Koh, and Teo [421] and Ma and Feng [1008] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher, and Wegner [292], who proved various special cases. In 1985 Koh and Yap [805] generalized this by defining a *cycle with a P_k-chord* to be a cycle with the path P_k joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when $k = 3$ and conjectured that all cycles with a P_k-chord are graceful. This was proved for $k \geq 4$ by Punnim and Pabhapote in 1987 [1168]. Chen [377] obtained the same result except for three cases which were then handled by Gao [601]. In 2005, Sethuraman and Elumalai [1301] defined a *cycle with parallel P_k-chords* as a graph obtained from a cycle C_n $(n \geq 6)$ with consecutive vertices $v_0, v_1, \ldots, v_{n-1}$ by adding disjoint paths P_k, $(k \geq 3)$, between each pair of nonadjacent vertices $v_1, v_{n-1}, v_2, v_{n-2}, \ldots, v_i, v_{n-i}, \ldots, v_\alpha, v_\beta$ where $\alpha = \lfloor n/2 \rfloor - 1$ and $\beta = \lfloor n/2 \rfloor + 2$ if n is odd or $\beta = \lfloor n/2 \rfloor + 1$ if n is even. They proved that every cycle C_n $(n \geq 6)$ with parallel P_k-chords is graceful for $k = 3, 4, 6, 8$, and 10 and they conjecture that the cycle C_n with parallel P_k-chords is graceful for all even k. Xu [1662] proved that all cycles with a chord are harmonious except for C_6 in the case where the distance in C_6 between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords has also been investigated. For $3 \leq p \leq n - r$, let $C_n(p, r)$ denote the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the r chords $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$ have been added. Koh and Punnin [795] and Koh, Rogers, Teo, and Yap [802] have handled the cases $r = 2, 3$ and $n-3$ where n is the length of the cycle. Goh and Lim [575] then proved that all remaining cases are graceful. Moreover, Ma [1011] has shown that $C_n(p, n-p)$ is graceful when $p \equiv 0, 3 \pmod 4$ and Ma, Liu, and Liu [1012] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph $C_n(3, n-3)$ any number k_i of paths of length 2 from the vertex v_1 to the vertex v_i for $i = 2, \ldots, n$, the resulting graph is graceful. Chen [377] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. This generalizes the result that a cycle plus a chord is graceful. Liu [978] has shown that the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords v_1v_k and v_1v_{k+2} $(2 \le k \le n-3)$ are adjoined is graceful.

In [419] Deb and Limaye use the notation $C(n, k)$ to denote the cycle C_n with k cords sharing a common endpoint called the *apex*. For certain choices of n and k there is a unique $C(n, k)$ graph and for other choices there is more than one graph possible. They call these *shell-type* graphs and they call the unique graph $C(n, n-3)$ a *shell*. Notice that the shell $C(n, n-3)$ is the same as the fan $F_{n-1} = P_{n-1} + K_1$. Deb and Limaye define a *multiple shell* to be a collection of edge disjoint shells that have their apex in common. A multiple shell is said to be *balanced* with width w if every shell has order w or every shell has order w or $w + 1$. Deb and Limaye [419] have conjectured that all multiple shells are harmonious, and have shown that the conjecture is true for the balanced double shells and balanced triple shells. Yang, Xu, Xi, and Qiao [1685] proved the conjecture is true for balanced quadruple shells. Liang [953] proved the conjecture is true when each shell has the same order and the number of copies is odd.

Sethuraman and Dhavamani [1298] use $H(n, t)$ to denote the graph obtained from the cycle C_n by adding t consecutive chords incident with a common vertex. If the common vertex is u and v is adjacent to u, then for $k \ge 1$, $n \ge 4$, and $1 \le t \le n-3$, Sethuraman and Dhavamani denote by $G(n, t, k)$ the graph obtained by taking the union of k copies of $H(n, k)$ with the edge uv identified. They conjecture that every graph $G(n, t, k)$ is graceful. They prove the conjecture for the case that $t = n-3$.

For $i = 1, 2, \ldots, n$ let $v_{i,1}, v_{i,2}, \ldots, v_{i,2m}$ be the successive vertices of n copies of C_{2m}. Sekar [1249] defines a *chain of cycles* $C_{2m,n}$ as the graph obtained by identifying $v_{i,m}$ and $v_{i+1,m}$ for $i = 1, 2, \ldots, n-1$. He proves that $C_{6,2k}$ and $C_{8,n}$ are graceful for all k and all n. Barrientos [233] proved that all $C_{8,n}$, $C_{12,n}$, and $C_{6,2k}$ are graceful.

Truszczyński [1500] studied unicyclic graphs and proved several classes of such graphs are graceful. Among these are what he calls dragons. A *dragon* is formed by joining the end point of a path to a cycle (Koh, et al. [802] call these *tadpoles*; Kim and Park [790] call them *kites*). This work led Truszczyński to conjecture that all unicyclic graphs except C_n, where $n \equiv 1$ or $2 \pmod 4$, are graceful. Guo [600] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 (mod 4). Lu [1005] uses $C_n^{+(m,t)}$ to denote the graph obtained by identifying one vertex of C_n with one endpoint of m paths each of length t. He proves that $C_n^{+(1,t)}$ (a tadpole) is not harmonious when $a + t$ is odd and $C_n^{+(2m,t)}$ is harmonious when $n = 3$ and when $n = 2k+1$ and $t = k-1, k+1$ or $2k-1$. In his Master's thesis, Doma [437] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Because of the immense diversity of unicyclic graphs, a proof of Truszczyński's conjecture seems out of reach in the near future.

Cycles that share a common edge or a vertex have received some attention. Murugan and Arumugan [1085] have shown that books with n pentagonal pages (i.e., n copies of C_5 with an edge in common) are graceful when n is even and not graceful when n is odd. Lu [1005] uses

$\Theta(C_m)^n$ to denote the graph made from n copies of C_m that share an edge (an n page book with m-polygonal pages). He proves $\Theta(C_{2m+1})^{2n+1}$ is harmonious for all m and n; $\Theta(C_{4m+2})^{4n+1}$ and $\Theta(C_{4m})^{4n+3}$ are not harmonious for all m and n. Xu [1662] proved that $\Theta(C_m)^2$ is harmonious except when $m = 3$. ($\Theta(C_m)^2$ is isomorphic to $C_{2(m-1)}$ with a chord "in the middle.")

A *kayak paddle* $KP(k, m, l)$ is the graph obtained by joining C_k and C_m by a path of length l. Litersky [966] proves that kayak paddles have graceful labelings in the following cases: $k \equiv 0$ mod 4, $m \equiv 0$ or 3 (mod 4); $k \equiv m \equiv 2$ (mod 4) for $k \geq 3$; and $k \equiv 1$ (mod 4), $m \equiv 3$ (mod 4). She conjectures that $KP(4k+4, 4m+2, l)$ with $2k < m$ is graceful when $l \leq 2m$ if l is even and when $l \leq 2m+1$ if l is odd; and $KP(10, 10, l)$ is graceful when $l \geq 12$. The cases are open: $KP(4k, 4m+1, l); KP(4k, 4m+2, l); KP(4k+1, 4m+1, l); KP(4k+1, 4m+2, l); KP(4k+2, 4m+3, l); KP(4k+3, 4m+3, l)$.

Let $C_n^{(t)}$ denote the one-point union of t cycles of length n. Bermond, Brouwer, and Germa [263] and Bermond, Kotzig, and Turgeon [265]) proved that $C_3^{(t)}$ (that is, the *friendship graph* or *Dutch t-windmill*) is graceful if and only if $t \equiv 0$ or 1 (mod 4) while Graham and Sloane [586] proved $C_3^{(t)}$ is harmonious if and only if $t \not\equiv 2$ (mod 4). Koh, Rogers, Lee, and Toh [796] conjecture that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0$ or 3 (mod 4). Yang and Lin [1677] have proved the conjecture for the case $n = 5$ and Yang, Xu, Xi, Li, and Haque [1683] did the case $n = 7$. Xu, Yang, Li and Xi [1666] did the case $n = 11$. Xu, Yang, Han and Li [1667] did the case $n = 13$. Qian [1173] verifies this conjecture for the case that $t = 2$ and n is even and Yang, Xu, Xi, and Li [1684] did the case $n = 9$. Figueroa-Centeno, Ichishima, and Muntaner-Batle [498] have shown that if $m \equiv 0$ (mod 4) then the one-point union of 2, 3, or 4 copies of C_m admits a special kind of graceful labeling called an α-labeling (see Section 3.1) and if $m \equiv 2$ (mod 4), then the one-point union of 2 or 4 copies of C_m admits an α-labeling. Bodendiek, Schumacher, and Wegner [298] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [1328] has proved that $C_4^{(t)}$ is graceful for all t. Seoud and Youssef [1292] have shown that the one-point union of a triangle and C_n is harmonious if and only if $n \equiv 1$ (mod 4) and that if the one-point union of two cycles is harmonious then the number of edges is divisible by 4. The question of whether this latter condition is sufficient is open. Figueroa-Centeno, Ichishima, and Muntaner-Batle [498] have shown that if G is harmonious then the one-point union of an odd number of copies of G using the vertex labeled 0 as the shared point is harmonious. Sethuraman and Selvaraju [1320] have shown that for a variety of choices of points, the one-point union of any number of non-isomorphic complete bipartite graphs is graceful. They raise the question of whether this is true for all choices of the common point.

Another class of cycle-related graphs is that of triangular cacti. The *block-cutpoint graph* of a graph G is a bipartite graph in which one partite set consists of the cut vertices of G, and the other has a vertex b_i for each block B_i of G. A *block* of a graph is a maximal connected subgraph that has no cut-vertex. A *triangular cactus* is a connected graph all of whose blocks are triangles. A *triangular snake* is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$). Rosa [1211] conjectured that all triangular cacti with $t \equiv 0$ or 1 (mod 4) blocks are graceful. (The cases where $t \equiv 2$ or 3 (mod 4) fail to be graceful because of the parity condition.) Moulton [1078] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular cacti) seems hopelessly difficult. Liu and Zhang [970] gave an incorrect proof that triangular snakes with an odd number of triangles are harmonious whereas triangular snakes with $n \equiv 2 \pmod 4$ triangles are not harmonious. Xu [1663] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to 2 (mod 4).

A *double triangular snake* consists of two triangular snakes that have a common path. That is, a double triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$ and to a new vertex u_i for $i = 1, 2, \ldots, n-1$. Xi, Yang, and Wang [1659] proved that all double triangular snakes are harmonious.

For any graph G defining G-snake analogous to triangular snakes, Sekar [1249] has shown that C_n-snakes are graceful when $n \equiv 0 \pmod 4$ $(n \geq 8)$ and when $n \equiv 2 \pmod 4$ and the number of C_n is even. Gnanajothi [572, pp. 31-34] had earlier shown that quadrilateral snakes are graceful. Grace [584] has proved that K_4-snakes are harmonious. Rosa [1211] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases. Yu, Lee, and Chin [1708] showed that Q_2-and Q_3-snakes are graceful and, when the number of blocks is greater than 1, Q_2-, Q_3- and Q_4-snakes are harmonious.

Barrientos [224] calls a graph a *kC_n-snake* if it is a connected graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to C_n. (When $n > 3$ and $k > 3$ there is more than one kC_n-snake.) If a kC_n-snake where the path of minimum length that contains all the cut-vertices of the graph has the property that the distance between any two consecutive cut-vertices is $\lfloor n/2 \rfloor$ it is called *linear*. Barrientos proves that kC_4-snakes are graceful and that the linear kC_6-snakes are graceful when k is even. He further proves that kC_8-snakes and kC_{12}-snakes are graceful in the cases where the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph are all even and that certain cases of kC_{4n}-snakes and kC_{5n}-snakes are graceful (depending on the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph).

Several people have studied cycles with pendent edges attached. Frucht [515] proved that any cycle with a pendent edge attached at each vertex (i.e., a *crown*) is graceful (see also [669]). If G has order n, the *corona of G with H, $G \odot H$* is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H. Barrientos [229] also proved: if G is a graceful graph of order m and size $m-1$, then $G \odot nK_1$ and $G + nK_1$ are graceful; if G is a graceful graph of order p and size q with $q > p$, then $(G \cup (q+1-p)K_1) \odot nK_1$ is graceful; and all unicyclic graphs, other than a cycle, for which the deletion of any edge from the cycle results in a caterpillar are graceful.

For a given cycle C_n with $n \equiv 0$ or $3 \pmod 4$ and a family of trees $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, let u_i and $v_i, 1 \leq i \leq n$, be fixed vertices of C_n and T_i, respectively. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [503] provide two construction methods that generate a graceful labeling of the unicyclic graphs obtained from C_n and $\mathcal{T}$ by amalgamating them at each u_i and v_i. Their results encompass all previously known results for unicyclic graphs whose cycle length is 0 or 3 (mod 4) and considerably extend the known classes of graceful unicyclic graphs.

In [226] Barrientos proved that helms (graphs obtained from a wheel by attaching one pendent edge to each vertex) are graceful. Grace [583] showed that an odd cycle with one or more pendent edges at each vertex is harmonious and conjectured that $C_{2n} \odot K_1$, an even cycle with one pendent edge attached at each vertex, is harmonious. This conjecture has been proved by Liu and Zhang [969], Liu [981] and [982], Hegde [630], Huang [670], and Bu [314]. Sekar [1249] has shown that the graph $C_m \odot P_n$ obtained by attaching the path P_n to each vertex of C_m is graceful. For any $n \geq 3$ and any t with $1 \leq t \leq n$, let C_n^{+t} denote the class of graphs formed by adding a single pendent edge to t vertices of a cycle of length n. Ropp [1208] proved that for every n and t the class C_n^{+t} contains a graceful graph. Gallian and Ropp [530] conjectured that for all n and t, all members of C_n^{+t} are graceful. This was proved by Qian [1173] and by Kang, Liang, Gao, and Yang [764]. Of course, such graphs are just a special case of the aforementioned conjecture of Truszczyński that all unicyclic graphs except C_n for $n \equiv 1$ or 2 (mod 4) are graceful. Sekar [1249] proved that the graph obtained by identifying an endpoint of a star with a vertex of a cycle is graceful. Lu [1005] shows that the graph obtained by identifying each vertex of an odd cycle with a vertex disjoint copy of C_{2m+1} is harmonious if and only if m is odd.

For given cycle C_n with $n \equiv 0$ or 3 (mod 4) and a family of trees $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, let u_i and $v_i, 1 \leq i \leq n$, be fixed vertices of C_n and T_i, respectively. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [503] provide two construction methods that generate a graceful labeling of the unicyclic graphs obtained from C_n and $\mathcal{T}$ by amalgamating them at each u_i and v_i. Their results encompass all previously known results for unicyclic graphs whose cycle length is 0 or 3 (mod 4) and considerably extend the known classes of graceful unicyclic graphs.

Solairaju and Chithra [1414] defined three classes of graphs obtained by connecting copies of C_4 in various ways. Denote the four consecutive vertices of ith copy of C_4 by $v_{i,1}, v_{i,2}, v_{i,3}, v_{i_4}$. They show that the graphs obtained by identifying $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ is graceful; the graphs obtained by joining $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ by an edge is graceful; and the graphs obtained by joining $v_{i,4}$ with $v_{i+1,2}$ for $i = 1, 2, \ldots, n-1$ with a path of length 2 is graceful.

In a paper published in 1985, Bloom and Hsu [289] say a directed graph D with e edges has a graceful labeling θ if for each vertex v there is a vertex labeling θ that assigns each vertex a distinct integer from 0 to e such that for each directed edge (u, v) the integers $\theta(v) - \theta(u)$ mod $(e + 1)$ are distinct and nonzero . They conjectured that digraphs whose underlying graphs are wheels and that have all directed edges joining the hub and the rim in the same direction and all directed edges in the same direction are graceful. This conjecture was proved in 2009 by Hegde and Shivarajkumarn [650].

Yao, Yao, and Cheng [1688] investigated the gracefulness for many orientations of undirected trees with short diameters and proved some directed trees do not have graceful labelings.

2.3 Product Related Graphs

Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids, $P_m \times P_n$ $(m, n \geq 2)$, are graceful was proved by Acharya and Gill [24] in 1978 although the much simpler labeling scheme given by Maheo [1018] in 1980 for $P_m \times P_2$ readily extends to all grids. Liu, T. Zou, Y. Lu [976] proved $P_m \times P_n \times P_2$ is graceful. In 1980 Graham and Sloane [586] proved ladders, $P_m \times P_2$, are harmonious when $m > 2$ and in 1992 Jungreis and Reid [756] showed that the grids $P_m \times P_n$ are harmonious when $(m, n) \neq (2, 2)$. A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [768] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful [770]. Acharya [15] has shown that certain subgraphs of grid graphs are graceful. Lee [863] defines a *Mongolian tent* as a graph obtained from $P_m \times P_n$, n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex. A *Mongolian village* is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A *Young tableau* is a subgraph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and Ng [885] have proved that all Young tableaus are graceful. Lee [863] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful. In [1413] and [1412] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are graceful.

Prisms are graphs of the form $C_m \times P_n$. These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher, and Wegner [292] proved that $C_m \times P_2$ is graceful when $m \equiv 0 \pmod 4$. According to the survey by Bermond [262], Gangopadhyay and Rao Hebbare did the case that m is even about the same time. In a 1979 paper, Frucht [515] stated without proof that he had done all $C_m \times P_2$. A complete proof of all cases and some related results were given by Frucht and Gallian [518] in 1988.

In 1992 Jungreis and Reid [756] proved that all $C_m \times P_n$ are graceful when m and n are even or when $m \equiv 0 \pmod 4$. They also investigated the existence of a stronger form of graceful labeling called an α-labeling (see Section 3.1) for graphs of the form $P_m \times P_n, C_m \times P_n$, and $C_m \times C_n$ (see also [532]).

Yang and Wang have shown that the prisms $C_{4n+2} \times P_{4m+3}$ [1682], $C_n \times P_2$ [1680], and $C_6 \times P_m (m \geq 2)$ (see [1682]) are graceful. Singh [1373] proved that $C_3 \times P_n$ is graceful for all n. In their 1980 paper Graham and Sloane [586] proved that $C_m \times P_n$ is harmonious when n is odd and they used a computer to show $C_4 \times P_2$, the cube, is not harmonious. In 1992 Gallian, Prout, and Winters [534] proved that $C_m \times P_2$ is harmonious when $m \neq 4$. In 1992, Jungreis and Reid [756] showed that $C_4 \times P_n$ is harmonious when $n \geq 3$. Huang and Skiena [672] have shown that $C_m \times P_n$ is graceful for all n when m is even and for all n with $3 \leq n \leq 12$ when m is odd. Abhyanker [2] proved that the graphs obtained from $C_{2m+1} \times P_5$ by adding a pendent edge to each vertex of an outer cycle is graceful.

Torus grids are graphs of the form $C_m \times C_n$ $(m > 2, n > 2)$. Very little success has been achieved with these graphs. The graceful parity condition is violated for $C_m \times C_n$ when m and n are odd and the harmonious parity condition [586, Theorem 11] is violated for $C_m \times C_n$ when $m \equiv 1, 2, 3 \pmod 4$ and n is odd. In 1992 Jungreis and Reid [756] showed that $C_m \times C_n$ is graceful when $m \equiv 0 \pmod 4$ and n is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout, and Winters [534] proved that all prisms $C_m \times P_2$ with a single vertex deleted or single edge deleted are graceful and harmonious. The *Möbius ladder* M_n is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n. In 1989 Gallian [529] showed that all Möbius ladders are graceful and all but M_3 are harmonious. Ropp [1208] has examined two classes of prisms with pendent edges attached. He proved that all $C_m \times P_2$ with a single pendent edge at each vertex are graceful and all $C_m \times P_2$ with a single pendent edge at each vertex of one of the cycles are graceful.

Another class of cartesian products that has been studied is that of books and "stacked" books. The *book* B_m is the graph $S_m \times P_2$ where St_m is the star with $m + 1$ vertices. In 1980 Maheo [1018] proved that the books of the form B_{2m} are graceful and conjectured that the books B_{4m+1} were also graceful. (The books B_{4m+3} do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [420] in 1980. Maheo [1018] also proved that $L_n \times P_2$ and

$B_{2m} \times P_2$ are graceful. Both Grace [582] and Reid (see [533]) have given harmonious labelings for B_{2m}. The books B_{4m+3} do not satisfy the harmonious parity condition [586, Theorem 11]. Gallian and Jungreis [533] conjectured that the books B_{4m+1} are harmonious. Gnanajothi [572] has verified this conjecture by showing B_{4m+1} has an even stronger form of labeling – see Section 4.1. Liang [949] also proved the conjecture. In 1988 Gallian and Jungreis [533] defined a *stacked book* as a graph of the form $S_m \times P_n$. They proved that the stacked books of the form $S_{2m} \times P_n$ are graceful and posed the case $S_{2m+1} \times P_n$ as an open question. The *n-cube* $K_2 \times K_2 \times \cdots \times K_2$ (n copies) was shown to be graceful by Kotzig [818]—see also [1018]. Although Graham and Sloane [586] used a computer in 1980 to show that the 3-cube is not harmonious (see also [1118]), Ichishima and Oshima [683] proved that the n-cube Q_n has a stronger form of harmonious labeling (see Section 4.1) for $n \geq 4$.

In 1986 Reid [1195] found a harmonious labeling for $K_4 \times P_n$. Petrie and Smith [1129] have investigated graceful labelings of graphs as an exercise in constraint programming satisfaction. They have shown that $K_m \times P_n$ is graceful for $(m, n) = (4, 2), (4, 3), (4, 4), (4, 5)$, (see also [1187]) and (5,2) but is not graceful for $(3, 3)$ and $(6, 2)$. Redl [1187] also proved that $K_4 \times P_n$ is graceful for $n = 1, 2, 3, 4$ and 5 using a constraint programming approach. Their labeling for $K_5 \times P_2$ is the unique graceful labeling. They also considered the graph obtained by identifying the hubs of two copies of W_n. The resulting graph is not graceful when $n = 3$ but is graceful when n is 4 and 5. Smith and Puget [1405] has used a computer search to prove that $K_m \times P_2$ is not graceful for $m = 7, 8, 9$, and 10. She conjectures that $K_m \times P_2$ is not graceful for $m > 5$. Redl [1187] asks if all graphs of the form $K_4 \times P_n$ are graceful.

Vaidya, Kaneria, Srivastav, and Dani [1528] proved that $P_n \cup P_t \cup (P_r \times P_s)$ where $t < \min\{r, s\}$ and $P_n \cup P_t \cup K_{r,s}$ where $t \leq \min\{r, s\}$ and $r, s \geq 3$ are graceful. Kaneria, Vaidya, Ghodasara, and Srivastav [758] proved $K_{mn} \cup (P_r \times P_s)$ where $m, n, r, s > 1$; $(P_r \times P_s) \cup P_t$ where $r, s > 1$ and $t \neq 2$; and $K_{mn} \cup (P_r \times P_s) \cup P_t$ where $m, n, r, s > 1$ and $t \neq 2$ are graceful.

The *composition* $G_1[G_2]$ is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2)| \ x_1x_2 \in E(G_1) \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(G_2)\}$. The *symmetric product* $G_1 \oplus G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2)| \ x_1x_2 \in E(G_1) \text{ or } y_1y_2 \in E(G_2) \text{ but not both}\}$. Seoud and Youssef [1293] have proved that $P_n \oplus \overline{K_2}$ is graceful when $n > 1$ and $P_n[P_2]$ is harmonious for all n. They also observe that the graphs $C_m \oplus C_n$ and $C_m[C_n]$ violate the parity conditions for graceful and harmonious graphs when m and n are odd.

2.4 Complete Graphs

The questions of the gracefulness and harmoniousness of the complete graphs K_n have been answered. In each case the answer is positive if and only if $n \leq 4$ ([576], [1372], [586], [268]). Both Rosa [1209] and Golomb [576] proved that the complete bipartite graphs $K_{m,n}$ are graceful while Graham and Sloane [586] showed they are harmonious if and only if m or $n = 1$. Aravamudhan and Murugan [98] have shown that the complete tripartite graph $K_{1,m,n}$ is both graceful and harmonious while Gnanajothi [572, pp. 25–31] has shown that $K_{1,1,m,n}$ is both graceful and harmonious and $K_{2,m,n}$ is graceful. Some of the same results have been obtained by Seoud and Youssef [1288] who also observed that when m, n, and p are congruent to 2 (mod 4), $K_{m,n,p}$ violates the parity conditions for harmonious graphs. Beutner and Harborth [268] give graceful labelings for $K_{1,m,n}, K_{2,m,n}, K_{1,1,m,n}$ and conjecture that these and $K_{m,n}$ are the only complete multipartite graphs that are graceful. They have verified this conjecture for graphs with up to

23 vertices via computer.

Beutner and Harborth [268] also show that $K_n - e$ (K_n with an edge deleted) is graceful only if $n \le 5$; any $K_n - 2e$ (K_n with two edges deleted) is graceful only if $n \le 6$; and any $K_n - 3e$ is graceful only if $n \le 6$. They also determine all graceful graphs of the form $K_n - G$ where G is $K_{1,a}$ with $a \le n-2$ and where G is a matching M_a with $2a \le n$.

The *windmill* graph $K_n^{(m)}$ ($n > 3$) consists of m copies of K_n with a vertex in common. A necessary condition for $K_n^{(m)}$ to be graceful is that $n \le 5$ – see [802]. Bermond [262] has conjectured that $K_4^{(m)}$ is graceful for all $m \ge 4$. The gracefulness of $K_4^{(m)}$ is equivalent to the existence of a $(12m+1, 4, 1)$-perfect difference family, which are known to exit for $m \le 1000$ (see [672], [1], [1630], and [559]). Bermond, Kotzig, and Turgeon [265] proved that $K_n^{(m)}$ is not graceful when $n = 4$ and $m = 2$ or 3, and when $m = 2$ and $n = 5$. In 1982 Hsu [666] proved that $K_4^{(m)}$ is harmonious for all m. Graham and Sloane [586] conjectured that $K_n^{(2)}$ is harmonious if and only if $n = 4$. They verified this conjecture for the cases that n is odd or $n = 6$. Liu [968] has shown that $K_n^{(2)}$ is not harmonious if $n = 2^a p_1^{a_1} \cdots p_s^{a_s}$ where $a, a_1, \ldots, a_s$ are positive integers and $p_1, \ldots, p_s$ are distinct odd primes and there is a j for which $p_j \equiv 3 \pmod 4$ and a_j is odd. He also shows that $K_n^{(3)}$ is not harmonious when $n \equiv 0 \pmod 4$ and $3n = 4^e(8k+7)$ or $n \equiv 5 \pmod 8$. Koh, Rogers, Lee, and Toh [796] and Rajasingh and Pushpam [1182] have shown that $K_{m,n}{}^{(t)}$, the one-point union of t copies of $K_{m,n}$, is graceful. Sethuraman and Selvaraju [1316] have proved that the one-point union of graphs of the form K_{2,m_i} for $i = 1, 2, \ldots, n$, where the union is taken at a vertex from the partite set with exactly 2 vertices is graceful if at most two of the m_i are equal. They conjecture that the restriction that at most two of the m_i are equal is not necessary. Koh, Rogers, Lee, and Toh [802] introduced the notation $B(n, r, m)$ for the graph consisting of m copies of K_n with a K_r in common ($n \ge r$). (We note that Guo [601] has used the notation $B(n, r, m)$ to denote the graph obtained by joining opposite endpoints of three disjoint paths of lengths n, r and m.) Bermond [262] raised the question: "For which m, n, and r is $B(n, r, m)$ graceful?" Of course, the case $r = 1$ is the same as $K_n^{(m)}$. For $r > 1$, $B(n, r, m)$ is graceful in the following cases: $n = 3,\ r = 2,\ m \ge 1$ [797]; $n = 4,\ r = 2,\ m \ge 1$ [420]; $n = 4,\ r = 3,\ m \ge 1$ (see [262]), [797]. Seoud and Youssef [1288] have proved $B(3, 2, m)$ and $B(4, 3, m)$ are harmonious. Liu [967] has shown that if there is a prime p such that $p \equiv 3 \pmod 4$ and p divides both n and $n-2$ and the highest power of p that divides n and $n-2$ is odd, then $B(n, 2, 2)$ is not graceful. Smith and Puget [1405] has shown that up to symmetry, $B(5, 2, 2)$ has a unique graceful labeling; $B(n, 3, 2)$ is not graceful for $n = 6, 7, 8, 9,$ and 10; $B(6, 3, 3)$ and $B(7, 3, 3)$ are not graceful; and $B(5, 3, 3)$ is graceful. Combining results of Bermond and Farhi [264] and Smith and Puget [1405] show that $B(n, 2, 2)$ is not graceful for $n > 5$. Lu [1005] obtained the following results: $B(m, 2, 3)$ and $B(m, 3, 3)$ are not harmonious when $m \equiv 1 \pmod 8$; $B(m, 4, 2)$ and $B(m, 5, 2)$ are not harmonious when m satisfies certain special conditions; $B(m, 1, n)$ is not harmonious when $m \equiv 5 \pmod 8$ and $n \equiv 1, 2, 3 \pmod 4$; $B(2m+1, 2m, 2n+1) \cong K_{2m} + \overline{K_{2n+1}}$ is not harmonious when $m \equiv 2 \pmod 4$.

More generally, Bermond and Farhi [264] have investigated the class of graphs consisting of m copies of K_n having exactly k copies of K_r in common. They proved such graphs are not graceful for n sufficiently large compared to r. Barrientos [230] proved that the graph obtained by performing the one-point union of any collection of the complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_t,n_t}$, where each K_{m_i,n_i} appears at most twice and $\gcd(n_1, n_2, \ldots, n_t) = 1$, is graceful.

Sethuraman and Elumalai [1300] have shown that $K_{1,m,n}$ with a pendent edge attached to each vertex is graceful and Jirimutu [748] has shown that the graph obtained by attaching a pendent edge to every vertex of $K_{m,n}$ is graceful (see also [84]). In [1313] Sethuraman and Kishore determine the graceful graphs that are the union of n copies of K_4 with i edges deleted for $1 \leq i \leq 5$ and with one edge in common. The only cases that are not graceful are those graphs where the members of the union are C_4 for $n \equiv 3 \pmod 4$ and where the members of the union are P_2. They conjecture that these two cases are the only instances of edge induced subgraphs of the union of n copies of K_4 with one edge in common that are not graceful.

Renuka, Balaganesan, Selvaraju [1196] proved the graphs obtained by joining a vertex of $K_{1,m}$ to a vertex of $K_{1,n}$ by a path are harmonious. Sethuraman and Selvaraju [1322] have shown that union of any number of copies of K_4 with an edge deleted and one edge in common is harmonious.

Clemens, Coulibaly, Garvens, Gonnering, Lucas, and Winters [405] investigated the gracefulness of the one-point and two-point unions of graphs. They show the following graphs are graceful: the one-point union of an end vertex of P_n and K_4; the graph obtained by taking the one-point union of K_4 with one end vertex of P_n and the one-point union of the other end vertex of P_n with the central vertex of $K_{1,r}$; the graph obtained by taking the one-point union of K_4 with one end vertex of P_n and the one-point union of the other end of P_n with a vertex from the partite set of order 2 of $K_{2,r}$; the graph obtained from the graph just described by appending any number of edges to the other vertex of the partite set of order 2; the two-point union of the two vertices of the partite set of order 2 in $K_{2,r}$ and two vertices from K_4; and the graph obtained from the graph just described by appending any number of edges to one of the vertices from the partite set of order 2.

A *Golomb ruler* is a marked straightedge such that the distances between different pairs of marks on the straightedge are distinct. If the set of distances between marks is every positive integer up to and including the length of the ruler, then ruler is a called a *perfect Golomb ruler*. Golomb [576] proved that perfect Golomb rulers exist only for rulers with at most 4 marks. Beavers [250] examines the relationship between Golomb rulers and graceful graphs through a correspondence between rulers and complete graphs. He proves that K_n is graceful if and only if there is a perfect Golomb ruler with n marks and Golomb rulers are equivalent to complete subgraphs of graceful graphs.

2.5 Disconnected Graphs

There have been many papers dealing with graphs that are not connected. For any graph G the graph mG denotes the disjoint union of m copies of G. In 1975 Kotzig [817] investigated the gracefulness of the graphs rC_s. When $rs \equiv 1$ or $2 \pmod 4$, these graphs violate the gracefulness parity condition. Kotzig proved that when $r = 3$ and $4k > 4$, then rC_{4k} has a stronger form of graceful labeling called α-labeling (see §3.1) whereas when $r \geq 2$ and $s = 3$ or 5, rC_s is not graceful. In 1984 Kotzig [819] once again investigated the gracefulness of rC_s as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of rC_s. M. He [619] has shown that graphs of the form $2C_{2m}$ and graphs obtained by connecting two copies of C_{2m} with an edge are graceful. Cahit [336] has shown that rC_s is harmonious when r and s are odd and Seoud, Abdel Maqsoud, and Sheehan [1260] noted that when r or s is even, rC_s is not harmonious. Seoud, Abdel Maqsoud, and Sheehan [1260] proved that $C_n \cup C_{n+1}$ is harmonious if and only if $n \geq 4$. They conjecture that

$C_3 \cup C_{2n}$ is harmonious when $n \geq 3$. This conjecture was proved when Yang, Lu, and Zeng [1678] showed that all graphs of the form $C_{2j+1} \cup C_{2n}$ are harmonious except for $(n, j) = (2, 1)$. As a consequence of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [502] have that $C_n \cup C_3$ is harmonious if and only if $n \geq 6$ and n is even. Renuka, Balaganesan, Selvaraju [1196] proved that for odd n $C_n \cup P_3$ and $C_n \odot \overline{K_m} \cup P_3$ are harmonious.

In 1978 Kotzig and Turgeon [822] proved that mK_n is graceful if and only if $m = 1$ and $n \leq 4$. Liu and Zhang [970] have shown that mK_n is not harmonious for n odd and $m \equiv 2 \pmod 4$ and is harmonious for $n = 3$ and m odd. They conjecture that mK_3 is not harmonious when $m \equiv 0 \pmod 4$. Bu and Cao [315] give some sufficient conditions for the gracefulness of graphs of the form $K_{m,n} \cup G$ and they prove that $K_{m,n} \cup P_t$ and the disjoint union of complete bipartite graphs are graceful under some conditions.

Recall a *Skolem sequence* of order n is a sequence $s_1, s_2, \ldots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. (A Skolem sequence of order n exists if and only if $n \equiv 0$ or $1 \pmod 4$). Abrham [6] has proved that any graceful 2-regular graph of order $n \equiv 0 \pmod 4$ in which all the component cycles are even or of order $n \equiv 3 \pmod 4$, with exactly one component an odd cycle, can be used to construct a Skolem sequence of order $n + 1$. Also, he showed that certain special Skolem sequences of order n can be used to generate graceful labelings on certain 2-regular graphs.

The graph H_n obtained from the cycle with consecutive vertices $u_1, u_2, \ldots, u_n$ $(n \geq 6)$ by adding the chords $u_2u_n, u_3u_{n-1}, \ldots, u_\alpha u_\beta$, where $\alpha = (n-1)/2$ for all n and $\beta = (n-1)/2 + 3$ if n is odd or $\beta = n/2 + 2$ if n is even is called the *cycle with parallel chords*. In Elumalai and Sethuraman [456] prove the following: for odd $n \geq 5$, $H_n \cup K_{p,q}$ is graceful; for even $n \geq 6$ and $m = (n-2)/2$ or $m = n/2$ $H_n \cup K_{1,m}$ is graceful; for $n \geq 6$, $H_n \cup P_m$ is graceful, where $m = n$ or $n - 2$ depending on $n \equiv 1$ or $3 \pmod 4$ or $m \equiv n - 1$ or $n - 3$ depending on $n \equiv 0$ or $2 \pmod 4$. Elumali and Sethuraman [459] proved that every n-cycle $(n \geq 6)$ with parallel chords is graceful and every n-cycle with parallel P_k-chords of increasing lengths is graceful for $n = 2 \pmod 4$ with $1 \leq k \leq (\lfloor n/2 \rfloor - 1)$.

In 1985 Frucht and Salinas [519] conjectured that $C_s \cup P_n$ is graceful if and only if $s + n \geq 6$ and proved the conjecture for the case that $s = 4$. The conjecture was proved by Traetta [1494] in 2012 who used his result to get a complete solution to the well known two-table Oberwolfach problem; that is, given odd number of people and two round tables when is it possible to arrange series of seatings so that each person sits next to each other person exactly once during the series. The t-table Oberwolfach problem $\mathrm{OP}(n_1, n_2, \ldots, n_t)$ asks to arrange a series of meals for an odd number $n = \sum n_i$ of people around t tables of sizes $n_1, n_2, \ldots, n_t$ so that each person sits next to each other exactly once. A solution to $\mathrm{OP}(n_1, n_2, \ldots, n_t)$ is a 2–factorization of K_n whose factors consists of t cycles of lengths $n_1, n_2, \ldots, n_t$. The λ–fold Oberwolfach problem $\mathrm{OP}_\lambda(n_1, n_2, \ldots, n_t)$ refers to the case where K_n is replaced by λK_n. Traetta used his proof of the Frucht and Salinas conjecture to provide a complete solutions to both $OP(2r + 1, 2s)$ and $OP(2r + 1, s, s)$, except possibly for $OP(3, s, s)$. He also gave a complete solution of the general λ-fold Oberwolfach problem $OP_\lambda(r, s)$.

Seoud and Youssef [1295] have shown that $K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q}$ $(m, n, p, q \geq 2), K_{m,n} \cup K_{p,q} \cup K_{r,s}$ $(m, n, p, q, r, s \geq 2, \ (p, q) \neq (2, 2))$, and $pK_{m,n}$ $(m, n \geq 2, (m, n) \neq (2, 2))$ are graceful. They also prove that $C_4 \cup K_{1,n}$ $(n \neq 2)$ is not graceful whereas Choudum and Kishore [397], [794] have proved that $C_s \cup K_{1,n}$ is graceful for $s \geq 7$ and $n \geq 1$. Lee, Quach, and

Wang [900] established the gracefulness of $P_s \cup K_{1,n}$. Seoud and Wilson [1287] have shown that $C_3 \cup K_4, C_3 \cup C_3 \cup K_4$, and certain graphs of the form $C_3 \cup P_n$ and $C_3 \cup C_3 \cup P_n$ are not graceful. Abrham and Kotzig [11] proved that $C_p \cup C_q$ is graceful if and only if $p+q \equiv 0$ or 3 (mod 4). Zhou [1725] proved that $K_m \cup K_n$ $(n > 1, m > 1)$ is graceful if and only if $\{m,n\} = \{4,2\}$ or $\{5,2\}$. (C. Barrientos has called to my attention that $K_1 \cup K_n$ is graceful if and only if $n = 3$ or 4.) Shee [1327] has shown that graphs of the form $P_2 \cup C_{2k+1}$ $(k > 1)$, $P_3 \cup C_{2k+1}$, $P_n \cup C_3$, and $S_n \cup C_{2k+1}$ all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [274] have shown that $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$ are graceful. Section 3.1 includes numerous families of disconnected graphs that have a stronger form of graceful labelings.

For $m = 2p+3$ or $2p+4$, Wang, Liu, and Li [1623] proved the following graphs are graceful: $W_m \cup K_{n,p}$ and $W_{m,2m+1} \cup K_{n,p}$; for $n \geq m$, $W_{m,2m+1} \cup K_{1,n}$; for $m = 2n+5$, $W_{m,2m+1} \cup (C_3 + \overline{K_n})$. If G_p is a graceful graph with p edges, they proved $W_{2p+3} \cup G_p$ is graceful.

In considering graceful labelings of the disjoint unions of two or three stars with e edges Yang and Wang [1681] permitted the vertex labels to range from 0 to $e+1$ and 0 to $e+2$, respectively. With these definitions of graceful, they proved that $S_m \cup S_n$ is graceful if and only if m or n is even and that $S_m \cup S_n \cup S_k$ is graceful if and only if at least one of m, n, or k is even $(m > 1, n > 1, k > 1)$.

Seoud and Youssef [1291] investigated the gracefulness of specific families of the form $G \cup K_{m,n}$. They obtained the following results: $C_3 \cup K_{m,n}$ is graceful if and only if $m \geq 2$ and $n \geq 2$; $C_4 \cup K_{m,n}$ is graceful if and only if $(m,n) \neq (1,1)$; $C_7 \cup K_{m,n}$ and $C_8 \cup K_{m,n}$ are graceful for all m and n; $mK_3 \cup nK_{1,r}$ is not graceful for all m, n and r; $K_i \cup K_{m,n}$ is graceful for $i \leq 4$ and $m \geq 2, n \geq 2$ except for $i = 2$ and $(m,n) = (2,2)$; $K_5 \cup K_{1,n}$ is graceful for all n; $K_6 \cup K_{1,n}$ is graceful if and only if n is not 1 or 3. Youssef [1698] completed the characterization of the graceful graphs of the form $C_n \cup K_{p,q}$ where $n \equiv 0$ or 3 (mod 4) by showing that for $n > 8$ and $n \equiv 0$ or 3 (mod 4), $C_n \cup K_{p,q}$ is graceful for all p and q (see also [228]). Note that when $n \equiv 1$ or 2 (mod 4) certain cases of $C_n \cup K_{p,q}$ violate the parity condition for gracefulness.

For $i = 1, 2, \ldots, m$ let $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ be a 4-cycle. Yang and Pan [1676] define $F_{k,4}$ to be the graph obtained by identifying $v_{i,3}$ and $v_{i+1,1}$ for $i = 1, 2, \ldots, k-1$. They prove that $F_{m_1,4} \cup F_{m_2,4} \cup \cdots \cup F_{m_n,4}$ is graceful for all n. Pan and Lu [1115] have shown that $(P_2 + \overline{K_n}) \cup K_{1,m}$ and $(P_2 + \overline{K_n}) \cup T_n$ are graceful.

Barrientos [228] has shown the following graphs are graceful: $C_6 \cup K_{1,2n+1}$; $\bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leq m_i < n_i$; and $C_m \cup \bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leq m_i < n_i, m \equiv 0$ or 3 (mod 4), $m \geq 11$.

Youssef [1696] has shown that if G is harmonious then mG is harmonious for all odd m.

Wang and Li [1622] use $St(n)$ to denote the star $K_{n,1}$, F_n to denote the fan $P_n \odot K_1$, and $F_{m,n}$ to denote the graph obtained by identifying the vertex of F_m with degree m and the vertex of F_n with degree n. They showed: for all positive integers n and p and $m \geq 2p+2$, $F_m \cup K_{n,p}$ and $F_{m,2m} \cup K_{n,p}$ are graceful; $F_m \cup St(n)$ is graceful; and $F_{m,2m} \cup St(n)$ and $F_{m,2m} \cup G_r$ are graceful. In [1626] Wang, Wang, and Li gave a sufficient condition for the gracefulness of graphs of the form $(P_3 + \overline{K_m}) \cup G$ and $(C_3 + \overline{K_m}) \cup G$. They proved the gracefulness of such graphs for a variety of cases when G involves stars and paths. More technical results like these are given in [1628] and [1627].

2.6 Joins of Graphs

A number of classes of graphs that are the join of graphs have been shown to be graceful or harmonious. Acharya [12] proved that if G is a connected graceful graph, then $G + \overline{K_n}$ is graceful. Redl [1187] showed that the double cone $C_n + \overline{K_2}$ is graceful for $n = 3, 4, 5, 7, 8, 9, 11$. That $C_n + \overline{K_2}$ is not graceful for $n \equiv 2 \pmod 4$ follows that Rosa's parity condition. Redl asks what other double cones are graceful. Reid [1195] proved that $P_n + \overline{K_t}$ is harmonious. Sethuraman and Selvaraju [1321] and [1252] have shown that $P_n + K_2$ is harmonious. They ask whether $S_n + P_n$ or $P_m + P_n$ is harmonious. Of course, wheels are of the form $C_n + K_1$ and are graceful and harmonious. In 2006 Chen [372] proved that multiple wheels $nC_m + K_1$ are harmonious for all $n \not\equiv 0$ mod 4. She believes that the $n \not\equiv 0 \pmod 4$ case is also harmonious. Chen also proved that if H has at least one edge, $H + K_1$ is harmonious, and if n is odd, then $nH + K$ is harmonious.

Shee [1327] has proved $K_{m,n} + K_1$ is harmonious and observed that various cases of $K_{m,n} + K_t$ violate the harmonious parity condition in [586]. Liu and Zhang [970] have proved that $K_2 + K_2 + \cdots + K_2$ is harmonious. Youssef [1696] has shown that if G is harmonious then G^m is harmonious for all odd m. He asks the question of whether G is harmonious implies G^m is harmonious when $m \equiv 0 \pmod 4$. Yuan and Zhu [1710] proved that $K_{m,n} + K_2$ is graceful and harmonious. Gnanajothi [572, pp. 80–127] obtained the following: $C_n + \overline{K_2}$ is harmonious when n is odd and not harmonious when $n \equiv 2, 4, 6 \pmod 8$; $S_n + \overline{K_t}$ is harmonious; and $P_n + \overline{K_t}$ is harmonious. Balakrishnan and Kumar [213] have proved that the join of $\overline{K_n}$ and two disjoint copies of K_2 is harmonious if and only if n is even. Ramírez-Alfonsín [1185] has proved that if G is graceful and $|V(G)| = |E(G)| = e$ and either 1 or e is not a vertex label then $G + \overline{K_t}$ is graceful for all t.

Seoud and Youssef [1293] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and $C_n + \overline{K_t}$ is harmonious for every t when n is odd. They also prove that if any edge is added to $K_{m,n}$ the resulting graph is harmonious if m or n is at least 2. Deng [422] has shown certain cases of $C_n + \overline{K_t}$ are harmonious. Seoud and Youssef [1290] proved: the graph obtained by appending any number of edges from the two vertices of degree $n \geq 2$ in $K_{2,n}$ is not harmonious; dragons $D_{m,n}$ (i.e., an endpoint of P_m is appended to C_n) are not harmonious when $m + n$ is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has odd order. Youssef [1695] has shown that if G is a graceful graph with p vertices and q edges with $p = q + 1$, then $G + S_n$ is graceful.

Sethuraman and Elumalai [1304] have proved that for every graph G with p vertices and q edges the graph $G + K_1 + \overline{K_m}$ is graceful when $m \geq 2^p - p - 1 - q$. As a corollary they deduce that every graph is a vertex induced subgraph of a graceful graph. Balakrishnan and Sampathkumar [214] ask for which $m \geq 3$ is the graph $mK_2 + \overline{K_n}$ graceful for all n. Bhat-Nayak and Gokhale [278] have proved that $2K_2 + \overline{K_n}$ is not graceful. Youssef [1695] has shown that $mK_2 + \overline{K_n}$ is graceful if $m \equiv 0$ or $1 \pmod 4$ and that $mK_2 + \overline{K_n}$ is not graceful if n is odd and $m \equiv 2$ or $3 \pmod 4$. Ma [1010] proved that if G is a graceful tree then, $G + K_{1,n}$ is graceful. Amutha and Kathiresan [84] proved that the graph obtained by attaching a pendent edge to each vertex of $2K_2 + \overline{K_n}$ is graceful.

Wu [1652] proves that if G is a graceful graph with n edges and $n + 1$ vertices then the join of G and $\overline{K_m}$ and the join of G and any star are graceful. Wei and Zhang [1636] proved that for $n \geq 3$ the disjoint union of $P_1 + P_n$ and a star, the disjoint union of $P_1 + P_n$ and $P_1 + P_{2n}$, and the disjoint union of $P_2 + \overline{K_n}$ and a graceful graph with n edges are graceful. More technical results on disjoint unions and joins are given in [1635],[1636], [1637],[1634], and [342].

2.7 Miscellaneous Results

It is easy to see that P_n^2 is harmonious [583] while a proof that P_n^2 is graceful has been given by Kang, Liang, Gao, and Yang [764]. (P_n^k, the kth *power of* P_n, is the graph obtained from P_n by adding edges that join all vertices u and v with $d(u, v) = k$.) This latter result proved a conjecture of Grace [583]. Seoud, Abdel Maqsoud, and Sheehan [1260] proved that P_n^3 is harmonious and conjecture that P_n^k is not harmonious when $k > 3$. The same conjecture was made by Fu and Wu [522]. However, Youssef [1704] has proved that P_8^4 is harmonious and P_n^k is harmonious when k is odd. Yuan and Zhu [1710] proved that P_n^{2k} is harmonious when $1 \leq k \leq (n-1)/2$. Selvaraju [1250] has shown that P_n^3 and the graphs obtained by joining the centers of any two stars with the end vertices of the path of length n in P_n^3 are harmonious.

Cahit [336] proves that the graphs obtained by joining p disjoint paths of a fixed length k to single vertex are harmonious when p is odd and when $k = 2$ and p is even. Gnanajothi [572, p. 50] has shown that the graph that consists of n copies of C_6 that have exactly P_4 in common is graceful if and only if n is even. For a fixed n, let v_{i1}, v_{i2}, v_{i3} and v_{i4} $(1 \leq i \leq n)$ be consecutive vertices of n 4-cycles. Gnanajothi [572, p. 35] also proves that the graph obtained by joining each v_{i1} to $v_{i+1,3}$ is graceful for all n and the generalized Petersen graph $P(n, k)$ is harmonious in all cases (see also [905]). Recall $P(n, k)$, where $n \geq 5$ and $1 \leq k \leq n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i a_{i+1} \mid i = 0, 1, \ldots, n-1\} \cup \{a_i b_i \mid i = 0, 1, \ldots, n-1\} \cup \{b_i b_{i+k} \mid i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo n [1633]. The standard Petersen graph is $P(5, 2)$.) Redl [1187] has used a constraint programming approach to show that $P(n, k)$ is graceful for $n = 5, 6, 7, 8, 9$, and 10. In [1575] and [1576] Vietri proved that $P(8t, 3)$ and $P(8t + 4, 3)$ are graceful for all t. He conjectures that the graphs $P(8t, 3)$ have a stronger form a graceful labeling called an α-labeling (see §3.1). The gracefulness of the generalized Petersen graphs is an open problem. A conjecture in the graph theory book by Chartrand and Lesniak [363, p. 266] that graceful graphs with arbitrarily large chromatic numbers do not exist was shown to be false by Acharya, Rao, and Arumugam [30] (see also Mahmoody [1020]).

Sethuraman and Selvaraju [1315] define a graph H to be a *supersubdivision* of a graph G, if every edge uv of G is replaced by $K_{2,m}$ (m may vary for each edge) by identifying u and v with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Sethuraman and Selvaraju prove that every supersubdivision of a path is graceful and every cycle has some supersubdivision that is graceful. They conjecture that every supersubdivision of a star is graceful and that paths and stars are the only graphs for which every supersubdivision is graceful. Barrientos [230] disproved this latter conjecture by proving that every supersubdivision of a y-trees is graceful (recall a y-tree is obtained from a path by appending an edge to a vertex of a path adjacent to an end point). Barrientos asks if paths and y-trees are the only graphs for which every supersubdivision is graceful. This seems unlikely to be the case. The conjecture that every supersubdivision of a star is graceful was proved by Kathiresan and Amutha [772]. In [1319] Sethuraman and Selvaraju prove that every connected graph has some supersubdivision that is graceful. They pose the question as to whether this result is valid for disconnected graphs. Barrientos and Barrientos [237] answered this question by proving that any disconnected graph has a supersubdivision that admits an α-labeling. Sethuraman and Selvaraju also asked if there is any graph other than $K_{2,m}$ that can be used to replace an edge of a connected graph to obtain a supersubdivision that is graceful.

Sethuraman and Selvaraju [1315] call superdivision graphs of G where every edge uv of G is replaced by $K_{2,m}$ and m is fixed an *arbitrary supersubdivision* of G. Barrientos and Barrientos [237] answered the question of Sethuraman and Selvaraju by proved that any graph obtained from $K_{2,m}$ by attaching k pendent edges and n pendent edges to the vertices of its 2-element stable set can be used instead of $K_{2,m}$ to produce an arbitrary supersubdivision that admits an α-labeling. K. Kathiresan and R. Sumathi [778] affirmatively answer the question posed by Sethuraman and Selvaraju in [1315] of whether there are graphs different from paths whose arbitrary supersubdivisions are graceful.

For a graph G Ambili and Singh [83] call the graph G^* a *strong supersubdivision* of G if G^* is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{r_i,s_i}. A strong supersubdivision G^* of G is said to be an *arbitrary strong supersubdivision* if G^* is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{r,s_i} (r is fixed and s_i may vary). They proved that arbitrary strong supersubdivisions of paths, cycles, and stars are graceful. They conjecture that every arbitrary strong supersubdivision of a tree is graceful and ask if it is true that for any non-trivial connected graph G, an arbitrary strong supersubdivision of G is graceful?

In [1318] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of graceful labeling called an α-labeling (see §3.1 for the definition).

Kathiresan [769] uses the notation $P_{a,b}$ to denote the graph obtained by identifying the end points of b internally disjoint paths each of length a. He conjectures that $P_{a,b}$ is graceful except when a is odd and $b \equiv 2 \pmod 4$ and proves the conjecture for the case that a is even and b is odd. Liang and Zuo [954] proved that the graph $P_{a,b}$ is graceful when both a and b are even. Daili, Wang and Xie [416] provided an algorithm for finding a graceful labeling of $P_{2r,2}$ and showed that a $P_{2r,2(2k+1)}$ is graceful for all positives r and k. Sekar [1249] has shown that $P_{a,b}$ is graceful when $a \neq 4r+1,\ r>1, b=4m$, and $m>r$. Yang (see [1679]) proved that $P_{a,b}$ is graceful when $a=3,5,7$, and 9 and b is odd and when $a=2,4,6$, and 8 and b is even (see [1679]). Yang, Rong, and Xu [1679] proved that $P_{a,b}$ is graceful when $a=10,12$, and 14 and b is even. Yan [1672] proved $P_{2r,2m}$ is graceful when r is odd. Yang showed that $P_{2r+1,2m+1}$ and $P_{2r,2m}$ ($r \leq 7$, and $r=9$) are graceful (see [1207]). Rong and Xiong [1207] showed that $P_{2r,b}$ is graceful for all positive integers r and b. Kathiresan also shows that the graph obtained by identifying a vertex of K_n with any noncenter vertex of the star with $2^{n-1}-n(n-1)/2$ edges is graceful.

For a family of graphs $G_1(u_1,u_2), G_2(u_2,u_3), \ldots, G_m(u_m,u_{m+1})$ where u_i and u_{i+1} are vertices in G_i Cheng, Yao, Chen, and Zhang [381] define a *graph-block chain* H_m as the graph obtained by identifying u_{i+1} of G_i with u_{i+1} of G_{i+1} for $i=1,2,\ldots,m$. They denote this graph by $H_m = G_1(u_1,u_2) \oplus G_2(u_2,u_3) \oplus \cdots \oplus G_m(u_m,u_{m+1})$. The case where each G_i has the form P_{a_i,b_i} they call a *path-block chain*. The vertex u_1 is called the *initial vertex* of H_m. They define a *generalized spider* S^*_m as a graph obtained by starting with an initial vertex u_0 and m path-block graphs and join u_0 with each initial vertex of each of the path-block graphs. Similarly, they define a *generalized caterpillar* T^*_m as a graph obtained by starting with m path-block chains $H_1, H_2, \ldots, H_m$ and a caterpillar T with m isolated vertices $v_1, v_2, \ldots, v_m$ and join each v_i with the initial vertex of each H_i. They prove several classes of path-block chains, generalized spiders, and generalized caterpillars are graceful.

The graph T_n with $3n$ vertices and $6n-3$ edges is defined as follows. Start with a triangle T_1 with vertices $v_{1,1}, v_{1,2}$ and $v_{1,3}$. Then T_{i+1} consists of T_i together with three new vertices $v_{i+1,1}, v_{i+1,2}, v_{i+1,3}$ and edges $v_{i+1,1}v_{i,2}$, $v_{i+1,1}v_{i,3}$, $v_{i+1,2}v_{i,1}$, $v_{i+1,2}v_{i,3}$, $v_{i+1,3}v_{i,1}$, $v_{i+1,3}v_{i,2}$.

Gnanajothi [572] proved that T_n is graceful if and only if n is odd. Sekar [1249] proved T_n is graceful when n is odd and T_n with a pendent edge attached to the starting triangle is graceful when n is even.

In [253] and [606] Begam, Palanivelrajan, Gunasekaran, and Hameed give graceful labelings for graphs constructed by combining theta graphs with paths and stars.

For a graph G, the *splitting graph of* G, $S^1(G)$, is obtained from G by adding for each vertex v of G a new vertex v^1 so that v^1 is adjacent to every vertex that is adjacent to v. Sekar [1249] has shown that $S^1(P_n)$ is graceful for all n and $S^1(C_n)$ is graceful for $n \equiv 0, 1 \pmod 4$. Vaidya and Shah [1547] proved that the square graph of a bistar, the splitting graph of a bistar, and the splitting graph of a star are graceful graphs.

The *duplication of an edge* $e = uv$ of a graph G is the graph G' obtained from G by adding an edge $e' = u'v'$ such that $N(u) = N(u')$ and $N(v) = N(v')$. The *duplication of a vertex* of a graph G is the graph G' obtained from G by adding a new vertex v' to G such that $N(v') = N(v)$. Kaneria, Vaidya, Ghodasara, and Srivastav [758] proved the duplication of a vertex of a cycle, the duplication of an edge of an even cycle, and the graph obtained by joining two copies of a fixed cycle by an edge are graceful.

For a bipartite graph G with partite sets X and Y let G' be a copy of G and X' and Y' be copies of X and Y. Lee and Liu [879] define the *mirror graph*, $M(G)$, of G as the disjoint union of G and G' with additional edges joining each vertex of Y to its corresponding vertex in Y'. The case that $G = K_{m,n}$ is more simply denoted by $M(m, n)$. They proved that for many cases $M(m, n)$ has a stronger form of graceful labeling (see §3.1 for details).

The *total graph* $T(P_n)$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n. Balakrishnan, Selvam, and Yegnanarayanan [215] have proved that $T(P_n)$ is harmonious.

For any graph G with vertices $v_1, \ldots, v_n$ and a vector $\mathbf{m} = (m_1, \ldots, m_n)$ of positive integers the corresponding *replicated graph*, $R_{\mathbf{m}}(G)$, of G is defined as follows. For each v_i form a stable set S_i consisting of m_i new vertices $i = 1, 2, \ldots, n$ (a *stable* set S consists of a set of vertices such that there is not an edge v_iv_j for all pairs v_i, v_j in S); two stable sets $S_i, S_j, i \neq j$, form a complete bipartite graph if each v_iv_j is an edge in G and otherwise there are no edges between S_i and S_j. Ramírez-Alfonsín [1185] has proved that $R_{\mathbf{m}}(P_n)$ is graceful for all $\mathbf{m}$ and all $n > 1$ (see §3.4 for a stronger result) and that $R_{(m,1,\ldots,1)}(C_{4n}), R_{(2,1,\ldots,1)}(C_n)$ $(n \geq 8)$ and, $R_{(2,2,1,\ldots,1)}(C_{4n})$ $(n \geq 12)$ are graceful.

For any permutation f on $1, \ldots, n$, the f*-permutation* graph on a graph G, $P(G, f)$, consists of two disjoint copies of G, G_1 and G_2, each of which has vertices labeled $v_1, v_2, \ldots, v_n$ with n edges obtained by joining each v_i in G_1 to $v_{f(i)}$ in G_2. In 1983 Lee (see [942]) conjectured that for all $n > 1$ and all permutations on $1, 2, \ldots, n$, the permutation graph $P(P_n, f)$ is graceful. Lee, Wang, and Kiang [942] proved that $P(P_{2k}, f)$ is graceful when $f = (12)(34)\cdots(k, k+1)\cdots(2k-1, 2k)$. They conjectured that if G is a graceful nonbipartite graph with n vertices, then for any permutation f on $1, 2, \ldots, n$, the permutation graph $P(G, f)$ is graceful. Fan and Liang [483] have shown that if f is a permutation in S_n where $n \geq 2(m-1) + 2l$ then the permutation graph $P(P_n, f)$ is graceful if the disjoint cycle form of f is $\prod_{k=0}^{l-1}(m + 2k, m + 2k + 1)$, and if $n \geq 2(m-1) + 4l$ the permutation graph $P(P_n, f)$ is graceful the disjoint cycle form of f is $\prod_{k=0}^{l-1}(m+4k, m+4k+2)(m+4k+1, m+4k+3)$. For any integer $n \geq 5$ and some permutations f in S(n), Liang and Y. Miao, [956] discuss gracefulness of the permutation graphs $P(P_n, f)$ if $f = (m, m+1, m+2, m+3, m+4), (m, m+2)(m+1, m+3), (m, m+1, m+2, m+4, m+3), (m, m+1, m+4, m+3, m+2), (m, m+2, m+3, m+4, m+1), (m, m+3, m+4, m+2, m+1)$

and $(m, m+4, m+3, m+2, m+1)$. Some families of graceful permutation graphs are given in [872], [951], and [607].

Gnanajothi [572, p. 51] calls a graph G *bigraceful* if both G and its line graph are graceful. She shows the following are bigraceful: P_m; $P_m \times P_n$; C_n if and only if $n \equiv 0, 3 \pmod 4$; S_n; K_n if and only if $n \leq 3$; and B_n if and only if $n \equiv 3 \pmod 4$. She also shows that $K_{m,n}$ is not bigraceful when $n \equiv 3 \pmod 4$. (Gangopadhyay and Hebbare [537] used the term "bigraceful" to mean a bipartite graceful graph.) Murugan and Arumugan [1083] have shown that graphs obtained from C_4 by attaching two disjoint paths of equal length to two adjacent vertices are bigraceful.

Several well-known isolated graphs have been examined. Graceful labelings have been found for the Petersen graph [515], the cube [546], the icosahedron and the dodecahedron. Graham and Sloane [586] showed that all of these except the cube are harmonious. Winters [1647] verified that the Grőtzsch graph (see [300, p. 118]), the Heawood graph (see [300, p. 236]), and the Herschel graph (see [300, p. 53]) are graceful. Graham and Sloane [586] determined all harmonious graphs with at most five vertices. Seoud and Youssef [1292] did the same for graphs with six vertices.

A number of authors have investigated the gracefulness of the directed graphs obtained from copies of directed cycles $\vec{C}_m$ that have a vertex in common or have an edge in common. A digraph $D(V, E)$ is said to be *graceful* if there exists an injection $f\colon V(G) \to \{0, 1, \ldots, |E|\}$ such that the induced function $f'\colon E(G) \to \{1, 2, \ldots, |E|\}$ that is defined by $f'(u, v) = (f(v) - f(u)) \pmod{|E|+1}$ for every directed edge uv is a bijection. The notations $n \cdot \vec{C}_m$ and $n - \vec{C}_m$ are used to denote the digraphs obtained from n copies of $\vec{C}_m$ with exactly one point in common and the digraphs obtained from n copies of $\vec{C}_m$ with exactly one edge in common. Du and Sun [447] proved that a necessary condition for $n - \vec{C}_m$ to be graceful is that mn is even and that $n \cdot \vec{C}_m$ is graceful when m is even. They conjectured that $n \cdot \vec{C}_m$ is graceful for any odd m and even n. This conjecture was proved by Jirimutu, Xu, Feng, and Bao in [753]. Xu, Jirimutu, Wang, and Min [1664] proved that $n - \vec{C}_m$ is graceful for $m = 4, 6, 8, 10$ and even n. Feng and Jirimutu (see [1719]) conjectured that $n - \vec{C}_m$ is graceful for even n and asked about the situation for odd n. The cases where $m = 5, 7, 9, 11$, and 13 and even n were proved Zhao and Jirimutu [1717]. The cases for $m = 15, 17$, and 19 and even n were proved by Zhao et al. in [1718], [1719], and [1389]. Zhao, Siqintuya, and Jirimutu [1719] also proved that a necessary condition for $n - \vec{C}_m$ to be graceful is that nm is even. A survey of results on graceful digraphs by Feng, Xu, and Jirimutu in given in [489].

Marr [1028] and [1027] summarizes previously known results on graceful directed graphs and presents some new results on directed paths, stars, wheels, and umbrellas.

In 2009 Zak [1713] defined the following generalization of harmonious labelings. For a graph $G(V, E)$ and a positive integer $t \geq |E|$ a function h from $V(G)$ to Z_t (the additive group of integers modulo t) is called a *t-harmonious labeling* of G if h is injective for $t \geq |V|$ or surjective for $t < |V|$, and $h(u) + h(v) \neq h(x) + h(y)$ for all distinct edges uv and xy. The smallest such t for which G has a t-harmonious labeling is called the *harmonious order of G*. Obviously, a graph $G(V, E)$with $|E| \geq |V|$ is harmonious if and only if the harmonious order of G is $|E|$. Zak determines the harmonious order of complete graphs, complete bipartite graphs, even cycles, some cases of P_n^k, and $2nK_3$. He presents some results about the harmonious order of the Cartesian products of graphs, the disjoint union of copies of a given graph, and gives an upper bound for the harmonious order of trees. He conjectures that the harmonious order of a tree of order n is $n + o(n)$.

For a graph with e edges Vietri [1577] generalizes the notion of a graceful labeling by allowing the vertex labels to be real numbers in the interval $[0, e]$. For a simple graph $G(V, E)$ he defines an injective map γ from V to $[0, e]$ to be a *real-graceful* labeling of G provided that $\sum 2^{\gamma(u)-\gamma(v)} + 2^{\gamma(v)-\gamma(u)} = 2^{e+1} - 2^{-e} - 1$, where the sum is taken over all edges uv. In the case that the labels are integers, he shows that a real-graceful labeling is equivalent to a graceful labeling. In contrast to the case for graceful labelings, he shows that the cycles C_{4t+1} and C_{4t+2} have real-graceful labelings. He also shows that the non-graceful graphs K_5, K_6 and K_7 have real-graceful labelings. With one exception, his real-graceful labels are integers.

2.8 Summary

The results and conjectures discussed above are summarized in the tables following. The letter G after a class of graphs indicates that the graphs in that class are known to be graceful; a question mark indicates that the gracefulness of the graphs in the class is an open problem; we put a question mark after a "G" if the graphs have been conjectured to be graceful. The analogous notation with the letter H is used to indicate the status of the graphs with regard to being harmonious. The tables impart at a glimpse what has been done and what needs to be done to close out a particular class of graphs. Of course, there is an unlimited number of graphs one could consider. One wishes for some general results that would handle several broad classes at once but the experience of many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture alone has withstood the efforts of scores of people over the past four decades. Analogous sweeping conjectures are probably true but appear hopelessly difficult to prove.

Table 1: **Summary of Graceful Results**

Graph	*Graceful*
trees	G if ≤ 35 vertices [484] G if symmetrical [266] G if at most 4 end-vertices [671] G? Ringel-Kotzig G caterpillars [1209] G firecrakers [371] G bananas [1312], citeSeje04B G? lobsters [262]
cycles C_n	G iff $n \equiv 0, 3 \pmod 4$ [1209]
wheels W_n	G [515], [660]
helms (see §2.2)	G [108]
webs (see §2.2)	G [764]
gears (see §2.2)	G [1009]

Table 1: **Summary of Graceful Results continued**

Graph	*Graceful*
cycles with P_k-chord (see §2.2)	G [421], [1008], [805], [1168]
C_n with k consecutive chords (see §2.2)	G if $k = 2, 3, n-3$ [795], [802]
unicyclic graphs	G? iff G $\neq C_n, n \equiv 1, 2 \pmod 4$ [1500]
P_n^k	G if $k = 2$ [764]
$C_n^{(t)}$ (see §2.2)	$n = 3$ G iff $t \equiv 0, 1 \pmod 4$ [263], [265] G? if $nt \equiv 0, 3 \pmod 4$ [796] G if $n = 6, t$ even [796] G if $n = 4, t > 1$ [1328] G if $n = 5, t > 1$ [1677] G if $n = 7$ and $t \equiv 0, 3 \pmod 4$ [1683] G if $n = 9$ and $t \equiv 0, 3 \pmod 4$ [1684] G if $t = 2$ $n \not\equiv 1 \pmod 4$ [1173], [298] G if $n = 11$ [1666]
triangular snakes (see §2.2)	G iff number of blocks $\equiv 0, 1 \pmod 4$ [1078]
K_4-snakes (see §2.2)	?
quadrilateral snakes (see §2.2)	G [572], [1173]
crowns $C_n \odot K_1$	G [515]
$C_n \odot P_k$	G [1249]
grids $P_m \times P_n$	G [24]
prisms $C_m \times P_n$	G if $n = 2$ [518], [1680] G if m even [672] G if m odd and $3 \leq n \leq 12$ [672] G if $m = 3$ [1373] G if $m = 6$ see [1682] G if $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$ [1682]

Table 1: **Summary of Graceful Results continued**

Graph	*Graceful*
$K_m \times P_n$	G if $(m,n) = (4,2),(4,3),(4,4),(4,5),(5,2)$ not G if $(m,n) = (3,3),(6,2),(7,2),(8,2),(9,2)(10,2)$ not G? for $(m,2)$ with $m > 5$ [1405]
$K_{m,n} \odot K_1$	G [748]
torus grids $C_m \times C_n$	G if $m \equiv 0 \pmod 4$, n even [756] not G if m, n odd (parity condition)
vertex-deleted $C_m \times P_n$	G if $n = 2$ [534]
edge-deleted $C_m \times P_n$	G if $n = 2$ [534]
Möbius ladders M_n (see §2.3)	G [529]
stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, G iff $m \not\equiv 3 \pmod 4$ [1018], [420], [533] G if m even [533]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	G [818]
$K_4 \times P_n$	G if $n = 2,3,4,5$ [1129]
K_n	G iff $n \leq 4$ [576], [1372]
$K_{m,n}$	G [1209], [576]
$K_{1,m,n}$	G [98]
$K_{1,1,m,n}$	G [572]
windmills $K_n^{(m)}(n > 3)$ (see §2.4)	G if $n = 4, m \leq 1000$ [672],[1],[1630],[559] G? if $n = 4, m \geq 4$ [262] not G if $n = 4, m = 2,3$ [262] not G if $(m,n) = (2,5)$ [265] not G if $n > 5$ [802]
$B(n,r,m)$ $r > 1$ (see §2.4)	G if $(n,r) = (3,2),(4,3)$ [797], (4,2) [420] G $(n,r,m) = (5,2,2)$ [1405] not G for $(n,2,2)$ for $n > 5$ [264], [1405]
mK_n (see §2.5)	G iff $m = 1, n \leq 4$ [822]

Table 1: **Summary of Graceful Results continued**

Graph	*Graceful*
$C_s \cup P_n$	G iff $s + n \geq 6$ [1494]
$C_p \cup C_q$	G iff $p + q \equiv 0, 3 \pmod 4$ [11]
$C_n \cup K_{p,q}$	for $n > 8$ G iff $n \equiv 0, 3 \pmod 4$ [1698] G $C_6 \times K_{1,2n+1}$ [228] G $C_3 \times K_{m,n}$ iff $m, n \geq 2$ [1291] G $C_4 \times K_{m,n}$ iff $(m, n) \neq (1, 1)$[1291] G $C_7 \times K_{m,n}$ [1291] G $C_8 \times K_{m,n}$ [1291]
$K_i \cup K_{m,n}$	G [228]
$\bigcup_{i=1}^{t} K_{m_i,n_i}$	G $2 \leq m_i < n_i$ [228]
$C_m \cup \bigcup_{i=1}^{t} K_{m_i,n_i}$	G $2 \leq m_i < n_i$, $m \equiv 0$ or $3 \pmod 4$, $m \geq 11$ [228]
$G + \overline{K_t}$	G for connected G
double cones $C_n + \overline{K_2}$	G for $n = 3, 4, 5, 7, 8, 9, 11, 12$ not G for $n \equiv 2 \pmod 4$ [1187]
t-point suspension $C_n + \overline{K_t}$	G if $n \equiv 0$ or $3 \pmod{12}$ [279] not G if t is even and $n \equiv 2, 6, 10 \pmod{12}$ G if $n = 4, 7, 11$ or 19 [279] G if $n = 5$ or 9 and $t = 2$ [279]
P_n^2 (see §2.7)	G [871]
Petersen $P(n, k)$ (see §2.7)	G for $n = 5, 6, 7, 8, 9, 10$ [1187], $(n, k) = (8t, 3)$[1575]

Table 2: **Summary of Harmonious Results**

Graph	*Harmonious*
trees	H if ≤ 26 vertices [65] H? [586] H caterpillars [586] ? lobsters
cycles C_n	H iff n is odd [586]
wheels W_n	H [586]
helms (see §2.2)	H [572], [982]
webs (see §2.2)	H if cycle is odd
gears (see §2.2)	H [372]
cycles with P_k-chord (see §2.2)	?
C_n with k consecutive chords (see §2.2)	?
unicyclic graphs	?
P_n^k	H if $k = 2$ [583], $k = 3$ [1260] H if k is even and $k/2 \leq (n-1)/2$ [1710]
$C_n^{(t)}$ (see §2.2)	$n = 3$ H iff $t \not\equiv 2 \pmod 4$ [586] H if $n = 4,\ t > 1$ [1328]
triangular snakes (see §2.2)	H if number of blocks is odd [1663] not H if number of blocks $\equiv 2$ (mod 4) [1663]
K_4-snakes (see §2.2)	H [584]
quadrilateral snakes (see §2.2)	?
crowns $C_n \odot K_1$	H [583], [969]
grids $P_m \times P_n$	H iff $(m, n) \neq (2, 2)$ [756]

Table 2: **Summary of Harmonious Results continued**

Graph	*Harmonious*
prisms $C_m \times P_n$	H if $n = 2, m \neq 4$ [534] H if n odd [586] H if $m = 4$ and $n \geq 3$ [756]
torus grids $C_m \times C_n$,	H if $m = 4$, $n > 1$ [756] not H if $m \not\equiv 0 \pmod 4$ and n odd [756]
vertex-deleted $C_m \times P_n$	H if $n = 2$ [534]
edge-deleted $C_m \times P_n$	H if $n = 2$ [534]
Möbius ladders M_n (see §2.3)	H iff $n \neq 3$ [529]
stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, H if m even [582], [1195] not H $m \equiv 3 \pmod 4$, $n = 2$, (parity condition) H if $m \equiv 1 \pmod 4$, $n = 2$ [572]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	H if and only if $n \geq 4$ [683]
$K_4 \times P_n$	H [1195]
K_n	H iff $n \leq 4$ [586]
$K_{m,n}$	H iff m or $n = 1$ [586]
$K_{1,m,n}$	H [98]
$K_{1,1,m,n}$	H [572]
windmills $K_n^{(m)}$ $(n > 3)$ (see §2.4)	 H if $n = 4$ [666] $m = 2,$ H? iff $n = 4$ [586] not H if $m = 2, n$ odd or 6 [586] not H for some cases $m = 3$ [968]

Table 2: **Summary of Harmonious Results continued**

Graph	*Harmonious*
$B(n,r,m)$ $r>1$ (see §2.4)	$(n,r)=(3,2),(4,3)$ [1288]
mK_n (see §2.5)	H $n=3,\ m$ odd [970] not H for n odd, $m \equiv 2 \pmod 4$ [970]
nG	H when G is harmonious and n odd [1696]
G^n	H when G is harmonious and n odd [1696]
$C_s \cup P_n$	?
fans $F_n = P_n + K_1$	H [586]
$nC_m + K_1$ $n \not\equiv 0 \bmod 4$	H [372]
double fans $P_n + \overline{K_2}$	H [586]
t-point suspension $P_n + \overline{K_t}$ of P_n	H [1195]
$S_m + K_1$	H [572], [354]
t-point suspension $C_n + \overline{K_t}$ of C_n	H if n odd and $t=2$ [1195], [572] not H if $n \equiv 2,4,6 \pmod 8$ and $t=2$ [572]
P_n^2 (see §2.7)	H [583], [969]
Petersen $P(n,k)$ (see §2.7)	H [572], [905]

3 Variations of Graceful Labelings

3.1 α-labelings

In 1966 Rosa [1209] defined an *α-labeling* (or *α-valuation*) as a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. (Other names for such labelings are *balanced*, *interlaced*, and *strongly graceful*.) It follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an α-labeling is necessarily bipartite and therefore can not contain a cycle of odd length. Wu [1655] has shown that a necessary condition for a bipartite graph with n edges and degree sequence $d_1, d_2, \ldots, d_p$ to have an α-labeling is that the $\gcd(d_1, d_2, \ldots, d_p, n)$ divides $n(n-1)/2$.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an α-labeling is a typical approach. (See [354], [583], [371], and [756].) Moreover, Jungreis and Reid [756] showed how sequential labelings of graphs (see Section 4.1) can often be obtained by modifying α-labelings of the graphs.

Graphs with α-labelings have proved to be useful in the development of the theory of graph decompositions. Rosa [1209], for instance, has shown that if G is a graph with q edges and has an α-labeling, then for every natural number p, the complete graph K_{2qp+1} can be decomposed into copies of G in such a way that the automorphism group of the decomposition itself contains the cyclic group of order p. In the same vein El-Zanati and Vanden Eynden [463] proved that if G has q edges and admits an α-labeling then $K_{qm,qn}$ can be partitioned into subgraphs isomorphic to G for all positive integers m and n. Although a proof of Ringel's conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have α-labelings are easy to construct (one exmple is the subdivision graph of $K_{1,3}$ — see [1209]). Kotzig [816] has shown however that almost all trees have α-labelings.

As to which graphs have α-labelings, Rosa [1209] observed that the n-cycle has an α-labeling if and only if $n \equiv 0 \pmod 4$ whereas P_n always has an α-labeling. Other familiar graphs that have α-labelings include caterpillars [1209], the n-cube [815], Möbius ladders M_n when n is odd (see §2.3) for the definition) [1122], B_{4n+1} (i.e., books with $4n+1$ pages) [533], $C_{2m} \cup C_{2m}$ and $C_{4m} \cup C_{4m} \cup C_{4m}$ for all $m > 1$ [817], $C_{4m} \cup C_{4m} \cup C_{4n}$ for all $(m, n) \neq 1, 1)$ [477], $P_n \times Q_n$ [1018], $K_{1,2k} \times Q_n$ [1018], $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$ [852], $C_{4m} \cup C_{4n+2} \cup C_{4r+2}, C_{4m} \cup C_{4n} \cup C_{4r}$ when $m + n \leq r$ [11], $C_{4m} \cup C_{4n} \cup C_{4r} \cup C_{4s}$ when $m \geq n + r + s$ [7], $C_{4m} \cup C_{4n} \cup C_{4r+2} \cup C_{4s+2}$ when $m \geq n + r + s + 1$ [7], $((m+1)^2 + 1)C_4$ for all m [1724], $k^2 C_4$ for all k [1724], and $(k^2 + k)C_4$ for all k [1724]. Abrham and Kotzig [9] have shown kC_4 has an α-labeling for $4 \leq k \leq 10$ and that if kC_4 has an α-labeling then so does $(4k+1)C_4, (5k+1)C_4$ and $(9k+1)C_4$. Eshghi [472] proved that $5C_{4k}$ has an α-labeling for all k. In [477] Eshghi and Carter show several families of graphs of the form $C_{4n_1} \cup C_{4n_2} \cup \cdots \cup C_{4n_k}$ have α-labelings. Pei-Shan Lee [862] proved that $C_6 \times P_{2t+1}$ and gear graphs have α-labelings. He raises the question of whether $C_{4m+2} \times P_{2t+1}$ has an α-labeling for all m. Brankovic, Murch, Pond, and Rosa [306] conjectured that all trees with maximum degree three and a perfect matching have an α-labeling.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [498] have shown that if $m \equiv 0 \pmod 4$ then the one-point union of 2, 3, or 4 copies of C_m admits an α-labeling, and if $m \equiv 2 \pmod 4$ then the one-point union of 2 or 4 copies of C_m admits an α-labeling. They conjecture that the one-point union of n copies of C_m admits an α-labeling if and only if $mn \equiv 0 \pmod 4$.

In his 2001 Ph. D. thesis Selvaraju [1250] investigated the one-point union of complete bipartite graphs. He proves that the one-point unions of the following forms have an α-labeling: K_{m,n_1} and K_{m,n_2}; K_{m_1,n_1}, K_{m_2,n_2}, and K_{m_3,n_3} where $m_1 \leq m_2 \leq m_3$ and $n_1 < n_2 < n_3$; $K_{m_1,n}$, $K_{m_2,n}$, and $K_{m_3,n}$ where $m_1 < m_2 < m_3 \leq 2n$.

Zhile [1724] uses $C_m(n)$ to denote the connected graph all of whose blocks are C_m and whose block-cutpoint-graph is a path. He proves that for all positive integers m and n, $C_{4m}(n)$ has an α-labeling but $C_m(n)$ does not have an α-labeling when m is odd.

Abrham and Kotzig [11] have proved that $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m + n \equiv 0 \pmod 4$. Kotzig [817] has also shown that $C_4 \cup C_4 \cup C_4$ does not have an α-labeling. He asked if $n = 3$ is the only integer such that the disjoint union of n copies of C_4 does not have an α-labeling. This was confirmed by Abrham and Kotzig in [10]. Eshghi [471] proved that every 2-regular bipartite graph with 3 components has an α-labeling if and only if the number of edges is a multiple of four except for $C_4 \cup C_4 \cup C_4$. In [474] Eshghi gives more results on the existence of α-labelings for various families of disjoint union of cycles.

Jungreis and Reid [756] investigated the existence of α-labelings for graphs of the form $P_m \times P_n, C_m \times P_n$, and $C_m \times C_n$ (see also [532]). Of course, the cases involving C_m with m odd are not bipartite, so there is no α-labeling. The only unresolved cases among these three families are $C_{4m+2} \times P_{2n+1}$ and $C_{4m+2} \times C_{4n+2}$. All other cases result in α-labelings. Balakrishman [209] uses the notation $Q_n(G)$ to denote the graph $P_2 \times P_2 \times \cdots \times P_2 \times G$ where P_2 occurs $n-1$ times. Snevily [1408] has shown that the graphs $Q_n(C_{4m})$ and the cycles C_{4m} with the path P_n adjoined at each vertex have α-labelings. He [1409] also has shown that compositions of the form $G[\overline{K_n}]$ (see §2.3 for the definition) have an α-labeling whenever G does (see §2.3 for the definition of composition). Balakrishman and Kumar [212] have shown that all graphs of the form $Q_n(G)$ where G is $K_{3,3}, K_{4,4}$, or P_m have an α-labeling. Balakrishman [209] poses the following two problems. For which graphs G does $Q_n(G)$ have an α-labeling? For which graphs G does $Q_n(G)$ have a graceful labeling?

Rosa [1209] has shown that $K_{m,n}$ has an α-labeling (see also [225]). In [686] Ichishima and Oshima proved that if m, s and t are integers with $m \geq 1, s \geq 2$, and $t \geq 2$, then the graph $mK_{s,t}$ has an α-labeling if and only if $(m, s, t) \neq (3, 2, 2)$. Barrientos [225] has shown that for n even the graph obtained from the wheel W_n by attaching a pendent edge at each vertex has an α-labeling. In [232] Barrientos shows how to construct graceful graphs that are formed from the one-point union of a tree that has an α-labeling, P_2, and the cycle C_n. In some cases, P_2 is not needed. Qian [1173] has proved that quadrilateral snakes have α-labelings. Yu, Lee, and Chin [1708] showed that Q_3-and Q_3-snakes have α-labelings. Fu and Wu [522] showed that if T is a tree that has an α-labeling with partite sets V_1 and V_2 then the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to every vertex of V_1 has an α-labeling. Similarly, they prove that the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to the vertices of V_1 and new vertices $u_1, u_2, \ldots, u_t$ to every vertex of V_2 has an α-labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of V_1 or the vertices of both V_1 and V_2, the resulting graphs have α-labelings. Fu and Wu [522] further show that if T is a tree with an α-labeling and the sizes of the two partite sets of T differ at by at most 1, then $T \times P_m$ has an α-labeling.

Selvaraju and G. Sethurman [1252] prove that the graphs obtained from a path P_n by joining all the pairs of vertices u, v of P_n with $d(u, v) = 3$ and the graphs obtained by identifying one of vertices of degree 2 of such graphs with the center of a star and the other vertex the graph of degree 2 with the center of another star (the two stars needs need not have the same size) have α labelings. They conjecture that the analogous graphs where 3 is replaced with any t with

$2 \leq t \leq n-2$ have α-labelings.

Lee and Liu [879] investigated the mirror graph $M(m,n)$ of $K_{m,n}$ (see §2.3 for the definition) for α-labelings. They proved: $M(m,n)$ has an α-labeling when n is odd or m is even; $M(1,n)$ has an α-labeling when $n \equiv 0 \pmod 4$; $M(m,n)$ does not have an α-labeling when m is odd and $n \equiv 2 \pmod 4$, or when $m \equiv 3 \pmod 4$ and $n \equiv 4 \pmod 8$.

Barrientos [226] defines a *chain graph* as one with blocks $B_1, B_2, \ldots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cutpoint graph is a path. He shows that if $B_1, B_2, \ldots, B_m$ are blocks that have α-labelings then there exists a chain graph G with blocks $B_1, B_2, \ldots, B_m$ that has an α-labeling. He also shows that if $B_1, B_2, \ldots, B_m$ are complete bipartite graphs, then any chain graph G obtained by concatenation of these blocks has an α-labeling.

Wu ([1654] and [1656]) has given a number of methods for constructing larger graceful graphs from graceful graphs. Let $G_1, G_2, \ldots, G_p$ be disjoint connected graphs. Let w_i be in G_i for $1 \leq i \leq p$. Let w be a new vertex not in any G_i. Form a new graph $\oplus_w(G_1, G_2, \ldots, G_p)$ by adjoining to the graph $G_1 \cup G_2 \cup \cdots \cup G_p$ the edges $ww_1, ww_2, \ldots, ww_p$. In the case where each of $G_1, G_2, \ldots, G_p$ is isomorphic to a graph G that has an α-labeling and each w_i is the isomorphic image of the same vertex in G_i, Wu shows that the resulting graph is graceful. If f is an α-labeling of a graph, the integer k with the property that for any edge uv either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$ is called the *boundary value* or *critical number* of f. Wu [1654] has also shown that if $G_1, G_2, \ldots, G_p$ are graphs of the same order and have α-labelings where the labelings for each pair of graphs G_i and G_{p-i+1} have the same boundary value for $1 \leq i \leq n/2$, then $\oplus_w(G_1, G_2, \ldots, G_p)$ is graceful. In [1652] Wu proves that if G has n edges and $n+1$ vertices and G has an α-labeling with boundary value λ, where $|n - 2\lambda - 1| \leq 1$, then $G \times P_m$ is graceful for all m.

Given graceful graphs H and G with at least one having an α-labeling Wu and Lu [1657] define four graph operations on H and G that when used repeatedly or in turns provide a large number of graceful graphs. In particular, if both H and G have α-labelings, then each of the graphs obtained by the four operations on H and G has an α-labeling.

Ajitha, Arumugan, and Germina [74] use a construction of Koh, Tan, and Rogers [804] to create trees with α-labelings from smaller trees with graceful labelings. These in turn allows them to generate large classes of trees that have a type of called edge-antimagic labelings (see §6.1). Shiue and Lu [1363] prove that the graph obtained from $K_{1,k}$ by replacing each edge with a path of length 3 has an α-labeling if and only if $k \leq 4$.

Seoud and Helmi [1274] have shown that all gear graphs have an α-labeling, all dragons with a cycle of order $n \equiv 0 \pmod 4$ have an α-labeling, and the graphs obtained by identifying an endpoint of a star S_m ($m \geq 3$) with a vertex of C_{4n} has an α-labeling.

Mavonicolas and Michael [1035] say that trees $\langle T_1, \theta_1, w_1 \rangle$ and $\langle T_2, \theta_2, w_2 \rangle$ with roots w_1 and w_2 and $|V(T_1)| = |V(T_2)|$ are *gracefully consistent* if either they are identical or they have α-labelings with the same boundary value and $\theta_1(w_1) = \theta_2(w_2)$. They use this concept to show that a number of known constructions of new graceful trees using several identical copies of a given graceful rooted tree can be extended to the case where the copies are replaced by a set of pairwise gracefully consistent trees. In particular, let $\langle T, \theta, w \rangle$ and $\langle T_0, \theta_0, w_0 \rangle$ be gracefully labeled trees rooted at w and w_0 respectively. They show that the following four constructions are adaptable to the case when a set of copies of $\langle T, \theta, w \rangle$ is replaced by a set of pairwise gracefully consistent trees. When $\theta(w) = |E(T)|$ the garland construction due to Koh, Rogers, and Tan [798] gracefully labels the tree consisting of h copies of $\langle T, w \rangle$ with their roots connected to a new vertex r. In the case when $\theta(w) = |E(T)|$ and whenever $uw \in E(T)$ and $\theta(u) \neq 0$, then $vw \in E(T)$ where $\theta(u) + \theta(v) = |E(T)|$, the attachment construction of Koh, Tan and Rogers [804] gracefully labels the tree formed by identifying the roots of h copies of $\langle T, w\rangle$. A construction given by Koh, Tan and Rogers [804] gracefully labels the tree formed by merging each vertex of $\langle T_0, w_0\rangle$ with the root of a distinct copy of $\langle T, w\rangle$. When $\theta_0(w_0) = |E(T_0)|$, let N be the set of neighbors of w_0 and let x be the vertex of T at even distance from w with $\theta(x) = 0$ or $\theta(x) = |E(T)|$. Then a construction of Burzio and Ferrarese [326] gracefully labels the tree formed by merging each non-root vertex of T_0 with the root of a distinct copy of $\langle T, w\rangle$ so that for each $v \in N$ the edge vw_0 is replaced with a new edge xw_0 (where x is in the corresponding copy of T).

Snevily [1409] says that a graph G *eventually has an α-labeling* provided that there is a graph H, called a *host* of G, which has an α-labeling and that the edge set of H can be partitioned into subgraphs isomorphic to G. He defines the *α-labeling number of G* to be $G_\alpha = \min\{t :$ there is a host H of G with $|E(H)| = t|G|\}$. Snevily proved that even cycles have α-labeling number at most 2 and he conjectured that every bipartite graph has an α-labeling number. This conjecture was proved by El-Zanati, Fu, and Shiue [460]. There are no known examples of a graph G with $G_\alpha > 2$. In [1409] Snevily conjectured that the α-labeling number for a tree with n edges is at most n. Ahmed and Snevily [55] further conjectured that the α-labeling number of any tree is at most 2. Shiue and Fu [1361] proved that the α-labeling number for a tree with n edges and radius r is at most $\lceil r/2\rceil n$. They also prove that a tree with n edges and radius r decomposes K_t for some $t \leq (r+1)n^2 + 1$.

Ahmed and Snevily [55] investigated the claim that for every tree T there exists an α-labeling of T, or else there exists a graph H_T with an α-labeling such that H_T can be decomposed into two edge-disjoint copies of T. They proved this claim is true for the graphs $C_{m,k}$ obtained from $K_{1,m}$ by replacing each edge in $K_{1,m}$ with a path of length k.

For a tree T with m edges, the *α-deficit* $\alpha_{def}(T)$ equals $m - \alpha(T)$ where $\alpha(T)$ is defined as the maximum number of distinct edge labels over all bipartite labelings of T. Rosa and Siran [1212] showed that for every $m \geq 1$, $\alpha_{def}(C_{m,2}) = \lfloor m/3\rfloor$, which implies that $(C_{m,2})_\alpha \geq 2$ for $m \geq 3$. Ahmed and Snevily [55] define the graph $C'_{m,j}$ as a comet-like tree with a central vertex of degree m where each neighbor of the central vertex is attached to j pendent vertices for $1 \leq j \leq (m-1)$. For $m \geq 3$ and $1 \leq j \leq (m-1)$ they prove: $(C'_{m,j})_\alpha \leq 2$; $(C'_{2k+1,j})_\alpha = 2$ for $1 \leq j \leq 2k$ and conjecture if $\Delta_T = (2k+1)$, then $\alpha_{def}(T) \leq k$.

Given two bipartite graphs G_1 and G_2 with partite sets H_1 and L_1 and H_2 and L_2, respectively, Snevily [1408] defines their *weak tensor product* $G_1 \overline{\bigotimes} G_2$ as the bipartite graph with vertex set $(H_1 \times H_2, L_1 \times L_2)$ and with edge $(h_1, h_2)(l_1, l_2)$ if $h_1 l_1 \in E(G_1)$ and $h_2 l_2 \in E(G_2)$. He proves that if G_1 and G_2 have α-labelings then so does $G_1 \overline{\bigotimes} G_2$. This result considerably enlarges the class of graphs known to have α-labelings.

The sequential join of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_n$ by adding edges joining each vertex of G_i with each vertex of G_{i+1} for $1 \leq i \leq n-1$. Lee and Wang [930] have shown that for all $n \geq 2$ and any positive integers $a_1, a_2, \ldots, a_n$ the sequential join of the graphs $\overline{K}_{a_1}, \overline{K}_{a_2}, \ldots, \overline{K}_{a_n}$ has an α-labeling.

In [530] Gallian and Ropp conjectured that every graph obtained by adding a single pendent edge to one or more vertices of a cycle is graceful. Qian [1173] proved this conjecture and in the case that the cycle is even he shows the graphs have an α-labeling. He further proves that for n even any graph obtained from an n-cycle by adding one or more pendent edges at some vertices has an α-labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

In [1123] Pasotti introduced the following generalization of a graceful labeling. Given a graph G with $e = d \cdot m$ edges, an injective function from $V(\Gamma)$ to the set $\{0, 1, 2, \ldots, d(m+1)-1\}$ such that $\{|f(x)-f(y)| \mid [x,y] \in E(\Gamma)\} = \{1, 2, 3, \ldots, d(m+1)-1\} - \{m+1, 2(m+1), \ldots, (d-1)(m+1)\}$ is called a *d-divisible graceful labeling* of G. Note that for $d = 1$ and of $d = e$ one obtains the classical notion of a graceful labeling and of an odd-graceful labeling (see §3.6 for the definition), respectively. A d-divisible graceful labeling of a bipartite graph G with the property that the maximum value on one of the two bipartite sets is less than the minimum value on the other one is called a *d-divisible α-labeling* of G. Pasotti proved that these new concepts allow to obtain certain cyclic graph decompositions. In particular, if there exists a d-divisible graceful labeling of a graph G of size $e = d \cdot m$ then there exists a cyclic G-decomposition of $K_{\left(\frac{e}{d}+1\right)\times 2d}$ and that if there exists a d-divisible α-labeling of a graph Γ of size e then there exists a cyclic G-decomposition of $K_{\left(\frac{e}{d}+1\right)\times 2dn}$ for any integer $n \geq 1$. She also it is proved the following: paths and stars admit a d-divisible α-labeling for any admissible d; C_{4k} admits a 2-divisible α-labeling and a 4-divisible α-labeling for any $k \geq 1$; C_{2k} admits a 2-divisible labeling for any odd integer $k > 1$; and the ladder graph L_{2k} has a 2-divisible α-labeling if and only if k is even.

In [1124], Pasotti proved the existence of d-divisible α-labelings for $C_{4k} \times P_m$ for any integers $k \geq 1$, $m \geq 2$ for $d = 2m-1,\ 2(2m-1),\ 4(2m-1)$.

Benini and Pasotti [255] proved that the generalized Petersen graph $P_{8n,3}$ admits an α-labeling for any integer $n \geq 1$ confirming that the conjecture posed by A. Vietri in [1575] is true.

For any tree $T(V,E)$ whose vertices are properly 2-colored Rosa and Širáň [1212] define a *bipartite labeling* of T as a bijection $f : V \to \{0, 1, 2, \ldots, |E|\}$ for which there is a k such that whenever $f(u) \leq k \leq f(v)$, then u and v have different colors. They define the *α-size* of a tree T as the maximum number of distinct values of the induced edge labels $|f(u) - f(v)|, uv \in E$, taken over all bipartite labelings f of T. They prove that the α-size of any tree with n edges is at least $5(n+1)/7$ and that there exist trees whose α-size is at most $(5n+9)/6$. They conjectured that minimum of the α-sizes over all trees with n edges is asymptotically $5n/6$. This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [327]. For trees with n vertices and maximum degree 3 Brankovic, Rosa, and Širáň [307] have shown that the α-size is at least $\lfloor \frac{6n}{7} \rfloor - 1$. In [306] Brankovic, Murch, Pond, and Rose provide a lower bound for the α-size trees with maximum degree three and a perfect matching as a function of a lower bound for minimum order of such a tree that does not have an α-labeling. Using a computer search they showed that all such trees on less than 30 vertices have an α-labeling. This brought the lower bound for the α-size to $14n/15$, for such trees of order n. They conjecture that all trees with maximum degree three and a perfect matching have an α-labeling. Heinrich and Hell [653] defined the *gracesize* of a graph G with n vertices as the maximum, over all bijections $f\colon V(G) \to \{1, 2, \ldots, n\}$, of the number of distinct values $|f(u) - f(v)|$ over all edges uv of G. So, from Rosa and Širáň's result, the gracesize of any tree with n edges is at least $5(n+1)/7$.

In [310] Brinkmann, Crevals, Mélot, Rylands, and Steffan define the parameter α_{def} which measures how far a tree is from having an α-labeling as it counts the minimum number of errors, that is, the minimum number of edge labels that are missing from the set of all possible labels. Trees with an α-labeling have deficit 0. For a tree $T = (V, E)$ with bipartition classes V_1 and V_2 and a bipartite labeling $f : V \to \{0, \ldots, |V|-1\}$ the *edge parity* of T is $(\sum_{i=1}^{|E|} i) \bmod 2 = \frac{1}{2}(|V|-1)|V| \bmod 2$. So if f is an α-labeling this is the sum of all edge labels modulo 2; it is 0 if $|V| \equiv 0, 1 \bmod 4$ and 1 if $|V| \equiv 2, 3 \bmod 4$. The *vertex parity* is the parity of the number of vertices of odd degree with odd label.

Brinkmann et al. [310] proved: in a tree T with α-deficit 0 the edge parity and the vertex parities are equal; and for all non-negative integers k and d and $n \geq k^2+k$, the number of trees T with n vertices, $\alpha_{\text{def}}(T) = d$ and maximum degree $n-k$ is the same. Furthermore, they provide computer results on the α-deficit of all trees with up to 26 vertices; with maximum degree 3 and up to 36 vertices, with maximum degree 4 and up to 32 vertices, and with maximum degree 5 and up to 31 vertices.

In [534] Gallian weakened the condition for an α-labeling somewhat by defining a *weakly α-labeling* as a graceful labeling for which there is an integer k so that for each edge xy either $f(x) \leq k \leq f(y)$ or $f(y) \leq k \leq f(x)$. Unlike α-labelings, this condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph; namely, that the vertex with the label k must be on every odd cycle. Gallian, Prout, and Winters [534] showed that the prisms $C_n \times P_2$ with a vertex deleted have α-labelings. The same paper reveals that $C_n \times P_2$ with an edge deleted from a cycle has an α-labeling when n is even and a weakly α-labeling when $n > 3$.

A special case of α-labeling called strongly graceful was introduced by Maheo [1018] in 1980. A graceful labeling f of a graph G is called *strongly graceful* if G is bipartite with two partite sets A and B of the same order s, the number of edges is $2t + s$, there is an integer k with $t - s \leq k \leq t + s - 1$ such that if $a \in A, f(a) \leq k$, and if $b \in B, f(b) > k$, and there is an involution π that is an automorphism of G such that: π exchanges A and B and the s edges $a\pi(a)$ where $a \in A$ have as labels the integers between $t + 1$ and $t + s$. Maheo's main result is that if G is strongly graceful then so is $G \times Q_n$. In particular, she proved that $(P_n \times Q_n) \times K_2$, B_{2n}, and $B_{2n} \times Q_n$ have strongly graceful labelings.

In 1999 Broersma and Hoede [311] conjectured that every tree containing a perfect matching is strongly graceful. Yao, Cheng, Yao, and Zhao [1686] proved that this conjecture is true for every tree with diameter at most 5 and provided a method for constructing strongly graceful trees.

El-Zanati and Vanden Eynden [464] call a strongly graceful labeling a *strong α-labeling*. They show that if G has a strong α-labeling, then $G \times P_n$ has an α-labeling. They show that $K_{m,2} \times K_2$ has a strong α-labeling and that $K_{m,2} \times P_n$ has an α-labeling. They also show that if G is a bipartite graph with one more vertex than the number of edges, and if G has an α-labeling such that the cardinalities of the sets of the corresponding bipartition of the vertices differ by at most 1, then $G \times K_2$ has a strong α-labeling and $G \times P_n$ has an α-labeling. El-Zanati and Vanden Eynden [464] also note that $K_{3,3} \times K_2$, $K_{3,4} \times K_2$, $K_{4,4} \times K_2$, and $C_{4k} \times K_2$ all have strong α-labelings. El-Zanati and Vanden Eynden proved that $K_{m,2} \times Q_n$ has a strong α-labeling and that $K_{m,2} \times P_n$ has an α-labeling for all n. They also prove that if G is a connected bipartite graph with partite sets of odd order such that in each partite set each vertex has the same degree, then $G \times K_2$ does not have a strong α-labeling. As a corollary they have that $K_{m,n} \times K_2$ does not have a strong α-labeling when m and n are odd.

An α-labeling f of a graph G is called *free* by El-Zanati and Vanden Eynden in [465] if the critical number k (in the definition of α-labeling) is greater than 2 and if neither 1 nor $k - 1$ is used in the labeling. Their main result is that the union of graphs with free α-labelings has an α-labeling. In particular, they show that $K_{m,n}$, $m > 1$, $n > 2$, has a free α-labeling. They also show that Q_n, $n \geq 3$, and $K_{m,2} \times Q_n$, $m > 1$, $n \geq 1$, have free α-labelings. El-Zanati [personal communication] has shown that the Heawood graph has a free α-labeling.

Wannasit and El-Zanati [1632] proved that if G is a cubic bipartite graph each of whose components is either a prism, a Möbius ladder, or has order at most 14, then G admits free -α-labeling. They conjecture that every bipartite cubic graph admits a free α-labeling.

For connected bipartite graphs Grannell, Griggs, and Holroyd [587] introduced a labeling that lies between α-labelings and graceful labelings. They call a vertex labeling f of a bipartite graph G with q edges and partite sets D and U *gracious* if f is a bijection from the vertex set of G to $\{0, 1, \ldots, q\}$ such that the set of edge labels induced by $f(u) - f(v)$ for every edge uv with $u \in U$ and $v \in D$ is $\{1, 2, \ldots, q\}$. Thus a gracious labeling of G with partite sets D and U is a graceful labeling in which every vertex in D has a label lower than every adjacent vertex. They verified by computer that every tree of size up to 20 has a gracious labeling. This led them to conjecture that every tree has a gracious labeling. For any $k > 1$ and any tree T Grannell et al. say that T has a *gracious k-labeling* if the vertices of T can be partitioned into sets D and U in such a way that there is a function f from the verticies of G to the integers modulo k such that the edge labels induced by $f(u) - f(v)$ where $u \in U$ and $v \in D$ have the following properties: the number of edges labeled with 0 is one less than the number of verticies labeled with 0 and for each nonzero integer t the number of edges labeled with t is the same as the number of verticies labeled with t. They prove that every nontrivial tree has a k-gracious labeling for $k = 2, 3, 4$, and 5 and that caterpillars are k-gracious for all $k \geq 2$.

The same labeling that is called gracious by Grannell, Griggs, and Holroyd is called a *near α-labeling* by El-Zanati, Kenig, and Vanden Eynden [462]. The latter prove that if G is a graph with n edges that has a near α-labeling then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x and a cyclic G-decomposition of $K_{n,n}$. They further prove that if G and H have near α-labelings, then so does their weak tensor product (see earlier part of this section) with respect to the corresponding vertex partitions. They conjecture that every tree has a near α-labeling.

Another kind of labelings for trees was introduced by Ringel, Llado, and Serra [1199] in an approach to proving their conjecture $K_{n,n}$ is edge-decomposable into n copies of any given tree with n edges. If T is a tree with n edges and partite sets A and B, they define a labeling f from the set of vertices to $\{1, 2, \ldots, n\}$ to be a *bigraceful* labeling of T if f restricted to A is injective, f restricted to B is injective, and the edge labels given by $f(y) - f(x)$ where yx is an edge with y in B and x in A is the set $\{0, 1, 2, \ldots, n-1\}$. (Notice that this terminology conflicts with that given in Section 2.7 In particular, the Ringel, Llado, and Serra bigraceful does not imply the usual graceful.) Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars $S_{k,m}$ with k spokes of paths of length m, and complete d-ary trees for d odd. They also prove that if T is a tree then there is a vertex v and a nonnegative integer m such that the addition of m leaves to v results in a bigraceful tree. They conjecture that all trees are bigraceful.

Table 3 summarizes some of the main results about α-labelings. α indicates that the graphs have an α-labeling.

Table 3: **Summary of Results on α-labelings**

Graph	*α-labeling*
cycles C_n	α iff $n \equiv 0 \pmod 4$ [1209]
caterpillars	α [1209]
n-cube	α [815]
books $B_{2n},\ B_{4n+1}$	α [1018],[533]
Möbius ladders M_{2k+1}	α [1122]
$C_m \cup C_n$	α iff m, n are even and $m + n \equiv 0 \pmod 4$[11]
$C_{4m} \cup C_{4m} \cup C_{4m}$ $(m > 1)$	α [817]
$C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$	α [817]
$mK_{s,t}$ $(m \geq 1, s, t \geq 2)$	iff $(m, s, t) \neq (3, 2, 2)$ [686]
$P_n \times Q_n$	α [1018]
$B_{2n} \times Q_n$	α [1018]
$K_{1,n} \times Q_n$	α [1018]
$K_{m,2} \times Q_n$	α [464]
$K_{m,2} \times P_n$	α [464]
$P_2 \times P_2 \times \cdots \times P_2 \times G$	α when $G = C_{4m},\ P_m,\ K_{3,3},\ K_{4,4}$ [1408]
$P_2 \times P_2 \times \cdots \times P_2 \times P_m$	α [1408]
$P_2 \times P_2 \times \cdots \times P_2 \times K_{m,m}$	α [1408] when $m = 3$ or 4
$G[\overline{K_n}]$	α when G is α [1409]

3.2 γ-Labelings

In 2004 Chartrand, Erwin, VanderJagt, and Zhang [355] define a γ-labeling of a graph G of size m as a one-to-one function f from the vertices of G to $\{0, 1, 2, \ldots, m\}$ that induces an edge labeling f' defined by $f'(uv) = |f(u) - f(v)|$ for each edge uv. They define the following parameters of a γ-labeling: $\mathrm{val}(f) = \Sigma f'(e)$ over all edges e of G; $\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f$ is a γ-labeling of $G\}$, $\mathrm{val}_{\min}(G) =$
$\min\{\mathrm{val}(f) : f$ is a $\gamma-$ labeling of $G\}$. Among their results are the following:
$\mathrm{val}_{\min}(P_n) = \mathrm{val}_{\max}(P_n) = \lfloor (n^2 - 2)/2 \rfloor; \mathrm{val}_{\min}(C_n) = 2(n-1)$; for $n \geq 4$, n even, $\mathrm{val}_{\max}(C_n) = n(n+2)/2$; for $n \geq 3$, n odd, $\mathrm{val}_{\max}(C_n) = (n-1)(n+3)/2; \mathrm{val}_{\min}(K_n) = \begin{pmatrix} n+1 \\ 3 \end{pmatrix}$; for odd n, $\mathrm{val}_{\max}(K_n) = (n^2-1)(3n^2 - 5n + 6)/24$; for even n, $\mathrm{val}_{\max}(K_n) = n(3n^3 - 5n^2 + 6n - 4)/24$; for every $n \geq 3$, $\mathrm{val}_{\min}(K_{1,n-1}) = \begin{pmatrix} \lfloor \frac{n+1}{2} \rfloor \\ 2 \end{pmatrix} + \begin{pmatrix} \lceil \frac{n+1}{2} \rceil \\ 2 \end{pmatrix}$;
$\mathrm{val}_{\max}(K_{1,n-1}) = \begin{pmatrix} n \\ 2 \end{pmatrix}$; for a connected graph of order n and size m, $\mathrm{val}_{\min}(G) = m$ if and only if G is isomorphic to P_n; if G is maximal outerplanar of order $n \geq 2$, $\mathrm{val}_{\min}(G) \geq 3n - 5$ and equality occurs if and only if $G = P_n^2$; if G is a connected r-regular bipartite graph of order n and size m where $r \geq 2$, then $\mathrm{val}_{\max}(G) = rn(2m - n + 2)/4$.

In another paper on γ-labelings of trees Chartrand, Erwin, VanderJagt, and Zhang [356] prove for $p, q \geq 2$, $\mathrm{val}_{\min}(S_{p,q})$ (that is, the graph obtained by joining the centers of $K_{1,p}$ and $K_{1,q}$ by an edge)$= (\lfloor p/2 \rfloor + 1)^2 + (\lfloor q/2 \rfloor + 1)^2 - (n_p \lfloor p/2 \rfloor + 1)^2 + (n_q \lfloor (q+2)/2 \rfloor + 1)^2)$, where n_i is 1 if i is even and n_i is 0 if n_i is odd; $\mathrm{val}_{\min}(S_{p,q}) = (p^2 + q^2 + 4pq - 3p - 3q + 2)/2$; for a connected graph G of order n at least 4, $\mathrm{val}_{\min}(G) = n$ if and only if G is a caterpillar with maximum degree 3 and has a unique vertex of degree 3; for a tree T of order n at least 4, maximum degree Δ, and diameter d, $\mathrm{val}_{\min}(T) \geq (8n + \Delta^2 - 6\Delta - 4d + \delta_\Delta)/4$ where δ_Δ is 0 if Δ is even and δ_Δ is 0 if Δ is odd. They also give a characterization of all trees of order n at least 5 whose minimum value is $n + 1$.

In [325] Bunge, Chantasartraaamee, El-Zanati, and Vanden Eynden generalized γ-labelings by introducing two labelings for tripartite graphs. Graphs G that admit either of these labelings guarantee the existence of cyclic G-decompositions of K_{2nx+1} for all positive integers x. They also proved that, except for $C_3 \cup C_3$, the disjoint union of two cycles of odd length admits one of these labelings.

3.3 Graceful-like Labelings

As a means of attacking graph decomposition problems, Rosa [1209] invented another analogue of graceful labelings by permitting the vertices of a graph with q edges to assume labels from the set $\{0, 1, \ldots, q+1\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, \ldots, q-1, q\}$ or $\{1, 2, \ldots, q-1, q+1\}$. He calls these *$\hat{\rho}$-labelings.* Frucht [517] used the term *nearly graceful labeling* instead of *$\hat{\rho}$-labelings.* Frucht [517] has shown that the following graphs have nearly graceful labelings with edge labels from $\{1, 2, \ldots, q-1, q+1\}$: $P_m \cup P_n$; $S_m \cup S_n$; $S_m \cup P_n$; $G \cup K_2$ where G is graceful; and $C_3 \cup K_2 \cup S_m$ where m is even or $m \equiv 3 \pmod{14}$. Seoud and Elsakhawi [1268] have shown that all cycles are nearly graceful. Barrientos [224] proved that C_n is nearly graceful with edge labels $1, 2, \ldots, n-1, n+1$ if and only if $n \equiv 1$ or $2 \pmod 4$. Gao [543] shows that a variation of banana trees is odd-graceful (see § 3.6 definitiion) and in some cases has a nearly graceful labeling. In 1988 Rosa [1211] conjectured that triangular snakes with $t \equiv 0$ or 1 (mod 4) blocks are graceful and those with $t \equiv 2$ or 3 (mod 4) blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [1078] proved Rosa's conjecture while introducing the slightly stronger concept of *almost graceful* by permitting the vertex labels to come from $\{0, 1, 2, \ldots, q-1, q+1\}$ while the edge labels are $1, 2, \ldots, q-1, q$, or $1, 2, \ldots, q-1, q+1$. More generally, Rosa [1211] conjectured that all triangular cacti are either graceful or near graceful and suggested the use of Skolem sequences to label some types of triangular cacti. Dyer, Payne, Shalaby, and Wicks [450] verified the conjecture for two families of triangular cacti using Langford sequences to obtain Skolem and hooked Skolem sequences with specific subsequences.

Seoud and Elsakhawi [1268] and [1269] have shown that the following graphs are almost graceful: $C_n; P_n + \overline{K_m}; P_n + K_{1,m}; K_{m,n}; K_{1,m,n}; K_{2,2,m}; K_{1,1,m,n}$; $P_n \times P_3$ $(n \geq 3)$; $K_5 \cup K_{1,n}; K_6 \cup K_{1,n}$, and ladders.

The *symmetric product* $G_1 \oplus G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2)\}$ where u_1u_2 is an edge in G_1 or v_1v_2 is an edge in G_2 but not both u_1u_2 is an edge in G_1 and v_1v_2 is an edge in G_2. In [1269] Seoud and Elsakhawi show that $P_2 \oplus \overline{K_2}$ $(n \geq 2)$ is arbitrarily graceful.

For a graph G with p vertices, q edges, and $1 \leq k \leq q$, Eshghi [473] defines a *holey α-labeling with respect to k* as an injective vertex labeling f for which $f(v) \in \{1, 2, \ldots, q+1\}$ for all v, $\{|f(u) - f(v)| \mid \text{for all edges } uv\} = \{1, 2, \ldots, k-1, k+1, \ldots, q+1\}$, and there exist an integer γ with $0 \leq \gamma \leq q$ such that $\min\{f(u), f(v)\} \leq \gamma \leq \max\{f(u), f(v)\}$. He proves the following: P_n has a holey α-labeling with respect to all k; C_n has a holey α-labeling with respect to k if and only if either $n \equiv 2 \pmod 4$, k is even, and $(n, k) \neq (10, 6)$, or $n \equiv 0 \pmod 4$ and k is odd.

Recall from Section 2.2 that a *kC_n-snake* is a connected graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to C_n. In addition to his results on the graceful kC_n-snakes given in Section 2.2, Barrientos [228] proved that when k is odd the linear kC_6-snake is nearly graceful and that $C_m \cup K_{1,n}$ is nearly graceful when $m = 3, 4, 5$, and 6.

Yet another kind of labeling introduced by Rosa in his 1967 paper [1209] is a ρ-labeling. (Sometimes called a *rosy* labeling). A *ρ-labeling* (or ρ-valuation) of a graph is an injection from the vertices of the graph with q edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, \ldots, a_q$, then $a_i = i$ or $a_i = 2q + 1 - i$. Rosa [1209] proved that a cyclic decomposition of the edge set of the complete graph K_{2q+1} into subgraphs isomorphic to a given graph G with q edges exists if and only if G has a ρ-labeling. (A decomposition of K_n into copies of G is called *cyclic* if the automorphism group of the decomposition itself contains the cyclic group of order n.) It is known that every graph with at most 11 edges has a ρ-labeling and that all lobsters have a ρ-labeling (see [348]). Donovan, El-Zanati, Vanden Eyden, and Sutinuntopas [438] prove that rC_m has a ρ-labeling (or a more restrictive labeling) when $r \leq 4$. They conjecture that every 2-regular graph has a ρ-labeling. Gannon and El-Zanati [538] proved that for any odd $n \geq 7, rC_n$ admits ρ-labelings. The cases $n = 3$ and $n = 5$ were done in [436] and [461]. Aguado, El-Zanati, Hake, Stob, and Yayla [40] give a ρ-labeling of $C_r \cup C_s \cup C_t$ for each of the cases where $r \equiv 0$, $s \equiv 1$, $t \equiv 1 \pmod 4$; $r \equiv 0$, $s \equiv 3$, $t \equiv 3 \pmod 4$; and $r \equiv 1$, $s \equiv 1$, $t \equiv 3 \pmod 4$; (iv) $r \equiv 1$, $s \equiv 2$, $t \equiv 3 \pmod 4$; (v) $r \equiv 3$, $s \equiv 3$, $t \equiv 3 \pmod 4$. Caro, Roditty, and Schőnheim [348] provide a construction for the adjacency matrix for every graph that has a ρ-labeling. They ask the following question: If H is a connected graph having a ρ-labeling and q edges and G is a new graph with q edges constructed by breaking H up into disconnected parts, does G also have a ρ-labeling? Kézdy

[784] defines a *stunted tree* as one whose edges can be labeled with $e_1, e_2, \ldots, e_n$ so that e_1 and e_2 are incident and, for all $j = 3, 4, \ldots, n$, edge e_j is incident to at least one edge e_k satisfying $2k \le j - 1$. He uses Alon's "Combinatorial Nullstellensatz" to prove that if $2n + 1$ is prime, then every stunted tree with n edges has a ρ-labeling.

Recall a kayak paddle $KP(k, m, l)$ is the graph obtained by joining C_k and C_m by a path of length l. Fronček and Tollefeson [512], [513] proved that $KP(r, s, l)$ has a ρ-labeling for all cases. As a corollary they have that the complete graph K_{2n+1} is decomposable into kayak paddles with n edges.

In [509] Fronček generalizes the notion of an α-labeling by showing that if a graph G on n edges allows a certain type of ρ-labeling), called α_2-labeling, then for any positive integer k the complete graph K_{2nk+1} can be decomposed into copies of G.

In their investigation of cyclic decompositions of complete graphs El-Zanati, Vanden Eynden, and Punnim [467] introduced two kinds of labelings. They say a bipartite graph G with n edges and partite sets A and B has a *θ-labeling* h if h is a one-to-one function from $V(G)$ to $\{0, 1, \ldots, 2n\}$ such that $\{|h(b) - h(a)|\ ab \in E(G), a \in A, b \in B\} = \{1, 2, \ldots, n\}$. They call h a *ρ^+-labeling* of G if h is a one-to-one function from $V(G)$ to $\{0, 1, \ldots, 2n\}$ and the integers $h(x) - h(y)$ are distinct modulo $2n + 1$ taken over all ordered pairs (x, y) where xy is an edge in G, and $h(b) > h(a)$ whenever $a \in A, b \in B$ and ab is an edge in G. Note that θ-labelings are ρ^+-labelings and ρ^+-labelings are ρ-labelings. They prove that if G is a bipartite graph with n edges and a ρ^+-labeling, then for every positive integer x there is a cyclic G-decomposition of K_{2nx+1}. They prove the following graphs have ρ^+-labelings: trees of diameter at most 5, C_{2n}, lobsters, and comets (that is, graphs obtained from stars by replacing each edge by a path of some fixed length). They also prove that the disjoint union of graphs with α-labelings have a θ-labeling and conjecture that all forests have ρ-labelings.

A *σ-labeling* of $G(V, E)$ is a one-to-one function f from V to $\{0, 1, \ldots, 2|E|\}$ such that $\{|f(u) - f(v)| \mid uv \in E(G)\} = \{1, 2, \ldots, |E|\}$. Such a labeling of G yields cyclic G-decompositions of K_{2n+1} and of $K_{2n+2} - F$, where F is a 1-factor of K_{2n+2}. El-Zanati and Vanden Eynden (see [39]) have conjectured that that every 2-regular graph with n edges has a ρ-labeling and, if $n \equiv 0$ or $3 \pmod 4$, then every 2-regular graph has a σ-labeling. Aguado and El-Zanati [39] have proved that the latter conjecture holds when the graph has at most three components.

Given a bipartite graph G with partite sets X and Y and graphs H_1 with p vertices and H_2 with q vertices, Fronček and Winters [514] define the *bicomposition* of G and H_1 and H_2, $G[H_1, H_2]$, as the graph obtained from G by replacing each vertex of X by a copy of H_1, each vertex of Y by a copy of H_2, and every edge xy by a graph isomorphic to $K_{p,q}$ with the partite sets corresponding to the vertices x and y. They prove that if G is a bipartite graph with n edges and G has a θ-labeling that maps the vertex set $V = X \cup Y$ into a subset of $\{0, 1, 2, \ldots, 2n\}$, then the bicomposition $G[\overline{K_p}, \overline{K_q}]$ has a θ-labeling for every $p, q \ge 1$. As corollaries they have: if a bipartite graph G with n edges and at most $n + 1$ vertices has a gracious labeling (see §3.1), then the bicomposition graph $G[\overline{K_p}, \overline{K_q}]$ has a gracious labeling for every $p, q \ge 1$, and if a bipartite graph G with n edges has a θ-labeling, then for every $p, q \ge 1$, the bicomposition $G[\overline{K_p}, \overline{K_q}]$ decomposes the complete graph K_{2npq+1}.

In a paper published in 2009 [466] El-Zannati and Vanden Eynden survey "Rosa-type" labelings. That is, labelings of a graph G that yield cyclic G-decompositions of K_{2n+1} or K_{2nx+1} for all natural numbers x. The 2009 survey by Fronček [508] includes generalizations of ρ- and α-labelings that have been used for finding decompositions of complete graphs that are not covered in [466].

Blinco, El-Zanati, and Vanden Eynden [282] call a non-bipartite graph *almost-bipartite* if the removal of some edge results in a bipartite graph. For these kinds of graphs G they call a labeling f a γ-*labeling* of G if the following conditions are met: f is a ρ-labeling; G is tripartite with vertex tripartition A, B, C with $C = \{c\}$ and $\bar{b} \in B$ such that $\{\bar{b}, c\}$ is the unique edge joining an element of B to c; if av is an edge of G with $a \in A$, then $f(a) < f(v)$; and $f(c) - f(\bar{b}) = n$. (In § 3.2 the term γ-labeling is used for a different kind of labeling.) They prove that if an almost-bipartite graph G with n edges has a γ-labeling then there is a cyclic G-decomposition of K_{2nx+1} for all x. They prove that all odd cycles with more than 3 vertices have a γ-labeling and that $C_3 \cup C_{4m}$ has a γ-labeling if and only if $m > 1$. In [324] Bunge, El-Zanati, and Vanden Eynden prove that every 2-regular almost bipartite graph other than C_3 and $C_3 \cup C_4$ have a γ-labeling.

In [282] Blinco, El-Zanati, and Vanden Eynden consider a slightly restricted ρ^+-labeling for a bipartite graph with partite sets A and B by requiring that there exists a number λ with the property that $\rho^+(a) \leq \lambda$ for all $a \in A$ and $\rho^+(b) > \lambda$ for all $b \in B$. They denote such a labeling by ρ^{++}. They use this kind of labeling to show that if G is a 2-regular graph of order n in which each component has even order then there is a cyclic G-decomposition of K_{2nx+1} for all x. They also conjecture that every bipartite graph has a ρ-labeling and every 2-regular graph has a ρ-labeling.

Dufour [449] and Eldergill [452] have some results on the decomposition of complete graphs using labeling methods. Balakrishnan and Sampathkumar [214] showed that for each positive integer n the graph $\overline{K_n} + 2K_2$ admits a ρ-labeling. Balakrishnan [209] asks if it is true that $\overline{K_n} + mK_2$ admits a ρ-labeling for all n and m. Fronček [507] and Fronček and Kubesa [511] have introduced several kinds of labelings for the purpose of proving the existence of special kinds of decompositions of complete graphs into spanning trees.

For (p, q)-graphs with $p = q + 1$, Frucht [517] has introduced a stronger version of almost graceful graphs by permitting as vertex labels $\{0, 1, \ldots, q-1, q+1\}$ and as edge labels $\{1, 2, \ldots, q\}$. He calls such a labeling *pseudograceful*. Frucht proved that P_n $(n \geq 3)$, combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), $C_3 \cup P_n$ $(n \neq 3)$, and $C_4 \cup P_n$ $(n \neq 1)$ are pseudograceful whereas $K_{1,n}$ $(n \geq 3)$ is not. Kishore [794] proved that $C_s \cup P_n$ is pseudograceful when $s \geq 5$ and $n \geq (s+7)/2$ and that $C_s \cup S_n$ is pseudograceful when $s = 3, s = 4$, and $s \geq 7$. Seoud and Youssef [1295] and [1291] extended the definition of pseudograceful to all graphs with $p \leq q+1$. They proved that K_m is pseudograceful if and only if $m = 1, 3$, or 4 [1291]; $K_{m,n}$ is pseudograceful when $n \geq 2$, and $P_m + \overline{K_n}$ $(m \geq 2)$ [1295] is pseudograceful. They also proved that if G is pseudograceful, then $G \cup K_{m,n}$ is graceful for $m \geq 2$ and $n \geq 2$ and $G \cup K_{m,n}$ is pseudograceful for $m \geq 2, n \geq 2$ and $(m, n) \neq (2, 2)$ [1291]. They ask if $G \cup K_{2,2}$ is pseudograceful whenever G is. Seoud and Youssef [1291] observed that if G is a pseudograceful Eulerian graph with q edges, then $q \equiv 0$ or 3 (mod 4). Youssef [1698] has shown that C_n is pseudograceful if and only if $n \equiv 0$ or 3 (mod 4), and for $n > 8$ and $n \equiv 0$ or 3 (mod 4), $C_n \cup K_{p,q}$ is pseudograceful for all $p, q \geq 2$ except $(p, q) = (2, 2)$. Youssef [1695] has shown that if H is pseudograceful and G has an α-labeling with k being the smaller vertex label of the edge labeled with 1 and if either $k + 2$ or $k - 1$ is not a vertex label of G, then $G \cup H$ is graceful. In [1699] Youssef shows that if G is (p, q) pseudograceful graph with $p = q + 1$, then $G \cup S_m$ is Skolem-graceful. As a corollary he obtains that for all $n \geq 2$, $P_n \cup S_m$ is Skolem-graceful if and only if $n \geq 3$ or $n = 2$ and m is even.

For a graph G without isolated vertices Ichishima, Muntaner-Batle, and Oshima [680] defined the *beta-number* of G to be either the smallest positive integer n for which there exists an injective function f from the vertices of G to $\{1, 2, \ldots n\}$ such that when each edge uv is labeled $|f(u) - f(v)|$ the resulting set of edge labels is $\{c, c+1, \ldots, c+|E(G)|-1\}$ for some positive integer c or $+\infty$ if there exists no such integer n. They defined the *strong beta-number* of G to be either the smallest positive integer n for which there exists an injective function f from the vertices of G to $\{1, 2, \ldots n\}$ such that when each edge uv is labeled $|f(u) - f(v)|$ the resulting set of edge labels is $\{1, 2, \ldots, |E(G)|\}$ or $+\infty$ if there exists no such integer n. They gave some necessary conditions for a graph to have a finite (strong) beta-number and some sufficient conditions for a graph to have a finite (strong) beta-number. They also determined formulas for the beta-numbers and strong beta-numbers of $C_n,\ 2C_n, K_n\ (n \geq 2),\ S_m \cup S_n,\ P_m \cup S_n$, and prove that nontrivial trees and forests without isolated vertices have finite strong beta-numbers.

McTavish [1044] has investigated labelings of graphs with q edges where the vertex and edge labels are from $\{0, \ldots, q, q+1\}$. She calls these *$\tilde{\rho}$-labelings*. Graphs that have $\tilde{\rho}$-labelings include cycles and the disjoint union of P_n or S_n with any graceful graph.

Frucht [517] has made an observation about graceful labelings that yields nearly graceful analogs of α-labelings and weakly α-labelings in a natural way. Suppose $G(V, E)$ is a graceful graph with the vertex labeling f. For each edge xy in E, let $[f(x), f(y)]$ (where $f(x) \leq f(y)$) denote the interval of real numbers r with $f(x) \leq r \leq f(y)$. Then the intersection $\cap[f(x), f(y)]$ over all edges $xy \in E$ is a unit interval, a single point, or empty. Indeed, if f is an α-labeling of G then the intersection is a unit interval; if f is a weakly α-labeling, but not an α-labeling, then the intersection is a point; and, if f is a graceful but not a weakly α-labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes.

A (p, q)-graph G is said to be a *super graceful* graph if there is a a bijective function $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, p+q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ labeling. Perumal, Navaneethakrishnan, Nagarajan, Arockiaraj [1128] show that the graphs $P_n, C_n, P_m \odot nK_1, P_n \odot K_1$ minus a pendent edge at an endpoint of P_n are super graceful graphs.

Singh and Devaraj [1380] call a graph G with p vertices and q edges *triangular graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, T_q\}$ where T_q is the qth triangular number and the labels induced on each edge uv by $|f(u) - f(v)|$ are the first q triangular numbers. They prove the following graphs are triangular graceful: paths, level 2 rooted trees, olive trees (see § 2.1 for the definition), complete n-ary trees, double stars, caterpillars, C_{4n}, C_{4n} with pendent edges, the one-point union of C_3 and P_n, and unicyclic graphs that have C_3 as the unique cycle. They prove that wheels, helms, flowers (see §2.2 for the definition) and K_n with $n \geq 3$ are not triangular graceful. They conjecture that all trees are triangular graceful. In [1324] Sethuraman and Venkatesh introduced a new method for combining graceful trees to obtain trees that have α-labelings.

Van Bussel [1567] considered two kinds of relaxations of graceful labelings as applied to trees. He called a labeling *range-relaxed graceful* it is meets the same conditions as a graceful labeling except the range of possible vertex labels and edge labels are not restricted to the number of edges of the graph (the edges are distinctly labeled but not necessarily labeled 1 to q where q is the number of edges). Similarly, he calls a labeling *vertex-relaxed graceful* if it satisfies the conditions of a graceful labeling while permitting repeated vertex labels. He proves that every tree T with q edges has a range-relaxed graceful labeling with the vertex labels in the range $0, 1, \ldots, 2q-d$ where d is the diameter of T and that every tree on n vertices has a vertex-relaxed graceful labeling such that the number of distinct vertex labels is strictly greater than $n/2$.

Sekar [1249] calls an injective function ϕ from the vertices of a graph with q edges to $\{0, 1, 3, 4, 6, 7, \ldots, 3(q-1), 3q-2\}$ *one modulo three graceful* if the edge labels induced by labeling each edge uv with $|\phi(u) - \phi(v)|$ is $\{1, 4, 7, \ldots, 3q-2\}$. He proves that the following graphs are one modulo three graceful: P_m; C_n if and only if $n \equiv 0$ mod 4; $K_{m,n}$; $C_{2n}^{(2)}$ (the one-point union of two copies of C_{2n}); $C_n^{(t)}$ for $n = 4$ or 8 and $t > 2$; $C_6^{(t)}$ and $t \geq 4$; caterpillars; stars; lobsters; banana trees; rooted trees of height 2; ladders; the graphs obtained by identifying the endpoints of any number of copies of P_n; the graph obtained by attaching pendent edges to each endpoint of two identical stars and then identifying one endpoint from each of these graphs; the graph obtained by identifying a vertex of C_{4k+2} with an endpoint of a star; n-polygonal snakes (see §2.2) for $n \equiv 0 \pmod 4$; n-polygonal snakes for $n \equiv 2 \pmod 4$ where the number of polygons is even; crowns $C_n \odot K_1$ for n even; $C_{2n} \odot P_m$ (C_{2n} with P_m attached at each vertex of the cycle) for $m \geq 3$; chains of cycles (see §2.2) of the form $C_{4,m}, C_{6,2m}$, and $C_{8,m}$. He conjectures that every one modulo three graceful graph is graceful.

Deviating from the standard definition of Fibonacci numbers, Kathiresan and Amutha [773] define $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$. They call a function $f : V(G) \to \{0, 1, 2, \ldots, F_q\}$ where F_q is their qth Fibonacci number, to be *Fibonacci graceful labeling* if the induced edge labeling $\overline{f}(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. If a graph admits a Fibonacci graceful labeling, it is is called a *Fibonacci graceful* graph. They prove the following: K_n is Fibonacci graceful if and only if $n \leq 3$; if an Eulerian graph with q edges is Fibonacci graceful then $q \equiv 0 \pmod 3$; paths are Fibonacci graceful; fans $P_n \odot K_1$ are Fibonacci graceful; squares of paths P_n^2 are Fibonacci graceful; and caterpillars are Fibonacci graceful. They define a function $f : V(G) \to \{0, F_1, F_2, \ldots, F_q\}$ where F_i is the ith Fibonacci number, to be *super Fibonacci graceful* labeling if the induced labeling $\overline{f}(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. They show that bistars $B_{n,n}$ are Fibonacci graceful but not super Fibonacci graceful for $n \geq 5$; cycles C_n are super Fibonacci graceful if and only if $n \equiv 0 \pmod 3$; if G is Fibonacci or super Fibonacci graceful then $G \odot K_1$ is Fibonacci graceful; if G_1 and G_2 are super Fibonacci graceful in which no two adjacent vertices have the labeling 1 and 2 then $G_1 \cup G_2$ is Fibonacci graceful; and if $G_1, G_2, \ldots, G_n$ are super Fibonacci graceful graphs in which no two adjacent vertices are labeled with 1 and 2 then the amalgamation of $G_1, G_2, \ldots, G_n$ obtained by identifying the vertices having labels 0 is also a super Fibonacci graceful.

Vaidya and Prajapati [1541] proved: the graphs obtained joining a vertex of C_{3m} and a vertex of C_{3n} by a path P_k are Fibonacci graceful; the graphs obtained by starting with any number of copies of C_{3m} and joining each copy with a copy of the next by identifying the end points of a path with a vertex of each succesive pair of C_{3m} (the paths need not be the same length) are Fibonacci graceful; the one point union of C_{3m} and C_{3n} is Fibonacci graceful; the one point union of k cycles C_{3m} is super Fibonacci graceful; every cycle C_n with $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$ is an induced subgraph of a super Fibonacci graceful graph; and every cycle C_n with $n \equiv 2 \pmod 3$ can be embedded as a subgraph of a Fibonacci graceful graph.

For a graph G with q edges an injective function f from the vertices of G to $\{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$, where F_i is the ith Fibonacci number (as defined by Kathiresan and Amuth above), is said to be *almost super Fibonacci graceful* if the induced edge labeling $f*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$ or $\{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$. Sridevi, Navaneethakrishnan and Nagarajan [1431] proved that paths, combs, graphs obtained by subdividing each edge of a star, and some special types of extension of cycle related graphs are almost super Fibonacci graceful labeling.

For a graph G and a vertex v of G, Vaidya, Srivastav, Kaneria, and Kanani [1551] define

Table 4: **Summary of Results on Graceful-like labelings**

Graph	α-labeling	β-labeling	σ-labeling	ρ-labeling
Cycle $C_n,\ n \equiv 0 \bmod 4$	Y [1209]	Y	Y	Y
Cycle $C_n,\ n \equiv 3 \bmod 4$	N [1209]	Y [1209]	Y	Y
Wheels	N	Y [515], [660]	Y	Y
Trees				
Yes, if order less than or equal to	5	35 [484]	54	
Paths	Y [1209]	Y	Y	Y
Caterpillars	Y [1209]	Y	Y	Y
Firecrackers	Y [371]	Y	Y	Y
Lobsters	N[283]	?C [262]	Y	Y [348]
Bananas	?	Y [1312], [1311]	Y	Y
Symmetrical trees	N [283]	Y [266]	Y	Y
Olive trees	?	Y [1125], [3]	Y	Y
Diameter < 6	N [283]	Y [1720], [664]	Y	Y
Diameter < 7	N [283]	C	Y	Y
< 5 end vertices	N [283]	Y [1209]	Y	Y
Max degree 3	N [1212]	C	C	C
Max degree 3 and perfect matching	C [306]	C	C	C

a *vertex switching* G_v as the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Vaidya and Vihol [1558] prove the following: trees are Fibonacci graceful; the graph obtained by switching of a vertex in cycle is Fibonacci graceful; wheels and helms are not Fibonacci graceful; the graph obtained by switching of a vertex in a cycle is super Fibonacci graceful except $n \geq 6$; the graph obtained by switching of a vertex in cycle C_n for $n \geq 6$ can be embedded as an induced subgraph of a super Fibonacci graceful graph; and the graph obtained by joining two copies of a fixed fan with an edge is Fibonacci graceful.

In [309] Brešar and Klavžar define a natural extension of graceful labelings of certain tree subgraphs of hypercubes. A subgraph H of a graph G is called *isometric* if for every two vertices $u,\ v$ of H, there exists a shortest u-v path that lies in H. The isometric subgraphs of hypercubes are called *partial cubes*. Two edges $xy,\ uv$ of G are in Θ-relation if $d_G(x,u)+d_G(y,v) \neq d_G(x,v)+d_G(y,u)$. A Θ-relation is an equivalence relation that partitions $E(G)$ into Θ-classes. A Θ-*graceful labeling* of a partial cube G on n vertices is a bijection $f\colon V(G) \to \{0,1,\ldots,n-1\}$ such that, under the induced edge labeling, all edges in each Θ-class of G have the same label and distinct Θ-classes get distinct labels. They prove that several classes of partial cubes are Θ-graceful and the Cartesian product of Θ-graceful partial cubes is Θ-graceful. They also show that if there exists a class of partial cubes that contains all trees and every member of the class admits a Θ-graceful labeling then all trees are graceful.

Table 4 provides a summary results about graceful-like labelings adapted from [308]. "Y" indicates that all graphs in that class have the labeling; "N" indicates that not all graphs in that class have the labeling; "?" means unknown; "C" means conjectured.

3.4 k-graceful Labelings

A natural generalization of graceful graphs is the notion of k-graceful graphs introduced independently by Slater [1398] in 1982 and by Maheo and Thuillier [1019] in 1982. A graph G with q edges is *k-graceful* if there is labeling f from the vertices of G to $\{0, 1, 2, \ldots, q+k-1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k+1, \ldots, q+k-1\}$. Obviously, 1-graceful is graceful and it is readily shown that any graph that has an α-labeling is k-graceful for all k. Graphs that are k-graceful for all k are sometimes called *arbitrarily graceful*. Ng [1092] has shown that there are graphs that are k-graceful for all k but do not have an α-labeling.

Results of Maheo and Thuillier [1019] together with those of Slater [1398] show that: C_n is k-graceful if and only if either $n \equiv 0$ or 1 (mod 4) with k even and $k \le (n-1)/2$, or $n \equiv 3$ (mod 4) with k odd and $k \le (n^2-1)/2$. Maheo and Thuillier [1019] also proved that the wheel W_{2k+1} is k-graceful and conjectured that W_{2k} is k-graceful when $k \ne 3$ or $k \ne 4$. This conjecture was proved by Liang, Sun, and Xu [957]. Kang [762] proved that $P_m \times C_{4n}$ is k-graceful for all k. Lee and Wang [928] showed that the graphs obtained from a nontrivial path of even length by joining every other vertex to one isolated vertex (a *lotus*), the graphs obtained from a nontrivial path of even length by joining every other vertex to two isolated vertices (a *diamond*), and the graphs obtained by arranging vertices into a finite number of rows with i vertices in the ith row and in every row the jth vertex in that row is joined to the jth vertex and $j+1$st vertex of the next row (a *pyramid*) are k-graceful. Liang and Liu [947] have shown that $K_{m,n}$ is k-graceful. Bu, Gao, and Zhang [318] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are k-graceful for all k. Acharya (see [15]) has shown that a k-graceful Eulerian graph with q edges must satisfy one of the following conditions: $q \equiv 0$ (mod 4), $q \equiv 1$ (mod 4) if k is even, or $q \equiv 3$ (mod 4) if k is odd. Bu, Zhang, and He [323] have shown that an even cycle with a fixed number of pendent edges adjoined to each vertex is k-graceful. Lu, Pan, and Li [1007] have proved that $K_{1,m} \cup K_{p,q}$ is k-graceful when $k > 1$, and p and q are at least 2. Jirimutu, Bao, and Kong [749] have shown that the graphs obtained from $K_{2,n}$ $(n \ge 2)$ and $K_{3,n}$ $(n \ge 3)$ by attaching $r \ge 2$ edges at each vertex is k-graceful for all $k \ge 2$. Seoud and Elsakhawi [1269] proved: paths and ladders are arbitrarily graceful; and for $n \ge 3$, K_n is k-graceful if and only if $k = 1$ and $n = 3$ or 4. Li, Li, and Yan [946] proved that $K_{m,n}$ is k-graceful graph.

Yao, Cheng, Zhongfu, and Yao [1687] have shown: a tree of order p with maximum degree at least $p/2$ is k-graceful for some k; if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$ and $d_{T_2}(u_2) \ge |T_2|/2$, then T is k-graceful for some positive k; if a tree T has two edges u_1u_2 and u_2u_3 such that the three components T_1, T_2, and T_3 of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$, $d_{T_2}(u_2) \ge |T_2|/2$, and $d_{T_3}(u_3) \ge |T_3|/2$, then T is k-graceful for some $k > 1$; and every Skolem-graceful (see 3.5 for the definition) tree is k-graceful for all $k \ge 1$. They conjecture that every tree is k-graceful for some $k > 1$.

Several authors have investigated the k-gracefulness of various classes of subgraphs of grid graphs. Acharya [13] proved that all 2-dimensional polyminoes that are convex and Eulerian are k-graceful for all k; Lee [863] showed that Mongolian tents and Mongolian villages are k-graceful for all k (see §2.3 for the definitions); Lee and K. C. Ng [885] proved that all Young tableaus (see §2.3 for the definitions) are k-graceful for all k. (A special case of this is $P_n \times P_2$.) Lee and H. K. Ng [885] subsequently generalized these results on Young tableaus to a wider class of planar graphs.

Duan and Qi [448] use $G_t(m_1, n_1; m_2, n_2; \dots; m_s, n_s)$ to denote the graph composed of the s complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \dots, K_{m_s,n_s}$ that have only t $(1 \le t \le \min\{m_1, m_2, \dots, m_s\})$ common vertices but no common edge and $G(m_1, n_1; m_2, n_2)$ to denote the graph composed of the complete bipartite graphs K_{m_1,n_1}, K_{m_2,n_2} with exactly one common edge. They prove that these graphs are k-graceful graphs for all k.

Let $c, m, p_1, p_2, \dots, p_m$ be positive integers. For $i = 1, 2, \dots, m$, let S_i be a set of $p_i + 1$ integers and let D_i be the set of positive differences of the pairs of elements of S_i. If all these differences are distinct then the system $D_1, D_2, \dots, D_m$ is called a *perfect system of difference sets starting at* c if the union of all the sets D_i is $c, c+1, \dots, c-1+\sum_{i=1}^{m}\binom{p_i+1}{2}$. There is a relationship between k-graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with c describes a c-graceful labeling of a graph that is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in [5].

Acharya and Hegde [27] generalized k-graceful labelings to (k, d)*-graceful labelings* by permitting the vertex labels to belong to $\{0, 1, 2, \dots, k + (q-1)d\}$ and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be $\{k, k+d, k+2d, \dots, k+(q-1)d\}$. They also introduce an analog of α-labelings in the obvious way. Notice that a (1,1)-graceful labeling is a graceful labeling and a $(k,1)$-graceful labeling is a k-graceful labeling. Bu and Zhang [322] have shown: $K_{m,n}$ is (k,d)-graceful for all k and d; for $n > 2, K_n$ is (k,d)-graceful if and only if $k = d$ and $n \le 4$; if $m_i, n_i \ge 2$ and $\max\{m_i, n_i\} \ge 3$, then $K_{m_1,n_1} \cup K_{m_2,n_2} \cup \dots \cup K_{m_r,n_r}$ is (k,d)-graceful for all k, d, and r; if G has an α-labeling, then G is (k,d)-graceful for all k and d; a k-graceful graph is a (kd,d)-graceful graph; a (kd,d)-graceful connected graph is k-graceful; and a (k,d)-graceful graph with q edges that is not bipartite must have $k \le (q-2)d$.

Let T be a tree with adjacent vertices u_0 and v_0 and pendent vertices u and v such that the length of the path $u_0 - u$ is the same as the length of the path $v_0 - v$. Hegde and Shetty [645] call the graph obtained from T by deleting u_0v_0 and joining u and v an *elementary parallel transformation* of T. They say that a tree T is a T_p*-tree* if it can be transformed into a path by a sequence of elementary parallel transformations. They prove that every T_p-tree is (k,d)-graceful for all k and d and every graph obtained from a T_p-tree by subdividing each edge of the tree is (k,d)-graceful for all k and d.

Yao, Cheng, Zhongfu, and Yao [1687] have shown: a tree of order p with maximum degree at least $p/2$ is (k,d)-graceful for some k and d; if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$ and T_2 is a caterpillar, then T is Skolem-graceful (see 3.5 for the definition); if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$ and $d_{T_2}(u_2) \ge |T_2|/2$, then T is (k,d)-graceful for some $k > 1$ and $d > 1$; if a tree T has two edges u_1u_2 and u_2u_3 such that the three components T_1, T_2, and T_3 of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \ge |T_1|/2$, $d_{T_2}(u_2) \ge |T_2|/2$, and $d_{T_3}(u_3) \ge |T_3|/2$, then T is (k,d)-graceful for some $k > 1$ and $d > 1$; and every Skolem-graceful tree is (k,d)-graceful for $k \ge 1$ and $d > 0$. They conjecture that every tree is (k,d)-graceful for some $k > 1$ and $d > 1$.

Hegde [633] has proved the following: if a graph is (k,d)-graceful for odd k and even d, then the graph is bipartite; if a graph is (k,d)-graceful and contains C_{2j+1} as a subgraph, then $k \le jd(q-j-1)$; K_n is (k,d)-graceful if and only if $n \le 4$; C_{4t} is (k,d)-graceful for all k and d; C_{4t+1} is $(2t,1)$-graceful; C_{4t+2} is $(2t-1,2)$-graceful; and C_{4t+3} is $(2t+1,1)$-graceful.

Hegde [631] calls a (k,d)-graceful graph (k,d)*-balanced* if it has a (k,d)-graceful labeling f with the property that there is some integer m such that for every edge uv either $f(u) \le m$

and $f(v) > m$, or $f(u) > m$ and $f(v) \leq m$. He proves that if a graph is $(1,1)$-balanced then it is (k,d)-graceful for all k and d and that a graph is $(1,1)$-balanced graph if and only if it is (k,k)-balanced for all k. He conjectures that all trees are (k,d)-balanced for some values of k and d.

Slater [1401] has extended the definition of k-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices, and the countably infinite Dutch windmill is k-graceful for all k.

More specialized results on k-graceful labelings can be found in [863], [885], [889], [1398], [317], [319], [318], and [369].

3.5 Skolem-Graceful Labelings

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [914]) calls a graph G with p vertices and q edges *Skolem-graceful* if there is an injection from the set of vertices of G to $\{1, 2, \ldots, p\}$ such that the edge labels induced by $|f(x) - f(y)|$ for each edge xy are $1, 2, \ldots, q$. A necessary condition for a graph to be Skolem-graceful is that $p \geq q + 1$. Lee and Wui [943] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. Yao, Cheng, Zhongfu, and Yao [1687] have shown that a tree of order p with maximum degree at least $p/2$ is Skolem-graceful. Although the disjoint union of trees cannot be graceful, they can be Skolem-graceful. Lee and Wui [943] prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. In [396] Choudum and Kishore show that the disjoint union of k copies of the star $K_{1,2p}$ is Skolem graceful if $k \leq 4p + 1$ and the disjoint union of any number of copies of $K_{1,2}$ is Skolem graceful. For $k \geq 2$, let $St(n_1, n_2, \ldots, n_k)$ denote the disjoint union of k stars with $n_1, n_2, \ldots, n_k$ edges. Lee, Wang, and Wui [936] showed that the 4-star $St(n_1, n_2, n_3, n_4)$ is Skolem-graceful for some special cases and conjectured that all 4-stars are Skolem-graceful. Denham, Leu, and Liu [423] proved this conjecture. Kishore [794] has shown that a necessary condition for $St(n_1, n_2, \ldots, n_k)$ to be Skolem graceful is that some n_i is even or $k \equiv 0$ or 1 (mod 4) (see also [1709] . He conjectures that each one of these conditions is sufficient. Yue, Yuan-sheng, and Xin-hong [1709] show that for k at most 5, a k-star is Skolem-graceful if at one star has even size or $k \equiv 0$ or 1 (mod 4). Choudum and Kishore [394] proved that all 5-stars are Skolem graceful.

Lee, Quach, and Wang [900] showed that the disjoint union of the path P_n and the star of size m is Skolem-graceful if and only if $n = 2$ and m is even or $n \geq 3$ and $m \geq 1$. It follows from the work of Skolem [1390] that nP_2, the disjoint union of n copies of P_2, is Skolem-graceful if and only if $n \equiv 0$ or 1 (mod 4). Harary and Hsu [613] studied Skolem-graceful graphs under the name *node-graceful*. Frucht [517] has shown that $P_m \cup P_n$ is Skolem-graceful when $m + n \geq 5$. Bhat-Nayak and Deshmukh [275] have shown that $P_{n_1} \cup P_{n_2} \cup P_{n_3}$ is Skolem-graceful when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1 + 2) + 1$ and n_1 is even and when $n_1 < n_2 \leq n_3$, $n_2 = t(n_1 + 3) + 1$ and n_1 is odd. They also prove that the graphs of the form $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i}$ where $i \geq 4$ are Skolem-graceful under certain conditions. In [427] Deshmukh states the following results: the sum of all the edges on any cycle in a Skolem graceful graph is even; $C_5 \cup K_{1,n}$ if and only if $n = 1$ or 2; $C_6 \cup K_{1,n}$ if and only if $n = 2$ or 4.

Youssef [1695] proved that if G is Skolem-graceful, then $G + \overline{K_n}$ is graceful. In [1699] Youssef shows that that for all $n \geq 2$, $P_n \cup S_m$ is Skolem-graceful if and only if $n \geq 3$ or $n = 2$ and m is even. Yao, Cheng, Zhongfu, and Yao [1687] have shown that if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and T_2 is a caterpillar or have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and $d_{T_2}(u_2) \geq |T_2|/2$, then T is Skolem-graceful.

Mendelsohn and Shalaby [1048] defined a *Skolem labeled* graph $G(V, E)$ as one for which there is a positive integer d and a function $L\colon V \to \{d, d+1, \ldots, d+m\}$, satisfying (a) there are exactly two vertices in V such that $L(v) = d + i$, $0 \leq i \leq m$; (b) the distance in G between any two vertices with the same label is the value of the label; and (c) if G' is a proper spanning subgraph of G, then L restricted to G' is not a Skolem labeled graph. Note that this definition is different from the Skolem-graceful labeling of Lee, Quach, and Wang. A *hooked Skolem sequence* of order n is a sequence $s_1, s_2, \ldots, s_{2n+1}$ such that $s_{2n} = 0$ and for each $j \in \{1, 2, \ldots, n\}$, there exists a unique $i \in \{1, 2, \ldots, 2n-1, 2n+1\}$ such that $s_i = s_{i+j} = j$. Mendelsohn [1047] established the following: any tree can be embedded in a Skolem labeled tree with $O(v)$ vertices; any graph can be embedded as an induced subgraph in a Skolem labeled graph on $O(v^3)$ vertices; for $d = 1$, there is a Skolem labeling or the minimum hooked Skolem (with as few unlabeled vertices as possible) labeling for paths and cycles; for $d = 1$, there is a minimum Skolem labeled graph containing a path or a cycle of length n as induced subgraph. In [1047] Mendelsohn and Shalaby prove that the necessary conditions in [1048] are sufficient for a Skolem or minimum hooked Skolem labeling of all trees consisting of edge-disjoint paths of the same length from some fixed vertex. Graham, Pike, and Shalaby [585] obtained various Skolem labeling results for grid graphs. Among them are $P_1 \times P_n$ and $P_2 \times P_n$ have Skolem labelings if and only if $n \equiv 0$ or 1 mod 4; and $P_m \times P_n$ has a Skolem labeling for all m and n at least 3.

3.6 Odd-Graceful Labelings

Gnanajothi [572, p. 182] defined a graph G with q edges to be *odd-graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. She proved that the class of odd-graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labeling has an odd-graceful labeling and every graph with an odd cycle is not odd-graceful. She also proved the following graphs are odd-graceful: P_n; C_n if and only if n is even; $K_{m,n}$; combs $P_n \odot K_1$ (graphs obtained by joining a single pendent edge to each vertex of P_n); books; crowns $C_n \odot K_1$ (graphs obtained by joining a single pendent edge to each vertex of C_n) if and only if n is even; the disjoint union of copies of C_4; the one-point union of copies of C_4; $C_n \times K_2$ if and only if n is even; caterpillars; rooted trees of height 2; the graphs obtained from P_n ($n \geq 3$) by adding exactly two leaves at each vertex of degree 2 of P_n; the graphs obtained from $P_n \times P_2$ by deleting an edge that joins to end points of the P_n paths; the graphs obtained from a star by adjoining to each end vertex the path P_3 or by adjoining to each end vertex the path P_4. She conjectures that all trees are odd-graceful and proves the conjecture for all trees with order up to 10. Barrientos [231] has extended this to trees of order up to 12. Eldergill [452] generalized Gnanajothi's result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd-graceful. He also proved that the one-point union of any number of copies of C_6 is odd-graceful. Kathiresan [771] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd-graceful. Barrientos [234] and [231] has proved the following graphs are odd-graceful: every forest whose components are caterpillars; every tree with diameter at most five is odd-graceful; and all disjoint unions of caterpillars. He conjectures that every bipartite graph is odd-graceful. Seoud, Diab, and Elsakhawi [1266] have shown that a connected complete r-partite graph is odd-graceful if and only if $r = 2$ and that the join of any two connected graphs is not odd-graceful. Yan [1673] proved that $P_m \times P_n$ is odd-graceful labeling. Vaidya and Shah [1547] prove that the splitting graph and the shadow graph of bistar are odd-graceful. Li, Li, and Yan [946] proved that $K_{m,n}$ is odd-graceful Liu, Wang, and Lu [977] that proved that a class of bicyclic graphs with a common edge is odd-graceful.

Sekar [1249] has shown the following graphs are odd-graceful: $C_m \odot P_n$ (the graph obtained by identifying an end point of P_n with every vertex of C_m) where $n \geq 3$ and m is even; $P_{a,b}$ when $a \geq 2$ and b is odd (see §2.7); $P_{2,b}$ and $b \geq 2$; $P_{4,b}$ and $b \geq 2$; $P_{a,b}$ when a and b are even and $a \geq 4$ and $b \geq 4$; $P_{4r+1,4r+2}$; $P_{4r-1,4r}$; all n-polygonal snakes with n even; $C_n^{(t)}$ (see §2.2 for the definition); graphs obtained by beginning with C_6 and repeatedly forming the one-point union with additional copies of C_6 in succession; graphs obtained by beginning with C_8 and repeatedly forming the one-point union with additional copies of C_8 in succession; graphs obtained from even cycles by identifying a vertex of the cycle with the endpoint of a star; $C_{6,n}$ and $C_{8,n}$ (see §2.7); the splitting graph of P_n (see §2.7) the splitting graph of C_n, n even; lobsters, banana trees, and regular bamboo trees (see §2.1).

Yao, Cheng, Zhongfu, and Yao [1687] have shown the following: if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and T_2 is a caterpillar, then T is odd-graceful; and if a tree T has a vertex of degree at least $|T|/2$, then T is odd-graceful. They conjecture that for trees the properties of being Skolem-graceful and odd-graceful are equivalent. Recall a banana tree is a graph obtained by starting with any number os stars and connecting one end-vertex from each to a new vertex. Zhenbin [1722] has shown that graphs obtained by starting with any number of stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are odd-graceful.

Gao [542] has proved the following graphs are odd-graceful: the union of any number of paths; the union of any number of stars; the union of any number of stars and paths; $C_m \cup P_n$; $C_m \cup C_n$; and the union of any number of cycles each of which has order divisible by 4.

If f is an odd-graceful labeling of a bipartite graph G with bipartition (V_1, V_2) such that $\max\{f(u) : u \in V_1\} < \min\{f(v) : v \in V_2\}$, Zhou, Yao, Chen, and Tao [1729] say that f is a *set-ordered odd-graceful labeling* of G. They proved that every lobster is odd-graceful and adding leaves to a connected set-ordered odd-graceful graph is an odd-graceful graph.

In [1256] Seoud and Abdel-Aal determined all odd-graceful graphs of order at most 6 and proved that if G is odd-graceful then $G \cup K_{m,n}$ is odd-graceful. In [1274] Seoud and Helmi proved: if G has an odd-graceful labeling f with bipartition (V_1, V_2) such that $\max\{f(x) : f(x)$ is even, $x \in V_1\} < \min\{f(x) : f(x)$ is odd, $x \in V_2\}$, then G has an α-labeling; if G has an α-labeling, then $G \odot \overline{K_n}$ is odd-graceful; and if G_1 has an α-labeling and G_2 is odd-graceful, then $G_1 \cup G_2$ is odd-graceful. They also proved the following graphs have odd-graceful labelings: dragons obtained from an even cycle; graphs obtained from a gear graph by attaching a fixed number of pendent edges to each vertex of degree 2 on rim of the wheel of the graph; $C_{2m} \odot \overline{K_n}$; graphs obtained from an even cycle by attaching a fixed number of pendent edges to every other vertex; graphs obtained by identifying an endpoint of a star S_n ($n \geq 3$) with a vertex of an even cycle; the graphs consisting of two even cycles of the same order that share a common vertex with any number of pendent edges attached at the common vertex; and the graphs obtained by joining two even cycles of the same order by an edge. Seoud, El Sonbaty, and Abd El Rehim [1267] proved that the conjunction $P_m \wedge P_n$ for all $n, m \geq 2$ and the conjunction $K_2 \wedge F_n$ for n

even are odd graceful.

In [1075] and [1076] Moussa proved that $C_m \cup P_n$ is odd graceful in some cases and gave algorithms to prove that for all $m \geq 2$ the graphs $P_{4r-1;m}$, $r = 1, 2, 3$ and $P_{4r+1;m}$, $r = 1, 2$ are odd graceful. ($P_{n;m}$ is the graph obtained by identifying the endpoints of m paths each of length n). He also presented an algorithm that showed that closed spider graphs and the graphs obtained by joining one or two copies of P_m to each vertex of the path P_n are odd graceful. Moussa and Brader [1074] proved that $C_m \odot P_n$ is odd graceful if and only if m is even.

Moussa [1077] defines the *tensor product*, $P_m \wedge P_n$, of P_m and P_n as the graph with vertices $v_i^j, i = 1, \ldots, n; j = 1, \ldots, m$ and edges $v_1^j v_2^{j+1}, v_2^{j+1} v_3^j, \ldots, v_{n-1}^j v_n^{j+1}$ for j odd and $v_1^j v_2^{j-1}, v_2^{j-1} v_3^j, \ldots, v_{n-1}^j v_n^{j-1}$ for j even. He proves that $P_m \wedge P_m$ is odd-graceful.

Vaidya and Bijukumar [1508] proved the following are odd-graceful: graphs obtained by joining two copies of C_n by a path; graphs that are two copies of an even cycle that share a common edge; graphs that are the splitting graph of a star; and graphs that are the tensor product of a star and P_2.

Acharya, Germina, Princy, and Rao [23] proved that every bipartite graph G can be embedded in an odd-graceful graph H. The construction is done in such a way that if G is planar and odd-graceful, then so is H.

In [366] Chawathe and Krishna extend the definition of odd-gracefulness to countably infinite graphs and show that all countably infinite bipartite graphs that are connected and locally finite have odd-graceful labelings.

Solairaju and Chithra [1415] defined a graph G with q edges to be *edge-odd graceful* if there is an bijection f from the edges of the graph to $\{1, 3, 5, \ldots, 2q-1\}$ such that, when each vertex is assigned the sum of all the edges incident to it mod $2q$, the resulting vertex labels are distinct. They prove they following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders $P_n \times P_2$ $(n \geq 3)$; stars with an even number of edges; and crowns $C_n \odot K_1$. In [1416] they prove the following graphs have edge-odd graceful labelings: P_n $(n > 1)$ with a pendent edge attached to each vertex (combs); the graph obtained by appending $2n+1$ pendent edges to each endpoint of P_2 or P_3; and the graph obtained by subdividing each edge of the star $K_{1,2n}$.

Singhun [1386] proved the following graphs have edge-odd graceful labelings: W_{2n}; $W_n \odot K_1$; and $W_n \odot K_m$, when n is odd, m is even, and n divides m.

In [1432] Sridevi, Navaeethakrishnan, Nagarajan, and Nagarajan call a graph G with q edges *odd-even graceful* if there is an injection f from the vertices of G to $\{1, 3, 5, \ldots, 2q+1\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $\{2, 4, 6, \ldots, 2q\}$. They proved that P_n, combs $P_n \odot K_1$, stars $K_{1,n}, K_{1,2,n}, K_{m,n}$, and bistars $B_{m,n}$ are odd-even graceful.

3.7 Cordial Labelings

Cahit [331] has introduced a variation of both graceful and harmonious labelings. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Call f a *cordial labeling* of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [332] proved the following: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of t 3-cycles) is cordial if and only if $t \not\equiv 2 \pmod 4$; all fans are cordial; the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$ (see also [445]); maximal outerplanar graphs are cordial;

and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang, and Kwong [840] determine all m and n for which mK_n is cordial. Youssef [1699] proved that every Skolem-graceful graph (see 3.5 for the definition) is cordial. Liu and Zhu [984] proved that a 3-regular graph of order n is cordial if and only if $n \not\equiv 4 \pmod 8$.

A *k-angular cactus* is a connected graph all of whose blocks are cycles with k vertices. In [332] Cahit proved that a k-angular cactus with t cycles is cordial if and only if $kt \not\equiv 2$ (mod 4). This was improved by Kirchherr [792] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [793] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho, Lee, and Shee [659] proved: $P_n \times C_{4m}$ is cordial for all m and all odd n; the composition G and H is cordial if G is cordial and H is cordial and has odd order and even size (see §2.3 for definition of composition); for $n \geq 4$ the composition $C_n[K_2]$ is cordial if and only if $n \not\equiv 2 \pmod 4$; the Cartesian product of two cordial graphs of even size is cordial. Ho, Lee, and Shee [658] showed that a unicyclic graph is cordial unless it is C_{4k+2} and that the generalized Petersen graph (see §2.7 for the definition) $P(n, k)$ is cordial if and only if $n \not\equiv 2 \pmod 4$. Khan [781] proved that a graph that consisting of a finite number of cycles of finite length joined at a common cut vertex is cordial if and only if the number of edges is not congruent to 2 mod 4.

Du [445] determines the maximal number of edges in a cordial graph of order n and gives a necessary condition for a k-regular graph to be cordial. Riskin [1200] proved that Möbius ladders M_n (see §2.3 for the definition) are cordial if and only if $n \geq 3$ and $n \not\equiv 2 \pmod 4$. (See also [1269].)

Seoud and Abdel Maqusoud [1258] proved that if G is a graph with n vertices and m edges and every vertex has odd degree, then G is not cordial when $m + n \equiv 2 \pmod 4$. They also prove the following: for $m \geq 2$, $C_n \times P_m$ is cordial except for the case $C_{4k+2} \times P_2$; P_n^2 is cordial for all n; P_n^3 is cordial if and only if $n \neq 4$; and P_n^4 is cordial if and only if $n \neq 4, 5$, or 6. Seoud, Diab, and Elsakhawi [1266] have proved the following graphs are cordial: $P_n + P_m$ for all m and n except $(m, n) = (2, 2)$; $C_m + C_n$ if $m \not\equiv 0 \pmod 4$ and $n \neq 2 \pmod 4$; $C_n + K_{1,m}$ for $n \not\equiv 3$ (mod 4) and odd m except $(n, m) = (3, 1)$; $C_n + \overline{K_m}$ when n is odd, and when n is even and m is odd; $K_{1,m,n}$; $K_{2,2,m}$; the n-cube; books B_n if and only if $n \not\equiv 3 \pmod 4$; $B(3, 2, m)$ for all m; $B(4, 3, m)$ if and only if m is even; and $B(5, 3, m)$ if and only if $m \not\equiv 1 \pmod 4$ (see §2.4 for the notation $B(n, r, m)$). In [1411] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are cordial.

Diab [430], [431], and [433] proved the following graphs are cordial: $C_m + P_n$ if and only if $(m, n) \neq (3, 3), (3, 2)$, or (3,1); $P_m + K_{1,n}$ if and only if $(m, n) \neq (1, 2)$; $P_m \cup K_{1,n}$ if and only if $(m, n) \neq (1, 2)$; $C_m \cup K_{1,n}$; $C_m + \overline{K_n}$ for all m and n except $m \equiv 3 \pmod 4$ and n odd, and $m \equiv 2$ (mod 4) and n even; $C_m \cup \overline{K_n}$ for all m and n except $m \equiv 2 \pmod 4$; $P_m + \overline{K_n}$; $P_m \cup \overline{K_n}$; $P_m^2 \cup P_n^2$ except for $(m, n) = (2, 2)$ or (3,3); $P_n^2 + P_m$ except for $(m, n) = (3, 1), (3, 2), (2, 2), (3, 3)$ and (4,2); $P_n^2 \cup P_m$ except for $(n, m) = (2, 2), (3, 3)$ and (4,2); $P_n^2 + C_m$ if and only if $(n, m) \neq (1, 3), (2, 3)$ and $(3, 3)$. $P_n + \overline{K_m}$; $C_n + K_{1,m}$ for all $n > 3$ and all m except $n \equiv 3 \pmod 4$; $C_n + K_{1,m}$ for $n \equiv 3 \pmod 4$ $(n \neq 3)$ and even $m \geq 2$; and $C_m \times C_n$ if and only if $2mn$ is not congruent to 2 (mod 4).

In [432] Diab proved the graphs $W_n + W_m$ are cordial if and only if one of the following conditions is not satisfied: (i) $(n, m) = (3, 3)$, (ii) $n = 3$ and $m \equiv 1 \pmod 4$, (iii) $n \equiv 1$ (mod 4) and $m \equiv 3 \pmod 4$; the graphs $W_n \cup W_m$ are cordial if and only if one of the following conditions is not satisfied: (i) $n = 3$ and $m \equiv 1 \pmod 4$, (ii) $n \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$; the graphs $W_n + P_m$ are cordial if and only if one of the following conditions is not satisfied: (i)

$(n,m) = (3,1), (3,2)$ and $(3,3)$, (ii) $n \equiv 3 \pmod 4$ and $m = 1$. They also prove that $W_n \cup P_m$ and $W_n \cup C_m$ are cordial for all m and n and $W_n + C_m$ is cordial if and only if $(m,n) \neq (3,3)$ and $(3,4)$.

Youssef [1701] has proved the following: If G and H are cordial and one has even size, then $G \cup H$ is cordial; if G and H are cordial and both have even size, then $G + H$ is cordial; if G and H are cordial and one has even size and either one has even order, then $G + H$ is cordial; $C_m \cup C_n$ is cordial if and only if $m + n \not\equiv 2 \pmod 4$; mC_n is cordial if and only if $mn \not\equiv 2$ (mod 4); $C_m + C_n$ is cordial if and only if $(m,n) \neq (3,3)$ and $\{m \pmod 4, n \pmod 4\} \neq \{0,2\}$; and if P_n^k is cordial, then $n \geq k+1+\sqrt{k-2}$. He conjectures that this latter condition is also sufficient. He confirms the conjecture for $k = 5, 6, 7, 8$, and 9.

In [1700] Youssef obtained the following results: C_{2k} with one pendnt edge is not $(2k+1)$-cordial for $k > 1$; K_n is 4-cordial if and only if $n \leq 6$; C_n^2 is 4-cordial if and only if $n \not\equiv 2$ (mod 4); and $K_{m,n}$ is 4-cordial if and only if $n \not\equiv 2 \pmod 4$; He also provides some necessary conditions for a graph to be k-cordial.

Lee and Liu [880] have shown that the complete n-partite graph is cordial if and only if at most three of its partite sets have odd cardinality (see also [445]). Lee, Lee, and Chang [857] prove the following graphs are cordial: the Cartesian product of an arbitrary number of paths; the Cartesian product of two cycles if and only if at least one of them is even; and the Cartesian product of an arbitrary number of cycles if at least one of them has length a multiple of 4 or at least two of them are even.

Shee and Ho [1329] have investigated the cordiality of the one-point union of n copies of various graphs. For $C_m^{(n)}$, the one-point union of n copies of C_m, they prove:

(i) If $m \equiv 0 \pmod 4$, then $C_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 1$ or $3 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(iii) If $m \equiv 2 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if n is even.

For $K_m^{(n)}$, the one-point union of n copies of K_m, Shee and Ho [1329] prove:

(i) If $m \equiv 0 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 3 \pmod 4$;

(ii) If $m \equiv 4 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 1 \pmod 4$;

(iii) If $m \equiv 5 \pmod 8$, then $K_m^{(n)}$ is not cordial for all odd n;

(iv) $K_4^{(n)}$ is cordial if and only if $n \not\equiv 1 \pmod 4$;

(v) $K_5^{(n)}$ is cordial if and only if n is even;

(vi) $K_6^{(n)}$ is cordial if and only if $n > 2$;

(vii) $K_7^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(viii) $K_n^{(2)}$ is cordial if and only if n has the form p^2 or $p^2 + 1$.

For $W_m^{(n)}$, the one-point union of n copies of the wheel W_m with the common vertex being the center, Shee and Ho [1329] show:

(i) If $m \equiv 0$ or $2 \pmod 4$, then $W_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 3 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 1 \pmod 4$;

(iii) If $m \equiv 1 \pmod 4$, then $W_m^{(n)}$ is cordial if $n \not\equiv 3 \pmod 4$. For all n and all $m > 1$ Shee and Ho [1329] prove $F_m^{(n)}$, the one-point union of n copies of the fan $F_m = P_m + K_1$ with the common point of the fans being the center, is cordial (see also [960]). The *flag* Fl_m is obtained by joining one vertex of C_m to an extra vertex called the *root*. Shee and Ho [1329] show all $Fl_m^{(n)}$, the one-point union of n copies of Fl_m with the common point being the root, are cordial. In his 2001 Ph. D. thesis Selvaraju [1250] proves that the one-point union of any number of copies of a complete bipartite graph is cordial. Benson and Lee [257] have investigated the regular windmill graphs $K_m^{(n)}$ and determined precisely which ones are cordial for $m < 14$.

Diab and Mohammedm [435] proved the following: the join of two fans $F_n + F_m$ is cordial if and only if $n, m \geq 4$; $F_n \cup F_m$ is cordial if and only if $(n, m) \neq$ (1,1) or (2,2); $F_n + P_m$ is cordial if and only if $(n, m) \neq$ (1,2), (2,1), (2,2) (2,3), or (3,2); $F_n \cup P_m$ is cordial if and only if $(n, m) \neq$ (1,2); $F_n + C_m$ is cordial if and only if $(n, m) \neq$ (1,3), (2,3) or (3,3); and $F_n \cup C_m$ is cordial if and only if $(n, m) \neq (2, 3)$.

Andar, Boxwala, and Limaye [86], [87], and [90] have proved the following graphs are cordial: helms; closed helms; generalized helms obtained by taking a web (see 2.2 for the definitions) and attaching pendent vertices to all the vertices of the outermost cycle in the case that the number cycles is even; flowers (graphs obtained by joining the vertices of degree one of a helm to the central vertex); sunflower graphs (that is, graphs obtained by taking a wheel with the central vertex v_0 and the n-cycle $v_1, v_2, \ldots, v_n$ and additional vertices $w_1, w_2, \ldots, w_n$ where w_i is joined by edges to v_i, v_{i+1}, where $i + 1$ is taken modulo n); multiple shells (see §2.2); and the one point unions of helms, closed helms, flowers, gears, and sunflower graphs, where in each case the central vertex is the common vertex.

Du [446] proved that the disjoint union of $n \geq 2$ wheels is cordial if and only if n is even or n is odd and the number of vertices of in each cycle is not 0 (mod 4) or n is odd and the number of vertices of in each cycle is not 3 (mod 4).

Elumalai and Sethurman [455] proved: cycles with parallel cords are cordial and n-cycles with parallel P_k-chords (see §2.2 for the definition) are cordial for any odd positive integer k at least 3 and any $n \not\equiv 2 \pmod 4$ of length at least 4. They call a graph H an *even-multiple subdivision graph* of a graph G if it is obtained from G by replacing every edge uv of G by a pair of paths of even length starting at u and ending at v. They prove that every even-multiple subdivision graph is cordial and that every graph is a subgraph of a cordial graph. In [1639] Wen proves that generalized wheels $C_n + mK_1$ are cordial when m is even and $n \not\equiv 2 \pmod 4$ and when m is odd and $n \not\equiv 3 \pmod 4$.

Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical graphs by a path. They prove: graphs obtained by joining two copies of the same cycle by a path are cordial [1519]; graphs obtained by joining two copies of the same cycle that has two chords with a common vertex with opposite ends of the chords joining two consecutive vertices of the cycle by a path are cordial [1519]; graphs obtained by joining two rim verticies of two copies of the same wheel by a path are cordial [1521]; and graphs obtained by joining two copies of the same Petersen graph by a path are cordial [1521]. They also prove that graphs obtained by replacing one vertex of a star by a fixed wheel or by replacing each vertex of a star by a fixed Petersen graph are cordial [1521]. In [1549] Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical cycles that have a chord are cordial and the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed cycle with a chord that forms a triangle with two consecutive edges of the cycle and joining each G_i to G_{i+1} $(i = 1, 2, \ldots, n-1)$ by an edge that is incident with the endpoints of the chords in G_i and G_{i+1} are cordial. Vaidya, Dani, Kanani, and Vihol [1514] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of G_i to the center of G_{i+1} $(i = 1, 2, \ldots, n-1)$ by an edge are cordial.

S. Vaidya, K. Kanani, S. Srivastav, and G. Ghodasara [1529] proved: graphs obtained by subdividing every edge of a cycle with exactly two extra edges that are chords with a common endpoint and whose other end points are joined by an edge of the cycle are cordial; graphs obtained by subdividing every edge of the graph obtained by starting with C_n and adding exactly three chords that result in two 3-cycles and a cycle of length $n-3$ are cordial; graphs obtained by subdividing every edge of a Petersen graph are cordial.

Recall the shell $C(n, n-3)$ is the cycle C_n with $n-3$ cords sharing a common endpoint. Vaidya, Dani, Kanani, and Vihol [1515] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed shell and joining common endpoint of the chords of G_i to the common endpoint of the chords of G_{i+1} $(i = 1, 2, \ldots, n-1)$ by an edge are cordial. Vaidya, Dani, Kanani and Vihol [1530] define $C_n(C_n)$ as the graph obtained by subdividing each edge of C_n and connecting the new n vertices to form a copy of C_n inscribed the original C_n. They prove that $C_n(C_n)$ is cordial if $n \not\equiv 2 \pmod 4$; the graphs obtained by starting with copies $G_1, G_2, \ldots, G_k$ of $C_n(C_n)$ the graph obtained by joining a vertex of degree 2 in G_i to a vertex of degree 2 in G_{i+1} $(i = 1, 2, \ldots, n-1)$ by an edge are cordial; and the graphs obtained by joining vertex of degree 2 from one copy of $C_n(C_n)$ to a vertex of degree 2 to another copy of $C_n(C_n)$ by any finite path are cordial.

A graph $C(2n, n-2)$ is called an *alternate shell* if $C(2n, n-2)$ is obtained from the cycle C_{2n} $(v_0, v_1, v_2, \ldots, v_{2n-1})$ by adding $n-2$ chords between the vertex v_0 and the vertices v_{2i+1}, for $1 \leq i \leq n-2$. Sethuraman and Sankar [1314] proved that some graphs obtained by merging alternate shells and joining certain vertices by a path have α-labelings.

Vaidya, Srivastav, Kaneria, and Ghodasara [1550] proved that a cycle with two chords that share a common vertex and the opposite ends of which join two consecutive vertices of the cycle is cordial. For a graph G Vaidya, Ghodasara, Srivastav, and Kaneria [1520] introduced a graph G^* called *star of a graph* as the graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of G and prove that ${C_n}^*$ admits cordial labeling. Vaidya and Dani [1510] proved that the graphs obtained by starting with n copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of G_i to the center of G_{i+1} by an edge as well as each of the centers to a new vertex x_i $(1 \leq i \leq n-1)$ by an edge admit cordial labelings. An *arbitrary supersubdivison* H of a graph G is the graph obtained from G by replacing every edge of G by $K_{2,m}$, where m may vary for each edge arbitrarily. Vaidya and Kanani [1522] proved that arbitrary supersubdivisions of paths and stars admit cordial labelings. Vaidya and Dani [1511] prove that arbitrary supersubdivisions of trees, $K_{m,n}$, and $P_m \times P_n$ are cordial. They also prove that an arbitrary supersubdivision of the graph obtained by identifying an end vertex of a path with every vertex of a cycle C_n is cordial except when n is odd, m_i $(1 \leq i \leq n)$ are odd, and m_i $(n+1 \leq i \leq mn)$ of the K_{2,m_i} are even. Recall for a graph G and a vertex v of G Vaidya, Srivastav, Kaneria, and Kanani [1551] define a *vertex switching* G_v as the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. They proved that the graphs obtained by the switching of a vertex in C_n admit cordial labelings. They also show that the graphs obtained by the switching of any arbitrary vertex of cycle C_n with one chord that forms a triangle with two consecutive edges of the cycle are cordial. Moreover they prove that the graphs obtained by the switching of any arbitrary vertex in cycle with two chords that share a common vertex the opposite ends of which join two consecutive vertices of the cycle are cordial.

The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it. Vaidya and Vihol [1553] prove that the middle graph $M(G)$ of an Eulerian graph is Eulerian with $|E(M(G))| = \sum_{i=1}^{n} (d(v_i)^2 + 2e)/2$.

They prove that middle graphs of paths, crowns $C_n \odot K_1$, stars, and tadpoles (that is, graphs obtained by appending a path to a cycle) admit cordial labelings.

Vaidya and Dani [1513] define the *duplication of an edge* $e = uv$ *of a graph* G *by a new vertex* w as the graph G' obtained from G by adding a new vertex w and the edges wv and wu. They prove that the graphs obtained by duplication of an arbitrary edge of a cycle and a wheel admit a cordial labeling. Starting with k copies of fixed wheel W_n, $W_n^{(1)}, W_n^{(2)}, \ldots W_n^{(k)}$, Vaidya, Dani, Kanani, and Vihol [1517] define $G =< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ as the graph obtained by joining the center vertices of each of $W_n^{(i)}$ and $W_n^{(i+1)}$ to a new vertex x_i where $1 \leq i \leq k-1$. They prove that $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ are cordial graphs. Kaneria and Vaidya [757] define the *index of cordiality* of G as n if the disjoint union of n copies of G is cordial but the disjoint union of fewer than n copies of G is not cordial. They obtain several results on index of cordiality of K_n. In the same paper they investigate cordial labelings of graphs obtained by replacing each vertex of $K_{1,n}$ by a graph G.

In [90] Andar et al. define a *t-ply graph* $P_t(u,v)$ as a graph consisting of t internally disjoint paths joining vertices u and v. They prove that $P_t(u,v)$ is cordial except when it is Eulerian and the number of edges is congruent to 2 (mod 4). In [91] Andar, Boxwala, and Limaye prove that the one-point union of any number of plys with an endpoint as the common vertex is cordial if and only if it is not Eulerian and the number of edges is congruent to 2 (mod 4). They further prove that the path union of shells obtained by joining any point of one shell to any point of the next shell is cordial; graphs obtained by attaching a pendent edge to the common vertex of the cords of a shell are cordial; and cycles with one pendent edge are cordial.

For a graph G and a positive integer t, Andar, Boxwala, and Limaye [88] define the *t-uniform homeomorph* $P_t(G)$ of G as the graph obtained from G by replacing every edge of G by vertex disjoint paths of length t. They prove that if G is cordial and t is odd, then $P_t(G)$ is cordial; if $t \equiv 2 \pmod 4$ a cordial labeling of G can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 0 in G is even; and when $t \equiv 0 \pmod 4$ a cordial labeling of G can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 1 in G is even. In [89] Ander et al. prove that $P_t(K_{2n})$ is cordial for all $t \geq 2$ and that $P_t(K_{2n+1})$ is cordial if and only if $t \equiv 0 \pmod 4$ or t is odd and $n \not\equiv 2 \pmod 4$, or $t \equiv 2 \pmod 4$ and n is even.

In [91] Andar, Boxwala, and Limaya show that a cordial labeling of G can be extended to a cordial labeling of the graph obtained from G by attaching $2m$ pendent edges at each vertex of G. For a binary labeling g of the vertices of a graph G and the induced edge labels given by $g(e) = |g(u) - g(v)|$ let $v_g(j)$ denote the number of vertices labeled with j and $e_g(j)$ denote the number edges labeled with j. Let $i(G) = \min\{|e_g(0) - e_g(1)|\}$ taken over all binary labelings g of G with $|v_g(0) - v_g(1)| \leq 1$. Andar et al. also prove that a cordial labeling g of a graph G with p vertices can be extended to a cordial labeling of the graph obtained from G by attaching $2m+1$ pendent edges at each vertex of G if and only if G does not satisfy either of the conditions: (1) G has an even number of edges and $p \equiv 2 \pmod 4$; (2) G has an odd number of edges and either $p \equiv 1 \pmod 4$ with $e_g(1) = e_g(0) + i(G)$ or $n \equiv 3 \pmod 4$ and $e_g(0) = e_g(1) + i(G)$. Andar, Boxwala, and Limaye [92] also prove: if g is a binary labeling of the n vertices of graph G with induced edge labels given by $g(e) = |g(u) - g(v)|$ then g can be extended to a cordial labeling of $G \odot \overline{K_{2m}}$ if and only if n is odd and $i(G) \equiv 2 \pmod 4$; $K_n \odot \overline{K_{2m}}$ is cordial if and only if $n \not\equiv 4 \pmod 8$; $K_n \odot \overline{K_{2m+1}}$ is cordial if and only if $n \not\equiv 7 \pmod 8$; if g is a binary labeling of the n vertices of graph G with induced edge labels given by $g(e) = |g(u) - g(v)|$ then g can be extended to a cordial labeling of $G \odot C_t$ if $t \neq 3$ mod 4, n is odd and $e_g(0) = e_g(1)$. For any binary labeling g of a graph G with induced edge labels given by $g(e) = |g(u) - g(v)|$ they also characterize in terms of $i(G)$ when g can be extended to graphs of the form $G \odot \overline{K_{2m+1}}$.

For graphs $G_1, G_2, \ldots, G_n$ $(n \geq 2)$ that are all copies of a fixed graph G, Shee and Ho [1330] call a graph obtained by adding an edge from G_i to G_{i+1} for $i = 1, \ldots, n-1$ a *path-union* of G (the resulting graph may depend on how the edges are chosen). Among their results they show the following graphs are cordial: path-unions of cycles; path-unions of any number of copies of K_m when $m = 4, 6$, or 7; path-unions of three or more copies of K_5; and path-unions of two copies of K_m if and only if $m-2, m$, or $m+2$ is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees, and various compositions.

Lee and Liu [880] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let H be a graph with an even number of edges and a cordial labeling such that the vertices of H can be divided into t parts $H_1, H_2, \ldots, H_t$ each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let G be any graph and $G_1, G_2, \ldots, G_t$ be any t subsets of the vertices of G. Let (G, H) be the graph that is the disjoint union of G and H augmented by edges joining every vertex in G_i to every vertex in H_i for all i. Then G is cordial if and only if (G, H) is. From this it follows that: all generalized fans $F_{m,n} = \overline{K_m} + P_n$ are cordial; the generalized bundle $B_{m,n}$ is cordial if and only if m is even or $n \not\equiv 2 \pmod 4$ ($B_{m,n}$ consists of $2n$ vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ with an edge from v_i to u_i and $2m$ vertices $x_1, x_2, \ldots x_m, y_1, y_2, \ldots, y_m$ with x_i joined to v_i and y_i joined to u_i); if m is odd the generalized wheel $W_{m,n} = \overline{K_m} + C_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$. If m is even, $W_{m,n}$ is cordial if and only if $n \not\equiv 2 \pmod 4$; a complete k-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Sethuraman and Selvaraju [1322] have shown that certain cases of the union of any number of copies of K_4 with one or more edges deleted and one edge in common are cordial. Youssef [1704] has shown that the kth power of C_n is cordial for all n when $k \equiv 2 \pmod 4$ and for all even n when $k \equiv 0 \pmod 4$. Ramanjaneyulu, Venkaiah, and Kothapalli [1184] give cordial labelings for a family of planar graphs for which each face is a 3-cycle and a family for which each face is a 4-cycle. Acharya, Germina, Princy, and Rao [23] prove that every graph G can be embedded in a cordial graph H. The construction is done in such a way that if G is planar or connected, then so is H.

Recall from §2.7 that a graph H is a *supersubdivision* of a graph G, if every edge uv of G is replaced by $K_{2,m}$ (m may vary for each edge) by identifying u and v with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Vaidya and Kanani [1522] prove that supersubdivisions of paths and stars are cordial. They also prove that supersubdivisions of C_n are cordial provided that n and the various values for m are odd.

Raj and Koilraj [1180] proved that the splitting graphs of $P_n, C_n, K_{m,n}, W_n, nK_2$, and the graphs obtained by starting with k copies of stars $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(k)}$ and joining the central vertex of $K_{1,n}^{(p-1)}$ and $K_{1,n}^{(p)}$ to a new vertex x_{p-1} for each $2 \leq p \leq k$ are cordial.

Seoud, El Sonbaty, and Abd El Rehim [1267] proved the following graphs are cordial: $K_{1,l,m,n}$ when mn is even; $P_m + K_{1,n}$ if n is even or n is odd and $(m \neq 2)$; the conjunction graph $P_4 \wedge C_n$ is cordial if n is even; and the join of the one-point union of two copies of C_n and K_1.

Cahit [337] calls a graph *H-cordial* if it is possible to label the edges with the numbers from the set $\{1, -1\}$ in such a way that, for some k, at each vertex v the sum of the labels on the edges incident with v is either k or $-k$ and the inequalities $|v(k) - v(-k)| \leq 1$ and $|e(1) - e(-1)| \leq 1$ are also satisfied, where $v(i)$ and $e(j)$ are, respectively, the number of vertices labeled with i and the number of edges labeled with j. He calls a graph *H_n-cordial* if it is possible to label the edges with the numbers from the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ in such a way that, at each vertex v the sum of the labels on the edges incident with v is in the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ and the inequalities $|v(i) - v(-i)| \leq 1$ and $|e(i) - e(-i)| \leq 1$ are also satisfied for each i with $1 \leq i \leq n$. Among Cahit's results are: $K_{n,n}$ is H-cordial if and only if $n > 2$ and n is even; and $K_{m,n}, m \neq n$, is H-cordial if and only if $n \equiv 0 \pmod 4$, m is even and $m > 2, n > 2$. Unfortunately, Ghebleh and Khoeilar [568] have shown that other statements in Cahit's paper are incorrect. In particular, Cahit states that K_n is H-cordial if and only if $n \equiv 0 \pmod 4$; W_n is H-cordial if and only if $n \equiv 1 \pmod 4$; and K_n is H_2-cordial if and only if $n \equiv 0 \pmod 4$ whereas Ghebleh and Khoeilar instead prove that K_n is H-cordial if and only if $n \equiv 0$ or $3 \pmod 4$ and $n \neq 3; W_n$ is H-cordial if and only if n is odd; K_n is H_2-cordial if $n \equiv 0$ or $3 \pmod 4$; and K_n is not H_2-cordial if $n \equiv 1 \pmod 4$. Ghebleh and Khoeilar also prove every wheel has an H_2-cordial labeling. In [506] Freeda and Chellathurai prove that the following graphs are H_2-cordial: the join of two paths, the join of two cycles, ladders, and the tensor product $P_n \otimes P_2$. They also prove that the join of W_n and W_m where $n + m \equiv 0 \pmod 4$ is H-cordial. Cahit generalizes the notion of H-cordial labelings in [337].

Cahit and Yilmaz [341] call a graph *E_k-cordial* if it is possible to label the edges with the numbers from the set $\{0, 1, 2, \ldots, k-1\}$ in such a way that, at each vertex v, the sum of the labels on the edges incident with v modulo k satisfies the inequalities $|v(i) - v(j)| \leq 1$ and $|e(i) - e(j)| \leq 1$, where $v(s)$ and $e(t)$ are, respectively, the number of vertices labeled with s and the number of edges labeled with t. Cahit and Yilmaz prove the following graphs are E_3-cordial: P_n $(n \geq 3)$; stars S_n if and only if $n \not\equiv 1 \pmod 3$; K_n $(n \geq 3)$; C_n $(n \geq 3)$; friendship graphs; and fans F_n $(n \geq 3)$. They also prove that S_n $(n \geq 2)$ is E_k-cordial if and only if $n \not\equiv 1 \pmod k$ when k is odd or $n \not\equiv 1 \pmod{2k}$ when k is even and $k \neq 2$.

Bapat and Limaye [222] provide E_3-cordial labelings for: K_n $(n \geq 3)$; snakes whose blocks are all isomorphic to K_n where $n \equiv 0$ or $2 \pmod 3$; the one-point union of any number of copies of K_n where $n \equiv 0$ or $2 \pmod 3$; graphs obtained by attaching a copy of K_n where $n \equiv 0$ or $3 \pmod 3$ at each vertex of a path; and $K_m \odot K_n$. Rani and Sridharan [1191] proved: for odd $n > 1$ and $k \geq 2$, $P_n \odot K_1$ is E_k-cordial; for n even and $n \neq k/2$, $P_n \odot K_1$ is E_k-cordial; and certain cases of fans are E_k-cordial. Youssef [1702] gives a necessary condition for a graph to be E_k-cordial for certain k. He also gives some new families of E_k-cordial graphs and proves Lee's [910] conjecture about the edge-gracefulness of the disjoint union of two cycles.

Hovey [663] has introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group A (under addition) and graph $G(V, E)$ he defines G to be *A-cordial* if there is a labeling of V with elements of A such that for all a and b in A when the edge ab is labeled with $f(a) + f(b)$, the number of vertices labeled with a and the number of vertices labeled b differ by at most one and the number of edges labeled with a and the number labeled with b differ by at most one. In the case where A is the cyclic group of order k, the labeling is called *k-cordial*. With this definition we have: $G(V, E)$ is harmonious if and only if G is $|E|$-cordial; G is cordial if and only if G is 2-cordial.

Hovey has obtained the following: caterpillars are k-cordial for all k; all trees are k-cordial for $k = 3, 4$, and 5; odd cycles with pendent edges attached are k-cordial for all k; cycles are k-cordial for all odd k; for k even, C_{2mk+j} is k-cordial when $0 \leq j \leq \frac{k}{2} + 2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial; K_m is 3-cordial; and, for k even, K_{mk} is k-cordial if and only if $m = 1$.

Hovey advances the following conjectures: all trees are k-cordial for all k; all connected graphs are 3-cordial; and C_{2mk+j} is k-cordial if and only if $j \neq k$, where k and j are even and $0 \leq j < 2k$. The last conjecture was verified by Tao [1484]. Tao's result combined with those of Hovey show that for all positive integers k the n-cycle is k-cordial with the exception that k is even and $n = 2mk + k$. Tao also proved that the crown with $2mk + j$ vertices is k-cordial unless $j = k$ is even, and for $4 \leq n \leq k$ the wheel W_n is k-cordial unless $k \equiv 5 \pmod 8$ and $n = (k+1)/2$.

In [1318] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph G and successively form supersubdivisions (see §2.7 for the definition) that are cordial in the case that every edge in G is replaced by $K_{2,m}$ where m is even. Sethuraman and Selvaraju [1317] also show that the one-vertex union of any number of copies of $K_{m,n}$ is cordial and that the one-edge union of k copies of shell graphs $C(n, n-3)$ (see §2.2) is cordial for all $n \geq 4$ and all k. They conjectured that the one-point union of any number of copies of graphs of the form $C(n_i, n_i - 3)$ for various $n_i \geq 4$ is cordial. This was proved by Yue, Yuansheng, and Liping in [1712]. Riskin [1202] claimed that K_n is $Z_2 \times Z_2$-cordial if and only if n is at most 3 and $K_{m,n}$ is $Z_2 \times Z_2$ cordial if and only if $(m, n) \neq (2, 2)$. However, Pechenik and Wise [1126] report that the correct statement for $K_{m,n}$ is $K_{m,n}$ is $Z_2 \times Z_2$ cordial if and only if mv and n are not both congruent to 2 mod 4. Seoud and Salim [1283] gave an upper bound on the number of edges of a graph that admits a $Z_2 \oplus Z_2$-cordial labeling in terms the number of vertices.

In [1126] Pechenik and Wise investigate $Z_2 \times Z_2$-cordiality of complete bipartite graphs, paths, cycles, ladders, prisms, and hypercubes. They proved that all complete bipartite graphs are $Z_2 \times Z_2$-cordial except $K_{m,n}$ where $m, n \equiv 2 \bmod 4$; all paths are $Z_2 \times Z_2$-cordial except P_4 and P_5; all cycles are $Z_2 \times Z_2$-cordial except C_4, C_5, C_k, where $k \equiv 2 \bmod 4$; and all ladders $P_2 \times P_k$ are $Z_2 \times Z_2$-cordial except C_4. They also introduce a generalization of A-cordiality involving digraphs and quasigroups, and we show that there are infinitely many Q-cordial digraphs for every quasigroup Q.

Cairnie and Edwards [344] have determined the computational complexity of cordial and k-cordial labelings. They prove the conjecture of Kirchherr [793] that deciding whether a graph admits a cordial labeling is NP-complete. As a corollary, this result implies that the same problem for k-cordial labelings is NP-complete. They remark that even the restricted problem of deciding whether connected graphs of diameter 2 have a cordial labeling is also NP-complete.

In [362] Chartrand, Lee, and Zhang introduced the notion of uniform cordiality as follows. Let f be a labeling from $V(G)$ to $\{0, 1\}$ and for each edge xy define $f^*(xy) = |f(x) - f(y)|$. For $i = 0$ and 1, let $v_i(f)$ denote the number of vertices v with $f(v) = i$ and $e_i(f)$ denote the number of edges e with $f^*(e) = i$. They call a such a labeling f *friendly* if $|v_0(f) - v_1(f)| \leq 1$. A graph G for which every friendly labeling is cordial is called *uniformly cordial*. They prove that a connected graph of order $n \geq 2$ is uniformly cordial if and only if $n = 3$ and $G = K_3$, or n is even and $G = K_{1,n-1}$.

In [1200] Riskin introduced two measures of the noncordiality of a graph. He defines the *cordial edge deficiency* of a graph G as the minimum number of edges, taken over all friendly labelings of G, needed to be added to G such that the resulting graph is cordial. If a graph G has a vertex labeling f using 0 and 1 such that the edge labeling f_e given by $f_e(xy) = |f(x) - f(y)|$ has the property that the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1, the *cordial vertex deficiency* defined as ∞. Riskin proved: the cordial edge deficiency of K_n $(n > 1)$ is $\lfloor \frac{n}{2} \rfloor - 1$; the cordial vertex deficiency of K_n is $j - 1$ if $n = j^2 + \delta$, when δ is

$-2, 0$ or 2, and ∞ otherwise. In [1200] Riskin determines the cordial edge deficiency and cordial vertex deficiency for the cases when the Möbius ladders and wheels are not cordial. In [1201] Riskin determines the cordial edge deficiencies for complete multipartite graphs that are not cordial and obtains a upper bound for their cordial vertex deficiencies.

If f is a binary vertex labeling of a graph G Lee, Liu, and Tan [881] defined a partial edge labeling of the edges of G by $f^*(uv) = 0$ if $f(u) = f(v) = 0$ and $f^*(uv) = 1$ if $f(u) = f(v) = 1$. They let $e_0(G)$ denote the number of edges uv for which $f^*(uv) = 0$ and $e_1(G)$ denote the number of edges uv for which $f^*(uv) = 1$. They say G is *balanced* if it has a friendly labeling f such that if $|e_0(f) - e_1(f)| \leq 1$. In the case that the number of vertices labeled 0 and the number of vertices labeled 1 are equal and the number of edges labeled 0 and the number of edges labeled 1 are equal they say the labeling is *strongly balanced*. They prove: P_n is balanced for all n and is strongly balanced if n is even; $K_{m,n}$ is balanced if and only if m and n are even, m and n are odd and differ by at most 2, or exactly one of m or n is even (say $n = 2t$) and $t \equiv -1, 0, 1 \pmod{|m-n|}$; a k-regular graph with p vertices is strongly balanced if and only if p is even and is balanced if and only if p is odd and $k = 2$; and if G is any graph and H is strongly balanced, the composition $G[H]$ (see §2.3 for the definition) is strongly balanced. In [813] Kong, Lee, Seah, and Tang show: $C_m \times P_n$ is balanced if m and n are odd and is strongly balanced if either m or n is even; and $C_m \odot K_1$ is balanced for all $m \geq 3$ and strongly balanced if m is even. They also provide necessary and sufficient conditions for a graph to be balanced or strongly balanced. Lee, Lee, and Ng [855] show that stars are balanced if and only if the number of edges of the star is at most 4. Kwong, Lee, Lo, and Wang [846] define a graph G to be *uniformly balanced* if $|e_0(f) - e_1(f)| \leq 1$ for every vertex labeling f that satisfies if $|v_0(f) - v_1(f)| \leq 1$. They present several ways to construct families of uniformly balanced graphs. Kim, Lee, and Ng [788] prove the following: for any graph G, mG is balanced for all m; for any graph G, mG is strongly balanced for all even m; if G is strongly balanced and H is balanced, then $G \cup H$ is balanced; mK_n is balanced for all m and strongly balanced if and only if $n = 3$ or mn is even; if H is balanced and G is any graph, the $G \times H$ is strongly balanced; if one of m or n is even, then $P_m[P_n]$ is balanced; if both m and n are even, then $P_m[P_n]$ is balanced; and if G is any graph and H is strongly balanced, then the tensor product $G \otimes H$ is strongly balanced. (The *tensor product* $G \otimes H$ of graphs G and H, has the vertex set $V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in $G \otimes H$ if and only if u' is adjacent with v' and u is adjacent with v.)

A graph G is *k-balanced* if there is a function f from the vertices of G to $\{0, 1, 2, \ldots, k-1\}$ such that for the induced function f^* from the edges of G to $\{0, 1, 2, \ldots, k-1\}$ defined by $f^*(uv) = |f(u) - f(v)|$ the number of vertices labeled i and the number of edges labeled j differ by at most 1 for each i and j. Seoud, El Sonbaty, and Abd El Rehim [1267] proved the following: if $|E| \geq 2k+1$ and $|V| \leq k$ then $G(V, E)$ is not k-balanced; if $|E| \geq 3k+1$, $(k \geq 2)$ and $3k-1 \geq |V| \geq 2k+1$ then $G(V, E)$ is not k-balanced; r-regular graphs with $3 \leq r \leq n-1$ are not r-balanced; if G_1 has m vertices and G_2 has n vertices then $G_1 + G_2$ is not $(m+n)$-balanced for $m, n \geq 5$; $P_3 \times P_n$ with edge set E is $3n$-balanced and $|E|$-balanced; $L_n \times P_2$ $(L_n = P_n \times P_2)$ with vertex set V and edge set E is $|V|$-balanced and k-balanced for $k \geq |E|$ but not n-balanced for $n \geq 2$; the one-point union of two copies of $K_{2,n}$ is $2n$-balanced, $|V|$-balanced, and $|E|$-balanced not is 3-balanced when $n \geq 4$. They also proved that the composition graph $P_n[P_2]$ is not n-balanced for $n \geq 3$, is not $2n$-balanced for $n \geq 5$, and is not $|E|$-balanced.

A graph whose edges are labeled with 0 and 1 so that the absolute difference in the number of edges labeled 1 and 0 is no more than one is called *edge-friendly*. We say an edge-friendly labeling induces a *partial vertex labeling* if vertices which are incident to more edges labeled 1 than 0, are labeled 1, and vertices which are incident to more edges labeled 0 than 1, are labeled 0. Vertices that are incident to an equal number of edges of both labels are called *unlabeled*. Call a procedure on a labeled graph a label switching algorithm if it consists of pairwise switches of labels. Krop, Lee, and Raridan [834] prove that given an *edge-friendly labeling* of K_n, we show a label switching algorithm producing an edge-friendly relabeling of K_n such that all the vertices are labeled.

3.8 The Friendly Index–Balance Index

Recall a function f from $V(G)$ to $\{0,1\}$ where for each edge xy, $f^*(xy) = |f(x) - f(y)|$, $v_i(f)$ is the number of vertices v with $f(v) = i$, and $e_i(f)$ is the number of edges e with $f^*(e) = i$ is called *friendly* if $|v_0(f) - v_1(f)| \leq 1$. Lee and Ng [888] define the *friendly index set* of a graph G as FI(G)= $\{|e_0(f) - e_1(f)|$ where f runs over all friendly labelings f of $G\}$. They proved: for any graph G with q edges FI(G) $\subseteq \{0, 2, 4, \ldots, q\}$ if q is even and FI(G)$\subseteq \{1, 3, \ldots, q\}$ if q is odd; for $1 \leq m \leq n$, FI($K_{m,n}$)= $\{(m-2i)^2 |\, 0 \leq i \leq \lfloor m/2 \rfloor\}$ if $m+n$ is even; and FI($K_{m,n}$)= $\{i(i+1) |\, 0 \leq i \leq m\}$ if $m + n$ is odd. In [892] Lee and Ng prove the following: FI(C_{2n}) $= \{0, 4, 8, \ldots, 2n\}$ when n is even; FI(C_{2n}) $= \{2, 6, 10, \ldots, 2n\}$ when n is odd; and FI(C_{2n+1}) $= \{1, 3, 5, \ldots, 2n-1\}$. Elumalai [454] defines a *cycle with a full set of chords* as the graph PC_n obtained from $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the cords $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$ when n is even and $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$ when n is odd. Lee and Ng [890] prove: FI(PC_{2m+1}) $= \{3m-2, 3m-4, 3m-6, \ldots, 0\}$ when m is even and FI(PC_{2m+1}) $= \{3m-2, 3m-4, 3m-6, \ldots, 1\}$ when m is odd; FI(PC_4) $= \{1, 3\}$; for $m \geq 3$, FI(PC_{2m}) $= \{3m-5, 3m-7, 3m-9, \ldots, 1\}$ when m is even; FI(PC_{2m}) $= \{3m-5, 3m-7, 3m-9, \ldots, 0\}$ when m is odd.

Salehi and Lee [1227] determined the friendly index for various classes of trees. Among their results are: for a tree with q edges that has a perfect matching, the friendly index is the odd integers from 1 to q and for $n \geq 2$, FI(P_n)= $\{n-1-2i |\, 0 \leq i\lfloor (n-1)/2 \rfloor$. Lee and Ng [890] define $PC(n,p)$ as the graph obtained from the cycle C_n with consecutive vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the p cords joining v_i to v_{n-i} for $1 \leq p\lfloor n/2 \rfloor - 1$. They prove FI($PC(2m+1,p)$) $= \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 1\}$ if p is even and FI($PC(2m+1,p)$) $= \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 0\}$ if p is odd; FI($PC(2m,1)$) $= \{2m-1, 2m-3, 2m-5, \ldots, 1\}$; for $m \geq 3$, and $p \geq 2$, FI($PC(2m,p)$) $= \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 0\}$ when p is even, and FI($PC(2m,p)$) $= \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 1\}$ when p is odd. More generally, they show that the integers in the friendly index of a cycle with an arbitrary nonempty set of parallel chords form an arithmetic progression with a common difference 2. Shiu and Kwong [1337] determine the friendly index of the grids $P_n \times P_2$. The maximum and minimum friendly indices for $C_m \times P_n$ were given by Shiu and Wong in [1358].

In [891] Lee and Ng prove: for $n \geq 2$, FI($C_{2n} \times P_2$) $= \{0, 4, 8, \ldots, 6n-8, 6n\}$ if n is even and FI($C_{2n} \times P_2$) $= \{2, 6, 10, \ldots, 6n-8, 6n\}$ if n is odd; FI($C_3 \times P_2$) $= \{1, 3, 5\}$; for $n \geq 2$, FI($C_{2m+1} \times P_2$) $= \{6n-1\} \cup \{6n-5-2k|$ where $k \geq 0$ and $6n-5-2k \geq 0\}$; FI(M_{4n}) (here M_{4n} is the Möbius ladder with $4n$ steps) $= \{6n-4-4k|$ where $k \geq 0$ and $6n-4-4k \geq 0\}$; FI(M_{4n+2}) $= \{6n+3\} \cup \{6n-5-2k|$ where $k \geq 0$ and $6n-5-2k > 0\}$. In [847] Kwong, Lee, and Ng completely determine the friendly index of all 2-regular graphs. As a corollary, they show that $C_m \cup C_n$ is cordial if and only if $m+n = 0, 1$ or $3 \pmod 4$. Ho, Lee, and Ng [656] determine the friendly index sets of stars and various regular windmills. In [1639] Wen determines the friendly index of generalized wheels $C_n + mK_1$ for all $m > 1$. In [1226] Salehi and De determine the friendly index sets of certain caterpillars of diameter 4 and disprove a conjecture of Lee and

Ng [892] that the friendly index sets of trees form an arithmetic progression. The maximum and minimum friendly indices for for $C_m \times P_n$ were given by Shiu and Wong in [1358]. Salehi and Bayot [1224] have determined the friendly index set of $P_m \times P_n$.

For positive integers $a \leq b \leq c$, Lee, Ng, amd Tong [896] define the *broken wheel* $W(a,b,c)$ with three spokes as the graph obtained from K_4 with vertices u_1, u_2, u_3, c by inserting vertices $x_{1,1}, x_{1,2}, \ldots x_{1,a-1}$ along the edge u_1u_2, $x_{2,1}, x_{2,2}, \ldots x_{2,b-1}$ along the edge u_2u_3, $x_{3,1}, x_{3,2}, \ldots x_{3,c-1}$ along the edge u_3u_1. They determine the friendly index set for broken wheels with three spokes.

Lee and Ng [890] define a *parallel chord* of C_n as an edge of the form v_iv_{n-i} $(i < n-1)$ that is not an edge of C_n. For $n \geq 6$, they call the cycle C_n with consecutive vertices $v_1, v_2, \ldots, v_n$ and the edges $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}v_{(n+2)/2}$ for n even and $v_2v_{n-1}, v_3v_{n-2}, \ldots, v_{(n-1)/2}v_{(n+3)/2}$ for n odd, C_n with a *full set of parallel chords*. They determine the friendly index of these graphs and show that for any cycle with an arbitrary non-empty set of parallel chords the numbers in its friendly index set form an arithmetic progression with common difference 2.

For a graph $G(V,E)$ and a graph H rooted at one of its vertices v, Ho, Lee, and Ng [655] define a *root-union of* (H,v) *by* G as the graph obtained from G by replacing each vertex of G with a copy of the root vertex v of H to which is appended the rest of the structure of H. They investigate the friendly index set of the root-union of stars by cycles.

For a graph $G(V,E)$, the *total graph* $T(G)$ of G, is the graph with vertex set $V \cup E$ and edge set $E \cup \{(v,uv)|\ v \in V, uv \in E\}$. Note that the total graph of the n-star is the friendship graph and the total graph of P_n is a triangular snake. Lee and Ng [887] use $SP(1^n,m)$ to denote the spider with one central vertex joining n isolated vertices and a path of length m. They show: $\mathrm{FI}(K_1 + 2nK_2)$ (friendship graph with $2n$ triangles) $= \{2n, 2n-4, 2n-8, \ldots, 0\}$ if n is even; $\{2n, 2n-4, 2n-8, \ldots, 2\}$ if n is odd; $\mathrm{FI}(K_1 + (2n+1)K_2) = \{2n+1, 2n-1, 2n-3, \ldots, 1\}$; for n odd, $\mathrm{FI}(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, z\}$ where $z = 0$ if $n \equiv 1 \pmod 4$ and $z = 2$ if $n \equiv 3 \pmod 4$; for n even, $\mathrm{FI}(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, n+1\} \cup \{n-1, n-3, n-5, \ldots, 1\}$; for $m \leq n-1$ and $m+n$ even, $\mathrm{FI}(T(SP(1^n,m))) = \{3(m+n)-4, 3(m+n)-8, 3(m+n)-12, \ldots, (m+n) \pmod 4\}$; for $m+n$ odd, $\mathrm{FI}(T(SP(1^n,m))) = \{3(m+n)-4, 3(m+n)-8, 3(m+n)-12, \ldots, m+n+2\} \cup \{m+n, m+n-2, m+n-4, \ldots, 1\}$; for $n \geq m$ and $m+n$ even, $\mathrm{FI}(T(SP(1^n,m))) = \{|4k-3(m+n)|\ |(n-m+2)/2 \leq k \leq m+n\}$; for $n \geq m$ and $m+n$ odd, $\mathrm{FI}(T(SP(1^n,m))) = \{|4k-3(m+n)|\ |(n-m+3)/2 \leq k \leq m+n\}$.

Kwong and Lee [843] determine the friendly index any number of copies of C_3 that share an edge in common and the friendly index any number of copies of C_4 that share an edge in common.

In [789] Kim, Lee, and Ng define the *balance index set* of a graph G as $\{|e_0(f) - e_1(f)|\}$ where f runs over all friendly labelings f of G. Zhang, Lee, and Wen [855] investigate the balance index sets for the disjoint union of up to four stars and Zhang, Ho, Lee, and Wen [1714] investigate the balance index sets for trees with diameter at most four. Kwong, Lee, and Sarvate [850] determine the balance index sets for cycles with one pendent edge, flowers, and regular windmills. Lee, Ng, and Tong [895] determine the balance index set of certain graphs obtained by starting with copies of a given cycle and successively identifying one particular vertex of one copy with a particular vertex of the next. For graphs G and H and a bijection π from G to H, Lee and Su [916] define $\mathrm{Perm}(G,\pi,H)$ as the graph obtaining from the disjoint union of G and H by joining each v in G to $\pi(v)$ with an edge. They determine the balanced index sets of the disjoint union of cycles and the balanced index sets for graphs of the form $\mathrm{Perm}(G,\pi,H)$ where G and H are regular graphs, stars, paths, and cycles with a chord. They conjecture that the balanced index set for every graph of the form $\mathrm{Perm}(G, \pi, H)$ is an arithmetic progression.

Wen [1638] determines the balance index set of the graph that is constructed by identifying the center of a star with one vertex from each of two copies of C_n and provides a necessary and sufficient for such graphs to be balanced. In [918] Lee, Su, and Wang determine the balance index sets of the disjoint union of a variety of regular graphs of the same order. Kwong [841] determines the balanced index sets of rooted trees of height at most 2, thereby settling the problem for trees with diameter at most 4. His method can be used to determine the balance index set of any tree. The *homeomorph* $Hom(G, p)$ of a graph G is the collection of graphs obtained from G by adding p ($p \geq 0$) additional degree 2 vertices to its edges. For any regular graph G, Kong, Lee, and Lee [808] studied the changes of the balance index sets of $Hom(G, p)$ with respect to the parameter p. They derived explicit formulas for their balance index sets provided new examples of uniformly balanced graphs. In [302] Bouchard, Clark, Lee, Lo, and Su investigate the balance index sets of generalized books and ear expansion graphs. In [1213] Rose and Su provided an algorithm to calculate the balance index sets of a graph.

In[1338] Shiu and Kwong made a major advance by introducing an easier approach to find the balance index sets of a large number of families of graphs in a unified and uniform manner. They use this method to determine the balance index sets for r-regular graphs, amalgamations of r-regular graphs, complete bipartite graphs, wheels, one point unions of regular graphs, sun graphs, generalized theta graphs, m-ary trees, spiders, grids $P_m \times P_n$, and cylinders $C_m \times P_n$. They provide a formula that enables one to determine the balance index sets of many biregular graphs (that is, graphs with the property that there exist two distinct positive integers r and s such that every vertex has degree r or s).

For a graph G and a connected graph H with a distinguished vertex s, the L-product of G and (H, s), $G \times_L (H, s)$, is the graph obtained by taking $|V(G)|$ copies of (H, s) and identifying each vertex of G with s of a single copy of H. In [392] and [304] Chou, Galiardi, Kong, Lee, Perry, Bouchard, Clark, and Su investigated the edge-balance index sets of L-product of cycles with stars. Bouchard, Clark, and Su [303] gave the exact values of the edge-balance index sets of L-product of cycles with cycles.

In [1337] Shiu and Kwong define the *full friendly index set* of a graph G as $\{e_0(f) - e_1(f)\}$ where f runs over all friendly labelings of G. The full friendly index for $P_2 \times P_n$ is given by Shiu and Kwong in [1337]. The full friendly index of $C_m \times C_n$ is given by Shiu and Ling in [1350]. In [1387] and [1388] Sinha and Kaur investgated the full friendly index sets complete graphs, cycles, fans, double fans, wheels, double stars, $P_3 \times P_n$, and the tensor product of P_2 and P_n.

The *twisted cylinder* graph is the permutation graph on $4n$ ($n \geq 2$) vertices, $P(2n; \sigma)$, where $\sigma = (1, 2)(3, 4) \cdots (2n-1, 2n)$ (the product of n transpositions). Shiu and Lee [1348] determined the full friendly index sets and the full product-cordial index sets of twisted cylinders.

In [390] and [844] Chopra, Lee and Su and Kwong and Lee introduce a dual of balance index sets as follows. For an edge labeling f using 0 and 1 they define a partial vertex labeling f^* by assigning 0 or 1 to $f^*(v)$ depending on whether there are more 0-edges or 1-edges incident to v and leaving $f^*(v)$ undefined otherwise. For $i = 0$ or 1 and a graph $G(V, E)$, let $e_f(i) = |\{uv \in E : f(uv) = i\}|$ and $v_f(i) = |\{v \in V : f^*(v) = i\}|$. They define the *edge-balance index* of G as $\mathrm{EBI}(G) = \{|v_f(0) - v_f(1)| : \text{the edge labeling } f \text{ satisfies } |e_f(0) - e_f(1)| \leq 1\}$. Among the graphs whose edge-balance index sets have been investigated by Lee and his colleagues are: fans and wheels [390]; generalized theta graphs [844]; flower graphs [845] and [845]; stars, paths, spiders, and double stars [925]; $(p, p+1)$-graphs [920]; prisms and Möbius ladders [1621]; 2-regular graphs, complete graphs [1620]; and the envelope graphs of stars, paths, and cycles [400]. (The *envelope* *graph* of $G(V,E)$ is the graph with vertex set $V(G) \cup E(G)$ and set $E(G) \cup \{(u,(u,v)) : U \in V, (u,v) \in E)\})$.

Lee, Kong, and Wang [5] found the EBI$(K_{m,n})$ for $m = 1,2,3,4,5$ and $m = n$. Krop and Sikes [837] determined EBI$(K_{m,m-2a})$ for $1 \leq a \leq (m-3)/4$ and m odd.

Chopra, Lee, and Su [393] prove that the edge-balanced index of the fan $P_3 + K_1$ is $\{0,1,2\}$ and edge-balanced index of the fan $P_n + K_1$, $n \geq 4$, is $\{0,1,2,\ldots,n-2\}$. They define the broken fan graphs $BF(a,b)$ as the graph with $V(BF(a,b)) = \{c\} \cup \{v_1,\ldots,v_a\} \cup \{u_1,\ldots,u_b\}$ and $E(BF(a,b)) = \{(c,v_i)|\ i = 1,\ldots,a\} \cup \{(c,u_i)|\ 1,\ldots,b\} \cup E(P_a) \cup E(P_b)$ ($a \geq 2$ and $b \geq 2$). They prove the edge-balance index set of $BF(a,b)$ is $\{0,1,2,\ldots,a+b-4\}$. In [856] Lee, Lee, and Su present a technique that determines the balance index sets of a graph from its degree sequence. In addition, they give an explicit formula giving the exact values of the balance indices of generalized friendship graphs, envelope graphs of cycles, and envelope graphs of cubic trees.

3.9 k-equitable Labelings

In 1990 Cahit [333] proposed the idea of distributing the vertex and edge labels among $\{0,1,\ldots,k-1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V,E)$ and any positive integer k, assign vertex labels from $\{0,1,\ldots,k-1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. Cahit has called a graph with such an assignment of labels *k-equitable*. Note that $G(V,E)$ is graceful if and only if it is $|E|+1$-equitable and $G(V,E)$ is cordial if and only if it is 2-equitable. Cahit [332] has shown the following: C_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; the triangular snake with n blocks is 3-equitable if and only if n is even; the friendship graph $C_3^{(n)}$ is 3-equitable if and only if n is even; an Eulerian graph with $q \equiv 3 \pmod 6$ edges is not 3-equitable; and all caterpillars are 3-equitable [332]. Cahit [332] claimed to prove that W_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$ but Youssef [1697] proved that W_n is 3-equitable for all $n \geq 4$. Youssef [1695] also proved that if G is a k-equitable Eulerian graph with q edges and $k \equiv 2$ or $3 \pmod 4$ then $q \not\equiv k \pmod{2k}$. Cahit conjectures [332] that a triangular cactus with n blocks is 3-equitable if and only if n is even. In [333] Cahit proves that every tree with fewer than five end vertices has a 3-equitable labeling. He conjectures that all trees are k-equitable [334]. In 1999 Speyer and Szaniszló [1430] proved Cahit's conjecture for $k = 3$.

Vaidya, Dani, Kanani and Vihol [1514] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of G_i to the center of G_{i+1} ($i = 1,2,\ldots,n-1$) by an edge are 3-equitable. Recall the shell $C(n,n-3)$ is the cycle C_n with $n-3$ cords sharing a common endpoint called the *apex*. Vaidya, Dani, Kanani, and Vihol [1515] proved that the graphs obtained by starting with copies $G_1, G_2, \ldots, G_n$ of a fixed shell and joining each apex of G_i to the apex of G_{i+1} ($i = 1,2,\ldots,n-1$) by an edge are 3-equitable. For a graph G and vertex v of G, Vaidya, Dani, Kanani, and Vihol [1516] prove that the graphs obtained from the wheel W_n, $n \geq 5$, by duplicating (see 3.7 for the definition) any rim vertex is 3-equitable and the graphs obtained from the wheel W_n by duplicating the center is 3-equitable when n is even and not 3-equitable when n is odd and at least 5. They also show that the graphs obtained from the wheel W_n, $n \neq 5$, by duplicating every vertex is 3-equitable.

Vaidya, Srivastav, Kaneria, and Ghodasara [1550] prove that cycle with two chords that share a common vertex with opposite ends that are incident to two consecutive vertices of the cycle is 3-equitable. Vaidya, Ghodasara, Srivastav, and Kaneria [1520] prove that star of cycle ${C_n}^*$ is 3-equitable for all n. Vaidya and Dani [1510] proved that the graphs obtained by starting with n copies $G_1, G_2, \ldots, G_n$ of a fixed star and joining the center of G_i to the center of G_{i+1} by an edge and each center to a new vertex x_i $(1 \leq i \leq n-1)$ by an edge have 3-equitable labeling. Vaidya and Dani [1513] prove that the graphs obtained by duplication of an arbitrary edge of a cycle or a wheel have 3-equitable labelings.

Recall $G =< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ 1s the graph obtained by joining the center vertices of each of $W_n^{(i)}$ and $W_n^{(i+1)}$ to a new vertex x_i where $1 \leq i \leq k-1$. Vaidya, Dani, Kanani, and Vihol [1517] prove that $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ is 3-equitable. Vaidya and Vihol [1554] prove that any graph G can be embedded as an induced subgraph of a 3-equitable graph thereby ruling out any possibiltity of obtaining any forbidden subgraph characterization for 3-equitable graphs.

The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, G' and G'' and joining each vertex u' in G'' to the neighbors of the corresponding vertex u'' in G''. Vaidya, Vihol, and Barasara [1557] prove that the shadow graph of C_n is 3-equitable except for $n = 3$ and 5 while the shadow graph of P_n is 3-equitable except for $n = 3$. They also prove that the middle graph of P_n is 3-equitable and the middle graph of C_n is 3-equitable for n even and not 3-equitable for n odd.

Bhut-Nayak and Telang have shown that crowns $C_n \odot K_1$, are k-equitable for $k = n, \ldots, 2n-1$ [280] and $C_n \odot K_1$ is k-equitable for all n when $k = 2, 3, 4, 5$, and 6 [281].

In [1257] Seoud and Abdel Maqsoud prove: a graph with n vertices and q edges in which every vertex has odd degree is not 3-equitable if $n \equiv 0 \pmod 3$ and $q \equiv 3 \pmod 6$; all fans except $P_2 + \overline{K_1}$ are 3-equitable; all double fans $P_n + \overline{K_2}$ except $P_4 + \overline{K_2}$ are 3-equitable; P_n^2 is 3-equitable for all n except 3; $K_{1,1,n}$ is 3-equitable if and only if $n \equiv 0$ or $2 \pmod 3$; $K_{1,2,n},\ n \geq 2$, is 3-equitable if and only if $n \equiv 2 \pmod 3$; $K_{m,n},\ 3 \leq m \leq n$, is 3-equitable if and only if $(m, n) = (4, 4)$; and $K_{1,m,n},\ 3 \leq m \leq n$, is 3-equitable if and only if $(m, n) = (3, 4)$. They conjectured that C_n^2 is not 3-equitable for all $n \geq 3$. However, Youssef [1703] proved that C_n^2 is 3-equitable if and only if n is at least 8. Youssef [1703] also proved that $C_n + \overline{K_2}$ is 3-equitable if and only if n is even and at least 6 and determined the maximum number of edges in a 3-equitable graph as a function of the number of its vertices. For a graph with n vertices to admit a k-equitable labeling, Seoud and Salim [1283] proved that the number of edges is at most $k\lceil(n/k)\rfloor^2 + k - 1$.

Bapat and Limaye [220] have shown the following graphs are 3-equitable: helms $H_n,\ n \geq 4$; flowers (see §2.2 for the definition); the one-point union of any number of helms; the one-point union of any number of copies of K_4; K_4-snakes (see §2.2 for the definition); C_t-snakes where $t = 4$ or 6; C_5-snakes where the number of blocks is not congruent to 3 modulo 6. A *multiple shell* $\mathrm{MS}\{n_1^{t_1}, \ldots, n_r^{t_r}\}$ is a graph formed by t_i shells each of order $n_i,\ 1 \leq i \leq r$, that have a common apex. Bapat and Limaye [221] show that every multiple shell is 3-equitable and Chitre and Limaye [382] show that every multiple shell is 5-equitable. In [383] Chitre and Limaye define the *H-union* of a family of graphs $G_1, G_2, \ldots, G_t$, each having a graph H as an induced subgraph, as the graph obtained by starting with $G_1 \cup G_2 \cup \cdots \cup G_t$ and identifying all the corresponding vertices and edges of H in each of $G_1, \ldots, G_t$. In [383] and [384] they proved that the $\overline{K_n}$-union of gears and helms H_n $(n \geq 6)$ are edge-3-equitable.

Szaniszló [1483] has proved the following: P_n is k-equitable for all k; K_n is 2-equitable if and only if $n = 1, 2$, or 3; K_n is not k-equitable for $3 \leq k < n$; S_n is k-equitable for all k; $K_{2,n}$ is k-equitable if and only if $n \equiv k-1 \pmod k$, or $n \equiv 0, 1, 2, \ldots, \lfloor k/2 \rfloor - 1 \pmod k$, or $n = \lfloor k/2 \rfloor$

and k is odd. She also proves that C_n is k-equitable if and only if k meets all of the following conditions: $n \neq k$; if $k \equiv 2, 3 \pmod 4$, then $n \neq k-1$ and $n \not\equiv k \pmod{2k}$.

Vickrey [1574] has determined the k-equitability of complete multipartite graphs. He shows that for $m \geq 3$ and $k \geq 3$, $K_{m,n}$ is k-equitable if and only if $K_{m,n}$ is one of the following graphs: $K_{4,4}$ for $k = 3$; $K_{3,k-1}$ for all k; or $K_{m,n}$ for $k > mn$. He also shows that when k is less than or equal to the number of edges in the graph and at least 3, the only complete multipartite graphs that are k-equitable are $K_{kn+k-1,2,1}$ and $K_{kn+k-1,1,1}$. Partial results on the k-equitability of $K_{m,n}$ were obtained by Krussel [838].

In [991] Lopez, Muntaner-Batle, and Rius-Font prove that if n is an odd integer and F is optimal k-equitable for all proper divisors k of $|E(F)|$, then nF is optimal k-equitable for all proper divisors k of $|E(F)|$. They also prove that if $m-1$ and n are odd, then then nC_m is optimal k-equitable for all proper divisors k of $|E(F)|$.

As a corollary of the result of Cairnie and Edwards [344] on the computational complexity of cordially labeling graphs it follows that the problem of finding k-equitable labelings of graphs is NP-complete as well.

Seoud and Abdel Maqsoud [1258] call a graph *k-balanced* if the vertices can be labeled from $\{0, 1, \ldots, k-1\}$ so that the number of edges labeled i and the number of edges labeled j induced by the absolute value of the differences of the vertex labels differ by at most 1. They prove that P_n^2 is 3-balanced if and only if $n = 2, 3, 4$, or 6; for $k \geq 4$, P_n^2 is not k-balanced if $k \leq n-2$ or $n+1 \leq k \leq 2n-3$; for $k \geq 4$, P_n^2 is k-balanced if $k \geq 2n-2$; for $k, m, n \geq 3$, $K_{m,n}$ is k-balanced if and only if $k \geq mn$; for $m \leq n$, $K_{1,m,n}$ is k-balanced if and only if (i) $m = 1$, $n = 1$ or 2, and $k = 3$; (ii) $m = 1$ and $k = n+1$ or $n+2$; or (iii) $k \geq (m+1)(n+1)$.

In [1703] Youssef gave some necessary conditions for a graph to be k-balanced and some relations between k-equitable labelings and k-balanced labelings. Among his results are: C_n is 3-balanced for all $n \geq 3$; K_n is 3-balanced if and only if $n \leq 3$; and all trees are 2-balanced and 3-balanced. He conjectures that all trees are k-balanced ($k \geq 2$).

Bloom has used the term k-equitable to describe another kind of labeling (see [1648] and [1649]). He calls a graph *k-equitable* if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label occurs exactly k times. Bloom calls a graph of order n *minimally k-equitable* if the vertex labels are 1, 2,..., n and it is k-equitable. Both Bloom and Wojciechowski [1648], [1649] proved that C_n is minimally k-equitable if and only if k is a proper divisor of n. Barrientos and Hevia [239] proved that if G is k-equitable of size $q = kw$ (in the sense of Bloom), then $\delta(G) \leq w$ and $\Delta(G) \leq 2w$. Barrientos, Dejter, and Hevia [238] have shown that forests of even size are 2-equitable. They also prove that for $k = 3$ or $k = 4$ a forest of size kw is k-equitable if and only if its maximum degree is at most $2w$ and that if 3 divides $mn+1$, then the double star $S_{m,n}$ is 3-equitable if and only if $q/3 \leq m \leq \lfloor (q-1)/2 \rfloor$. ($S_{m,n}$ is P_2 with m pendent edges attached at one end and n pendent edges attached at the other end.) They discuss the k-equitability of forests for $k \geq 5$ and characterize all caterpillars of diameter 2 that are k-equitable for all possible values of k. Acharya and Bhat-Nayak [33] have shown that coronas of the form $C_{2n} \odot K_1$ are minimally 4-equitable. In [223] Barrientos proves that the one-point union of a cycle and a path (dragon) and the disjoint union of a cycle and a path are k-equitable for all k that divide the size of the graph. Barrientos and Havia [239] have shown the following: $C_n \times K_2$ is 2-equitable when n is even; books B_n $(n \geq 3)$ are 2-equitable when n is odd; the vertex union of k-equitable graphs is k-equitable; and wheels W_n are 2-equitable when $n \not\equiv 3 \pmod 4$. They conjecture that W_n is 2-equitable when $n \equiv 3 \pmod 4$ except when $n = 3$. Their 2-equitable labelings of $C_n \times K_2$

and the n-cube utilized graceful labelings of those graphs.

M. Acharya and Bhat-Nayak [34] have proved the following: the crowns $C_{2n} \odot K_1$ are minimally 2-equitable, minimally $2n$-equitable, minimally 4-equitable, and minimally n-equitable; the crowns $C_{3n} \odot K_1$ are minimally 3-equitable, minimally $3n$-equitable, minimally n-equitable, and minimally 6-equitable; the crowns $C_{5n} \odot K_1$ are minimally 5-equitable, minimally $5n$-equitable, minimally n-equitable, and minimally 10-equitable; the crowns $C_{2n+1} \odot K_1$ are minimally $(2n+1)$-equitable; and the graphs P_{kn+1} are k-equitable.

In [225] Barrientos calls a k-equitable labeling *optimal* if the vertex labels are consecutive integers and *complete* if the induced edge labels are $1, 2, \ldots, w$ where w is the number of distinct edge labels. Note that a graceful labeling is a complete 1-equitable labeling. Barrientos proves that $C_m \odot nK_1$ (that is, an m-cycle with n pendent edges attached at each vertex) is optimal 2-equitable when m is even; $C_3 \odot nK_1$ is complete 2-equitable when n is odd; and that $C_3 \odot nK_1$ is complete 3-equitable for all n. He also shows that $C_n \odot K_1$ is k-equitable for every proper divisor k of the size $2n$. Barrientos and Havia [239] have shown that the n-cube ($n \geq 2$) has a complete 2-equitable labeling and that $K_{m,n}$ has a complete 2-equitable labeling when m or n is even. They conjecture that every tree of even size has an optimal 2-equitable labeling.

3.10 Hamming-graceful Labelings

Mollard, Payan, and Shixin [1068] introduced a generalization of graceful graphs called Hamming-graceful. A graph $G = (V, E)$ is called *Hamming-graceful* if there exists an injective labeling g from V to the set of binary $|E|$-tuples such that $\{d(g(v), g(u))|\ uv \in E\} = \{1, 2, \ldots, |E|\}$ where d is the Hamming distance. Shixin and Yu [1364] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful; C_n is Hamming-graceful if and only if $n \equiv 0$ or 3 (mod 4); if K_n is Hamming-graceful, then n has the form k^2 or k^2+2; and K_n is Hamming-graceful for $n = 2, 3, 4, 6, 9, 11, 16$, and 18. They conjecture that K_n is Hamming-graceful for n of the forms k^2 and $k^2 + 2$ for $k \geq 5$.

4 Variations of Harmonious Labelings

4.1 Sequential and Strongly c-harmonious Labelings

Chang, Hsu, and Rogers [354] and Grace [582], [583] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling f of a graph G with q vertices to be *strongly c-harmonious* if the vertex labels are from $\{0, 1, \ldots, q-1\}$ and the edge labels induced by $f(x) + f(y)$ for each edge xy are $c, \ldots, c+q-1$. Grace called such a labeling *sequential*. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices whereas Grace allows the vertex labels to range from 0 to q with no vertex label being used twice. For graphs other than trees, we use the term c-sequential labelings interchangeably with strongly c-harmonious labelings. By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [583] proved that caterpillars, caterpillars with a pendent edge, odd cycles with zero or more pendent edges, trees with α-labelings, wheels W_{2n+1}, and P_n^2 are sequential. Liu and Zhang [969] finished off the crowns $C_{2n} \odot K_1$. (The case $C_{2n+1} \odot K_1$ was a special case of Grace's results. Liu [981] proved crowns are harmonious.) Bu [314] also proved that crowns are sequential as are all even cycles with m pendent edges attached at each vertex. Figueroa-Centeno, Ichishima, and Muntaner-Batle [497] proved that all cycles with m pendent edges attached at each vertex are sequential. Wu [1653] has shown that caterpillars with m pendent edges attached at each vertex are sequential.

Singh has proved the following: $C_n \odot K_2$ is sequential for all odd $n > 1$ [1375]; $C_n \odot P_3$ is sequential for all odd n [1376]; $K_2 \odot C_n$ (each vertex of the cycle is joined by edges to the end points of a copy of K_2) is sequential for all odd n [1376]; helms H_n are sequential when n is even [1376]; and $K_{1,n} + K_2$, $K_{1,n} + \overline{K}_2$, and ladders are sequential [1378]. Santhosh [1238] has shown that $C_n \odot P_4$ is sequential for all odd $n \geq 3$. Both Grace [582] and Reid (see [533]) have found sequential labelings for the books B_{2n}. Jungreis and Reid [756] have shown the following graphs are sequential: $P_m \times P_n$ $(m, n) \neq (2, 2)$; $C_{4m} \times P_n$ $(m, n) \neq (1, 2)$; $C_{4m+2} \times P_{2n}$; $C_{2m+1} \times P_n$; and $C_4 \times C_{2n}$ $(n > 1)$. The graphs $C_{4m+2} \times C_{2n+1}$ and $C_{2m+1} \times C_{2n+1}$ fail to satisfy a necessary parity condition given by Graham and Sloane [586]. The remaining cases of $C_m \times P_n$ and $C_m \times C_n$ are open. Gallian, Prout, and Winters [534] proved that all graphs $C_n \times P_2$ with a vertex or an edge deleted are sequential.

Gnanajothi [572, pp. 68–78] has shown the following graphs are sequential: $K_{1,m,n}$; mC_n, the disjoint union of m copies of C_n if and only if m and n are odd; books with triangular pages or pentagonal pages; and books of the form B_{4n+1}, thereby answering a question and proving a conjecture of Gallian and Jungreis [533]. Sun [1455] has also proved that B_n is sequential if and only if $n \not\equiv 3 \pmod 4$. Ichishima and Oshima [686] pose determining whether or not $mK_{s,t}$ is sequential as a problem.

Yuan and Zhu [1710] have shown that mC_n is sequential when m and n are odd. Although Graham and Sloane [586] proved that the Möbius ladder M_3 is not harmonious, Gallian [529] established that all other Möbius ladders are sequential (see §2.3 for the definition of Möbius ladder). Chung, Hsu, and Rogers [354] have shown that $K_{m,n} + K_1$, which includes $S_m + K_1$, is sequential. Seoud and Youssef [1290] proved that if G is sequential and has the same number of edges as vertices, then $G + \overline{K_n}$ is sequential for all n. Recall that $\Theta(C_m)^n$ denotes the book with n m-polygonal pages. Lu [1005] proved that $\Theta(C_{2m+1})^{2n}$ is $2mn$-sequential for all n and $m = 1, 2, 3, 4$ and $\Theta(C_m)^2$ is $(m-2)$-sequential if $m \geq 3$ and $m \equiv 2, 3, 4, 7 \pmod 8$.

Zhou and Yuan [1727] have shown that for every c-sequential graph G with p vertices and q edges and any positive integer m the graph $(G + \overline{K_m}) + \overline{K_n}$ is also k-sequential when $q - p + 1 \leq m \leq q - p + c$. Zhou [1726] has shown that the analogous results hold for strongly c-harmonious graphs. Zhou and Yuan [1727] have shown that for every c-sequential graph G with p vertices and q edges and any positive integer m the graph $(G + \overline{K_m}) + \overline{K_n}$ is c-sequential when $q - p + 1 \leq m \leq q - p + c$.

Shee [905] proved that every graph is a subgraph of a sequential graph. Acharya, Germina, Princy, and Rao [23] prove that every connected graph can be embedded in a strongly c-harmonious graph for some c. Miao and Liang [1049] use $C_n(d; i, j; P_k)$ to denote a cycle C_n with path P_k joining two nonconsecutive vertices x_i and x_j of the cycle, where d is the distance between x_i and x_j on C_n. They proved that the graph $C_n(d; i, j; P_k)$ is strongly c-harmonious when $k = 2, 3$ and integer $n \geq 6$. Lu [1004] provides three techniques for constructing larger sequential graphs from some smaller one: an attaching construction, an adjoining construction, and the join of two graphs. Using these, he obtains various families of sequential or strongly c-indexable graphs.

For $1 \leq s \leq n_3$, let $C_n(i : i_1, i_2 \ldots i_s)$ denote an n-cycle with consecutive vertices $x_1, x_2, \ldots x_n$ to which the s chords $x_i x_{i_1}, x_i x_{i_2}, \ldots, x_i x_{i_s}$ have been added. Liang [953] proved a variety of graphs of the form $C_n(i : i_1, i_2 \dots i_s)$ are strongly c-harmonius.

Youssef [1700] observed that a strongly c-harmonious graph with q edges is c-cordial for all $c \geq q$ and a strongly k-indexable graph is k-cordial for every k. The converse of this latter result is not true.

In [683] Ichishima and Oshima show that the hypercube Q_n ($n \geq 2$) is sequential if and only if $n \geq 4$. They also introduce a special kind of sequential labeling of a graph G with size $2t+s$ by defining a sequential labeling f to be a *partitional labeling* if G is bipartite with partite sets X and Y of the same cardinality s such that $f(x) \leq t+s-1$ for all $x \in X$ and $f(y) \geq t-s$ for all $y \in Y$, and there is a positive integer m such that the induced edge labels are partitioned into three sets $[m, m+t-1]$, $[m+t, m+t+s-1]$, and $[m+t+s, m+2t+s-1]$ with the properties that there is an involution π, which is an automorphism of G such that π exchanges X and Y, $x\pi(x) \in E(G)$ for all $x \in X$, and $\{f(x)+f(\pi(x)) | \, x \in X\} = [m+t, m+t+s-1]$. They prove if G has a partitional labeling, then $G \times Q_n$ has a partitional labeling for every nonnegative integer n. Using this together with existing results and the fact that every graph that has a partitional labeling is sequential, harmonious, and felicitous (see §4.5) they show that the following graphs are partitional, sequential, harmonious, and felicitous: for $n \geq 4$, hypercubes Q_n; generalized books $S_{2m} \times Q_n$; and generalized ladders $P_{2m+1} \times Q_n$.

In [684] Ichishma and Oshima proved the following: if G is a partitional graph, then $G \times K_2$ is partitional, sequential, harmonious and felicitous; if G is a connected bipartite graph with partite sets of distinct odd order such that in each partite set each vertex has the same degree, then $G \times K_2$ is not partitional; for every positive integer m, the book B_m is partitional if and only if m is even; the graph $B_{2m} \times Q_n$ is partitional if and only if $(m, n) \neq (1, 1)$; the graph $K_{m,2} \times Q_n$ is partitional if and only if $(m, n) \neq (2, 1)$; for every positive integer n, the graph $K_{m,3} \times Q_n$ is partitional when $m = 4, 8, 12$, or 16. As open problems they ask which m and n is $K_{m,n} \times K_2$ partitional and for which l, m and n is $K_{l,m} \times Q_n$ partitional?

Ichishma and Oshima [684] also investigated the relationship between partitional graphs and strongly graceful graphs (see §3.1 for the definition) and partitional graphs and strongly felicitous graphs (see §4.5 for the definition). They proved the following. If G is a partitional graph, then $G \times K_2$ is partitional, sequential, harmonious and felicitous. Assume that G is a partitional graph of size $2t+s$ with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that $\lambda_1 = \max\{f(x) : x \in X\}$ and $\lambda_2 = \max\{f(y) : y \in Y\}$. If $\lambda_1 + 1 = m + 2t + s - \lambda_2$, where $m = \min\{f(x) + f(y) : xy \in E(G)\} = \min\{f(y) : y \in Y\}$, then G has a strong α-valuation. Assume that G is a partitional graph of size $2t+s$ with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that $\lambda_1 = \max\{f(x) : x \in X\}$ and $\lambda_2 = \max\{f(y) : y \in Y\}$. If $\lambda_1 + 1 = m + 2t + s - \lambda_2$, where $m = \min\{f(x) + f(y) : xy \in E(G)\} = \min\{f(y) : y \in Y\}$, then G is strongly felicitous. Assume that G is a partitional graph of size $2t+s$ with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that $\mu_1 = f(x_1) = \min\{f(x) : x \in X\}$ and $\mu_2 = f(y_1) = \min\{f(y) : y \in Y\}$. If $t+s = m+1$ and $\mu_1 + \mu_2 = m$, where $m = \min\{f(x) + f(y) : xy \in E(G)\}$ and $x_1y_1 \in E(G)$, then G has a strong α-valuation and strongly felicitous labeling.

Singh and Varkey [1382] call a graph with q edges *odd sequential* if the vertices can be labeled with distinct integers from the set $\{0, 1, 2, \dots, q\}$ or, in the case of a tree, from the set $\{0, 1, 2, \dots, 2q-1\}$, such that the edge labels induced by addition of the labels of the endpoints take on the values $\{1, 3, 5, \dots, 2q-1\}$. They prove that combs, grids, stars, and rooted trees of level 2 are odd sequential whereas odd cycles are not. Singh and Varkey call a graph G

bisequential if both G and its line graph have a sequential labeling. They prove paths and cycles are bisequential.

Among the strongly 1-harmonious (also called *strongly harmonious*) graphs are: fans F_n with $n \geq 2$ [354]; wheels W_n with $n \not\equiv 2 \pmod 3$ [354]; $K_{m,n} + K_1$ [354]; French windmills $K_4^{(t)}$ [666], [765]; the friendship graphs $C_3^{(n)}$ if and only if $n \equiv 0$ or $1 \pmod 4$ [666], [765], [1671]; $C_{4k}^{(t)}$ [1456]; and helms [1181].

Seoud, Diab, and Elsakhawi [1266] have shown that the following graphs are strongly harmonious: $K_{m,n}$ with an edge joining two vertices in the same partite set; $K_{1,m,n}$; the composition $P_n[P_2]$ (see §2.3 for the definition); $B(3,2,m)$ and $B(4,3,m)$ for all m (see §2.4 for the notation); P_n^2 $(n \geq 3)$; and P_n^3 $(n \geq 3)$. Seoud et al. [1266] have also proved: B_{2n} is strongly $2n$-harmonious; P_n is strongly $\lfloor n/2 \rfloor$-harmonious; ladders L_{2k+1} are strongly $(k+1)$-harmonious; and that if G is strongly c-harmonious and has an equal number of vertices and edges, then $G + \overline{K_n}$ is also strongly c-harmonious.

Sethuraman and Selvaraju [1321] have proved that the graph obtained by joining two complete bipartite graphs at one edge is graceful and strongly harmonious. They ask whether these results extend to any number of complete bipartite graphs.

For a graph $G(V,E)$ Gayathri and Hemalatha [556] define an *even sequential harmonious labeling* f of G as an injection from V to $\{0,1,2,\ldots,2|E|\}$ with the property that the induced mapping f^+ from E to $\{2,4,6,\ldots,2|E|\}$ defined by $f^+(uv) = f(u)+f(v)$ when $f(u)+f(v)$ is even, and $f^+(uv) = f(u)+f(v)+1$ when $f(u)+f(v)$ is odd, is an injection. They prove the following have even sequential harmonious labelings (all cases are the nontrivial ones): $P_n, P_n^+, C_n (n \geq 3)$, triangular snakes, quadrilateral snakes, Möbius ladders, $P_m \times P_n$ $(m \geq 2, n \geq 2)$, $K_{m,n}$; crowns $C_m \odot K_1$, graphs obtained by joining the centers of two copies of $K_{1,n}$ by a path; banana trees (see §2.1), P_n^2, closed helms (see §2.2), $C_3 \odot nK_1 (n \geq 2)$; $D \odot K_{1,n}$ where D is a dragon (see §2.2); $\langle K_{1,n} : m \rangle$ $(m, n \geq 2)$ (see §4.5); the wreath product $P_n * \overline{K_2}$ $(n \geq 2)$ (see §4.5); combs $P_n \odot K_1$; the one-point union of the end point of a path to a vertex of a cycle (tadpole); the one-point union of the end point of a tadpole and the center of a star; the graphs PC_n obtained from $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$ by adding the cords $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$ when n is even and $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$ when n is odd (that is, cycles with a full set of cords); $P_m \cdot nK_1$; the one-point union of a vertex of a cycle and the center of a star; graphs obtained by joining the centers of two stars with an edge; graphs obtained by joining two disjoint cycles with an edge (dumbbells); graphs consisting of two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex (butterflies).

4.2 (k,d)-arithmetic Labelings

Acharya and Hegde [27] have generalized sequential labelings as follows. Let G be a graph with q edges and let k and d be positive integers. A labeling f of G is said to be (k,d)-*arithmetic* if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x)+f(y)$ for each edge xy are $k, k+d, k+2d, \ldots, k+(q-1)d$. They obtained a number of necessary conditions for various kinds of graphs to have a (k,d)-arithmetic labeling. The case where $k=1$ and $d=1$ was called *additively graceful* by Hegde [627]. Hegde [627] showed: K_n is additively graceful if and only if $n = 2, 3$, or 4; every additively graceful graph except K_2 or $K_{1,2}$ contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendent edge attached. Jinnah and Singh [746] noted that P_n^2 is additively graceful.

Hegde [628] proved that if G is strongly k-indexable, then G and $G+\overline{K_n}$ are (kd,d)-arithmetic. Acharya and Hegde [29] proved that K_n is (k,d)-arithmetic if and only if $n \geq 5$ (see also [320]). They also proved that a graph with an α-labeling is a (k,d)-arithmetic for all k and d. Bu and Shi [320] proved that $K_{m,n}$ is (k,d)-arithmetic when k is not of the form id for $1 \leq i \leq n-1$. For all $d \geq 1$ and all $r \geq 0$, Acharya and Hegde [27] showed the following: $K_{m,n,1}$ is $(d+2r,d)$-arithmetic; C_{4t+1} is $(2dt+2r,d)$-arithmetic; C_{4t+2} is not (k,d)-arithmetic for any values of k and d; C_{4t+3} is $((2t+1)d+2r,d)$-arithmetic; W_{4t+2} is $(2dt+2r,d)$-arithmetic; and W_{4t} is $((2t+1)d+2r,d)$-arithmetic. They conjecture that C_{4t+1} is $(2dt+2r,d)$-arithmetic for some r and that C_{4t+3} is $(2dt+d+2r,d)$-arithmetic for some r. Hegde and Shetty [643] proved the following: the generalized web $W(t,n)$ (see §2.2 for the definition) is $((n-1)d/2,d)$-arithmetic and $((3n-1)d/2,d)$-arithmetic for odd n; the join of the generalized web $W(t,n)$ with the center removed and $\overline{K_p}$ where n is odd is $((n-1)d/2,d)$-arithmetic; every T_p-tree (see §3.2 for the definition) with q edges and every tree obtained by subdividing every edge of a T_p-tree exactly once is $(k+(q-1)d,d)$-arithmetic for all k and d. Lu, Pan, and Li [1007] proved that $K_{1,m} \cup K_{p,q}$ is (k,d)-arithmetic when $k > (q-1)d+1$ and $d > 1$.

Yu [1706] proved that a necessary condition for C_{4t+1} to be (k,d)-arithmetic is that $k = 2dt+r$ for some $r \geq 0$ and a necessary condition for C_{4t+3} to be (k,d)-arithmetic is that $k = (2t+1)d+2r$ for some $r \geq 0$. These conditions were conjectured by Acharya and Hegde [27]. Singh proved that the graph obtained by subdividing every edge of the ladder L_n is $(5,2)$-arithmetic [1374] and that the ladder L_n is $(n,1)$-arithmetic [1377]. He also proves that $P_m \times C_n$ is $((n-1)/2,1)$-arithmetic when n is odd [1377]. Acharya, Germina, and Anandavally [21] proved that the subdivision graph of the ladder L_n is (k,d)-arithmetic if either d does not divide k or $k = rd$ for some $r \geq 2n$ and that $P_m \times P_n$ and the subdivision graph of the ladder L_n are (k,k)-arithmetic if and only if k is at least 3. Lu, Pan, and Li [1007] proved that $S_m \cup K_{p,q}$ is (k,d)-arithmetic when $k > (q-1)d+1$ and $d > 1$.

A graph is called *arithmetic* if it is (k,d)-arithmetic for some k and d. Singh and Vilfred [1384] showed that various classes of trees are arithmetic. Singh [1377] has proved that the union of an arithmetic graph and an arithmetic bipartite graph is arithmetic. He conjectures that the union of arithmetic graphs is arithmetic. He provides an example to show that the converse is not true.

Germina and Anandavally [565] investigated embedding of graphs in arithmetic graphs. They proved: every graph can be embedded as an induced subgraph of an arithmetic graph; every bipartite graph can be embedded in a (k,d)-arithmetic graph for all k and d such that d does not divide k; and any graph containing an odd cycle cannot be embedded as an induced subgraph of a connected (k,d)-arithmetic with $k < d$.

4.3 (k,d)-indexable Labelings

Acharya and Hegde [27] call a graph with p vertices and q edges (k,d)-*indexable* if there is an injective function from V to $\{0,1,2,\ldots,p-1\}$ such that the set of edge labels induced by adding the vertex labels is a subset of $\{k,k+d,k+2d,\ldots,k+q(d-1)\}$. When the set of edges is $\{k,k+d,k+2d,\ldots,k+q(d-1)\}$ the graph is said to be *strongly* (k,d)-*indexable*. A $(k,1)$-graph is more simply called k-*indexable* and strongly 1-indexable graphs are simply called *strongly indexable*. Notice that strongly indexable graphs are a stronger form of sequential graphs and for trees and unicyclic graphs the notions of sequential labelings and strongly k-indexable labelings coincide. Hegde and Shetty [648] have shown that the notions of $(1,1)$-strongly indexable graphs and super edge-magic total labelings (see §5.2) are equivalent.

Zhou [1726] has shown that for every k-indexable graph G with p vertices and q edges the graph $(G + \overline{K_{q-p+k}}) + \overline{K_1}$ is strongly k-indexable. Acharaya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are K_2, K_3, and $K_2 \times K_3$, and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [27] call a graph with p vertices *indexable* if there is an injective labeling of the vertices with labels from $\{0, 1, 2, \ldots, p-1\}$ such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [102] who also proved that all trees are indexable. Bu and Shi [321] also proved that all trees are indexable and that all unicyclic graphs with the cycle C_3 are indexable. Hegde [628] has shown the following: every graph can be embedded as an induced subgraph of an indexable graph; if a connected graph with p vertices and q edges ($q \geq 2$) is (k,d)-indexable, then $d \leq 2$; $P_m \times P_n$ is indexable for all m and n; if G is a connected $(1,2)$-indexable graph, then G is a tree; the minimum degree of any $(k,1)$-indexable graph with at least two vertices is at most 3; a caterpillar with partite sets of orders a and b is strongly $(1,2)$-indexable if and only if $|a-b| \leq 1$; in a connected strongly k-indexable graph with p vertices and q edges, $k \leq p-1$; and if a graph with p vertices and q edges is (k,d)-indexable, then $q \leq (2p-3-k+d)/d$. As a corollary of the latter, it follows that K_n ($n \geq 4$) and wheels are not (k,d)-indexable.

Lee and Lee [854] provide a way to construct a (k,d)-strongly indexable graph from two given (k,d)-strongly indexable graphs. Lee and Lo [882] show that every given (1,2)-strongly indexable spider can extend to an (1,2)-strongly indexable spider with arbitrarily many legs.

Seoud, Abd El Hamid, and Abo Shady [1255] proved the following graphs are indexable: $P_m \times P_n$ ($m, n \geq 2$); the graphs obtained from $P_n + K_1$ by inserting one vertex between every two consecutive vertices of P_n; the one-point union of any number of copies of $K_{2,n}$; and the graphs otained by identifying a vertex of a cycle with the center of a star. They showed P_n is strongly $\lceil n/2 \rceil$-indexable; odd cycles C_n are strongly $\lceil n/2 \rceil$-indexable; $K_(m,n)$ ($m, n > 2$) is indexable if and only if m or n is at most 2. For a simple indexable graph $G(V,E)$ they proved $|E| \leq 2|V| - 3$. Also, they determine all indexable graphs of order at most 6.

Hegde and Shetty [647] also prove that if G is strongly k-indexable Eulerian graph with q edges then $q \equiv 0, 3 \pmod 4$ if k is even and $q \equiv 0, 1 \pmod 4$ if k is odd. They further showed how strongly k-indexable graphs can be used to construct polygons of equal internal angles with sides of different lengths.

Germina [562] has proved the following: fans $P_n + K_1$ are strongly indexable if and only if $n = 1, 2, 3, 4, 5, 6$; $P_n + K_2$ is strongly indexable if and only if $n \leq 2$; the only strongly indexable complete m-partite graphs are $K_{1,n}$ and $K_{1,1,n}$; ladders $P_n \times P_2$ are $\lceil \frac{n}{2} \rceil$-strongly indexable, if n is odd; $K_n \times P_k$ is a strongly indexable if and only if $n = 3$; $C_m \times P_n$ is 2-strongly indexable if m is odd and $n \geq 2$; $K_{1,n} + K_i$ is not strongly indexable for $n \geq 2$; for $G_i \cong K_{1,n}$, $1 \leq i \leq n$, the sequential join $G \cong (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$ is strongly indexable if and only if, either $i = n = 1$ or $i = 2$ and $n = 1$ or $i = 1, n = 3$; $P_1 \cup P_n$ is strongly indexable if and only if $n \leq 3$; $P_2 \cup P_n$ is not strongly indexable; $P_2 \cup P_n$ is $\lceil \frac{n+3}{2} \rceil$-strongly indexable; mC_n is k-strongly indexable if and only if m and n are odd; $K_{1,n} \cup K_{1,n+1}$ is strongly indexable; and $mK_{1,n}$ is $\lceil \frac{3m-1}{2} \rceil$-strongly indexable when m is odd.

Acharya and Germina [18] proved that every graph can be embedded in a strongly indexable graph and gave an algorithmic characterization of strongly indexable unicyclic graphs. In [19] they provide necessary conditions for an Eulerian graph to be strongly k-indexable and investigate strongly indexable (p,q)-graphs for which $q = 2p - 3$.

Hegde and Shetty [643] proved that for n odd the generalized web graph $W(t,n)$ with the center removed is strongly $(n-1)/2$-indexable. Hegde and Shetty [648] define a *level joined planar grid* as follows. Let u be a vertex of $P_m \times P_n$ of degree 2. For every pair of distinct vertices v and w that do not have degree 4, introduce an edge between v and w provided that the distance from u to v equals the distance from u to w. They prove that every level joined planar grid is strongly indexable. For any sequence of positive integers $(a_1, a_2, \ldots, a_n)$ Lee and Lee [873] show how to associate a strongly indexible $(1,1)$-graph. As a corollary, they obtain the aforementioned result Hegde and Shetty on level joined planar grids.

Section 5.2 of this survey includes a discussion of a labeling method called super edge-magic. In 2002 Hegde and Shetty [648] showed that a graph has a strongly k-indexable labeling if and only if it has a super edge-magic labeling.

4.4 Elegant Labelings

In 1981 Chang, Hsu, and Rogers [354] defined an *elegant labeling* f of a graph G with q edges as an injective function from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $f(x) + f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and nonzero. An injective labeling f of a graph G with q vertices is called *strongly k-elegant* if the vertex labels are from $\{0, 1, \ldots, q\}$ and the edge labels induced by $f(x) + f(y) \pmod{(q+1)}$ for each edge xy are $k, \ldots, k+q-1$. Note that in contrast to the definition of a harmonious labeling, for an elegant labeling it is not necessary to make an exception for trees.

Whereas the cycle C_n is harmonious if and only if n is odd, Chang et al. [354] proved that C_n is elegant when $n \equiv 0$ or $3 \pmod 4$ and not elegant when $n \equiv 1 \pmod 4$. Chang et al. further showed that all fans are elegant and the paths P_n are elegant for $n \not\equiv 0 \pmod 4$. Cahit [330] then showed that P_4 is the only path that is not elegant. Balakrishnan, Selvam, and Yegnanarayanan [216] have proved numerous graphs are elegant. Among them are $K_{m,n}$ and the mth-subdivision graph of $K_{1,2n}$ for all m. They prove that the bistar $B_{n,n}$ (K_2 with n pendent edges at each endpoint) is elegant if and only if n is even. They also prove that every simple graph is a subgraph of an elegant graph and that several families of graphs are not elegant. Deb and Limaye [417] have shown that triangular snakes (see §2.2 for the definition) are elegant if and only if the number of triangles is not equal to 3 (mod 4). In the case where the number of triangles is 3 (mod 4) they show the triangular snakes satisfy a weaker condition they call *semi-elegant* whereby the edge label 0 is permitted. In [418] Deb and Limaye define a graph G with q edges to be *near-elegant* if there is an injective function f from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $f(x) + f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and not equal to q. Thus, in a near-elegant labeling, instead of 0 being the missing value in the edge labels, q is the missing value. Deb and Limaye show that triangular snakes where the number of triangles is 3 (mod 4) are near-elegant. For any positive integers $\alpha \le \beta \le \gamma$ where β is at least 2, the *theta graph* $\theta_{\alpha,\beta,\gamma}$ consists of three edge disjoint paths of lengths α, β, and γ having the same end points. Deb and Limaye [418] provide elegant and near-elegant labelings for some theta graphs where $\alpha = 1, 2$, or 3. Seoud and Elsakhawi [1268] have proved that the following graphs are elegant: $K_{1,m,n}$; $K_{1,1,m,n}$; $K_2 + \overline{K_m}$; $K_3 + \overline{K_m}$; and $K_{m,n}$ with an edge joining two vertices of the same partite set. Elumalai and Sethuraman [457] proved ${P_2}^n$, $P_m^2 + \overline{K_n}, S_m + S_n$, $S_m + \overline{K_m}$, $C_3 \times P_m$, and even cycles C_{2n} with vertices $a_0, a_1, \ldots, a_{2n-1}, a_0$ and $2n-3$ chords $a_0a_2, a_0a_3, \ldots, a_0a_{2n-2}$ ($n \ge 2$) are elegant. Zhou [1726] has shown that for every strongly k-elegant graph G with p vertices and q edges and any positive integer m the graph $(G + \overline{K_m}) + \overline{K_n}$ is also strongly k-elegant when $q - p + 1 \leq m \leq q - p + k$.

Sethuraman and Elumalai [1304] proved that every graph is a vertex induced subgraph of a elegant graph and present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a strong form of elegant labeling. Acharya, Germina, Princy, and Rao [23] prove that every (p, q)-graph G can be embedded in a connected elegant graph H. The construction is done in such a way that if G is planar and elegant (harmonious), then so is H.

In [1303] Sethuraman and Elumalai define a graph H to be a $K_{1,m}$-*star extension* of a graph G with p vertices and q edges at a vertex v of G where $m > p - 1 - deg(v)$ if H is obtained from G by merging the center of the star $K_{1,m}$ with v and merging $p - 1 - deg(v)$ pendent vertices of $K_{1,m}$ with the $p - 1 - deg(v)$ nonadjacent vertices of v in G. They prove that for every graph G with p vertices and q edges and for every vertex v of G and every $m \geq 2^{p-1} - 1 - q$, there is a $K_{1,m}$-star extension of G that is both graceful and harmonious. In the case where $m \geq 2^{p-1} - q$, they show that G has a $K_{1,m}$-star extension that is elegant. Sethuraman and Selvaraju [1322] have shown that certain cases of the union of any number of copies of K_4 with one or more edges deleted and one edge in common are elegant.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let G be a graph with q edges and H a finite Abelian group (under addition) of order q. Define G to be H-*harmonious* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x) + f(y)$ the resulting edge labels are distinct. When G is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley, and Jungreis [246] have shown that if H is a finite Abelian group of order $n > 1$ then C_n is H-harmonious if and only if H has a non-cyclic or trivial Sylow 2-subgroup and H is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$. Thus, for example, C_{12} is not Z_{12}-harmonious but is $(Z_2 \times Z_2 \times Z_3)$-harmonious. Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let G be a graph with q edges and H a finite Abelian group (under addition) with $q + 1$ elements. We say G is H-*elegant* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x) + f(y)$ the resulting set of edge labels is the non-identity elements of H. Beals et al. [246] proved that if H is a finite Abelian group of order n with $n \neq 1$ and $n \neq 3$, then C_{n-1} is H-elegant using only the non-identity elements of H as vertex labels if and only if H has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu, and Rogers [354] by showing that C_n is elegant when $n \equiv 2 \pmod 4$. Mollard and Payan [1067] also proved that C_n is elegant when $n \equiv 2 \pmod 4$ and gave another proof that P_n is elegant when $n \neq 4$.

A function f is said to be an *odd elegant* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q - 1$ such that the induced mapping $f^*(uv) = f(u) + f(v) \pmod{2q}$ from the edges of G to the odd integers between 1 to $2q - 1$ is a bijection. Zhou, Yao, and Chen [1728] proved that every lobster is odd-elegant.

For a graph $G(V, E)$ and an Abelian group H Valentin [1566] defines a *polychrome* labeling of G by H to be a bijection f from V to H such that the edge labels induced by $f(uv) = f(v) + f(u)$ are distinct. Valentin investigates the existence of polychrome labelings for paths and cycles for various Abelian groups.

4.5 Felicitous Labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, \ldots, q\}$ is called *felicitous* if the edge labels induced by $f(x) + f(y) \pmod q$ for each edge xy are distinct. (Recall a harmonious labeling only allows the vertex labels $0, 1, \ldots, q-1$.) This definition first appeared in a paper by Lee, Schmeichel, and Shee in [905] and is attributed to E. Choo. Acharya, Germina, Princy, and Rao [23] observed that every harmonious labeling of a graph is also a felicitous labeling of the graph. Balakrishnan and Kumar [213] proved the conjecture of Lee, Schmeichel, and Shee [905] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are: C_n except when $n \equiv 2 \pmod 4$ [905]; $K_{m,n}$ when $m, n > 1$ [905]; $P_2 \cup C_{2n+1}$ [905]; $P_2 \cup C_{2n}$ [1488]; $P_3 \cup C_{2n+1}$ [905]; $S_m \cup C_{2n+1}$ [905]; K_n if and only if $n \leq 4$ [458]; $P_n + \overline{K_m}$ [458]; the friendship graph $C_3^{(n)}$ for n odd [905]; $P_n \cup C_3$ [1331]; $P_n \cup C_{n+3}$ [1488]; and the one-point union of an odd cycle and a caterpillar [1331]. Shee [1327] conjectured that $P_m \cup C_n$ is felicitous when $n > 2$ and $m > 3$. Lee, Schmeichel, and Shee [905] ask for which m and n is the one-point union of n copies of C_m felicitous. They showed that in the case where mn is twice an odd integer the graph is not felicitous. In contrast to the situation for felicitous labelings, we remark that C_{4k} and $K_{m,n}$ where $m, n > 1$ are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel, and Shee [905] conjectured that the n-cube is felicitous. This conjecture was proved by Figueroa-Centeno and Ichishima in 2001 [492].

Balakrishnan, Selvam, and Yegnanarayanan [215] obtained numerous results on felicitous labelings. The *wreath product*, $G * H$, of graphs G and H has vertex set $V(G) \times V(H)$ and (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. They define $H_{n,n}$ as the graph with vertex set $\{u_1, \ldots, u_n; v_1, \ldots, v_n\}$ and edge set $\{u_iv_j | \ 1 \leq i \leq j \leq n\}$. They let $\langle K_{1,n} : m\rangle$ denote the graph obtained by taking m disjoint copies of $K_{1,n}$, and joining a new vertex to the centers of the m copies of $K_{1,n}$. They prove the following are felicitous: $H_{n,n}$; $P_n * \overline{K_2}$; $\langle K_{1,m} : m\rangle$; $\langle K_{1,2} : m\rangle$ when $m \not\equiv 0 \pmod 3$, or $m \equiv 3 \pmod 6$, or $m \equiv 6 \pmod{12}$; $\langle K_{1,2n} : m\rangle$ for all m and $n \geq 2$; $\langle K_{1,2t+1} : 2n+1\rangle$ when $n \geq t$; P_n^k when $k = n-1$ and $n \not\equiv 2 \pmod 4$, or $k = 2t$ and $n \geq 3$ and $k < n-1$; the join of a star and $\overline{K_n}$; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [1689] conjectures that the graphs obtained from an even cycle by attaching n new vertices to each vertex of the cycle is felicitous. This conjecture was verified by Figueroa-Centeno, Ichishima, and Muntaner-Batle in [497]. In [1318] Sethuraman and Selvaraju [1322] have shown that certain cases of the union of any number of copies of K_4 with 3 edges deleted and one edge in common are felicitous. Sethuraman and Selvaraju [1318] present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a felicitous labeling. Krisha and Dulawat [832] give algorithms for finding graceful, harmonious, sequential, felicitous, and antimagic (see §5.7) labelings of paths.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [498] define a felicitous graph to be *strongly felicitous* if there exists an integer k so that for every edge uv, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. For a graph with p vertices and q edges with $q \geq p-1$ they show that G is strongly felicitous if and only if G has an α-labeling (see §3.1). They also show that for graphs G_1 and G_2 with strongly felicitous labelings f_1 and f_2 the graph obtained from G_1 and G_2 by identifying the vertices u and v such that $f_1(u) = 0 = f_2(v)$ is strongly felicitous and that the one-point union of two copies of C_m where $m \geq 4$ and m is even is strongly felicitous. As a corollary they have that the one-point union of n copies of C_m where m is even and at least 4 and $n \equiv 2 \pmod 4$ is felicitous. They conjecture that the one-point union of n copies of C_m is felicitous if and only if $mn \equiv 0, 1$, or $3 \pmod 4$. In [502] Figueroa-Centeno, Ichishima, and Muntaner-Batle prove that $2C_n$ is strongly felicitous if and only if n is even and at least 4. They conjecture [502] that mC_n is felicitous if and only if $mn \not\equiv 2 \pmod 4$ and that $C_m \cup C_n$ is felicitous if and only if $m + n \not\equiv 2 \pmod 4$.

As consequences of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [502] have the following corollaries: if m and n are odd with $m \geq 1$ and $n \geq 3$, then mC_n is felicitous; $3C_n$ is felicitous if and only if $n \not\equiv 2 \pmod 4$; and $C_5 \cup P_n$ is felicitous for all n.

In [1023] Manickam, Marudai, and Kala prove the following graphs are felicitous: the one-point union of m copies of C_n if $mn \equiv 1, 3 \bmod 4$; the one-point union of m copies of C_4; mC_n if $mn \equiv 1, 3 \pmod 4$; and mC_4. These results partially answer questions raised by Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima in [498] and [502].

Chang, Hsu, and Rogers [354] have given a sequential counterpart to felicitous labelings. They call a graph with q edges *strongly c-elegant* if the vertex labels are from $\{0, 1, \ldots, q\}$ and the edge labels induced by addition are $\{c, c+1, \ldots, c+q-1\}$. (A strongly 1-elegant labeling has also been called a *consecutive* labeling.) Notice that every strongly c-elegant graph is felicitous and that strongly c-elegant is the same as $(c, 1)$-arithmetic in the case where the vertex labels are from $\{0, 1, \ldots, q\}$. Chang et al. [354] have shown: K_n is strongly 1-elegant if and only if $n = 2, 3, 4$; C_n is strongly 1-elegant if and only if $n = 3$; and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [1328] has proved that $K_{m,n}$ is strongly c-elegant for a particular value of c and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

Seoud and Elsakhawi [1270] have shown: $K_{m,n}$ $(m \leq n)$ with an edge joining two vertices of the same partite set is strongly c-elegant for $c = 1, 3, 5, \ldots, 2n + 2$; $K_{1,m,n}$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 2m$ when $m = n$, and for $c = 1, 3, 5, \ldots, m + n + 1$ when $m \neq n$; $K_{1,1,m,m}$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 2m + 1$; $P_n + \overline{K_m}$ is strongly $\lfloor n/2 \rfloor$-elegant; $C_m + \overline{K_n}$ is strongly c-elegant for odd m and all n for $c = (m - 1)/2, (m - 1)/2 + 2, \ldots, 2m$ when $(m - 1)/2$ is even and for $c = (m - 1)/2, (m - 1)/2 + 2, \ldots, 2m - (m - 1)/2$ when $(m - 1)/2$ is odd; ladders L_{2k+1} $(k > 1)$ are strongly $(k + 1)$-elegant; and $B(3, 2, m)$ and $B(4, 3, m)$ (see §2.4 for notation) are strongly 1-elegant and strongly 3-elegant for all m; the composition $P_n[P_2]$ (see §2.3 for the definition) is strongly c-elegant for $c = 1, 3, 5, \ldots, 5n - 6$ when n is odd and for $c = 1, 3, 5, \ldots, 5n - 5$ when n is even; P_n is strongly $\lfloor n/2 \rfloor$-elegant; P_n^2 is strongly c-elegant for $c = 1, 3, 5, \ldots, q$ where q is the number of edges of P_n^2; and P_n^3 $(n > 3)$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 6k - 1$ when $n = 4k$; $c = 1, 3, 5, \ldots, 6k + 1$ when $n = 4k + 1$; $c = 1, 3, 5, \ldots, 6k + 3$ when $n = 4k + 2$; $c = 1, 3, 5, \ldots, 6k + 5$ when $n = 4k + 3$.

4.6 Odd Harmonious and Even Harmonious Labelings

A function f is said to be an *odd harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q - 1$ such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q - 1$ is a bijection. A function f is said to be an *strongly odd harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to q such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection. Liang and Bai [955] have shown the following: odd harmonious graphs are bipartite; if a (p,q)-graph is odd harmonious, then $2\sqrt{q} \le p \le 2q-1$; if a (p,q)-graph with degree sequence $(d_1, d_2, \ldots, d_p)$ is odd harmonious, then $\gcd(d_1, d_2, \ldots, d_p)$ divides q^2; P_n $(n > 1)$ is odd harmonious and strongly odd harmonious; C_n is odd harmonious if and only if $n \equiv 0 \bmod 4$; K_n is odd harmonious if and only if $n = 2$; $K_{n_1,n_2,\ldots,n_k}$ is odd harmonious if and only if $k = 2$; K_n^t is odd harmonious if and only if $n = 2$; $P_m \times P_n$ is odd harmonious; the tadpole graph obtained by identifying the endpoint of a path with a vertex of an n-cycle is odd harmonious if $n \equiv 0 \bmod 4$; the graph obtained by appending two or more pendent edges to each vertex of C_{4n} is odd harmonious; the graph obtained by subdividing every edge of the cycle of a wheel (gear graphs) is odd harmonious; the graph obtained by appending an edge to each vertex of a strongly odd harmonious graph is odd harmonious; and caterpillars and lobsters are odd harmonious. They conjecture that every tree is odd harmonious.

Vaidya and Shah [1544] prove that the shadow graphs (see §3.8 for the definition) of path P_n and star K_{1n} are odd harmonious. They also show that the splitting graphs (see §2.7 for the definition)) of path P_n and star K_{1n} are odd harmonious. In [1545] Vaidya and Shah proved the following graphs are odd harmonious: the shadow graph and the splitting graph of bistar $B_{n,n}$; the arbitrary supersubdivision of paths; graphs obtained by joining two copies of cycle C_n for $n \equiv 0 (\bmod\ 4)$ by an edge; and the graphs $H_{n,n}$, where $V(H_{n,n}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, v_n\}$ and $E(H_{n,n}) = \{v_i u_j :\ 1 \le i \le n,\ n-i+1 \le j \le n\}$. In [1673] Yan proves that $P_m \times P_n$ is odd strongly harmonious. Koppendrayer [809] has proved that every graph with an α-labeling is odd harmonious. Li, Li, and Yan [946] proved that $K_{m,n}$ is odd strongly harmonious.

Saputri, Sugeng, and Froncek [1244] proved that the graph obtained by joining C_n to C_k by an edge (dumbbell graph $D_{n,k,2}$) is odd harmonious for $n \equiv k \equiv 0 (\bmod\ 4)$ and $n \equiv k \equiv 2$ (mod 4), and $C_n \times P_m$ is odd harmonious if and only if $n \equiv 0$ (mod 4). They also observe that $C_n \odot K_1$ with $n \equiv 0$ (mod 4) is odd harmonious.

Sarasija and Binthiya [1245] say a function f is an *even harmonious* labeling of a graph G with q edges if $f : V \to \{0, 1, \ldots, 2q\}$ is injective and the induced function $f^* : E \to \{0, 2, \ldots, 2(q-1)\}$ defined as $f^*(uv) = f(u) + f(v) \pmod{2q}$ is bijective. They proved the following graphs are even harmonious: non-trivial paths; complete bipartite graphs; odd cycles; bistars $B_{m,n}$; $K_2 + \overline{K_n}$; P_n^2; and the friendship graphs F_{2n+1}. López, Muntaner-Batle and Rious-Font [990] proved that every super edge-magic graph (see Section 5.2 for the definition of super edge-magic) with p vertices and q edges where $q \ge p - 1$ has an even harmonious labeling.

Because $2q$ is 0 modulo $2q$, Gallian and Schoenhard [535] gave the following equivalent definition of an even harmonious labeling. A function f is said to be an *even harmonious* labeling of a graph G with q edges if f is a function from the vertices of G to $\{0, 1, \ldots, 2(q-1)\}$ and the induced function f^* from the edges of G to $\{0, 2, \ldots, 2(q-1)\}$ defined by $f^*(uv) = f(u) + f(v)$ (mod $2q$) has at most one label used twice. In the case of harmonious labelings for connected graphs there is no loss of generality to assume that all the vertex labels are even integers and the duplicate vertex is 0. Gallian and Schoenhard also call an even harmonious labeling a *properly even harmonious* if no vertex label is duplicated and say an even harmonious labeling of a graph with q edges is *strongly even harmonious* if it satisfies the additional condition that for any two adjacent vertices with labels u and v, $0 < u + v \le 2q$.

Jared Bass [245] has observed that for connected graphs any harmonious labeling of a graph with q edges yields an even harmonious labeling by simply multiplying each vertex label by 2 and adding the vertex labels modulo $2q$. Thus we know that every connected harmonious graph is an even harmonious graph and every connected graph that is not a tree that has a harmonious labeling also has a properly even harmonious labeling. Conversely, a properly even harmonious labeling of a connected graph with q edges (assuming that the vertex labels are even) yields a harmonious labeling of the graph by dividing each vertex label by 2 and adding the vertex labels modulo q.

Gallian and Schoenhard [535] proved the following: wheels W_n and helms H_n are properly even harmonious when n is odd; nP_2 is even harmonious for n odd; nP_2 is properly even harmonious if and only if n is even; K_n is not even harmonius when $n > geq5$. C_{2n} is not even harmonius when n is odd; $C_n \cup P_3$ is properly even harmonious when odd $n \geq 3$; $C_4 \cup P_n$ is even harmonious when $n \geq 2$; $C_4 \cup F_n$ is even harmonious when $n \geq 2$; $S_m \cup P_n$ is even harmonious when $n \geq 2$; $K_4 \cup S_n$ is properly even harmonious; $P_m \cup P_n$ is properly even harmonious for all $m \geq 2$ and $n \geq 2$; $C_3 \cup P_n^2$ is even harmonious when $n \geq 2$; $C_4 \cup P_n^2$ is even harmonious when $n \geq 2$; the disjoint union of two or three stars where each star has at least two edges and one has at least three edges is properly even harmonious; $P_m^2 \cup P_n$ is even harmonious for $m \geq 2$ and $2 \leq n < 4m - 5$; the one-point union of two complete graphs each with at least 3 vertices is not even harmonious; $S_m \cup P_n$ is strongly even harmonious if $n \geq 2$; and $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious for $n_1 \geq n_2, \geq \ldots \geq n_t$ and $t < \frac{n_1}{2} + 2$. They conjecture that $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$ is strongly even harmonious if at least one star has more than 2 edges. They also note that $C_4, C_8, C_{12}, C_{16}, C_{20}, C_{24}$ are even harmonious and conjecture that C_{4n} is even harmonius for all n. Hall, Hillesheim, Kocina, and Schmit [605] proved that nC_{2m+1} is properly even harmonious for all n and m.

5 Magic-type Labelings

5.1 Magic Labelings

Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedláček [1247] in 1963. Responding to a problem raised by Sedláček, Stewart [1436] and [1437] studied various ways to label the edges of a graph in the mid 1960s. Stewart calls a connected graph *semi-magic* if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. (Berge [258] used the term "regularisable" for this notion.) A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic* labeling. Stewart calls a magic labeling *supermagic* if the set of edge labels consists of consecutive positive integers. The classic concept of an $n \times n$ magic square in number theory corresponds to a supermagic labeling of $K_{n,n}$. Stewart [1436] proved the following: K_n is magic for $n = 2$ and all $n \geq 5$; $K_{n,n}$ is magic for all $n \geq 3$; fans F_n are magic if and only if n is odd and $n \geq 3$; wheels W_n are magic for $n \geq 4$; and W_n with one spoke deleted is magic for $n = 4$ and for $n \geq 6$. Stewart [1436] also proved that $K_{m,n}$ is semi-magic if and only if $m = n$. In [1437] Stewart proved that K_n is supermagic for $n \geq 5$ if and only if $n > 5$ and $n \not\equiv 0 \pmod 4$. Sedláček [1248] showed that Möbius ladders M_n (see §2.3 for the definition) are supermagic when $n \geq 3$ and n is odd and that $C_n \times P_2$ is magic, but not supermagic, when $n \geq 4$ and n is even. Shiu, Lam, and Lee [1344] have proved: the composition of C_m and $\overline{K}_n$ (see §2.3 for the definition) is supermagic when $m \geq 3$ and $n \geq 2$; the complete m-partite graph $K_{n,n,\ldots,n}$ is supermagic when $n \geq 3$, $m > 5$ and $m \not\equiv 0 \pmod 4$; and if G is an r-regular supermagic graph, then so is the composition of G and $\overline{K}_n$ for $n \geq 3$. Ho and Lee [654] showed that the composition of K_m and $\overline{K_n}$ is supermagic for $m = 3$ or 5 and

$n = 2$ or n odd. Bača, Holländer, and Lih [173] have found two families of 4-regular supermagic graphs. Shiu, Lam, and Cheng [1341] proved that for $n \geq 2$, $mK_{n,n}$ is supermagic if and only if n is even or both m and n are odd. Ivančo [690] gave a characterization of all supermagic regular complete multipartite graphs. He proved that Q_n is supermagic if and only if $n = 1$ or n is even and greater than 2 and that $C_n \times C_n$ and $C_{2m} \times C_{2n}$ are supermagic. He conjectures that $C_m \times C_n$ is supermagic for all m and n. Trenklér [1496] has proved that a connected magic graph with p vertices and q edges other than P_2 exits if and only if $5p/4 < q \leq p(p-1)/2$. In [1457] Sun, Guan, and Lee give an efficient algorithm for finding a magic labeling of a graph. In [1642] Wen, Lee, and Sun show how to construct a supermagic multigraph from a given graph G by adding extra edges to G.

In [825] Kovář provides a general technique for constructing supermagic labelings of copies of certain kinds of regular supermagic graphs. In particular, he proves: if G is a supermagic r-regular graph ($r \geq 3$) with a proper edge r coloring, then nG is supermagic when r is even and supermagic when r and n are odd; if G is a supermagic r-regular graph with m vertices and has a proper edge r coloring and H is a supermagic s-regular graph with n vertices and has a proper edge s coloring, then $G \times H$ is supermagic when r is even or n is odd and is supermagic when s or m is odd.

In [442] Drajnová, Ivančo, and Semaničová proved that the maximal number of edges in a supermagic graph of order n is 8 for $n = 5$ and $\frac{n(n-1)}{2}$ for $6 \leq n \not\equiv 0 \pmod 4$, and $\frac{n(n-1)}{2} - 1$ for $8 \leq n \equiv 0 \pmod 4$. They also establish some bounds for the minimal number of edges in a supermagic graph of order n. Ivančo, and Semaničová [697] proved that every 3-regular triangle-free supermagic graph has an edge such that the graph obtained by contracting that edge is also supermagic and the graph obtained by contracting one of the edges joining the two n-cycles of $C_n \times K_2$ ($n \geq 3$) is supermagic.

Ivančo [692] proved: the complement of a d-regular bipartite graph of order $8k$ is supermagic if and only if d is odd; the complement of a d-regular bipartite graph of order $2n$ where n is odd and d is even is supermagic if and only if $(n, d) \neq (3, 2)$; if G_1 and G_2 are disjoint d-regular Hamiltonian graphs of odd order and $d \geq 4$ and even, then the join $G_1 \oplus G_2$ is supermagic; and if G_1 is d-regular Hamiltonian graph of odd order n, G_2 is $d-2$-regular Hamiltonian graph of order n and $4 \leq d \equiv 0 \pmod 4$, then the join $G_1 \oplus G_2$ is supermagic.

In [269] Bezegová and Ivančo [271] extended the notion of supermagic regular graphs by defining a graph to be *degree-magic* if the edges can be labeled with $\{1, 2, \ldots, |E(G)|\}$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1+|E(G))/\deg(v)$. They used this notion to give some constructions of supermagic graphs and proved that for any graph G there is a supermagic regular graph which contains an induced subgraph isomorphic to G. In [271] they gave a characterization of complete tripartite degree-magic graphs and in [272] they provided some bounds on the number of edges in degree-magic graphs. They say a graph G is *conservative* if it admits an orientation and a labeling of the edges by $\{1, 2, \ldots, |E(G)|\}$ such that at each vertex the sum of the labels on the incoming edges is equal to the sum of the labels on the outgoing edges. In [270] Bezegová and Ivančo introduced some constructions of degree-magic labelings for a large family of graphs using conservative graphs. Using a connection between degree-magic labelings and supermagic labelings they also constructed supermagic labelings for the disjoint union of some regular non-isomorphic graphs. Among their results are: If G is a δ-regular graph where δ is even and at least 6, and each component of G is a complete multipartite graph of even size, then G is a supermagic graph; for any δ-regular supermagic graph H, the union of disjoint graphs H and G is supermagic; if G is a δ-regular graph with $\delta \equiv 0$ (mod

8) and each component is a circulant graph, then G is a supermagic graph; for any δ-regular supermagic graph H, the union of disjoint graphs H and G is a supermagic graph; and that the complement of the union of disjoint cycles $C_{n_1}, \ldots, C_{n_k}$ is supermagic when $k \equiv 1 \pmod 4$ and $11 \le n_i \equiv 3 \pmod 8$ for all $i = 1, \ldots, k$.

Sedláček [1248] proved that graphs obtained from an odd cycle with consecutive vertices $u_1, u_2, \ldots, u_m, u_{m+1}, v_m, \ldots, v_1$ ($m \ge 2$) by joining each u_i to v_i and v_{i+1} and u_1 to v_{m+1}, u_m to v_1 and v_1 to v_{m+1} are magic. Trenklér and Vetchý [1499] have shown that if G has order at least 5, then G^n is magic for all $n \ge 3$ and G^2 is magic if and only if G is not P_5 and G does not have a 1-factor whose every edge is incident with an end-vertex of G. Avadayappan, Jeyanthi, and Vasuki [106] have shown that k-sequential trees are magic (see §4.1 for the definition). Seoud and Abdel Maqsoud [1257] proved that $K_{1,m,n}$ is magic for all m and n and that P_n^2 is magic for all n. However, Serverino has reported that P_n^2 is not magic for $n = 2, 3$, and 5 [567]. Jeurissan [710] characterized magic connected bipartite graphs. Ivančo [691] proved that bipartite graphs with $p \ge 8$ vertices, equal sized partite sets, and minimum degree greater than p are magic. Bača [137] characterizes the structure of magic graphs that are formed by adding edges to a bipartite graph and proves that a regular connected magic graph of degree at least 3 remains magic if an arbitrary edge is deleted. In [1411] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are magic.

A *prime-magic labeling* is a magic labeling for which every label is a prime. Sedláček [1248] proved that the smallest magic constant for prime-magic labeling of $K_{3,3}$ is 53 while Bača and Holländer [169] showed that the smallest magic constant for a prime-magic labeling of $K_{4,4}$ is 114. Letting σ_n be the smallest natural number such that $n\sigma_n$ is equal to the sum of n^2 distinct prime numbers we have that the smallest magic constant for a prime-magic labeling of $K_{n,n}$ is σ_n. Bača and Hollaänder [169] conjecture that for $n \ge 5$, $K_{n,n}$ has a prime-magic labeling with magic constant σ_n. They proved the conjecture for $5 \le n \le 17$ and confirmed the conjecture for $n = 5$, 6 and 7.

Characterizations of regular magic graphs were given by Doob [441] and necessary and sufficient conditions for a graph to be magic were given in [710], [743], and [426]. Some sufficient conditions for a graph to be magic are given in [439], [1495], and [1079]. Bertault, Miller, Pé-Rosés, Feria-Puron, and Vaezpour [267] provided a heuristic algorithm for finding magic labelings for specific families of graphs. The notion of magic graphs was generalized in [440] and [1234].

Let $m, n, a_1, a_2, \ldots, a_m$ be positive integers where $1 \le a_i \le \lfloor n/2 \rfloor$ and the a_i are distinct. The *circulant graph* $C_n(a_1, a_2, \ldots, a_m)$ is the graph with vertex set $\{v_1, v_2, \ldots, v_m\}$ and edge set $\{v_i v_{i+a_j} \mid 1 \le i \le n,\ 1 \le j \le m\}$ where addition of indices is done modulo n. In [1253] Semaničová characterizes magic circulant graphs and 3-regular supermagic circulant graphs. In particular, if $G = C_n(a_1, a_2, \ldots, a_m)$ has degree r at least 3 and $d = \gcd(a_1, n/2)$ then G is magic if and only if $r = 3$ and $n/d \equiv 2 \pmod 4$, $a_1/d \equiv 1 \pmod 2$, or $r \ge 4$ (a necessary condition for $C_n(a_1, a_2, \ldots, a_m)$ to be 3-regular is that n is even). In the 3-regular case, $C_n(a_1, n/2)$ is supermagic if and only $n/d \equiv 2 \pmod 4$, $a_1/d \equiv 1 \pmod 2$ and $d \equiv 1 \pmod 2$. Semaničová also notes that a bipartite graph that is decomposable into an even number of Hamilton cycles is supermagic. As a corollary she obtains that $C_n(a_1, a_2, \ldots, a_{2k})$ is supermagic in the case that n is even, every a_i is odd, and $\gcd(a_{2j-1}, a_{2j}, n) = 1$ for $i = 1, 2, \ldots, 2k$ and $j = 1, 2, \ldots, k$.

Ivančo, Kovář, and Semaničová-Feňovčková [694] characterize all pairs n and r for which an r-regular supermagic graph of order n exists. They prove that for positive integers r and n with $n \ge r+1$ there exists an r-regular supermagic graph of order n if and only if one of the following statements holds: $r = 1$ and $n = 2$; $3 \leq r \equiv 1 \pmod 2$ and $n \equiv 2 \pmod 4$; and $4 \leq r \equiv 0 \pmod 2$ and $n > 5$. The proof of the main result is based on finding supermagic labelings of circulant graphs. The authors construct supermagic labelings of several circulant graphs.

In [690] Ivančo completely determines the supermagic graphs that are the disjoint unions of complete k-partite graphs where every partite set has the same order.

Trenklér [1497] extended the definition of supermagic graphs to include hypergraphs and proved that the complete k-uniform n-partite hypergraph is supermagic if $n \neq 2$ or 6 and $k \geq 2$ (see also [1498]).

For connected graphs of size at least 5, Ivančo, Lastivkova, and Semaničová [696] provide a forbidden subgraph characterization of the line graphs that can be magic. As a corollary they obtain that the line graph of every connected graph with minimum degree at least 3 is magic. They also prove that the line graph of every bipartite regular graph of degree at least 3 is supermagic.

In 1976 Sedláček [1248] defined a connected graph with at least two edges to be *pseudo-magic* if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when $n \geq 4$ and n is even, the Möbius ladder M_n is not pseudo-magic and when $m \geq 3$ and m is odd, $C_m \times P_2$ is not pseudo-magic.

Kong, Lee, and Sun [814] used the term "magic labeling" for a labeling of the edges with nonnegative integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. In particular, the edge labels need not be distinct. They let $M(G)$ denote the set of all such labelings of G. For any L in $M(G)$, they let $s(L) = \max\{L(e)\colon e \text{ in } E\}$ and define the *magic strength* of G as $m(G) = \min\{s(L)\colon L \text{ in } M(G)\}$. To distinguish these notions from others with the same names and notation, which we will introduced in the next section for labelings from the set of vertices and edges, we call the Kong, Lee, and Sun version the *edge magic strength* and use $em(G)$ for $\min\{s(L)\colon L \text{ in } M(G)\}$ instead of $m(G)$. Kong, Lee, and Sun [814] use $DS(k)$ to denote the graph obtained by taking two copies of $K_{1,k}$ and connecting the k pairs of corresponding leafs. They show: for $k > 1$, $em(DS(k)) = k - 1$; $em(P_k + K_1) = 1$ for $k = 1$ or 2, $em(P_k + K_1) = k$ if k is even and greater than 2, and 0 if k is odd and greater than 1; for $k \geq 3$, $em(W(k)) = k/2$ if k is even and $em(W(k)) = (k-1)/2$ if k is odd; $em(P_2 \times P_2) = 1$, $em(P_2 \times P_n) = 2$ if $n > 3$, $em(P_m \times P_n) = 3$ if m or n is even and greater than 2; $em(C_3^{(n)}) = 1$ if $n = 1$ (Dutch windmill, – see §2.4), and $em(C_3^{(n)}) = 2n - 1$ if $n > 1$. They also prove that if G and H are magic graphs then $G \times H$ is magic and $em(G \times H) = \max\{em(G), em(H)\}$ and that every connected graph is an induced subgraph of a magic graph (see also [468] and [495]). They conjecture that almost all connected graphs are not magic. In [902] Lee, Saba, and Sun show that the edge magic strength of P_n^k is 0 when k and n are both odd. Sun and Lee [1458] show that the Cartesian, conjunctive, normal, lexicographic, and disjunctive products of two magic graphs are magic and the sum of two magic graphs is magic. They also determine the edge magic strengths of the products and sums in terms of the edge magic strengths of the components graphs.

In [57] Akka and Warad define the *super magic strength* of a graph G, sms(G) as the minimum of all magic constants $c(f)$ where the minimum is taken over all super magic labeling f of G if there exist at least one such super magic labeling. They determine the super magic strength of paths, cycles, wheels, stars, bistars, P_n^2, $< K_{1,n} : 2 >$ (the graph obtained by joining the centers of two copies of $K_{1,n}$ by a path of length 2), and $(2n+1)P_2$.

A *Halin graph* ia a planar 3-connected graphs that consist of a tree and a cycle connecting the end vertices of the tree. Let G be a (p,q)-graph in which the edges are labeled $k, k+1, \ldots, k+q-1$, where $k \geq 0$. In [919] Lee, Su, and Wang define a graph with p vertices to be *k-edge-magic* for every vertex v the sum of the labels of the incident edges at v are constant modulo p. They investigate some classes of Halin graphs that are k-edge-magic. Lee, Su, and Wang [921] investigated some classes of cubic graphs that are k-edge-magic and provided a counterexample to a conjecture that any cubic graph of order $p \equiv 2 \pmod 4$ is k-edge-magic for all k.

S. M. Lee and colleagues [940] and [875] call a graph *G k-magic* if there is a labeling from the edges of G to the set $\{1, 2, \ldots, k-1\}$ such that for each vertex v of G the sum of all edges incident with v is a constant independent of v. The set of all k for which G is k-magic is denoted by $\mathrm{IM}(G)$ and called the *integer-magic spectrum* of G. In [940] Lee and Wong investigate the integer-magic spectrum of powers of paths. They prove: $\mathrm{IM}(P_4^2)$ is $\{4, 6, 8, 10, \ldots\}$; for $n > 5$, $\mathrm{IM}(P_n^2)$ is the set of all positive integers except 2; for all odd $d > 1$, $\mathrm{IM}(P_{2d}^d)$ is the set of all positive integers except 1; $\mathrm{IM}(P_4^3)$ is the set of all positive integers; for all odd $n \geq 5$, $\mathrm{IM}(P_n^3)$ is the set of all positive integers except 1 and 2; and for all even $n \geq 6$, $\mathrm{IM}(P_n^3)$ is the set of all positive integers except 2. For $k > 3$ they conjecture: $\mathrm{IM}(P_n^k)$ is the set of all positive integers when $n = k+1$; the set of all positive integers except 1 and 2 when n and k are odd and $n \geq k$; the set of all positive integers except 1 and 2 when n and k are even and $k \geq n/2$; the set of all positive integers except 2 when n is even and k is odd and $n \geq k$; and the set of all positive integers except 2 when n and k are even and $k \leq n/2$. In [917] Lee, Su, and Wang showed that besides the natural numbers there are two types of the integer-magic spectra of honeycomb graphs. Fu, Jhuang and Lin [520] determine the integer-magic spectra of graphs obtained from attaching a path of length at least 2 to the end vertices of each edge of a cycle.

In [875] Lee, Lee, Sun, and Wen investigated the integer-magic spectrum of various graphs such as stars, double stars (trees obtained by joining the centers of two disjoint stars $K_{1,m}$ and $K_{1,n}$ with an edge), wheels, and fans. In [1225] Salehi and Bennett report that a number of the results of Lee et al. are incorrect and provide a detailed accounting of these errors as well as determine the integer-magic spectra of caterpillars.

Lee, Lee, Sun, and Wen [875] use the notation $C_m@C_n$ to denote the graph obtained by starting with C_m and attaching paths P_n to C_m by identifying the endpoints of the paths with each successive pairs of vertices of C_m. They prove that $\mathrm{IM}(C_m@C_n)$ is the set of all positive integers if m or n is even and $\mathrm{IM}(C_m@C_n)$ is the set of all even positive integers if m and n are odd.

Lee, Valdés, and Ho [927] investigate the integer magic spectrum for special kinds of trees. For a given tree T they define the *double tree DT* of T as the graph obtained by creating a second copy T^* of T and joining each end vertex of T to its corresponding vertex in T^*. They prove that for any tree T, $\mathrm{IM}(DT)$ contains every positive integer with the possible exception of 2 and $\mathrm{IM}(DT)$ contains all positive integers if and only if the degree of every vertex that is not an end vertex is even. For a given tree T they define ADT, the *abbreviated double tree of T,* as the the graph obtained from DT by identifying the end vertices of T and T^*. They prove that for every tree T, $\mathrm{IM}(ADT)$ contains every positive integer with the possible exceptions of 1 and 2 and $\mathrm{IM}(ADT)$ contains all positive integers if and only if T is a path.

Lee, Salehi, and Sun [904] have investigated the integer-magic spectra of trees with diameter at most four. Among their findings are: if $n \geq 3$ and the prime power factorization of $n-1 = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, then $\mathrm{IM}(K_{1,n}) = p_1\mathbb{N} \cup p_2\mathbb{N} \cup \cdots \cup p_k\mathbb{N}$ (here $p_i\mathbb{N}$ means all positive integer multiples of p_i); for $m, n \geq 3$, the double star $\mathrm{IM}(DS(m,m))$ (that is, stars $K_{m,1}$ and $K_{n,1}$ that have an edge in common) is the set of all natural numbers excluding all divisors of $m-2$ greater than 1; if the prime power factorization of $m-n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ and the prime power factorization of $n-2 = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, (the exponents are permitted to be 0) then $\mathrm{IM}(DS(m,n)) = A_1 \cup A_2 \cup \cdots \cup A_k$ where $A_i = p_i^{1+s_i}\mathbb{N}$ if $r_i > s_i \geq 0$ and $A_i = \emptyset$ if $s_i \geq r_i \geq 0$; for $m,n \geq 3$, $\mathrm{IM}(DS(m,n)) = \emptyset$ if and only if $m-n$ divides $n-2$; if $m,n \geq 3$ and $|m-n| = 1$, then $DS(m,n)$ is not magic. Lee and Salehi [903] give formulas for the integer-magic spectra of trees of diameter four but they are too complicated to include here.

For a graph $G(V,E)$ and a function f from the V to the positive integers, Salehi and Lee [1228] define the *functional extension of G by f*, as the graph H with $V(H) = \cup\{u_i| \ u \in V(G) \text{ and } i = 1,2,\ldots,f(u)\}$ and $E(H) = \cup\{u_iu_j| \ uv \in E(G), i = 1,2,\ldots,f(u); j = 1,2,\ldots,f(v)\}$. They determine the integer-magic spectra for P_2, P_3 and P_4.

More specialized results about the integer-magic spectra of amalgamations of stars and cycles are given by Lee and Salehi in [903].

Table 5 summarizes the state of knowledge about magic-type labelings. In the table, **SM** means semi-magic, **M** means magic, and **SPM** means supermagic. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 5: **Summary of Magic Labelings**

Graph	*Types*	*Notes*
K_n	M	if $n=2$, $n \geq 5$ [1436]
	SPM	for $n \geq 5$ iff $n > 5$
		$n \not\equiv 0 \pmod 4$ [1437]
$K_{m,n}$	SM	if $n \geq 3$ [1436]
$K_{n,n}$	M	if $n \geq 3$ [1436]
fans f_n	M	iff n is odd, $n \geq 3$ [1436]
	not SM	if $n \geq 2$ [567]
wheels W_n	M	if $n \geq 4$ [1436]
	SM	if $n = 5$ or 6 [567]
wheels with one spoke deleted	M	if $n = 4$, $n \geq 6$ [1436]
null graph with n vertices		

Table 5: **Summary of Magic Labelings continued**

Graph	*Types*	*Notes*
Möbius ladders M_n	SPM	if $n \geq 3$, n is odd [1248]
$C_n \times P_2$	not SPM	for $n \geq 4$, n even [1248]
$C_m[\overline{K}_n]$	SPM	if $m \geq 3$, $n \geq 2$ [1344]
$K_{\underbrace{n,n,\ldots,n}_{p}}$	SPM	$n \geq 3$, $p > 5$ and $p \not\equiv 0 \pmod 4$ [1344]
composition of r-regular SPM graph and $\overline{K}_n$	SPM	if $n \geq 3$ [1344]
$K_k[\overline{K}_n]$	SPM	if $k = 3$ or 5, $n = 2$ or n odd [654]
$mK_{n,n}$	SPM	for $n \geq 2$ iff n is even or both n and m are odd [1341]
Q_n	SPM	iff $n = 1$ or $n > 2$ even [690]
$C_m \times C_n$	SPM	$m = n$ or m and n are even [690]
$C_m \times C_n$	SPM?	for all m and n [690]
connected (p,q)-graph other than P_2	M	iff $5p/4 < q \leq p(p-1)/2$ [1496]
G^i	M	$\|G\| \geq 5$, $i \geq 3$ [1499]
G^2	M	$G \neq P_5$ and G does not have a 1-factor whose every edge is incident with an end-vertex of G [1499]
$K_{1,m,n}$	M	for all m, n [1257]
P_n^2	M	for all n except 2, 3, 5 [1257], [567]
$G \times H$	M	iff G and H are magic [814]

5.2 Edge-magic Total and Super Edge-magic Total Labelings

In 1970 Kotzig and Rosa [820] defined a *magic valuation* of a graph $G(V,E)$ as a bijection f from $V \cup E$ to $\{1,2,\dots,|V \cup E|\}$ such that for all edges xy, $f(x)+f(y)+f(xy)$ is constant (called the *magic constant*). This notion was rediscovered by Ringel and Lladó [1198] in 1996 who called this labeling *edge-magic*. To distinguish between this usage from that of other kinds of labelings that use the word magic we will use the term *edge-magic total* labeling as introduced by Wallis [1602] in 2001. (We note that for 2-regular graphs a vertex-magic total labeling is an edge-magic total labeling and vice versa.) Inspired by Kotzig-Rosa notion, Enomoto, Lladó, Nakamigawa, and Ringel [468] called a graph $G(V,E)$ with an edge-magic total labeling that has the additional property that the vertex labels are 1 to $|V|$ *super edge-magic total* labeling. Kotzig and Rosa proved: $K_{m,n}$ has an edge-magic total labeling for all m and n; C_n has an edge-magic total labeling for all $n \geq 3$ (see also [573], [1205], [261], and [468]); and the disjoint union of n copies of P_2 has an edge-magic total labeling if and only if n is odd. They further state that K_n has an edge-magic total labeling if and only if $n = 1,2,3,5$ or 6 (see [821], [407], and [468]) and ask whether all trees have edge-magic total labelings. Wallis, Baskoro, Miller, and Slamin [1606] enumerate every edge-magic total labeling of complete graphs. They also prove that the following graphs are edge-magic total: paths, crowns, complete bipartite graphs, and cycles with a single edge attached to one vertex. Enomoto, Llado, Nakamigana, and Ringel [468] prove that all complete bipartite graphs are edge-magic total. They also show that wheels W_n are not edge-magic total when $n \equiv 3 \pmod 4$ and conjectured that all other wheels are edge-magic total. This conjecture was proved when $n \equiv 0,1 \pmod 4$ by Phillips, Rees, and Wallis [1135] and when $n \equiv 6 \pmod 8$ by Slamin, Bača, Lin, Miller, and Simanjuntak [1393]. Fukuchi [526] verified all cases of the conjecture independently of the work of others. Slamin et al. further show that all fans are edge-magic total.

Ringel and Llado [1198] prove that a graph with p vertices and q edges is not edge-magic total if q is even and $p+q \equiv 2 \pmod 4$ and each vertex has odd degree. Ringel and Llado conjecture that trees are edge-magic total. In [125] Babujee and Rao show that the path with n vertices has an edge-magic total labeling with magic constant $(5n+2)/2$ when n is even and $(5n+1)/2$ when n is odd. For stars with n vertices they provide an edge-magic total labeling with magic constant $3n$. In [476] Eshghi and Azimi discuss a zero-one integer programming model for finding edge-magic total labelings of large graphs.

Santhosh [1241] proved that for n odd and at least 3, the crown $C_n \odot P_2$ has an edge-magic total labeling with magic constant $(27n+3)/2$ and for n odd and at least 3, $C_n \odot P_3$ has an edge-magic total labeling with magic constant $(39n+3)/2$.

Ahmad, Baig, and Imran [48] define a *zig-zag triangle* as the graph obtained from the path $x_1,x_2,\dots,x_n$ by adding n new vertices $y_1,y_2,\dots,y_n$ and new edges y_1x_1, y_nx_{n-1}; x_iy_i for $1 \leq i \leq n$; $y_ix_{i-1}y_ix_{i+1}$ for $2 \leq i \leq n-1$. They define a graph Cb_n as one obtained from the path $x_1,x_2,\dots,x_n$ adding $n-1$ new vertices $y_1,y_2,\dots,y_{n-1}$ and new edges y_ix_{i+1} for $1 \leq i \leq n-1$. The graph Cb_n^* is obtained from the Cb_n by joining a new edge x_1y_1. They prove that zig-zag triangles, graphs that are the disjoint union of a star and a banana tree, certain disjoint unions of stars, and for $n \geq 4$, $Cb_n^* \cup Cb_{n-1}$ are super edge-magic total.

Beardon [248] extended the notion of edge-magic total to countable infinite graphs $G(V,E)$ (that is, $V \cup E$ is countable). His main result is that a countably infinite tree that processes an infinite simple path has a bijective edge-magic total labeling using the integers as labels. He asks whether all countably infinite trees have an edge-magic total labeling with the integers as labels and whether the graph with the integers as vertices and an edge joining every two distinct vertices has a bijective edge-magic total labeling using the integers.

Cavenagh, Combe, and Nelson [351] investigate edge-magic total labelings of countably infinite graphs with labels from a countable Abelian group A. Their main result is that if G is a countable graph that has an infinite set of mutually disjoint edges and A is isomorphic to a countable subgroup of the real numbers under addition then for any k in A there is an edge-magic labeling of G with elements from A that has magic constant k.

Balakrishnan and Kumar [213] proved that the join of $\overline{K_n}$ and two disjoint copies of K_2 is edge-magic total if and only if $n = 3$. Yegnanarayanan [1690] has proved the following graphs have edge-magic total labelings: nP_3 where n is odd; $P_n + K_1; P_n \times C_3$ $(n \geq 2)$; the crown $C_n \odot K_1$; and $P_m \times C_3$ with n pendent vertices attached to each vertex of the outermost C_3. He conjectures that for all n, $C_n \odot \overline{K_n}$, the n-cycle with n pendent vertices attached at each vertex of the cycle, and nP_3 have edge-magic total labelings. In fact, Figueroa-Centeno, Ichishima, and Muntaner-Batle, [502] have proved the stronger statement that for all $n \geq 3$, the corona $C_n \odot \overline{K_m}$ admits an edge-magic labeling where the set of vertex labels is $\{1, 2, \ldots, |V|\}$. (See also [1022].)

Yegnanarayanan [1690] also introduces several variations of edge-magic labelings and provides some results about them. Kotzig [1604] provides some necessary conditions for graphs with an even number of edges in which every vertex has odd degree to have an edge-magic total labeling. Craft and Tesar [407] proved that an r-regular graph with r odd and $p \equiv 4 \pmod 8$ vertices can not be edge-magic total. Wallis [1602] proved that if G is an edge-magic total r-regular graph with p vertices and q edges where $r = 2^t s + 1$ $(t > 0)$ and q is even, then 2^{t+2} divides p.

Kojima [806] proved the following. Let G be a C_4-free super edge-magic (p, q)-graph with the minimum degree at least one and $m \geq 2$. If q odd and $m = 2$ or $|p - q| \geq 2$, then $P_m \times G$ is C_4-supermagic; if p is odd and $m = 2$ or $|p-q| = 1$ and $m \leq 5$, then $P_m \times G$ is C_4-supermagic; if $n \geq 3$ is odd and m is even, then $P_2 \times (C_n \odot \overline{K_m})$ is C_4-supermagic; if $n \geq 3$ is odd and m is odd, then $P_2 \times (C_n \odot \overline{K_m})$ is not C_4-supermagic; if G is a caterpillar, then $P_m \times G$ is C_4-supermagic for $m \geq 2$; and $P_m \times C_n$ is C_4-supermagic for $m \geq 2$ and $n \geq 3$. The latter result solved an open problem in [1102]. Kojma also proved that if a C_4-free bipartite $(p, p-1)$-graph G with the minimum degree at least one and partite sets U and V has a super edge-magic labeling f of G such that $f(U) = \{1, 2, \ldots, |U|\}$, then $P_m \times (2G)$ is C_4-supermagic.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [496] have proved the following graphs are edge-magic total: $P_4 \cup nK_2$ for n odd; $P_3 \cup nK_2$; $P_5 \cup nK_2$; nP_i for n odd and $i = 3, 4, 5$; $2P_n$; $P_1 \cup P_2 \cup \cdots \cup P_n$; $mK_{1,n}$; $C_m \odot nK_1$; $K_1 \odot nK_2$ for n even; W_{2n}; $K_2 \times \overline{K}_n$, nK_3 for n odd (the case nK_3 for n even and larger than 2 is done in [1039]); binary trees, generalized Petersen graphs (see also [1095]), ladders (see also [1644]), books, fans, and odd cycles with pendent edges attached to one vertex.

In [502] Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima, investigate super edge-magic total labelings of graphs with two components. Among their results are: $C_3 \cup C_n$ is super edge-magic total if and only if $n \geq 6$ and n is even; $C_4 \cup C_n$ is super edge-magic total if and only if $n \geq 5$ and n is odd; $C_5 \cup C_n$ is super edge-magic total if and only if $n \geq 4$ and n is even; if m is even with $m \geq 4$ and n is odd with $n \geq m/2 + 2$, then $C_m \cup C_n$ is super edge-magic total; for $m = 6, 8$, or 10, $C_m \cup C_n$ is super edge-magic total if and only if $n \geq 3$ and n is odd; $2C_n$ is strongly felicitous if and only if $n \geq 4$ and n is even (the converse was proved by Lee, Schmeichel, and Shee in [905]); $C_3 \cup P_n$ is super edge-magic total for $n \geq 6$; $C_4 \cup P_n$ is super edge-magic total if and only if $n \neq 3$; $C_5 \cup P_n$ is super edge-magic total for $n \geq 4$; if m is even with $m \geq 4$ and $n \geq m/2 + 2$ then $C_m \cup P_n$ is super edge-magic total; $P_m \cup P_n$ is super edge-magic total if and only $(m,n) \neq (2,2)$ or(3,3); and $P_m \cup P_n$ is edge-magic total if and only $(m,n) \neq (2,2)$.

Enomoto, Llado, Nakamigawa, and Ringel [468] conjecture that if G is a graph of order $n+m$ that contains K_n, then G is not edge-magic total for $n \gg m$. Wijaya and Baskoro [1644] proved that $P_m \times C_n$ is edge-magic total for odd n at least 3. Ngurah and Baskoro [1095] state that $P_2 \times C_n$ is not edge-magic total. Hegde and Shetty [639] have shown that every T_p-tree (see §4.4 for the definition) is edge-magic total. Ngurah, Simanjuntak, and Baskoro [1103] show that certain subdivisions of the star $K_{1,3}$ have edge-magic total labelings. In [1100] Ngurah, Baskoro, Tomescu, gave methods for construction new (super) edge-magic total graphs from old ones by adding some new pendent edges. They also proved that $K_{1,m} \cup P_n{}^m$ is super edge-magic total. Wallis [1602] proves that a cycle with one pendent edge is edge-magic total. In [1602] Wallis poses a large number of research problems about edge-magic total graphs.

For $n \geq 3$, López, Muntaner-Batle, and Rius-Font [991] (see [992] for (corrigendum) let S_n denote the set of all super edge-magic total 1-regular labeled digraphs of order n where each vertex takes the name of the label that has been assigned to it. For $\pi \in S_n$. they define a generalization of generalized Petersen graphs that they denote by $GGP(n;\pi)$, which consists of an outer n-cycle $x_0, x_1, \ldots, x_{n-1}, x_0$, a set of n-spokes $x_i y_i$, $0 \leq i \leq n-1$, and n inner edges defined by $y_i y_{\pi(i)}, i = 0, \ldots, n-1$. Notice that, for the permutation π defined by $\pi(i) = i + k \pmod{n}$ we have $GGP(n;\pi) = P(n;k)$. They define a second generalization of generalized Petersen graphs, $GGP(n;\pi_2, \ldots, \pi_m)$, as the graphs with vertex sets $\cup_{j=1}^{m}\{x_i^j : i = 0, \ldots, n-1\}$, an outer n-cycle $x_0^1, x_1^1, \ldots, x_{n-1}^1, x_0^1$, and inner edges $x_i^{j-1} x_i^j$ and $x_i^j x_{\pi_j(i)}^j$, for $j = 2, \ldots, m$, and $i = 0, \ldots, n-1$. Notice that, $GGP(n;\pi_2, \ldots, \pi_m) = P_m \times C_n$, when $\pi_j(i) = i + 1 \pmod{n}$ for every $j = 2, \ldots, m$. Among their results are the Petersen graphs are super edge-magic total; for each m with $1 < l \leq m$ and $1 \leq k \leq 2$, the graph $GGP(5;\pi_2, \ldots, \pi_m)$, where $\pi_i = \sigma_1$ for $i \neq l$ and $\pi_l = \sigma_k$, is super edge-magic total; for each $1 \leq k \leq 2$, the graph $P(5n; k + 5r)$ where r is the smallest integer such that $k + 5r = 1 \pmod{n}$ is super edge-magic total.

A w-graph, $W(n)$, has vertices $\{(c_1, c_2, b, w, d) \cup (x^1, x^2, \ldots, x^n) \cup (y^1, y^2, \ldots, y^n)\}$ and edges $\{(c_1x^1, c_1x^2, \ldots, c_1x^n) \cup (c_2y^1, c_2y^2, \ldots, c_2y^n) \cup (c_1b, c_1w) \cup (c_2w, c_2d)\}$. A w-tree, $WT(n,k)$, is a tree obtained by taking k copies of a w-graph $W(n)$ and a new vertex **a** and joining **a** with in each copy d where $n \geq 2$ and $k \geq 3$. An *extended w-tree* $Ewt(n,k,r)$ is a tree obtained by taking k copies of an extended w-graph $Ew(n,r)$ and a new vertex **a** and joining **a** with the vertex d in each of the k copies for $n \geq 2$, $k \geq 3$ and $r \geq 2$. Super edge-magic total labelings for w-trees, extended w-trees, and disjoint unions of extended w-trees are given in [706], [705], and [71]. Javaid, Hussain, Ali, and Shaker [707] provided super edge-magic total labelings of subdivisions of $K_{1,4}$ and w-trees.

In 1988 Godbod and Slater [573] made the following conjecture. If n is odd, $n \neq 5$, C_n has an edge magic labeling with valence k, when $(5n+3)/2 \leq k \leq (7n+3)/2$. If n is even, C_n has an edge-magic labeling with valence k when $5n/2 + 2 \leq k \leq 7n/2 + 1$. Except for small values of n, very few valences for edge-magic labelings of C_n are known. In [996] López, Muntaner-Batle, and Rius-Font use the $\otimes_h$-product in order to prove the following two results. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique prime factorization of an odd number n. Then C_n admits at least $1 + \sum_{i=1}^{k} \alpha_i$ edge-magic labelings with at least $1 + \sum_{i=1}^{k} \alpha_i$ mutually different valences. Let $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique prime factorization of an even number n, with $p_1 > p_2 > \cdots > p_k$. Then C_n

admits at least $\sum_{i=1}^{k} \alpha_i$ edge-magic labelings with at least $\sum_{i=1}^{k} \alpha_i$ mutually different valences. If $\alpha \geq 2$ this lower bound can be improved to $1 + \sum_{i=1}^{k} \alpha_i$.

In 1996 Erdős asked for $M(n)$, the maximum number of edges that an edge-magic total graph of order n can have (see [407]). In 1999 Craft and Tesar [407] gave the bound $\lfloor n^2/4 \rfloor \leq M(n) \leq \lfloor n(n-1)/2 \rfloor$. For large n this was improved by Pikhurko [1138] in 2006 to $2n^2/7 + O(n) \leq M(n) \leq (0.489 + \cdots + o(1)n^2)$.

Enomoto, Lladó, Nakamigawa, and Muntaner-Batle [468] proved that a super edge-magic total graph $G(V,E)$ with $|V| \geq 4$ and with girth at least 4 has at most $2|V| - 5$ edges. They prove this bound is tight for graphs with girth 4 and 5 in [468] and [682].

In his Ph.D. thesis, Barrientos [227] introduced the following notion. Let $L_1, L_2, \ldots, L_h$ be ordered paths in the grid $P_r \times P_t$ that are maximal straight segments such that the end vertex of L_i is the beginning vertex of L_{i+1} for $i = 1, 2, \ldots h-1$. Suppose for some i with $1 < i < h$ we have $V(L_i) = \{u_0, v_0\}$ where u_0 is the end vertex of L_{i-1} and the beginning vertex of L_i and v_0 is the end vertex of L_1 and the beginning vertex of L_{i+1}. Let $u \in V(L_{i-1}) - \{u_0\}$ and $v \in V(L_{i+1}) - \{v_0\}$. The replacement of the edge u_0v_0 by a new edge uv is called an *elementary transformation* of the path P_n. A tree is called a *path-like tree* if it can be obtained from P_n by a sequence of elementary transformations on an embedding of P_n in a 2-dimensional grid. In [185] Bača, Lin, and Muntaner-Batle proved that if $T_1, T_2, \ldots, T_m$ are path-like trees each of order $n \geq 4$ where m is odd and at least 3, then $T_1 \cup T_2, \cup \cdots \cup T_m$ has a super edge-magic labeling. In [184] Bača, Lin, Muntaner-Batle and Rius-Font proved that the number of such trees grows at least exponentially with m. As an open problem Bača, Lin, Muntaner-Batle and Rius-Font ask if graphs of the form $T_1 \cup T_2 \cup \cdots \cup T_m$ where $T_1, T_2, \ldots, T_m$ are path-like trees each of order $n \geq 2$ and m is even have a super edge-magic labeling. In [227] Barrientos proved that all path-like trees admit an α-valuation. Using Barrientos's result, it is very easy to obtain that all path-like trees are a special kind of super edge-magic by using a super edge-magic labeling of the path P_n, and hence they are also super edge-magic. Furthermore in [7] Figueroa-Centeno at al. proved that if a tree is super edge-magic, then it is also harmonious. Therefore all path-like trees are also harmonious. In [988] López, Muntaner-Batle, and Rius-Font also use a variation of the Kronecker product of matrices in order to obtain lower bounds for the number of non isomorphic super edge-magic labeling of some types of path-like trees. As a corollary they obtain lower bounds for the number of harmonious labelings of the same type of trees. López, Muntaner-Batle, and Rius-Font [997] proved that if $m \geq 4$ is an even integer and $n \geq 3$ is an odd divisor of m, then $C_m \cup C_n$ is super edge-magic

In [990] López, Muntaner-Batle and Rius-Font proved that every super edge-magic graph with p vertices and q edges where $q \geq p-1$ has an even harmonious labeling (See Section 4.6.) In [995] they stated some open problems concerning relationships among super edge-magic labelings and graceful and harmonious labelings.

Let $G = (V,E)$ be a (p,q)-linear forest. In [184] Bača, Lin, Muntaner-Batle, and Rius-Font call a labeling f a *strong super edge-magic* labeling of G and G a *strong super edge-magic* graph if $f : V \cup E \to \{1, 2, \ldots, p+q\}$ with the extra property that if $uv \in E, u', v' \in V(G)$ and $d_G(u,u') = d_G(v,v') < +\infty$, then we have that $f(u) + f(v) = f(u') + f(v')$. In [51] Ahmad, López, Muntaner-Batle, and Rius-Font define the concept of strong super edge-magic labeling of a graph with respect to a linear forest as follows. Let $G = (V,E)$ be a (p,q)-graph and let F be any linear forest contained in G. A *strong super edge-magic labeling of G with respect to F* is a super edge-magic labeling f of G with the extra property with if $uv \in E(F), u', v' \in V(F)$ and $d_F(u,u') = d_F(v,v') < +\infty$ then we have that $f(u) + f(v) = f(u') + f(v')$. If a graph G

admits a strong super edge-magic labeling with respect to some linear forest F, they say that G is a *strong super edge-magic graph with respect to* F. They prove that if m is odd and G is an acyclic graph which is strong super edge-magic with respect to a linear forest F, then mG is strong super edge-magic with respect to $F_1 \cup F_2 \cup \cdots \cup F_m$, where $F_i \simeq F$ for $i = 1, 2, \ldots, m$ and every regular caterpillar is strong super edge-magic with respect to its spine.

Noting that for a super edge-magic labeling f of a graph G with p vertices and q edges, the magic constant k is given by the formula: $k = (\sum_{u \in V} \deg(u) f(u) + \sum_{i=p+1}^{p+q} i)/q$, López, Muntaner-Batle and Rius-Font [989] define the set

$$S_G = \left\{ \frac{\sum_{u \in V} \deg(u) g(u) + \sum_{i=p+1}^{p+q} i}{q} : \text{ the function } g : V \to \{i\}_{i=1}^{p} \text{ is bijective} \right\}.$$

If $\lceil \min S_G \rceil \leq \lfloor \max S_G \rfloor$ then the *super edge-magic interval* of G is the set $I_G = [\lceil \min S_G \rceil, \lfloor \max S_G \rfloor] \cap \mathbb{N}$. The *super edge-magic set* of G is $\sigma_G = \{k \in I_G :$ there exists a super edge-magic labeling of G with valence $k\}$. López et al. call a graph G *perfect super edge-magic* if $I_G = \sigma_G$. They show that the family of paths P_n is a family of perfect super edge-magic graphs with $|I_{P_n}| = 1$ if n is even and $|I_{P_n}| = 2$ if n is odd and raise the question of whether there is an infinite family $F_1, F_2, \ldots$ of graphs such that each member of the family is perfect super edge-magic and $\lim_{i \to +\infty} |I_{F_i}| = +\infty$. They show that graphs $G \cong C_{p^k} \odot \overline{K_n}$ where $p > 2$ is a prime is such a family.

In [990] López et al. define the *irregular crown* $C(n; j_1, j_2, \ldots, j_n) = (V, E)$, where $n > 2$ and $j_i \geq 0$ for all $i \in \{1, 2, \ldots, n\}$ as follows: $V = \{v_i\}_{i=1}^{n} \cup V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_k = \{v_k^1, v_k^2, \ldots, v_k^{j_k}\}$, if $j_k \neq 0$ and $V_k = \emptyset$ if $j_k = 0$, for each $k \in \{1, 2, \ldots, n\}$ and $E = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_1 v_n\} \cup (\cup_{k=1, j_k \neq 0}^{n} \{v_k v_k^l\}_{l=1}^{j_k})$. In particular, they denote $C_m^n \cong C(m; j_1, j_2, \ldots, j_m)$, where $j_{2i-1} = n$, for each i with $1 \leq i \leq (m+1)/2$, and $j_{2i} = 0$, for each i, $1 \leq i \leq (m-1)/2$. They prove that the graphs C_3^n and C_5^n are perfect edge-magic for all $n > 1$.

López et al. [993] define $\mathfrak{F}^k$-family and $\mathfrak{E}^k$-family of graphs as follows. The infinite family of graphs $(F_1, F_2, \ldots)$ is an $\mathfrak{F}^k$-*family* if each element F_n admits exactly k different valences for super edge-magic labelings, and $\lim_{n \to +\infty} |I(F_n)| = +\infty$. The infinite family of graphs $(F_1, F_2, \ldots)$ is an $\mathfrak{E}^k$-*family* if each element F_n admits exactly k different valences for edge-magic labelings, and $\lim_{n \to +\infty} |J(F_n)| = +\infty$.

An easy observation is that $(K_{1,2}, K_{1,3}, \ldots)$ is an $\mathfrak{F}^2$-family and an $\mathfrak{E}^3$-family. They pose the two problems: for which positive integers k is it possible to find $\mathfrak{F}^k$-families and $\mathfrak{E}^k$-families? Their main results in [993] are that an $\mathfrak{F}^k$-family exits for each $k = 1, 2, 3$; and an $\mathfrak{E}^k$-family exits for each $k = 3, 4$ and 7.

McSorley and Trono [1043] define a relaxed version of edge-magic total labelings of a graph as follows. An *edge-magic injection* μ of a graph G is an injection μ from the set of vertices and edges of G to the natural numbers such that for every edge uv the sum $\mu(u) + \mu(v) + \mu(uv)$ is some constant k_μ. They investigate $\kappa(G)$, the smallest k_μ among all edge-magic injections of a graph G. They determine $\kappa(G)$ in the cases that G is K_2, K_3, K_5, K_6 (recall that these are the only complete graphs that have edge-magic total labelings), a path, a cycle, or certain types of trees. They also show that every graph has an edge-magic injection and give bounds for $\kappa(K_n)$.

Avadayappan, Vasuki, and Jeyanthi [107] define the *edge-magic total strength* of a graph G as the minimum of all constants over all edge-magic total labelings of G. We denote this by $emt(G)$. They use the notation $< K_{1,n} : 2 >$ for the tree obtained from the bistar $B_{n,n}$ (the graph obtained by joining the center vertices of two copies of $K_{1,n}$ with an edge) by subdividing the edge joining the two stars. They prove: $emt(P_{2n}) = 5n+1$; $emt(P_{2n+1}) = 5n+3$; $emt(< K_{1,n} : 2 >) = 4n+9$; $emt(B_{n,n}) = 5n+6$; $emt((2n+1)P_2) = 9n+6$; $emt(C_{2n+1}) = 5n+4$; $emt(C_{2n}) = 5n+2$; $emt(K_{1,n}) = 2n+4$; $emt(P_n^2) = 3n$; and $emt(K_{n,m}) \leq (m+2)(n+1)$ where $n \leq m$. Using an analogous definition for super edge-magic total strength, Swaninathan and Jeyanthi [1477], [1477], [1478] provide results about the super edge-magic strength of trees, fire crackers, unicyclic graphs, and generalized theta graphs. Ngurah, Simanjuntak, and Baskoro [1103] show that certain subdivisions of the star $K_{1,3}$ have super edge-magic total labelings. In [468] Enomoto, Lladó, Nakamigawa and Ringel conjectured that all trees have a super edge-magic total labeling. Ichishima, Muntaner-Batle, and Rius-Font [681] have shown that any tree of order p is contained in a tree of order at most $2p-3$ that has a super edge-magic total labeling.

In [184] Bača, Lin, Muntaner-Batle, and Rius-Font call a super edge-magic labeling f of a linear forest G of order p and size q satisfying $f\colon V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ with the additional property that if $uv \in E(G)$, $u'v' \notin E(G)$ and $d_G(u,u') = d_G(v,v') < \infty$, then $f(u)+f(v) = f(u')+f(v')$ a *strong super edge-magic labeling* of G. They use a generalization of the Kronecker product of matrices introduced by Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [504] to obtain an exponential lower bound for the number of non-isomorphic strong super edge-magic labelings of the graph mP_n, for m odd and any n, starting from the strong super edge-magic labeling of P_n. They prove that the number of non-isomorphic strong super edge-magic labelings of the graph mP_n, $n \geq 4$, is at least $\frac{5}{2}2^{\lfloor \frac{m}{2} \rfloor}+1$ where $m \geq 3$ is an odd positive integer. This result allows them to generate an exponential number of non-isomorphic super edge-magic labelings of the forest $F \cong \bigcup_{j=1}^{m} T_j$, where each T_j is a path-like tree of order n and m is an odd integer.

López, Muntaner-Batle, and Rius-Font [987] introduced a generalization of super edge-magic graphs called *super edge-magic models* and prove some results about them.

Yegnanarayanan and Vaidhyanathan [1691] use the term *nice* $(1,1)$ *edge-magic labeling* for a super edge-magic total labeling. They prove: a super edge-magic total labeling f of a (p,q)-graph G satisfies $2\sum_{v \in V(G)} f(v)deg(v) \equiv 0 \mod q$; if G is (p,q) r-regular graph $(r > 1)$ with a super edge-magic total labeling then q is odd and the magic constant is $(4p+q+3)/2$; every super edge-magic total labeling has at least two vertices of degree less than 4; fans $P_n + K_1$ are edge-magic total for all n and super edge-magic total if and only if n is at most 6; books B_n are edge-magic total for all n; a super edge-magic total (p,q)-graph with $q \geq p$ is sequential; a super edge-magic total tree is sequential; and a super edge-magic total tree is cordial.

In [1690] Yegnanarayanan conjectured that the disjoint union of $2t$ copies of P_3, $t \geq 1$, has a $(1,1)$ edge-magic labeling posed the problem of determining the values of m and n such that mP_n has a $(1,1)$ edge-magic labeling Manickam and Marudai [1022] proved the the conjectures and partially settle the open problem.

Hegde and Shetty [645] (see also [644]) define the *maximum magic strength* of a graph G as the maximum magic constant over all edge-magic total labelings of G. We use $eMt(G)$ to denote the maximum magic strength of G. Hegde and Shetty call a graph G with p vertices *strong magic* if $eMt(G) = emt(G)$; *ideal magic* if $1 \leq eMt(G) - emt(G) \leq p$; and *weak magic* if $eMt(G) - emt(G) > p$. They prove that for an edge-magic total graph G with p vertices and q edges, $eMt(G) = 3(p+q+1) - emt(G)$. Using this result they obtain: P_n is ideal magic for $n > 2$; $K_{1,1}$ is strong magic; $K_{1,2}$ and $K_{1,3}$ are ideal magic; and $K_{1,n}$ is weak magic for $n > 3$; $B_{n,n}$ is ideal magic; $(2n+1)P_2$ is strong magic; cycles are ideal magic; and the generalized web $W(t,3)$ (see §2.2 for the definition) with the central vertex deleted is weak magic.

Santhosh [1241] has shown that for n odd and at least 3, $eMt(C_n \odot P_2) = (27n+3)/2$ and for n odd and at least 3, $(39n+3)/2 \leq eMt(C_n \odot P_2) \leq (40n+3)/2$. Moreover, he proved that for n odd and at least 3 both $C_n \odot P_2$ and $C_n \odot P_3$ are weak magic. In [387] Chopra and Lee provide an number of families of super edge-magic graphs that are weak magic.

In [1081] Murugan introduces the notions of *almost-magic labeling, relaxed-magic labeling, almost-magic strength*, and *relaxed-magic strength* of a graph. He determines the magic strength of Huffman trees and twigs of odd order and the almost-magic strength of nP_2 (n is even) and twigs of even order. Also, he obtains a bound on the magic strength of the path-union $P_n(m)$ and on the relaxed-magic strength of kS_n and kP_n.

Enomoto, Llado, Nakamigawa, and Ringel [468] call an edge-magic total labeling *super edge-magic* if the set of vertex labels is $\{1, 2, \ldots, |V|\}$ (Wallis [1602] calls these labelings *strongly edge-magic*). They prove the following: C_n is super edge-magic if and only if n is odd; caterpillars are super edge-magic; $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$; and K_n is super edge-magic if and only if $n = 1, 2$, or 3. They also prove that if a graph with p vertices and q edges is super edge-magic then, $q \leq 2p - 3$. In [1017] MacDougall and Wallis study super edge-magic (p, q)-graphs where $q = 2p - 3$. Enomoto et al. [468] conjecture that every tree is super edge-magic. Lee and Shan [913] have verified this conjecture for trees with up to 17 vertices with a computer. Fukuchi, and Oshima, [528] have shown that if T is a tree of order $n \geq 2$ such that T has diameter greater than or equal to $n - 5$, then T has a super edge-magic labeling.

Various classes of banana trees that have super edge-magic total labelings have been found by Swaminathan and Jeyanthi [1477] and Hussain, Baskoro, and Slamin [677]. In [43] Ahmad, Ali, and Baskoro [43] investigate the existence of super edge-magic labelings of subdivisions of banana trees and disjoint unions of banana trees. They pose three open problems.

Kotzig and Rosa's ([820] and [821]) proof that nK_2 is edge-magic total when n is odd actually shows that it is super edge-magic. Kotzig and Rosa also prove that every caterpillar is super-edge magic. Figueroa-Centeno, Ichishima, and Muntaner-Batle prove the following: if G is a bipartite or tripartite (super) edge-magic graph, then nG is (super) edge-magic when n is odd [499]; if m is a multiple of $n+1$, then $K_{1,m} \cup K_{1,n}$ is super edge-magic [499]; $K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 3; $K_{1,m} \cup K_{1,n}$ is edge-magic if and only if mn is even [499]; $K_{1,3} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 4 [499]; $P_m \cup K_{1,n}$ is super edge-magic when $m \geq 4$ [499]; $2P_n$ is super edge-magic if and only if n is not 2 or 3; $K_{1,m} \cup 2nK_2$ is super edge-magic for all m and n [499]; $C_3 \cup C_n$ is super edge-magic if and only if $n \geq 6$ and n is even [502] (see also [588]); $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and n is odd [502] (see also [588]); $C_5 \cup C_n$ is super edge-magic if and only if $n \geq 4$ and n is even [502]; if m is even and at least 6 and n is odd and satisfies $n \geq m/2 + 2$, then $C_m \cup C_n$ is super edge-magic [502]; $C_4 \cup P_n$ is super edge-magic if and only if $n \neq 3$ [502]; $C_5 \cup P_n$ is super edge-magic if $n \geq 4$ [502]; if m is even and at least 6 and $n \geq m/2 + 2$, then $C_m \cup P_n$ is super edge-magic [502]; and $P_m \cup P_n$ is super edge-magic if and only if $(m, n) \neq (2, 2)$ or (3,3) [502]. They [499] conjecture that $K_{1,m} \cup K_{1,n}$ is super edge-magic only when m is a multiple of $n + 1$ and they prove that if G is a super edge-magic graph with p vertices and q edges with $p \geq 4$ and $q \geq 2p - 4$, then G contains triangles. In [502] Figueroa-Centeno et al. conjecture that $C_m \cup C_n$ is super edge-magic if and only if $m + n \geq 9$ and $m + n$ is odd.

In [527] Fukuchi and Oshima describe a construction of super-edge-magic labelings of some families of trees with diameter 4. Salman, Ngurah, and Izzati [1232] use S_n^m ($n \geq 3$) to denote the graph obtained by inserting m vertices in every edge of the star S_n. They prove that S_n^m is super edge-magic when $m = 1$ or 2.

Muntaner-Batle calls a bipartite graph with partite sets V_1 and V_2 *special super edge-magic* if is has a super edge-magic total labeling f with the property that $f(V_1) = \{1, 2, \ldots, |V_1|\}$. He proves that a tree has a special super edge-magic labeling if and only if it has an α-labeling (see §3.1 for the definition). Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [504] use matrices to generate edge-magic total labeling and define the concept of super edge-magic total labelings for digraphs. They prove that if G is a graph with a super edge-magic total labeling then for every natural number d there exists a natural number k such that G has a (k, d)-arithmetic labeling (see §4.2 for the definition). In [854] Lee and Lee prove that a graph is super edge-magic if and only if it is $(k, 1)$-strongly indexable (see §4.3 for the definition of (k, d)-strongly indexable graphs). They also provide a way to construct (k, d)-strongly indexable graphs from two given (k, d)-strongly indexable graphs. This allows them to obtain several existing results about super edge-magic graphs as special cases of their constructions. Acharya and Germina [18] proved that the class of strongly indexable graphs is a proper subclass of super edge-magic graphs.

In [678] Ichishima, López, Muntaner-Batle and Rius-Font show how one can use the product $\otimes_h$ of super edge-magic 1-regular labeled digraphs and digraphs with harmonious, or sequential labelings to create new undirected graphs that have harmonious, sequential labelings or partitional labelings (see §4.1 for the definition). They define the product $\otimes_h$ as follows. Let $\overrightarrow{D} = (V, E)$ be a digraph with adjacency matrix $A(\overrightarrow{D}) = (a_{ij})$ and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of m digraphs all with the same set of vertices V'. Assume that $h : E \longrightarrow \Gamma$ is any function that assigns elements of Γ to the arcs of D. Then the digraph $\overrightarrow{D} \otimes_h \Gamma$ is defined by $V(D \otimes_h \Gamma) = V \times V'$ and $((a_1, b_1), (a_2, b_2)) \in E(D \otimes_h \Gamma) \iff [(a_1, a_2) \in E(D) \wedge (b_1, b_2) \in E(h(a_1, a_2))]$. An alternative way of defining the same product is through adjacency matrices, since one can obtain the adjacency matrix of $\overrightarrow{D} \otimes_h \Gamma$ as follows: if $a_{ij} = 0$ then a_{ij} is multiplied by the $p' \times p'$ 0-square matrix, where $p' = |V'|$. If $a_{ij} = 1$ then a_{ij} is multiplied by $A(h(i, j))$ where $A(h(i, j))$ is the adjacency matrix of the digraph $h(i, j)$. They prove the following. Let $\overrightarrow{D} = (V, E)$ be a harmonious (p, q)-digraph with $p \leq q$ and let h be any function from E to the set of all super edge-magic 1-regular labeled digraphs of order n, which we denote by S_n. Then the undirected graph $und(\overrightarrow{D} \otimes_h S_n)$ is harmonious. Let $\overrightarrow{D} = (V, E)$ be a sequential digraph and let $h : E \longrightarrow S_n$ be any function. Then $und(\overrightarrow{D} \otimes_h S_n)$ is sequential. Let D be a partitional graph and let $h : E \longrightarrow S_n$ be any function, where $\overrightarrow{D} = (V, E)$ is the digraph obtained by orienting all edges from one stable set to the other one. Then

In [994] López, Muntaner-Batle and Rius-Font introduce the concept of $\{H_i\}_{i \in I}$*-super edge-magic decomposable* as follows: Let $G = (V, E)$ be any graph and let $\{H_i\}_{i \in I}$ be a set of graphs such that $G = \oplus_{i \in I} H_i$ (that is, G decomposes into the graphs in the set $\{H_i\}_{i \in I}$). Then we say that G is $\{H_i\}_{i \in I}$-super edge-magic decomposable if there is a bijection $\beta : V \to [1, |V|]$ such that for each $i \in I$ the subgraph H_i meets the following two requirements: (i) $\beta(V(H_i)) = [1, |V(H_i)|]$ and (ii) $\{\beta(a) + \beta(b) : \ ab \in E(H_i)\}$ is a set of consecutive integers. Such function β is called an $\{H_i\}_{i \in I}$*-super edge-magic labeling* of G. When $H_i = H$ for every $i \in I$ we just use the notation H-super edge-magic decomposable labeling.

Among their results are the following. Let $G = (V, E)$ be a (p, q)-graph which is $\{H_1, H_2\}$-super edge-magic decomposable for a pair of graphs H_1 and H_2. Then G is super edge-bimagic; Let n be an even integer. Then the cycle C_n is $(n/2)K_2$-super edge-magic decomposable if and only if $n \equiv 2 \pmod 4$. Let n be odd. Then for any super edge-magic tree T there exists a bipartite connected graph $G = G(T, n)$ such that G is (nT)-super edge-magic decomposable. Let G be a $\{H_i\}_{i \in I}$-super edge magic decomposable graph, where H_i is an acyclic digraph for each $i \in I$. Assume that $\overrightarrow{G}$ is any orientation of G and $h : E(\overrightarrow{G}) \to S_p$ is any function. Then $\mathrm{und}(\overrightarrow{G} \otimes_h S_p)$ is $\{pH_i\}_{i\in I}$-super edge magic decomposable.

As a corollary of the last result they have that if G is a 2-regular, (1-factor)-super edge-magic decomposable graph and $\overrightarrow{G}$ is any orientation of G and $h : E(\overrightarrow{G}) \to S_p$ is any function, then $\mathrm{und}(\overrightarrow{G} \otimes_h S_p)$ is a 2-regular, (1-factor)-super edge-magic decomposable graph. Moreover, if we denote the 1-factor of G by F then pF is the 1-factor of $\mathrm{und}(\overrightarrow{G} \otimes_h S_p)$.

They pose the following two open questions: Fix $p \in \mathbb{N}$. Find the maximum $r \in \mathbb{N}$ such that there is a r-regular graph of order p which is $(p/2)K_2$-super edge-magic decomposable: and characterize the set of 2-regular graphs of order n, $n \equiv 2 \pmod 4$, such that each component has even order and admits an $(n/2)K_2$-super edge-magic decomposition.

In connection to open question 1 they prove: For all $r \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that there exists a k-regular bipartite graph $B(n)$, with $k > r$ and $|V(B(n))| = 2 \cdot 3^n$, such that $B(n)$ is (3^nK_2)-super edge-magic decomposable.

Avadayappan, Jeyanthi, and Vasuki [106] define the *super magic strength* of a graph G as $sm(G) = \min\{s(L)\}$ where L runs over all super edge-magic labelings of G. They use the notation $< K_{1,n} : 2 >$ for the tree obtained from the bistar $B_{n,n}$ (the graph obtained by joining the center vertices of two copies of $K_{1,n}$ with an edge) by subdividing the edge joining the two stars. They prove: $sm(P_{2n}) = 5n + 1$; $sm(P_{2n+1}) = 5n + 3$; $sm(< K_{1,n} : 2 >) = 4n + 9$; $sm(B_{n,n}) = 5n + 6$; $sm((2n + 1)P_2) = 9n + 6$; $sm(C_{2n+1}) = 5n + 4$; $emt(C_{2n}) = 5n+2$; $sm(K_{1,n}) = 2n+4$; and $sm(P_n^2) = 3n$. Note that in each case the super magic strength of the graph is the same as its magic strength.

Santhosh and Singh [1240] proved that $C_n \odot P_2$ and $C_n \odot P_3$ are super edge-magic for all odd $n \geq 3$ and prove for odd $n \geq 3$, $sm(C_n \odot P_2) = (15n+3)/2$ and $(20n+3) \leq sm(C_n \odot P_3) \leq (21n + 3)/2$.

In his Ph.D. thesis [589] Gray proves that $C_3 \cup C_n$ is super edge-magic if and only if $n \geq 6$ and $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$. His computer search shows that $C_5 \cup 2C_3$ does not have a super edge-magic labeling.

In [1602] Wallis posed the problem of investigating the edge-magic properties of C_n with the path of length t attached to one vertex. Kim and Park [790] call such a graph an (n, t)-*kite*. They prove that an $(n, 1)$-kite is super edge-magic if and only if n is odd and an $(n, 3)$-kite is super edge-magic if and only if n is odd and at least 5. Park, Choi, and Bae [1121] show that $(n, 2)$-kite is super edge-magic if and only if n is even. Wallis [1602] also posed the problem of determining when $K_2 \cup C_n$ is super edge-magic. In [1121] and [790] Park et al. prove that $K_2 \cup C_n$ is super edge-magic if and only if n is even. Kim and Park [790] show that the graph obtained by attaching a pendent edge to a vertex of degree one of a star is super-edge magic and that a super edge-magic graph with edge magic constant k and q edges satisfies $q \leq 2k/3 - 3$.

Lee and Kong [871] use $\mathrm{St}(a_1, a_2, \ldots, a_n)$ to denote the disjoint union of the n stars $\mathrm{St}(a_1)$, $\mathrm{St}(a_2), \ldots, \mathrm{St}(a_n)$. They prove the following graphs are super edge-magic: $\mathrm{St}(m, n)$ where $n \equiv 0 \bmod (m + 1)$; $\mathrm{St}(1, 1, n)$; $\mathrm{St}(1, 2, n)$; $\mathrm{St}(1, n, n)$; $\mathrm{St}(2, 2, n)$; $\mathrm{St}(2, 3, n)$; $\mathrm{St}(1, 1, 2, n)$ $(n \geq 2)$; $\mathrm{St}(1, 1, 3, n)$; $\mathrm{St}(1, 2, 2, n)$; and $\mathrm{St}(2, 2, 2, n)$. They conjecture that $\mathrm{St}(a_1, a_2, \ldots, a_n)$ is super edge-magic when $n > 1$ is odd. Gao and Fan [544] proved that $\mathrm{St}(1, m, n)$; $\mathrm{St}(3, m, m + 1)$; and $\mathrm{St}(n, n + 1, n + 2)$ are super edge-magic, and under certain conditions $\mathrm{St}(a_1, a_2, \ldots, a_{2n+1})$, $\mathrm{St}(a_1, a_2, \ldots, a_{4n+1})$, and $\mathrm{St}(a_1, a_2, \ldots, a_{4n+3})$ are also super edge magic.

In [1016] MacDougall and Wallis investigate the existence of super edge-magic labelings of cycles with a chord. They use C_v^t to denote the graph obtained from C_v by joining two vertices that are distance t apart in C_v. They prove: C^t_{4m+1} $(m \geq 3)$ has a super edge-magic labeling for every t except $4m-4$ and $4m-8$; C^t_{4m} $(m \geq 3)$ has a super edge-magic labeling when $t \equiv 2$ mod 4; and that C^t_{4m+2} $(m > 1)$ has a super edge-magic labeling for all odd t other than 5, and for $t = 2$ and 6. They pose the problem of what values of t does C^t_{2n} have a super edge-magic labeling.

Enomoto, Masuda, and Nakamigawa [469] have proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph. Slamin, Bača, Lin, Miller, Simanjuntak [1393] proved that the friendship graph consisting of n triangles is super edge-magic if and only if n is 3, 4, 5 or 7. Fukuchi proved [525] the generalized Petersen graph $P(n, 2)$ (see §2.7 for the definition) is super edge-magic if n is odd and at least 3 while Xu, Yang, Xi, Haque, and Shen [1669] showed that $P(n, 3)$ is super edge-magic for odd n is odd and at least 5. Baskoro and Ngurah [243] showed that nP_3 is super edge-magic for $n \geq 4$ and n even.

Hegde and Shetty [648] showed that a graph is super edge-magic if and only if it is strongly k-indexable (see §4.1 for the definition). Figueroa-Centeno, Ichishima, and Muntaner-Batle [495] proved that a graph is super edge-magic if and only if it is strongly 1-harmonious and that every super edge-magic graph is cordial. They also proved that P^2_n and $K_2 \times C_{2n+1}$ are super edge-magic. In [496] Figueroa-Centeno et al. show that the following graphs are super edge-magic: $P_3 \cup kP_2$ for all k; kP_n when k is odd; $k(P_2 \cup P_n)$ when k is odd and $n = 3$ or $n = 4$; and fans F_n if and only if $n \leq 6$. They conjecture that kP_2 is not super edge-magic when k is even. This conjecture has been proved by Z. Chen [375] who showed that kP_2 is super edge-magic if and only if k is odd. Figueroa-Centeno et al. proved that the book B_n is not super edge-magic when $n \equiv 1, 3, 7 \pmod 8$ and when $n = 4$. They proved that B_n is super edge-magic for $n = 2$ and 5 and conjectured that for every $n \geq 5$, B_n is super edge-magic if and only if n is even or $n \equiv 5 (\mathrm{mod}\ 8)$. Yuansheng, Yue, Xirong, and Xinhong [1711] proved this conjecture for the case that n is even. They prove that every tree with an α-labeling is super edge-magic. Yokomura (see [468]) has shown that $P_{2m+1} \times P_2$ and $C_{2m+1} \times P_m$ are super edge-magic (see also [495]). In [497], Figueroa-Centeno et al. proved that if G is a (super) edge-magic 2-regular graph, then $G \odot \overline{K}_n$ is (super) edge-magic and that $C_m \odot \overline{K}_n$ is super edge-magic. Fukuchi [524] shows how to recursively create super edge-magic trees from certain kinds of existing super edge-magic trees. Ngurah, Baskoro, and Simanjuntak [1099] provide a method for constructing new (super) edge-magic graphs from existing ones. One of their results is that if G has an edge-magic total labeling and G has order p and size p or $p-1$, then $G \odot nK_1$ has an edge-magic total labeling.

Ichishima, Muntaner-Batle, Oshima [679] enlarged the classes of super edge-magic 2-regular graphs by presenting some constructions that generate large classes of super edge-magic 2-regular graphs from previously known super edge-magic 2-regular graphs or pseudo super edge-magic graphs. By virtue of known relationships among other classes of labelings the 2-regular graphs obtained from their constructions are also harmonious, sequential, felicitous and equitable. Their results add credence to the conjecture of Holden et al. [661] that all 2-regular graphs of odd order with the exceptions of $C_3 \cup C_4$, $3C_3 \cup C_4$, and $2C_3 \cup C_5$ possess a strong vertex-magic total labeling, which is equivalent to super edge-magic labelings for 2-regular graphs.

Lee and Lee [874] investigate the existence of total edge-magic labelings and super edge-magic labelings of unicylic graphs. They obtain a variety of positive and negative results and conjecture that all unicyclic are edge-magic total.

Shiu and Lee [1347] investigated edge labelings of multigraphs. Given a multigraph G with q edges they call a bijection from the set of edges of G to $\{1, 2, \ldots, q\}$ with the property that for each vertex v the sum of all edge labels incident to v is a constant independent of v a *supermagic* labeling of G. They use $K_2[n]$ to denote the multigraph consisting of n edges joining 2 vertices and $mK_2[n]$ to denote the disjoint union of m copies of $K_2[n]$. They prove that for m and n at least 2, $mK_2[n]$ is supermagic if and only if n is even or if both m and n are odd.

In 1970 Kotzig and Rosa [820] defined the *edge-magic deficiency*, $\mu(G)$, of a graph G as the minimum n such that $G \cup nK_1$ is edge-magic total. If no such n exists they define $\mu(G) = \infty$. In 1999 Figueroa-Centeno, Ichishima, and Muntaner-Batle [501] extended this notion to *super edge-magic deficiency*, $\mu_s(G)$, is the analogous way. They prove the following: $\mu_s(nK_2) = \mu(nK_2) = n-1 \pmod 2$; $\mu_s(C_n) = 0$ if n is odd; $\mu_s(C_n) = 1$ if $n \equiv 0 \pmod 4$; $\mu_s(C_n) = \infty$ if $n \equiv 2 \pmod 4$; $\mu_s(K_n) = \infty$ if and only if $n \geq 5$; $\mu_s(K_{m,n}) \leq (m-1)(n-1)$; $\mu_s(K_{2,n}) = n-1$; and $\mu_s(F)$ is finite for all forests F. They also prove that if a graph G has q edges with $q/2$ odd, and every vertex is even, then $\mu_s(G) = \infty$ and conjecture that $\mu_s(K_{m,n}) \leq (m-1)(n-1)$. This conjecture was proved for $m = 3, 4$, and 5 by Hegde, Shetty, and Shankaran [649] using the notion of strongly k-indexable labelings.

For an (n,t)-kite graph (a path of length t attached to a vertex of an n-cycle) G Ahmad, Siddiqui, Nadeem, and Imran [53] proved the following: for odd $n \geq 5$ and even $t \geq 4, \mu_s(G) = 1$; for odd $n \geq 5$ $t \geq 5, t \neq 11$, and $t \equiv 3, 7 \pmod 8$, $\mu_s(G) \leq 1$; for $n \geq 10$, $n \equiv 2 \pmod 4$ and $t = 4$, $\mu_s(G) \leq 1$; and for $t = 5$, $\mu_s(G) = 1$.

In [202] Baig, Ahmad, Baskoro, and Simanjuntak provide an upper bound for the super edge-magic deficiency of a forest formed by paths, stars, combs, banana trees, and subdivisions of $K_{1,3}$. Baig, Baskoro, and Semaničová-Feňovčíková [203] investigate the super edge-magic deficiency of forests consisting of stars. Among their results are: a forest consisting of $k \geq 3$ stars has super edge-magic deficiency at most $k-2$; for every positive integer n a forest consisting of 4 stars with exactly 1, n, n, and $n+2$ leaves has a super edge-magic total labeling; for every positive integer n a forest consisting of 4 stars with exactly 1, $n+5, 2n+6$, and $n+1$ leaves has a super edge-magic total labeling; and for every positive integers n and k a forest consisting of k identical stars has super edge-magic deficiency at most 1 when k is even and deficiency 0 when k is odd.

The *generalized Jahangir* graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices, consisting of a cycle C_{nm} with one additional vertex that is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}. In [204] Baig, Imran, Javaid, and Semaničová-Feňovčiková study the super edge-magic deficiencies of the web graph $Wb_{n,m}$, the generalized Jahangir graph $J_{2,n}$, crown products $L_n \odot K_1$, $K_4 \odot nK_1$ and gave the exact value of super edge-magic deficiency for one class of lobsters.

In [500] Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that $\mu_s(P_m \cup K_{1,n}) = 1$ if $m = 2$ and n is odd, or $m = 3$ and n is not congruent to 0 mod 3, whereas in all other cases $\mu_s(P_m \cup K_{1,n}) = 0$. They also proved that $\mu_s(2K_{1,n}) = 1$ when n is odd and $\mu_s(2K_{1,n}) \leq 1$ when n is even. They conjecture that $\mu_s(2K_{1,n}) = 1$ in all cases. Other results in [500] are: $\mu_s(P_m \cup P_n) = 1$ when $(m,n) = (2,2)$ or $(3,3)$ and $\mu_s(P_m \cup P_n) = 0$ in all other cases; $\mu_s(K_{1,m} \cup K_{1,n}) = 0$ when mn is even and $\mu_s(K_{1,m} \cup K_{1,n}) = 1$ when mn is odd; $\mu(P_m \cup K_{1,n}) = 1$ when $m = 2$ and n is odd and $\mu(P_m \cup K_{1,n}) = 0$ in all other cases; $\mu(P_m \cup P_n) = 1$ when $(m,n) = (2,2)$ and $\mu(P_m \cup P_n) = 0$ in all other cases; $\mu_s(2C_n) = 1$ when n is even and ∞ when n is odd; $\mu_s(3C_n) = 0$ when n is odd; $\mu_s(3C_n) = 1$ when $n \equiv 0 \pmod 4$; $\mu_s(3C_n) = \infty$ when $n \equiv 2 \pmod 4$; and $\mu_s(4C_n) = 1$ when $n \equiv 0 \pmod 4$. They conjecture the following: $\mu_s(mC_n) = 0$ when mn is odd; $\mu_s(mC_n) = 1$ when $mn \equiv 0 \pmod 4$; $\mu_s(mC_n) = \infty$ when $mn \equiv 2 \pmod 4$; $\mu_s(2K_{1,n}) = 1$; and if F is a forest with two components, then $\mu(F) \leq 1$ and $\mu_s(F) \leq 1$. Santhosh and Singh [1239] proved: for n odd at least 3, $\mu_s(K_2 \odot C_n) \leq (n-3)/2$;

for $n > 1,\ 1 \le \mu_s(P_n[P_2]) = \lceil (n-1)/2 \rceil$; and for $n \ge 1,\ 1 \le \mu_s(P_n \times K_4) \le n$.

Ichishima and Oshima [686] prove the following: if a graph $G(V,E)$ has an α-labeling and no isolated vertices, then $\mu_s(G) \le |E| - |V| + 1$; if a graph $G(V,E)$ has an α-labeling, is not sequential, and has no isolated vertices, then $\mu_s(G) = |E| - |V| + 1$; and, if m is even, then $\mu_s(mK_{1,n}) \le 1$. As corollaries of the last result they have: $\mu_s(2K_{1,n}) = 1$; when $m \equiv 2 \pmod 4$ and n is odd, $\mu_s(mK_{1,n}) = 1$; $\mu_s(mK_{1,3}) = 0$ when $m \equiv 4 \pmod 8$ or m is odd; $\mu_s(mK_{1,3}) = 1$ when $m \equiv 2 \pmod 4$; $\mu_s(mK_{2,2}) = 1$; for $n \ge 4,\ (n-4)2^{n-2} + 3 \le \mu_s(Q_n) \le (n-2)2^{n-1} - 4$; and for $s \ge 2$ and $t \ge 2,\ \mu_s(mK_{s,t}) \le m(st - s - t) + 1$. They conjecture that for $s \ge 2$ and $t \ge 2,\ \mu_s(mK_{s,t}) = m(st - s - t) + 1$ and pose as a problem determining the exact value of $\mu_s(Q_n)$.

Ichishima and Oshima [684] determined the super edge-magic deficiency of graphs of the form $C_m \cup C_n$ for m and n even and for arbitrary n when $m = 3, 4, 5$, and 7. They state a conjecture for the super edge-magic deficiency of $C_m \cup C_n$ in the general case.

A *block* of a graph is a maximal subgraph with no cut-vertex. The *block-cut-vertex graph* of a graph G is a graph H whose vertices are the blocks and cut-vertices in G; two vertices are adjacent in H if and only if one vertex is a block in G and the other is a cut-vertex in G belonging to the block. A *chain graph* is a graph with blocks $B_1, B_2, B_3, \ldots, B_k$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. The chain graph with k blocks where each block is identical and isomorphic to the complete graph K_n is called the *kK_n-path*.

Ngurah, Baskoro, and Simanjuntak [1098] investigate the exact values of $\mu_s(kK_n\text{-path})$ when $n = 2$ or 4 for all values of k and when $n = 3$ for $k \equiv 0, 1, 2 \pmod 4$, and give an upper bound for $k \equiv 3 \pmod 4$. They determine the exact super edge-magic deficiencies for fans, double fans, wheels of small order and provide upper and lower bounds for the general case as well as bounds for some complete partite graphs. They also include some open problems. Lee and Wang [932] show that various chain graphs with blocks that are complete graphs are super edge-magic. In [50] investigate the super edge-magic deficiency of some kites and $C_n \cup K_2$.

Figueroa-Centeno and Ichishima [493] introduce the notion of the *sequential number* $\sigma(G)$ of a graph G without isolated vertices to be either the smallest positive integer n for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, n\}$ in such a way that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels are $|E(G)|$ consecutive integers or $+\infty$ if there exists no such integer n. They prove that $\sigma(G) = \mu_s(G) + |V(G)| - 1$ for any graph G without isolated vertices, and $\sigma(K_{m,n)} = mn$, which settles the conjecture of Figueroa-Centeno, Ichishima, and Muntaner-Batle [501] that $\mu_s(K_{m,n}) = (m-1)(n-1)$.

Z. Chen [375] has proved: the join of K_1 with any subgraph of a star is super edge-magic; the join of two nontrivial graphs is super edge-magic if and only if at least one of them has exactly two vertices and their union has exactly one edge; and if a k-regular graph is super edge-magic, then $k \le 3$. Chen also obtained the following: there is a connected super edge-magic graph with p vertices and q edges if and only if $p - 1 \le q \le 2p - 3$; there is a connected 3-regular super edge-magic graph with p vertices if and only if $p \equiv 2 \pmod 4$; and if G is a k-regular edge-magic total graph with p vertices and q edges then $(p+q)(1+p+q) \equiv 0 \pmod{2d}$ where $d = \gcd(k-1, q)$. As a corollary of the last result, Chen observes that $nK_2 + nK_2$ is not edge-magic total.

Another labeling that has been called "edge-magic" was introduced by Lee, Seah, and Tan in 1992 [911]. They defined a graph $G = (V, E)$ to be *edge-magic* if there exists a bijection $f\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^+\colon V \to N$ defined by $f^+(u) = \sum_{(u,v)\in E} f(u,v)$

(mod $|V|$) is a constant map. Lee (see [899]) conjectured that a cubic graph with p vertices is edge-magic if and only if $p \equiv 2 \pmod 4$. Lee, Pigg, and Cox [899] verified this conjecture for prisms and several other classes of cubic graphs. They also show that $C_n \times K_2$ is edge-magic if and only if n is odd. Shiu and Lee [1347] showed that the conjecture is not true for multigraphs and disconnected graphs. In [1347] Lee's conjecture was modified by restricting it to simple connected cubic graphs. A computer search by Lee, Wang, and Wen [935] showed that the new conjecture was false for a graph of order 10. Shiu [1336] gave a proof that is was false.

Lee, Seah, and Tan [911] establish that a necessary condition for a multigraph with p vertices and q edges to be edge-magic is that p divides $q(q+1)$ and they exhibit several new classes of cubic edge-magic graphs. They also proved: $K_{n,n}$ $(n \geq 3)$ is edge-magic and K_n is edge-magic for $n \equiv 1, 2 \pmod 4$ and for $n \equiv 3 \pmod 4$ $(n \geq 7)$. Lee, Seah, and Tan further proved that following graphs are not edge-magic: all trees except P_2; all unicyclic graphs; and K_n where $n \equiv 0 \pmod 4$. Schaffer and Lee [1246] have proved that $C_m \times C_n$ is always edge-magic. Lee, Tong, and Seah [926] have conjectured that the total graph of a (p,p)-graph is edge-magic if and only if p is odd. They prove this conjecture for cycles. Lee, Kitagaki, Young, and Kocay [870] proved that a maximal outerplanar graph with p vertices is edge-magic if and only if $p = 6$. Shiu [1335] used matrices with special properties to prove that the composition of P_n with $\overline{K_n}$ and the composition of P_n with $\overline{K_{kn}}$ where kn is odd and n is at least 3 have edge-magic labelings.

Chopra, Dios, and Lee [386] investigated the edge-magicness of joins of graphs. Among their results are: $K_{2,m}$ is edge-magic if and only if $m = 4$ or 10; the only possible edge-magic graphs of the form $K_{3,m}$ are those with $m = 3, 5, 6, 15, 33$, and 69; for any fixed m there are only finitely many n such that $K_{m,n}$ is edge-magic; for any fixed m there are only finitely many trees T such that $T + \overline{K_m}$ is edge-magic; and wheels are not edge-magic.

Lee, Ho, Tan, and Su [869] define the *edge-magic index* of a graph G to be the smallest positive integer k such that the graph kG is edge-magic. They completely determined the edge-magic indices of graphs which are stars.

For any graph G and any positive integer k the graph $G[k]$, called the *k-fold* G, is the hypergraph obtained from G by replacing each edge of G with k parallel edges. Lee, Seah, and Tan [911] proved that for any graph G with p vertices, $G[2p]$ is edge-magic and, if p is odd, $G[p]$ is edge-magic. Shiu, Lam, and Lee [1345] show that if G is an $(n+1,n)$-multigraph, then G is edge-magic if and only if n is odd and G is isomorphic to the disjoint union of K_2 and $(n-1)/2$ copies of $K_2[2]$. They also prove that if G is a $(2m+1, 2m)$-multigraph and $k \geq 2$, then $G[k]$ is edge-magic if and only if $2m+1$ divides $k(k-1)$. For a $(2m, 2m-1)$-multigraph G and k at least 2, they show that $G[k]$ is edge-magic if $4m$ divides $(2m-1)k((2m-1)k+1)$ or if $4m$ divides $(2m+k-1)k$. In [1343] Shiu, Lam, and Lee characterize the (p,p)-multigraphs that are edge-magic as $mK_2[2]$ or the disjoint union of $mK_2[2]$ and two particular multigraphs or the disjoint union of K_2, $mK_2[2]$, and four particular multigraphs. They also show for every $(2m+1, 2m+1)$-multigraph G, $G[k]$ is edge-magic for all k at least 2. Lee, Seah, and Tan [911] prove that the multigraph $C_n[k]$ is edge-magic for $k \geq 2$.

Tables 6 and 7 summarize what is known about edge-magic total labelings and super edge-magic total labelings. We use **SEM** to indicate the graphs have super edge-magic total labelings and **EMT** to indicate the graphs have edge-magic total labelings. A question mark following SEM or EMT indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 6: **Summary of Edge-magic Total Labelings**

Graph	*Types*	*Notes*
P_n	EMT	[1606]
trees	EMT?	[821], [1198]
C_n	EMT	for $n \geq 3$ [820], [573], [1205], [261]
K_n	EMT	iff $n = 1, 2, 3, 4, 5$, or 6 [821], [407], [468] enumeration of all EMT of K_n [1606]
$K_{m,n}$	EMT	for all m and n [820]
crowns $C_n \odot K_1$	EMT	[1606]
$K_{m,n}$	EMT	[1606]
C_n with a single edge attached to one vertex	EMT	[1606]
wheels W_n	EMT	iff $n \not\equiv 3 \pmod 4$ [468],[525]
fans	EMT	[1393], [495], [496]
(p, q)-graph nP_2	not EMT EMT	if q even and $p + q \equiv 2 \pmod 4$ [1198] iff n odd [820]
$P_n + K_1$	EMT	[1690]
$P_n \times C_3$	EMT	$n \geq 2$ [1690]
crown $C_n \odot K_1$	EMT	[1690]
r-regular graph	not EMT	r odd and $p \equiv 4 \pmod 8$ [407]
$P_3 \cup nK_2$ and $P_5 \cup nK_2$	EMT	[495], [496]
$P_4 \cup nK_2$	EMT	n odd [495], [496]

Table 6: **Summary of Edge-magic Total continued**

Graph	*Types*	*Notes*
nP_i	EMT	n odd, $i = 3, 4, 5$ [1690] [495],[496]
nP_3	EMT?	[1690]
$2P_n$	EMT	[495], [496]
$P_1 \cup P_2 \cup \cdots \cup P_n$	EMT	[495], [496]
$mK_{1,n}$	EMT	[495], [496]
$C_m \odot \overline{K}_n$	EMT	[495], [496]
unicylic graphs	EMT?	[874]
$K_1 \odot nK_2$	EMT	n even [495], [496]
$K_2 \times \overline{K}_n$	EMT	[495], [496]
nK_3	EMT	iff $n \neq 2$ odd [495], [496], [1039]
binary trees	EMT	[495], [496]
$P(m, n)$ (generalized Petersen graph see §2.7)	EMT	[495], [496], [1095]
ladders	EMT	[495], [496]
books	EMT	[495], [496]
odd cycle with pendent edges attached to one vertex	EMT	[495], [496]
$P_m \times C_n$	EMT	n odd $n \geq 3$ [1644]
$P_m \times P_2$	EMT	m odd $m \geq 3$ [1644]
$P_2 \times C_n$	not EMT	[1095]
$K_{1,m} \cup K_{1,n}$	EMT	iff mn is even [499]
$G \odot \overline{K}_n$	EMT	if G is EMT 2-regular graph [497]

Table 7: **Summary of Super Edge-magic Labelings**

Graph	*Types*	*Notes*
C_n	SEM	iff n is odd [468]
caterpillars	SEM	[468], [820], [821]
$K_{m,n}$	SEM	iff $m = 1$ or $n = 1$ [468]
K_n	SEM	iff $n = 1, 2$ or 3 [468]
trees	SEM?	[468]
nK_2	SEM	iff n odd [375]
nG	SEM	if G is a bipartite or tripartite SEM graph and n odd [499]
$K_{1,m} \cup K_{1,n}$	SEM	if m is a multiple of $n+1$ [499]
$K_{1,m} \cup K_{1,n}$	SEM?	iff m is a multiple of $n+1$ [499]
$K_{1,2} \cup K_{1,n}$	SEM	iff n is a multiple of 3 [499]
$K_{1,3} \cup K_{1,n}$	SEM	iff n is a multiple of 4 [499]
$P_m \cup K_{1,n}$	SEM	if $m \geq 4$ is even [499]
$2P_n$	SEM	iff n is not 2 or 3 [499]
$2P_{4n}$	SEM	for all n [499]
$K_{1,m} \cup 2nK_{1,2}$	SEM	for all m and n [499]
$C_3 \cup C_n$	SEM	iff $n \geq 6$ even [502], [588]
$C_4 \cup C_n$	SEM	iff $n \geq 5$ odd [502], [588]
$C_5 \cup C_n$	SEM	iff $n \geq 4$ even [502]
$C_m \cup C_n$	SEM	if $m \geq 6$ even and n odd $n \geq m/2 + 2$ [502]
$C_m \cup C_n$	SEM?	iff $m + n \geq 9$ and $m + n$ odd [502]

Table 7: **Summary of Super Edge-magic Labelings continued**

Graph	*Types*	*Notes*
$C_4 \cup P_n$	SEM	iff $n \neq 3$ [502]
$C_5 \cup P_n$	SEM	if $n \neq 4$ [502]
$C_m \cup P_n$	SEM	if $m \geq 6$ even and $n \geq m/2+2$ [502]
$P_m \cup P_n$	SEM	iff $(m,n) \neq (2,2)$ or $(3,3)$ [502]
corona $C_n \odot \overline{K}_m$	SEM	$n \geq 3$ [502]
$St(m,n)$	SEM	$n \equiv 0 \pmod{m+1}$ [871]
$St(1,k,n)$	SEM	$k = 1,2$ or n [871]
$St(2,k,n)$	SEM	$k = 2,3$ [871]
$St(1,1,k,n)$	SEM	$k = 2,3$ [871]
$St(k,2,2,n)$	SEM	$k = 1,2$ [871]
$St(a_1,\ldots,a_n)$	SEM?	for $n > 1$ odd [871]
C_{4m}^t	SEM	[1016]
C_{4m+1}^t	SEM	[1016]
friendship graph of n triangles	SEM	iff $n = 3,4,5$, or 7 [1393]
generalized Petersen graph $P(n,2)$ (see §2.7)	SEM	if $n \geq 3$ odd [524]
nP_3	SEM	if $n \geq 4$ even [243]
P_n^2	SEM	[495]
$K_2 \times C_{2n+1}$	SEM	[495]
$P_3 \cup kP_2$	SEM	for all k [496]
kP_n	SEM	if k is odd [496]
$k(P_2 \cup P_n)$	SEM	if k is odd and $n = 3,4$ [496]

Table 7: **Summary of Super Edge-magic Labelings continued**

Graph	*Types*	*Notes*
fans F_n	SEM	iff $n \leq 6$ [496]
books B_n	SEM	if n even [1711]
books B_n	SEM?	if $n \equiv 5 \pmod 8$[496]
trees with α-labelings	SEM	[496]
$P_{2m+1} \times P_2$	SEM	[468], [495]
$C_{2m+1} \times P_m$	SEM	[468], [495]
$G \odot \overline{K}_n$	SEM	if G is SEM 2-regular graph [497]
$C_m \odot \overline{K}_n$	SEM	[497]
join of K_1 with any subgraph of a star	SEM	[375]
if G is k-regular SEM graph		then $k \leq 3$ [375]
G is connected (p,q)-graph	SEM	G exists iff $p-1 \leq q \leq 2p-3$ [375]
G is connected 3-regular graph on p vertices	SEM	iff $p \equiv 2 \pmod 4$ [375]
$nK_2 + nK_2$	not SEM	[375]

5.3 Vertex-magic Total Labelings

MacDougall, Miller, Slamin, and Wallis [1013] introduced the notion of a vertex-magic total labeling in 1999. For a graph $G(V, E)$ an injective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a *vertex-magic total labeling* if there is a constant k, called the *magic constant*, such that for every vertex v, $f(v) + \sum f(vu) = k$ where the sum is over all vertices u adjacent to v (some authors use the term "vertex-magic" for this concept). They prove that the following graphs have vertex-magic total labelings: C_n; P_n $(n > 2)$; $K_{m,m}$ $(m > 1)$; $K_{m,m} - e$ $(m > 2)$; and K_n for n odd. They also prove that when $n > m + 1$, $K_{m,n}$ does not have a vertex-magic total labeling. They conjectured that $K_{m,m+1}$ has a vertex-magic total labeling for all m and that K_n has vertex-magic total labeling for all $n \geq 3$. The latter conjecture was proved by Lin and Miller [962] for the case that n is divisible by 4 while the remaining cases were done by MacDougall, Miller, Slamin, and Wallis [1013]. McQuillan [1038] provided many vertex-magic total labelings for cycles C_{nk} for $k \geq 3$ and odd $n \geq 3$ using given vertex-magic labelings for C_k. Gray, MacDougall, and Wallis [598] then gave a simpler proof that all complete graphs are vertex-magic total. Krishnappa, Kothapalli, and Venkaiah [812] gave another proof that all complete graphs are vertex-magic total.

In [1013] MacDougall, Miller, Slamin, and Wallis conjectured that for $n \geq 5$, K_n has a vertex-magic total labeling with magic constant h if and only if h is an integer satisfying $n^3+3n \leq 4h \leq n^3 + 2n^2 + n$. In [1040] McQuillan and Smith proved that this conjecture is true when n is odd. Armstrong and McQuillan [100] proved that if $n \equiv 2 \pmod 4$ $(n \geq 6)$ then K_n has a vertex-magic total labeling with magic constant h for each integer h satisfying $n^3+6n \leq 4h \leq n^3+2n^2-2n$. If, in addition, $n \equiv 2 \pmod 8$, then K_n has a vertex-magic total labeling with magic constant h for each integer h satisfying $n^3+4n \leq 4h \leq n^3+2n^2$. They further showed that for each odd integer $n \geq 5$, $2K_n$ has a vertex-magic total labeling with magic constant h for each integer h such that $n^3 + 5n \leq 2h \leq n^3 + 2n^2 - 3n$. If, in addition, $n \equiv 1 (\bmod\ 4)$, then $2K_n$ has a vertex-magic total labeling with magic constant h for each integer h such that $n^3 + 3n \leq 2h \leq n^3 + 2n^2 - n$.

In [1039] McQuillan and McQuillan investigate the existence of vertex-magic labelings of nC_3. They prove: for every even integer $n \geq 4$, nC_3 is vertex-magic (and therefore also edge-magic); for each even integer $n \geq 6$, nC_3 has vertex-magic total labelings with at least $2n - 2$ different magic constants; if $n \equiv 2 \bmod 4$, two extra vertex-magic total labelings with the highest possible and lowest possible magic constants exist; if $n = 2 \cdot 3^k$, $k > 1$, nC_3 has a vertex-magic total labeling with magic constant k if and only if $(1/2)(15n + 4) \leq k \leq (1/2)(21n + 2)$; if n is odd, there are vertex-magic total labelings for nC_3 with $n + 1$ different magic constants. In [1037] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if m is an odd positive integer, $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$ has a vertex-magic total labeling, and J is any subset of $I = \{1, 2, \ldots, k\}$ then $(\cup_{i \in J}\ mC_{n_i}) \cup (\cup_{i \in I-J}\ mC_{n_i})$ has a vertex-magic total labeling.

Lin and Miller [962] have shown that $K_{m,m}$ is vertex-magic total for all $m > 1$ and that K_n is vertex-magic total for all $n \equiv 0 \pmod 4$. Phillips, Rees, and Wallis [1136] generalized the Lin and Miller result by proving that $K_{m,n}$ is vertex-magic total if and only if m and n differ by at most 1. Cattell [349] has shown that a necessary condition for a graph of the form $H + \overline{K_n}$ to be vertex-magic total is that the number of vertices of H is at least $n - 1$. As a corollary he gets that a necessary condition for $K_{m_1,m_2,\ldots,m_r,n}$ where n is the largest size of any partite set to be vertex-magic total is that $m_1 + m_2 + \cdots + m_r \geq n$. He poses as an open question whether graphs that meet the conditions of the theorem are vertex-magic total. Cattell also proves that

$K_{1,n,n}$ has a vertex-magic total labeling when n is odd and $K_{2,n,n}$ has a vertex-magic total labeling when $n \equiv 3 \pmod 4$. In [1178] Rahim and Slamin proved the disjoint union of coronas $C_{t_1} \odot K_1 \cup C_{t_2} \odot K_1 \cup \cdots \cup C_{t_n} \odot K_1$ has a vertex-magic total labeling with magic constant $6\sum_{k=1}^{n} t_k + 1$.

Miller, Bača, and MacDougall [1053] have proved that the generalized Petersen graphs $P(n,k)$ (see §2.7) for the definition) are vertex-magic total when n is even and $k \leq n/2-1$. They conjecture that all $P(n,k)$ are vertex-magic total when $k \leq (n-1)/2$ and all prisms $C_n \times P_2$ are vertex-magic total. Bača, Miller, and Slamin [195] proved the first of these conjectures (see also [1395] for partial results) while Slamin and Miller prove the second. Slamin, Prihandoko, Setiawan, Rosita and Shaleh [1396] constructed vertex-magic total labelings for the disjoint union of two copies of $P(n,k)$ and Silaban, Parestu, Herawati, Sugeng, and Slamin [1369] extended this to any number of copies of $P(n,k)$. More generally, they proved that for $n_j \geq 3$ and $1 \leq k_j \leq \lfloor (n_j-1)/2 \rfloor$, the union $P(n_1,k_1) \cup P(n_2,k_2) \cup \cdots \cup P(n_t,k_t)$ has a vertex-magic total labeling with vertex magic constant $10(n_1+n_2+\cdots+n_t)+2$. In the same article Silaban et al. define the union of t special circulant graphs $\cup_{j=1}^{t} C_n(1,m_j)$ as the graph with vertex set $\{v_i^j |\ 0 \leq i \leq n-1, 1 \leq j \leq t\}$ and edge set $\{v_i^j v_{i+1}^j |\ 0 \leq i \leq n-1, 1 \leq j \leq t\} \cup \{v_i^j v_{i+m_j}^j |\ 0 \leq i \leq n-1, 1 \leq j \leq t\}$. They prove that for odd n at least 5 and $m_j \in \{2,3,\ldots,(n-1)/2\}$, the disjoint union $\cup_{j=1}^{t} C_n(1,m_j)$ has a vertex-magic total labeling with constant $8tn+(n-10/2+3$.

MacDougall et al. ([1013], [1015] and [596]) have shown: W_n has a vertex-magic total labeling if and only if $n \leq 11$; fans F_n have a vertex-magic total labelings if and only if $n \leq 10$; friendship graphs have vertex-magic total labelings if and only if the number of triangles is at most 3; $K_{m,n}$ $(m > 1)$ has a vertex-magic total labeling if and only if m and n differ by at most 1. Wallis [1602] proved: if G and H have the same order and $G \cup H$ is vertex-magic total then so is $G+H$; if the disjoint union of stars is vertex-magic total, then the average size of the stars is less than 3; if a tree has n internal vertices and more than $2n$ leaves then it does not have a vertex-magic total labeling. Wallis [1603] has shown that if G is a regular graph of even degree that has a vertex-magic total labeling then the graph consisting of an odd number of copies of G is vertex-magic total. He also proved that if G is a regular graph of odd degree (not K_1) that has a vertex-magic total labeling then the graph consisting of any number of copies of G is vertex-magic total.

Gray, MacDougall, McSorley, and Wallis [597] investigated vertex-magic total labelings of forests. They provide sufficient conditions for the nonexistence of a vertex-magic total labeling of forests based on the maximum degree and the number of internal vertices, and leaves or the number of components. They also use Skolem sequences to prove a star forest with each component a $K_{1,2}$ has a vertex-magic total labeling.

Recall a helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel. A *generalized helm* $H(n,t)$ is a graph obtained from a wheel W_n by attaching a path on t vertices at each vertex of the n-cycle. A *generalized web* $\mathrm{W}(n,t)$ is a graph obtained from a generalized helm $H(n,t)$ by joining the corresponding vertices of each path to form an n-cycle. Thus $\mathrm{W}(n,t)$ has $(t+1)n+1$ vertices and $2(t+1)n$ edges. A *generalized Jahangir graph* $J_{k,s}$ is a graph on $ks+1$ vertices consisting of a cycle C_{ks} and one additional vertex that is adjacent to k vertices of C_{ks} at distance s to each other on C_{ks}. Rahim, Tomescu, and Slamin [1179] prove: H_n has no vertex-magic total labeling for any $n \geq 3$; $\mathrm{W}(n,t)$ has a vertex-magic total labeling for $n=3$ or $n=4$ and $t=1$, but it is not vertex-magic total for $n \geq 17t+12$ and $t \geq 0$; and $J_{n,t+1}$ is vertex-magic total for $n=3$ and $t=1$, but it does not have this property for $n \geq 7t+11$ and $t \geq 1$. Recall a flower is the graph obtained from a helm by joining each pendent vertex to the central vertex of the helm. Ahmad and Tomescu [54] proved that flower graph is vertex-magic if and only if the underlying cycle is C_3.

Fronček, Kovář, and Kovářová [510] proved that $C_n \times C_{2m+1}$ and $K_5 \times C_{2n+1}$ are vertex-magic total. Kovář [823] furthermore proved some general results about products of certain regular vertex-magic total graphs. In particular, if G is a $(2r+1)$-regular vertex-magic total graph that can be factored into an $(r+1)$-regular graph and an r-regular graph, then $G \times K_5$ and $G \times C_n$ for n even are vertex-magic total. He also proved that if G an r-regular vertex-magic total graph and H is a $2s$-regular supermagic graph that can be factored into two s-regular factors, then their Cartesian product $G \times H$ is vertex-magic total if either r is odd, or r is even and $|H|$ is odd.

MacDougall, Miller, and Sugeng [1014] define a *super vertex-magic total labeling* of a graph $G(V, E)$ as a vertex-magic total labeling f of G with the additional property that $f(V) = \{1, 2, \ldots, |V|\}$ and $f(E) = \{|V|+1, |V|+2, \ldots, |V|+|E|\}$ (some authors use the term "super vertex-magic" for this concept). They show that a (p, q)-graph that has a super vertex-magic total labeling with magic constant k satisfies the following conditions: $k = (p+q)(p+q+1)/v - (v+1)/2$; $k \geq (41p+21)/18$; if G is connected, $k \geq (7p-5)/2$; p divides $q(q+1)$ if p is odd, and p divides $2q(q+1)$ if p is even; if G has even order either $p \equiv 0 \pmod 8$ and $q \equiv 0$ or $3 \pmod 4$ or $p \equiv 4 \pmod 8$ and $q \equiv 1$ or $2 \pmod 4$; if G is r-regular and p and r have opposite parity then $p \equiv 0 \pmod 8$ implies $q \equiv 0 \pmod 4$ and $p \equiv 4 \pmod 8$ implies $q \equiv 2 \pmod 4$. They also show: C_n has a super vertex-magic total labeling if and only if n is odd; and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjecture that no tree has a super vertex-magic total labeling and that K_{4n} has a super vertex-magic total labeling when $n > 1$. The latter conjecture was proved by Gómez in [578]. In [579] Gómez proved that if G is a d-regular graph that has a vertex-magic total labeling and k is a positive integer such that $(k-1)(d+1)$ is even, then kG has a super vertex-magic total labeling. As a corollary, we have that if n and k are odd or if $n \equiv 0 \pmod 4$ and $n > 4$, then kK_n has a super vertex-magic total labeling. Gómez also shows how graphs with super vertex-magic total labeling can be constructed from a given graph G with super vertex-magic total labeling by adding edges to G in various ways.

Gray and MacDougall [595] establish the existence of vertex-magic total labelings for several infinite classes of regular graphs. Their method enables them to begin with any even-regular graph and from it construct a cubic graph possessing a vertex-magic total labeling. A feature of the construction is that it produces strong vertex-magic total labelings many even order regular graphs. The construction also extends to certain families of non-regular graphs. MacDougall has conjectured (see [824]) that every r-regular $(r > 1)$ graph with the exception of $2K_3$ has a vertex-magic total labeling. As a corollary of a general result Kovář [824] has shown that every $2r$-regular graph with an odd number of vertices and a Hamiltonian cycle has a vertex-magic total labeling.

Beardon [247] has shown that a necessary condition for a graph with c components, p vertices, q edges and a vertex of degree d to be vertex-magic total is $(d+2)^2 \leq (7q^2+(6c+5)q+c^2+3c)/p$. When the graph is connected this reduces to $(d+2)^2 \leq (7q^2+11q+4)/p$. As a corollary, the following are not vertex-magic total: wheels W_n when $n \geq 12$; fans F_n when $n \geq 11$; and friendship graphs $C_3^{(n)}$ when $n \geq 4$.

Beardon [249] has investigated how vertices of small degree effect vertex-magic total labelings. Let $G(p, q)$ be a graph with a vertex-magic total labeling with magic constant k and let d_0 be the minimum degree of any vertex. He proves $k \leq (1+d_0)(p+q-d_0/2)$ and $q < (1+d_0)q$.

He also shows that if $G(p, q)$ is a vertex-magic graph with a vertex of degree one and t is the number of vertices of degree at least two, then $t > q/3 \geq (p-1)/3$. Beardon [249] has shown that the graph obtained by attaching a pendent edge to K_n is vertex-magic total if and only if $n = 2, 3$, or 4.

Meissner and Zwierzyński [1045] used finding vertex-magic total labelings of graphs as a way to compare the efficiency of parallel execution of a program versus sequential processing.

Swaminathan and Jeyanthi [1475] prove the following graphs are super vertex-magic total: P_n if and only if n is odd and $n \geq 3$; C_n if and only if n is odd; the star graph if and only if it is P_2; and mC_n if and only if m and n are odd. In [1476] they prove the following: no super vertex-magic total graph has two or more isolated vertices or an isolated edge; a tree with n internal edges and tn leaves is not super vertex-magic total if $t > (n+1)/n$; if Δ is the largest degree of any vertex in a tree T with p vertices and $\Delta > (-3 + \sqrt{1+16p})/2$, then T is not super vertex-magic total; the graph obtained from a comb by appending a pendent edge to each vertex of degree 2 is super vertex-magic total; the graph obtained by attaching a path with t edges to a vertex of an n-cycle is super vertex-magic total if and only if $n + t$ is odd. Ali, Bača, and Bashir [68] proved that mP_3 and mP_4 have no super vertex-magic total labeling

For $n > 1$ and distinct odd integers x, y and z in $[1,n-1]$ Javaid, Ismail, and Salman [701] define the *chordal ring* of order n $CR_n(x, y, z)$, as the graph with vertex set Z_n, the additive group of integers modulo n, and edges $(i, i+x), (i, i+y), (i, i+z)$ for all even i. They prove that $CR_n(1, 3, n-1)$ has a super vertex-magic total labeling when $n \equiv 0 \bmod 4$ and $n \geq 8$ and conjecture that for an odd integer Δ, $3 \leq \Delta \leq n-3, n \equiv 0 \bmod 4$, $CR_n(1, \Delta, n-1)$ has a super vertex-magic total labeling with magic constant $23n/4 + 2$.

The *Knödel graphs* $W_{\Delta,n}$ with n even and degree Δ, where $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ have vertices pairs (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$ where for every $0 \leq j \leq n/2 - 1$ and there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod n/2)$, for $k = 0, 1, \ldots, \Delta - 1$. Xi, Yang, Mominul, and Wong [1658] have shown that $W_{3,n}$ is super vertex-magic total when $n \equiv 0 \bmod 4$.

A vertex magic total labeling of $G(V, E)$ is said to be E-super if $f(E(G)) = \{1, 2, 3, \ldots, |E(G)|\}$. The *cocktail party* graph, $H_{m,n}$ $(m, n \geq 2)$, is the graph with a vertex set $V = \{v_1, v_2, \ldots, v_{mn}\}$ partitioned into n independent sets $V = \{I_1, I_2, \ldots, I_n\}$ each of size m such that $v_i v_j \in E$ for all $i, j \in \{1, 2, \ldots, mn\}$ where $i \in I_p$, $j \in I_q$, $p \neq q$. The graph $H_{m,n}$ is the complement of the ladder graph and the dual graph of the hypercube. Marimuthu and Balakrishnan [1025] gave some basic properties of such labelings and proved that $H_{m,n}$ is E-super vertex magic.

Balbuena, Barker, Das, Lin, Miller, Ryan, and Slamin [205] call a vertex-magic total labeling of $G(V, E)$ a *strongly vertex-magic total labeling* if the vertex labels are $\{1, 2, \ldots, |V|\}$. They prove: the minimum degree of a strongly vertex-magic total graph is at least 2; for a strongly vertex-magic total graph G with n vertices and e edges, if $2e \geq \sqrt{10n^2 - 6n + 1}$ then the minimum degree of G is at least 3; and for a strongly vertex-magic total graph G with n vertices and e edges if $2e < \sqrt{10n^2 - 6n + 1}$ then the minimum degree of G is at most 6. They also provide strongly vertex-magic total labelings for certain families of circulant graphs. In [1037] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if m is an odd positive integer, $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$ has a strongly vertex-magic total labeling, and J is any subset of $I = \{1, 2, \ldots, k\}$ then $(\cup_{i \in J}\, mC_{n_i}) \cup (\cup_{i \in I-J}\, mC_{n_i})$ has a strongly vertex-magic total labeling.

Gray [589] proved that if G is a graph with a spanning subgraph H that possesses a strongly vertex-magic total labeling and $G - E(H)$ is even regular, then G also possesses a strongly vertex-magic total labeling. As a corollary one has that regular Hamiltonian graphs of odd order have a strongly vertex-magic total labelings.

In a series of papers Gray and MacDougall expand on McQuillan's technique to obtain a variety of results. In [592] Gray and MacDougall show that for any $r \geq 4$, every r-regular graph of odd order at most 17 has a strong vertex-magic total labeling. They also show that several large classes of r-regular graphs of even order, including some Hamiltonian graphs, have vertex-magic total labelings. They conjecture that every 2-regular graph of odd order possesses a strong vertex-magic total labeling if and only if it is not of the form $(2t-1)C_3 \cup C_4$ or $2tC_3 \cup C_5$. They include five open problems.

In [594] Gray and MacDougall introduce a procedure called a *mutation* that transforms one vertex-magic totaling labeling into another one by swapping sets of edges among vertices that may result in different labeling of the same graph or a labeling of a different graph. Among their results are: a description of all possible mutations of a labeling of the path and the cycle; for all $n \geq 2$ and all i from 1 to $n-1$ the graphs obtained by identifying an end points of paths of lengths $i, i+1$, and $2n-2i-1$ have a vertex-magic total labeling; for odd n, the graph obtained by attaching a path of length $n-m$ to an m cycle, (such graphs are called $(m; n-m)$-*kites*) have strong vertex-magic total labelings for $m = 3, \ldots, n-2$; $C_{2n+1} \cup C_{4n+4}$ and $3C_{2n+1}$ have a strong vertex-magic total labeling; and for $n \geq 2$, $C_{4n} \cup C_{6n-1}$ has a strong vertex-magic total labeling. They conclude with three open problems.

Kimberley and MacDougall [791] studied mutations that involve labelings of regular graphs into labelings of other regular graphs. They present results of extensive computations which confirm how prolific this procedure is. These computations add weight to MacDougall's conjecture that all non-trivial regular graphs are vertex-magic.

Gray and MacDougall [593] show how to construct vertex-magic total labelings for several families of non-regular graphs, including the disjoint union of two other graphs already possessing vertex-magic total labelings. They prove that if G is a d-regular graph of order v and H a t-regular graph of order u with each having a strong vertex magic total labeling and $vd^2 + 2d + 2v + 2u = 2tvd + 2t + ut^2$ then $G \cup H$ possesses a strong vertex-magic total labeling. They also provide bounds on the minimum degree of a graph with a vertex-magic total labeling.

In [595] Gray and MacDougall establish the existence of vertex-magic total labelings for several infinite classes of regular graphs. Their method enables them to begin with any even-regular graph and construct a cubic graph possessing a vertex-magic total labeling that produces strong vertex-magic total labelings for many even order regular graphs. The construction also extends to certain families of non-regular graphs.

Rahim and Slamin [1177] give the bounds for the number of vertices for Jahangir graphs, helms, webs, flower graphs and sunflower graphs when the graphs considered are not vertex-magic total.

Thirusangu, Nagar, and Rajeswari [1490] show that certain Cayley digraphs of cyclic groups have vertex-magic total labelings.

Balbuena, Barker, Lin, Miller, and Sugeng [210] call vertex-magic total labeling an *a-vertex consecutive magic* labeling if the vertex labels are $\{a, a+1, \ldots, a+|V|\}$. For an a-vertex consecutive magic labeling of a graph G with p vertices and q edges they prove: if G has one isolated vertex, then $a = q$ and $(p-1)^2 + p^2 = (2q+1)^2$; if $q = p-1$, then p is odd and $a = p-1$; if $p = q$, then p is odd and if G has minimum degree 1, then $a = (p+1)/2$ or $a = p$; if G is 2-regular, then p is odd and $a = 0$ or p; and if G is r-regular, then p and r have opposite parities. They also define an b-*edge consecutive magic* labeling analogously and state some results for these labelings.

Wood [1650] generalizes vertex-magic total and edge-magic total labelings by requiring only that the labels be positive integers rather than consecutive positive integers. He gives upper bounds for the minimum values of the magic constant and the largest label for complete graphs, forests, and arbitrary graphs.

Exoo, Ling, McSorley, Phillips, and Wallis [481] call a function λ a *totally magic labeling* of a graph G if λ is both an edge-magic total and a vertex-magic total labeling of G. A graph with such a labeling is called *totally magic*. Among their results are: P_3 is the only connected totally magic graph that has a vertex of degree 1; the only totally magic graphs with a component K_1 are K_1 and $K_1 \cup P_3$; the only totally magic complete graphs are K_1 and K_3; the only totally magic complete bipartite graph is $K_{1,2}$; nK_3 is totally magic if and only if n is odd; $P_3 \cup nK_3$ is totally magic if and only if n is even. In [1605] Wallis asks: Is the graph $K_{1,m} \cup nK_3$ ever totally magic? That question was answered by Calhoun, Ferland, Lister, and Polhill [343] who proved that if $K_{1,m} \cup nK_3$ is totally magic then $m = 2$ and $K_{1,2} \cup nK_3$ is totally magic if and only if n is even.

McSorley and Wallis [1042] examine the possible totally magic labelings of a union of an odd number of triangles and determine the spectrum of possible values for the sum of the label on a vertex and the labels on its incident edges and the sum of an edge label and the labels of the endpoints of the edge for all known totally magic graphs.

Gray and MacDougall [590] define an *order n sparse semi-magic square* to be an $n \times n$ array containing the entries $1, 2, \ldots, m$ once (for some $m < n^2$), has its remaining entries equal to 0, and whose rows and columns have a constant sum of k. They prove some basic properties of such squares and provide constructions for several infinite families of squares, including squares of all orders $n \geq 3$. Moreover, they show how such arrays can be used to construct vertex-magic total labelings for certain families of graphs.

In Tables 8, 9 and 10, **VMT** means vertex-magic total labeling, **SVMT** means super vertex magic total, and **TM** means totally magic labeling. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2007.

Table 8: **Summary of Vertex-magic Total Labelings**

Graph	*Labeling*	*Notes*
C_n	VMT	[1013]
P_n	VMT	$n > 2$ [1013]
$K_{m,m} - e$	VMT	$m > 2$ [1013]
$K_{m,n}$	VMT	iff $\lvert m - n\rvert \leq 1$ [1136],[1013],[1015]
K_n	VMT	for n odd [1013] for $n \equiv 2 \pmod 4$,$n > 2$ [962]

Table 8: **Summary of Vertex-magic Total Labelings continued**

Graph	*Labeling*	*Notes*
nK_3	VMT	iff $n \neq 2$ [495], [496], [1039]
mK_n	VMT	$m \geq 1, \ n \geq 4$ [1041]
Petersen $P(n,k)$	VMT	[195]
prisms $C_n \times P_2$	VMT	[1395]
W_n	VMT	iff $n \leq 11$ [1013],[1015]
F_n	VMT	iff $n \leq 10$ [1013],[1015]
friendship graphs	VMT	iff # of triangles ≤ 3 [1013],[1015]
$G + H$	VMT	$\lvert V(G)\rvert = \lvert V(H)\rvert$ and $G \cup H$ is VMT [1602]
unions of stars	VMT	[1602]
tree with n internal vertices and more than $2n$ leaves	not VMT	[1602]
nG	VMT	n odd, G regular of even degree, VMT [1603] G is regular of odd degree, VMT, but not K_1 [1603]
$C_n \times C_{2m+1}$	VMT	[510]
$K_5 \times C_{2n+1}$	VMT	[510]
$G \times C_{2n}$	VMT	G $2r+1$-regular VMT [823]
$G \times K_5$	VMT	G $2r+1$-regular VMT [823]
$G \times H$	VMT	G r-regular VMT, r odd or r even and $\lvert H\rvert$ odd, H $2s$-regular supermagic [823]

Table 9: **Summary of Super Vertex-magic Total Labelings**

Graph	*Labeling*	*Notes*
P_n	SVMT	iff $n > 1$ is odd [1475]
C_n	SVMT	iff n is odd [1475] and [1014]
$K_{1,n}$	SVMT	iff $n = 1$ [1475]
mC_n	SVMT	iff m and n are odd [1475]
W_n	not SVMT	[1014]
ladders	not SVMT	[1014]
friendship graphs	not SVMT	[1014]
$K_{m,n}$	not SVMT	[1014]
dragons (see §2.2)	SVMT	iff order is even [1476], [1476]
Knödel graphs $W_{3,n}$	SVMT	$n \equiv 0 \pmod 4$ [1658]
graphs with minimum degree 1	not SVMT	[1014]
K_{4n}	SVMT	$n > 1$ [578]

Table 10: **Summary of Totally Magic Labelings**

Graph	*Labeling*	*Notes*
P_3	TM	the only connected TM graph with vertex of degree 1 [481]
K_n	TM	iff $n = 1, 3$ [481]
$K_{m,n}$	TM	iff $K_{m,n} = K_{1,2}$ [481]
nK_3	TM	iff n is odd [481]
$P_3 \cup nK_3$	TM	iff n is even [481]
$K_{1,m} \cup nK_3$	TM	iff $m = 2$ and n is even [343]

5.4 Magic Labelings of Type (a, b, c)

A magic-type method for labeling the vertices, edges, and faces of a planar graph was introduced by Lih [959] in 1983. Lih defines a *magic labeling of type* (1,1,0) of a planar graph $G(V, E)$ as an injective function from $\{1, 2, \dots, |V| + |E|\}$ to $V \cup E$ with the property that for each interior face the sum of the labels of the vertices and the edges surrounding that face is some fixed value. Similarly, Lih defines a *magic labeling of type* $(1, 1, 1)$ of a planar graph $G(V, E)$ with face set F as an injective function from $\{1, 2, \dots, |V| + |E| + |F|\}$ to $V \cup E \cup F$ with the property that for each interior face the sum of the labels of the face and the vertices and the edges surrounding that face is some fixed value. Lih calls a labeling involving the faces of a plane graph *consecutive* if for every integer s the weights of all s-sided faces constitute a set of consecutive integers. Lih gave consecutive magic labelings of type $(1, 1, 0)$ for wheels, friendship graphs, prisms, and some members of the Platonic family. In [138] Bača shows that the cylinders $C_n \times P_m$ have magic labelings of type $(1, 1, 0)$ when $m \geq 2, n \geq 3, n \neq 4$. In [148] Bača proves that the generalized Petersen graph $P(n, k)$ (see §2.7 for the definition) has a consecutive magic labeling if and only if n is even and at least 4 and $k \leq n/2 - 1$.

Bača gave magic labelings of type $(1, 1, 1)$ for fans [132], ladders [132], planar bipyramids (that is, 2-point suspensions of paths) [132], grids [141], hexagonal lattices [140], Möbius ladders [135], and $P_n \times P_3$ [136]. Kathiresan and Ganesan [774] show that the graph $P_{a,b}$ consisting of $b \geq 2$ internally disjoint paths of length $a \geq 2$ with common end points has a magic labeling of type $(1, 1, 1)$ when b is odd, and when $a = 2$ and $b \equiv 0 \pmod 4$. They also show that $P_{a,b}$ has a consecutive labeling of type $(1, 1, 1)$ when b is even and $a \neq 2$.

Bača [134], [133], [144], [142], [136], [143] and Bača and Holländer [170] gave magic labelings of type $(1, 1, 1)$ and type $(1, 1, 0)$ for certain classes of convex polytopes. Kathiresan and Gokulakrishnan [776] provided magic labelings of type $(1, 1, 1)$ for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces, and one external infinite face. Bača [139] also provides consecutive and magic labelings of type $(0, 1, 1)$ (that is, an injective function from $\{1, 2, \dots, |E| + |F|\}$ to $E \cup F$ with the property that for each interior face the sum of the labels of the face and the edges surrounding that face is some fixed value) and a consecutive labeling of type $(1, 1, 1)$ for a kind of planar graph with hexagonal faces.

A *magic labeling of type* (1,0,0) of a planar graph G with vertex set V is an injective function from $\{1, 2, \dots, |V|\}$ to V with the property that for each interior face the sum of the labels of the vertices surrounding that face is some fixed value. Kathiresan, Muthuvel, and Nagasubbu [777] define a *lotus inside a circle* as the graph obtained from the cycle with consecutive vertices $a_1, a_2, \dots, a_n$ and the star with central vertex b_0 and end vertices $b_1, b_2, \dots, b_n$ by joining each b_i to a_i and a_{i+1} $(a_{n+1} = a_1)$. They prove that these graphs $(n \geq 5)$ and subdivisions of ladders have consecutive labelings of type $(1, 0, 0)$. Devaraj [429] proves that graphs obtained by subdividing each edge of a ladder exactly the same number of times has a magic labeling of type $(1, 0, 0)$.

In Table 11 we use following abbreviations

M(a, b, c) magic labeling of type (a, b, c)

CM(a, b, c) consecutive magic labeling of type (a, b, c).

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Table 11: **Summary of Magic Labelings of Type** (a, b, c)

Graph	*Labeling*	*Notes*
W_n	CM(1,1,0)	[959]
friendship graphs	CM(1,1,0)	[959]
prisms	CM(1,1,0)	[959]
cylinders $C_n \times P_m$	M(1,1,0)	$m \geq 2$, $n \geq 3$, $n \neq 4$ [138]
fans F_n	M(1,1,1)	[132]
ladders	M(1,1,1)	[132]
planar bipyramids (see §5.3)	M(1,1,1)	[132]
grids	M(1,1,1)	[141]
hexagonal lattices	M(1,1,1)	[140]
Möbius ladders	M(1,1,1)	[135]
$P_n \times P_3$	M(1,1,1)	[136]
certain classes of convex polytopes	M(1,1,1) M(1,1,0)	[134], [144], [142], [136] [143], [170]
certain classes of planar graphs with hexagonal faces	M(0,1,1) CM(0,1,1) CM(1,1,1)	[139]
lotus inside a circle (see §5.3)	CM(1,0,0)	$n \geq 5$ [777]
subdivisions of ladders	M(1,0,0) CM(1,0,0)	[429] [777]

5.5 Sigma Labelings/1-vertex magic labelings/Distance Magic

In 1987 Vilfred [1586] (see also [1587]) defined a *sigma-labeling* of a graph G with n vertices as a bijection f from the vertices of G to $\{1, 2, \ldots, n\}$ such that there is a constant k with the property that, at any vertex v the sum $\sum f(u)$ taken over all neighbors u of v is k. The concept of sigma labeling was independently studied in 2003 by Miller, Rodger, and Simanjuntak in [1058] under the name 1-*vertex magic vertex*. In a 2009 article Sugeng, Froncek, Miller, Ryan, and Walker [1444] used the term *distance magic labeling* . For convenience, we will use the term distance magic. In [1588] Vilfred and Jinnah give a number of necessary conditions for a graph to have a distance magic labeling. One of them is that if u and v are vertices of a graph with a distance labeling, then the order of the symmetric difference of $N(u)$ and $N(v)$ (neighborhoods of u and v) is not 1 or 2. This condition rules out a large class of graphs as having distance magic labelings. Rao, Singh, and Parameswaran [1193] have shown $C_m \times C_n$ has a distance magic labeling if and only if $m = n \equiv 2 \pmod 4$ and $K_m \times K_n,\ m \geq 2, n \geq 3$ does not have a distance magic labeling. In [252] Benna gives necessary and sufficient condition for $K_{m,n}$ to be a distance magic graph and proves that if G_1 and G_2 are connected graphs with minimum degree 1 and at least three vertices, then $G_1 \times G_2$ does not have a distance magic labeling. Rao, Sighn, and Parameswaran [32] prove that every graph is an induced subgraph of a regular graph that has a distance magic labeling. As open problems, Rao [1192] asks for a characterize 4-regular graphs that have distance magic labelings and which graphs of the form $C_m \times C_n,\ m = n \equiv 2 \pmod 4$ have distance magic labelings. Kovář, Frončk, and Kovářová [826] classified all orders n for which a 4-regular distance magic graph exists and also showed that there exists a distance magic graph with $k = 2t$ for every integer $t \geq 6$. Acharaya, Rao, Signh, and Parameswaran [31] proved $P_m \times C_n$ does not have a distance magic labeling when m is at least 3 and provide necessary and sufficient conditions for $K_{m,n}$ to have a distance magic labeling. Kovár and Silber [827] proved that an $(n-3)$-regular distance magic graph with n vertices exists if and only if $n \equiv 3 \pmod 6$ and that its structure is determined uniquely. Moreover, they reduce constructions of Frončk to a single construction and provide another sufficient condition for the existence a distance magic graph with an odd number of vertices.

Among the results of Miller, Rodger, and Simanjuntak in [1058]: the only trees that have a distance magic labeling are P_1 and P_3; C_n has a distance magic labeling if and only if $n = 4$; K_n has a distance magic labeling if and only if $n = 1$; the wheel $W_n = C_n + P_1$ has a distance magic labeling if and only if $n = 4$; the complete graph $K_{n,n,\ldots,n}$ with p partite sets has a distance magic labeling if and only if n is even or both n and p are odd; an r-regular graph where n is odd does not have a distance magic labeling; and $G \times \overline{K_{2n}}$ has a distance magic labeling for any regular graph G. They also give necessary and sufficient conditions for complete tripartite graphs to have a distance magic labeling.

In [1259] Seoud, Maqsoud, and Aldiban determined whether or not the following families of graphs have a distance magic vertex labeling: $K_n - \{e\}$; $K_n - \{2e\}$; P_n^k; C_n^2; $K_m \times C_n$; $C_m + P_n$; $C_m + C_n$; $P_m + P_n$; $K_{1,r,s}$; $K_{1,r,m,n}$; $K_{2,r,m,n}$; $K_{m,n} + P_k$; $K_{m,n} + C_k$; $C_m + \overline{K_n}$; $P_m + \overline{K_n}$; $P_m \times P_n$; $K_{m,n} \times P_k$; $K_m \times P_n$; the splitting graph of $K_{m,n}$; $K_n + G$; $K_m + \overline{K_n}$; $K_m + C_n$; $K_m + P_n$; $K_{m,n} + K_r$; $C_m \times P_n$; $C_m \times K_{1,n}$; $C_m \times K_{n,n}$; $C_m \times K_{n,n+1}$; $K_m \times K_{n,r}$; and $K_m \times K_n$. Typically, distance magic labelings exist only a few low parameter cases.

A survey of results on distance magic (sigma, 1-vertex) labelings through 2009 is given in [101].

5.6 Other Types of Magic Labelings

In 2004 Babujee [109] and [110] introduced the notion of *bimagic* labeling in which there exist two constants k_1 and k_2 such that the sums involved in a specified type of magic labeling is k_1 or k_2. Thus a vertex-bimagic total labeling with bimagic constants k_1 and k_2 is the same as a vertex-magic total labeling except for each vertex v the sum of the label of v and all edges adjacent to v may be k_1 or k_2. A bimagic labeling is of interest for graphs that do not have a magic labeling of a particular type. Bimagic labelings for which the number of sums equal to k_1 and the number of sums equal to k_2 differ by at most 1 are called *equitable*. When all sums except one are the same the labeling is called *almost magic*. Although the wheel W_n does not have an edge-magic total labeling when when $n \equiv 3 \pmod 4$, Marr, Phillips and Wallis [1029] showed that these wheels have both equitable bimagic and almost magic labelings. They also show that whereas nK_2 has an edge-magic total labeling if and only if n is odd, nK_2 has an edge-bimagic total labeling when n is even and although even cycles do not have super edge-magic total labelings all cycles have super edge-bimagic total labelings. They conjecture that there is a constant N such that K_n has a edge-bimagic total labeling if and only if n is at most N. They show that such an N must be at least 8. They also prove that if G has an edge-magic total labeling then $2G$ has an edge-bimagic total equitable labeling.

Babujee and Jagadesh [110], [116], [117], and [115] proved the following graphs have super edge bimagic labelings: cycles of length 3 with a nontrivial path attached; $P_3 \odot K_{1,n}$ n even; $P_n + \overline{K_2}$ (n odd);$P_2 + mK_1$ ($m \geq 2$); $2P_n$ ($n \geq 2$); the disjoint union of two stars; $3K_{1,n}$ ($n \geq 2$); $P_n \cup P_{n+1}$ ($n \geq 2$); $C_3 \cup K_{1,n}$;P_n; $K_{1,n}$; $K_{1,n,n}$; the graphs obtained by joining the centers of any two stars with an edge or a path of length 2; the graphs obtained by joining the centers of two copies of $K_{1,n}$ ($n \geq 3$) with a path of length 2 then joining the center one of copies of $K_{1,n}$ to the center of a third copy of $K_{1,n}$ with a path of length 2; combs $P_n \odot K_1$; cycles; wheels; fans; gears; K_n if and only if $n \leq 5$.

In [991] López, Muntaner-Batle, and Rius-Font give a necessary condition for a complete graph to be edge bimagic in the case that the two constants have the same parity.

In [113] Babujee, Babitha, and Vishnupriya make the following definitions. For any natural number a, a graph $G(p,q)$ is said to be *a-additive super edge bimagic* if there exists a bijective function f from $V(G) \cup E(G)$ to $\{a+1, a+2, \ldots, a+p+q\}$ such that for every edge uv, $f(u) + f(v) + f(uv) = k_1$ or k_2. For any natural number a, a graph $G(p,q)$ is said to be *a-multiplicative super edge bimagic* if there exists a bijective f from $V(G) \cup E(G)$ to $\{a, 2a, \ldots, (p+q)a\}$ such that for every edge uv, $f(u) + f(v) + f(uv) = k_1$ or k_2. A graph $G(p,q)$ is said to be *super edge-odd bimagic* if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 3, 5, \ldots, 2(p+q)-1\}$ such that for every edge uv $f(u) + f(v) + f(uv) = k_1$ or k_2. If f is a super edge bimagic labeling, then a function g from $E(G)$ to $\{0, 1\}$ with the property that for every edge uv, $g(uv) = 0$ if $f(u)+f(v)+f(uv) = k_1$ and $g(uv) = 1$ if $f(u)+f(v)+f(uv) = k_2$ is called a *super edge bimagic cordial labeling* if the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. They prove: super edge bimagic graphs are a-additive super edge bimagic; super edge bimagic graphs are a-multiplicative super edge bimagic; if G is super edge-magic, then $G + K_1$ is super edge bimagic labeling; the union of two super edge magic graphs is super edge bimagic; and P_n, C_{2n} and $K_{1,n}$ are super edge bimagic cordial.

For any nontrivial Abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling f of the edges of G with the nonzero elements of A such that the vertex labeling f^+ defined by $f^+(v) = \Sigma f(vu)$ over all edges vu is a constant. In [1433] and [1434] Stanley noted that Z-magic graphs can be viewed in the more general context of linear homogeneous diophantine equations. Shiu, Lam, and Sun [1346] have shown the following: the union of two edge-disjoint A-magic graphs with the same vertex set is A-magic; the Cartesian product of two A-magic graphs is A-magic; the lexicographic product of two A-magic connected graphs is A-magic; for an Abelian group A of even order a graph is A-magic if and only if the degrees of all of its vertices have the same parity; if G and H are connected and A-magic, G composed with H is A-magic; $K_{m,n}$ is A-magic when $m, n \geq 2$ and A has order at least 4; K_n with an edge deleted is A-magic when $n \geq 4$ and A has order at least 4; all generalized theta graphs (§4.4 for the definition) are A-magic when A has order at least 4; $C_n + \overline{K_m}$ is A-magic when $n \geq 3, m \geq 2$ and A has order at least 2; wheels are A-magic when A has order at least 4; flower graphs $C_m@C_n$ are A-magic when $m, n \geq 2$ and A has order at least 4 ($C_m@C_n$ is obtained from C_n by joining the end points of a path of length $m-1$ to each pair of consecutive vertices of C_n).

In [901] Lee, Saba, Salehi, and Sun investigate graphs that are A-magic where $A = V_4 \approx Z_2 \oplus Z_2$ is the Klein four-group. Many of theorems are special cases of the results of Shiu, Lam, and Sun [1346] given in the previous paragraph. They also prove the following are V_4-magic: a tree if and only if every vertex has odd degree; the star $K_{1,n}$ if and only if n is odd; $K_{m,n}$ for all $m, n \geq 2; K_n - e$ (edge deleted K_n) when $n > 3$; even cycles with k pendent edges if and only if k is even; odd cycles with k pendent edges if and only if k is odd; wheels; $C_n + \overline{K_2}$; generalized theta graphs; graphs that are copies of C_n that share a common edge; and $G + \overline{K_2}$ whenever G is V_4-magic.

In [385] Choi, Georges, and Mauro explore Z_2^k-magic graphs in terms of even edge-coverings, graph parity, factorability, and nowhere-zero 4-flows. They prove that the minimum k such that bridgeless G is zero-sum Z_2^k-magic is equal to the minimum number of even subgraphs that cover the edges of G, known to be at most 3. They also show that bridgeless G is zero-sum Z_2^k-magic for all $k \geq 2$ if and only if G has a nowhere-zero 4-flow, and that G is zero-sum Z_2^k-magic for all $k \geq 2$ if G is Hamiltonian, bridgeless planar, or isomorphic to a bridgeless complete multipartite graph, and establish equivalent conditions for graphs of even order with bridges to be Z_2^k-magic for all $k \geq 4$. In [560] Georges, Mauro, and Wang utilized well-known results on edge-colorings in order to construct infinite families that are V_4-magic but not Z_4-magic.

For $k \geq 2$ and graphs G and H, the graph $G \odot^k H$ defined as $(G \odot^{k-1} H) \odot H$ (where $G \odot^1 H = G \odot H$) is called the *k-multilevel corona* of G with H. Marbun and Salman [1024] proved $(W_n \odot^{k-1}) \odot C_n$ is W_n-edge magic.

Babujee and Shobana [128] prove that the following graphs have Z_3-magic labelings: C_{2n}; K_n $(n \geq 4)$; $K_{m,2m}$ $(m \geq 3)$; ladders $P_n \times P_2$ $(n \geq 4)$; bistars $B_{3n-1,3n-1}$; and cyclic, dihedral and symmetric Cayley digraphs for certain generating sets. Siddiqui [1367] proved that generalized prisms, generalized antiprisms, fans and friendship graphs are Z_{3k}-magic for $k \geq 1$. In [391] Chou and Lee investigated Z_3-magic graphs.

Chou and Lee [391] showed that every graph is an induced subgraph of an A-magic graph for any non-trivial Abelian group A. Thus it is impossible to find a Kuratowski type characterization of A-magic graphs. Low and Lee [1000] have shown that if a graph is A_1-magic then it is A_2-magic for any subgroup A_2 of A_1 and for any nontrivial Abelian group A every Eulerian graph of even size is A-magic. For a connected graph G, Low and Lee define $T(G)$ to be the graph obtained from G by adding a disjoint uv path of length 2 for every pair of adjacent vertices u and v. They prove that for every finite nontrivial Abelian group A the graphs $T(P_{2k})$ and $T(K_{1,2n+1})$ are A-magic. Shiu and Low [1353] show that $K_{k_1,k_2,\ldots,k_n} (k_i \geq 2)$ is A-magic, for all A where $|A| \geq 3$. Lee, Salehi and Sun [904] have shown that for $m, n \geq 3$ the double star $DS(m, n)$ is Z-magic if and only if $m = n$.

S. M. Lee [866] calls a graph G *fully magic* if it is A-magic for all non-trivial abelian groups A. Low and Lee [1000] showed that if G is an eulerian graph of even size, then G is fully magic. In [866] Lee gives several constructions that produce infinite families of fully magic graphs and proves that every graph is an induced subgraph of a fully magic graph.

In [842] Kwong and Lee call the set of all k for which a graph is Z_k-magic the *integer-magic spectrum* of the graph. They investigate the integer-magic spectra of the coronas of some specific graphs including paths, cycles, complete graphs, and stars. Low and Sue [1003] have obtained some results on the integer-magic spectra of tessellation graphs. Shiu and Low [1354] provide the integer-magic spectra of sun graphs. Chopra and Lee [389] determined the integer-magic spectra of all graphs consisting of any number of pairwise disjoint paths with common end vertices (that is, generalized theta graphs). Low and Lee [1000] show that Eulerian graphs of even size are A-magic for every finite nontrivial Abelian group A whereas Wen and Lee [1640] provide two families of Eularian graphs that are not A-magic for every finite nontrivial Abelian group A and eight infinite families of Eulerian graphs of odd sizes that are A-magic for every finite nontrivial Abelian group A. Low and Lee [1000] also prove that if A is an Abelian group and G and H are A-magic, then so are $G \times H$ and the lexicographic product of G and H. Low and Shiu [1002] prove: $K_{1,n} \times K_{1,n}$ has a Z_{n+1}-magic labeling with magic constant 0; if $G \times H$ is Z_2-magic, then so are G and H; if G is Z_m-magic and H is Z_n-magic, then the integer-magic spectra of $G \times H$ contains all common multiples of m and n; if n is even and $k_i \geq 3$ then the integer-magic spectra of $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_n} = \{3, 4, 5, \ldots\}$. In [1356] Shiu and Low determine all positive integers k for which fans and wheels have a Z_k-magic labeling with magic constant 0. Shiu and Low [1357] determined for which $k \geq 2$ a connected bicyclic graph without a pendent has a Z_k-magic labeling.

Shiu and Low [1355] have introduced the notion of ring-magic as follows. Given a commutative ring R with unity, a graph G is called *R-ring-magic* if there exists a labeling f of the edges of G with the nonzero elements of R such that the vertex labeling f^+ defined by $f^+(v) = \Sigma f(vu)$ over all edges vu and vertex labeling $f^\times$ defined by $f^\times(v) = \Pi f(vu)$ over all edges vu are constant. They give some results about R-ring-magic graphs.

In [338] Cahit says that a graph $G(p, q)$ is *total magic cordial* (TMC) provided there is a mapping f from $V(G) \cup E(G)$ to $\{0, 1\}$ such that $(f(a) + f(b) + f(ab)) \bmod 2$ is a constant modulo 2 for all edges $ab \in E(G)$ and $|f(0) - f(1)| \leq 1$ where $f(0)$ denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and $f(1)$ denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He says a graph G is *total sequential cordial* (TSC) if there is a mapping f from $V(G) \cup E(G)$ to $\{0, 1\}$ such that for each edge $e = ab$ with $f(e) = |f(a) - f(b)|$ it is true that $|f(0) - f(1)| \leq 1$ where $f(0)$ denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and $f(1)$ denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He proves that the following graphs have a TMC labeling: $K_{m,n}$ $(m, n > 1)$, trees, cordial graphs, and K_n if and only if $n = 2, 3, 5$, or 6. He also proves that the following graphs have a TSC labeling: trees; cycles; complete bipartite graphs; friendship graphs; cordial graphs; cubic graphs other than K_4; wheels W_n $(n > 3)$; K_{4k+1} if and only if $k \geq 1$ and $\sqrt{k}$ is an integer; K_{4k+2} if and only if $\sqrt{4k+1}$ is an integer; K_{4k} if and only if $\sqrt{4k+1}$ is an integer; and K_{4k+3} if and only if $\sqrt{k+1}$ is an integer.

Jeyanthi and Benseera [718] investigated the existence of totally magic cordial labelings of the one-point unions of copies of cycles, complete graphs and wheels.

In 2001, Simanjuntak, Rodgers, and Miller [1058] defined a *1-vertex magic (also known as distance magic labeling vertex labeling* of $G(V,E)$ as a bijection from V to $\{1,2,\ldots,|V|\}$ with the property that there is a constant k such that at any vertex v the sum $\sum f(u)$ taken over all neighbors of v is k. Among their results are: $H \times \overline{K}_{2k}$ has a 1-vertex-magic vertex labeling for any regular graph H; the symmetric complete multipartite graph with p parts, each of which contains n vertices, has a 1-vertex-magic vertex labeling if and only if whenever n is odd, p is also odd; P_n has a 1-vertex-magic vertex labeling if and only if $n = 1$ or 3; C_n has a 1-vertex-magic vertex labeling if and only if $n = 4$; K_n has a 1-vertex-magic vertex labeling if and only if $n = 1$; W_n has a 1-vertex-magic vertex labeling if and only if $n = 4$; a tree has a 1-vertex-magic vertex labeling if and only if it is P_1 or P_3; and r-regular graphs with r odd do not have a 1-vertex-magic vertex labeling.

Miller, Rogers, and Simanjuntak [1058] the complete p-partite ($p > 1$) graph $K_{n,n,\ldots,n}$ ($n > 1$) has a 1-vertex-magic vertex labeling if and only if either n is even or np is odd. Shafiq, Ali, Simanjuntak [1325] proved $mK_{n,n,\ldots,n}$ has a 1-vertex-magic vertex labeling if n is even or mnp is odd and $m \geq 1, n > 1$, and $p > 1$ and $mK_{n,n,\ldots,n}$ does not have a 1-vertex-magic vertex labeling if np is odd, $p \equiv 3$ (mod 4, and m is even.

Recall if $V(G) = \{v_1, v_2, \ldots, v_p\}$ is the vertex set of a graph G and $H_1, H_2, \ldots, H_p$ are isomorphic copies of a graph H, then $G[H]$ is the graph obtained from G by replacing each vertex v_i of G by H_i and joining every vertex in H_i to every neighbor of v_i. Shafiq, Ali, Simanjuntak [1325] proved if G is an r-regular graph ($r \geq 1$) then $G[C_n]$ has a 1-vertex-magic vertex labeling if and only if $n = 4$. They also prove that for $m \geq 1$ and $n > 1$, $mC_p[\overline{K_n}]$ has 1-vertex-magic vertex labeling if and only if either n is even or mnp is odd or n is odd and $p \equiv 3$ (mod 4).

For a graph G Jeyanthi and Benseera [716] define a function f from $V(G) \cup E(G)$ to $\{0,1\}$ to be a *totally vertex-magic cordial labeling* (TVMC) with a constant C if $f(a) + \sum_{b \in N(a)} f(ab) \equiv C$ (mod 2) for all vertices $a \in V(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $N(a)$ is the set of vertices adjacent to the vertex a and $n_f(i)$ is the sum of the number of vertices and edges with label i. They prove the following graphs have totally vertex-magic cordial labelings: vertex-magic total graphs; trees; K_n; $K_{m,n}$ whenever $|m - n| \leq 1$; $P_n + P_2$; friendship graphs with C $= 0$; and flower graphs Fl_n for $n \geq 3$ with C$= 0$. They also proved that if G is TVMC with $C = 1$, then the graph obtained by identifying any vertex of G with any vertex of a tree is TVMC with $C = 1$; if G is a (p,q) graph with $|p - q| \leq 1$, then G is TVMC with $C = 1$; and if $G(p,q)$ is a TVMC graph with constant $C = 0$ where p is odd, then $G + \overline{K_{2m}}$ is TVMC with $C = 1$ if m is odd and with $C = 0$ if m is even.

Jeyanthi, Benseera and Mary [717] showed that the following graphs have totally magic cordial labelings: (p,q) graphs with $|p - q| \leq 1$; flower graphs Fl_n for $n \geq 3$; ladders; and graphs obtained by identifying a vertex of C_m with each vertex of C_n. They also proved that if $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two disjoint totally magic cordial graphs with $p_1 = q_1$ or $p_2 = q_2$ then $G_1 \cup G_2$ is totally magic cordial. In Theorem 10 in [338] Cahit stated that K_n is totally magic cordial if and only if $n \in \{2,3,5,6\}$. Jeyanthi and Benseera proved that K_n is totally magic cordial if and only if $\sqrt{4k+1}$ has an integer value when $n = 4k$, $\sqrt{k+1}$ or $\sqrt{k}$ have an integer value when $n = 4k+1$, $\sqrt{4k+5}$ or $\sqrt{4k+1}$ have an integer value when $n = 4k+2$, or $\sqrt{k+1}$ has an integer value when $n = 4k+3$.

Balbuena, Barker, Lin, Miller, and Sugeng [217] call a vertex-magic total labeling of a graph $G(V,E)$ an *a-vertex consecutive magic labeling* if the vertex labels are $\{a+1, a+2, \ldots, a+|V|\}$

where $0 \leq a \leq |E|$. They prove: if a tree of order n has an a-vertex consecutive magic labeling then n is odd and $a = n-1$; if G has an a-vertex consecutive magic labeling with n vertices and $e = n$ edges, then n is odd and if G has minimum degree 1, then $a = (n+1)/2$ or $a = n$; if G has an a-vertex consecutive magic labeling with n vertices and e edges such that $2a \leq e$ and $2e \geq \sqrt{6}n - 1$, then the minimum degree of G is at least 2; if a 2-regular graph of order n has an a-vertex consecutive magic labeling, then n is odd and $a = 0$ or n; and if a r-regular graph of order n has an a-vertex consecutive magic labeling, then n and r have opposite parities.

Balbuena et al. also call a vertex-magic total labeling of a graph $G(V,E)$ a *b-edge consecutive magic labeling* if the edge labels are $\{b+1, b+2, \ldots, b+|E|\}$ where $0 \leq b \leq |V|$. They prove: if G has n vertices and e edges and has a b-edge consecutive magic labeling and one isolated vertex, then $b = 0$ and $(n-1)^2 + n^2 = (2e+1)^2$; if a tree with odd order has a b-edge consecutive magic labeling then $b = 0$; if a tree with even order has a b-edge consecutive magic labeling then it is P_4; a graph with n vertices and e edges such that $e \geq 7n/4$ and $b \geq n/4$ and a b-edge consecutive magic labeling has minimum degree 2; if a 2-regular graph of order n has a b-edge consecutive magic labeling, then n is odd and $b = 0$ or $b = n$; and if a r-regular graph of order n has an b-edge consecutive magic labeling, then n and r have opposite parities.

Sugeng and Miller [1446] prove: If (V,E) has an a-vertex consecutive edge magic labeling, where $a \neq 0$ and $a \neq |E|$, then G is disconnected; if (V,E) has an a-vertex consecutive edge magic labeling, where $a \neq 0$ and $a \neq |E|$, then G cannot be the union of three trees with more than one vertex each; for each nonnegative a and each positive n, there is an a-vertex consecutive edge magic labeling with n vertices; the union of r stars and a set of $r-1$ isolated vertices has an s-vertex consecutive edge magic labeling, where s is the minimum order of the stars; for every b every caterpillar has a b-edge consecutive edge magic labeling; if a connected graph G with n vertices has a b-edge consecutive edge magic labeling where $1 \leq b \leq n-1$, then G is a tree; the union of r stars and a set of $r-1$ isolated vertices has an r-edge consecutive edge magic labeling.

Babujee, Vishnupriya, and Jagadesh [131] introduced a labeling called *a-vertex consecutive edge bimagic total* as a graph $G(V,E)$ for which there are two positive integers k_1 and k_2 and a bijection f from $V \cup E$ to $\{1, 2, \ldots, |V|+|E|\}$ such that $f(u)+f(v)+f(uv) = k_1$ or k_2 for all edges uv and $f(V) = \{a+1, a+2, \ldots, a+|V|\},\ 0 \leq a \leq |V|$. They proved the following graphs have such labelings: $P_n,\ K_{1,n}$, combs, bistars $B_{m,n}$, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from P_5 by identifying the center of a copy $K_{1,n}$ with the two end vertices and the middle vertex. In [121] Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have a-vertex consecutive edge bimagic total labelings. Babujee, Jagadesh, Vishnupriya [123] study the properties of a-vertex consecutive edge bimagic total labeling for $P_3 \odot K_{1,2n},\ P_n + \overline{K_2}$ (n is odd and $n \geq 3$), $(P_2 \cup mK_1) + \overline{K_2},\ (P_2 + mK_1)$ $(m \geq 2), C_n$, fans $P_n + K_1$, double fans $P_n + 2K_1$, and graphs obtained by appending a path of length at least 2 to a vertex of C_3. Babujee, Jagadesh [122] prove the following graphs have a-vertex consecutive edge bimagic total labelings: $2P_n$ $(n \geq 2)$, $P_n \cup P_n + 1 (n \geq 2)$, $K_{2,n},\ C_n \odot K_1$, and that $C_3 \cup K_{1,n}$ an a-vertex consecutive edge bimagic labeling for $a = n+3$.

In 2005 Gutiérrez and Lladó [602] introduced the notion of an H-magic labeling of a graph, which generalizes the concept of a magic valuation. Let H and $G = (V,E)$ be finite simple graphs with the property that every edge of G belongs to at least one subgraph isomorphic to H. A bijection $f\colon V \cup E \to \{1, \ldots, |V|+|E|\}$ is an *H-magic labeling* of G if there exists a positive integer $m(f)$, called the *magic sum*, such that for any subgraph $H'(V', E')$ of G isomorphic to

H, the sum $\sum_{v\in V'} f(v) + \sum_{e\in E'} f(e)$ is equal to the magic sum, $m(f)$. A graph is *H-magic* if it admits an H-magic labeling. If, in addition, the H-magic labeling f has the property that $\{f(v)\}_{v\in V} = \{1,\ldots,|V|\}$, then the graph is *$H$-supermagic*. A K_2-magic labeling is also known as an edge-magic total labeling. Gutiérrez and Lladó investigate the cases where $G = K_n$ or $G = K_{m,n}$ and H is a star or a path. Among their results are: a d-regular graph is not $K_{1,h}$ for any $1 < h < d$; $K_{n,n}$ is $K_{1,n}$-magic for all n; $K_{n,n}$ is not $K_{1,n}$-supermagic for $n > 1$; for any integers $1 < r < s$, $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$; P_n is P_h-supermagic for all $2 \leq h \leq n$; K_n is not P_h-magic for any $2 < h \leq n$; C_n is P_h-magic for any $2 \leq h < n$ such that $\gcd(n, h(h-1)) = 1$. They also show that by uniformly gluing copies of H along edges of another graph G, one can construct connected H-magic graphs from a given 2-connected graph H and an H-free supermagic graph G.

Lladó and Moragas [985] studied cycle-magic graphs. They proved: wheels W_n are C_3-magic for odd n at least 5; for $r \geq 3$ and $k \geq 2$ the windmill graphs $C_r^{(k)}$ (the one-point union of k copies of C_r) are C_r-supermagic; and if G is C_4-free supermagic graph of odd size, then $G \times K_2$ is C_4-supermagic. As corollaries of the latter result, they have that for n odd, prisms $C_n \times K_2$ and books $K_{1,n} \times K_2$ are C_4-magic. They define a subdivided wheel $W_n(r,k)$ as the graph obtained from a wheel W_n by replacing each radial edge $vv_i, 1 \leq i \leq n$ by a vv_i-path of size $r \geq 1$, and every external edge v_iv_{i+1} by a v_iv_{i+1}-path of size $k \geq 1$. They prove that $W_n(r,k)$ is C_{2r+k}-magic for any odd $n \neq 2r/k + 1$ and that $W_n(r,1)$ is C_{2r+1}-supermagic. They also prove that the graph obtained by joining the end points of any number of internally disjoint paths of length $p \geq 2$ is C_{2p}-supermagic.

Liang [952] proved the following: if there exist an even integer k and $m_i \equiv 0 \pmod{k}$ for every i in $[1,n]$, then there exist $K_{k,k}$- and C_{2k}-supermagic decompositions of $K_{m_1,\ldots,m_n}$; if k and $t_n \geq k$ are even integers, then for any positive integers $t_i \equiv 0 \pmod{k}$, i in $[1, n-1]$, there exists a C_{2k}-supermagic decomposition of $K_{t_1,\ldots,t_{n-1},t_n}$; if there exists an even integer k and $K_{m,n}$ is C_{2k}-decomposable, then there exists a C_{2k}-supermagic decomposition of $K_{m,n}$; and if G is a graph with p vertices and p edges, H is a graph with q vertices and q edges, and there is an H-supermagic decomposition of G, then there exists an H-supermagic decomposition of nG.

In [1031] Maryati, Baskoro, and Salman provided P_h-(super) magic labelings of subdivisions of stars, shrubs and banana trees. Ngurah, Salman, and Sudarsana [1101] construct C_h-(super) magic labelings for some fans and ladders. For any connected graph H, Maryati, Salman, Baskoro, and Irawati [1033] proved that the disjoint union of k isomorphic copies of a connected graph H is a H-supermagic graph if and only if $|V(H)| + |E(H)|$ is even or k is odd.

[1034] Maryati, Salman, Baskoro, Ryan, and Miller define a *shackle* as a graph obtained from non-trivial connected graphs $G_1, G_2, \ldots, G_k$ ($k \geq 2$) such that G_s and G_t have no common vertex for every s and t in $[1,k]$ with $|s-t| \geq 2$, and for every i in $[1, k-1]$, G_i and G_{i+1} share exactly one common vertex that are all distinct. They prove that shackles and amalgamations constructed from copies of a connected graph H is H-supermagic. (Recall for finite collection of graph $G_1, G_2, \ldots, G_k$ with a fixed vertex v_i from each G_i, an *amalgamation*, AmalG_i, v_i), is the graph obtained by identifying the v_i.)

Ngurah, Salman, and Susilowati [1102] proved the following: chain graphs with identical blocks each isomorphic to C_n are C_n-supermagic; fans are C_3-supermagic; ladders and books are C_4-supermagic; $K_{1,n} + K_1$ are C_3-supermagic; grids $P_m \times P_n$ are C_4-supermagic for $m \geq 3$ and $n = 3, 4$, and 5. They pose the case that $P_m \times P_n$ are C_4-supermagic for $n > 5$ as an open problem. They also have some results on P_t-(super) magic labelings of cycles.

Roswitha, Baskoro, Maryati, Kurdhi, and Susanti [1214] proved: the generalized Jahangir graph $J_{k,s}$ is C_{s+2}-supermagic; $K_{2,n}$ is C_4-supermagic; and W_n for n even and $n \geq 4$ is C_3-supermagic. As an open problem they asked if $K_{m,n},\ 2 < m \leq n$, admits a C_{2m}-supermagic labeling. Roswitha and Baskoro [1215] proved that double stars, caterpillars, firecrackers, and banana trees admit star-supermagic labelings.

Maryati, Salman, and Baskoro [1032] characterized all graphs G such that the disjoint union of copies of G is G-supermagic. They also showed: the disjoint union of any paths is mP_n-supermagic for certain values of m and n; some subgraph amalgamations of graphs G are G-supermagic; and for any subgraph H of G Amal$(G,\ H,\ k)$ is G-supermagic. Salman and Maryati [1231] proved that Amal(G, P_n, k) is G-supermagic.

Selvagopal and Jeyanthi proved: for any positive integer n, a the k-polygonal snake of length n is C_k-supermagic [1251]; for $m \geq 2$, $n = 3$, or $n > 4$, $C_n \times P_m$ is C_4-supermagic [740]; $P_2 \times P_n$ and $P_3 \times P_n$ are C_4-supermagic for all $n \geq 2$ [740]; the one-point union of any number of copies of a 2-connected H is H-magic [738]; graphs obtained by taking copies $H_1, H_2, \ldots, H_n$ of a 2-connected graph H and two distinct edges e_i, e'_i from each H_i and identifying e'_i of H_i with e_{i+1} of H_{i+1} where $|V(H)| \geq 4, |E(H)| \geq 4$ and n is odd or both n and $|V(H)| + |E(H)|$ are even are H-supermagic [738]. For simple graphs H and G the *H-supermagic strength of G* is the minimum constant value of all H-magic total labelings of G for which the vertex labels are $\{1, 2, \ldots, |V|\}$. Jeyanthi and Selvagopal [739] found the C_n-supermagic strength of n-polygonal snakes of any length and the H-supermagic strength of a chain of an arbitrary 2-connected simple graph.

Let $H_1, H_2, \ldots, H_n$ be copies of a graph H. Let u_i and v_i be two distinct vertices of H_i for $i = 1, 2, \ldots, n$. The *chain graph* H_n of H of length n is the graph obtained by identifying the vertices u_i and v_{i+1} for $i = 1, 2, \ldots, n-1$. In [737] Jayanthi and Selvagopal show that a chain graph of any 2-connected simple graph H is H-supermagic and if H is a 2-connected (p, q) simple graph, then H_n is H-supermagic if $p + q$ is even or $p + q + n$ is even.

The *antiprism* on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ (subscripts are taken modulo n). Jeyanthi, Selvagopal, and Sundaram [742] proved the following graphs are C_3-supermagic: antiprisms, fans, and graphs obtained from the ladders $P_2 \times P_n$ with the two paths $v_{1,1}, \ldots, v_{1,n}$ and $v_{2,1}, \ldots, v_{2,n}$ by adding the edges $v_{1,j}v_{2,j+1}$.

Jeyanthi and Selvagopal [741] show that for any 2-connected simple graph H the edge amalogamation of a finite number of copies of H is H-supermagic. They also show that the graph obtained by picking one endpoint v_i from each of k copies of $K_{1,k}$ then creating a new graph by joining each v_i to a fixed new vertex v is $K_{1,k}$-supermagic.

Vishnupriya, Manimekalai, and Babujee [1600] define a labeling f of a graph $G(p, q)$ to be a *edge bimagic total labeling* if there exists a bijection f from $V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that for each edge $e = (u, v) \in E(G)$ we have $f(u) + f(e) + f(v) = k_1$ or k_2, where k_1 and k_2 are two constants. They provide edge bimagic total labelings for $B_{m,n}$, $K_{1,n,n}$, and trees obtained from a path by appending an edge to one of the vertices adjacent to an endpoint of the path. An edge bimagic total labeling is $G(V, E)$ is called an *a-vertex consecutive edge bimagic total labeling* if the vertex labels are $\{a+1, a+2, \ldots, a+|V|\}$ where $0 \leq a \leq |E|$. Babujee and Jagadesh [119] prove the following graphs a-vertex consecutive edge-bimagic total labelings: the trees obtained from $K_{1,n}$ by adding a new pendent edge to each of the existing n pendent vertices; the trees obtained by adding a pendent path of length 2 to each of the n pendent vertices of $K_{1,n}$; the graphs obtained by joining the centers of two copies of identical stars by a path of length 2; and the trees obtained from a path by adding new pendent edges to one pendent vertex of the path. Babujee, Vishnupriya, and Jagadesh [131] proved the following graphs have such labelings: $P_n,\ K_{1,n}$, combs, bistars $B_{m,n}$, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from P_5 by identifying the center of a copy $K_{1,n}$ with the two end vertices and the middle vertex. In [121] Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have a-vertex consecutive edge bimagic total labelings. Babujee, Jagadesh, Vishnupriya [123] study the properties of a-vertex consecutive edge bimagic total labeling for $P_3 \odot K_{1,2n},\ P_n + \overline{K_2}$ (n is odd and $n \geq 3$), $(P_2 \cup mK_1) + \overline{K_2},\ (P_2 + mK_1)\ (m \geq 2), C_n$, fans $P_n + K_1$, double fans $P_n + 2K_1$, and graphs obtained by appending a path of length at least 2 to a vertex of C_3. J. Babujee, R. Jagadesh [122] prove the following graphs have a-vertex consecutive edge bimagic total labelings: $2P_n\ (n \geq 2)$, $P_n \cup P_n + 1 (n \geq 2)$, $K_{2,n},\ C_n \odot K_1$, and that $C_3 \cup K_{1,n}$ an a-vertex consecutive edge bimagic labeling for $a = n + 3$ Vishnupriya, Manimekalai, and Babujee [1600] prove that bistars, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, and trees obtained subdividing each edge of a star have edge bimagic total labelings. Prathap and Babujee [1166] obtain all possible edge magic total labelings and edge bimagic total labelings for the star $K_{1,n}$. Magic labelings of directed graphs are discussed in [1027] and [290].

6 Antimagic-type Labelings

6.1 Antimagic Labelings

Hartsfield and Ringel [616] introduced antimagic graphs in 1990. A graph with q edges is called *antimagic* if its edges can be labeled with $1, 2, \ldots, q$ such that the sums of the labels of the edges incident to each vertex are distinct. Among the graphs they prove are antimagic are: P_n ($n \geq 3$), cycles, wheels, and K_n ($n \geq 3$). T. Wang [1616] has shown that the toroidal grids $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$ are antimagic and, more generally, graphs of the form $G \times C_n$ are antimagic if G is an r-regular antimagic graph with $r > 1$. Cheng [380] proved that all Cartesian products or two or more regular graphs of positive degree are antimagic and that if G is j-regular and H has maximum degree at most k, minimum degree at least one (G and H need not be connected), then $G \times H$ is antimagic provided that j is odd and $j^2 - j \geq 2k$, or j is even and $j^2 > 2k$. Wang and Hsiao [1617] prove the following graphs are antimagic: $G \times P_n$ ($n > 1$) where G is regular; $G \times K_{1,n}$ where G is regular; compositions $G[H]$ (see §2.3 for the definition) where H is d-regular with $d > 1$; and the Cartesian product of any double star (two stars with an edge joining their centers) and a regular graph. In [379] Cheng proved that $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_t}$ ($t \geq 2$) is antimagic. In [1411] Solairaju and Arockiasamy prove that various families of subgraphs of grids $P_m \times P_n$ are antimagic. Liang and Zhu [948] proved that if G is k-regular ($k \geq 2$), then for any graph H with $|E(H)| \geq |V(H)| - 1 \geq 1$, the Cartesian product $H \times G$ is anti-magic. They also showed that if $|E(H)| \geq |V(H)| - 1$ and each connected component of H has a vertex of odd degree, or H has at least $2|V(H)| - 2$ edges, then the prism of H is anti-magic. Lee, Lin and Tsai [860] proved that C_n^2 is antimagic and the vertex sums form a set of successive integers when n is odd.

For a graph G and a vertex v of G, the *vertex switching* graph G_v is the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Vaidya and Vyas [1564] proved that the graphs obtained by the switching of a pendent vertex of a path, a vertex of a cycle, a rim vertex of a wheel, the center vertex of a helm, or a vertex of degree 2 of a fan are antimagic graphs.

Phanalasy, Miller, Rylands and Lieby [1134] in 2011 showed that there is a relationship between completely separating systems and labeling of regular graphs. Based on this relationship they proved that some regular graphs are antimagic. Phanalasy, Miller, Iliopoulos, Pissis and Vaezpour [1132] proved the Cartesian product of regular graphs obtained from [1134] is antimagic. Ryan, Phanalasy, Miller and Rylands introduced the generalized web and flower graphs in [1217] and proved that these families of graphs are antimagic. Rylands, Phanalasy, Ryan and Miller extended the concept of generalized web graphs to the single apex multi-generalized web graphs and they proved these graphs to be antimagic in [1220]. Ryan, Phanalasy, Rylands and Miller extended the concept of generalized flower to the single apex multi-(complete) generalized flower graphs and constructed antimagic labeling for this family of graphs in [1218]. For more about antimagicness of generalized web and flower graphs see [1055]. Phanalasy, Ryan, Miller and Arumugam [1133] introduced the concept of generalized pyramid graphs and they constructed antimagic labeling for these graphs. Bača, Miller, Phanalasy and Feňovčíková proved that some join graphs and incomplete join graphs are antimagic in [193].

A *split* graph is a graph that has a vertex set that can be partitioned into a clique and an independent set. Tyshkevich (see [241]) defines a canonically decomposable graph as follows. For a split graph S with a given partition of its vertex set into an independent set A and a clique B (denoted by $S(A, B)$), and an arbitrary graph H the composition $S(A, B) \circ H$ is the graph obtained by taking the disjoint union of $S(A, B)$ and H and adding to it all edges having an endpoint in each of B and $V(H)$. If G contains nonempty induced subgraphs H and S and vertex subsets A and B such that $G = S(A, B) \circ H$, then G is *canonically decomposable*; otherwise G is *canonically indecomposable*. Barrus [241] proved that every connected graph on at least 3 vertices that is split or canonically decomposable is antimagic.

Hartsfield and Ringel [616] conjecture that every tree except P_2 is antimagic and, moreover, every connected graph except P_2 is antimagic. Alon, Kaplan, Lev, Roditty, and Yuster [78] use probabilistic methods and analytic number theory to show that this conjecture is true for all graphs with n vertices and minimum degree $\Omega(\log n)$. They also prove that if G is a graph with $n \geq 4$ vertices and $\Delta(G) \geq n-2$, then G is antimagic and all complete partite graphs except K_2 are antimagic. Slíva Sl12 proved the conjecture for graphs with a regular dominating subgraph. Chawathe and Krishna [367] proved that every complete m-ary tree is antimagic. Yilma [1692] extended results on antimagic graphs that contain vertices of large degree by proving that a connected graph with $\Delta(G) \geq |V(G)|-3$, $|V(G)| \geq 9$ is antimagic and that if G is a graph with $\Delta(G) = \deg(u) = |V(G)| - k$, where $k \leq |V(G)|/3$ and there exists a vertex v in G such that the union of neighborhoods of the vertices u and v forms the whole vertex set $V(G)$, then G is antimagic.

In [408] Cranston used the Marriage Theorem to prove that every regular bipartite graph with degree at least 2 is anitmagic and in [409] he proved that regular graphs of odd degree are antimagic. Beck and Jackanich [251] showed that every connected bipartite graph except P_2 with $|E|$ edges admits an edge labeling with labels from $\{1, 2, \ldots, |E|\}$, with repetition allowed, such that the sums of the labels of the edges incident to each vertex are distinct. They call such a graph *weak antimagic*.

Wang, Liu, and Li [1624] proved: mP_3 $(m \geq 2)$ is not antimagic; $P_n \cup P_n$ $(n \geq 4)$ is antimagic; $S_n \cup P_n$ is antimagic; $S_n \cup P_{n+1}$ is antimagic; $C_n \cup S_m$ is antimagic for $m \geq 2\sqrt{n} + 2$; mS_n is antimagic; if G and H are graphs of the same order and $G \cup H$ is antimagic, then so is $G + H$; and if G and H are r-regular graphs of even order, then $G + H$ is antimagic. In [1625] Wang, Liu, and Li proved that if G is an n-vertex graph with minumum degree at least r and H is an m-vertex graph with miximum degree at most $2r - 1$ $(m \geq n)$, then $G + H$ is antimagic.

For any given degree sequence pertaining to a tree, Miller, Phanalasy, Ryan, and Rylands [1056] gave a construction for two vertex antimagic edge trees with the given degree sequence and provided a construction to obtain an antimagic unicyclic graph with a given degree sequence pertaining to a unicyclic graph.

Kaplan, Lev, and Roditty [767] prove that every non-trivial rooted tree for which every vertex that is a not a leaf has at least two children is antimagic. For a graph H with m vertices and an Abelian group G they define H to be *G-antimagic* if there is a one-to-one mapping from the edges of H to the nonzero elements of G such that the sums of the labels of the edges incident to v, taken over all vertices v of H, are distinct. For any $n \geq 2$ they show that a non-trivial rooted tree with n vertices for which every vertex that is a not a leaf has at least two children is Z_n-antimagic if and only if n is odd. They also show that these same trees are G-antimagic for elementary Abelian groups G with prime exponent congruent to 1 (mod 3).

In [1565] Vaidya and Vyas proved that the middle graphs, total graphs, and shadow graphs of paths and cycles are antimagic. Krishnaa [831] provided some results for antimagic labelings for graphs derived from wheels.

Bertault, Miller, Pé-Rosés, Feria-Puron, and Vaezpour [267] approached labeling problems as combinatorial optimization problems. They developed a general algorithm to determine whether a graph has a magic labeling, antimagic labeling, or an (a, d)-antimagic labeling. They verified that all trees with fewer than 10 vertices are super edge magic and all graphs of the form $P_2^r \times P_3^s$ with less than 50 vertices are antimagic. In [186] Bača, MacDougall, Miller, Slamin, and Wallis survey results on antimagic, edge-magic total, and vertex-magic total labelings.

In [625] Hefetz, Mütze, and Schwartz investigate antimagic labelings of directed graphs. An *antimagic* labeling of a directed graph D with n vertices and m arcs is a bijection from the set of arcs of D to the integers $\{1, \ldots, m\}$ such that all n oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of all edges entering that vertex minus the sum of labels of all edges leaving it. Hefetz et al. raise the questions "Is every orientation of any simple connected undirected graph antimagic? and "Given any undirected graph G, does there exist an orientation of G which is antimagic?" They call such an orientation an *antimagic orientation* of G. Regarding the first question, they state that, except for $K_{1,2}$ and K_3, they know of no other counterexamples. They prove that there exists an absolute constant C such that for every undirected graph on n vertices with minimum degree at least $C\log n$ every orientation is antimagic. They also show that every orientation of $S_n,\ n \neq 2$, is antimagic; every orientation of W_n is antimagic; and every orientation of $K_n,\ n \neq 3$, is antimagic. For the second question they prove: for odd r, every undirected r-regular graph has an antimagic orientation; for even r every undirected r-regular graph that admits a matching that covers all but at most one vertex has an antimagic orientation; and if G is a graph with $2n$ vertices that admits a perfect matching and has an independent set of size n such that every vertex in the independent set has degree at least 3, then G has an antimagic orientation. They conjecture that every connected undirected graph admits an antimagic orientation and ask if it true that every connected directed graph with at least 4 vertices is antimagic. Sonntag [1424] investagated antimagic labelings of hypergraphs. He shows that certain classes of cacti, cycle, and wheel hypergraphs have antimagic labelings. Javaid and Bhatti [703] extended some of Sonntag's results to disjoint unions of hypergraphs.

Hefetz [624] calls a graph with q edges k-*antimagic* if its edges can be labeled with $1, 2, \ldots, q+k$ such that the sums of the labels of the edges incident to each vertex are distinct. In particular, antimagic is the same as 0-antimagic. More generally, given a weight function ω from the vertices to the natural numbers Hefetz calls a graph with q edges (ω, k)-*antimagic* if its edges can be labeled with $1, 2, \ldots, q + k$ such that the sums of the labels of the edges incident to each vertex and the weight assigned to each vertex by ω are distinct. In particular, antimagic is the same as $(\omega, 0)$-antimagic where ω is the zero function. Using Alon's combinatorial nullstellensatz [77] as his main tool, Hefetz has proved the following: a graph with 3^m vertices and a K_3 factor is antimagic; a graph with q edges and at most one isolated vertex and no isolated edges is $(\omega, 2q - 4)$-antimagic; a graph with $p > 2$ vertices that admits a 1-factor is $(p - 2)$-antimagic; a graph with p vertices and maximum degree $n - k$, where $k \geq 3$ is any function of p is $(3k - 7)$-antimagic and, in the case that $p \geq 6k^2$, is $(k - 1)$-antimagic. Hefetz, Saluz, and Tran [626] improved the first of Hefetz's results by showing that a graph with p^m vertices, where p is an odd prime and m is positive, and a C_p factor is antimagic.

Bača, Baskoro, Jendroľ, and Miller [157] investigated various k-antimagic labelings for graphs in the shape of hexagonal honeycombs. They use H_n^m to denote the honeycomb graph with m rows, n columns, and mn 6-sided faces. They prove: for n odd H_n^m, has a 0-antimagic vertex labeling and a 2-antimagic edge labeling, and if n is odd and $mn > 1$, H_n^m has a 1-antimagic face labeling.

Huang, Wong, and Zhu [673] say a graph G is *weighted-k-antimagic* if for any vertex weight function w from the vertices of G to the natural numbers there is an injection f from the edges

of G to $\{1, 2, \ldots, |E| + k\}$ such that for any two distinct vertices u and v, $\sum(f(e) + w(v)) \neq \sum(f(e) + w(u))$ over all edges incidence to v. They proved that if G has odd prime power order p^z and has total domination number 2 with the degree of one vertex in the total dominating set not a multiple of p, then G is weighted-1-antimagic, and if G has odd prime power order p^z, $p \neq 3$ and has maximum degree at least $|V(G)| - 3$, then G is weighted-1-antimagic.
Wong and Zhu [674] proved: graphs that have a vertex that is adjacent to all other vertices are weighted-2-antimagic; graphs with a prime number of vertices that have a Hamiltonian path are weighted-1-antimagic; and connected graphs $G \neq K_2$ on n vertices are weighted-$\lfloor 3n/2 \rfloor$-antimagic.

In [104] Arumugam and Kamatchi introduced the notion of (a, d)-distance antimagic graphs as follows. Let G be a graph with vertex set V and $f : V \to \{1, 2, \ldots, |V|\}$ be a bijection. If for all v in G the set of sums $\sum f(u)$ taken over all neighbors u of v is the arithmetic progression $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$, f is called an (a, d)-*distance antimagic* labeling and G is called a (a, d)-*distance antimagic graph*. Arumugam and Kamatchi [104] proved: C_n is (a, d)-distance antimagic if and only if n is odd and $d = 1$; there is no $(1, d)$-distance antimagic labeling for P_n when $n \geq 3$; a graph G is $(1, d)$-distance antimagic graph if and only if every component of G is K_2; $C_n \times K_2$ is $(n+2, 1)$-distance antimagic; and the graph obtained from $C_{2n} = (v_1, v_2, \ldots, v_{2n})$ by adding the edges $v_1 v_{n+1}$ and $v_i v_{2n+2-i}$ for $i = 2, 3, \ldots, n$ is $(2n + 2, 1)$-distance antimagic.

In Table 12 we use the abbreviations **A** to mean antimagic. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 12: **Summary of Antimagic Labelings**

Graph	*Labeling*	*Notes*
P_n	A	for $n \geq 3$ [616]
C_n	A	[616]
W_n	A	[616]
K_n	A	for $n \geq 3$ [616]
every tree except K_2	A?	[616]
every connected graph except K_2	A?	[616]
$n \geq 4$ vertices $\Delta(G) \geq n - 2$	A	[78]
all complete partite graphs except K_2	A	[78]

6.2 (a, d)-Antimagic Labelings

The concept of an (a, d)-antimagic labelings was introduced by Bodendiek and Walther [293] in 1993. A connected graph $G = (V, E)$ is said to be (a, d)-*antimagic* if there exist positive integers a, d and a bijection $f\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $g_f\colon V \to N$, defined by $g_f(v) = \sum\{f(uv)|\ uv \in E(G)\}$, is injective and $g_f(V) = \{a, a + d, \ldots, a + (|V| - 1)d\}$. (In [963] Lin, Miller, Simanjuntak, and Slamim called these (a, d)-*vertex-antimagic edge labelings*). Bodendick and Walther ([295] and [296]) prove the Herschel graph is not (a, d)-antimagic and obtain both positive and negative results about (a, d)-antimagic labelings for various cases of graphs called *parachutes* $P_{g,p}$. ($P_{g,p}$ is the graph obtained from the wheel W_{g+p} by deleting p consecutive spokes.) In [171] Bača and Holländer prove that necessary conditions for $C_n \times P_2$ to be (a, d)-antimagic are $d = 1,\ a = (7n + 4)/2$ or $d = 3,\ a = (3n + 6)/2$ when n is even, and $d = 2,\ a = (5n + 5)/2$ or $d = 4,\ a = (n + 7)/2$ when n is odd. Bodendiek and Walther [294] conjectured that $C_n \times P_2$ $(n \geq 3)$ is $((7n+4)/2, 1)$-antimagic when n is even and is $((5n+5)/2, 2)$-antimagic when n is odd. These conjectures were verified by Bača and Holländer [171] who further proved that $C_n \times P_2$ $(n \geq 3)$ is $((3n + 6)/2, 3)$-antimagic when n is even. Bača and Holländer [171] conjecture that $C_n \times P_2$ is $((n+7)/2, 4)$-antimagic when n is odd and at least 7. Bodendiek and Walther [294] also conjectured that $C_n \times P_2$ $(n \geq 7)$ is $((n + 7)/2, 4)$-antimagic. Miller and Bača [1051] prove that the generalized Petersen graph $P(n, 2)$ is $((3n + 6)/2, 3)$-antimagic for $n \equiv 0 \pmod 4$, $n \geq 8$ and conjectured that $P(n, k)$ is $((3n + 6)/2, 3)$-antimagic for even n and $2 \leq k \leq n/2 - 1$ (see §2.7 for the definition of $P(n, k)$). This conjecture was proved for $k = 3$ by Xu, Yang, Xi, and Li [1670]. Jirimutu and Wang proved that $P(n, 2)$ is $((5n + 5)/2, 2)$-antimagic for $n \equiv 3 \pmod 4$ and $n \geq 7$. Xu, Xu, Lü, Baosheng, and Nan [1665] proved that $P(n, 2)$ is $((3n + 6)/2, 2)$-antimagic for $n \equiv 2 \pmod 4$ and $n \geq 10$. Xu, Yang, Xi, and Li [1668] proved that $P(n, 3)$ is $((3n + 6)/2, 3)$-antimagic for even $n \geq 10$.

Bodendiek and Walther [297] proved that the following graphs are not (a, d)-antimagic: even cycles; paths of even order; stars; $C_3^{(k)}$; $C_4^{(k)}$; trees of odd order at least 5 that have a vertex that is adjacent to three or more end vertices; n-ary trees with at least two layers when $d = 1$; the Petersen graph; K_4 and $K_{3,3}$. They also prove: P_{2k+1} is $(k, 1)$-antimagic; C_{2k+1} is $(k + 2, 1)$-antimagic; if a tree of odd order $2k + 1$ $(k > 1)$ is (a, d)-antimagic, then $d = 1$ and $a = k$; if K_{4k} $(k \geq 2)$ is (a, d)-antimagic, then d is odd and $d \leq 2k(4k - 3) + 1$; if K_{4k+2} is (a, d)-antimagic, then d is even and $d \leq (2k + 1)(4k - 1) + 1$; and if K_{2k+1} $(k \geq 2)$ is (a, d)-antimagic, then $d \leq (2k + 1)(k - 1)$. Lin, Miller, Simanjuntak, and Slamin [963] show that no wheel W_n $(n > 3)$ has an (a, d)-antimagic labeling.

In [698] Ivančo, and Semaničová show that a 2-regular graph is super edge-magic if and only if it is $(a, 1)$-antimagic. As a corollary we have that each of the following graphs are $(a, 1)$-antimagic: kC_n for n odd and at least 3; $k(C_3 \cup C_n)$ for n even and at least 6; $k(C_4 \cup C_n)$ for n odd and at least 5; $k(C_5 \cup C_n)$ for n even and at least 4; $k(C_m \cup C_n)$ for m even and at least 6, n odd, and $n \geq m/2 + 2$. Extending a idea of Kovář they prove if G is $(a_1, 1)$-antimagic and H is obtained from G by adding an arbitrary $2k$-factor then H is $(a_2, 1)$-antimagic for some a_2. As corollaries they observe that the following graphs are $(a, 1)$-antimagic: circulant graphs of odd order; $2r$-regular Hamiltonian graphs of odd order; and $2r$-regular graphs of odd order $n < 4r$. They further show that if G is an $(a, 1)$-antimagic r-regular graph of order n and $n - r - 1$ is a divisor of the non-negative integer $a + n(1 + r - (n + 1)/2)$, then $G \oplus K_1$ is supermagic. As a corollary of this result they have if G is $(n - 3)$-regular for n odd and $n \geq 7$ or $(n - 7)$-regular for n odd and $n \geq 15$, then $G \oplus K_1$ is supermagic.

Bertault, Miller, Feria-Purón, and Vaezpour [267] approached labeling problems as combinatorial optimization problems. They developed a general algorithm to determine whether a graph has a magic labeling, antimagic labeling, or an (a,d)-antimagic labeling. They verified that all trees with fewer than 10 vertices are super edge magic and all graphs of the form $P_2^r \times P_3^s$ with less than 50 vertices are antimagic. Javaid, Hussain, Ali, and Dar [706] and Javaid, Bhatti, and Hussain [704] constructed super (a,d)-edge-antimagic total labelings for w-trees and extended w-trees (see 5.2 for the definitions) as well as super (a,d)-edge-antimagic total labelings for disjoint union of isomorphic and non-isomorphic copies of extended w-trees.

Yegnanarayanan [1690] introduced several variations of antimagic labelings and provides some results about them.

The *antiprism* on $2n$ vertices has vertex set $\{x_{1,1},\ldots,x_{1,n},x_{2,1},\ldots,x_{2,n}\}$ and edge set $\{x_{j,i},x_{j,i+1}\} \cup \{x_{1,i},x_{2,i}\} \cup \{x_{1,i},x_{2,i-1}\}$ (subscripts are taken modulo n). For $n \geq 3$ and $n \not\equiv 2 \pmod 4$ Bača [146] gives $(6n+3,2)$-antimagic labelings and $(4n+4,4)$-antimagic labelings for the antiprism on $2n$ vertices. He conjectures that for $n \equiv 2 \pmod 4$, $n \geq 6$, the antiprism on $2n$ vertices has a $(6n+3,2)$-antimagic labeling and a $(4n+4,4)$-antimagic labeling.

Nicholas, Somasundaram, and Vilfred [1106] prove the following: If $K_{m,n}$ where $m \leq n$ is (a,d)-antimagic, then d divides $((m-n)(2a+d(m+n-1)))/4+dmn/2$; if $m+n$ is prime, then $K_{m,n}$, where $n>m>1$, is not (a,d)-antimagic; if $K_{n,n+2}$ is (a,d)-antimagic, then d is even and $n+1 \leq d < (n+1)^2/2$; if $K_{n,n+2}$ is (a,d)-antimagic and n is odd, then a is even and d divides a; if $K_{n,n+2}$ is (a,d)-antimagic and n is even, then d divides $2a$; if $K_{n,n}$ is (a,d)-antimagic, then n and d are even and $0<d<n^2/2$; if G has order n and is unicylic and (a,d)-antimagic, then $(a,d)=(2,2)$ when n is even and $(a,d)=(2,2)$ or $(a,d)=((n+3)/2,1)$ when n is odd; a cycle with m pendent edges attached at each vertex is (a,d)-antimagic if and only if $m=1$; the graph obtained by joining an endpoint of P_m with one vertex of the cycle C_n is $(2,2)$-antimagic if $m=n$ or $m=n-1$; if $m+n$ is even the graph obtained by joining an endpoint of P_m with one vertex of the cycle C_n is (a,d)-antimagic if and only if $m=n$ or $m=n-1$. They conjecture that for n odd and at least 3, $K_{n,n+2}$ is $((n+1)(n^2-1)/2,n+1)$-antimagic and they have obtained several results about (a,d)-antimagic labelings of caterpillars.

In [1590] Vilfred and Florida proved the following: the one-sided infinite path is $(1,2)$-antimagic; P_{2n} is not (a,d)-antimagic for any a and d; P_{2n+1} is (a,d)-antimagic if and only if $(a,d)=(n,1)$; C_{2n+1} has an $(n+2,1)$-antimagic labeling; and that a 2-regular graph G is (a,d)-antimagic if and only if $|V(G)|=2n+1$ and $(a,d)=(n+2,1)$. They also prove that for a graph with an (a,d)-antimagic labeling, q edges, minimum degree δ and maximum degree Δ, the vertex labels lie between $\delta(\delta+1)/2$ and $\Delta(2q-\Delta+1)/2$.

Chelvam, Rilwan, and Kalaimurugan [368] proved that Cayley digraph of any finite group admits a super vertex (a,d)-antimagic labeling depending on d and the size of the generating set. They provide algorithms for constructing the labelings.

For $n>1$ and distinct odd integers x,y and z in $[1,n-1]$ Javaid, Ismail, and Salman [701] define the *chordal ring* of order n, $CR_n(x,y,z)$, as the graph with vertex set Z_n, the additive group of integers modulo n, and edges $(i,i+x),(i,i+y),(i,i+z)$ for all even i. They prove that $CR_n(1,3,7)$ and $CR_n(1,5,n-1)$ have (a,d)-antimagic labelings when $n \equiv 0 \bmod 4$ and conjecture that for an odd integer Δ, $3 \leq \Delta \leq n-3, n \equiv 0 \bmod 4$, $CR_n((1,\Delta,n-1)$ has an $((7n+8)/4,1)$-antimagic labeling.

In [1591] Vilfred and Florida call a graph $G=(V,E)$ *odd antimagic* if there exist a bijection $f\colon E \to \{1,3,5,\ldots,2|E|-1\}$ such that the induced mapping $g_f\colon V \to N$, defined by $g_f(v) = \sum\{f(uv)|\ uv \in E(G)\}$, is injective and *odd (a,d)-antimagic* if there exist positive integers a, d

and a bijection $f\colon E \to \{1,3,5,\ldots,2|E|-1\}$ such that the induced mapping $g_f\colon V \to N$, defined by $g_f(v) = \sum\{f(uv)|\ uv \in E(G)\}$, is injective and $g_f(V) = \{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. Although every (a,d)-antimagic graph is antimagic, C_4 has an antimagic labeling but does not have an (a,d)-antimagic labeling. They prove: P_{2n+1} is not odd (a,d)-antimagic for any a and d; C_{2n+1} has an odd $(2n+2,2)$-antimagic labeling; if a 2-regular graph G has an odd (a,d)-antimagic labeling, then $|V(G)| = 2n+1$ and $(a,d) = (2n+2,2)$; C_{2n} is odd magic; and an odd magic graph with at least three vertices, minimum degree δ, maximum degree Δ, and $q \geq 2$ edges has all its vertex labels between δ^2 and $\Delta(2q-\Delta)$.

In Table 13 we use the abbreviations (a,d)-**A** to mean that the graph has an (a,d)-antimagic labeling. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 13: **Summary of (a, d)-Antimagic Labelings**

Graph	*Labeling*	*Notes*
C_{2n+1}	not $(n+2,1)$-A	n even [297]
P_{2n}	not (a,d)-A	[297]
P_{2n+1}	$(n,1)$-A	[297]
stars	not (a,d)-A	[297]
$C_3^{(k)}, C_4^{(k)}$	not (a,d)-A	[297]
$K_{n,n+2}$	$\left(\frac{(n+1)(n^2-1)}{2}, n+1\right)$-A	$n \geq 3$, n odd [297]
$K_{3,3}$	not (a,d)-A	[297]
K_4	not (a,d)-A	[297]
Petersen graph	not (a,d)-A	[297]
W_n	not (a,d)-A	$n > 3$ [963]
antiprism on $2n$ vertices (see §6.2)	$(6n+3,2)$-A	$n \geq 3$, $n \not\equiv 2 \pmod 4$ [146]
	$(4n+4,4)$-A	$n \geq 3$, $n \not\equiv 2 \pmod 4$ [146]
	$(2n+5,6)$-A?	$n \geq 4$ [146]
	$(6n+3,2)$-A?	$n \geq 6$, $n \not\equiv 2 \pmod 4$ [146]
	$(4n+4,4)$-A?	$n \geq 6$, $n \not\equiv 2 \pmod 4$ [146]
Hershel graph (see [363])	not (a,d)-A	[293], [295]
parachutes $P_{g,p}$ (see §6.2)	(a,d)-A	for certain classes [293], [295]
C_n	not (a,d)-A	n even [297]
prisms $C_n \times P_2$	$((7n+4)/2,1)$-A	$n \geq 3$, n even [294], [171]
	$((5n+5)/2,2)$-A	$n \geq 3$, n odd [294], [171]
	$((3n+6)/2,3)$-A	$n \geq 3$, n even [171]
	$((n+7)/2,4)$-A?	$n \geq 7$, [295], [171]
generalized Petersen graph $P(n,2)$	$((3n+6)/2,3)$-A	$n \geq 8$, $n \equiv 0 \pmod 4$ [172]

6.3 (a,d)-Antimagic Total Labelings

Bača, Bertault, MacDougall, Miller, Simanjuntak, and Slamin [162] introduced the notion of a (a,d)-vertex-antimagic total labeling in 2000. For a graph $G(V,E)$, an injective mapping f from $V \cup E$ to the set $\{1,2,\ldots,|V|+|E|\}$ is a (a,d)-*vertex-antimagic total labeling* if the set $\{f(v)+\sum f(vu)\}$ where the sum is over all vertices u adjacent to v for all v in G is $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. In the case where the vertex labels are 1,2, …, $|V|$, (a,d)-vertex-antimagic total labeling is called a *super* (a,d)-*vertex-antimagic total labeling.* Among their results are: every super-magic graph has an $(a,1)$-vertex-antimagic total labeling; every (a,d)-antimagic graph $G(V,E)$ is $(a+|E|+1,d+1)$-vertex-antimagic total; and, for $d>1$, every (a,d)-antimagic graph $G(V,E)$ is $(a+|V|+|E|,d-1)$-vertex-antimagic total. They also show that paths and cycles have (a,d)-vertex-antimagic total labelings for a wide variety of a and d. In [163] Bača et al. use their results in [162] to obtain numerous (a,d)-vertex-antimagic total labelings for prisms, and generalized Petersen graphs (see §2.7 for the definition). (See also [174] and [1448] for more results on generalized Petersen graphs.)

Sugeng, Miller, Lin, and Bača [1448] prove: C_n has a super (a,d)-vertex-antimagic total labeling if and only if $d=0$ or 2 and n is odd, or $d=1$; P_n has a super (a,d)-vertex-antimagic total labeling if and only if $d=2$ and $n\geq 3$ is odd, or $d=3$ and $n\geq 3$; no even order tree has a super $(a,1)$-vertex antimagic total labeling; no cycle with at least one tail and an even number of vertices has a super $(a,1)$-vertex-antimagic labeling; and the star $S_n,\ n\geq 3$, has no super (a,d)-super antimagic labeling. As open problems they ask whether $K_{n,n}$ has a super (a,d)-vertex-antimagic total labeling and the generalized Petersen graph has a super (a,d)-vertex-antimagic total labeling for specific values a,d, and n. Lin, Miller, Simanjuntak, and Slamin [963] have shown that for $n>20$, W_n has no (a,d)-vertex-antimagic total labeling. Tezer and Cahit [1489] proved that neither P_n nor C_n has (a,d)-vertex-antimagic total labelings for $a\geq 3$ and $d\geq 6$. Kovář [824] has shown that every $2r$-regular graph with n vertices has an $(s,1)$-vertex antimagic total labeling for $s\in\{(rn+1)(r+1)+tn \mid t=0,1,\ldots,r\}$.

Several papers have been written about vertex-antimagic total labeling of graphs that are the disjoint union of suns. The sun graph S_n is $C_n\odot K_1$. Rahim and Sugeng [1176] proved that $S_{n_1}\cup S_{n_2}\cup\cdots\cup S_{n_t}$ is $(a,0)$-vertex-antimagic total (or vertex magic total); Parestu, Silaban, and Sugeng [1119] and [1120] proved $S_{n_1}\cup S_{n_2}\cup\cdots\cup S_{n_t}$ is (a,d)-vertex-antimagic total for $d=1,2,3,4$, and 6 and particular values of a. In [1174] Rahim, Ali, Kashif, and Javaid provide (a,d)-vertex antimagic total labelings of disjoint unions of cycles, sun graphs, and disjoint unions of sun graphs.

In [1096] Ngurah, Baskova, and Simanjuntak provide (a,d)-vertex-antimagic total labelings for the generalized Petersen graphs $P(n,m)$ for the cases: $n\geq 3,\ 1\leq m\leq\lfloor(n-1)/2\rfloor,\ (a,d)=(8n+3,2)$; odd $n\geq 5,\ m=2,\ (a,d)=((15n+5)/2,1)$; odd $n\geq 5,\ m=2,\ (a,d)=((21n+5)/2,1)$; odd $n\geq 7,\ m=3,\ (a,d)=((15n+5)/2,1)$; odd $n\geq 7,\ m=3,\ (a,d)=((21n+5)/2,1)$; odd $n\geq 9,\ m=4,\ (a,d)=((15n+5)/2,1)$; and $(a,d)=((21n+5)/2,1)$. They conjecture that for n odd and $1\leq m\leq\lfloor(m-1)/2\rfloor$, $P(n,m)$ has an $((21n+5)/2,1)$-vertex-antimagic labeling. In [1453] Sugeng and Silaban show: the disjoint union of any number of odd cycles of orders $n_1,n_2,\ldots,n_t$, each at least 5, has a super $(3(n_1+n_2+\cdots+n_t)+2,1)$-vertex-antimagic total labeling; for any odd positive integer t, the disjoint union of t copies of the generalized Petersen graph $P(n,1)$ has a super $(10t+2)n-\lfloor n/2\rfloor+2,1)$-vertex-antimagic total labeling; and for any odd positive integers t and n $(n\geq 3)$, the disjoint union of t copies of the generalized Petersen graph $P(n,2)$ has a super $(21tn+5)/2,1)$-vertex-antimagic total labeling.

Ail, Bača, Lin, and Semaničová-Feňovčiková [70] investigated super-(a,d)-vertex antimagic total labelings of disjoint unions of regular graphs. Among their results are: if m and $(m-1)(r+1)/2$ are positive integers and G is an r-regular graph that admits a super-vertex magic total labeling, then mG has a super-$(a,2)$-vertex antimagic total labeling; if G has a 2-regular super-$(a,1)$-vertex antimagic total labeling, then mG has a super-$(m(a-2)+2,1),1)$-vertex antimagic total labeling; mC_n has a super-(a,d)-vertex antimagic total labeling if and only if either d is 0 or 2 and m and n are odd and at least 3 or $d=1$ and $n \geq 3$; and if G is an even regular Hamilton graph, then mG has a super-$(a,1)$-vertex antimagic total labeling for all positive integers m.

Sugeng and Bong [1443] show how to construct super (a,d)-vertex antimagic total labelings for the circulant graphs $C_n(1,2,3)$, for $d = 0,1,2,3,4,8$. Thirusangu, Nagar, and Rajeswari [1490] show that certain Cayley digraphs of dihedral groups have (a,d)-vertex-magic total labelings.

For a simple graph H we say that $G(V,E)$ admits an *H-covering*, if every edge in $E(G)$ belongs to a subgraph of G that is isomorphic to H. Inayah, Salman, and Simanjuntak [687] define an *(a,d)-H-antimagic total labeling* of G as a bijective function ξ from $V \cup E \to \{1,2,\ldots,|V|+|E|\}$ such that for all subgraphs H' isomorphic to H, the H-weights $w(H') = \sum_{v\in V(H')}\xi(v)+\sum_{e\in E(H')}\xi(e)$ constitute an arithmetic progression $a, a+d, a+2d, \ldots, a+(t-1)d$ where a and d are positive integers and t is the number of subgraphs of G isomorphic to H. Such a labeling ξ is called a *super (a,d)-H-antimagic total labeling*, if $\xi(V) = \{1,2,\ldots,|V|\}$. Inayah et al. study some basic properties of such labeling and give (a,d)-cycle-antimagic labelings of fans.

A graph G is said to have an *$(H_1,H_2,\ldots,H_k)$-covering* if every edge in G belongs to at least one of the H_i's. Susilowati, Sania, and Estuningsih [1469] investigated such labelings for the ladders $P_n \times P_2$ with C_4-, C_6- and C_8- coverings for some value of d.

Simanjuntak, Bertault, and Miller [1371] define an *(a,d)-edge-antimagic vertex labeling* for a graph $G(V,E)$ as an injective mapping f from V onto the set $\{1,2,\ldots,|V|\}$ such that the set $\{f(u)+f(v)|uv \in E\}$ is $\{a, a+d, a+2d, \ldots, a+(|E|-1)d\}$. (The equivalent notion of *(a,d)-indexable* labeling was defined by Hegde in 1989 in his Ph. D. thesis–see [628].) Similarly, Simanjuntak et al. define an *(a,d)-edge-antimagic total labeling* for a graph $G(V,E)$ as an injective mapping f from $V\cup E$ onto the set $\{1,2,\ldots,|V|+|E|\}$ such that the set $\{f(v)+f(vu)+f(v)|uv \in E\}$ where v ranges over all of V is $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. Among their results are: C_{2n} has no (a,d)-edge-antimagic vertex labeling; C_{2n+1} has a $(n+2,1)$-edge-antimagic vertex labeling and a $(n+3,1)$-edge-antimagic vertex labeling; P_{2n} has a $(n+2,1)$-edge-antimagic vertex labeling; P_n has a $(3,2)$-edge-antimagic vertex labeling; C_n has $(2n+2,1)$- and $(3n+2,1)$-edge-antimagic total labelings; C_{2n} has $(4n+2,2)$- and $(4n+3,2)$-edge-antimagic total labelings; C_{2n+1} has $(3n+4,3)$- and $(3n+5,3)$-edge-antimagic total labelings; P_{2n+1} has $(3n+4,2)$-, $(3n+4,3)$-, $(2n+4,4)$-, $(5n+4,2)$-, $(3n+5,2)$-, and $(2n+6,4)$-edge-antimagic total labelings; P_{2n} has $(6n,1)$- and $(6n+2,2)$-edge-antimagic total labelings; and several parity conditions for (a,d)-edge-antimagic total labelings. They conjecture: C_{2n} has a $(2n+3,4)$- or a $(2n+4,4)$-edge-antimagic total labeling; C_{2n+1} has a $(n+4,5)$- or a $(n+5,5)$-edge-antimagic total labeling; paths have no (a,d)-edge-antimagic vertex labelings with $d > 2$; and cycles have no (a,d)-antimagic total labelings with $d > 5$. The first and last of these conjectures were proved by Zhenbin in [1723] and the last two were verified by Bača, Lin, Miller, and Simanjuntak [181] who proved that a graph with v vertices and e edges that has an (a,d)-edge-antimagic vertex labeling must satisfy $d(e-1) \leq 2v-1-a \leq 2v-4$. As a consequence, they obtain: for every path there is no (a,d)-edge-antimagic vertex labeling with $d > 2$; for every cycle there is no (a,d)-edge-antimagic vertex labeling with $d > 1$; for K_n $(n > 1)$ there is no (a,d)-edge-antimagic vertex labeling (the cases for $n = 2$ and $n = 3$ are handled individually); $K_{n,n}$ $(n > 3)$ has no (a,d)-edge-antimagic vertex labeling; for every wheel there is no (a,d)-edge-antimagic vertex labeling; for every generalized Petersen graph there is no (a,d)-edge-antimagic vertex labeling with $d > 1$. They also study the relationship between graphs with (a,d)-edge-antimagic labelings and magic and antimagic labelings. They conjecture that every tree has an $(a,1)$-edge-antimagic total labeling.

Baca and Barrientos [150] prove that if a tree T has an α-labeling and $\{A, B\}$ is the bipartition of the vertices of T, then T also admits an $(a,1)$-edge-antimagic vertex labeling and it admits a $(3,2)$-edge-antimagic vertex labeling if and only if $||A| - |B|| \leq 1$.

In [181] Bača, Lin, Miller, and Simanjuntak prove: if P_n has an (a,d)-edge-antimagic total labeling, then $d \leq 6$; P_n has $(2n+2,1)$-, $(3n,1)$-, $(n+4,3)$-, and $(2n+2,3)$-edge-antimagic total labelings; P_{2n+1} has $(3n+4,2)$-,$(5n+4,3)$-, $(2n+4,4)$-, and $(2n+6,4)$-edge-antimagic total labelings; and P_{2n} has $(3n+3,2)$- and $(5n+1,2)$-edge-antimagic total labelings. Ngurah [1094] proved P_{2n+1} has $(4n+4,1)$-, $(6n+5,3)$-,$(4n+4,2)$-,$(4n+5,2)$-edge-antimagic total labelings and C_{2n+1} has $(4n+4,2)$- and $(4n+5,2)$-edge-antimagic total labelings. Silaban and Sugeng [1370] prove: P_n has $(n+4,4)$- and $(6,6)$-edge-antimagic total labelings; if $C_m \odot \overline{K_n}$ has an (a,d)-edge-antimagic total labeling, then $d \leq 5$; $C_m \odot \overline{K_n}$ has (a,d)-edge-antimagic total labelings for $m > 3, n > 1$ and $d = 2$ or 4; and $C_m \odot \overline{K_n}$ has no (a,d)-edge-antimagic total labelings for m and d and $n \equiv 1 \bmod 4$. They conjecture that P_n $(n \geq 3)$ has $(a,5)$-edge-antimagic total labelings. In [1454] Sugeng and Xie use adjacency methods to construct super edge magic graphs from (a,d)-edge-antimagic vertex graphs. Pushpam and Saibulla [1169] determined super (a,d)-edge antimagic total labelings for graphs derived from copies of generalized ladders, fans, generalized prisms and web graphs.

In [200] Bača and Youssef used parity arguments to find a large number of conditions on p, q and d for which a graph with p vertices and q edges cannot have an (a,d)-edge-antimagic total labeling or vertex-antimagic total labeling. Bača and Youssef [200] made the following connection between (a,d)-edge-antimagic vertex labelings and sequential labelings: if G is a connected graph other than a tree that has an (a,d)-edge-antimagic vertex labeling, then $G+K_1$ has a sequential labeling.

In [1442] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every $n \geq 2$, $P_n \cup P_{n+1}$ has a $(6n+1,1)$- and a $(4n+3,3)$-edge-antimagic total labeling, for every odd $n \geq 3$, $P_n \cup P_{n+1}$ has a $(6n,1)$- and a $(5n+1,2)$-edge-antimagic total labeling, for every $n \geq 2$, $nP_2 \cup P_n$ has a $(7n,1)$- and a $(6n+1,2)$-edge-antimagic total labeling. In [1439] the same authors show that $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ $(n \geq 2)$, and $nP_2 \cup P_{n+2}$ are super edge-magic total. They also show that under certain conditions one can construct new super edge-magic total graphs from existing ones by joining a particular vertex of the existing super edge-magic total graph to every vertex in a path or every vertex of a star and by joining one extra vertex to some vertices of the existing graph. Baskoro, Sudarsana, and Cholily [244] also provide algorithms for constructing new super edge-magic total graphs from existing ones by adding pendent vertices to the existing graph. A corollary to one of their results is that the graph obtained by attaching a fixed number of pendent edges to each vertex of a path of even length is super edge-magic. Baskoro and Cholily [242] show that the graphs obtained by attaching any numbers of pendent edges to a single vertex or a fix number of pendent edges to every vertex of the following graphs are super edge-magic total graphs: odd cycles, the generalized Petersen graphs $P(n,2)$ (n odd and at least

5), and $C_n \times P_m$ (n odd, $m \geq 2$).

Arumugam and Nalliah [105] proved: the friendship graph $C_3^{(n)}$ with $n \equiv 0, 8 \pmod{12}$ has no super $(a,2)$-edge-antimagic total labeling; $C_n^{(n)}$ with $n \equiv 2 \pmod 4$ has no super $(a,2)$-edge-antimagic total labeling; and the generalized friendship graph $F_{2,p}$ consisting of 2 cycles of various lengths, having a common vertex, and having order p where $p \geq 5$, has a super $(2p+2,1)$-edge-antimagic total labeling if and only if p is odd.

An (a,d)-edge-antimagic total labeling of $G(V,E)$ is called a *super (a,d)-edge-antimagic total* if the vertex labels are $\{1,2,\ldots,|V(G)|\}$ and the edge labels are $\{|V(G)|+1, |V(G)|+2, \ldots, |V(G)|+|E(G)|\}$. Bača, Baskoro, Simanjuntak, and Sugeng [161] prove the following: C_n has a super (a,d)-edge-antimagic total labeling if and only if either d is 0 or 2 and n is odd, or $d=1$; for odd $n \geq 3$ and $m=1$ or 2, the generalized Petersen graph $P(n,m)$ has a super $(11n+3)/2,0)$-edge-antimagic total labeling and a super $((5n+5)/2,2)$-edge-antimagic total labeling; for odd $n \geq 3$, $P(n,(n-1)/2)$ has a super $((11n+3)/2,0)$-edge-antimagic total labeling and a super $((5n+5)/2,2)$-edge-antimagic total labeling. They also prove: if $P(n,m), n \geq 3$, $1 \leq m \leq \lfloor (n-1)/2 \rfloor$ is super (a,d)-edge-antimagic total, then $(a,d)=(4n+2,1)$ if n is even, and either $(a,d)=((11n+3)/2,0)$, or $(a,d)=(4n+2,1)$, or $(a,d)=((5n+5)/2,2)$, if n is odd; and for odd $n \geq 3$ and $m=1,2$, or $(n-1)/2$, $P(n,m)$ has an $(a,0)$-edge-antimagic total labeling and an $(a,2)$-edge-antimagic total labeling. (In a personal communication MacDougall argues that "edge-magic" is a better term than "$(a,0)$-edge-antimagic" for while the latter is technically correct, "antimagic" suggests different weights whereas "magic" emphasizes equal weights and that the edge-magic case is much more important, interesting, and fundamental rather than being just one subcase of equal value to all the others.) They conjecture that for odd $n \geq 9$ and $3 \leq m \leq (n-3)/2$, $P(n,m)$ has a $(a,0)$-edge-antimagic total labeling and an $(a,2)$-edge-antimagic total labeling. Ngurah and Baskoro [1095] have shown that for odd $n \geq 3$, $P(n,1)$ and $P(n,2)$ have $((5n+5)/2,2)$-edge-antimagic total labelings and when $n \geq 3$ and $1 \leq m < n/2, P(n,m)$ has a super $(4n+2,1)$-edge-antimagic total labeling. In [1096] Ngurah, Baskova, and Simanjuntak provide (a,d)-edge-antimagic total labelings for the generalized Petersen graphs $P(n,m)$ for the cases $m=1$ or 2, odd $n \geq 3$, and $(a,d)=((9n+5)/2,2)$.

In [1440] Sudarsana, Baskoro, Uttunggadewa, and Ismaimuza show how to construct new larger super (a,d)-edge-antimagic-total graphs from existing smaller ones.

In [1097] Ngurah, Baskoro, and Simanjuntak prove that mC_n ($n \geq 3$) has an (a,d)-edge-antimagic total in the following cases: $(a,d)=(5mn/2+2,1)$ where m is even; $(a,d)=(2mn+2,2)$; $(a,d)=((3mn+5)/2,3)$ for m and n odd; and $(a,d)=((mn+3),4)$ for m and n odd; and mC_n has a super $(2mn+2,1)$-edge-antimagic total labeling.

Bača and Barrientos [151] have shown that mK_n has a super (a,d)-edge-antimagic total labeling if and only if (i) $d \in \{0,2\}$, $n \in \{2,3\}$ and $m \geq 3$ is odd, or (ii) $d=1$, $n \geq 2$ and $m \geq 2$, or (iii) $d \in \{3,5\}$, $n=2$ and $m \geq 2$, or (iv) $d=4$, $n=2$, and $m \geq 3$ is odd. In [150] Bača and Barrientos proved the following: if a graph with q edges and $q+1$ vertices has an α-labeling, than it has an $(a,1)$-edge-antimagic vertex labeling; a tree has a $(3,2)$-edge-antimagic vertex labeling if and only if it has an α-labeling and the number of vertices in its two partite sets differ by at most 1; if a tree with at least two vertices has a super (a,d)-edge-antimagic total labeling, then d is at most 3; if a graph has an $(a,1)$-edge-antimagic vertex labeling, then it also has a super $(a_1,0)$-edge-antimagic total labeling and a super $(a_2,2)$-edge-antimagic total labeling.

Bača and Youssef [199] proved the following: if G is a connected (a,d)-edge-antimagic vertex graph that is not a tree, then $G+K_1$ is sequential; mC_n has an (a,d)-edge-antimagic vertex labeling if and only if m and n are odd and $d = 1$; an odd degree (p, q)-graph G cannot have a (a, d)-edge-antimagic total labeling if $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$, or $p \equiv 0 \pmod 4$, $q \equiv 2 \pmod 4$, and d is even; a (p, q)-graph G cannot have a super (a, d)-edge-antimagic total labeling if G has odd degree, $p \equiv 2 \pmod 4$, q is even, and d is odd, or G has even degree, $q \equiv 2 \pmod 4$, and d is even; C_n has a $(2n + 2, 3)$- and an $(n + 4, 3)$-edge-antimagic total labeling; a (p, q)-graph is not super (a, d)-vertex-antimagic total if: $p \equiv 2 \pmod 4$ and d is even; $p \equiv 0 \pmod 4$, $q \equiv 2 \pmod 4$, and d is odd; $p \equiv 0 \pmod 8$ and $q \equiv 2 \pmod 4$.

In [1442] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every $n \geq 2$, $P_n \cup P_{n+1}$ has super $(n+4, 1)$- and $(2n+6, 3)$-edge antimagic total labelings; for every odd $n \geq 3$, $P_n \cup P_{n+1}$ has super $(4n + 5, 1)$-,$(3n + 6, 2)$-, $(4n + 3, 1)$- and $(3n + 4, 2)$-edge antimagic total labelings; for every $n \geq 2$, $nP_2 \cup P_n$ has super $(6n + 2, 1)$- and $(5n + 3, 2)$-edge antimagic total labelings; and for every $n \geq 1$, $nP_2 \cup P_{n+2}$ has super $(6n + 6, 1)$- and $(5n + 6, 2)$-edge antimagic total labelings. They pose a number of open problems about constructing (a, d)-edge antimagic labelings and super (a, d)-edge antimagic labelings for the graphs $P_n \cup P_{n+1}$, $nP_2 \cup P_n$, and $nP_2 \cup P_{n+2}$ for specific values of d.

Dafik, Miller, Ryan, and Bača [412] investigated the super edge-antimagicness of the disconnected graph mC_n and mP_n. For the first case they prove that mC_n, $m \geq 2$, has a super (a, d)-edge-antimagic total labeling if and only if either d is 0 or 2 and m and n are odd and at least 3, or $d = 1$, $m \geq 2$, and $n \geq 3$. For the case of the disjoint union of paths they determine all feasible values for m, n and d for mP_n to have a super (a, d)-edge-anti-magic total labeling except when m is even and at least 2, $n \geq 2$, and d is 0 or 2. In [414] Dafik, Miller, Ryan, and Bača obtain a number of results about super edge-antimagicness of the disjoint union of two stars and state three open problems.

Sudarsana, Hendra, Adiwijaya, and Setyawan [1441] show that the t-joint copies of wheel W_n have a super edge anti-magic $((2n + 2)t + 2, 1)$-total labeling for $n \geq 4$ and $t \geq 2$.

In [177] Bača, Lascsáková, and Semaničová investigated the connection between graphs with α-labelings and graphs with super (a, d)-edge-antimagic total labelings. Among their results are: If G is a graph with n vertices and $n - 1$ edges $(n \geq 3)$ and G has an α-labeling, then mG is super (a, d)-edge-antimagic total if either d is 0 or 2 and m is odd, or $d = 1$ and n is even; if G has an α-labeling and has n vertices and $n - 1$ edges with vertex bipartition sets V_1 and V_2 where $|V_1|$ and $|V_2|$ differ by at most 1, then mG is super (a, d)–edge-antimagic total for $d = 1$ and $d = 3$. In the same paper Bača et al. prove: caterpillars with odd order at least 3 have super $(a, 1)$-edge-antimagic total labelings; if G is a caterpillar of odd order at least 3 and G has a super $(a, 1)$-edge-antimagic total labeling, then mG has a super $(b, 1)$-edge-antimagic total labeling for some b that is a function of a and m.

In [411] Dafik, Miller, Ryan, and Bača investigated the existence of antimagic labelings of disjoint unions of s-partite graphs. They proved: if $s \equiv 0$ or $1 \pmod 4$, $s \geq 4, m \geq 2, n \geq 1$ or mn is even , $m \geq 2$, $n \geq 1$, $s \geq 4$, then the complete s-partite graph $mK_{n,n,\ldots,n}$ has no super $(a, 0)$-edge-antimagic total labeling; if $m \geq 2$ and $n \geq 1$, then $mK_{n,n,n,n}$ has no super $(a, 2)$-antimagic total labeling; and for $m \geq 2$ and $n \geq 1$, $mK_{n,n,n,n}$ has an $(8mn + 2, 1)$-edge-antimagic total labeling. They conjecture that for $m \geq 2$, $n \geq 1$ and $s \geq 5$, the complete s-partite graph $mK_{n,n,\ldots,n}$ has a super $(a, 1)$-antimagic total labeling.

In [196] Bača, Muntaner-Batle, Semaničová-Feňovčiková, and Shafiq investigate super (a, d)-edge-antimagic total labelings of disconnected graphs. Among their results are: If G is a (super) $(a, 2)$-edge-antimagic total labeling and m is odd, then mG has a (super) $(a', 2)$-edge-antimagic-total labeling where $a' = m(a - 3) + (m + 1)/2 + 2$; and if d a positive even integer and k

a positive odd integer, G is a graph with all of its vertices having odd degree, and the order and size of G have opposite parity, then $2kG$ has no (a,d)-edge-antimagic total labeling. Bača and Brankovic [164] have obtained a number of results about the existence of super (a,d)-edge-antimagic totaling of disjoint unions of the form $mK_{n,n}$. In [167] Bača, Dafik, Miller, and Ryan provide (a,d)-edge-antimagic vertex labelings and super (a,d)-edge-antimagic total labelings for a variety of disjoint unions of caterpillars. Bača and Youssef [200] proved that mC_n has an (a,d)-edge-antimagic vertex labeling if and only if m and n are odd and $d=1$. Bača, Dafik, Miller, and Ryan [168] constructed super (a,d)-edge-antimagic total labeling for graphs of the form $m(C_n \odot \overline{K}_s)$ and $mP_n \cup kC_n$ while Dafik, Miller, Ryan, and Bača [413] do the same for graphs of the form $mK_{n,n,n}$ and $K_{1,m} \cup 2sK_{1,n}$. Both papers provide a number of open problems. In [185] Bača, Lin, and Muntaner-Batle provide super (a,d)-edge-antimagic total labeling of forests in which every component is a specific kind of tree. In [176] Bača, Kovǎr, Semaničová-Feňovčiková, and Shafiq prove that every even regular graph and every odd regular graph with a 1-factor are super $(a,1)$-edge-antimagic total and provide some constructions of non-regular super $(a,1)$-edge-antimagic total graphs. Bača, Lin and Semaničová-Feňovčiková [187] show: the disjoint union of m graphs with super $(a,1)$-edge antimagic total labelings have super $(m(a-2)+2,1)$-edge antimagic total labelings; the disjoint union of m graphs with super $(a,3)$-edge antimagic total labelings have super $(m(a-3)+3,3)$-edge antimagic total labelings; if G has a $(a,1)$-edge antimagic total labelings then mG has an $(b,1)$-edge antimagic total labeling for some b; and if G has a $(a,3)$-edge antimagic total labelings then mG has an $(b,3)$-edge antimagic total labeling for some b.

For $t \geq 2$ and $n \geq 4$ the *Harary graph*, C_p^t, is the graph obtained by joining every two vertices of C_p that are at distance t in C_p. In [1174] Rahim, Ali, Kashif, and Javaid provide super (a,d)-edge antimagic total labelings for disjoint unions of Harary graphs and disjoint unions of cycles. In [675] Hussain, Ali, Rahim, and Baskoro construct various (a,d)-vertex-antimagic labelings for Harary graphs and disjoint unions of identical Harary graphs. For p odd and at least 5, Balbuena, Barker, Das, Lin, Miller, Ryan, Slamin, Sugeng, and Tkac [205] give a super $((17p+5)/2)$-vertex-antimagic total labeling of C_p^t. MacDougall and Wallis [1016] have proved the following: $C_{4m+3}^t,\ m \geq 1$, has a super $(a,0)$-edge-antimagic total labeling for all possible values of t with $a = 10m+9$ or $10m+10$; $C_{4m+1}^t,\ m \geq 3$, has a super $(a,0)$-edge-antimagic total labeling for all possible values except $t = 5, 9, 4m-4$, and $4m-8$ with $a = 10m+4$ and $10m+5$; $C_{4m+1}^t,\ m \geq 1$, has a super $(10m+4,0)$-edge-antimagic total labeling for all $t \equiv 1 \pmod 4$ except $4m-3$; $C_{4m}^t,\ m > 1$, has a super $(10m+2,0)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$; $C_{4m+2}^t,\ m > 1$, has a super $(10m+7,0)$-edge-antimagic total labeling for all odd t other than 5 and for $t = 2$ or 6. In [676] Hussain, Baskoro, and Ali prove the following: for any $p \geq 4$ and for any $t \geq 2$, C_p^t admits a super $(2p+2,1)$-edge-antimagic total labeling; for $n \geq 4$, $k \geq 2$ and $t \geq 2$, kC_n^t admits a super $(2nk+2,1)$-edge-antimagic total labeling; and for $p \geq 5$ and $t \geq 2$, C_p^t admits a super $(8p+3,1)$-vertex-antimagic total labeling, provided if $p \neq 2t$.

Bača and Murugan [197] have proved: if $C_n^t,\ n \geq 4, 2 \leq t \leq n-2$, is super (a,d)-edge-antimagic total, then $d = 0, 1$, or 2; for $n = 2k+1 \geq 5$, C_n^t has a super $(a,0)$-edge-antimagic total labeling for all possible values of t with $a = 5k+4$ or $5k+5$; for $n = 2k+1 \geq 5$, C_n^t has a super $(a,2)$-edge-antimagic total labeling for all possible values of t with $a = 3k+3$ or $3k+4$; for $n \equiv 0 \pmod 4$, C_n^t has a super $(5n/2+2,0)$-edge-antimagic total labeling and a super $(3n/2+2,0)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$; for $n = 10$ and $n \equiv 2 \pmod 4$, $n \geq 18$, C_n^t has a super $(5n/2+2,0)$-edge-antimagic total labeling and a super $(3n/2+2,0)$-

edge-antimagic total labeling for all $t \equiv 3 \pmod 4$ and for $t = 2$ and 6; for odd $n \geq 5$, C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling for all possible values of t; for even $n \geq 6$, C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling for all odd $t \geq 3$; and for even $n \equiv 0 \pmod 4$, $n \geq 4$, C_n^t has a super $(2n+2,1)$-edge-antimagic total labeling for all $t \equiv 2 \pmod 4$. They conjecture that there is a super $(2n+2,1)$-edge-antimagic total labeling of C_n^t for $n \equiv 0 \pmod 4$ and for $t \equiv 0 \pmod 4$ and for $n \equiv 2 \pmod 4$ and for t even.

In [182] Bača, Lin, Miller, and Youssef prove: if the friendship $C_3^{(n)}$ is super (a,d)-antimagic total, then $d < 3$; $C_3^{(n)}$ has an $(a,1)$-edge antimagic vertex labeling if and only if $n = 1,3,4,5$, and 7; $C_3^{(n)}$ has a super (a,d)-edge-antimagic total labelings for $d = 0$ and 2; $C_3^{(n)}$ has a super $(a,1)$-edge-antimagic total labeling; if a fan F_n $(n \geq 2)$ has a super (a,d)-edge-antimagic total labeling, then $d < 3$; F_n has a super (a,d)-edge-antimagic total labeling if $2 \leq n \leq 6$ and $d = 0,1$ or 2; the wheel W_n has a super (a,d)-edge-antimagic total labeling if and only if $d = 1$ and $n \not\equiv 1 \pmod 4$; K_n, $n \geq 3$, has a super (a,d)-edge-antimagic total labeling if and only if either $d = 0$ and $n = 3$, or $d = 1$ and $n \geq 3$, or $d = 2$ and $n = 3$; and $K_{n,n}$ has a super (a,d)-edge antimagic total labeling if and only if $d = 1$ and $n \geq 2$.

Bača, Lin, and Muntaner-Batle [183] have shown that if a tree with at least two vertices has a super (a,d)-edge-antimagic total labeling, then d is at most three and P_n, $n \geq 2$, has a super (a,d)-edge-antimagic total labeling if and only if $d = 0,1,2$, or 3. They also characterize certain path-like graphs in a grid that have super(a,d)-edge-antimagic total labelings.

In [1447] Sugeng, Miller, and Bača prove that the ladder, $P_n \times P_2$, is super (a,d)-edge-antimagic total if n is odd and $d = 0,1$, or 2 and $P_n \times P_2$ is super $(a,1)$-antimagic total if n is even. They conjecture that $P_n \times P_2$ is super $(a,0)$- and $(a,2)$-edge-antimagic when n is even. Sugeng, Miller, and Bača [1447] prove that $C_m \times P_2$ has a super (a,d)-edge-antimagic total labeling if and only if either $d = 0,1$ or 2 and m is odd and at least 3, or $d = 1$ and m is even and at least 4. They conjecture that if m is even, $m \geq 4$, $n \geq 3$, and $d = 0$ or 2, then $C_m \times P_n$ has a super (a,d)-edge-antimagic total labeling. In [859] M.-J. Lee studied super $(a,1)$-edge-antimagic properties of $m(P_4 \times P_n)$ for $m,n \geq 1$ and $m(C_n \odot \overline{K_t})$ for n even and $m,t \geq 1$. He also proved that for $n \geq 2$ the graph $P_4 \times P_n$ has a super $(8n+2,1)$-edge antimagic total labeling.

Sugeng, Miller, and Bača [1447] define a variation of a ladder, $\mathbb{L}_n$, as the graph obtained from $P_n \times P_2$ by joining each vertex u_i of one path to the vertex v_{i+1} of the other path for $i = 1,2,\ldots,n-1$. They prove $\mathbb{L}_n$, $n \geq 2$, has a super (a,d)-edge-antimagic total labeling if and only if $d = 0,1$, or 2.

In [410] Dafik, Miller, and Ryan investigate the existence of super (a,d)-edge-antimagic total labelings of $mK_{n,n,n}$ and $K_{1,m} \cup 2sK_{1,n}$. Among their results are: for $d = 0$ or 2, $mK_{n,n,n}$ has a super (a,d)-edge-antimagic total labeling if and only if $n = 1$ and m is odd and at least 3; $K_{1,m} \cup 2sK_{1,n}$ has a super (a,d)-edge-antimagic labeling for $(a,d) = (4n+5)s+2m+4,0), ((2n+5)s+m+5,2), ((3n+5)s+(3m+9)/2,1)$ and $(5s+7,4)$.

In [153] Bača, Bashir, and Semaničová showed that for $n \geq 4$ and $d = 0,1,2,3,4,5$, and 6 the antiprism A_n has a super d-antimagic labeling of type $(1,1,1)$. The *generalized antiprism* A_m^n is obtained from $C_m \times P_n$ by inserting the edges $\{v_{i,j+1}, v_{i+1,j}\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n-1$ where the subscripts are taken modulo m. Sugeng et al. prove that A_m^n, $m \geq 3$, $n \geq 2$, is super (a,d)-edge-antimagic total if and only if $d = 1$.

Baca, Miller, Phanalasy, and A. Semaničová-Feňovčíková [192] investigated the existence of (super) 1-antimagic labelings of type $(1,1,1)$ for disjoint union of plane graphs. They prove that if a plane graph $G(V,E,F)$ has a (super) 1-antimagic labeling h of type $(1,1,1)$ such that $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$ where z_{ext} denotes the unique external face then, for every positive integer m, the graph mG also admits a (super) 1-antimagic labeling of type $(1,1,1)$; and if a plane graph $G(V,E,F)$ has 4-sided inner faces and h is a (super) d-antimagic labeling of type $(1,1,1)$ of G such that $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$ where $d = 1,3,5,7,9$ then, for every positive integer m, the graph mG also admits a (super) d- antimagic labeling of type $(1,1,1)$. They also give a similar result about plane graphs with inner faces that are 3-sided.

Sugeng, Miller, Slamin, and Bača [1450] proved: the star S_n has a super (a,d)-antimagic total labeling if and only if either $d = 0,1$ or 2, or $d = 3$ and $n = 1$ or 2; if a nontrivial caterpillar has a super (a,d)-edge-antimagic total labeling, then $d \leq 3$; all caterpillars have super $(a,0)$-, $(a,1)$- and $(a,2)$-edge-antimagic total labelings; all caterpillars have a super $(a,1)$-edge-antimagic total labeling; if m and n differ by at least 2 the double star $S_{m,n}$ (that is, the graph obtained by joining the centers of $K_{1,m}$ and $K_{1,n}$ with an edge) has no $(a,3)$-edge-antimagic total labeling.

Sugeng and Miller [1445] show how to manipulate adjacency matrices of graphs with (a,d)-edge-antimagic vertex labelings and super (a,d)-edge-antimagic total labelings to obtain new (a,d)-edge-antimagic vertex labelings and super (a,d)-edge-antimagic total labelings. Among their results are: every graph can be embedded in a connected (a,d)-edge-antimagic vertex graph; every (a,d)-edge-antimagic vertex graph has a proper (a,d)-edge-antimagic vertex subgraph; if a graph has a $(a,1)$-edge-antimagic vertex labeling and an odd number of edges, then it has a super $(a,1)$-edge-antimagic total labeling; every super edge magic total graph has an $(a,1)$-edge-antimagic vertex labeling; and every graph can be embedded in a connected super (a,d)-edge-antimagic total graph.

Ajitha, Arumugan, and Germina [74] show that $(p,p-1)$ graphs with α-labelings (see §3.1) and partite sets with sizes that differ by at most 1 have super (a,d)-edge antimagic total labelings for $d = 0,1,2$ and 3. They also show how to generate large classes of trees with super (a,d)-edge-antimagic total labelings from smaller graceful trees.

Bača, Lin, Miller, and Ryan [180] define a *Möbius grid*, M_n^m, as the graph with vertex set $\{x_{i,j}|\ i = 1,2,\ldots,m+1, j = 1,2,\ldots,n\}$ and edge set $\{x_{i,j}x_{i,j+1}|\ i = 1,2,\ldots,m+1, j = 1,2,\ldots,n-1\} \cup \{x_{i,j}x_{i+1,j}|\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n\} \cup \{x_{i,n}x_{m+2-i,1}|\ i = 1,2,\ldots,m+1\}$. They prove that for $n \geq 2$ and $m \geq 4$, M_n^m has no d-antimagic vertex labeling with $d \geq 5$ and no d-antimagic-edge labeling with $d \geq 9$.

Ali, Bača, and Bashir, [68] investigated super (a,d)-vertex-antimagic total labelings of the disjoint unions of paths. They prove: mP_2 has a super (a,d)-vertex-antimagic total labeling if and only if m is odd and $d = 1$; mP_3, $m > 1$, has no super $(a,3)$-vertex-antimagic total labeling; mP_3 has a super $(a,2)$-vertex-antimagic total labeling for $m \equiv 1 \pmod 6$; and mP_4 has a super $(a,2)$-vertex-antimagic total labeling for $m \equiv 3 \pmod 4$.

Lee, Tsai, and Lin [861] denote the subdivision of a star S_n obtained by inserting m vertices into every edge of the star S_n by S_m^n. They proved that for $n \geq 3$, the graph kS_m^n is super (a,d)-edge antimagic total for certain values. In [678] Ichishima, López, Muntaner-Batle and Rius-Font proved that if G is tripartite and has a (super) (a,d)-edge antimagic total labeling, then nG $(n \geq 3)$ has a (super) (a,d)-edge antimagic total labeling for $d = 1$ and for $d = 0,2$ when n is odd.

The book [191] by Bača and Miller has a wealth of material and open problems on super edge-antimagic labelings. In [160] Bača, Baskoro, Miller, Ryan, Simanjuntak, and Sugeng provide detailed survey of results on edge antimagic labelings and include many conjectures and open problems.

In Tables 14 and 15 we use the abbreviations

(a, d)**-VAT** (a, d)-vertex-antimagic total labeling

(a, d)**-SVAT** super (a, d)-vertex-antimagic total labeling

(a, d)**-EAT** (a, d)-edge-antimagic total labeling

(a, d)**-SEAT** super (a, d)-edge-antimagic total labeling

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 14: **Summary of (a, d)-Vertex-Antimagic Total and Super (a, d)-Vertex-Antimagic Total Labelings**

Graph	*Labeling*	*Notes*
P_n	(a, d)-VAT	wide variety of a and d [162]
P_n	(a, d)-SVAT	iff $d = 3$, $d = 2, n \geq 3$ odd or $d = 3$, $n \geq 3$ [1448]
C_n	(a, d)-VAT	wide variety of a and d [161]
C_n	(a, d)-SVAT	iff $d = 0, 2$ and n odd or $d = 1$ [1448]
generalized Petersen graph $P(n, k)$	(a, d)-VAT $(a, 1)$-VAT	[163] $n \geq 3$, $1 \leq k \leq n/2$ [1449]
prisms $C_n \times P_2$	(a, d)-VAT	[163]
antiprisms	(a, d)-VAT	[163]
$S_{n_1} \cup \ldots \cup S_{n_t}$	(a, d)-VAT	$d = 1, 2, 3, 4, 6$ [1120], citeRahSl
W_n	not (a, d)-VAT	for $n > 20$ [963]
$K_{1,n}$	not (a, d)-SVAT	$n \geq 3$ [1448]

Table 15: **Summary of (a, d)-Edge-Antimagic Total Labelings**

Graph	*Labeling*	*Notes*
trees	$(a, 1)$-EAT?	[181]
P_n	not (a, d)-EAT	$d > 2$ [181]
P_{2n}	$(6n, 1)$-EAT	[1371]
	$(6n + 2, 2)$-EAT	[1371]
P_{2n+1}	$(3n + 4, 2)$-EAT	[1371]
	$(3n + 4, 3)$-EAT	[1371]
	$(2n + 4, 4)$-EAT	[1371]
	$(5n + 4, 2)$-EAT	[1371]
	$(3n + 5, 2)$-EAT	[1371]
	$(2n + 6, 4)$-EAT	[1371]
C_n	$(2n + 2, 1)$-EAT	[1371]
	$(3n + 2, 1)$-EAT	[1371]
	not (a, d)-EAT	$d > 5$ [181]
C_{2n}	$(4n + 2, 2)$-EAT	[1371]
	$(4n + 3, 2)$-EAT	[1371]
	$(2n + 3, 4)$-EAT?	[1371]
	$(2n + 4, 4)$-EAT?	[1371]
C_{2n+1}	$(3n + 4, 3)$-EAT	[1371]
	$(3n + 5, 3)$-EAT	[1371]
	$(n + 4, 5)$-EAT?	[1371]
	$(n + 5, 5)$-EAT?	[1371]
K_n	not (a, d)-EAT	$d > 5$ [181]
$K_{n,n}$	(a, d)-EAT	iff $d = 1, n \geq 2$ [182]
caterpillars	(a, d)-EAT	$d \leq 3$ [1450]
W_n	not (a, d)-EAT	$d > 4$ [181]
generalized Petersen	not (a, d)-EAT	$d > 4$ [181]
graph $P(n, k)$	$((5n + 5)/2, 2)$-EAT	for n odd, $n \geq 3$ and $k = 1, 2$ [1095]
	super $(4n + 2, 1)$-EAT	for $n \geq 3$, and $1 \leq k \leq n/2$ [1095]

Table 16: **Summary of (a,d)-Edge-Antimagic Vertex Labelings**

Graph	*Labeling*	*Notes*
P_n	$(3,2)$-EAV	[1371]
	not (a,d)-EAV	$d > 2$ [1371]
P_{2n}	$(n+2,1)$-EAV	[1371]
C_n	not (a,d)-EAV	$d > 1$ [181]
C_{2n}	not (a,d)-EAV	[1371]
C_{2n+1}	$(n+2,1)$-EAV	[1371]
	$(n+3,1)$-EAV	[1371]
K_n	not (a,d)-EAV	for $n > 1$ [181]
$K_{n,n}$	not (a,d)-EAV	for $n > 3$ [181]
W_n	not (a,d)-EAV	[181]
$C_3^{(n)}$ (friendship graph)	$(a,1)$-EAV	iff $n = 1,3,4,5,7$ [182]
generalized Petersen graph $P(n,k)$	not (a,d)-EAV	$d > 1$ [181]

Table 17: **Summary of (a, d)-Super-Edge-Antimagic Total Labelings**

Graph	*Labeling*	*Notes*
C_n^+ (see §2.2)	(a, d)-SEAT	variety of cases [143], [197]
$P_n \times P_2$ (ladders)	(a, d)-SEAT	n odd, $d \leq 2$ [1447]
		n even, $d = 1$ [1447]
	(a, d)-SEAT?	$d = 0, 2,\ n$ even [1447]
$C_n \times P_2$	(a, d)-SEAT	iff $d \leq 3$ n odd [1447]
		or $d = 1,\ n \geq 4$ even [1447]
$C_m \times P_n$	(a, d)-SEAT?	$m \geq 4$ even, $n \geq 3,\ d = 0, 2$ [1447]
caterpillars	$(a, 1)$-SEAT	[1450]
$C_3^{(n)}$ (friendship graphs)	(a, d)-SEAT	$d = 0, 1, 2$ [182]
F_n $(n \geq 2)$ (fans)	(a, d) SEAT	only if $d < 3$ [182]
	(a, d)-SEAT	$2 \leq n \leq 6, d = 0, 1, 2$ [182]
W_n	(a, d)-SEAT	iff $d = 1, n \not\equiv 1 \pmod 4$ [182]
K_n $(n \geq 3)$	(a, d) SEAT	iff $d = 0, n = 3$ [182]
		$d = 1, n \geq 3$ [182]
		$d = 2, n = 3$ [182]
trees	(a, d)-SEAT	only if $d \leq 3$ [183]
P_n $(n > 1)$	(a, d)-SEAT	iff $d \leq 3$ [183]
mK_n	(a, d)-SEAT	iff $d \in \{0, 2\}, n \in \{2, 3\}, m \geq 3$ odd [151]
		$d = 1, m, n \geq 2$ [151]
		$d = 3$ or $5, n = 2, m \geq 2$ [151]
		$d = 4, n = 2,\ m \geq 3$ odd [151]
C_n	(a, d)-SEAT	iff $d = 0$ or 2, n odd [183]
		$d = 1$ [161]
$P(m, n)$	(a, d)-SEAT	many cases [161]

6.4 Face Antimagic Labelings and d-antimagic Labeling of Type (1,1,1)

Bača [145] defines a connected plane graph G with edge set E and face set F to be (a, d)-*face antimagic* if there exist positive integers a and d and a bijection $g\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $\psi_g\colon F \to \{a, a + d, \ldots, a + (|F(G)| - 1)d\}$, where for a face f, $\psi_g(f)$ is the sum of all $g(e)$ for all edges e surrounding f is also a bijection. In [147] Bača proves that for n even and at least 4, the prism $C_n \times P_2$ is $(6n + 3, 2)$-face antimagic and $(4n + 4, 4)$-face antimagic. He also conjectures that $C_n \times P_2$ is $(2n + 5, 6)$-face antimagic. In [178] Bača, Lin, and Miller investigate (a, d)-face antimagic labelings of the convex polytopes $P_{m+1} \times C_n$. They show that if these graphs are (a, d)-face antimagic then either $d = 2$ and $a = 3n(m + 1) + 3$, or $d = 4$ and $a = 2n(m + 1) + 4$, or $d = 6$ and $a = n(m + 1) + 5$. They also prove that if n is even, $n \geq 4$ and $m \equiv 1 \pmod 4$, $m \geq 3$, then $P_{m+1} \times C_n$ has a $(3n(m+1)+3, 2)$-face antimagic labeling and if n is at least 4 and even and m is at least 3 and odd, or if $n \equiv 2 \pmod 4$, $n \geq 6$ and m is even, $m \geq 4$, then $P_{m+1} \times C_n$ has a $(3n(m + 1) + 3, 2)$-face antimagic labeling and a $(2n(m+1)+4, 4)$-face antimagic labeling. They conjecture that $P_{m+1} \times C_n$ has $(3n(m+1)+3, 2)$- and $(2n(m+1)+4, 4)$-face antimagic labelings when $m \equiv 0 \pmod 4$, $n \geq 4$, and for m even and $m \geq 4$, that $P_{m+1} \times C_n$ has a $(n(m + 1) + 5, 6)$-face antimagic labeling when n is even and at least 4. Bača, Baskoro, Jendroľ, and Miller [157] proved that graphs in the shape of hexagonal honeycombs with m rows, n columns, and mn 6-sided faces have d-antimagic labelings of type $(1, 1, 1)$ for $d = 1, 2, 3$, and 4 when n odd and $mn > 1$.

In [189] Bača and Miller define the class Q_n^m of convex polytopes with vertex set $\{y_{j,i} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m + 1\}$ and edge set $\{y_{j,i}y_{j,i+1} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m + 1\} \cup \{y_{j,i}y_{j+1,i} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\} \cup \{y_{j,i+1}y_{j+1,i} : 1+1, 2, \ldots, n; j = 1, 2, \ldots, m, j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, j \text{ even}\}$ where $y_{j,n+1} = y_{j,1}$. They prove that for m odd, $m \geq 3, n \geq 3$, Q_n^m is $(7n(m + 1)/2 + 2, 1)$-face antimagic and when m and n are even, $m \geq 4, n \geq 4$, Q_n^m is $(7n(m + 1)/2 + 2, 1)$-face antimagic. They conjecture that when n is odd, $n \geq 3$, and m is even, then Q_n^m is $((5n(m+1)+5)/2, 2)-$face antimagic and $((n(m+1)+7)/2, 4)$-face antimagic. They further conjecture that when n is even, $n > 4, m > 1$ or n is odd, $n > 3$ and m is odd, $m > 1$, then Q_n^m is $(3n(m+1)/2+3, 3)$-face antimagic. In [149] Bača proves that for the case $m = 1$ and $n \geq 3$ the only possibilities for (a, d)-antimagic labelings for Q_n^m are $(7n+2, 1)$ and $(3n+3, 3)$. He provides the labelings for the first case and conjectures that they exist for the second case. Bača [145] and Bača and Miller [188] describe (a, d)-face antimagic labelings for a certain classes of convex polytopes.

In [156] Bača et al. provide a detailed survey of results on face antimagic labelings and include many conjectures and open problems.

For a plane graph G, Bača and Miller [190] call a bijection h from $V(G) \cup E(G) \cup F(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)| \cup |F(G)|\}$ a *d-antimagic labeling of type* $(1, 1, 1)$ if for every number s the set of s-sided face weights is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some integers a_s and d, where f_s is the number of s-sided faces (W_s varies with s). They show that the prisms $C_n \times P_2$ $(n \geq 3)$ have a 1-antimagic labeling of type $(1, 1, 1)$ and that for $n \equiv 3 \pmod 4$, $C_n \times P_2$ have a d-antimagic labeling of type $(1, 1, 1)$ for $d = 2, 3, 4$, and 6. They conjecture that for all $n \geq 3$, $C_n \times P_2$ has a d-antimagic labeling of type $(1, 1, 1)$ for $d = 2, 3, 4, 5$, and 6. This conjecture has been proved for the case $d = 3$ and $n \neq 4$ by Bača, Miller, and Ryan [194] (the case $d = 3$ and $n = 4$ is open). The cases for $d = 2, 4, 5$, and 6 were done by Lin, Slamin, Bača, and Miller [964]. Bača, Lin, and Miller [179] prove: for $m, n > 8$, $P_m \times P_n$ has no d-antimagic edge labeling of type $(1, 1, 1)$ with $d \geq 9$; for $m \geq 2, n \geq 2$, and $(m, n) \neq (2, 2)$, $P_m \times P_n$ has

d-antimagic labelings of type $(1,1,1)$ for $d = 1, 2, 3, 4$, and 6. They conjecture the same is true for $d = 5$.

Bača, Miller, and Ryan [194] also prove that for $n \geq 4$ the antiprism (see §6.1 for the definition) on $2n$ vertices has a d-antimagic labeling of type $(1,1,1)$ for $d = 1, 2$, and 4. They conjecture the result holds for $d = 3, 5$, and 6 as well. Lin, Ahmad, Miller, Sugeng, and Bača [961] did the cases that $d = 7$ for $n \geq 3$ and $d = 12$ for $n \geq 11$. Sugeng, Miller, Lin, and Bača [1449] did the cases: $d = 7, 8, 9, 10$ for $n \geq 5$; $d = 15$ for $n \geq 6$; $d = 18$ for $n \geq 7$; $d = 12, 14, 17, 20, 21, 24, 27, 30, 36$ for n odd and $n \geq 7$; and $d = 16, 26$ for n odd and $n \geq 9$.

Ali, Bača, Bashir, and Semaničová-Feňovčíková [69] investigated antimagic labelings for disjoint unions of prisms and cycles. They prove: for $m \geq 2$ and $n \geq 3$, $m(C_n \times P_2)$ has no super d-antimagic labeling of type $(1,1,1)$ with $d \geq 30$; for $m \geq 2$ and $n \geq 3$, $n \neq 4$, $m(C_n \times P_2)$ has super d-antimagic labeling of type $(1,1,1)$ for $d = 0, 1, 2, 3, 4$, and 5; and for $m \geq 2$ and $n \geq 3$, mC_n has $(m(n+1)+3, 3)$- and $(2mn+2, 2)$-vertex-antimagic total labeling. Bača and Bashir [152] proved that for $m \geq 2$ and $n \geq 3$, $n \neq 4$, $m(C_n \times P_2)$ has super 7-antimagic labeling of type $(1,1,1)$ and for $n \geq 3$, $n \neq 4$ and $2 \leq m \leq 2n$ $m(C_n \times P_2)$ has super 6-antimagic labeling of type $(1,1,1)$.

In [165] Bača, Brankovic, and A. Semaničová-Feňovčiková provide super d-antimagic labelings of type (1,1,1) for friendship graphs F_n $(n \geq 2)$ and several other families of planar graphs.

Bača, Jendraľ, Miller, and Ryan [174] prove: for n even, $n \geq 6$, the generalized Petersen graph $P(n,2)$ has a 1-antimagic labeling of type $(1,1,1)$; for n even, $n \geq 6$, $n \neq 10$, and $d = 2$ or 3, $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$; and for $n \equiv 0 \pmod 4$, $n \geq 8$ and $d = 6$ or 9, $P(n,2)$ has a d-antimagic labeling of type $(1,1,1)$. They conjecture that there is an d-antimagic labeling of type (1,1,1) for $P(n,2)$ when $n \equiv 2 \pmod 4$, $n \geq 6$, and $d = 6$ or 9.

Bača, Baskoro, and Miller [158] have proved that hexagonal planar honeycomb graphs with an even number of columns have 2-antimagic and 4-antimagic labelings of type $(1,1,1)$. They conjecture that these honeycombs also have d-antimagic labelings of type $(1,1,1)$ for $d = 3$ and 5. They pose the odd number of columns case for $1 \leq d \leq 5$ as an open problem. Bača, Baskoro, and Miller [159] give d-antimagic labelings of a special class of plane graphs with 3-sided internal faces for $d = 0$, 2, and 4. Bača, Lin, Miller, and Ryan [180] prove for odd $n \geq 3$, $m \geq 1$ and $d = 0, 1, 2$ or 4, the Möbius grid M_n^m has an d-antimagic labeling of type $(1,1,1)$.

Kathiresan and Ganesan [775] define a class of plane graphs denoted by P_a^b $(a \geq 3, b \geq 2)$ as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1, 2 \ldots, a-1$ joining v_i and v_{i+1} with b internally disjoint paths of length $i+1$. They prove that P_a^b has d-antimagic labelings of type $(1,1,1)$ for $d = 0, 1, 2, 3, 4$, and 6. Lin and Sugen [965] prove that P_a^b has a d-antimagic labeling of type $(1,1,1)$ for $d = 5, 7a-2, a+1, a-3, a-7, a+5, a-4, a+2, 2a-3, 2a-1, a-1, 3a-3, a+3, 2a+1, 2a+3, 3a+1, 4a-1, 4a-3, 5a-3, 3a-1, 6a-5, 6a-7, 7a-7$, and $5a-5$. Similarly, Bača, Baskoro, and Cholily [155] define a class of plane graphs denoted by C_a^b as the graph obtained by starting with vertices $v_1, v_2, \ldots, v_a$ and for each $i = 1, 2 \ldots, a$ joining v_i and v_{i+1} with b internally disjoint paths of length $i+1$ (subscripts are taken modulo a). In [155] and [154] they prove that for $a \geq 3$ and $b \geq 2$, C_a^b has a d-antimagic labeling of type $(1,1,1)$ for $d = 0, 1, 2, 3, a+1, a-1, a+2$, and $a-2$.

In [166] Ba[ca, Brankovic, and Semaničová-Feňovčikovǎ investigated the existence of super d-antimagic labelings of type (1,1,1) for plane graphs containing a special kind of Hamilton path. They proved: if there exists a Hamilton path in a plane graph G such that for every face except the external face, the Hamilton path contains all but one of the edges surrounding that face, then G is super d-antimagic of type (1,1,1) for $d = 0, 1, 2, 3, 5$; if there exists a Hamilton path in a plane graph G such that for every face except the external face, the Hamilton path contains all but one of the edges surrounding that face and if $2(|F(G)| - 1) \le |V(G)|$, then G is super d-antimagic of type $(1, 1, 1)$ for $d = 0, 1, 2, 3, 4, 5, 6$; if G is a plane graph with $M = \lfloor \frac{|V(G)|}{|F(G)|-1} \rfloor$ and a Hamilton path such that for every face, except the external face, the Hamilton path contains all but one of the edges surrounding that face, then for $M = 1$, G admits a super d-antimagic labeling of type (1,1,1) for $d = 0, 1, 2, 3, 5$; and for $M \ge 2$, G admits a super d-antimagic labeling of type (1,1,1) for $d = 0, 1, 2, 3, \dots, M + 4$. They also proved that $P_n \times P_2$ $(n \ge 3)$ admits a super d-antimagic labeling of type (1,1,1) for $d \in \{0, 1, 2, \dots, 15\}$ and the graph obtained from $P_n \times P_m$ $(n \ge 2)$ by adding a new edge in every 4-sided face such that the added edges are "parallel" admits a super d-antimagic labeling of type (1,1,1) for $d \in \{0, 1, 2, \dots, 9\}$.

In the table following we use the abbreviations

(a, d)**-FA** (a, d)-face antimagic labeling

d-**AT(1,1,1)** d-antimagic labeling of type $(1, 1, 1)$.

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Table 18: **Summary of Face Antimagic Labelings**

Graph	*Labeling*	*Notes*
Q_n^m (see §6.4)	$(7n(m+1)/2+2,1)$-FA	$m \ge 3$, $n \ge 3$, m odd [189]
	$(7n(m+1)/2+2,1)$-FA	$m \ge 4$, $n \ge 4$, m, n even [189]
	$((5n(m+1)+5)/2,2)$-FA?	$m \ge 2$, $n \ge 3$, m even, n odd [189]
	$((n(m+1)+7)/2,4)$-FA?	$m \ge 2$, $n \ge 3$, m even, n odd [189]
	$(3n(m+1)/2+3,3)$-FA?	$m > 1$, $n > 4$, n even [189]
	$(3n(m+1)/2+3,3)$-FA?	$m > 1$, $n > 3$, m odd, n odd [189]
$C_n \times P_2$	$(6n+3,2)$-FA	$n \ge 4$, n even [147]
	$(4n+4,4)$-FA	$n \ge 4$, n even [147]
	$(2n+5,6)$-FA?	[147]
$P_{m+1} \times C_n$	$(3n(m+1)+3,2)$-FA	$n \ge 4$, n even and [178] $m \ge 3$, $m \equiv 1 \pmod 4$,
	$(3n(m+1)+3,2)$-FA and $(2n(m+1)+4,4)$-FA	$n \ge 4$, n even and [178] $m \ge 3$, m odd [178], or $n \ge 6$, $n \equiv 2 \pmod 4$ and $m \ge 4$, m even
	$(3n(m+1)+3,2)$-FA?	$m \ge 4$, $n \ge 4$, $m \equiv 0 \pmod 4$ [178]
	$(2n(m+1)+4,4)$-FA?	$m \ge 4$, $n \ge 4$, $m \equiv 0 \pmod 4$ [178]
	$(n(m+1)+5,6)$-FA?	$n \ge 4$, n even [178]

Table 19: **Summary of d-antimagic Labelings of Type (1,1,1)**

Graph	*Labeling*	*Notes*
$P_m \times P_n$	not d-AT(1,1,1)	$m, n, d \geq 9$, [179]
$P_m \times P_n$	d-AT(1,1,1)	$d = 1, 2, 3, 4, 6;\ m, n \geq 2,\ (m, n) \neq (2, 2)$ [179]
$P_m \times P_n$	5-AT(1,1,1)	$m, n \geq 2,\ (m, n) \neq (2, 2)$ [179]
$C_n \times P_2$	1-AT(1,1,1) d-AT(1,1,1) d-AT(1,1,1) d-AT(1,1,1)	[190] $d = 2, 3, 4$ and 6 [190] for $n \equiv 3 \pmod 4$ $d = 2, 4, 5, 6$ for $n \geq 3$ [964] $d = 3$ for $n \geq 5$ [194]
$P_m \times P_n$	5-AT(1,1,1)? not d-AT	[964] $m, n > 8,\ d \geq 9$ [964]
antiprism on $2n$ vertices	d-AT(1,1,1) d-AT(1,1,1)?	$d = 1, 2$ and 4 for $n \geq 4$ [194] $d = 3, 5$ and 6 for $n \geq 4$ [194]
M_n^m (Möbius grids)	d-AT(1,1,1)	$n \geq 3$ odd, $d = 0, 1, 2, 4$ [180] $d = 7,\ n \geq 3$ [961] $d = 12,\ n \geq 11$ [961] $d = 7, 8, 9, 10,\ n \geq 5$ [1449] $d = 15,\ n \geq 6$ [1449] $d = 18\ n \geq 7$ [1449]
$P(n, 2)$	d-AT(1,1,1)	$d = 1;\ d = 2, 3,\ n \geq 6,\ n \neq 10$ [174]
$P(4n, 2)$	d-AT(1,1,1)	$d = 6, 9,\ n \geq 2,\ n \neq 10$ [174]
$P(4n + 2, 2)$	d-AT(1,1,1)?	$d = 6, 9,\ n \geq 1,\ n \neq 10$ [174]
honeycomb graphs with even number of columns	d-AT(1,1,1) d-AT(1,1,1)?	$d = 2, 4$ [158] $d = 3, 5$ [158]
$C_n \times P_2$	d-AT(1,1,1)	$d = 1, 2, 4, 5, 6$ [964], [190]
$C_n \times P_2$	3-AT(1,1,1)	$n \neq 4$ [194]

6.5 Product Antimagic Labelings

Figueroa-Centeno, Ichishima, and Muntaner-Batle [494] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph G of size q to be *product magic* if there is a labeling from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex v, the product of the labels on the edges incident with v is the same. They call a graph G of size q *product antimagic* if there is a labeling f from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that the products of the labels on the edges incident at each vertex v are distinct. They prove: a graph of size q is product magic if and only if $q \leq 1$ (that is, if and only if it is $K_2, \overline{K_n}$ or $K_2 \cup \overline{K_n}$); P_n $(n \geq 4)$ is product antimagic; every 2-regular graph is product antimagic; and, if G is product antimagic, then so are $G + K_1$ and $G \odot \overline{K}_n$. They conjecture that a connected graph of size q is product antimagic if and only if $q \geq 3$. Kaplan, Lev, and Roditty [766] proved the following graphs are product anti-magic: the disjoint union of cycles and paths where each path has least three edges; connected graphs with n vertices and m edges where $m \geq 4n\ln n$; graphs $G = (V, E)$ where each component has at least two edges and the minimum degree of G is at least $8\sqrt{\ln |E| \ln (\ln |E|)}$; all complete k-partite graphs except K_2 and $K_{1,2}$; and $G \odot H$ where G has no isolated vertices and H is regular.

In [1140] Pikhurko characterizes all large graphs that are product anti-magic graphs. More precisely, it is shown that there is an n_0 such that a graph with $n \geq n_0$ vertices is product anti-magic if and only if it does not belong to any of the following four classes: graphs that have at least one isolated edge; graphs that have at least two isolated vertices; unions of vertex-disjoint of copies of $K_{1,2}$; graphs consisting of one isolated vertex; and graphs obtained by subdividing some edges of the star $K_{1,k+l}$.

In [494] Figueroa-Centeno, Ichishima, and Muntaner-Batle also define a graph G with p vertices and q edges to be *product edge-magic* if there is a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, \ldots, p + q\}$ such that $f(u) \cdot f(v) \cdot f(uv)$ is a constant for all edges uv and *product edge-antimagic* if there is a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, \ldots, p+q\}$ such that for all edges uv the products $f(u) \cdot f(v) \cdot f(uv)$ are distinct. They prove $K_2 \cup \overline{K}_n$ is product edge-magic, a graph of size q without isolated vertices is product edge-magic if and only if $q \leq 1$ and every graph other than K_2 and $K_2 \cup \overline{K}_n$ is product edge-antimagic.

7 Miscellaneous Labelings

7.1 Sum Graphs

In 1990, Harary [610] introduced the notion of a sum graph. A graph $G(V, E)$ is called a *sum graph* if there is an bijection f from V to a set of positive integers S such that $xy \in E$ if and only if $f(x) + f(y) \in S$. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In 1991 Harary, Hentzel, and Jacobs [612] defined a *real sum graph* in an analogous way by allowing S to be any finite set of positive real numbers. However, they proved that every real sum graph is a sum graph. Bergstrand, Hodges, Jennings, Kuklinski, Wiener, and Harary [260] defined a *product graph* analogous to a sum graph except that 1 is not permitted to belong to S. They proved that every product graph is a sum graph and vice versa.

For a connected graph G, let $\sigma(G)$, the *sum number of* G, denote the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph (some authors use $s(G)$ for the sum number of G). A labeling that makes G together with $\sigma(G)$ isolated points a sum graph is called an *optimal sum graph* labeling. Ellingham [453] proved the conjecture of Harary [610] that $\sigma(T) = 1$ for every tree $T \neq K_1$. Smyth [1406] proved that there is no graph G with e edges and $\sigma(G) = 1$ when $n^2/4 < e \leq n(n-1)/2$. Smyth [1407] conjectures that the disjoint union of graphs with sum number 1 has sum number 1. More generally, Kratochvil, Miller, and Nguyen [829] conjecture that $\sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1$. Hao [608] has shown that if $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of a graph G, then $\sigma(G) > \max(d_i - i)$ where the maximum is taken over all i. Bergstand et al. [259] proved that $\sigma(K_n) = 2n - 3$. Hartsfield and Smyth [617] claimed to have proved that $\sigma(K_{m,n}) = \lceil 3m + n - 3\rceil/2$ when $n \geq m$ but Yan and Liu [1674] found counterexamples to this assertion when $m \neq n$. Pyatkin [1170], Liaw, Kuo, and Chang [958], Wang and Liu [1631], and He, Shen, Wang, Chang, Kang, and Yu [622] have shown that for $2 \leq m \leq n$, $\sigma(K_{m,n}) = \lceil \frac{n}{p} + \frac{(p+1)(m-1)}{2} \rceil$ where $p = \lceil \sqrt{\frac{2n}{m-1} + \frac{1}{4}} - \frac{1}{2} \rceil$ is the unique integer such that $\frac{(p-1)p(m-1)}{2} < n \leq \frac{(p+1)p(m-1)}{2}$.

Miller, Ryan, Slamin, and Smyth [1060] proved that $\sigma(W_n) = \frac{n}{2} + 2$ for n even and $\sigma(W_n) = n$ for $n \geq 5$ and n odd (see also [1473]). Miller, Ryan, and Smyth [1062] prove that the complete n-partite graph on n sets of 2 nonadjacent vertices has sum number $4n-5$ and obtain upper and lower bounds on the complete n-partite graph on n sets of m nonadjacent vertices. Fernau, Ryan, and Sugeng [491] proved that the generalized friendship graphs $C_n^{(t)}$ (see §2.2) has sum number 2 except for C_4. Gould and Rödl [581] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [580] proved that $\sigma(K_n - \text{edge}) \leq 2n - 4$. The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the *spum*. They proved spum of K_n is $4n - 6$ and the spum of C_n is at most $4n - 10$. Kratochvil, Miller, and Nguyen [829] have proved that every sum graph on n vertices has a sum labeling such that every label is at most 4^n.

At a conference in 2000 Miller [1050] posed the following two problems: Given any graph G, does there exist an optimal sum graph labeling that uses the label 1; Find a class of graphs G that have sum number of the order $|V(G)|^s$ for $s > 1$. (Such graphs were shown to exist for $s = 2$ by Gould and Rödl in [581]).

In [1392] Slamet, Sugeng, and Miller show how one can use sum graph labelings to distribute secret information to set of people so that only authorized subsets can reconstruct the secret.

Chang [353] generalized the notion of sum graph by permitting $x = y$ in the definition of sum graph. He calls graphs that have this kind of labeling *strong sum graphs* and uses $i^*(G)$ to denote the minimum positive integer m such that $G \cup mK_1$ is a strong sum graph. Chang proves that $i^*(K_n) = \sigma(K_n)$ for $n = 2, 3$, and 4 and $i^*(K_n) > \sigma(K_n)$ for $n \geq 5$. He further shows that for $n \geq 5$, $3n^{\log_2 3} > i^*(K_n) \geq 12\lfloor n/5 \rfloor - 3$.

In 1994 Harary [611] generalized sum graphs by permitting S to be any set of integers. He calls these graphs *integral sum graphs.* Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [1326] has shown that C_n and W_n are integral sum graphs for all $n \neq 4$. Chen [374] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. In [374] and [376] Chen gives methods for constructing new connected integral sum graphs from given integral sum graphs by identifying vertices. Chen [376] has shown that every graph is an induced subgraph of a connected integral sum graph. Chen [376] calls a vertex of a graph *saturated* if it is adjacent to every other vertex of the graph. He proves that every integral sum graph except K_3 has at most two saturated vertices and gives the exact structure of all integral sum graphs that have exactly two saturated vertices. Chen [376] also proves that a connected integral sum graph with $p > 1$ vertices and q edges and no saturated vertices satisfies $q \leq p(3p - 2)/8 - 2$. Wu, Mao, and Le [1651] proved that mP_n are integral sum graphs. They also show that the conjecture of Harary [611] that the sum number of C_n equals the integral sum number of C_n if and only if $n \neq 3$ or 5 is false and that for $n \neq 4$ or 6 the integral sum number of C_n is at most 1. Vilfred and Nicholas [1583] prove that graphs G of order n with $\Delta(G) = n - 1$ and $|V_\Delta(G)| > 2$ are not integral sum graphs, except K_3, and that integral sum graphs G of order n with $\Delta(G) = n - 1$ and $|V_\Delta(G)| = 2$ exist and are unique up to isomorphism. Chen [378] proved that if $G(V, E)$ is an integral sum other than K_3 that has vertex of degree $|V| - 1$, then the edge-chromatic number of G is $|V| - 1$.

He, Wang, Mi, Shen, and Yu [620] say that a graph has a *tail* if the graph contains a path for which each interior vertex has degree 2 and an end vertex of degree at least 3. They prove that every tree with a tail of length at least 3 is an integral sum graph.

B. Xu [1661] has shown that the following are integral sum graphs: the union of any three stars; $T \cup K_{1,n}$ for all trees T; mK_3 for all m; and the union of any number of integral sum trees. Xu also proved that if $2G$ and $3G$ are integral sum graphs, then so is mG for all $m > 1$. Xu poses the question as to whether all disconnected forests are integral sum graphs. Nicholas and Somasundaram [1104] prove that all banana trees (see Section 2.1 for the definition) and the union of any number of stars are integral sum graphs.

Liaw, Kuo, and Chang [958] proved that all caterpillars are integral sum graphs (see also [1651] and [1661] for some special cases of caterpillars). This shows that the assertion by Harary in [611] that $K(1, 3)$ and $S(2, 2)$ are not integral sum graphs is incorrect. They also prove that all cycles except C_4 are integral sum graphs and they conjecture that every tree is an integral sum graph. Singh and Santhosh show that the crowns $C_n \odot K_1$ are integral sum graphs for $n \geq 4$ [1381] and that the subdivision graphs of $C_n \odot K_1$ are integral sum graphs for $n \geq 3$ [1242].

For graphs with n vertices, Tiwari and Tripathi [1491] show that there exist sum graphs with m edges if and only if $m \leq \lfloor (n-1^2)/4 \rfloor$ and that there exists integral sum graphs with m edges if and only if $m \leq \lceil 3(n-1)^2/8 \rceil + \lfloor (n-1)/2 \rfloor$, except for $m = \lceil 3(n-1)^2/8 \rceil + \lfloor (n-1)/2 \rfloor - 1$ when n is of the form $4k + 1$. They also characterize sets of positive integers (respectively, integers) that are in bijection with sum graphs (respectively, integral sum graphs) of maximum size for a given order.

The *integral sum number*, $\zeta(G)$, of G is the minimum number of isolated vertices that must be added to G so that the resulting graph is an integral sum graph. Thus, by definition, G is a integral sum graph if and only if $\zeta(G) = 0$. Harary [611] conjectured that $\zeta(K_n) = 2n-3$ for $n \geq 4$. This conjecture was verified by Chen [373], by Sharary [1326], and by B. Xu [1661]. Yan and Liu proved: $\zeta(K_n - E(K_r)) = n-1$ when $n \geq 6,\ n \equiv 0 \pmod 3$ and $r = 2n/3 - 1$ [1675]; $\zeta(K_{m.m}) = 2m-1$ for $m \geq 2$ [1675]; $\zeta(K_n \backslash -\text{edge}) = 2n-4$ for $n \geq 4$ [1675], [1661]; if $n \geq 5$ and $n-3 \geq r$, then $\zeta(K_n \backslash E(K_r)) \geq n-1$ [1675]; if $\lceil 2n/3 \rceil - 1 > r \geq 2$, then $\zeta(K_n \backslash E(K_r)) \geq 2n-r-2$ [1675]; and if $2 \leq m < n$, and $n = (i+1)(im-i+2)/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = (m-1)(i+1)+1$ while if $(i+1)(im-i+2)/2 < n < (i+2)[(i+1)m-i+1]/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = \lceil((m-1)(i+1)(i+2)+2n)/(2i+2)\rceil$ [1675]. Wang [1608] proved that $\sigma(K_{n+1} \backslash E(K_{1,r})) = \zeta(K_{n+1} \backslash E(K_{1,r})) = 2n-2$ when $r+1,\ 2n-3$ when $2 \leq r \leq n-1$, and $2n-4$ when $r = n$.

Nagamochi, Miller, and Slamin [1087] have determined upper and lower bounds on the sum number a graph. For most graphs $G(V,E)$ they show that $\sigma(G) = \Omega(|E|)$. He, Yu, Mi, Sheng, and Wang [621] investigated $\zeta(K_n \backslash E(K_r))$ where $n \geq 5$ and $r \geq 2$. They proved that
$\zeta(K_n \backslash E(K_r)) = 0$ when $r = n$ or $n-1$;
$\zeta(K_n \backslash E(K_r)) = n-2$ when $r = n-2$;
$\zeta(K_n \backslash E(K_r)) = n-1$ when $n-3 \geq r \geq \lceil 2n/3 \rceil - 1; \zeta(K_n \backslash E(K_r)) = 3n-2r-4$ when $\lceil 2n/3 \rceil - 1 > r \geq n/2$;
$\zeta(K_n \backslash E(K_r)) = 2n-4$ when $\lceil 2n/3 \rceil - 1 \geq n/2 > r \geq 2$. Moreover, they prove that if $n \geq 5, r \geq 2$, and $r \neq n-1$, then $\sigma(K_n \backslash E(K_r)) = \zeta(K_n \backslash E(K_r))$.

Dou and Gao [443] prove that for $n \geq 3$, the fan $F_n = P_n + K_1$ is an integral sum graph, $\rho(F_4) = 1, \rho(F_n) = 2$ for $n \neq 4$, and $\sigma(F_4) = 2, \sigma(F_n) = 3$ for $n = 3$ or $n \geq 6$ and n even, and $\sigma(F_n) = 4$ for $n \geq 6$ and n odd.

Wang and Gao [1609] and [1610] determined the sum numbers and the integral sum numbers of the complements of paths, cycles, wheels, and fans as follows.
$0 = \zeta(\overline{P_4}) < \sigma(\overline{P_4}) = 1$; $1 = \zeta(\overline{P_5}) < \sigma(\overline{P_5}) = 2$;
$3 = \zeta(\overline{P_6}) < \sigma(\overline{P_6}) = 4$; $\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 0,\ n = 1,2,3$;
$\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 2n-7,\ n \geq 7$.
$\zeta(\overline{C_n}) = \sigma(\overline{C_n}) = 2n-7,\ n \geq 7$.
$\zeta(\overline{W_n}) = \sigma(\overline{W_n}) = 2n-8,\ n \geq 7$.
$0 = \zeta(\overline{F_5}) < \sigma(\overline{F_5}) = 1$;
$2 = \zeta(\overline{F_6}) < \sigma(\overline{F_6}) = 3$; $\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 0, n = 3,4$;
$\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 2n-8,\ n \geq 7$.

Wang, Yang and Li [1613] proved:
$\zeta(K_n \backslash E(C_{n-1}) = 0$ for $n = 4,5,6,7$;
$\zeta(K_n \backslash E(C_{n-1}) = 2n-7$ for $n \geq 8$;
$\sigma(K_4 \backslash E(C_{n-1}) = 1$;
$\sigma(K_5 \backslash E(C_{n-1}) = 2$;
$\sigma(K_6 \backslash E(C_{n-1}) = 5$;
$\sigma(K_7 \backslash E(C_{n-1}) = 7$;
$\sigma(K_n \backslash E(C_{n-1}) = 2n-7$ for $n \geq 8$.

Wang and Li [1612] proved: a graph with $n \geq 6$ vertices and degree greater than $(n+1)/2$ is not an integral sum graph; for $n \geq 8,\ \zeta(K_n \setminus E(2P_3)) = \sigma(K_n \setminus E(2P_3)) = \epsilon(K_n \setminus E(2P_3)) = \epsilon(K_n \setminus E(2P_3)) = 2n-7$; for $n \geq 7,\ \zeta(K_n \setminus E(K_2)) = \sigma(K_n \setminus E(K_2)) = 2n-4$; and for $n \geq 7$ and $1 \leq r \leq \lceil \frac{n}{2} \rceil,\ \zeta(K_n \setminus E(rK_2)) = \sigma(K_n \setminus E(rK_2)) = 2n-5$.

Chen [373] has given some properties of integral sum labelings of graphs G with $\Delta(G) <$

$|V(G)|-1$ whereas Nicholas, Somasundaram, and Vilfred [1106] provided some general properties of connected integral sum graphs G with $\Delta(G) = |V(G)| - 1$. They have shown that connected integral sum graphs G other than K_3 with the property that G has exactly two vertices of maximum degree are unique and that a connected integral sum graph G other than K_3 can have at most two vertices with degree $|V(G)| - 1$ (see also [1596]).

Vilfred and Florida [1593] have examined one-point unions of pairs of small complete graphs. They show that the one-point union of K_3 and K_2 and the one-point union of K_3 and K_3 are integral sum graphs whereas the one-point union of K_4 and K_2 and the one-point union of K_4 and K_3 are not integral sum graphs. In [1594] Vilfred and Florida defined and investigated properties of maximal integral sum graphs.

Vilfred and Nicholas [1597] have shown that the following graphs are integral sum graphs: banana trees, the union of any number of stars, fans $P_n + K_1$ $(n \geq 2)$, Dutch windmills $K_3^{(m)}$, and the graph obtained by starting with any finite number of integral sum graphs $G_1, G_2, \ldots, G_n$ and any collections of n vertices with $v_i \in G_i$ and creating a graph by identifying $v_1, v_2, \ldots, v_n$. The same authors [1598] also proved that $G + v$ where G is a union of stars is an integral sum graph.

Melnikov and Pyatkin [1046] have shown that every 2-regular graph except C_4 is an integral sum graph and that for every positive integer r there exists an r-regular integral sum graph. They also show that the cube is not an integral sum graph. For any integral sum graph G, Melnikov and Pyatkin define the *integral radius of* G as the smallest natural number $r(G)$ that has all its vertex labels in the interval $[-r(G), r(G)]$. For the family of all integral sum graphs of order n they use $r(n)$ to denote maximum integral radius among all members of the family. Two questions they raise are: Is there a constant C such that $r(n) \leq C_n$ and for $n > 2$, is $r(n)$ equal to the $(n-2)$th prime?

The concepts of sum number and integral sum number have been extended to hypergraphs. Sonntag and Teichert [1426] prove that every hypertree (i.e., every connected, non-trivial, cycle-free hypergraph) has sum number 1 provided that a certain cardinality condition for the number of edges is fulfilled. In [1427] the same authors prove that for $d \geq 3$ every d-uniform hypertree is an integral sum graph and that for $n \geq d+2$ the sum number of the complete d-uniform hypergraph on n vertices is $d(n-d)+1$. They also prove that the integral sum number for the complete d-uniform hypergraph on n vertices is 0 when $d = n$ or $n-1$ and is between $(d-1)(n-d-1)$ and $d(n-d)+1$ for $d \leq n-2$. They conjecture that for $d \leq n-2$ the sum number and the integral sum number of the complete d-uniform hypergraph are equal. Teichert [1486] proves that hypercycles have sum number 1 when each edge has cardinality at least 3 and that hyperwheels have sum number 1 under certain restrictions for the edge cardinalities. (A *hypercycle* $\mathcal{C}_n = (\mathcal{V}_n, \mathcal{E}_n)$ has $\mathcal{V}_n = \cup_{i=1}^n \{v_1^i, v_2^i, \ldots, v_{d_i-1}^i\}, \mathcal{E}_n = \{e_1, e_2, \ldots, e_n\}$ with $e_i = \{v_1^i, \ldots, v_{d_i}^i = v_1^{i+1}\}$ where $i+1$ is taken modulo n. A *hyperwheel* $\mathcal{W}_n = (\mathcal{V}'_n, \mathcal{E}'_n)$ has $\mathcal{V}'_n = \mathcal{V}_n \cup \{c\} \cup_{i=1}^n \{{v_2}^{n+i}, \ldots, {v_{d_{n+i}-1}}^{n+i}\}, \mathcal{E}'_n = \mathcal{E}_n \cup \{e_{n+1}, \ldots, e_{2n}\}$ with $e_{n+i} = \{{v_1}^{n+i} = c, {v_2}^{n+i}, \ldots, {v_{d_{n+i}-1}}^{n+i}, {v_{d_{n+i}}}^{n+i} = {v_1}^i\}$.)

Teichert [1485] determined an upper bound for the sum number of the d-partite complete hypergraph $K^d_{n_1,\ldots,n_d}$. In [1487] Teichert defines the *strong hypercycle* $\mathcal{C}^d_n$ to be the d-uniform hypergraph with the same vertices as C_n where any d consecutive vertices of C_n form an edge of $\mathcal{C}^d_n$. He proves that for $n \geq 2d+1 \geq 5$, $\sigma(\mathcal{C}^d_n) = d$ and for $d \geq 2$, $\sigma(\mathcal{C}^d_{d+1}) = d$. He also shows that $\sigma(\mathcal{C}^3_5) = 3$; $\sigma(\mathcal{C}^3_6) = 2$, and he conjectures that $\sigma(\mathcal{C}^d_n) < d$ for $d \geq 4$ and $d+2 \leq n \leq 2d$.

In [1107] Nicholas and Vilfred define the *edge reduced sum number* of a graph as the minimum number of edges whose removal from the graph results in a sum graph. They show that for K_n, $n \geq 3$, this number is $(n(n-1)/2 + \lfloor n/2 \rfloor)/2$. They ask for a characterization of graphs for which the edge reduced sum number is the same as its sum number. They conjecture that an integral sum graph of order p and size q exists if and only if $q \leq 3(p^2-1)/8 - \lfloor (p-1)/4 \rfloor$ when p is odd and $q \leq 3(3p-2)/8$ when p is even. They also define the *edge reduced integral sum number* in an analogous way and conjecture that for K_n this number is $(n-1)(n-3)/8 + \lfloor (n-1)/4 \rfloor$ when n is odd and $n(n-2)/8$ when n is even.

For certain graphs G Vilfred and Florida [1592] investigated the relationships among $\sigma(G), \zeta(G), \chi(G)$, and $\chi'(G)$ where $\chi(G)$ is the chromatic number of G and $\chi'(G)$ is the edge chromatic number of G. They prove: $\sigma(C_4) = \zeta(C_4) > \chi(C_4) = \chi'(C_4)$; for $n \geq 3$, $\zeta(C_{2n}) < \sigma(C_{2n}) = \chi(C_{2n}) = \chi'(C_{2n})$; $\zeta(C_{2n+1}) < \sigma(C_{2n+1}) < \chi(C_{2n+1}) = \chi'(C_{2n+1})$; for $n \geq 4$, $\chi'(K_n) \leq \chi(K_n) < \zeta(K_n) = \sigma(K_n)$; and for $n \geq 2$, $\chi(P_n \times P_2) < \chi'(P_n \times P_2) = \zeta(P_n \times P_2) = \sigma(P_n \times P_2)$.

Alon and Scheinermann [79] generalized sum graphs by replacing the condition $f(x)+f(y) \in S$ with $g(f(x), f(y)) \in S$ where g is an arbitrary symmetric polynomial. They called a graph with this property a *g-graph* and proved that for a given symmetric polynomial g not all graphs are g-graphs. On the other hand, for every symmetric polynomial g and every graph G there is some vertex labeling such that G together with at most $|E(G)|$ isolated vertices is a g-graph.

Boland, Laskar, Turner, and Domke [299] investigated a modular version of sum graphs. They call a graph $G(V,E)$ a *mod sum graph* (MSG) if there exists a positive integer n and an injective labeling from V to $\{1, 2, \ldots, n-1\}$ such that $xy \in E$ if and only if $(f(x)+f(y))$ (mod n) $= f(z)$ for some vertex z. Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [299] have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and $K_{2,n}$. They further proved that K_p ($p \geq 2$) is not MSG (see also [571]) and that W_4 is MSG. They conjecture that W_p is MSG for $p \geq 4$. This conjecture was refuted by Sutton, Miller, Ryan, and Slamin [1474] who proved that for $n \neq 4$, W_n is not MSG (the case where n is prime had been proved in 1994 by Ghoshal, Laskar, Pillone, and Fricke [571]. In the same paper Sutton et al. also showed that for $n \geq 3$, $K_{n,n}$ is not MSG. Ghoshal, Laskar, Pillone, and Fricke [571] proved that every connected graph is an induced subgraph of a connected MSG graph and any graph with n vertices and at least two vertices of degree $n-1$ is not MSG.

Sutton, Miller, Ryan, and Slamin [1474] define the *mod sum number*, $\rho(G)$, of a connected graph G to be the least integer r such that $G \cup \overline{K_r}$ is MSG. Recall the cocktail party graph $H_{m,n}$, $m, n \geq 2$, as the graph with a vertex set $V = \{v_1, v_2, \ldots, v_{mn}\}$ partitioned into n independent sets $V = \{I_1, I_2, \ldots, I_n\}$ each of size m such that $v_i v_j \in E$ for all $i, j \in \{1, 2, \ldots, mn\}$ where $i \in I_p$, $j \in I_q$, $p \neq q$. The graphs $H_{m,n}$ can be used to model relational database management systems (see [1470]). Sutton and Miller [1472] prove that $H_{m,n}$ is not MSG for $n > m \geq 3$ and $\rho(K_n) = n$ for $n \geq 4$. In [1471] Sutton, Draganova, and Miller prove that for n odd and $n \geq 5$, $\rho(W_n) = n$ and when n is even, $\rho(W_n) = 2$. Wang, Zhang, Yu, and Shi [1629] proved that fan $F_n (n \geq 2)$ are not mod sum graphs and $\rho(F_n) = 2$ for even n at least 6. They also prove that $\rho(K_{n,n}) = n$ for $n \geq 3$.

Dou and Gao [444] obtained exact values for $\rho(K_{m,n})$ and $\rho(K_m - E(K_n))$ for some cases of m and n and bounds in the remaining cases. They call a graph $G(V,E)$ a *mod integral sum graph* if there exists a positive integer n and an injective labeling from V to $\{0, 1, 2, \ldots, n-1\}$ (note that 0 is included) such that $xy \in E$ if and only if $(f(x)+f(y))$ (mod n) $= f(z)$ for some vertex z. They define the *mod integral sum number*, $\psi(G)$, of a connected graph G to be the least integer r such that $G \cup \overline{K_r}$ is a mod integral sum graph. They prove that for $m+n \geq 3$, $\psi(K_{m,n}) = \rho(K_{m,n})$ and obtained exact values for $\psi(K_m - E(K_n))$ for some cases of m and n and bounds in the remaining cases.

Wallace [1601] has proved that $K_{m,n}$ is MSG when n is even and $n \geq 2m$ or when n is odd and $n \geq 3m-3$ and that $\rho(K_{m,n}) = m$ when $3 \leq m \leq n < 2m$. He also proves that the complete m-partite $K_{n_1,n_2,\ldots,n_m}$ is not MSG when there exist n_i and n_j such that $n_i < n_j < 2n_i$. He poses the following conjectures: $\rho(K_{m,n}) = n$ when $3m-3 > n \geq m \geq 3$; if $K_{n_1,n_2,\ldots,n_m}$ where $n_1 > n_2 > \cdots > n_m$, is not MSG, then $(m-1)n_m \leq \rho(K_{n_1,n_2,\ldots,n_m}) \leq (m-1)n_1$; if G has n vertices, then $\rho(G) \leq n$; and determining the mod sum number of a graph is NP-complete (Sutton has observed that Wallace probably meant to say 'NP-hard'). Miller [1050] has asked if it is possible for the mod sum number of a graph G be of the order $|V(G)|^2$.

In a sum graph G, a vertex w is called a *working vertex* if there is an edge uv in G such that $w = u+v$. If $G = H \cup \overline{H_r}$ has a sum labeling such that H has no working vertex the labeling is called an *exclusive sum labeling of H with respect G*. The *exclusive sum number*, $\epsilon(H)$, of a graph H is the smallest integer r such that $G \cup \overline{K_r}$ has an exclusive sum labeling. The exclusive sum number is known in the following cases (see [1054] and [1061]): for $n \geq 3$, $\epsilon(P_n) = 2$; for $n \geq 3$, $\epsilon(C_n) = 3$; for $n \geq 3$, $\epsilon(K_n) = 2n-3$; for $n \geq 4$, $\epsilon(F_n) = n$ (fan of order $n+1$); for $n \geq 4$, $\epsilon(W_n) = n$; $\epsilon(C_3^{(n)}) = 2n$ (friendship graph–see §2.2); $m \geq 2$, $n \geq 2$, $\epsilon(K_{m,n}) = m+n-1$; for $n \geq 2$, $S_n = n$ (star of order $n+1$); $\epsilon(S_{m,n}) = \max\{m,n\}$ (double star); $H_{2,n} = 4n-5$ (cocktail party graph); and $\epsilon(\text{caterpillar } G) = \Delta(G)$. Vilfred and Florida [1595] proved that $\epsilon(P_3 \times P_3) = 4$ and $\epsilon(P_n \times P_2) = 3$. In [652] Hegde and Vasudeva provide an $O(n^2)$ algorithm that produces an exclusive sum labeling of a graph with n vertices given its adjacency matrix.

In 2001 Kratochvil, Miller, and Nguyen proved that $\sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1$. In 2003 Miller, Ryan, Slamin, Sugeng, and Tuga [1057] posed the problem of finding the exclusive sum number of the disjoint union of graphs. In 2010 Wang and Li [1611] proved the following. Let G_1 and G_2 be graphs without isolated vertices, L_i be an exclusive sum labeling of $G_i \cup \epsilon(G_i)K_1$, and C_i be the isolated set of L_i for $i = 1$ and 2. If $\max C_1$ and $\min C_2$ are relatively prime, then $\epsilon(G_1 \cup G_2) \leq \epsilon(G_1) + \epsilon(G_2) - 1$. Wang and Li also proved the following: $\epsilon(K_{r,s}) = s+r-1$; $\epsilon(K_{r,s} - E(K_2)) = s-1$; for $s \geq r \geq 2$, $\epsilon(K_{r,s} - E(rK_2)) = s+r-3$. For $n \geq 5$ they prove: $\epsilon(K_n - E(K_n)) = 0$; $\epsilon(K_n - E(K_{n-1})) = n-1$; for $2 \leq r < n/2$, $\epsilon(K_n - E(K_r)) = 2n-4$; for $n/2 \leq r \leq n-2$, $\epsilon(K_n - E(K_r)) = 3n-2r-4$, and $\epsilon(C_n \odot K_1)$ is 3 or 4. They show that $\epsilon(C_3 \odot K_1) = 3$ and guess that for $n \geq 4$, $\epsilon(C_n \odot K_1) = 4$. A survey of exclusive sum labelings of graphs is given by Ryan in [1216].

If $\epsilon(G) = \Delta(G)$, then G is said to be an Δ-*optimum summable graph.* An exclusive sum labeling of a graph G using $\Delta(G)$ isolates is called a Δ-*optimum exclusive sum labeling* of G. Tuga, Miller, Ryan, and Ryjáček [1502] show that some families of trees that are Δ-optimum summable and some that are not. They prove that if G is a tree that has at least one vertex that has two or more neighbors that are not leaves then $\epsilon(G) = \Delta(G)$.

Grimaldi [599] has investigated labeling the vertices of a graph $G(V,E)$ with n vertices with distinct elements of the ring Z_n so that $xy \in E$ whenever $(x+y)^{-1}$ exists in Z_n.

In his 2001 Ph. D. thesis Sutton [1470] introduced two methods of graph labelings with applications to storage and manipulation of relational database links specifically in mind. He calls a graph $G = (V_p \cup V_i, E)$ a *sum* graph* of $G_p = (V_p, E_p)$ if there is an injective labeling λ of the vertices of G with non-negative integers with the property that $uv \in E_p$ if and only if $\lambda(u) + \lambda(v) = \lambda(z)$ for some vertex $z \in G$. The *sum* number*, $\sigma^*(G_p)$, is the minimum cardinality of a set of new vertices V_i such that there exists a sum* graph of G_p on the set of vertices $V_p \cup V_i$. A *mod sum* graph* of G_p is defined in the identical fashion except the sum $\lambda(u) + \lambda(v)$ is taken modulo n where the vertex labels of G are restricted to $\{0, 1, 2, \ldots, n-1\}$. The *mod sum* number*, $\rho^*(G_p)$, of a graph G_p is defined in the analogous way. Sum* graphs are a generalization of sum graphs and mod sum* graphs are a generalization of mod sum graphs. Sutton shows that every graph is an induced subgraph of a connected sum* graph. Sutton [1470] poses the following conjectures: $\rho(H_{m,n}) \leq mn$ for $m, n \geq 2$; $\sigma^*(G_p) \leq |V_p|$; and $\rho^*(G_p) \leq |V_p|$.

The following table summarizing what is known about sum graphs, mod sum graphs, sum* graphs, and mod sum* graphs is reproduced from Sutton's Ph. D. thesis [1470]. It was updated by J. Gallian in 2006. A question mark indicates the value is unknown. The results on sum* and mod sum* graphs are found in [1470].

Table 20: **Summary of Sum Graph Labelings**

Graph	$\sigma(G)$	$\rho(G)$	$\sigma^*(G)$	$\rho^*(G)$
$K_2 = S_1$	1	1	0	0
stars, $S_n,\ n \geq 2$	1	0	0	0
trees $T_n,\ n \geq 3$ when $T_n \neq S_n$	1	0	1	0
C_3	2	1	1	0
C_4	3	0	2	0
$C_n,\ n > 4$	2	0	2	0
W_4	4	0	2	0
$W_n,\ n \geq 5,\ n$ odd	n	n	2	0
$W_n,\ n \geq 6,\ n$ even	$\frac{n}{2}+2$	2	2	0
fan, F_4,	2	1	1	0
fans, $F_n,\ n \geq 5,\ n$ odd	?	2	1	0
fans, $F_n,\ n \geq 6,\ n$ even	3	2	1	0
$K_n,\ n \geq 4$	$2n-3$	n	$n-2$	0
cocktail party graphs, $H_{2,n}$	$4n-5$	0	?	0
$C_n^{(t)}$ $(n,t) \neq (4,1)$ (see §2.2)	2	?	?	?
$K_{n,n}$	$\lceil \frac{4n-3}{2} \rceil$	$n (n \geq 3)$	?	?
$K_{m,n},\ 2nm \geq n \geq 3$	?	n	?	?
$K_{m,n}$ $m \geq 3n-3$, $n \geq 3$, m odd	?	0	?	0
$K_{m,n},\ m \geq 2n$, $n \geq 3$, m even	?	0	?	0
$K_{m,n},\ m < n$ $k = \lceil \sqrt{1+(8m+n-1)(n-1)/2}\ \rceil$	$\lceil (kn-k)/2 + m/(k-1) \rceil$	?	?	?
$K_{n,n} - E(nK_2),\ n \geq 6$	$2n-3$	$n-2$	?	?

7.2 Prime and Vertex Prime Labelings

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla [1493]. A graph with vertex set V is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge xy the labels assigned to x and y are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. Little progress was made on this conjecture until 2011 when Haxell, Pikhurko, Taraz [618] proved that all large trees are prime. Also, their method allowed them to determine the smallest size of a non-prime connected order-n graph for all large n, proving a conjecture of Rao [1194] in this range. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with one vertex of degree at least 3 and with all other vertices with degree at most 2), olive trees (i.e., a rooted tree consisting of k branches such that the ith branch is a path of length i), all trees of order up to 50, palm trees (i.e., trees obtained by appending identical stars to each vertex of a path), banana trees, and binomial trees (the binomial tree B_0 of order 0 consists of a single vertex; the binomial tree B_n of order n has a root vertex whose children are the roots of the binomial trees of order $0, 1, 2, \ldots, n-1$ (see [1137], [1139], [1493], [521] and [1204]). Seoud, Sonbaty, and Mahran [1286] provide necessary and sufficient conditions for a graph to be prime. They also give a procedure to determine whether or not a graph is prime.

Other graphs with prime labelings include all cycles and the disjoint union of C_{2k} and C_n [425]. The complete graph K_n does not have a prime labeling for $n \geq 4$ and W_n is prime if and only if n is even (see [944]).

Seoud, Diab, and Elsakhawi [1266] have shown the following graphs are prime: fans; helms; flowers (see §2.2); stars; $K_{2,n}$; and $K_{3,n}$ unless $n = 3$ or 7. They also shown that $P_n + \overline{K_m}$ $(m \geq 3)$ is not prime. Tout, Dabboucy, and Howalla [1493] proved that $C_m \odot \overline{K_n}$ is prime for all m and n.

For m and n at least 3, Seoud and Youssef [1289] define $S_n^{(m)}$, the (m,n)*-gon star*, as the graph obtained from the cycle C_n by joining the two end vertices of the path P_{m-2} to every pair of consecutive vertices of the cycle such that each of the end vertices of the path is connected to exactly one vertex of the cycle. Seoud and Youssef [1289] have proved the following graphs have prime labelings: books; $S_n^{(m)}$; $C_n \odot P_m$; $P_n + \overline{K_2}$ if and only if $n = 2$ or n is odd; and $C_n \odot K_1$ with a complete binary tree of order $2^k - 1$ $(k \geq 2)$ attached at each pendent vertex. They also prove that every spanning subgraph of a prime graph is prime and every graph is a subgraph of a prime graph. They conjecture that all unicycle graphs have prime labelings. Seoud and Youssef [1289] proved the following graphs are not prime: $C_m + C_n$; C_n^2 for $n \geq 4$; P_n^2 for $n = 6$ and for $n \geq 8$; and Möbius ladders M_n for n even (see §2.3 for the definition). They also give an exact formula for the maximum number of edges in a prime graph of order n and an upper bound for the chromatic number of a prime graph.

Youssef and Elsakhawi [1705] have shown: the union of stars $S_m \cup S_n$, are prime; the union of cycles and stars $C_m \cup S_n$ are prime; $K_m \cup P_n$ is prime if and only if m is at most 3 or if $m = 4$ and n is odd; $K_n \odot K_1$ is prime if and only if $n \leq 7$; $K_n \odot \overline{K_2}$ is prime if and only if $n \leq 16$; $6K_m \cup S_n$ is prime if and only if the number of primes less than or equal to $m + n + 1$ is at least m; and that the complement of every prime graph with order at least 20 is not prime.

Salmasian [1233] has shown that every tree with n vertices $(n \geq 50)$ can be labeled with n integers between 1 and $4n$ such that every two adjacent vertices have relatively prime labels. Pikhurko [1139] has improved this by showing that for any $c > 0$ there is an N such that any tree of order $n > N$ can be labeled with n integers between 1 and $(1 + c)n$ such that labels of adjacent vertices are relatively prime.

Varkey and Singh (see [1570]) have shown the following graphs have prime labelings: ladders, crowns, cycles with a chord, books, one point unions of C_n, and $L_n + K_1$. Varkey [1570] has shown that graph obtained by connecting two points with internally disjoint paths of equal length are prime. Varkey defines a *twig* as a graph obtained from a path by attaching exactly two pendent edges to each internal vertex of the path. He proves that twigs obtained from a path of odd length (at least 3) and lotus inside a circle (see §5.1 for the definition) graphs are prime.

Babujee and Vishnupriya [129] proved the following graphs have prime labelings: $nP_2, P_n \cup P_n \cup \cdots P_n$, bistars (that is, the graphs obtained by joining the centers of two identical stars with an edge), and the graph obtained by subdividing the edge joining edge of a bistar. Babujee [112] obtained prime labelings for the graphs: $(P_m \cup nK_1) + \overline{K_2}, (C_m \cup nK_1) + \overline{K_2}, (P_m \cup C_n \cup \overline{K}_r) + \overline{K_2}, C_n \cup C_{n+1}, (2n-2)C_{2n}$ $(n > 1), C_n \cup mP_k$ and the graph obtained by subdividing each edge of a star once. In [120] Babujee and Jagadesh prove the following graphs have prime labelings: bistars $B_m, n; P_3 \odot K_{1,n}$; the union of $K_{1,n}$ and the graph obtained from $K_{1,n}$ by appending a pendent edge to every pendent edge of $K_{1,n}$; and the graph obtained by identifying the center of $K_{1,n}$ with the two end points and the middle vertex of P_5.

In [1538] Vaidya and Prajapati prove the following graphs have prime labelings: a t-ply graph of prime order; graphs obtained by joining center vertices of wheels W_m and W_n to a new vertex w where m and n are even positive integers such that $m + n + 3 = p$ and p and $p - 2$ are twin primes; the disjoint union of the wheel W_{2n} and a path; the graph obtained by identifying any vertex of a wheel W_{2n} with an end vertex of a path; the graph obtained from a prime graph of order n by identifying an end vertex of a path with the vertex labeled with 1 or n; the graph obtained by identifying the center vertices of any number of fans (that is, a "multiple shell"); the graph obtained by identifying the center vertices of m wheels $W_{n_1}, W_{n_2}, \ldots, W_{n_m}$ where each $n_i \geq 4$ is an even integer and each n_i is relatively prime to $2 + \sum_{k=1}^{i-1} n_k$ for each $i \in \{2, 3, \ldots, m\}$.

The *Knödel graphs* $W_{\Delta,n}$ with n even and degree Δ, where $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ have vertices pairs (i, j) with $i = 1, 2$ and $0 \leq j \leq n/2 - 1$ where for every $0 \leq j \leq n/2 - 1$ and there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod n/2)$, for $k = 0, 1, \ldots, \Delta - 1$. Haque, Lin, Yang, and Zhao [609] have shown that $W_{3,n}$ is prime when $n \leq 130$.

Sundaram, Ponraj, and Somasundaram [1464] investigated the prime labeling behavior of all graphs of order at most 6 and established that only one graph of order 4, one graph of order 5, and 42 graphs of order 6 are not prime.

Given a collection of graphs $G_1, \ldots, G_n$ and some fixed vertex v_i from each G_i, Lee, Wui, and Yeh [944] define $Amal\{(G_i, v_i)\}$, the amagamation of $\{(G_i, v_i)|\ i = 1, \ldots, n\}$, as the graph obtained by taking the union of the G_i and identifying $v_1, v_2, \ldots, v_n$. Lee, Wui, and Yeh [944] have shown $Amal\{(G_i, v_i)\}$ has a prime labeling when G_i are paths and when G_i are cycles. They also showed that the amagamation of any number of copies of W_n, n odd, with a common vertex is not prime. They conjecture that for any tree T and any vertex v from T, the amagamation of two or more copies of T with v in common is prime. They further conjecture that the amagamation of two or more copies of W_n that share a common point is prime when n is even $(n \neq 4)$. Vilfred, Somasundaram, and Nicholas [1589] have proved this conjecture for the case that $n \equiv 2 \pmod 4$ where the central vertices are identified.

Vilfred, Somasundaram, and Nicholas [1589] have also proved the following: helms are prime; the grid $P_m \times P_n$ is prime when $m \leq 3$ and n is a prime greater than m; the double cone $C_n + \overline{K_2}$ is prime only for $n = 3$; the double fan $P_n \times \overline{K_2}$ $(n \neq 2)$ is prime if and only if n is odd or $n = 2$; and every cycle with a P_k-chord is prime. They conjecture that the grid $P_m \times P_n$ is prime when n is prime and $n > m$. This conjecture was proved by Sundaram, Ponraj, and Somasundaram [1462]. In the same article they also showed that $P_n \times P_n$ is prime when n is prime. Kanetkar [760] proved: $P_6 \times P_6$ is prime; that $P_{n+1} \times P_{n+1}$ is prime when n is a prime with $n \equiv 3$ or 9 (mod 10) and $(n+1)^2 + 1$ is also prime; and $P_n \times P_{n+2}$ is prime when n is an odd prime with $n \not\equiv 2 \pmod 7$.

Seoud, El Sonbaty, and Abd El Rehim [1267] proved that for $m = p_{n+t-1} - (t+n)$ where p_i is the i^{th} prime number in the natural order $K_n \cup K_{t,m}$ is prime and graphs obtained from $K_{2,n}$, $(n \geq 2)$ by adding p and q edges out from the two vertices of degree n of $K_{2,n}$ are prime. They also proved that if G is not prime, then $G \cup K_{1,n}$ is prime if $\pi(n+m+1) \geq m$ where m is the order of G and $\pi(x)$ is the number of primes less than or equal to x.

For any finite collection $\{G_i, u_iv_i\}$ of graphs G_i, each with a fixed edge u_iv_i, Carlson [347] defines the edge amalgamation *Edgeamal*$\{(G_i, u_iv_i)\}$ as the graph obtained by taking the union of all the G_i and identifying their fixed edges. The case where all the graphs are cycles she calls *generalized books*. She proves that all generalized books are prime graphs. Moreover, she shows that graphs obtained by taking the union of cycles and identifying in each cycle the path P_n are also prime. Carlson also proves that C_m-snakes are prime (see §2.2) for the definition).

In [111] Babujee proves that the maximum number of edges in a simple graph with n vertices that has a prime labeling is $\sum_{k=2}^{n} \phi(k)$. He also shows that the planar graphs having n vertices and $3(n-2)$ edges (i.e., the maximum number of edges for a planar graph with n vertices) obtained from K_n $(n \geq 5)$ with vertices $v_1, v_2, \ldots, v_n$ by deleting the edges joining v_s and v_t for all s and t satisfying $3 \leq s \leq n-2$ and $s+2 \leq t \leq n$ has a prime labeling if and only if n is odd.

Yao, Cheng, Zhongfu, and Yao [1687] have shown: a tree of order p with maximum degree at least $p/2$ is prime; a tree of order p with maximum degree at least $p/2$ has a vertex subdivision that is prime; if a tree T has an edge u_1u_2 such that the two components T_1 and T_2 of $T - u_1u_2$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$ and $d_{T_2}(u_2) \geq |T_2|/2$, then T is prime when $|T_1| + |T_2|$ is prime; if a tree T has two edges u_1u_2 and u_2u_3 such that the three components T_1, T_2, and T_3 of $T - \{u_1u_2, u_2u_3\}$ have the properties that $d_{T_1}(u_1) \geq |T_1|/2$, $d_{T_2}(u_2) \geq |T_2|/2$, and $d_{T_3}(u_3) \geq |T_3|/2$, then T is prime when $|T_1| + |T_2| + |T_3|$ is prime.

Vaidya and Prajapati [1539] define a *vertex switching* G_v of a graph G as the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex that is not adjacent to v in G. They say a prime graph G is *switching invariant* if for every vertex v of G, the graph G_v obtained by switching the vertex v in G is also a prime graph. They prove: P_n and $K_{1,n}$ are switching invariant; the graph obtained by switching the center of a wheel is a prime graph; and the graph obtained by switching a rim vertex of W_n is a prime graph if $n+1$ is a prime. They also prove that the graph obtained by switching a rim vertex in W_n is not a prime graph if $n+1$ is an even integer greater than 9.

In [1272] Seoud, El-Sonbaty, and Mahran discuss the primality of some corona graphs $G \odot H$ and conjecture that $K_n \odot \overline{K_m}$ is prime if and only if $n \leq \pi(nm+n)+1$, where $\pi(x)$ is the number of primes less than or equal to x. For $m \leq 20$ they give the exact values of n for which $K_n \odot \overline{K_m}$ is prime. They also show that $K_{m,n}$ is prime if and only if $\min\{m, n\} \leq \pi(m+n) - \pi((m+n)/2) + 1$.

Vaidya and Prajapati [1538] have introduced the notion of k-prime labeling. A k-*prime* *labeling* of a graph G is an injective function $f : V(G) \to \{k+1, k+2, k+3, \ldots, k+|V(G)|-1\}$ for some positive integer k that induces a function f^+ on the edges of G defined by $f^+(uv) = \gcd(f(u), f(v))$ such that $\gcd(f(u), f(v)) = 1$ for all edges uv. A graph that admits a k-prime labeling is called a *k-prime* graph. They prove the following are prime graphs: a tadpole (that is, a graph obtained by identifying a vertex of a cycle to an end vertex of a path); the union of a prime graph of order n and a $(n+1)$-prime graph; the graph obtained by identifying the vertex labeled with n in an n-prime graph with either of the vertices labeled with 1 or n in a prime graph of order n.

A dual of prime labelings has been introduced by Deretsky, Lee, and Mitchem [425]. They say a graph with edge set E has a *vertex prime labeling* if its edges can be labeled with distinct integers $1, \ldots, |E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee, and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs; $C_{2k} \cup C_n$; $C_{2m} \cup C_{2n} \cup C_{2k+1}$; $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$; and $5C_{2m}$. They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Let $G = \bigcup_{i=1}^{t} C_{2n_i}$ and $N = \sum_{i=1}^{t} n_i$. In [301] Borosh, Hensley and Hobbs proved that there is a positive constant n_0 such that the conjecture of Deretsky et al. is true for the following cases: G is the disjoint union of at most seven cycles; G is a union of cycles all of the same even length $2n$ where $n \leq 150\,000$ or where $n \geq n_0$; $n_i \geq (\log N)^{4 \log\log\log n}$ for all $i = 1, \ldots, t$; and when each C_{2n_i} is repeated at most n_i times. They end their paper with a discussion of graphs whose components are all even cycles, and of graphs with some components that are not cycles and some components that are odd cycles.

Jothi [754] calls a graph G *highly vertex prime* if its edges can be labeled with distinct integers $\{1, 2, \ldots, |E|\}$ such that the labels assigned to any two adjacent edges are relatively prime. Such labeling is called a *highly vertex prime* labeling. He proves: if G is highly vertex prime then the line graph of G is prime; cycles are highly vertex prime; paths are highly vertex prime; K_n is highly vertex prime if and only if $n \leq 3$; $K_{1,n}$ is highly vertex prime if and only if $n \leq 2$; even cycles with a chord are highly vertex prime; $C_p \cup C_q$ is not highly vertex prime when both p and q are odd; and crowns $C_n \odot K_1$ are highly vertex prime.

The tables following summarize the state of knowledge about prime labelings and vertex prime labelings. In the table, **P** means prime labeling exists, and **VP** means vertex prime labeling exists. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.

Table 21: **Summary of Prime Labelings**

Graph	*Types*	*Notes*
P_n	P	[521]
stars	P	[521]
caterpillars	P	[521]
complete binary trees	P	[521]
spiders	P	[521]
trees	P?	[944]
C_n	P	[425]
$C_n \cup C_{2m}$	P	[425]
K_n	P	iff $n \leq 3$ [944]
W_n	P	iff n is even [944]
helms	P	[1266], [1589]
fans	P	[1266]
flowers	P	[1266]
$K_{2,n}$	P	[1266]
$K_{3,n}$	P	$n \neq 3, 7$ [1266]
$P_n + \overline{K_m}$	not P	$n \geq 3$ [1266]
$P_n + \overline{K_2}$	P	iff $n = 2$ or n is odd [1266]
books	P	[1289]
$C_n \odot P_m$	P	[1289]
unicyclic graphs	P?	[1289]
$C_m + C_n$	not P	[1289]

Table 21: **Summary of Prime Labelings continued**

Graph	*Types*	*Notes*
C_n^2	not P	$n \geq 4$ [1289]
P_n^2	not P	$n \geq 6,\ n \neq 7$ [1289]
M_n (Möbius ladders)	not P	n even [1289]
$S_m \cup S_n$	P	[1705]
$C_m \cup S_n$	P	[1705]
$K_m \cup S_n$	P	iff number of primes $\leq m+n+1$ is at least m [1705]
$K_n \cdot K_1$	P	iff $n \leq 7$ [1705]
$P_n \times P_2$ (ladders)	P	[1570]
$P_m \times P_n$ (grids)	P	$m \leq 3,\ m > n,\ n$ prime [1589]
$C_n \odot K_1$ (crowns)	P	[1570]
cycles with a chord	P	[1570]
wheels	P	[1570]
$C_n \odot \overline{K_2}$	P	iff $n = 3$ [1589]
$P_n \odot \overline{K_2}$	P	iff $n \neq 2$ [1589]
C_m-snakes (see §2.2)	P	[347]
unicyclic	P?	[1266]
$C_m \odot P_n$	P	[1289]
$K_{1,n} + \overline{K_2}$	P	[1383]
$K_{1,n} + K_2$	P	n prime, $n \geq 4$ [1383]
$P_n \odot K_1$ (combs)	P	$n \geq 2$ [1383]

Table 22: **Summary of Vertex Prime Labelings**

Graph	*Types*	*Notes*
$P_1 \cup P_2 \cup \cdots \cup P_n$	P	[1383]
$P_n \times P_2$ (ladders)	P P?	$n \geq 3,\ 2n+1$ prime [1383] $n \geq 3$ [1383]
$C_m^{(n)}$ (see §2.2	P	$n(m-1)+1$ prime [1383]
triangular snakes	P	[1383]
quadrilateral snakes	P	[1383]
$C_m + C_n$	not P	[1289]
C_n^2	not P	$n \geq 4$ [1289]
P_n	not P	$n = 6,\ n \geq 8$ [1289]
M_{2n} (Möbius ladders)	not P	[1289]
connected graphs	VP	[425]
forests	VP	[425]
$C_{2m} \cup C_n$	VP	[425]
$C_{2m} \cup C_{2n} \cup C_{2k+1}$	VP	[425]
$C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$	VP	[425]
$5C_{2m}$	VP	[425]
$G \cup H$	VP	if $G,\ H$ are connected and one is not an odd cycle [425]
2-regular graph G	not VP VP?	G has at least 2 odd cycles [425] iff G has at most 1 odd cycle [425]

7.3 Edge-graceful Labelings

In 1985, Lo [986] introduced the notion of edge-graceful graphs. A graph $G(V, E)$ is said to be *edge-graceful* if there exists a bijection f from E to $\{1, 2, \ldots, |E|\}$ such that the induced mapping f^+ from V to $\{0, 1, \ldots, |V| - 1\}$ given by $f^+(x) = (\sum f(xy)) \pmod{|V|}$ taken over all edges xy is a bijection. Note that an edge-graceful graph is antimagic (see §6.1). A necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv p(p+1)/2 \pmod{p}$. Lee [865] notes that this necessary condition extends to any multigraph with p vertices and q edges. It was conjectured by Lee [865] that any connected simple (p, q)-graph with $q(q + 1) \equiv p(p - 1)/2 \pmod{p}$ vertices is edge-graceful. Lee, Kitagaki, Young, and Kocay [870] prove that the conjecture is true for maximal outerplanar graphs. Lee and Murthy [858] proved that K_n is edge-graceful if and only if $n \not\equiv 2 \pmod 4$. (An edge-graceful labeling given in [986] for K_n for $n \not\equiv 2 \pmod 4$ is incorrect.) Lee [865] notes that a multigraph with $p \equiv 2 \pmod 4$ vertices is not edge-graceful and conjectures that this condition is sufficient for the edge-gracefulness of connected graphs. Lee [864] has conjectured that all trees of odd order are edge-graceful. Small [1403] has proved that spiders for which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [783] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low, and Mitchem [328] have shown that regular spiders of odd order are edge-graceful.

Lee and Seah [907] have shown that $K_{n,n,\ldots,n}$ is edge-graceful if and only if n is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [906] have also proved that C_n^k (the kth power of C_n) is edge-graceful for $k < \lfloor n/2 \rfloor$ if and only if n is odd and C_n^k is edge-graceful for $k \geq \lfloor n/2 \rfloor$ if and only if $n \not\equiv 2 \pmod 4$ (see also [328]). Lee, Seah, and Wang [912] gave a complete characterization of edge-graceful P_n^k graphs. Shiu, Lam, and Cheng [1342] proved that the composition of the path P_3 and any null graph of odd order is edge-graceful.

Lo [986] proved that all odd cycles are edge-graceful and Wilson and Riskin [1646] proved the Cartesian product of any number of odd cycles is edge-graceful. Lee, Ma, Valdes, and Tong [883] investigated the edge-gracefulness of grids $P_m \times P_n$. The necessity condition of Lo [986] that a (p, q) graph must satisfy $q(q+1) \equiv 0$ or $p/2 \pmod{p}$ severely limits the possibilities. Lee et al. prove the following: $P_2 \times P_n$ is not edge-graceful for all $n > 1$; $P_3 \times P_n$ is edge-graceful if and only if $n = 1$ or $n = 4$; $P_4 \times P_n$ is edge-graceful if and only if $n = 3$ or $n = 4$; $P_5 \times P_n$ is edge-graceful if and only if $n = 1$; $P_{2m} \times P_{2n}$ is edge-graceful if and only if $m = n = 2$. They conjecture that for all $m, n \geq 10$ of the form $m = (2k + 1)(4k + 1)$, $n = (2k + 1)(4k + 3)$, the grids $P_m \times P_n$ are edge-graceful. Riskin and Weidman [1203] proved: if G is an edge-graceful $2r$-regular graph with p vertices and q edges and $(r, kp) = 1$, then kG is edge-graceful when k is odd; when n and k are odd, kC_n^r is edge-graceful; and if G is the cartesian product of an odd number of odd cycles and k is odd, then kG is edge-graceful. They conjecture that the disjoint union of an odd number of copies of a $2r$-regular edge-graceful graph is edge-graceful.

Shiu, Lee, and Schaffer [1349] investigated the edge-gracefulness of multigraphs derived from paths, combs, and spiders obtained by replacing each edge by k parallel edges. Lee, Ng, Ho, and Saba [893] construct edge-graceful multigraphs starting with paths and spiders by adding certain edges to the original graphs. Lee and Seah [908] have also investigated edge-gracefulness of various multigraphs.

Lee and Seah (see [865]) define a *sunflower* graph $SF(n)$ as the graph obtained by starting with an n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ and creating new vertices $w_1, w_2, \ldots, w_n$

with w_i connected to v_i and v_{i+1} (v_{n+1} is v_1). In [909] they prove that $SF(n)$ is edge-graceful if and only if n is even. In the same paper they prove that C_3 is the only triangular snake that is edge-graceful. Lee and Seah [906] prove that for $k \le n/2$, C_n^k is edge-graceful if and only if n is odd, and for $k \ge n/2$, C_n^k is edge-graceful if and only if $n \not\equiv 2 \pmod 4$. Lee, Seah, and Lo (see [865]) have proved that for n odd, $C_{2n} \cup C_{2n+1}, C_n \cup C_{2n+2}$, and $C_n \cup C_{4n}$ are edge-graceful. They also show that for odd k and odd n, kC_n is edge-graceful. Lee and Seah (see [865]) prove that the generalized Petersen graph $P(n,k)$ (see Section 2.7 for the definition) is edge-graceful if and only if n is even and $k < n/2$. In particular, $P(n,1) = C_n \times P_2$ is edge-graceful if and only if n is even.

Schaffer and Lee [1246] proved that $C_m \times C_n$ $(m > 2, n > 2)$ is edge-graceful if and only if m and n are odd. They also showed that if G and H are edge-graceful regular graphs of odd order then $G \times H$ is edge-graceful and that if G and H are edge-graceful graphs where G is c-regular of odd order m and H is d-regular of odd order n, then $G \times H$ is edge-magic if $\gcd(c,n) = \gcd(d,m) = 1$. They further show that if H has odd order, is $2d$-regular and edge-graceful with $\gcd(d,m) = 1$, then $C_{2m} \times H$ is edge-magic, and if G is odd-regular, edge-graceful of even order m that is not divisible by 3, and G can be partitioned into 1-factors, then $G \times C_m$ is edge-graceful.

In 1987 Lee (see [910]) conjectured that $C_{2m} \cup C_{2n+1}$ is edge-graceful for all m and n except for $C_4 \cup C_3$. Lee, Seah, and Lo [910] have proved this for the case that $m = n$ and m is odd. They also prove: the disjoint union of an odd number copies of C_m is edge-graceful when m is odd; $C_n \cup C_{2n+2}$ is edge-graceful; and $C_n \cup C_{4n}$ is edge-graceful for n odd. Bu [313] gave necessary and sufficient conditions for graphs of the form $mC_n \cup P_{n-1}$ to be edge-graceful.

Kendrick and Lee (see [865]) proved that there are only finitely many n for which $K_{m,n}$ is edge-graceful and they completely solve the problem for $m = 2$ and $m = 3$. Ho, Lee, and Seah [657] use $S(n; a_1, a_2, \dots, a_k)$ where n is odd and $1 \le a_1 \le a_2 \le \cdots \le a_k < n/2$ to denote the (n, nk)-multigraph with vertices $v_0, v_1, \dots, v_{n-1}$ and edge set $\{v_iv_j |\ i \ne j, i - j \equiv a_t \pmod n \text{ for } t = 1, 2, \dots, k\}$. They prove that all such multigraphs are edge-graceful. Lee and Pritikin (see [865]) prove that the Möbius ladders (see §2.2 for definition) of order $4n$ are edge-graceful. Lee, Tong, and Seah [926] have conjectured that the total graph of a (p,p)-graph is edge-graceful if and only if p is even. They have proved this conjecture for cycles. In [787] Khodkar and Vinhage proved that there exists a super edge-graceful labeling of the total graph of $K_{1,n}$ and the total graph of C_n.

Kuang, Lee, Mitchem, and Wang [839] have conjectured that unicyclic graphs of odd order are edge-graceful. They have verified this conjecture in the following cases: graphs obtained by identifying an end point of a path P_m with a vertex of C_n when $m+n$ is even; crowns with one pendent edge deleted; graphs obtained from crowns by identifying an endpoint of P_m, m odd, with a vertex of degree 1; amalgamations of a cycle and a star obtained by identifying the center of the star with a cycle vertex where the resulting graph has odd order; graphs obtained from C_n by joining a pendent edge to $n-1$ of the cycle vertices and two pendent edges to the remaining cycle vertex.

Gayathri and Subbiah [557] say a graph $G(V,E)$ has a *strong edge-graceful* labeling if there is an injection f from the E to $\{1, 2, 3, \dots, \langle 3|E|/2\rangle\}$ such that the induced mapping f^+ from V defined by $f^+(u) = (\Sigma f(uv)) \pmod{2|V|}$ taken all edges uv is an injection. They proved the following graphs have strong edge graceful labelings: $P_n (n \ge 3), C_n, K_{1,n} (n \ge 2)$, crowns $C_n \odot K_1$, and fans $P_n + K_1 (n \ge 2)$. In his Ph.D. thesis [1438] Subbiah provided edge-graceful and strong edge-graceful labelings for a large variety of graphs. Among them are bistars, twigs, y-

trees, spiders, flags, kites, friendship graphs, mirror of paths, flowers, sunflowers, graphs obtained by identifying a vertex of a cycle with an end point of a star, and $K_2 \odot C_n$, and various disjoint unions of path, cycles, and stars.

Hefetz [624] has shown that a graph $G = (V, E)$ of the form $G = H \cup f_1 \cup f_2 \cup \cdots \cup f_r$ where $H = (V, E')$ is edge-graceful and the f_i's are 2-factors is also edge-graceful and that a regular graph of even degree that has a 2-factor consisting of k cycles each of length t where k and t are odd is edge-graceful.

Bača and Holländer [172] investigated a generalization of edge-graceful labeling called (a, b)-consecutive labelings. A connected graph $G(V, E)$ is said to have an (a, b)-*consecutive labeling* where a is a nonnegative integer and b is a positive proper divisor of $|V|$, if there is a bijection from E to $\{1, 2, \ldots, |E|\}$ such that if each vertex v is assigned the sum of all edges incident to v the vertex labels are distinct and they can be partitioned into $|V|/b$ intervals $W_j = [w_{\text{min}} = (j-1)b + (j-1)a, w_{\text{min}} + jb + (j-1)a - 1]$, where $1 \leq j \leq p/b$ and w_{min} is the minimum value of the vertices. They present necessary conditions for (a, b)-consecutive labelings and describe (a, b)-consecutive labelings of the generalized Petersen graphs for some values of a and b.

A graph with p vertices and q edges is said to be k-*edge-graceful* if its edges can be labeled with $k, k+1, \ldots, k+q-1$ such that the sums of the edges incident to each vertex are distinct modulo p. In [929] Lee and Wang show that for each $k \neq 1$ there are only finitely many trees that are k-edge graceful (there are infinitely many 1-edge graceful trees). They describe completely the k-edge-graceful trees for $k = 0, 2, 3, 4$, and 5.

In 1991 Lee [865] defined the *edge-graceful spectrum* of a graph G as the set of all nonnegative integers k such that G has a k-edge graceful labeling. In [933] Lee, Wang, Ng, and Wang determine the edge-graceful spectrum of the following graphs: $G \odot K_1$ where G is an even cycle with one chord; two even cycles of the same order joined by an edge; and two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex (*butterfly graph*). Lee, Chen, and Wang [868] have determined the edge-graceful spectra for various cases of cycles with a chord and for certain cases of graphs obtained by joining two disjoint cycles with an edge (i.e., *dumbbell graphs*). More generally, Shiu, Ling, and Low [1351] call a connected with p vertices and $p+1$ edges *bicyclic*. In particular, the family of bicyclic graphs includes the one-point union of two cycles, two cycles joined by a path and cycles with one cord. In [1352] they determine the edge-graceful spectra of bicyclic graphs that do not have pendent edges. Kang, Lee, and Wang [763] determined the edge-graceful spectra of wheels and Wang, Hsiao, and Lee [1618] determined the edge-graceful spectra of the square of P_n for odd n (see also Lee, Wang, and Hsiao [931]). Results about the edge-graceful spectra of three types of $(p, p+1)$-graphs are given by Chen, Lee, and Wang [370]. In [1619] Wang and Lee determine the edge-graceful spectra of the one-point union of two cycles, the corona product of the one-point union of two cycles with K_1, and the cycles with one chord.

Lee, Levesque, Lo, and Schaffer [878] investigate the edge-graceful spectra of cylinders. They prove: for odd $n \geq 3$ and $m \equiv 2 \pmod 4$, the spectra of $C_n \times P_m$ is $\emptyset$; for $m = 3$ and $m \equiv 0, 1$ or 3 (mod 4), the spectra of $C_4 \times P_m$ is $\emptyset$; for even $n \geq 4$, the spectra of $C_n \times P_2$ is all natural numbers; the spectra of $C_n \times P_4$ is all odd positive integers if and only if $n \equiv 3 \pmod 4$; and $C_n \times P_4$ is all even positive integers if and only if $n \equiv 1 \pmod 4$. They conjecture that $C_4 \times P_m$ is k-edge-graceful for some k if and only if $m \equiv 2 \pmod 4$. Shiu, Ling, and Low [1352] determine the edge-graceful spectra of all connected bicyclic graphs without pendent edges.

A graph $G(V, E)$ is called *super edge-graceful* if there is a bijection f from E to

$\{0, \pm 1, \pm 2, \ldots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from E to $\{\pm 1, \pm 2, \ldots, \pm |E|/2\}$ when $|E|$ is even such that the induced vertex labeling f^* defined by $f^*(u) = \Sigma f(uv)$ over all edges uv is a bijection from V to $\{0, \pm 1, \pm 2, \ldots, \pm(p-1)/2\}$ when p is odd and from V to $\{\pm 1, \pm 2, \ldots, \pm p/2\}$ when p is even. Lee, Wang, Nowak, and Wei [934] proved the following: $K_{1,n}$ is super-edge-magic if and only if n is even; the double star $DS(m, n)$ (that is, the graph obtained by joining the centers of $K_{1,m}$ and $K_{1,n}$ by an edge) is super edge-graceful if and only if m and n are both odd. They conjecture that all trees of odd order are super edge-graceful. In [402] Chung, Lee, Gao and Schaffer pose the problems of characterizing the paths and tress of diameter 4 that are super edge-graceful.

In [401] Chung, Lee, Gao, prove various classes of caterpillars, combs, and amagamations of combs and stars of even order are super edge-graceful. Lee, Sun, Wei, Wen, and Yiu [922] proved that trees obtained by starting with the paths the P_{2n+2} or P_{2n+3} and identifying each internal vertex with an end point of a path of length 2 are super edge-graceful.

Shiu [1334] has shown that $C_n \times P_2$ is super-edge-graceful for all $n \geq 2$. More generally, he defines a family of graphs that includes $C_n \times P_2$ and generalized Petersen graphs are follows. For any permutation θ on n symbols without a fixed point the *θ-Petersen graph* $P(n; \theta)$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i u_{i+1}, u_i w_i, w_i w_{\theta(i)} \mid 1 \leq i \leq n\}$ where addition of subscripts is done modulo n. (The graph $P(n; \theta)$ need not be simple.) Shiu proves that $P(n; \theta)$ is super-edge-graceful for all $n \geq 2$. He also shows that certain other families of connected cubic multigraphs are super-edge-graceful and conjectures that every connected cubic of multigraph except K_4 and the graph with 2 vertices and 3 edges is super-edge-graceful.

In [1340] Shiu and Lam investigated the super-edge-gracefulness of fans and wheel-like graphs. They showed that fans F_{2n} and wheels W_{2n} are super-edge-graceful. Although F_3 and W_3 are not super-edge-graceful the general cases F_{2n+1} and W_{2n+1} are open. For a positive integer n_1 and even positive integers $n_2, n_3, \ldots, n_m$ they define an *m-level wheel* as follows. A wheel is a 1-level wheel and the cycle of the wheel is the 1-level cycle. An i-level wheel is obtained from an $(i-1)$-level wheel by appending $n_i/2$ pairs of edges from any number of vertices of the $i-1$-level cycle to n_i new vertices that form the vertices in the i-level cycle. They prove that all m-level wheels are super-edge-graceful. They also prove that for n odd $C_m \odot \overline{K_n}$ is super-edge-graceful, for odd $m \geq 3$ and even $n \geq 2$ $C_m \odot \overline{K_n}$ is edge-graceful, and for $m \geq 2$ and $n \geq 1$ $C_m \odot \overline{K_n}$ is super-edge-graceful. For a cycle C_m with consecutive vertices $v_1, v_2, \ldots, v_m$ and nonnegative integers $n_1, n_2, \ldots, n_m$ they define the graph $A(m; n_1, n_2, \ldots, n_m)$ as the graph obtained from C_m by attaching n_i edges to the vertex v_i for $1 \leq i \leq m$. They prove $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if m is odd and $A(m; n_1, n_2, \ldots, n_m)$ is super-edge-graceful if m is even and all the n_i are positive and have the same parity. Chung, Lee, Gao, and Schaffer [402] provide super edge-graceful labelings for various even order paths, spiders and disjoint unions of two stars. In [399] Chung and Lee characterize spiders of even orders that are not super-edge-graceful and exhibit some spiders of even order of diameter at most four that are super-edge-graceful. They raised the question of which paths are super edge-graceful. This was answered by Cichacz, Frončk and Xu [404] who showed that the only paths that are not super edge-graceful are P_2 and P_4. Gao and Zhang [545] proved that some cases of caterpillars are super edge-graceful.

In [402] Chung, Lee, Gao, and Schaffer asked for a characterize trees of diameter 4 that are super edge-graceful. Krop, Mutiso, and Raridan [836] provide a super edge-graceful labelings for all caterpillars and even size lobsters of diameter 4 that permit such labelings. They also provide super edge-graceful labelings for several families of odd size lobsters of diameter 4. They were unable to find general methods that describe super edge-graceful labelings for a few families of odd size lobsters of diameter 4, although they are able to show that certain lobsters in these families are super-edge graceful. They conclude with three conjectures about rooted trees of height 2 and diameter 4.

Although it is not the case that a super edge-graceful graph is edge-graceful, Lee, Chen, Yera, and Wang [867] proved that if G is a super edge-graceful with p vertices and q edges and $q \equiv -1 \pmod p$ when q is even, or $q \equiv 0 \pmod p$ when q is odd, then G is also edge-graceful. They also prove: the graph obtained from a connected super edge-graceful unicyclic graph of even order by joining any two nonadjacent vertices by an edge is super edge-graceful; the graph obtained from a super edge-graceful graph with p vertices and $p+1$ edges by appending two edges to any vertex is super edge-graceful; and the one-point union of two identical cycles is super edge-graceful.

Gayathri, Duraisamy, and Tamilselvi [550] calls a (p,q)-graph with $q \geq p$ *even edge-graceful* if there is an injection f from the set of edges to $\{1,2,3,\ldots,2q\}$ such that the values of the induced mapping f^+ from the vertex set to $\{0,1,2,\ldots,2q-1\}$ given by $f^+(x) = (\Sigma f(xy))(\mathrm{mod}\ 2q)$ over all edges xy are distinct and even. In [550] and [549] Gayathri et al. prove the following: cycles are even edge-graceful if and only if the cycles are odd; even cycles with one pendent edge are even edge-graceful; wheels are even edge-graceful; gears (see §2.2 for the definition) are not even edge-graceful; fans $P_n + K_1$ are even edge-graceful; $C_4 \cup P_m$ for all m are even edge-graceful; $C_{2n+1} \cup P_{2n+1}$ are even edge-graceful; crowns $C_n \odot K_1$ are even edge-graceful; $C_n^{(m)}$ (see §2.2 for the definition) are even edge-graceful; sunflowers (see §3.7 for the definition) are even edge-graceful; triangular snakes (see §2.2 for the definition) are even edge-graceful; closed helms (see §2.2 for the definition) with the center vertex removed are even edge-graceful; graphs decomposable into two odd Hamiltonian cycles are even edge-graceful; and odd order graphs that are decomposable into three Hamiltonian cycles are even edge-graceful.

In [549] Gayathri and Duraisamy generalized the definition of even edge-graceful to include (p,q)-graphs with $q < p$ by changing the modulus from $2q$ the maximum of $2q$ and $2p$. With this version of the definition, they have shown that trees of even order are not even edge-graceful whereas, for odd order graphs, the following are even edge-graceful: banana trees (see §2.1 for the definition); graphs obtained joining the centers of two stars by a path; $P_n \odot K_{1,m}$; graphs obtained by identifying an end point from each of any number of copies of P_3 and P_2; bistars (that is, graphs obtained by joining the centers of two stars with an edge); and graphs obtained by appending the end point of a path to the center of a star. They define odd edge-graceful graphs in the analogous way and provide a few results about such graphs.

Lee, Pan, and Tsai [898] call a graph G with p vertices and q edges *vertex-graceful* if there exists a labeling $f\ V(G) \to \{1,2,\ldots,p\}$ such that the induced labeling f^+ from $E(G)$ to Z_q defined by $f^+(uv) = f(u)+f(v) \pmod q$ is a bijection. Vertex-graceful graphs can be viewed the dual of edge-graceful graphs. They call a vertex-graceful graph *strong vertex-graceful* if the values of $f^+(E(G)$ are consecutive. They observe that the class of vertex-graceful graphs properly contains the super edge-magic graphs and strong vertex-graceful graphs are super edge-magic. They provide vertex-graceful and strong vertex-graceful labelings for various $(p,p+1)$-graphs of small order and their amalgamations.

Shiu and Wong [1359] proved the one-point union of an m-cycle and an n-cycle is vertex-graceful only if $m+n \equiv 0 \pmod 4$; for $k \geq 2$, $C(3,4k-3)$ is strong vertex-graceful; $C(2n+3,2n+1)$ is strong vertex-graceful for $n \geq 1$; and if the one-point union of two cycles is vertex-graceful, then it is also strong vertex-graceful. In [1417] Somashekara and Veena find the number of $(n, 2n-3)$ strong vertex graceful graphs.

As a dual to super edge-graceful graphs Lee and Wei [937] define a graph $G(V, E)$ to be *super vertex-graceful* if there is a bijection f from V to $\{\pm 1, \pm 2, \ldots, \pm(|V|-1)/2\}$ when $|V|$ is odd and from V to $\{\pm 1, \pm 2, \ldots, \pm |V|/2\}$ when $|V|$ is even such that the induced edge labeling f^* defined by $f^+(uv) = f(u) + f(v)$ over all edges uv is a bijection from E to $\{0, \pm 1, \pm 2, \ldots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from E to $\{\pm 1, \pm 2, \ldots, \pm |E|/2\}$ when $|E|$ is even. They show: for m and $n_1, n_2, \ldots, n_m$ each at least 3, $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m}$ is not super vertex-graceful; for n odd, books $K_{1,n} \times P_2$ are not super vertex-graceful; for $n \geq 3$, $P_n^2 \times P_2$ is super vertex-graceful if and only if $n = 3, 4$, or 5; and $C_m \times C_n$ is not super vertex-graceful. They conjecture that $P_n \times P_n$ is super vertex-graceful for $n \geq 3$.

In [941] Lee and Wong generalize super edge-vertex graphs by defining a graph $G(V, E)$ to be $P(a)Q(1)$*-super vertex-graceful* if there is a bijection f from V to $\{0, \pm a, \pm(a+1), \ldots, \pm(a-1+(|V|-1)/2)\}$ when $|V|$ is odd and from V to $\{\pm a, \pm(a+1), \ldots, \pm(a-1+|V|/2)\}$ when $|V|$ is even such that the induced edge labeling f^* defined by $f^+(uv) = f(u) + f(v)$ over all edges uv is a bijection from E to $\{0, \pm 1, \pm 2, \ldots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from E to $\{\pm 1, \pm 2, \ldots, \pm |E|/2\}$ when $|E|$ is even. They show various classes of unicyclic graphs are $P(a)Q(1)$-super vertex-graceful. In [877] Lee, Leung, and Ng more simply refer to $P(1)Q(1)$-super vertex-graceful graphs as *super vertex-graceful* and show how to construct a variety of unicyclic graphs that are super vertex-graceful. They conjecture that every unicyclic graph is an induced subgraph of a super vertex-graceful unicyclic graph. Lee and Leung [876] determine which trees of diameter at most 6 are super vertex-graceful graphs and propose two conjectures. Lee, Ng, and Sun [894] found many classes of caterpillars that are super vertex-graceful.

In [388] Chopra and Lee define a graph $G(V, E)$ to be $Q(a)P(b)$*-super edge-graceful* if there is a bijection f from E to $\{\pm a, \pm(a+1), \ldots, \pm(a+(|E|-2)/2)\}$ when $|E|$ is even and from E to $\{0, \pm a, \pm(a+1), \ldots, \pm(a+(|E|-3)/2)\}$ when $|E|$ is odd and $f^+(u)$ is equal to the sum of $f(uv)$ over all edges uv is a bijection from V to $\{\pm b, \pm(b+1), \ldots, (|V|-2)/2\}$ when $|V|$ is even and from V to $\{0, \pm b, \pm(b+1), \ldots, \pm(|V|-3)/2\}$ when $|V|$ is odd. They say a graph is *strongly super edge-graceful* if it is $Q(a)P(b)$-super edge-graceful for all $a \geq 1$. Among their results are: a star with n pendent edges is strongly super edge-graceful if and only if n is even; wheels with n spokes are strongly super edge-graceful if and only if n is even; coronas $C_n \odot K_1$ are strongly super edge-graceful for all $n \geq 3$; and double stars $DS(m, n)$ are strongly super edge-graceful in the case that m is odd and at least 3 and n is even and at least 2 and in the case that both m and n are odd and one of them is at least 3. Lee, Song, and Valdés [915] investigate the $Q(a)P(b)$-super edge-gracefulness of wheels W_n for $n = 3, 4, 5$, and 6.

In [938] Lee, Wang, and Yera proved that some Eulerian graphs are super edge-graceful, but not edge-graceful, and that some are edge-graceful, but not super edge-graceful. They also showed that a Rosa-type condition for Eulerian super edge-graceful graphs does not exist and pose some conjectures, one of which was: For which n, is K_n is super edge-graceful? It was known that the complete graphs K_n for $n = 3, 5, 6, 7, 8$ are super edge-graceful and K_4 is not super edge-graceful. Khodkar, Rasi, and Sheikholeslami, [786] answered this question by proving that all complete graphs of order $n \geq 3$, except 4, are super edge-graceful.

In 1997 Yilmaz and Cahit [1693] introduced a weaker version of edge-graceful called E-cordial. Let G be a graph with vertex set V and edge set E and let f a function from E to $\{0, 1\}$. Define f on V by $f(v) = \sum\{f(uv) | uv \in E\} \pmod 2$. The function f is called an *E-cordial* labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an E-cordial labeling is called *E-cordial*. Yilmaz and Cahit prove the following graphs are E-cordial: trees with n vertices if and only if $n \not\equiv 2 \pmod 4$; K_n if and only if $n \not\equiv 2 \pmod 4$; $K_{m,n}$ if and only if $m + n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; regular graphs of degree 1 on $2n$ vertices if and only if n is even; friendship graphs $C_3^{(n)}$ for all n (see §2.2 for the definition); fans F_n if and only if $n \not\equiv 1 \pmod 4$; and wheels W_n if and only if $n \not\equiv 1 \pmod 4$. They observe that graphs with $n \equiv 2 \pmod 4$ vertices can not be E-cordial. They generalized E-cordial labelings to E_k-cordial ($k > 1$) labelings by replacing $\{0, 1\}$ by $\{0, 1, 2, \ldots, k-1\}$. Of course, E_2-cordial is the same as E-cordial (see §3.7).

In [1561] Vaidya and Lekha prove that the following graphs are E-cordial: the mirror graphs (see §2.3 for the definition) even paths, even cycles, and the hypercube are E-cordial. In [1536] they show that the middle graph, the total graph, and the splitting graph of a path are E-cordial and the composition of P_2n with P_2. (See §2.7 for the definitions of middle, total and splitting graphs.) In [1537] Vaidya and Lekha [1537] prove the following graphs are E-cordial: the graph obtained by duplication of a vertex (see §2.7 for the definition) of a cycle; the graph obtained by duplication of an edge (see §2.7 for the definition) of a cycle; the graph obtained by joining of two copies of even cycle by an edge; the splitting graph of an even cycle; and the shadow graph (see §3.8 for the definition) of a path of even order.

Vaidya and Vyas [1562] proved the following graphs have E-cordial labelings: $K_{2n} \times P_2$; $P_{2n} \times P_2$; $W_n \times P_2$ for odd n; and $K_{1,n} \times P_2$ for odd n. Vaidya and Vyas [1563] proved that the Möbius ladders, the middle graph of C_n, and crowns $C_n \odot K_1$ are E-cordial graphs for even n while bistars $B_{n,n}$ and its square graph $B^2_{n,n}$ are E-cordial graphs for odd n.

Devaraj [428] has shown that $M(m, n)$, the mirror graph of $K(m, n)$, is E-cordial when $m+n$ is even and the generalized Petersen graph $P(n, k)$ is E-cordial when n is even. (Recall that $P(n, 1)$ is $C_n \times P_2$.)

The table following summarizes the state of knowledge about edge-graceful labelings. In the table **EG** means edge-graceful labeling exists. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.

Table 23: **Summary of Edge-graceful Labelings**

Graph	*Types*	*Notes*
K_n	EG	iff $n \not\equiv 2 \pmod 4$ [858]
odd order trees	EG?	[864]
$K_{n,n,\ldots,n}$ (k terms)	EG	iff n is odd or $k \not\equiv 2 \pmod 4$ [907]
$C_n^k,\ k < \lfloor n/2 \rfloor$	EG	iff n is odd [906]
$C_n^k,\ k \geq \lfloor n/2 \rfloor$	EG	iff $n \not\equiv 2 \pmod 4$ [906]
$P_3[K_n]$	EG	n is odd [906]
M_{4n} (Möbius ladders)	EG	[865]
odd order dragons	EG	[839]
odd order unicycilc graphs	EG?	[839]
$P_{2m} \times P_{2n}$	EG	iff $m = n = 2$ [883]
$C_n \cup P_2$	EG	n even [910]
$C_{2n} \cup C_{2n+1}$	EG	n odd [910]
$C_n \cup C_{2n+2}$	EG	[910]
$C_n \cup C_{4n}$	EG	n odd [910]
$C_{2m} \cup C_{2n+1}$	EG?	$(m,n) \neq (4,3)$ odd [911]
$P(n,k)$ generalized Petersen graph	EG	n even, $k < n/2$ [865]
$C_m \times C_n$	EG?	$(m,n) \neq (4,3)$ [911]

7.4 Radio Labelings

In 2001 Chartrand, Erwin, Zhang, and Harary [359] were motivated by regulations for channel assignments of FM radio stations to introduce radio labelings of graphs. A *radio labeling* of a connected graph G is an injection c from the vertices of G to the natural numbers such that $d(u,v)+|c(u)-c(v)| \geq 1+diam(G)$ for every two distinct vertices u and v of G. The *radio number* of c, $rn(c)$, is the maximum number assigned to any vertex of G. The *radio number* of G, $rn(G)$, is the minimum value of $rn(c)$ taken over all radio labelings c of G. Chartrand et al. and Zhang [1716] gave bounds for the radio numbers of cycles. The exact values for the radio numbers for paths and cycles were reported by Liu and Zhu [975] as follows: for odd $n \geq 3$, $rn(P_n) = (n-1)^2/2+2$; for even $n \geq 4$, $rn(P_n) = n^2/2-n+1; rn(C_{4k}) = (k+2)(k-2)/2+1$; $rn(C_{4k+1}) = (k+1)(k-1)/2$; $rn(C_{4k+2}) = (k+2)(k-2)/2+1$; and $rn(C_{4k+3}) = (k+2)(k-1)/2$. However, Chartrand, Erwin, and Zhang [358] obtained different values than Liu and Zhu for P_4 and P_5. Chartrand, Erwin, and Zhang [358] proved: $rn(P_n) \leq (n-1)(n-2)/2+n/2+1$ when n is even; $rn(P_n) \leq n(n-1)/2+1$ when n is odd; $rn(P_n) < rn(P_{n+1})$ $(n > 1)$; for a connected graph G of diameter d, $rn(G) \geq (d+1)^2/4+1$ when d is odd; and $rn(G) \geq d(d+2)/4+1$ when d is even. Benson, Porter, and Tomova [256] have determined the radio numbers of all graphs of order n and diameter $n-2$. In [971] Liu obtained lower bounds for the radio number of trees and the radio number of spiders (trees with at most one vertex of degree greater than 2) and characterized the graphs that achieve these bounds.

Chartrand, Erwin, Zhang, and Harary [359] proved: $rn(K_{n_1,n_2,\ldots,n_k}) = n_1+n_2+\cdots+n_k+k-1$; if G is a connected graph of order n and diameter 2, then $n \leq rn(G) \leq 2n-2$; and for every pair of integers k and n with $n \leq k \leq 2n-2$, there exists a connected graph of order n and diameter 2 with $rn(G) = k$. They further provide a characterization of connected graphs of order n and diameter 2 with prescribed radio number.

Fernandez, Flores, Tomova, and Wyels [490] proved $rn(K_n) = n$; $rn(W_n) = n+2$; and the radio number of the gear graph obtained from W_n by inserting a vertex between each vertex of the rim is $4n+2$. Morris-Rivera, Tomova, Wyels, and Yeager [1073] determine the radio number of $C_n \times C_n$. Martinez, Ortiz, Tomova, and Wyels [1030] define *generalized prisms,* denoted $Z_{n,s}$, $s \geq 1$, $n \geq s$, as the graphs with vertex set $\{(i,j) \,|\, i = 1,2 \text{ and } j = 1,...,n\}$ and edge set $\{((i,j),(i,j\pm 1))\} \cup \{((1,i),(2,i+\sigma)) \,|\, \sigma = -\left\lfloor \frac{s-1}{2} \right\rfloor \ldots, 0, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}$. They determine the radio number of $Z_{n,s}$ for $s = 1,2$ and 3.

The generalized gear graph $J_{t,n}$ is obtained from a wheel W_n by introducing t-vertices between every pair (v_i, v_{i+1}) of adjacent vertices on the n-cycle of wheel. Ali, Rahim, Ali, and Farooq [73] gave an upper bound for the radio number of generalized gear graph, which coincided with the lower bound found in [72] and [1175]. They proved for $t < n-1$ and $n \geq 7$, $\mathrm{rn}(J_{t,n}) = (nt^2+4nt+3n+4)/2$. They pose the determination of the radio number of $J_{t,n}$ when $n \leq 7$ and $t > n-1$ as an open problem.

Liu and Xie [973] determined the radio numbers of squares of cycles for most values of n. In [974] Liu and Xie proved that $rn(P_n^2)$ is $\lfloor n/2 \rfloor + 2$ if $n \equiv 1 \pmod 4$ and $n \geq 9$ and $rn(P_n^2)$ is $\lfloor n/2+1 \rfloor$ otherwise. In [972] Liu found a lower bound for the radio number of trees and characterizes the trees that achieve the bound. She also provides a lower bound for the radio number of spiders in terms of the lengths of their legs and characterizes the spiders that achieve this bound. Sweetly and Joseph [1482] prove that the radio number of the graph obtained from the wheel W_n by subdividing each edge of the rim exactly twice is $5n-3$. Marinescu-Ghemeci [1026] determined the radio number of the caterpillar obtained from a path by attaching a new terminal vertex to each non-terminal vertex of the path and the graph obtained from a star by attaching k new terminal vertices to each terminal vertex of the star.

Sooryanarayana and Raghunath. P [1428] determined the radio number of C_n^3, for $n \leq 20$ and for $n \equiv 0$ or 2 or 4 (mod 6). Sooryanarayana, Vishu Kumar, Manjula [1429] determine the radio number of P_n^3, for $n \geq 4$. Wang, Xu, Yang, Zhang, Luo, and Wang [1614] determine the radio number of ladder graphs. In [1559] Vaidya and Vihol determined upper bounds on radio numbers of cycles with chords and determined the exact radio numbers for the splitting graph and the middle graph of C_n.

In [346] Canales, Tomova, and Wyels investigated the question of which radio numbers of graphs of order n are achievable. They proved that the achievable radio numbers of graphs of order n must lie in the interval $[n, rn(P_n)]$, and that these bounds are the best possible. They also show that for odd n, the integer $rn(P_n)-1 = \frac{(n-1)^2}{2}+2$ is an unachievable radio number for any graph of order n. In [1410] Sokolowsky settled the question of exactly which radio numbers are achievable for a graph of order n.

For any connected graph G and positive integer k Chartrand, Erwin, and Zhang, [357] define a *radio k-coloring* as an injection f from the vertices of G to the natural numbers such that $d(u, v) + |f(u) - f(v)| \geq 1 + k$ for every two distinct vertices u and v of G. Using $rc_k(f)$ to denote the maximum number assigned to any vertex of G by f, the *radio k-chromatic number* of G, $rc_k(G)$, is the minimum value of $rc_k(f)$ taken over all radio k-colorings of G. Note that $rc_1(G)$ is $\chi(G)$, the chromatic number of G, and when $k = diam(G)$, $rc_k(G)$ is $rn(G)$, the radio number of G. Chartrand, Nebesky, and Zang [365] gave upper and lower bounds for $rc_k(P_n)$ for $1 \leq k \leq n-1$. Kchikech, Khennoufa, and Togni [779] improved Chartrand et al.'s lower bound for $rc_k(P_n)$ and Kola and Panigrahi [807] improved the upper bound for certain special cases of n. The exact value of $rc_{n-2}(P_n)$ for $n \geq 5$ was given by Khennoufa and Togni in [785] and the exact value of $rc_{n-3}(P_n)$ for $n \geq 8$ was given by Kola and Panigrahi in [807]. Kola and Panigrahi [807] gave the exact value of $rc_{n-4}(P_n)$ when n is odd and $n \geq 11$ and an upper bound for $rc_{n-4}(P_n)$ when n is even and $n \geq 12$. In [1221] Saha and Panigrahi provided an upper and a lower bound for $rc_k(C_n^r)$ for all possible values of n, k and r and showed that these bounds are sharp for antipodal number of C_n^r for several values of n and r. Kchikech, Khennoufa, and Togni [780] gave upper and lower bounds for $rc_k(G \times H)$ and $rc_k(Q_n)$. In [779] the same authors proved that $rc_k(K_{1,n}) = n(k-1)+2$ and for any tree T and $k \geq 2$, $rc_k(T) \leq (n-1)(k-1)$.

A radio k-coloring of G when $k = \text{diam}(G) - 1$ is called a *radio antipodal labeling*. The minimum span of a radio antipodal labeling of G is called the *radio antipodal number* of G and is denoted by $an(G)$. Khennoufa and Togni [782] determined the radio number and the radio antipodal number of the hypercube by using a generalization of binary Gray codes. They proved that $rn(Q_n) = (2^{n-1}-1)\lceil \frac{n+3}{2} \rceil + 1$ and $an(Q_n) = (2^{n-1}-1)\lceil \frac{n}{2} \rceil + \varepsilon(n)$, with $\varepsilon(n) = 1$ if $n \equiv 0$ mod 4, and $\varepsilon(n) = 0$ otherwise.

Sooryanarayana and Raghunath [1428] say a graph with n vertices is *radio graceful* if $rn(G) = n$. They determine the values of n for which C_n^3 is radio graceful.

The survey article by Panigrahi [1116] includes background information and further results about radio k-colorings.

7.5 Line-graceful Labelings

Gnanajothi [572] has defined a concept similar to edge-graceful. She calls a graph with n vertices *line-graceful* if it is possible to label its edges with $0, 1, 2, \ldots, n$ such that when each vertex is assigned the sum modulo n of all the edge labels incident with that vertex the resulting vertex labels are $0, 1, \ldots, n-1$. A necessary condition for the line-gracefulness of a graph is that its order is not congruent to 2 (mod 4). Among line-graceful graphs are (see [572, pp. 132–181]) P_n if and only if $n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; $K_{1,n}$ if and only if $n \not\equiv 1 \pmod 4$; $P_n \odot K_1$ (combs) if and only if n is even; $(P_n \odot K_1) \odot K_1$ if and only if $n \not\equiv 2 \pmod 4$; (in general, if G has order n, $G \odot H$ is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H); mC_n when mn is odd; $C_n \odot K_1$ (crowns) if and only if n is even; mC_4 for all m; complete n-ary trees when n is even; $K_{1,n} \cup K_{1,n}$ if and only if n is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete n-ary tree when n is even; trees for which exactly one vertex has even degree. She conjectures that all trees with $p \not\equiv 2 \pmod 4$ vertices are line-graceful and proved this conjecture for $p \leq 9$.

Gnanajothi [572] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from $K_{1,4}$ by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has $p \not\equiv 2 \pmod 4$ vertices; the graph obtained from $K_{1,n}$ by replacing each spoke with a path of length m (counting the center vertex) is line-graceful in the following cases: $n = 2$; $n = 3$ and $m \not\equiv 3 \pmod 4$; and m is even and $mn + 1 \equiv 0 \pmod 4$.

Gnanajothi studied graphs obtained by joining disjoint graphs G and H with an edge. She proved such graphs are line-graceful in the following circumstances: $G = H$; $G = P_n, H = P_m$ and $m + n \not\equiv 0 \pmod 4$; and $G = P_n \odot K_1$, $H = P_m \odot K_1$ and $m + n \not\equiv 0 \pmod 4$.

Vaidya and Kothari [1531] proved following graphs are line graceful: fans F_n for $n \not\equiv 1$ (mod 4); W_n $\not\equiv 1$ (mod 4); bistars $B_{n,n}$ if and only if for $n \equiv 1, 3 \pmod 4$; and helms H_n for all n.

7.6 Representations of Graphs modulo n

In 1989 Erdős and Evans [470] defined a *representation* modulo n of a graph G with vertices $v_1, v_2, \ldots, v_r$ as a set $\{a_1, \ldots, a_r\}$ of distinct, nonnegative integers each less than n satisfying $\gcd(a_i - a_j, n) = 1$ if and only if v_i is adjacent to v_j. They proved that every finite graph can be represented modulo some positive integer. The *representation number*, Rep(G), is smallest such integer. Obviously the representation number of a graph is prime if and only if a graph is complete. Evans, Fricke, Maneri, McKee, and Perkel [479] have shown that a graph is representable modulo a product of a pair of distinct primes if and only if the graph does not contain an induced subgraph isomorphic to $K_2 \cup 2K_1$, $K_3 \cup K_1$, or the complement of a chordless cycle of length at least five. Nešetřil and Pultr [1091] showed that every graph can be represented modulo a product of some set of distinct primes. Evans et al. [479] proved that if G is representable modulo n and p is a prime divisor of n, then $p \geq \chi(G)$. Evans, Isaak, and Narayan [480] determined representation numbers for specific families as follows (here we use q_i to denote the ith prime and for any prime p_i we use $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$ to denote the next k primes larger than p_i): $\mathrm{Rep}(P_n) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil}$; $\mathrm{Rep}(C_4) = 4$ and for $n \geq 3$, $\mathrm{Rep}(C_{2n}) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil + 1}$; $\mathrm{Rep}(C_5) = 3 \cdot 5 \cdot 7 = 105$ and for $n \geq 4$ and not a power of 2, $\mathrm{Rep}(C_{2n+1}) = 3 \cdot 5 \cdot \cdots \cdot q_{\lceil \log_2 n \rceil + 1}$; if $m \geq n \geq 3$, then $\mathrm{Rep}(K_m - P_n) = p_i p_{i+1}$ where p_i is the smallest prime greater than or equal to $m - n + \lceil n/2 \rceil$; if $m \geq n \geq 4$, and p_i is the smallest prime greater than or equal to $m - n + \lceil n/2 \rceil$, then $\mathrm{Rep}(K_m - C_n) = q_i q_{i+1}$ if n is even and

$\mathrm{Rep}(K_m - C_n) = q_i q_{i+1} q_{i+2}$ if n is odd; if $n \leq m-1$, then $\mathrm{Rep}(K_m - K_{1,n}) = p_s p_{s+1} \cdots p_{s+n-1}$ where p_s is the smallest prime greater than or equal to $m-1$; $\mathrm{Rep}(K_m)$ is the smallest prime greater than or equal to m; $\mathrm{Rep}(nK_2) = 2 \cdot 3 \cdot \ \cdots \ \cdot q_{\lceil \log_2 n \rceil + 1}$; if $n, m \geq 2$, then $\mathrm{Rep}(nK_m) = p_i p_{i+1} \cdots p_{i+m-1}$, where p_i is the smallest prime satisfying $p_i \geq m$, if and only if there exists a set of $n-1$ mutually orthogonal Latin squares of order m; $\mathrm{Rep}(mK_1) = 2m$; and if $t \leq (m-1)!$, then $\mathrm{Rep}(K_m + tK_1) = p_s p_{s+1} \cdots p_{s+m-1}$ where p_s is the smallest prime greater than or equal to m. Narayan [1090] proved that for $r \geq 3$ the maximum value for $\mathrm{Rep}(G)$ over all graphs of order r is $p_s p_{s+1} \cdots p_{s+r-2}$, where p_s is the smallest prime that is greater than or equal to $r-1$.

Evans [478] used matrices over the additive group of a finite field to obtain various bounds for the representation number of graphs of the form nK_m. Among them are $\mathrm{Rep}(4K_3) = 3{\cdot}5{\cdot}7{\cdot}11$; $\mathrm{Rep}(7K_5) = 5{\cdot}7{\cdot}11{\cdot}13{\cdot}17{\cdot}19{\cdot}23$; and $\mathrm{Rep}((3q-1)/2)K_q) \leq p_q p_{q+1} \cdots p_{(3q-1)/2)}$ where q is a prime power with $q \equiv 3 \pmod 4$, p_q is the smallest prime greater than or equal to q, and the remaining terms are the next consecutive $(3q-3)/2$ primes; $\mathrm{Rep}(2q-2)K_q) \leq p_q p_{q+1} \cdots p_{(3q-3)/2)}$ where q is a prime power with $q \equiv 3$ mod 4, and p_q is the smallest prime greater than or equal to q; $\mathrm{Rep}((2q-2)K_q) \leq p_q p_{q+1} \cdots p_{2q-3}$.

In [1089] Narayan asked for the values of $\mathrm{Rep}(C_{2^k+1})$ when $k \geq 3$ and $\mathrm{Rep}(G)$ when G is a complete multipartite graph or a disjoint union of complete graphs. He also asked about the behavior of the representation number for random graphs.

Akhtar, Evans, and Pritikin [58] characterized the representation number of $K_{1,n}$ using Euler's phi function, and conjectured that this representation number is always of the form 2^a or $2^a p$, where $a \geq 1$ and p is a prime. They proved this conjecture for "small" n and proved that for sufficiently large n, the representation number of $K_{1,n}$ is of the form $2^a, 2^a p$, or $2^a pq$, where $a \geq 1$ and p and q are primes. In [59] they showed that for sufficiently large $n \geq m$, $\mathrm{rep}(K_{m,n}) = 2^a, 3^a, 2^a p^b$, or $2^a pq$, where $a, b \geq 1$ and p and q are primes; and for sufficiently large order, $\mathrm{rep}(K_{n_1,n_2,\ldots,n_t} = p^a, p^a q^b$, or $p^a q^b u$, where p, q, u are primes with $p, q < u$. Akhtar [60] determined the representation number of graphs of the form $K_2 \cup nK_1$ (he uses the notation $K_2 + nK_1$) and studies their prime decompositions. Using relations between representation modulo r and product representations, he determined representation number of binary trees and gave an improved lower bound for hypercubes.

7.7 k-sequential Labelings

In 1981 Bange, Barkauskas, and Slater [218] defined a *k-sequential* labeling f of a graph $G(V, E)$ as one for which f is a bijection from $V \cup E$ to $\{k, k+1, \ldots, |V \cup E| + k - 1\}$ such that for each edge xy in E, $f(xy) = |f(x) - f(y)|$. This generalized the notion of *simply sequential* where $k = 1$ introduced by Slater. Bange, Barkauskas, and Slater showed that cycles are 1-sequential and if G is 1-sequential, then $G + K_1$ is graceful. Hegde and Shetty [639] have shown that every T_p-tree (see §4.4 for the definition) is 1-sequential. In [1397], Slater proved: K_n is 1-sequential if and only if $n \leq 3$; for $n \geq 2$, K_n is not k-sequential for any $k \geq 2$; and $K_{1,n}$ is k-sequential if and only if k divides n. Acharya and Hegde [25] proved: if G is k-sequential, then k is at most the independence number of G; P_{2n} is n-sequential for all n and P_{2n+1} is both n-sequential and $(n+1)$-sequential for all n; $K_{m,n}$ is k-sequential for $k = 1, m$, and n; $K_{m,n,1}$ is 1-sequential; and the join of any caterpillar and $\overline{K_t}$ is 1-sequential. Acharya [13] showed that if $G(E, V)$ is an odd graph with $|E| + |V| \equiv 1$ or $2 \pmod 4$ when k is odd or $|E| + |V| \equiv 2$ or $3 \pmod 4$ when k is even, then G is not k-sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig, and Turgeon [265] we have: mK_4 is not k-sequential for any k when m is odd and mK_2 is not k-sequential for any odd k when $m \equiv 2$ or 3 (mod 4) or for any even k when $m \equiv 1$ or 2 (mod 4). He further noted that $K_{m,n}$ is not k-sequential when k is even and m and n are odd, whereas $K_{m,k}$ is k-sequential for all k. Acharya points out that the following result of Slater's [1398] for $k = 1$ linking k-graceful graphs and k-sequential graphs holds in general: A graph is k-sequential if and only if $G + v$ has a k-graceful labeling f with $f(v) = 0$. Slater [1397] also proved that a k-sequential graph with p vertices and $q > 0$ edges must satisfy $k \leq p - 1$. Hegde [629] proved that every graph can be embedded as an induced subgraph of a simply sequential graph. In [13] Acharya conjectured that if G is a connected k-sequential graph of order p with $k > \lfloor p/2 \rfloor$, then $k = p - 1$ and $G = K_{1,p-1}$ and that, except for $K_{1,p-1}$, every tree in which all vertices are odd is k-sequential for all odd positive integers $k \leq p/2$. In [629] Hegde gave counterexamples for both of these conjectures.

In [638] Hegde and Miller prove the following: for $n > 1$, K_n is k-sequentially additive if and only if $(n, k) = (2, 1), (3, 1)$ or (3,2); $K_{1,n}$ is k-sequentially additive if and only if k divides n; caterpillars with bipartition sets of sizes m and n are k-sequentially additive for $k = m$ and $k = n$; and if an odd-degree (p, q)-graph is k-sequentially additive, then $(p+q)(2k+p+q-1) \equiv 0$ mod 4. As corollaries of the last result they observe that when m and n are odd and k is even $K_{m,n}$ is not k-sequentially additive and if an odd-degree tree is k-sequentially additive then k is odd.

In [1276] Seoud and Jaber proved the following graphs are 1-sequentially additive: graphs obtained by joining the centers of two identical stars with an edge; $S_n \cup S_m$ if and only if nm is even; $C_n \odot \overline{K_m}$; $P_n \odot \overline{K_m}$; kP_3; graphs obtained by joining the centers of k copies of P_3 to each vertex in $\overline{K_m}$; and trees obtained from $K_{1,n}$ by replacing each edge by a path of length 2 when $n \equiv 0, 1 \pmod 4$. They also determined all 1-sequentially additive graphs of order 6.

7.8 IC-colorings

For a subgraph H of a graph G with vertex set V and a coloring f from V to the natural numbers define $f_s(H) = \Sigma f(v)$ over all $v \in H$. The coloring f is called an IC-*coloring* if for any integer k between 1 and $f_s(G)$ there is a connected subgraph H of G such that $f_s(H) = k$. The IC-*index* of a graph G, $M(G)$, is $\max\{f_s| \ f_s$ is an IC-coloring of $G\}$. Salehi, Lee, and Khatirinejad [1229] obtained the following: $M(K_n) = 2^n - 1$; for $n \geq 2$, $M(K_{1,n}) = 2^n + 2$; if Δ is the maximum degree of a connected graph G, then $M(G) \geq 2^{\Delta} + 2$; if $ST(n; 3^n)$ is the graph obtained by identifying the end points of n paths of length 3, then $ST(n; 3^n)$ is at least $3^n + 3$ (they conjecture that equality holds for $n \geq 4$); for $n \geq 2$, $M(K_{2,n}) = 3 \cdot 2^n + 1$; $M(P_n) \geq (2 + \lfloor n/2 \rfloor)(n - \lfloor n/2 \rfloor) + \lfloor n/2 \rfloor - 1$; for $m, n \geq 2$, the IC-index of the double star $DS(m, n)$ is at least $(2^{m-1}+1)(2^{n-1}+1)$ (they conjecture that equality holds); for $n \geq 3$, $n(n + 1)/2 \leq M(C_n) \leq n(n - 1) + 1$; and for $n \geq 3$, $2^n + 2 \leq M(W_n) \leq 2^n + n(n - 1) + 1$. They pose the following open problems: find the IC-index of the graph obtained by identifying the end points of n paths of length b; find the IC-index of the graph obtained by identifying the end points of n paths; and find the IC-index of $K_{m,n}$. Shiue and Fu [1362] completed the partial results by Penrice [1127] Salehi, Lee, and Khatirinejad [1229] by proving $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$ for any $2 \leq m \leq n$.

7.9 Product and Divisor Cordial Labelings

Sundaram. Ponraj, and Somasundaram [1459] introduced the notion of product cordial labelings. A *product cordial labeling* of a graph G with vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a *product cordial* graph.

In [1459] and [1468] Sundaram, Ponraj, and Somasundaram prove the following graphs are product cordial: trees; unicyclic graphs of odd order; triangular snakes; dragons; helms; $P_m \cup P_n$; $C_m \cup P_n$; $P_m \cup K_{1,n}$; $W_m \cup F_n$ (F_n is the fan $P_n + K_1$); $K_{1,m} \cup K_{1,n}$; $W_m \cup K_{1,n}$; $W_m \cup P_n$; $W_m \cup C_n$; the total graph of P_n (the total graph of P_n has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n); C_n if and only if n is odd; $C_n^{(t)}$, the one-point union of t copies of C_n, provided t is even or both t and n are even; $K_2 + mK_1$ if and only if m is odd; $C_m \cup P_n$ if and only if $m+n$ is odd; $K_{m,n} \cup P_s$ if $s > mn$; $C_{n+2} \cup K_{1,n}$; $K_n \cup K_{n,(n-1)/2}$ when n is odd; $K_n \cup K_{n-1,n/2}$ when n is even; and P_n^2 if and only if n is odd. They also prove that $K_{m,n}$ $(m, n > 2)$, $P_m \times P_n$ $(m, n > 2)$ and wheels are not product cordial and if a (p, q)-graph is product cordial graph, then $q \leq (p-1)(p+1)/4 + 1$.

In [1273] Seoud and Helmi obtained the following results: K_n is not product cordial for all $n \geq 4$; C_m is product cordial if and only if m is odd; the gear graph G_m is product cordial if and only if m is odd; all web graphs are product cordial; the corona of a triangular snake with at least two triangles is product cordial; the C_4-snake is product cordial if and only if the number of 4-cycles is odd; $C_m \odot \overline{K_n}$ is product cordial; and they determine all graphs of order less than 7 that are not product cordial. Seoud and Helmi define the *conjunction* $G_1 \hat{\ } G_2$ of graphs G_1 and G_2 as the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2) | \ u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. They prove: $P_m \hat{\ } P_n$ $(m, n \geq 2)$ and $P_m \hat{\ } S_n$ $(m, n \geq 2)$ are product cordial.

Vaidya and Kanani [1523] prove the following graphs are product cordial: the path union of k copies of C_n except when k is odd and n is even; the graph obtained by joining two copies of a cycle by path; the path union of an odd number copies of the shadow of a cycle (see §3.8 for the definition); and the graph obtained by joining two copies of the shadow of a cycle by a path of arbitrary length. In [1526] Vaidya and Kanani prove the following graphs are product cordial: the path union of an even number of copies of $C_n(C_n)$; the graph obtained by joining two copies of $C_n(C_n)$ by a path of arbitrary length; the path union of any number of copies of the Petersen graph; and the graph obtained by joining two copies of the Petersen graph by a path of arbitrary length.

Vaidya and Barasara [1503] prove that the following graphs are product cordial: friendship graphs; the middle graph of a path; odd cycles with one chord except when the chord joins the vertices at a diameter distance apart; and odd cycles with two chords that share a common vertex and form a triangle with an edge of the cycle and neither chord joins vertices at a diameter apart.

In [1512] Vaidya and Dani prove the following graphs are product cordial: $< S_n^{(1)} : S_n^{(2)} : \ldots : S_n^{(k)} >$ except when k odd and n even; $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$; and $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ if and only if k is even or k is odd and n is even with $k > n$. (See §3.7 for the definitions.)

Vaidya and Barasara [1504] proved the following graphs are product cordial: closed helms, web graphs, flower graphs, double triangular snakes obtained from the path P_n if and only if n is odd, and gear graphs obtained from the wheel W_n if and only if n is odd. Vaidya and Barasara [1505] proved that the graphs obtained by the duplication of an edge of a cycle, the mutual duplication of pair of edges of a cycle, and mutual duplication of pair of vertices between two copies of C_n admit product cordial labelings. Moreover, if G and G' are the graphs such that their orders or sizes differ at most by 1 then the new graph obtained by joining G and G' by a path P_k of arbitrary length admits product cordial labeling.

Vaidya and Barasara [1506] define the *duplication of a vertex* v of a graph G *by a new edge* $u'v'$ as the graph G' obtained from G by adding the edges $u'v'$, vu' and vv' to G. They define the *duplication of an edge* uv of a graph G *by a new vertex* v' as the graph G' obtained from G by adding the edges uv' and vv' to G. They proved the following graphs have product cordial labelings: the graph obtained by duplication of an arbitrary vertex by a new edge in C_n or P_n $(n > 2)$; the graph obtained by duplication of an arbitrary edge by a new vertex in C_n $(n > 3)$ or P_n $(n > 3)$; and the graph obtained by duplicating all the vertices by edges in path P_n. They also proved that the graph obtained by duplicating all the vertices by edges in C_n $(n > 3)$ and the graph obtained by duplicating all the edges by vertices in C_n are not product cordial.

In [1223] Salehi called the set $\{|e_f(0) - e_f(1)| : f \text{ is a friendly labeling of } G\}$ the *product-cordial set* of G. He determines the product-cordial sets for paths, cycles, wheels, complete graphs, bipartite complete graphs, double stars, and complete graphs with an edge deleted. Salehi and Mukhin [1230] say a graph G of size q is *fully product-cordial* if its product cordial set is $\{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$. They proved: P_n $(n \geq 2)$ is fully product-cordial; trees with a perfect matching are fully product-cordial; $P_2 \times P_n$ is not fully product-cordial; and $P_m \times P_n$ has the maximum product cordial -index $2mn - m - n$. They determine the product-cordial sets of $P_2 \times P_n$, $P_n \times P_{2m}$ and $P_n \times P_{2m+1}$, where $m \geq n$. Because the product-cordial set is the multiplicative version of the friendly index set, Kwong, Lee, and Ng [848] called it the *product-cordial index set* of G and determined the exact values of the product-cordial index set of C_m and $C_m \times P_n$. In [849] Kwong, Lee, and Ng determined the friendly index sets and product-cordial index sets of 2-regular graphs and the graphs obtained by identifying the centers of any number of wheels.

In [1339] Shiu and Kwong define the *full product-cordial index* of G under f as FPCI(G) $= \{i^*_f(G) \mid f \text{ is a friendly labeling of } G\}$. They provide a relation between the friendly index and the product-cordial index of a regular graph. As applications, they determine the full product-cordial index sets of C_m and $C_m \times C_n$, which was asked by Kwong, Lee, and Ng in [848]. Recently Gao [539] determined the full friendly index set of $P_m \times P_n$, but he used the terms “edge difference set” instead of “full friendly index set” and “direct product” instead of “Cartesian product.”

Jeyanthi and Maheswari define a mapping $f : V(G) \to \{0, 1, 2\}$ to be a 3-*product cordial* labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with $i, e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i \pmod 3$. A graph with a 3-product cordial labeling is called a 3-*product cordial* graph. In [723] they prove that for a (p, q) 3-product cordial graph: $p \equiv 0 \pmod 3$ implies $q \leq \frac{p^2-3p+6}{3}$; $p \equiv 1 \pmod 3$ implies $q \leq \frac{p^2-2p+7}{3}$; and $p \equiv 2 \pmod 3$ implies $q \leq \frac{p^2-p+4}{3}$. They prove the following graphs are 3-product cordial: paths; stars; C_n if and only if $n \equiv 1, 2 \pmod 3$; $C_n \cup P_n$, $C_m \odot \overline{K_n}$; $P_m \odot \overline{K_n}$ for $m \geq 3$ and $n \geq 1$; W_n when $n \equiv 1 \pmod 3$; and the graph obtained by joining the centers of two identical stars to a new vertex. They also prove that K_n is not 3-product cordial for $n \geq 3$ and if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph, then $G_1 \cup G_2$ is a 3-product cordial graph. In [724] they prove that ladders, $< W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} >$ (see §3.7 for the definition), graphs obtained by duplicating an arbitrary edge of a wheel, graphs obtained by duplicating an arbitrary vertex of a cycle or a wheel are 3-product cordial. They also prove that the graphs obtained by from the ladders $L_n = P_n \times P_2$ $(n \geq 2)$ by adding the edges u_iv_{i+1} for $1 \leq i \leq n-1$, where the consecutive vertices of two copies of P_n are $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ and the edges are u_iv_i. They call these graphs *triangular ladders* .

Sundaram and Somasundaram [1463] also have introduced the notion of total product cordial labelings. A *total product cordial labeling* of a graph G with vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$ the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a *total product cordial* graph. In [1463] and [1461] Sundaram, Ponraj, and Somasundaram prove the following graphs are total product cordial: every product cordial graph of even order or odd order and even size; trees; all cycles except C_4; $K_{n,2n-1}$; C_n with m edges appended at each vertex; fans; double fans; wheels; helms; $C_n \times P_2$; $K_{2,n}$ if and only if $n \equiv 2 \pmod 4$; $P_m \times P_n$ if and only if $(m, n) \neq (2, 2)$; $C_n + 2K_1$ if and only if n is even or $n \equiv 1 \pmod 3$; $\overline{K_n} \times 2K_2$ if n is odd, or $n \equiv 0$ or $2 \pmod 6$, or $n \equiv 2$ (mod 8). Y.-L. Lai, the reviewer for MathSciNet [851], called attention to some errors in [1461].

Vaidya and Vihol [1552] prove the following graphs have total product labelings: a split graph; the total graph of C_n; the star of C_n (recall that the *star of a graph* G is the graph obtained fron G by replacing each vertex of star K_{1n} by a graph G); the friendship graph F_n; the one point union of k copies of a cycle; and the graph obtained by the switching of an arbitrary vertex in C_n.

Ramanjaneyulu, Venkaiah, and Kothapalli [1184] give total product cordial labeling for a family of planar graphs for which each face is a 4-cycle.

Sundaram, Ponraj, and Somasundaram [1466] introduced the notion of EP-*cordial labeling* (extended product cordial) labeling of a graph G as a function f from the verticies of a graph to $\{-1, 0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, then $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ where $i, j \in \{-1, 0, 1\}$ and $v_f(k)$ and $e_f(k)$ denote the number of vertices and edges respectively labeled with k. An EP-*cordial graph* is one that admits an EP-cordial labeling. In [1466] Sundaram, Ponraj, and Somasundaram prove the following: every graph is an induced subgraph of an EP-cordial graph, K_n is EP-cordial if and only if $n \leq 3$; C_n is EP-cordial if and only if $n \equiv 1, 2 \pmod 3$, W_n is EP-cordial if and only if $n \equiv 1 \pmod 3$; and caterpillars are EP-cordial. They prove that all $K_{2,n}$, paths, stars and the graphs obtained by subdividing each edge of of a star exactly once are EP-cordial. They also prove that if a (p, q) graph is EP-cordial, then $q \leq 1 + p/3 + p^2/3$. They conjecture that every tree is EP-cordial.

Ponraj, Sivakumar, and Sundaram [1155] introduced the notion of k-product cordial labeling of graphs. Let f be a map from $V(G)$ to $\{0, 1, 2, \ldots, k-1\}$, where $2 \leq k \leq |V|$. For each edge uv assign the label $f(u)f(v) \pmod k$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x. A graph with a k-product cordial labeling is called a *k-product cordial graph.* Observe that 2-product cordial labeling is simply a product cordial labeling and 3-product cordial labeling is an EP-cordial labeling. In [1155] and [1156] Ponraj et al. prove the following are 4-product cordial: P_n iff $n \leq 11$, C_n iff $n = 5, 6, 7, 8, 9$, or 10, K_n iff $n \leq 2$, $P_n \odot K_1$, $P_n \odot 2K_1$, $K_{2,n}$ iff $n \equiv 0, 3 \pmod 4$, W_n iff $n = 5$ or 9, $\overline{K_n} + 2K_2$ iff $n \leq 2$, and the subdivision graph of $K_{1,n}$.

Let f be a map from $V(G)$ to $\{0, 1, 2, \ldots, k-1\}$ where $2 \leq k \leq |V|$. For each edge uv assign the label $f(u)f(v) \pmod k$. Ponraj, Sivakumar, and Sundaram [1157] define f to be a *k-total* *product cordial labeling* if $|f(i)-f(j)| \leq 1$, $i,j \in \{0,1,2,\ldots,k-1\}$, where $f(x)$ denote the number of vertices and edges labeled with x. A graph with a k-total product cordial labeling is called a *k-total product cordial graph* . A 2-total product cordial labeling is simply a total product cordial labeling. In [1157], [1158], [1159], [1160] and [1161], Ponraj et al. proved the following graphs are 3-total product cordial: P_n, C_n iff $n \neq 3$ or 6, $K_{1,n}$ iff $n \equiv 0,2 \pmod 3$, $P_n \odot K_1$, $P_n \odot 2K_1$, $K_2 + mK_1$ iff $m \equiv 2 \pmod 3$, helms, wheels, $C_n \odot 2K_1$, $C_n \odot K_2$, dragons $C_m@P_n$, $C_n \odot K_1$, $B_{m,n}$, and the subdivision graphs of $K_{1,n}$, $C_n \odot K_1$, $K_{2,n}$, $P_n \odot K_1$, $P_n \odot 2K_1$, $C_n \odot K_2$, wheels and helms. Also they proved that every graph is a subgraph of a connected k-total product cordial graph, $B_{m,n}$ is $(n+2)$-total product cordial, and $K_{m,n}$ is $(n+2)$-total product cordial,

For a graph G Sundaram, Ponraj, and Somasundaram [1467] defined the *index of product cordiality*, $i_p(G)$, of G as the minimum of $\{|e_f(0)-e_f(1)|\}$ taken over all the 0-1 binary labelings f of G with $|v_f(i)-v_f(j)| \leq 1$ and $f(uv) = f(u)f(v)$, where $e_f(k)$ and $v_f(k)$ denote the number of edges and the number of vertices labeled with k. They established that $i_p(K_n) = \lfloor n/2 \rfloor^2$; $i_p(C_n) = 2$ if n is even; $i_p(W_n) = 2$ or 4 according as n is even or odd; $i_p(K_{2,n}) = 4$ or 2 according as n is even or odd; $i_p(K_2 + nK_1) = 3$ if n is even; $i_p(G \times P_2) \leq 2i_p(G)$; $i_p(G_1 \cup G_2) \leq i_p(G_1) + i_p(G_2) + 2\min\{\Delta(G_1), \Delta(G_2)\}$ where G_1 and G_2 are graphs of odd order; and $i_p(G_1 \odot G_2) \leq i_p(G_1) + i_p(G_2) + 2\delta(G_2) + 3$ where G_1 and G_2 have odd order.

Vaidya and Vyas [1560] define the *tensor product* $G_1(T_p)G_2$ of graphs G_1 and G_2 as the graph with vertex set $V(G_1) \times V(G_2)$ and edge set
$\{(u_1,v_1)(u_2,v_2)|\ u_1u_2 \in E(G_1), v_1v_2 \in V(G_2)\}$. They proved the following graphs are product cordial: $P_m(T_p)P_n$; $C_{2m}(T_p)P_{2n}$; $C_{2m}(T_p)C_{2n}$; the graph obtained by joining two components of $P_m(T_p)P_n$ an by arbitrary path; the graph obtained by joining two components of $C_{2m}(T_p)P_{2n}$ by an arbitrary path; and and the graph obtained by joining two components of $C_{2m}(T_p)C_{2n}$ by an arbitrary path.

In [1142] Ponraj introduced the notion of an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-cordial labeling of a graph. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a finite set of distinct integers and f be a function from a vertex set $V(G)$ to S. For each edge uv of G assign the label $f(u)f(v)$. He calls f an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-*cordial labeling* of G if $|v_f(\alpha_i) - v_f(\alpha_j)| \leq 1$ for all $i,j \in \{1,2,\ldots,k\}$ and $|e_f(\alpha_i\alpha_j) - e_f(\alpha_r\alpha_s)| \leq 1$ for all $i,j,r,s \in \{1,2,\ldots,k\}$, where $v_f(t)$ and $e_f(t)$ denote the number of vertices labeled with t and the number of edges labeled with t, respectively. A graph that admits an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-cordial labeling is called an $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-*cordial graph* Note that an $(-\alpha, \alpha)$-cordial graph is simply a cordial graph and a $(0, \alpha)$-cordial graph is a product cordial graph. Ponraj proved that $K_{1,n}$ is $(\alpha_1, \alpha_2, \ldots, \alpha_k)$-cordial if and only if $n \leq k$ and for $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_1 + \alpha_2 \neq 0$ proved the following: K_n is (α_1, α_2)-cordial if and only if $n \leq 2$; P_n is (α_1, α_2)-cordial; C_n is (α_1, α_2)-cordial if and only if $n > 3$; $K_{m,n}$ $(m, n > 2)$ is not (α_1, α_2)-cordial; the bistar $B_{n,n+1}$ is (α_1, α_2)-cordial; $B_{n+2,n}$ is (α_1, α_2)-cordial if and only if $n \equiv 1,2 \pmod 3$; $B_{n+3,n}$ is (α_1, α_2)-cordial if and only if $n \equiv 0,2 \pmod 3$; and $B_{n+r,n}, r > 3$ is not (α_1, α_2)-cordial. He also proved that if G is an (α_1, α_2)-cordial graph with p vertices and q edges, then $q \leq 3p^2/8 - p/2 + 9/8$. In [1142] Ponraj proved that combs $P_n \odot K_1$ are (α_1, α_2)-cordial; coronas $C_n \odot K_1$ are (α_1, α_2)-cordial for $n \equiv 0,2,4,5 \pmod 6$; $C_3^{(t)}$ is not (α_1, α_2)-cordial; W_n is not (α_1, α_2)-cordial; and $\overline{K_n} + 2K_2$ is (α_1, α_2)-cordial if and only if $n = 2$.

In [1568] Varatharajan, Navanaeethakrishnan Nagarajan define a *divisor cordial* labeling of a graph G with vertex set V as a bijection f from V to $\{1,2,\ldots,|V|\}$ such that an edge uv is assigned the label 1 if one $f(u)$ or $f(v)$ divides the other and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If graph that has a divisor cordial labeling, it is called a *divisor cordial* graph. They proved the standard graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial. They also proved that complete graphs are not divisor cordial. In [1569] they proved dragons, coronas, wheels, and complete binary trees are divisor cordial. For t copies $S_1, S_2, \ldots, S_t$ of an n-star $K_{1,n}$ they define $\langle S_1, S_2, \ldots, S_t \rangle$ as the graph obtained by starting with $S_1, S_2, \ldots, S_t$ and joining the central vertices of S_{k-1} and S_k to a new vertex x_{k-1}. They prove that $\langle S_1, S_2 \rangle$ and $\langle S_1, S_2, S_3 \rangle$ are divisor cordial.

Vaidya and Shah [1548] proved that the splitting graphs of stars and bistars are divisor cordial. They also showed that the shadow graphs and the squares of bistars are divisor cordial.

7.10 Difference Cordial Labelings

Ponraj, Sathish Narayanan and Kala [1153] introduced the notion of difference cordial labelings. A *difference cordial labeling* of a graph G is an injective function f from $V(G)$ to $\{0, 1, \ldots, |V(G)|\}$ such that if each edge uv is assigned the label $|f(u) - f(v)|$, the number of edges labeled with 1 and the number of edges not labeled with 1 differ by at most 1. A graph with a difference cordial labeling is called a *difference cordial* graph.

The following definitions appear in [1154] and [1152]: A *double triangular snake* DT_n consists of two triangular snakes that have a common path; a *double quadrilateral snake* DQ_n consists of two quadrilateral snakes that have a common path; an *alternate triangular snake* $A(T_n)$ is the graph obtained from a path $u_1u_2\ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertex v_i (that is, every alternate edge of a path is replaced by C_3); a *double alternate triangular snake* $DA(T_n)$ consists of two triangular snakes that have a common path (that is, a double alternate triangular snake is obtained from a path $u_1u_2\ldots u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i; an *alternate quadrilateral snake* $A(Q_n)$ is obtained from a path $u_1u_2\ldots u_n$ by joining u_i, u_{i+1} (alternatively) to new vertices v_i, w_i respectively and then joining v_i and w_i (that is, every alternate edge of a path is replaced by a cycle C_4); a *double alternate quadrilateral snake* $DA(Q_n)$ consists of two alternate quadrilateral snakes that have a common path (that is, a double alternate quadrilateral snake is obtained from a path $u_1u_2\ldots u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i and w_i, y_i respectively and then joining v_i, w_i and x_i, y_i.

In [1153], [1151], [1154] and [1152] Ponraj et al. proved the following graphs have difference cordial labelings: paths; cycles; wheels; fans; gears; helms; $K_{1,n}$ if and only if $n \le 5$; K_n if and only if $n \le 4$; $K_{2,n}$ if and only if $n \le 4$; $K_{3,n}$ if and only if $n \le 4$; $B_{1,n}$ if and only if $n \le 5$; $B_{2,n}$ if and only if $n \le 6$; $B_{3,n}$ if and only if $n \le 5$; $DT_n \odot K_1$; $DT_n \odot 2K_1$; $DT_n \odot K_2$; $DQ_n \odot K_1$; $DQ_n \odot 2K_1$; $DQ_n \odot K_2$; $DA(T_n) \odot K_1$; $DA(T_n) \odot 2K_1$; $DA(T_n) \odot K_2$; $DA(Q_n) \odot K_1$; $DA(Q_n) \odot 2K_1$; and $DA(Q_n) \odot K_2$. They also proved that if G is a (p, q) difference cordial graph, then $q \le 2p - 1$; if G is a r-regular graph with $r \ge 4$ then G is not difference cordial; if $m \ge 4$ and $n \ge 4$, then $K_{m,n}$ is not difference cordial; if $m + n \ge 9$ then $B_{m,n}$ is not difference cordial; and every graph is a subgraph of a connected difference cordial graph. If G is a book, sunflower, lotus inside a circle, or square of a path, they prove that $G \odot mK_1$ $(m = 1, 2)$ and $G \odot K_2$ is difference cordial.

7.11 Prime Cordial Labelings

Sundaram, Ponraj, and Somasundaram [1460] have introduced the notion of prime cordial labelings. A *prime cordial labeling* of a graph G with vertex set V is a bijection f from V to $\{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if

$\gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In [1460] Sundaram, Ponraj, and Somasundram prove the following graphs are prime cordial: C_n if and only if $n \geq 6$; P_n if and only if $n \neq 3$ or 5; $K_{1,n}$ (n odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geq 3$; bistars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders; $K_{1,n}$ if n is even and there exists a prime p such that $2p < n+1 < 3p$; $K_{2,n}$ if n is even and if there exists a prime p such that $3p < n+2 < 4p$; and $K_{3,n}$ if n is odd and if there exists a prime p such that $5p < n+3 < 6p$. They also prove that if G is a prime cordial graph of even size, then the graph obtained by identifying the central vertex of $K_{1,n}$ with the vertex of G labeled with 2 is prime cordial, and if G is a prime cordial graph of odd size, then the graph obtained by identifying the central vertex of $K_{1,2n}$ with the vertex of G labeled with 2 is prime cordial. They further prove that $K_{m,n}$ is not prime cordial for a number of special cases of m and n. Sundaram and Somasundaram [1463] and Youssef [1704] observed that for $n \geq 3$, K_n is not prime cordial provided that the inequality $\phi(2) + \phi(3) + \cdots + \phi(n) \geq n(n-1)/4 + 1$ is valid for $n \geq 3$. This inequality was proved by Yufei Zhao [1721].

Seoud and Salim [1281] give an upper bound for the number of edges of a graph with a prime cordial labeling as a function of the number of vertices. For bipartite graphs they give a stronger bound. They prove that K_n does not have a prime cordial labeling for $2 < n < 500$ and conjecture that K_n is not prime cordial for all $n > 2$. They determine all prime cordial graphs of order at most 6. For a graph with n vertices to admit a prime cordial labeling, Seoud and Salim [1283] proved that the number of edges must be less than $n(n-1) - 6n^2/\pi^2 + 3$. As a corollary they get that K_n ($n > 2$) is not prime cordial thereby proving their eariler conjecture.

In [126] Babujee and Shobana proved sun graphs $C_n \odot K_1$; C_n with a path of length $n-3$ attached to a vertex; and P_n ($n \geq 6$) with $n-3$ pendent edges attached to a pendent vertex of P_n have prime cordial labelings. Additional results on prime cordial labelings are given in [127].

In [1555] and [1556] Vaidya and Vihol prove following graphs are prime cordial: the total graph of P_n and the total graph of C_n for $n \geq 5$ (see §2.7 for the definition); $P_2[P_m]$ for all $m \geq 5$; the graph obtained by joining two copies of a fixed cycle by a path; and the graph obtained by switching of a vertex of C_n except for $n = 5$ (see §3.6 for the definition); the graph obtained by duplicating each edge by a vertex in C_n except for $n = 4$ (see §2.7 for the definition); the graph obtained by duplicating a vertex by an edge in cycle C_n (see §2.7 for the definition); the path union of any number of copies of a fixed cycle (see §3.7 for the definition); and the friendship graph F_n for $n \geq 3$. Vaidya and Shah [1543] prove following results: P_n^2 is prime cordial for $n = 6$ and $n \geq 8$; C_n^2 is prime cordial for $n \geq 10$; the shadow graphs of $K_{1,n}$ (see §3.8 for the definition) for $n \geq 4$ and the bistar $B_{n,n}$ are prime cordial graphs.

Let G_n be a simple nontrival connected cubic graph with vertex set $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \leq i \leq n-1\}$, and edge set $E(G_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_ia_i, d_ib_i, d_ic_i : 0 \leq i \leq n-1\}$, where the edge labels are taken modulo n. Let H_n be a graph obtained from G_n by replacing the edges $b_{n-1}b_0$ and $c_{n-1}c_0$ with $b_{n-1}c_0$ and $c_{n-1}b_0$ respectively. For odd $n \geq 5$, H_n is called a *flower snark* whereas G_n, H_3 and all H_n with even $n \geq 4$, are called the *related graphs* of a flower snark. Mominul Haque, Lin, Yang, and Zhang [1069] proved that flower snarks and related graphs are prime cordial for all $n \geq 3$.

In [1546] Vaidya and Shah prove that the following graphs are prime cordial: split graphs of $K_{1,n}$ and $B_{n,n}$; the square graph of $B_{n,n}$; the middle graph of P_n for $n \geq 4$; and W_n if and only if $n \geq 8$. Vaidya and Shah [1546] prove following graphs are prime cordial: the splitting graphs of $K_{1,n}$ and $B_{n,n}$; the square of $B_{n,n}$; the middle graph of P_n for $n \geq 4$; and wheels W_n for

$n \geq 8$. Vaidya and Prajapati [1540] call a graph *strongly prime cordial* if for any vertex v there is a prime labeling f of G such that $f(v) = 1$. They prove the following: the graphs obtained by identifying any two vertices of $K_{1,n}$ are prime cordial; the graphs obtained by identifying any two vertices of P_n are prime cordial; C_n, P_n, and $K_{1,n}$ are strongly prime cordial; and W_n is a strongly prime cordial for every even integer $n \geq 4$.

7.12 Geometric Labelings

If a and r are positive integers at least 2, we say a (p, q)-graph G is (a, r)-*geometric* if its vertices can be assigned distinct positive integers such that the value of the edges obtained as the product of the end points of each edge is $\{a, ar, ar^2, \ldots, ar^{q-1}\}$. Hegde [632] has shown the following: no connected bipartite graph, except the star, is (a, a)-geometric where a is a prime number or square of a prime number; any connected (a, a)-geometric graph where a is a prime number or square of a prime number, is either a star or has a triangle; $K_{a,b},\ 2 \leq a \leq b$ is (k, k)-geometric if and only if k is neither a prime number nor the square of a prime number; a caterpillar is (k, k)-geometric if and only if k is neither a prime number nor the square of a prime number; $K_{a,b,1}$ is (k, k)-geometric for all integers $k \geq 2$; C_{4t} is (a, a)-geometric if and only if a is neither a prime number nor the square of a prime number; for any positive integers t and $r \geq 2$, C_{4t+1} is (r^{2t}, r)-geometric; for any positive integer t, C_{4t+2} is not geometric for any values of a and r; and for any positive integers t and $r \geq 2$, C_{4t+3} is (r^{2t+1}, r)-geometric. Hegde [634] has also shown that every T_p-tree and the subdivision graph of every T_p-tree are (a, r)-geometric for some values of a and r (see Section 3.2 for the definition of a T_p-tree). He conjectures that all trees are (a, r)-geometric for some values of a and r.

Hegde and Shankaran [640] prove: a graph with an α-labeling (see §3.1 for the definition) where m is the fixed integer that is between the end points of each edge has an (a^{m+1}, a)-geometric for any $a > 1$; for any integers m and n both greater than 1 and m odd, mP_n is (a^r, a)-geometric where $r = (mn+3)/2$ if n is odd and (a^r, a)-geometric where $r = (m(n+1)+3)/2$ if n is even; for positive integers $k > 1, d \geq 1$, and odd n, the generalized closed helm (see §5.3 for the definition) $CH(t, n)$ is (k^r, k^d)-geometric where $r = (n-1)d/2$; for positive integers $k > 1, d \geq 1$, and odd n, the generalized web graph (see §5.3 for the definition) $W(t, n)$ is (k^r, a)-geometric where $a = k^d$ and $r = (n-1)d/2$; for positive integers $k > 1, d \geq 1$, the generalized n-crown $(P_m \times K_3) \odot K_{1,n}$ is (a, a)-geometric where $a = k^d$; and $n = 2r+1$, $C_n \odot P_3$ is (k^r, k)-geometric.

If a and r are positive integers and r is at least 2 Arumugan, Germina, and Anadavally [103] say a (p, q)-graph G is *additively* (a, r)-*geometric* if its vertices can be assigned distinct integers such that the value of the edges obtained as the sum of the end points of each edge is $\{a, ar, ar^2, \ldots, ar^{q-1}\}$. In the case that the vertex labels are nonnegative integers the labeling is called *additively* $(a, r)*$-*geometric*. They prove: for all a and r every tree is additively $(a, r)*$-geometric; a connected additively (a, r)-geometric graph is either a tree or unicyclic graph with the cycle having odd size; if G is a connected unicyclic graph and not a cycle, then G is additively (a, r)-geometric if and only if either a is even or a is odd and r is even; connected unicyclic graphs are not additively $(a, r)*$-geometric; if a disconnected graph is additively (a, r)-geometric, then each component is a tree or a unicyclic graph with an odd cycle; and for all even a at least 4, every disconnected graph for which every component is a tree or unicyclic with an odd cycle has an additively (a, r)-geometric labeling.

Vijayakumar [1581] calls a graph G (not necessarily finite) *arithmetic* if its vertices can be assigned distinct natural numbers such that the value of the edges obtained as the sum of the end points of each edge is an arithmetic progression. He proves [1580] and [1581] that a graph is arithmetic if and only if it is (a, r)-geometric for some a and r.

7.13 Strongly Multiplicative Graphs

Beineke and Hegde [254] call a graph with p vertices *strongly multiplicative* if the vertices of G can be labeled with distinct integers $1, 2, \ldots, p$ such that the labels induced on the edges by the product of the end vertices are distinct. They prove the following graphs are strongly multiplicative: trees; cycles; wheels; K_n if and only if $n \leq 5$; $K_{r,r}$ if and only if $r \leq 4$; and $P_m \times P_n$. They then consider the maximum number of edges a strongly multiplicative graph on n vertices can have. Denoting this number by $\lambda(n)$, they show: $\lambda(4r) \leq 6r^2; \lambda(4r+1) \leq 6r^2 + 4r; \lambda(4r+2) \leq 6r^2 + 6r + 1$; and $\lambda(4r+3) \leq 6r^2 + 10r + 3$. Adiga, Ramaswamy, and Somashekara [37] give the bound $\lambda(n) \leq n(n+1)/2 + n - 2 - \lfloor (n+2)/4 \rfloor - \sum_{i=2}^{n} i/p(i)$ where $p(i)$ is the smallest prime dividing i. For large values of n this is a better upper bound for $\lambda(n)$ than the one given by Beineke and Hegde. It remains an open problem to find a nontrivial lower bound for $\lambda(n)$.

Seoud and Zid [1296] prove the following graphs are strongly multiplicative: wheels; rK_n for all r and n at most 5; rK_n for $r \geq 2$ and $n = 6$ or 7; rK_n for $r \geq 3$ and $n = 8$ or 9; $K_{4,r}$ for all r; and the corona of P_n and K_m for all n and $2 \leq m \leq 8$. In [1278] Seoud and Mahran [1278] give some necessary conditions for a graph to be strongly multiplicative.

Germina and Ajitha [564] (see also [20]) prove that $K_2 + \overline{K_t}$, quadrilateral snakes, Petersen graphs, ladders, and unicyclic graphs are strongly multiplicative. Acharya, Germina, and Ajitha [20] have shown that $C_k^{(n)}$ (see §2.2 for the definition) is strongly multiplicative and that every graph can be embedded as an induced subgraph of a strongly multiplicative graph. Germina and Ajitha [564] define a graph with q edges and a strongly multiplicative labeling to be *hyper strongly multiplicative* if the induced edge labels are $\{2, 3, \ldots, q+1\}$. They show that every hyper strongly multiplicative graph has exactly one nontrivial component that is either a star or has a triangle and every graph can be embedded as an induced subgraph of a hyper strongly multiplicative graph.

Vaidya, Dani, Vihol, and Kanani [1518] prove that the arbitrary supersubdivisions of tree, K_{mn}, $P_n \times P_m$, $C_n \odot P_m$, and C_n^m are strongly multiplicative. Vaidya and Kanani [1524] prove that the following graphs are strongly multiplicative: a cycle with one chord; a cycle with twin chords (that is, two chords that share an endpoint and with opposite endpoints that join two consecutive vertices of the cycle; the cycle C_n with three chords that form a triangle and whose edges are the edges of two 3-cycles and a $n-3$-cycle. duplication of an vertex in cycle (see §2.7 for the definition); and the graphs obtained from C_n by identifying of two vertices v_i and v_j where $d(v_i, v_j) \geq 3$. In [1527] the same authors prove that the graph obtained by an arbitrary supersubdivision of path, a star, a cycle, and a tadpole (that is, a cycle with a path appended to a vertex of the cycle.

Krawec [811] calls a graph G on n edges *modular multiplicative* if the vertices of G can be labeled with distinct integers $0, 1, \ldots, n-1$ (with one exception if G is a tree) such that the labels induced on the edges by the product of the end vertices modulo n are distinct. He proves that every graph can be embedded as an induced subgraph of a modular multiplicative graph on prime number of edges. He also shows that if G is a modular multiplicative graph on prime number of edges p then for every integer $k \geq 2$ there exist modular multiplicative graphs on p^k and kp edges that contain G as a subgraph. In the same paper, Krawec also calls a graph

G on n edges *k-modular multiplicative* if the vertices of G can be labeled with distinct integers $0, 1, \ldots, n+k-1$ such that the labels induced on the edges by the product of the end vertices modulo $n+k$ are distinct. He proves that every graph is k-modular multiplicative for some k and also shows that if $p = 2n+1$ is prime then the path on n edges is $(n+1)$-modular multiplicative. He also shows that if $p = 2n+1$ is prime then the cycle on n edges is $(n+1)$-modular multiplicative if there does not exist $\alpha \in \{2, 3, \ldots, n\}$ such that $\alpha^2 + \alpha - 1 \equiv 0 \bmod p$. He concludes with four open problems.

7.14 Mean Labelings

Somasundaram and Ponraj [1420] have introduced the notion of mean labelings of graphs. A graph G with p vertices and q edges is called a *mean graph* if there is an injective function f from the vertices of G to $\{0, 1, 2, \ldots, q\}$ such that when each edge uv is labeled with $(f(u) + f(v))/2$ if $f(u)+f(v)$ is even, and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd, then the resulting edge labels are distinct. In [1420], [1421], [1422], [1423], [1163], and [1164] they prove the following graphs are mean graphs: $P_n,\ C_n,\ K_{2,n},\ K_2 + mK_1,\ \overline{K_n} + 2K_2,\ C_m \cup P_n,\ P_m \times P_n,\ P_m \times C_n,\ C_m \odot K_1,\ P_m \odot K_1$, triangular snakes, quadrilateral snakes, K_n if and only if $n < 3,\ K_{1,n}$ if and only if $n < 3$, bistars $B_{m,n}$ $(m > n)$ if and only if $m < n + 2$, the subdivision graph of the star $K_{1,n}$ if and only if $n < 4$, the friendship graph $C_3^{(t)}$ if and only if $t < 2$, the one point union of two copies a fixed cycle, dragons (the one point union of C_m and P_n, where the chosen vertex of the path is an end vertex), the one point union of a cycle and $K_{1,n}$ for small values of n, and the arbitrary super subdivision of a path, which is obtained by replacing each edge of a path by $K_{2,m}$. They also prove that W_n is not a mean graph for $n > 3$ and enumerate all mean graphs of order less than 5.

Lourdusamy and Seenivasan [999] prove that kC_n-snakes are means graphs and every cycle has a super subdivision that is a mean graph. They define a generalized *kC_n-snake* in the same way as a C_n-snake except that the sizes of the cycle blocks can vary (see Section 2.2). They prove that generalized kC_n-snakes are mean graphs. Vasuki and Nagarajan [1571] proved that the following graphs admit mean labelings: $P_{r,2m+1}$ for all r and m; $P_{r,2m}$ for all m and $2 \le r \le 6$; P_r^{2m+1} for all r and m; and P_r^{2m} for all m and $2 \le r \le 6$.

Barrientos and Krop [240] proved that there exist $n!$ graphs of size n that admit mean labelings. They give two necessary conditions for the existence of a mean labeling of a graph G with m vertices and n edges: if G is a mean graph, then $n + 1 \ge m$; if G is a mean graph with n edges and maximum degree $\Delta(G)$, then $\Delta(G) \le \frac{n+3}{2}$ when n is odd and $\Delta(G) \le \frac{n+2}{2}$ when n is even. They proved that the disjoint union of n copies of C_3 is a mean graph and if a mean r-regular graph has n vertices, then $r < n - 2$. They established a connection between α-labelings and mean labelings by proving that every tree that admits an α-labeling is a mean graph when the size of its stable sets differ by at most one. When the tree is a caterpillar, this difference can be up to two. Barrientos and Krop call a mean labeling of a bipartite graph an *α-mean labeling* if the labels assigned to vertices of the same color have the same parity. They show that the complementary labeling of a α-mean labeling is also an α-mean labeling. They use graphs with α-mean labelings to construct new mean graphs. One construction consists of connecting a pair of corresponding vertices of two copies of an α-mean graph by an edge. The other construction identifies a pair of suitable vertices from two α-mean graphs. Barrientos and Krop also proved that every quadrilateral snake admits an α-mean labeling. They conjecture that all trees of size n and maximum degree at most $\lceil (n+1)/2 \rceil$ are mean graphs and state some open problems. In [236] Barrientos proves that all trees with up to four end-vertices are mean graphs.

In [235], Barrientos studies several operations with mean graphs. He proves that the coronas $G \odot K_1$ and $G \odot K_2$ are mean graphs when G is an α-mean graph. Also, if G and H are mean graphs with n vertices and $n-1$ edges and H is an α-mean graph, then $G \times H$ is a mean graph. He proves that given two mean graphs G and H, there exists a mean graph obtained by identifying an edge from G with an edge from H and uses this result to prove that the graphs R_n $(n \geq 2)$ of order $2n$ and size $4n-3$ with vertex set $V(R_n) = \{v_1, v_2, \ldots, v_{2n}\}$ and edge set $E(R_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1 \text{ and } n+1 \leq i \leq 2n-1\} \cup \{v_i v_{n+i} \mid 1 \leq i \leq n\} \cup \{v_i v_{n+i-1} \mid 2 \leq i \leq n\}$ (*rigid ladders*) are mean graphs.

In [1509] Vaidya and Bijukumar define two methods of creating new graphs from cycles as follows. For two copies of a cycle C_n the *mutual duplication* of a pair of vertices v_k and v'_k respectively from each copy of C_n is the new graph G such that $N(v_k) = N(v'_k)$. For two copies of a cycle C_n and an edge $e_k = v_k v_{k+1}$ from one copy of C_n with incident edges $e_{k-1} = v_{k-1} v_k$ and $e_{k+1} = v_{k+1} v_{k+2}$ and an edge $e'_m = u_m u_{m+1}$ in the second copy of C_n with incident edges $e'_{m-1} = u_{m-1} u_m$ and $e'_{m+1} = u_{m+1} u_{m+2}$, the *mutual duplication* of a pair of edges e_k and e'_m respectively from two copies of C_n is the new graph G such that $N(v_k) - v_{k+1} = N(u_m) - u_{m+1} = \{v_{k-1}, u_{m-1}\}$ and $N(v_{k+1}) - v_k = N(u_{m+1}) - u_m = \{v_{k+2}, u_{m+2}\}$. They proved that the graph obtained by mutual duplication of a pair of vertices each from each copy of a cycle and the mutual duplication of a pair of edges from each copy of a cycle are mean graphs. Moreover they proved that the shadow graphs of the stars $K_{1,n}$ and bistars $B_{n,n}$ are mean graphs.

Vasuki and Nagarajan [1572] proved the following graphs are admit mean labelings: the splitting graphs of paths and even cycles; $C_m \odot P_n$; $C_m \odot 2P_n$; $C_n \cup C_n$; disjoint unions of any nimber of copies of the hypercube Q_3; and the graphs obtained from by starting with m copies of C_n and identifying one vertex of one copy of C_n with the corresponding vertex in the next copy of C_n.) Jeyanthi and Ramya [733] define the jewel graph J_n as the graph with vertex set $\{u, x, v, y, u_i : 1 \leq i \leq n\}$ and edge set $\{ux, vx, uy, vy, xy, uu_i, vu_i : 1 \leq i \leq n\}$. They proved that the jewel graphs, jelly fish graphs (see 7.24 for the definition), and the graph obtained by joining any number of isolated vertices to the two end points of P_3 are mean graphs. Ramya and Jeyanthi [1188] proved several families of graphs constructed from T_p-tree are mean graphs.

Ramya, Ponraj, and Jeyanthi [1190] called a mean graph *super mean* if vertex labels and the edge labels are $\{1, 2, \ldots, p+q\}$. They prove following graphs are super mean: paths, combs, odd cycles, P_n^2, $L_n \odot K_1$, $C_m \cup P_n$ $(n \geq 2)$, the bistars $B_{n,n}$ and $B_{n+1,n}$. They also prove that unions of super mean graphs are super mean and K_n and $K_{1,n}$ are not super mean when $n > 3$. In [734] Jeyanthi, Ramya, and Thangavelu prove the following are super mean: $nK_{1,4}$; the graphs obtained by identifying an endpoint of P_m $(m \geq 2)$ with each vertex of C_n; the graphs obtained by identifying an endpoint of two copies of P_m $(m \geq 2)$ with each vertex of C_n; the graphs obtained by identifying an endpoint of three copies of P_m $(m \geq 2)$; and the graphs obtained by identifying an endpoint of four copies of P_m $(m \geq 2)$. In [730] Jeyanthi and Ramya prove the following graphs have super mean labelings: the graph obtained by identifying the endpoints of two or more copies of P_5; the graph obtained from C_n $(n \geq 4)$ by joining two vertices of C_n distance 2 apart with a path of length 2 or 3; Jeyanthi and Rama [732] use $S(G)$ to denote the graph obtained from a graph G by subdividing each edge of G by inserting a vertex. They prove the following graphs have super mean labelings: $S(P_n \odot K_1), S(B_{n,n}), C_n \odot \overline{K_2}$; the graphs obtained by joining the central vertices of two copies of $K_{1,m}$ by a path P_n (denoted by $\langle B_{m,m} : P_n \rangle$); generalized antiprisms (see §6.2 for the definition), and the graphs obtained from the paths $v_1, v_2, v_3, \ldots, v_n$ by joining each v_i and v_{i+1} to two new vertices u_i and w_i (double triangular snakes. Jeyanthi, Ramya, Thangavelu [735] give super mean labelings for $C_m \cup C_n$ and k-super mean labelings for a variety of graphs.

In [207] and [208] Balaji, Ramesh and Subramanian use the term "Skolem mean" labeling for super mean labeling. They prove: P_n is Skolem mean; $K_{1,m}$ is not Skolem mean if $m \geq 4$; $K_{1,m} \cup K_{1,n}$ is Skolem mean if and only if $|m-n| \leq 4$; $K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| = 4+l$ for $l = 1,2,3,\ldots, m = 1,2,3,\ldots,$ and $l \leq m < n$; $K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 4+l$ for $l = 1,2,3,\ldots, m = 1,2,3,\ldots, n \geq l+m+5$ and $l \leq m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| = 4+2l$ for $l = 2,\ldots, m = 2,3,4\ldots, n = 2l+m+4$ and $l \leq m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 4+l$ for $l = 1,2,3,\ldots, m = 1,2,3,\ldots, n \geq l+m+5$ and $l \leq m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 4+2l$ for $l = 2,\ldots, m = 2,3,4\ldots, n \geq 2l+m+5$ and $l \leq m < n$; $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| = 7$ for $m = 1,2,3,\ldots, n = m+7$ and $1 \leq m < n$; and $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is not Skolem mean if $|m-n| > 7$ for $m = 1,2,3,\ldots, n \geq m+8$ and $1 \leq m < n$. Balaji [206] proved that $K_{1,l} \cup K_{1,m} \cup K_{1,n}$ is Skolem mean if $|m-n| < 4+l$ for integers $1, m \geq 1$ and $l \leq m < n$.

In [736] Jeyanthi, Ramya, and Thangavelu proved the following graphs have super mean labelings: the one point union of any two cycles, graphs obtained by joining any two cycles by an edge (dumbbell graphs), $C_{2n+1} \odot C_{2m+1}$, graphs obtained by identifying a copy of an odd cycle C_m with each vertex of C_n, the quadrilateral snake Q_n, where n is odd, and the graphs obtained from an odd cycle $u_1, u_2, \ldots, u_n$ by joining the vertices u_i and u_{i+1} by the path P_m (m is odd) for $1 \leq i \leq n-1$ and joining vertices u_n and u_1 by the path P_m. Jeyanthi, Ramya, Thangavelu, and Aditanar [734] give super mean labelings of $C_m \cup C_n$ and T_p-trees.

Jeyanthi, Ramya, Maheswari [731] prove that T_p-trees (see §3.2 for the definition), graphs of the form $T \odot \overline{K_n}$ where T is a T_p-tree, and the graph obtained from P_m and m copies of $K_{1,n}$ by identifying a noncentral vertex of ith copy of $K_{1,n}$ with ith vertex of P_m are mean graphs.

In [729] Jeyanthi and Ramya define $S_{m,n}$ as the graph obtained by identifying one endpoint of each of n copies of P_m and $< S_{m,n} : P_m >$ as a graph obtained by identifying one endpoint of a path P_m with the vertex of degree n of a copy of $S_{m,n}$ and the other endpoint of the same path to the vertex of degree n of another copy of $S_{m,n}$. They prove the following graphs have super mean labelings: caterpillars, $< S_{m,n} : P_{m+1} >$, and the graphs obtained from P_{2m} and $2m$ copies of $K_{1,n}$ by identifying a leaf of ith copy of $K_{1,n}$ with ith vertex of P_{2m}. They further establish that if T is a T_p-tree, then $T \odot K_1$, $T \odot \overline{K_2}$, and, when T has an even number of vertices, $T \odot \overline{K_n}$ $(n \geq 3)$ are super mean graphs.

Let G be a graph and let $f : V(G) \to \{1,2,\ldots,n\}$ be a function such that the label of the edge uv is $(f(u)+f(v))/2$ or $(f(u)+f(v)+1)/2$ according as $f(u)+f(v)$ is even or odd and $f(V(G)) \cup \{f^*(e) : e \in E(G)\} \subseteq \{1,2,\ldots,n\}$. If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex and edge labels are distinct and there is no common vertex and edge labels, then n is called the *super mean number* of a graph G and it is denoted by $S_m(G)$. Nagarajan, Vasuki, and Arockiaraj [1088] proved that for any graph of order p, $S_m(G) \leq 2^p - 2$ and provided an upper bound of the super mean number of the graphs: $K_{1,n}$ $n \geq 7$; $tK_{1,n}$, $n \geq 5$, $t > 1$; the bistar $B(p,n)$, $p > n$; the graphs obtained by identifying a vertex of C_m and the center of $K_{1,n}$, $n \geq 5$; and the graphs obtained by identifying a vertex of C_m and the vertex of degree 1 of $K_{1,n}$. They also gave the super mean number for the graphs $C_n, tK_{1,4}$, and $B(p,n)$ for $p = n$ and $n+1$.

Manickam and Marudai [1021] defined a graph G with q edges to be an *odd mean graph* if there is an injective function f from the vertices of G to $\{1, 3, 5, \ldots, 2q-1\}$ such that when each edge uv is labeled with $(f(u)+f(v))/2$ if $f(u)+f(v)$ is even, and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd, then the resulting edge labels are distinct. Such a function is called a *odd mean* labeling. For integers a and b at least 2, Vasuki and Nagarajan [1573] use P_a^b to denote the graph obtained by starting with verticies $y_1, y_2, \ldots, y_a$ and connecting y_i to y_{i+1} with b internally disjoint paths of length $i+1$ for $i = 1, 2, \ldots, a-1$ and $j = 1, 2, \ldots, b$. For integers $a \geq 1$ and $b \geq 2$ they use $P^b_{\langle 2a \rangle}$ to denote the graph obtained by starting with verticies $y_1, y_2, \ldots, y_{a+1}$ and connecting y_i to y_{i+1} with b internally disjoint paths of length $2i$ for $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$. They proved that the graphs $P_{2r,m}, P_{2r+1,2m+1}$, and $P^m_{\langle 2r \rangle}$ are odd mean graphs for all values of r and m.

Gayathri and Amuthavalli [547] (see also [85]) say a (p, q)-graph G has a (k, d)-*odd mean* labeling if there exists an injection f from the vertices of G to $\{0, 1, 2, \ldots, 2k-1+2(q-1)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = \lceil (f(u)+f(v))/2 \rceil$ is a bijection from edges of G to $\{2k-1, 2k-1+2d, 2k-1+4d, \ldots, 2k-1+2(q-1)d\}$. When $d = 1$ a (k, d)-odd mean labeling is called *k-odd mean*. For $n \geq 2$ they prove the following graphs are k-odd mean for all k: P_n; combs $P_n \odot K_1$; crowns $C_n \odot K_1$ $(n \geq 4)$; bistars $B_{n,n}$; $P_m \odot \overline{K_n}$ $(m \geq 2)$; $C_m \odot \overline{K_n}; K_{2,n}$; C_n except for $n = 3$ or 6; the one-point union of C_n $(n \geq 4)$ and an end point of any path; grids $P_m \times P_n$ $(m \geq 2)$; $(P_n \times P_2) \odot K_1$; arbitrary unions of paths; arbitrary unions of stars; arbitrary unions of cycles; the graphs obtained by joining two copies of C_n $(n \geq 4)$ by any path; and the graph obtained from $P_m \times P_n$ by replacing each edge by a path of length 2. They prove the following graphs are not k-odd mean for any k: $K_n; K_n$ with an edge deleted; $K_{3,n}$ $(n \geq 3)$; wheels; fans; friendship graphs; triangular snakes; Möbius ladders; books $K_{1,m} \times P_2$ $(m \geq 4)$; and webs. For $n \geq 3$ they prove $K_{1,n}$ is k-odd mean if and only if $k \geq n-1$. Gayathri and Amuthavalli [548] prove that the graph obtained by joining the centers of stars $K_{1,m}$ and $K_{1,n}$ are k-odd mean for $m = n, n+1, n+2$ and not k-odd mean for $m > n+2$. For $n \geq 2$ the following graphs have a (k, d)-mean labeling [558]: $C_m \cup P_n$ $(m \geq 4)$ for all k; arbitrary unions of cycles for all $k; P_{2m}$; P_{2m+1} for $k \geq d$ (P_{2m+1} is not (k, d)-mean when $k < d$); combs $P_n \odot K_1$ for all k; $K_{1,n}$ for $k \geq d$; $K_{2,n}$ for $k \geq d$; bistars for all k; nC_4 for all k; and quadrilateral snakes for $k \geq d$.

In [1283] Seoud and Salim [1284] proved that a graph has a k-odd mean labeling if and only if it has a mean labeling. In [1283] Seoud and Salim give upper bounds of the number of edges of graphs with a (k, d)-odd mean labeling

Gayathri and Gopi [551] defined a graph G with q edges to be an *k-even mean graph* if there is an injective function f from the vertices of G to $\{0, 1, 2, \ldots, 2k+2(q-1)\}$ such that when each edge uv is labeled with $(f(u)+f(v))/2$ if $f(u)+f(v)$ is even, and $(f(u)+f(v)+1)/2$ if $f(u)+f(v)$ is odd, then the resulting edge labels are $\{2k, 2k+2, 2k+4, \ldots, 2k+2(q-1)\}$. Such a function is called a *k-even mean* labeling. In [551] they proved that the graphs obtained by joining two copies of C_n with a path P_m are k-even mean for all k and all $m, n \geq 3$ when $n \equiv 0, 1$ mod 4 and for all $k \geq 1$, $m \geq 7$, and $n \geq 3$. In [552] Gayathri and Gopi proved that various graphs obtained by joining two copies of stars $K_{1,m}$ and $K_{1,n}$ with a path by identifying the one endpoint of the path with the center of one star and the other end point of the path with the center of the other star are k-even mean. In [553] they proved that various graphs obtained by appending a path to a vertex of a cycle are k-even mean. In [554] they proved that $C_n \cup P_m$, $n \geq 4$, $m \geq 2$, is k-even mean for for all k.

Gayathri and Gopi [555] say graph G with q edges has a (k, d)-*even mean* labeling if there exists an injection f from the vertices of G to $\{0, 1, 2, \ldots, 2k+2(q-1)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = (f(u) + f(v))/2$ if $f(u) + f(v)$ is even and $f^*(uv) = (f(u) + f(v) + 1)2$ if $f(u) + f(v)$ is odd is a bijection from edges of G to $\{2k, 2k + 2d, 2k + 4d, \ldots, 2k + 2(q-1)d\}$. A graph that has a (k, d)-even mean labeling is called a (k, d)-*even mean graph.* They proved that $P_m \oplus nK_1 (m \geq 3, n \geq 2)$ has a (k, d)-even mean labeling in the following cases: all (k, d) when m is even; all (k, d) when m is odd and n is odd; and m is odd, n is even and $k \geq d$.

Kalaimathy [759] investigated conditions under which a mean labeling for a graph G will yield a (k, d)-even mean labeling for G and vice versa. He also gave conditions under which two graphs that have $(1, 1)$-mean labelings can be joined by an single edge to obtain a new graph that has a $(1, 1)$-even mean labeling.

Murugan and Subramanian [1082] say a (p, q)-graph G has a *Skolem difference mean* labeling if there exists an injection f from the vertices of G to $\{1, 2, \ldots, p + q\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = (|f(u) - f(v)|)/2$ if $|f(u) - f(v)|$ is even and $f^*(uv) = (|f(u) - f(v)| + 1)/2$ if $|f(u) + f(v)|$ is odd is a bijection from edges of G to $\{1, 2, \ldots, q\}$. A graph that has a Skolem difference mean labeling is called a *Skolem difference mean* graph. They show that the graphs obtained by starting with two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ and joining the vertices $v_{(n+1)/2}$ and $u_{(n+1)/2}$ if n is odd and the vertices $v_{n/2+1}$ and $u_{n/2}$ if n is even are Skolem difference mean.

Gayathri and Tamilselvi [558] say a (p, q)-graph G has a (k, d)-*super mean* labeling if there exists an injection f from the vertices of G to $\{k, k + d, \ldots, k + (p + q)d\}$ such that the induced map f^* defined on the edges of G by $f^*(uv) = \lceil (f(u) + f(v))/2 \rceil$ has the property that the vertex labels and the edge labels together are the integers from k to $k + (p + q)d$. When $d = 1$ a (k, d)-super mean labeling is called *k-super mean.* For $n \geq 2$ they prove the following graphs are k-super mean for all k: odd cycles; P_n; $C_m \cup P_n$; the one-point union of a cycle and the end point of P_n; the union of any two cycles excluding C_4; and triangular snakes. For $n \geq 2$ they prove the following graphs are (k, d)-super mean for all k and d: P_n; odd cycles; combs $P_n \odot K_1$; and bistars. In [736] Jeyanthi, Ramya, and Thangavelu proved the following graphs have k-super mean labelings: C_{2n}, $C_{2n+1} \times P_m$, grids $P_m \times P_n$ with one arbitrary crossing edge in every square, and antiprisms on $2n$ vertices $(n > 4)$. (Recall an antiprism on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ where subscripts are taken modulo n). Jeyanthi, Ramya, Thangavelu [735] give k-super mean labelings for a variety of graphs. Jeyanthi, Ramya, Thangavelu, and Aditanar [734] show how to construct k-super mean graphs from existing ones.

Gayathri and Tamilselvi [558] say a (p, q)-graph G has a *k-super edge mean* labeling if there exists an injection f from the edges of G to $\{k, k + 1, \ldots, k + 2(p + q)\}$ such that the induced map f^* from the vertices of G to $\{k, k + 1, \ldots, k + 2(p + q)\}$ defined by $f^*(v) = \lceil (\Sigma f(vu))/2 \rceil$ taken all edges vu incident to v is an injection. For $n \geq 3$ they prove the following graphs are k-super edge mean for all k: paths; cycles; combs $P_n \odot K_1$; triangular snakes; crowns $C_n \odot K_1$; the one-point union of C_3 and an end point of P_n; and $P_n \odot K_2$.

In [1237] Sandhya, Somasundaram, and Ponraj call a graph with q edges a *harmonic mean* graph if there is an injective function f from the vertices of the graph to the integers from 1 to $q+1$ such that when each edge uv is labeled with $\lceil 2f(u)f(v)/(f(u)+f(v)) \rceil$ or $\lfloor 2f(u)f(v)/(f(u)+f(v)) \rfloor$ the edge labels are distinct. They prove the following graphs have such a labeling: paths, ladders, triangular snakes, quadrilateral snakes, $C_m \cup P_n$ $(n > 1)$; $C_m \cup C_n$; nK_3; $mK_3 \cup P_n$ $(n > 1)$; mC_4; $mC_4 \cup P_n$; $mK_3 \cup nC_4$; and $C_n \odot K_1$ (crowns). They also prove that wheels, prisms, and K_n $(n > 4)$ with an edge deleted are not harmonic mean graphs. In [1235] Sandhya, Somasundaram, and Ponraj investigated the harmonic mean labeling for a polygonal chain, square of the path and dragon and enumerate all harmonic mean graph of order at most 5.

Sandhya, Somasundaram, Ponraj [1236] proved that the following graphs have harmonic mean labelings: graphs obtained by duplicating an arbitrary vertex or an arbitrary edge of a cycle; graphs obtained by joining two copies of a fixed cycle by an edge; the one-point union of two copies of a fixed cycle; and the graphs obtained by starting with a path and replacing every other edge by a triangle or replacing every other edge by a quadrilateral.

Vaidya and Barasara [1507] proved that the following graphs have harmonic mean labelings: graphs obtained by the duplication of an arbitrary vertex or arbitrary edge of a path or a cycle; the graphs obtained by the duplication of an arbitrary vertex of a path or cycle by a new edge; and the graphs obtained by the duplication of an arbitrary edge of a path or cycle by a new vertex.

In [1465] Sundaram, Ponraj, and Somasundaram introduced a new labeling parameter called the *mean number* of a graph. Let f be a function from the vertices of a graph to the set $\{0, 1, 2, \ldots, n\}$ such that the label of any edge uv is $(f(u) + f(v))/2$ if $f(u) + f(v)$ is even and $(f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd. The smallest integer n for which the edge labels are distinct is called the *mean number* of a graph G and is denoted by $m(G)$. They proved that for a graph G with p vertices $m(tK_{1,n}) \leq t(n+1) + n - 4$; $m(G) \leq 2^{p-1} - 1$; $m(K_{1,n}) = 2n - 3$ if $n > 3$; $m(B(p, n)) = 2p - 1$ if $p > n + 2$ where $B(p, n)$ is a bistar; $m(kT) = kp - 1$ for a mean tree T, $m(W_n) \leq 3n - 1$, and $m(C_3^{(t)}) \leq 4t - 1$.

Let f be a function from $V(G)$ to $\{0, 1, 2\}$. For each edge uv of G, assign the label $\lceil \frac{f(u)+f(v)}{2} \rceil$. Ponraj, Sivakumar, and Sundaram [1162] say that f is a *mean cordial labeling* of G if $|v_{f(i)} - v_{f(j)}| \leq 1$ for i and j in $\{0, 1, 2\}$ where $v_{f(x)}$ and $e_{f(x)}$ denote the number of vertices and edges labeled with x, respectively. A graph with a mean cordial labeling is called a *mean cordial graph*. Observe that if the range set of f is restricted to $\{0, 1\}$, a mean cordial labeling coincides with that of a product cordial labeling. Ponraj, Sivakumar, and Sundaram [1162] prove the following: every graph is a subgraph of a connected mean cordial graph; $K_{1,n}$ is mean cordial if and only $n \leq 2$; C_n is mean cordial if and only $n \equiv 1, 2 \pmod 3$; K_n is mean cordial if and only $n \leq 2$; W_n is not mean cordial for all $n \geq 3$; the subdivision graph of $K_{1,n}$ is mean cordial; the comb $P_n \odot K_1$ is mean cordial; $P_n \odot 2K_1$ is mean cordial; and $K_{2,n}$ is a mean cordial if and only $n \leq 2$.

7.15 Pair Sum and Pair Mean Graphs

For a (p, q) graph G Ponraj and Parthipan [1144] define an injective map f from $V(G)$ to $\{\pm 1, \pm 2, \ldots, \pm p\}$ to be a *pair sum* labeling if the induced edge function f_{em} from $E(G)$ to the nonzero integers defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$, according as q is even or odd. A graph with a pair sum labeling is called *pair sum graph*. In [1144] and [1145] they proved the following are pair sum graphs: P_n, C_n, K_n iff $n \leq 4$, $K_{1,n}$, $K_{2,n}$, bistars $B_{m,n}$, combs $P_n \odot K_1$, $P_n \odot 2K_1$, and all trees of order up to 9. Also they proved that $K_{m,n}$ is not pair sum graph if $m, n \geq 8$ and enumerated all pair sum graphs of order at most 5.

In [1147], [1148], [1149], and [1150] Ponraj, Parthipan, and Kala proved the following are pair sum graphs: $K_{1,n} \cup K_{1,m}$, $C_n \cup C_n$, mK_n if $n \leq 4$, $(P_n \times K_1) \odot K_1$, $C_n \odot K_2$, dragons $D_{m,n}$ for n even, $\overline{K_n} + 2K_2$ for n even, $P_n \times P_n$ for n even, $C_n \times P_2$ for n even, $(P_n \times P_2) \odot K_1$, $C_n \odot K_2$ and the subdivision graphs of $P_n \times P_2, C_n \odot K_1, P_n \odot K_1$, triangular snakes, and quadrilateral snakes.

Jeyanthi and Devi [719] define an injective map f from $E(G)$ to $\{\pm1, \pm2, \dots, \pm q\}$ as an *edge pair sum labeling* of a graph $G(p, q)$ if the induced function of f^* from $V(G)$ to $Z - \{0\}$ defined by $f^*(v) = \sum f(e)$ taken over all edges e incident to v is one-one and $f^*(V(G))$ is either of the form $\{\pm k_1, \pm k_2, \dots, \pm k_{p/2}\}$ or $\{\pm k_1, \pm k_2, \dots, \pm k_{(p-1)/2}\} \cup \{k_{p/2}\}$ according as p is even or odd. A graph with an edge pair sum labeling is called an *edge pair sum graph.* They proved that P_n, C_n, triangular snakes, $P_m \cup K_{1,n}$, and $C_n \odot \overline{K_m}$ are edge pair sum graphs.

For a (p, q) graph G Ponraj and Parthipan [1146] define an injective map f from $V(G)$ to $\{\pm1, \pm2, \dots, \pm p\}$ to be a *pair mean* labeling if the induced edge function f_{em} from $E(G)$ to the nonzero integers defined by $f_{em}(uv) = (f(u) + f(v))/2$ if $f(u) + f(v)$ is even and $f_{em}(uv) = (f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd is one-one and $f_{em}(E(G)) = \{\pm k_1, \pm k_2, \dots, \pm k_{q/2}\}$ or $f_{em}(E(G)) = \{\pm k_1, \pm k_2, \dots, \pm k_{(q-1)/2}\} \cup \{k_{(q+1)/2}\}$, according as q is even or odd. A graph with a pair mean labeling is called a *pair mean* graph. They proved the following graphs have pair mean labelings: P_n, C_n if and only if $n \leq 3$, K_n if and only if $n \leq 2$, $K_{2,n}$, bistars $B_{m,n}$, $P_n \odot K_1$, $P_n \odot 2K_1$, and the subdivision graph of $K_{1,n}$. Also they found the relation between pair sum labelings and pair mean labelings.

The graph $G@P_n$ is obtained by identifying an end vertex of a path P_n with any vertex of G. A graph $G = (V, E)$ with q edges is called a $(k + 1)$-*equitable mean* graph if there is a function f from V to $\{0, 1, 2, \dots, k\}$ $(1 \leq k \leq q)$ such that the induced edge that labeling f^* from E to $\{0, 1, 2, \dots, k\}$ given by $f^*(uv) - \lceil (f(u) + f(v))/2 \rceil$ has the properties $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ for $i, j = 0, 1, 2, \dots, k$ where $v_f(x)$ and $e_{f^*}(x)$ are the number of vertices and edges of G respectively with the label x. In [715] Jeyanthi proved the following: a connected graph with q edges is a $(q+1)$-equitable mean graph if and only if it is a mean graph; a graph is 2-equitable mean graph if and only if it is a product cordial graph; for every graph G, the graph $3mG$ is a 3-equitable mean graph; for every 3-equitable mean graph G, the graph $(3m + 1)G$ is a 3-equitable mean graph; C_n is a 3-equitable mean graph if and only if $n \not\equiv 0 \pmod 3$; P_n is a 3-equitable mean graph for all $n \geq 2$; if G is a 3-equitable mean graph then $G@P_n$ is a 3-equitable mean graph for $n \equiv 1 \pmod 3$; the bistar $B(m, n)$ with $m \geq n$ is a 3-equitable mean graph if and only if $n \geq \lfloor q/3 \rfloor$; $K_{1,n}$ is a 3-equitable mean graph if and only if $n \leq 2$; and for any graph H and $3m$ copies $H_1, H_2, \dots, H_{3m}$ of H, the graph obtained by identifying a vertex of H_i with a vertex of H_{i+1} for $1 \leq i \leq 3m - 1$ is a 3-equitable mean graph.

7.16 Irregular Total Labelings

In 1988 Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba [361] defined an *irregular labeling* of a graph G with no isolated vertices as an assignment of positive integer weights to the edges of G in such a way that the sums of the weights of the edges at each vertex are distinct. The minimum of the largest weight of an edge over all irregular labelings is called the *irregularity strength* $s(G)$ of G. If no such weight exists, $s(G) = \infty$. Chartrand et al. gave a lower bound for $s(mK_n)$. Faudree, Jacobson, and Lehel [488] gave an upper bound for $s(mK_n)$ when $n \geq 5$ and proved that for graphs G with $\delta(G) \geq n-2 \geq 1$, $s(G) \leq 3$. They also proved that if G has order n and $\delta(G) = n - t$ and $1 \leq t \leq \sqrt{n/18}$, $s(G) \leq 3$. Aigner and Triesch proved $s(G) \leq n + 1$ for any graph G with $n \geq 4$ vertices for which $s(G)$ is finite. In [1172] Przybylo proved that $s(G) < 112n/\delta + 28$, where δ is the minimum degree of G. In [486] Faudree and Lehel conjectured that for each $d \geq 2$, there exists an absolute constant c such that $s(G) \leq n/d + c$ for each d-regular graph of order n. In Przybylo [1171] showed that for d-regular graphs $s(G) < 16n/d + 6$.

In 1991 Cammack, Schelp and Schrag [345] proved that the irregularity strength of a full d-ary tree ($d = 2, 3$) is its number of pendent vertices and conjctures that the irregularity strength of a tree with no vertices of degree two is its number of pendent vertices. This conjecture was proved by Amar and Togni [82] in 1998. In [747] Jinnah and Kumar determined the irregularity strength of triangular snakes and double triangular snakes.

Motivated by the notion of the irregularity strength of a graph and various kinds of other total labelings, Bača, Jendroľ, Miller, and Ryan [175] introduced the *total edge irregularity strength* of a graph as follows. For a graph $G(V, E)$ a labeling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ is called an *edge irregular total k-labeling* if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. Similarly, ∂ is called an *vertex irregular total k-labeling* if for every pair of distinct vertices u and v, $\partial(u) + \sum \partial(e)$ over all edges e incident to $u \neq \partial(v) + \sum \partial(e)$ over all edges e incident to v. The minimum k for which G has an edge (vertex) irregular total k-labeling is called the *total edge (vertex) irregularity strength* of G. The total edge (vertex) irregular strength of G is denoted by tes(G) (tvs(G)). They prove: for $G(V, E)$, E not empty, $\lceil (|E| + 2)/3 \rceil \leq$ tes(G)$\leq |E|$; tes(G)$\geq \lceil (\Delta(G) + 1)/2 \rceil$ and tes(G)$\leq |E| - \Delta(G)$, if $\Delta(G) \leq (|E|-1)/2$; tes(P_n) = tes(C_n)= $\lceil (n+2)/3 \rceil$; tes(W_n)= $\lceil (2n+2)/3 \rceil$; tes(C_3^m) (friendship graph) = $\lceil (3n + 2)/3 \rceil$; tvs(C_n) = $\lceil (n + 2)/3 \rceil$; for $n \geq 2$, tvs(K_n)= 2; tvs($K_{1,n}$) = $\lceil (n + 1)/2 \rceil$; and tvs($C_n \times P_2$)= $\lceil (2n+3)/4 \rceil$. Jendroľ, Miškul, and Soták [708] (see also [709]) proved: tes(K_5) = 5; for $n \geq 6$, tes(K_n)= $\lceil (n^2 - n + 4)/6 \rceil$; and that tes($K_{m,n}$)= $\lceil (mn + 2)/3 \rceil$. They conjecture that for any graph G other than K_5, tes(G) = $\max\{\lceil (\Delta(G) + 1)/2 \rceil, \lceil (|E| + 2)/3 \rceil\}$. Ivančo and Jendroľ [693] proved that this conjecture is true for all trees. Jendroľ, Miškuf, and Soták [708] prove the conjecture for complete graphs and complete bipartite graphs. The conjecture has been proven for the categorical product of two paths [46], the categorical product of a cycle and a path [1365], and the subdivision of a star [1366]. In [52] Ahmad, Siddiqui, and Afzal proved the conjecture is true for graphs obtained by starting with m vertex disjoint copies of P_n ($m, n \geq 2$) arranged in m horizontal rows with the jth vertex of row $i+1$ directly below the jth vertex row i for $1 = 1, 2, \ldots, m - 1$ and joining the jth vertex of row i to the $j + 1$th vertex of row $i + 1$ for $1 = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots, n - 1$ (the *zigzag* graph). Siddiqui, Ahmad, Nadeem, and Bashir [1368] proved the conjecture for the disjoint union of p isomorphic sun graphs (i. e., $C_n \odot K_1$) and the disjoint union of p sun graphs in which the orders of the n-cycles are consecutive integers. They pose as an open problem the determination of the total edge irregularity strength of disjoint union of any number of sun graphs. Brandt, Misškuf, and Rautenbach [305] proved the conjecture for large graphs whose maximum degree is not too large relative to its order and size. In particular, using the probabilistic method they prove that if $G(V, E)$ is a multigraph without loops and with nonzero maximum degree less than $|E|/10^3\sqrt{8|V|}$, then tes(G) = $(\lceil |E| + 2)/3 \rceil$. As corollaries they have: if $G(V, E)$ satisfies $|E| \geq 3 \cdot 10^3 |V|^{3/2}$, then tes($G$) = $\lceil (|E| + 2)/3 \rceil$; if $G(V, E)$ has minimum degree $\delta > 0$ and maximum degree Δ such that $\Delta < \delta\sqrt{|V|}/10^3 \cdot 4\sqrt{2}$ then tes(G) = $\lceil (|E| + 2)/3 \rceil$; and for every positive integer Δ there is some $n(\Delta)$ such that every graph $G(V, E)$ without isolated vertices with $|V| \geq n(\Delta)$ and maximum degree at most Δ satisfies tes(G) = $\lceil (|E| + 2)/3 \rceil$. Notice that this last result includes d-regular graphs of large order. They also prove that if $G(V, E)$ has maximum degree $\Delta \geq 2|E|/3$, then G has an edge irregular total k-labeling with $k = \lceil (\Delta + 1)/2 \rceil$. Pfender [1131] proved the conjecture for graphs with at least 7×10^{10} edges and proved for graphs $G(V, E)$ with $\Delta(G) \leq E(G)/4350$ we have tes(G) = $(\lceil |E| + 2)/3 \rceil$.

A generalized helm H_n^m is a graph obtained by inserting m vertices in every pendant edge of a helm H_n. Indriati, Widodo, and Sugeng [688] proved that for $n \geq 3$,

$tes(H_n^1) = \lceil(4n+2)/3\rceil$, $tes(H_n^2) = \lceil(5n+2)/3\rceil$, and $tes(H_n^m) = \lceil((m+3)n+2))/3\rceil$ for $m \equiv 0$ mod 3. They conjecture that $tes(H_n^m) = \lceil((m+3)n+2))/3\rceil$, for all $n \geq 3$ and $m \geq 10$.

The *strong product* of graphs G_1 and G_2 has as vertices the pairs (x, y) where $x \in V(G_1)$ and $y \in V(G_2)$. The vertices (x_1, y_1) and (x_2, y_2) are adjacent if either x_1x_2 is an edge of G_1 and $y_1 = y_2$ or if $x_1 = x_2$ and y_1y_2 is an edge of G_2. For $m, n \geq 2$ Ahmad, Bača, Bashir, Siddiqui [47] proved that the total edge irregular strength of the strong product of P_m and P_n is $\lceil 4(mn+1)/3\rceil - (m+n)$.

Nurdin, Baskoro, Salman, and Gaos [1110] determine the total vertex irregularity strength of trees with no vertices of degree 2 or 3; improve some of the bounds given in [175]; and show that $\mathrm{tvs}(P_n) = \lceil(n+1)/3)\rceil$. In [1113] Nurdin, Salman, Gaos, and Baskoro prove that for $t \geq 2$, $\mathrm{tvs}(tP_1)= t$; $\mathrm{tvs}(tP_2)= t+1$; $\mathrm{tvs}(tP_3)= t+1$; and for $n \geq 4$, $\mathrm{tvs}(tP_n)= \lceil(nt+1)/3)\rceil$. Anholcer, Kalkowski, and Przybylo [96] prove that for every graph with $\delta(G) > 0$, $\mathrm{tvs}(G) \leq \lceil 3n/\delta\rceil + 1$. Anholcer, Karonński, and Pfender [95] prove that for every forest F with no vertices of degree 2 and no isolated vertices $\mathrm{tvs}(F)= \lceil(n_1+1)/2\rceil$, where n_1 is the number of vertices in F ofdegree 1. They also prove that for every forest with no isolated vertices and at most one vertex of degree 2, $\mathrm{tvs}(F) = \lceil(n_1+1)/2\rceil$. Anholcer and Palmer [97] determined the total vertex irregularity strength C_n^k, which is a generalization of the circulant graphs $C_n(1, 2, \ldots, k)$. They prove that for $k \geq 2$ and $n \geq 2k + 1$, $\mathrm{tvs}(C_n^k = \lceil(n + 2k)/(2k + 1)\rceil$. Przybylo [1172] obtained a variety of upper bounds for the total irregularity strength of graphs as a function of the order and minimum degree of the graph.

In [1492] Tong, Lin, Yang, and Wang give the exact values of the total edge irregularity strength and total vertex irregularity strength of the toroidal grid $C_m \times C_n$. Chunling, Xiaohui, Yuansheng, and Liping, [403] showed $\mathrm{tvs}(K_p) = 2$ $(p \geq 2)$ and for the generalized Petersen graph $P(n, k)$ they proved $\mathrm{tvs}(P(n, k)) = \lceil n/2\rceil + 1$ if $k \leq n/2$ and $\mathrm{tvs}(P(n, n/2))= n/2 + 1$. They also obtained the exact values for the total vertex strengths for ladders, Möbius ladders, and Knödel graphs. For graphs with no isolated vertices, Przybylo [1171] gave bounds for $\mathrm{tvs}(G)$ in terms of the order and minimum and maximum degrees of G. For d-regular $(d > 0)$ graphs, Przybylo [1172] gave bounds for $\mathrm{tvs}(G)$ in terms d and the order of G. Ahmad, Ahtsham, Imran, and Gaig [42] determined the exact values of the total vertex irregularity strength for five families of cubic plane graphs. In [44] Ahmad and Bača determine that the total edge-irregular strength of the categorical product of C_n and P_m where $m \geq 2$, $n \geq 4$ and n and m are even is $\lceil(2n(m-1)+2)/3\rceil$. They leave the case where at least one of n and m is odd as an open problem. Al-Mushayt, Ahmad, and Siddiqui [75] determined the exact values of the total edge-irregular strength of hexagonal grid graphs. Rajasingh, Rajan, and Annamma [1183] obtain bounds for the total vertex irregularity strength of three families of triangle related graphs.

In [1112] Nurdin, Salman, and Baskoro determine the total edge-irregular strength of the following graphs: for any integers $m \geq 2$, $n \geq 2$, $\mathrm{tes}(P_m \odot P_n)= \lceil(2mn+1)/3\rceil$; for any integers $m \geq 2$, $n \geq 3$, $\mathrm{tes}(P_m \odot C_n)= \lceil((2n+1)m+1)/3\rceil$; for any integers $m \geq 2$, $n \geq 2$, $\mathrm{tes}(P_m \odot K_{1,n})= \lceil(2m(n+1)+1)/3\rceil$; for any integers $m \geq 2$ and $n \geq 3$, $\mathrm{tes}(P_m \odot G_n)= \lceil(m(5n+2)+1)/3\rceil$ where G_n is the gear graph obtained from the wheel W_n by subdividing every edge on the n-cycle of the wheel; for any integers $m \geq 2$, $n \geq 2$, $\mathrm{tes}(P_m \odot F_n)= \lceil m(5n+2)+1\rceil$, where F_n is the friendship graph obtained from W_{2m} by subdividing every other rim edge; for any integers $m \geq 2$ and $n \geq 3$: and $\mathrm{tes}(P_m \odot W_n)= \lceil((3n+2)m+1)/3\rceil$.

In [1111] Nurdin, Baskoro, Salman and Gaos proved: the total vertex-irregular strength of the complete k-ary tree $(k \geq 2)$ with depth $d \geq 1$ is $\lceil(k^d+1)/2\rceil$ and the total vertex-irregular strength of the subdivision of $K_{1,n}$ for $n \geq 3$ is $\lceil (n+1)/3 \rceil$. They also determined that if G is isomorphic to the caterpillar obtained by starting with P_m and m copies of P_n denoted by $P_{n,1}, P_{n,2}, \ldots, P_{n,m}$, where $m \geq 2,\ n \geq 2$, then joining the i-th vertex of P_m to an end vertex of the path $P_{n,i}$, tvs(G)= $\lceil (mn+3)/3 \rceil$.

Ahmad and Bača [45] proved tvs($J_{n,2}$)= $\lceil (n+1)/2) \rceil$ ($n \geq 4$) and conjectured that for $n \geq 3$ and $m \geq 3$, tvs($J_{n,m}$)= $\max\{\lceil (n(m-1)+2)/3 \rceil, \lceil (nm+2)/4 \rceil\}$. They also proved that for the circulant graph (see § 5.1 for the definition) $C_n(1,2),\ n \geq 5$, tvs($C_n(1,2)$)= $\lceil (n+4)/5 \rceil$. They conjecture that for the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ with degree r at least 5 and $n \geq 5,\ 1 \leq a_i \leq \lfloor n/2 \rfloor$, tvs($C_n(a_1, a_2, \ldots, a_m)$= $\lceil (n+r)/(1+r) \rceil$.

Slamin, Dafik, and Winnona [1394] consider the total vertex irregularity strengths of the disjoint union of isomorphic sun graphs, the disjoint union of consecutive nonisomorphic sun graphs, tvs($\cup_{i=1}^{t} S_{i+2}$), and disjoint union of any two nonisomorphic sun graphs. (Recall $S_n = C_n \odot K_1$.)

In [41] Ahmad shows that the total vertex irregularity strength of the antiprism graph A_n ($n \geq 3$) is $\lceil (2n+4)/5 \rceil$ (see 5.6 for the definition and gives the vertex irregularity strength of three other families convex polytope graphs. Al-Mushayt, Arshad, and Siddiqui [76] determined an exact value of the total vertex irregularity strength of some convex polytope graphs. Ahmad, Baskoro, and Imran [49] determined the exact value of the total vertex irregularity strength of disjoint union of Helm graphs.

Marzuki, Salman, and Miller [1036] introduced a new irregular total k-labeling of a graph G called *total irregular total k-labeling*, denoted by $ts(G)$, which is required to be at the same time both vertex and edge irregular. They gave an upper bound and a lower bound of $ts(G)$; determined the total irregularity strength of cycles and paths; and proved $ts(G) \geq \max\{tes(G), tvs(G)\}$. For $n \geq 3$, Ramdani and Salman [1186] proved $ts(S_n \times P_2) = n+1$; $ts((P_n + P_1) \times P_2) = \lceil (5n+1)/3 \rceil$, $ts(P_n \times P_2) = n$; and $ts(C_n \times P_2) = n$.

An edge $e \in \overline{G}$ is called a *total positive edge* or *total negative edge* or *total stable edge* of G if tvs($G+e$) > tvs(G) or tvs($G+e$) < tvs(G) or tvs($G+e$) = tvs(G), respectively. If all edges of $\overline{G}$ are total stable (total negative) edges of G, then G is called a *total stable* (*total negative*) graph. Otherwise G is called a *total mixed* graph. Packiam and K. Kathiresan [1114] showed that $K_{1,n}$ $n \geq 4$, and the disjoint union of t copies of K_3, $t \geq 2$, are total negative graphs and that the disjoint union of t copies of P_3, $t \geq 2$, is a total mixed graph.

For a simple graph G with no isolated edges and at most one isolated vertex Anholcer [93] calls a labeling $w : E(G) \rightarrow \{1, 2, \ldots, m\}$ *product-irregular*, if all product degrees $pd_G(v) = \prod_{e \ni v} w(e)$ are distinct. Analogous to the notion of irregularity strength the goal is to find a product-irregular labeling that minimizes the maximum label. This minimum value is called the *product irregularity strength* of G and is denoted by $ps(G)$. He provides bounds for the product irregularity strength of paths, cycles, cartesian products of paths, and cartesian products of cycles. In [94] Anholcer gives the exact values of $ps(G)$ for $K_{m,n}$ where $2 \leq m \leq n \leq (m+2)(m+1)/2$, some families of forests including complete d-ary trees, and other graphs with $d(G) = 1$. Skowronek-Kaziów [1391] proves that for the complete graphs $ps(K_n) = 3$.

7.17 Minimal k-rankings

A *k-ranking* of a graph is a labeling of the vertices with the integers 1 to k inclusively such that any path between vertices of the same label contains a vertex of greater label. The *rank number of a graph G*, $\chi_r(G)$, is the smallest possible number of labels in a ranking. A k-ranking is *minimal* if no label can be replaced by a smaller label and still be a k-ranking. The concept of the rank number arose in the study of the design of very large scale integration (VLSI) layouts and parallel processing (see [415], [945] and [1254]). Ghoshal, Laskar, and Pillone [570] were the first to investigate minimal k-rankings from a mathematical perspective. Laskar and Pillone [853] proved that the decision problem corresponding to minimal k-rankings is NP-complete. It is HP-hard even for bipartite graphs [424]. Bodlaender, Deogun, Jansen, Kloks, Kratsch, Müller, Tuza [291] proved that the rank number of P_n is $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1$ and satisfies the recursion $\chi_r(P_n) = 1 + \chi_r(P_{\lceil (n-1)/2 \rceil})$ for $n > 1$. The following results are given in [424]: $\chi_r(S_n) = 2$; $\chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2$; $\chi_r(W_n) = \lfloor \log_2(n-3) \rfloor + 3 (n > 3)$; $\chi_r(K_n) = n$; the complete t-partite graph with n vertices has ranking number $n+1$ - the cardinality of the largest partite set; and a split graph with n vertices has ranking number $n+1$ - the cardinality of the largest independent set (a *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set.) Wang proved that for any graphs G and H $\chi_r(G+H) = \min\{|V(G)| + \chi_r(H), |V(H) + \chi_r(G)\}$.

In 2009 Novotny, Ortiz, and Narayan [1108] determined the rank number of P_n^2 from the recursion $\chi_r(P_n^2) = 2 + \chi_r(P(_{\lceil (n-2)/2 \rceil})$ for $n > 2$. They posed the problem of determining $\chi_r(P_m \times P_n)$ and $\chi(P_n^k)$. In 2009 [81] and [80] Alpert determined the rank numbers of P_n^k, C_n^k, $P_2 \times C_n$, $K_m \times P_n$, $P_3 \times P_n$, Möbius ladders and found bounds for rank numbers of general grid graphs $P_m \times P_n$. About the same time as Alpert and independently, Chang, Kuo, and Lin [352] determined the rank numbers of P_n^k, C_n^k, $P_2 \times P_n, P_2 \times C_n$. Chang et al. also determined the rank numbers of caterpillars and proved that for any graphs G and H $\chi_r(G[H]) = \chi_r(H) + |V(H)|(\chi_r(G) - 1)$.

In 2010 J. Jacob, D. Narayan, E. Sergel, P. Richter, and A. Tran [699] investigated k-rankings of paths and cycles with pendent paths of length 1 or 2. Among their results are: for any caterpillar G $\chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n) + 1$ and both cases occur; if $2^m \leq n \leq 2^{m+1}$ then for any graph G obtained by appending edges to an n-cycle we have $m + 2 \leq \chi_r(G) \leq m + 3$ and both cases occur; if G is a lobster with spine P_n then $\chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n) + 2$ and all three cases occur; if G a graph obtained from the cycle C_n by appending paths of length 1 or 2 to any number of the vertices of the cycle then $\chi_r(P_n) \leq \chi(G) \leq \chi(P_n) + 2$ and all three cases occur; and if G the graph obtained from the comb obtained from P_n by appending one path of length m to each vertex of P_n then $\chi_r(G) = \chi_r(P_n) + \chi_r(P_{m+1}) - 1$.

Sergel, Richter, Tran, Curran, Jacob, and Narayan [1297] investigated the rank number of a cycle C_n with pendent edges, which they denote by CC_n, and call a *caterpillar cycle*. They proved that $\chi(CC_n) = \chi_r(C_n))$ or $\chi(CC_n) = \chi_r(C_n)) + 1$ and showed that both cases occur. A *comb tree*, denoted by $C(n, m)$, is a tree that has a path P_n such that every vertex of P_n is adjacent to an end vertex of a path P_m. In the comb tree $C(n, m)$ $(n \geq 3)$ there are 2 pendent paths P_{m+2} and $n-2$ paths P_{m+1}. They proved $\chi_r(C(n, m)) = \chi_r(P_{m+1}) - 1$. They define a *circular lobster* as a graph where each vertex is either on a cycle C_n or at most distance two from a vertex on C_n. They proved that if G is a lobster with longest path P_n, then $\chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n) + 2$ and determined the conditions under which each true case occurs. If G is circular lobster with cycle C_n, they showed that $\chi_r(C_n) \leq \chi_r(G) \leq \chi_r(C_n) + 2$ and determined the conditions under which each true case occurs. An *icicle graph* I_n $(n \geq 3)$ has three pendent paths P_2 and is comprised of a path P_n with vertices $v_1, v_2, \ldots, v_n$ where a path P_{i-1} is appended to vertex v_i. They determine the rank number for icicle graphs.

The *arank number* of a graph G is the maximum value of k such that G has a minimal k-ranking. Eyabi, Jacob, Laskar, Narayan, and Pillone [482] determine the arank number of the

Cartesian product $K_n \square K_n$, and investigated the arank number of $K_m \square K_n$.

7.18 Set Graceful and Set Sequential Graphs

The notions of set graceful and set sequential graphs were introduced by Acharaya in 1983 [14]. A graph is called *set graceful* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge, the assignment of sets to the vertices is injective, and the assignment to the edges is bijective. A graph is called *set sequential* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge and the the assignment of sets to the vertices and the edges is bijective. The following has been shown: P_n $(n > 3)$ is not set graceful [633]; C_n is not set sequential [26]; C_n is set graceful if and only if $n = 2^m - 1$ [635] and [14]; K_n is set graceful if and only if $n = 2, 3$ or 6 [1068]; K_n $(n \geq 2)$ is set sequential if and only if $n = 2$ or 5 [635]; $K_{a,b}$ is set sequential if and only if $(a+1)(b+1)$ is a positive power of 2 [635]; a necessary condition for $K_{a,b,c}$ to be set sequential is that a, b, and c cannot have the same parity [633]; $K_{1,b,c}$ is not set sequential when b and c even [635]; $K_{2,b,c}$ is not set sequential when b and c are odd [633]; no theta graph is set graceful [633]; the complete nontrivial n-ary tree is set sequential if and only if $n+1$ is a power of 2 and the number of levels is 1 [633]; a tree is set sequential if and only if it is set graceful [633]; the nontrivial plane triangular grid graph G_n is set graceful if and only if $n = 2$ [635]; every graph can be embedded as an induced subgraph of a connected set sequential graph [633]; every graph can be embedded as an induced subgraph of a connected set graceful graph [633], every planar graph can be embedded as an induced subgraph of a set sequential planar graph [635]; every tree can be embedded as an induced subgraph of a set sequential tree [635]; and every tree can be embedded as an induced subgraph of a set graceful tree [635]. Hegde conjectures [635] that no path is set sequential. Hegde's conjecture [636] every complete bipartite graph that has a set graceful labeling is a star was proved by Vijayakumar [1582]

Germina, Kumar, and Princy [563] prove: if a (p, q)-graph is set-sequential with respect to a set with n elements, then the maximum degree of any vertex is $2^{n-1} - 1$; if G is set-sequential with respect to a set with n elements other than K_5, then for every edge uv with $d(u) = d(v)$ one has $d(u) + d(v) < 2^{n-1} - 1$; $K_{1,p}$ is set-sequential if and only if p has the form $2^{n-1} - 1$ for some $n \geq 2$; binary trees are not set-sequential; hypercubes Q_n are not set-sequential for $n > 1$; wheels are not set-sequential; and uniform binary trees with an extra edge appended at the root are set-graceful and set graceful.

Acharya [14] has shown: a connected set graceful graph with q edges and $q+1$ vertices is a tree of order $p = 2^m$ and for every positive integer m such a tree exists; if G is a connected set sequential graph, then $G + K_1$ is set graceful; and if a graph with p vertices and q edges is set sequential, then $p + q = 2^m - 1$. Acharya, Germina, Princy, and Rao [22] proved: if G is set graceful, then $G \cup \overline{K_t}$ is set sequential for some t; if G is a set graceful graph with n edges and $n+1$ vertices, then $G + \overline{K_t}$ is set graceful if and only if m has the form $2^t - 1$; $P_n + \overline{K_m}$ is set graceful if $n = 1$ or 2 and m has the form $2^t - 1$; $K_{1,m,n}$ is set graceful if and only if m has the form $2^t - 1$ and n has the form $2^s - 1$; $P_4 + \overline{K_m}$ is not set graceful when m has the form $2^t - 1$ $(t \geq 1)$; $K_{3,5}$ is not set graceful; if G is set graceful, then graph obtained from G by adding for each vertex v in G a new vertex v' that is adjacent to every vertex adjacent to v is not set graceful; and $K_{3,5}$ is not set graceful.

7.19 Vertex Equitable Graphs

Given a graph G with q edges and a labeling f from the vertices of G to the set $\{0, 1, 2, \ldots, \lceil q/2 \rceil\}$ define a labeling f^* on the edges by $f^*(uv) = f(u) + f(v)$. If for all i and j and each vertex the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the edge labels induced by f^* are $1, 2, \ldots, q$, Lourdusamy and Seenivasan [998] call a f a *vertex equitable* labeling of G. They proved the following graphs are vertex equitable: paths, bistars, combs, n-cycles for $n \equiv 0$ or 3 (mod 4), $K_{2,n}$, ${C_3}^t$ for $t \geq 2$, quadrilateral snakes, $K_2 + mK_1$, $K_{1,n} \cup K_{1,n+k}$ if and only if $1 \leq k \leq 3$, ladders, arbitrary super divisions of paths, and n-cycles with $n \equiv 0$ or 3 (mod 4). They further proved that $K_{1,n}$ for $n \geq 4$, Eulerian graphs with n edges where $n \equiv 1$ or 2 (mod 4), wheels, K_n for $n > 3$, triangular cacti with $q \equiv 0$ or 6 or 9 (mod 12), and graphs with p vertices and q edges, where q is even and $p < \lceil q/2 \rceil + 2$ are not vertex equitable.

Jeyanthi and Maheswari [727] and [728] proved that the following graphs have vertex equitable labeling: the square of the bistar $B_{n,n}$; the splitting graph of the bistar $B_{n,n}$; C_4-snakes; connected graphs for in which each block is a cycle of order divisible by 4 (they need not be the same order) and whose block-cut point graph is a path; $C_m \odot P_n$; tadpoles; the one-point union of two cycles; and the graph obtained by starting friendship graphs, $C_{n_1}^{(2)}, C_{n_2}^{(2)}, \ldots, C_{n_k}^{(2)}$ where each $n_i \equiv 0 \pmod 4$ and joining the center of $C_{n_i}^{(2)}$ to the center of $C_{i+1}^{(2)}$ with an edge for $i = 1, 2, \ldots, k-1$. In [722] Jeyanthi and Maheswari prove that T_p trees, bistars $B(n, n+1), C_n \odot K_m, {P_n}^2$, tadpoles, certain classes of caterpillars, and $T \odot \overline{K_n}$ where T is a T_p tree with an even number of vertices are vertex equitable.

Jeyanthi and Maheswari [721] proved that graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle, total graphs of a paths, splitting graphs of paths, and the graphs obtained identifying an edge of one cycle with an edge of another cycle are vertex equitable (see §2.7 for the definitions of duplicating vertices and edges, a total graph, and a splitting graph.)

7.20 Sequentially Additive Graphs

Bange, Barkauskas, and Slater [219] defined a *k-sequentially additive labeling* f of a graph $G(V, E)$ to be a bijection from $V \cup E$ to $\{k, \ldots, k + |V \cup E| - 1\}$ such that for each edge xy, $f(xy) = f(x) + f(y)$. They proved: K_n is 1-sequentially additive if and only if $n \leq 3$; C_{3n+1} is not k-sequentially additive for $k \equiv 0$ or 2 (mod 3); C_{3n+2} is not k-sequentially additive for $k \equiv 1$ or 2 (mod 3); C_n is 1-sequentially additive if and only if $n \equiv 0$ or 1 (mod 3); and P_n is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive. Hegde [631] proved that $K_{1,n}$ is k-sequentially additive if and only if k divides n.

Hajnal and Nagy [604] investigated 1-sequentially additive labelings of 2-regular graphs. They prove: kC_3 is 1-sequentially additive for all k; kC_4 is 1-sequentially additive if and only if $k \equiv 0$ or 1 (mod 3); $C_{6n} \cup C_{6n}$ and $C_{6n} \cup C_{6n} \cup C_3$ are 1-sequentially additive for all n; C_{12n} and $C_{12n} \cup C_3$ are 1-sequentially additive for all n. They conjecture that every 2-regular simple graph on n vertices is 1-sequentially additive where $n \equiv 0$ or 1 (mod 3).

Acharya and Hegde [27] have generalized k-sequentially additive labelings by allowing the image of the bijection to be $\{k, k+d, \ldots, (k+|V \cup E|-1)d\}$. They call such a labeling *additively (k, d)-sequential*.

7.21 Difference Graphs

Analogous to a sum graph, Harary [610] calls a graph a *difference graph* if there is an bijection f from V to a set of positive integers S such that $xy \in E$ if and only if $|f(x) - f(y)| \in S$. Bloom, Hell, and Taylor [286] have shown that the following graphs are difference graphs: trees, $C_n, K_n, K_{n,n}, K_{n,n-1}$, pyramids, and n-prisms. Gervacio [566] proved that wheels W_n are difference graphs if and only if $n = 3, 4$, or 6. Sonntag [1425] proved that cacti (that is, graphs in which every edge is contained in at most one cycle) with girth at least 6 are difference graphs and he conjectures that all cacti are difference graphs. Sugeng and Ryan [1451] provided difference labelings for cycles, fans, cycles with chords, graphs obtained by the one-point union of K_n and P_m; and graphs made from any number of copies of a given graph G that has a difference labeling by identifying one vertex the first with a vertex of the second, a different vertex of the second with the third and so on.

Hegde and Vasudeva [651] call a simple digraph a *mod difference digraph* if there is a positive integer m and a labeling L from the vertices to $\{1, 2, \ldots, m\}$ such that for any vertices u and v, (u, v) is an edge if and only if there is a vertex w such that $L(v) - L(u) \equiv L(w) \pmod{m}$. They prove that the complete symmetric digraph and unidirectional cycles and paths are mod difference digraphs.

In [1275] Seoud and Helmi provided a survey of all graphs of order at most 5 and showed the following graphs are difference graphs: $K_n,\ (n \geq 4)$ with two deleted edges having no vertex in common; $K_n,\ (n \geq 6)$ with three deleted edges having no vertex in common; gear graphs G_n for $n \geq 3$; $P_m \times P_n$ $(m, n \geq 2)$; triangular snakes; C_4-snakes; dragons (that is, graphs formed by identifying the end vertex of a path and any vertex in a cycle); graphs consisting of two cycles of the same order joined by an edge; and graphs obtained by identifying the center of a star with a vertex of a cycle.

7.22 Square Sum Labelings

Ajitha, Arumugam, and Germina [74] call a labeling f from a graph $G(p, q)$ to $\{1, 2, \ldots, q\}$ a *square sum labeling* if the induced edge labeling $f^*(uv) = (f(u))^2 + (f(v))^2$ is injective. They say a square sum labeling is a *strongly square sum labeling* if the q edge labels are the first q consecutive integers of the form $a^2 + b^2$ where a and b are less than p and distinct. They prove the following graphs have square sum labelings: trees; cycles; $K_2 + mK_1$; K_n if and only if $n \leq 5$; $C_n^{(t)}$ (the one-point union of t copies of C_n); grids $P_m \times P_n$; and $K_{m,n}$ if $m \leq 4$. They also prove that every strongly square sum graph except K_1, K_2, and K_3 contains a triangle.

In [1418] Somashekara and Veena used the term "square sum labeling" to mean "strongly square sum labeling." They proved that the following graphs have strongly square sum labelings: paths, $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$, complete n-ary trees, and lobsters obtained by joining centers of any number of copies of a star to a new vertex. They observed that that if every edge of a graph is an edge of a triangle then the graph does not have strongly square sum labeling. As a consequence the following graphs do not have a strongly square sum labelings: $K_n, n \geq 3$, wheels, fans $P_n + K_1,\ n \geq 2$, double fans $P_n + K_2,\ n \geq 2$, friendship graphs $C_3^{(n)}$, windmills $K_m^{(n)},\ m > 3$, triangular ladders, triangular snakes, double triangular snakes, and flowers. They also proved that helms are not strongly square sum graphs and the graphs obtained by joining the centers of two wheels to a new vertex are not strongly square sum graphs.

7.23 Permutation and Combination Graphs

Hegde and Shetty [646] define a graph G with p vertices to be a *permutation graph* if there exists a injection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f defined by $g_f(uv) = f(u)!/|f(u) - f(v)|!$ is injective. They say a graph G with p vertices is a *combination graph* if there exists a injection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f defined as $g_f(uv) = f(u)!/|f(u) - f(v)|!f(v)!$ is injective. They prove: K_n is a permutation graph if and only if $n \leq 5$; K_n is a combination graph if and only if $n \leq 5$; C_n is a combination graph for $n > 3$; $K_{n,n}$ is a combination graph if and only if $n \leq 2$; W_n is a not a combination graph for $n \leq 6$; and a necessary condition for a (p, q)-graph to be a combination graph is that $4q \leq p^2$ if p is even and $4q \leq p^2 - 1$ if p is odd. They strongly believe that W_n is a combination graph for $n \geq 7$ and all trees are combinations graphs. Babujee and Vishnupriya [130] prove the following graphs are permutation graphs: P_n; C_n; stars; graphs obtained adding a pendent edge to each edge of a star; graphs obtained by joining the centers of two identical stars with an edge or a path of length 2); and complete binary trees with at least three vertices. Seoud and Salim [1285] determine all permutation graphs of order at most 9 and prove that every bipartite graph of order at most 50 is a permutation graph. Seoud and Mahran [1277] give an upper bound on the number of edges of a permutation graph and introduce some necessary conditions for a graph to be a permutation graph. They show that these conditions are not sufficient for a graph to be a permutation graph.

Hegde and Shetty [646] say a graph G with p vertices and q edges is a *strong k-combination graph* if there exists a bijection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f from the edges to $\{k, k+1, \ldots, k+q-1\}$ defined by $g_f(uv) = f(u)!/|f(u) - f(v)|!f(v)!$ is a bijection. They say a graph G with p vertices and q edges is a *strong k-permutation graph* if there exists a bijection f from the vertices of G to $\{1, 2, 3, \ldots, p\}$ such that the induced edge function g_f from the edges to $\{k, k+1, \ldots, k+q-1\}$ defined by $g_f(uv) = f(u)!/|f(u) - f(v)|!$ is a bijection. Seoud and Anwar [1265] provided necessary conditions for combination graphs, permutation graphs, strong k-combination graphs, and strong k-permutation graphs.

Seoud and Al-Harere [1264] showed that the following families are combination graphs: graphs that are two copies of C_n sharing a common edge; graphs consisting of two cycles of the same order joined by a path; graphs that are the union of three cycles of the same order; wheels W_n ($n \geq 7$); coronas $T_n \odot K_1$, where T_n is the triangular snake; and the graphs obtained from the gear G_m by attaching n pendent vertices to each vertex which is not joined to the center of the gear. They proved that a graph $G(n, q)$ having at least 6 vertices such that 3 vertices are of degree $1, n-1, n-2$ is not a combination graph, and a graph $G(n, q)$ having at least 6 vertices such that there exist 2 vertices of degree $n-3$, two vertices of degree 1 and one vertex of degree $n-1$ is not a combination graph.

Seoud and Al-Harere [1263] proved that the following families are combination graphs: unions of four cycles of the same order; double triangular snakes; fans F_n if and only if $n \geq 6$; caterpillars; complete binary trees; ternary trees with at least 4 vertices; and graphs obtained by identifying the pendent vertices of stars S_m with the paths P_{n_i}, for $1 \leq n_i \leq m$. They include a survey of trees of order at most 10 that are combination graphs and proved the following graphs are not combination graphs: bipartite graphs with two partite sets with $n \geq 6$ elements such that $n/2$ elements of each set have degree n; the splitting graph of $K_{n,n}$ ($n \geq 3$); and certain chains of two and three complete graphs. Seoud and Anwar [1265] proved the following graphs are combination graphs: dragon graphs (the graphs obtained from by joining the end point of a path to a vertex of a cycle); triangular snakes T_n $(n \geq 3)$; wheels; and the graphs obtained by adding k pendant edges to every vertex of C_n for certain values of k.

In [1262] and [1263] Seoud and Al-Harere proved the following graphs are non-combination graphs: $G_1 + G_2$ if $|V(G_1)|, |V(G_2)| \geq 2$ and at least one of $|V(G_1)|$ and $|V(G_2)|$ is greater than 2; the double fan $\overline{K_2} + P_n$; $K_{l,m,n}$; $K_{k,l,m,n}$; $P_2[G]$; $P_3[G]$; $C_3[G]$; $C_4[G]$; $K_m[G]$; $W_m[G]$; the splitting graph of $K_n (n \geq 3)$; K_n $(n \geq 4)$ with an edge deleted; K_n $(n \geq 5)$ with three edges deleted; and $K_{n,n}$ $(n \geq 3)$ with an edge deleted. They also proved that a graph $G(n, q)$ $(n \geq 3)$ is not a combination graph if it has more than one vertex of degree $n - 1$.

7.24 Strongly *-graphs

A variation of strong multiplicity of graphs is a strongly *-graph. A graph of order n is said to be a *strongly *-graph* if its vertices can be assigned the values $1, 2, \ldots, n$ in such a way that, when an edge whose vertices are labeled i and j is labeled with the value $i + j + ij$, all edges have different labels. Adiga and Somashekara [38] have shown that all trees, cycles, and grids are strongly *-graphs. They further consider the problem of determining the maximum number of edges in any strongly *-graph of given order and relate it to the corresponding problem for strongly multiplicative graphs. In [1279] and [1280] Seoud and Mahan give some technical necessary conditions for a graph to be strongly *-graph,

Babujee and Vishnupriya [130] have proved the following are strongly *-graphs: $C_n \times P_2, (P_2 \cup \overline{K}_m) + \overline{K}_2$, windmills $K_3^{(n)}$, and *jelly fish* graphs $J(m, n)$ obtained from a 4-cycle v_1, v_2, v_3, v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.

Babujee and Beaula [114] prove that cycles and complete bipartite graphs are vertex strongly *-graphs. Babujee, Kannan, and Vishnupriya [124] prove that wheels, paths, fans, crowns, $(P_2 \cup mK_1) + \overline{K_2}$, and umbrellas (graphs obtained by appending a path to the central vertex of a fan) are vertex strongly *-graphs.

7.25 Triangular Sum Graphs

S. Hegde and P. Shankaran [641] call a labeling of graph with q edges a *triangular sum labeling* if the vertices can be assigned distinct non-negative integers in such a way that, when an edge whose vertices are labeled i and j is labeled with the value $i + j$, the edges labels are $\{k(k + 1)/2 |\ k = 1, 2, \ldots, q\}$. They prove the following graphs have triangular sum labelings: paths, stars, complete n-ary trees, and trees obtained from a star by replacing each edge of the star by a path. They also prove that K_n has a triangular sum labeling if and only if n is 1 or 2 and the friendship graphs $C_3^{(t)}$ do not have a triangular sum labeling. They conjecture that K_n $(n \geq 5)$ are forbidden subgraphs of graph with triangular sum labelings. They conjectured that every tree admits a triangular sum labeling. They show that some families of graphs can be embedded as induced subgraphs of triangular sum graphs. They conclude saying "as every graph cannot be embedded as an induced subgraph of a triangular sum graph, it is interesting to embed families of graphs as an induced subgraph of a triangular sum graph". In response, Seoud and Salim [1282] showed the following graphs can be embedded as an induced subgraph of a triangular sum graph: trees, cycles, nC_4, and the one-point union of any number of copies of C_4 (friendship graphs).

Vaidya, Prajapati, and Vihol [1542] showed that cycles, cycles with exactly one chord, and cycles with exactly two chords that form a triangle with an edge of the cycle can be embedded as an induced subgraph of a graph with a triangular sum labeling.

Vaidya, Prajapati, and Vihol [1542] proved that several classes of graphs do not have triangular sum labelings. Among them are: helms, graphs obtained by joining the centers of two wheels to a new vertex, and graphs in which every edge is an edge of a triangle. As a corollary of the latter result they have that $P_m + \overline{K_n}$, $W_m + \overline{K_n}$, wheels, friendship graphs, flowers, triangular ladders, triangular snakes, double triangular snakes, and flowers. do not have triangular sum labelings.

Seoud and Salim [1282] proved the following are triangular sum graphs: $P_m \cup P_n,\ m \geq 4$; the union of any number of copies of $P_n,\ n \geq 5$; $P_n \odot \overline{K_m}$; symmetrical trees; the graph obtained from a path by attaching an arbitrary number of edges to each vertex of the path; the graph obtained by identifying the centers of any number of stars; and all trees of order at most 9.

For a positive integer i the ith *pentagonal number* is $i(3i-1)/2$. Somashekara and Veena [1419] define a *pentagonal sum labeling* of a graph $G(V,E)$ as one for which there is a one-to-one function f from $V(G)$ to the set of nonnegative integers that induces a bijection f^+ from $E(G)$ to the set of the first $|E|$ pentagonal numbers. A graph that admits such a labeling is called a *pentagonal sum graph*. Somashekara and Veena [1419] proved that the following graphs have pentagonal sum labelings: paths, $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$, complete n-ary trees, and lobsters obtained by joining centers of any number of copies of a star to a new vertex. They conjecture that every tree has a pentagonal sum labeling and as an open problem they ask for a proof or disprove that cycles have pentagonal labelings.

Somashekara and Veena [1419] observed that that if every edge of a graph is an edge of a triangle then the graph does not have pentagonal sum labeling. As was the case for triangular sum labelings the following graphs do not have a pentagonal sum labeling: $P_m + \overline{K_n}$, and $W_m + \overline{K_n}$ wheels, friendship graphs, flowers, triangular ladders, triangular snakes, double triangular snakes, and flowers. Somashekara and Veena [1419] also proved that helms and the graphs obtained by joining the centers of two wheels to a new vertex are not pentagonal sum graphs.

7.26 Divisor Graphs

G. Santhosh and G. Singh [1243] call a graph $G(V,E)$ a *divisor graph* if V is a set of integers and $uv \in E$ if and only if u divides v or vice versa. They prove the following are divisor graphs: trees; mK_n; induced subgraphs of divisor graphs; cocktail party graphs $H_{m,n}$ (see Section 7.1 for the definition); the one-point union of complete graphs of different orders; complete bipartite graphs; W_n for n even and $n > 2$; and $P_n + \overline{K_t}$. They also prove that C_n $(n \geq 4)$ is a divisor graph if and only if n is even and if G is a divisor graph then for all n so is $G + K_n$.

Chartrand, Muntean, Saenpholphat, and Zhang [364] proved complete graphs, bipartite graphs, complete multipartite graphs, and joins of divisor graphs are divisor graphs. They also proved if G is a divisor graph, then $G \times K_2$ is a divisor graph if and only if G is a bipartite graph; a triangle-free graph is a divisor graph if and only if it is bipartite; no divisor graph contains an induced odd cycle of length 5 or more; and that a graph G is divisor graph if and only if there is an orientation D of G such that if (x,y) and (y,z) are edges of D then so is (x,z).

In [61] and [63] Al-Addasi, AbuGhneim, and Al-Ezeh determined precisely the values of n for which P_n^k $(k \geq 2)$ are divisor graphs and proved that for any integer $k \geq 2$, C_n^k is a divisor graph if and only if $n \leq 2k+2$. In [64] they gave a characterization of the graphs G and H

for which $G \times H$ is a divisor graph and a characterization of which block graphs are divisor graphs. (Recall a graph is a *block graph* if every one of its blocks is complete.) They showed that divisor graphs form a proper subclass of perfect graphs and showed that cycle permutation graphs of order at least 8 are divisor graphs if and only if they are perfect. (Recall a graph is *perfect* if every subgraph has chromatic number equal to the order of its maximal clique.) In [62] Al-Addasi, AbuGhneim, and Al-Ezeh proved that the contraction of a divisor graph along a bridge is a divisor graph; if e is an edge of a divisor graph that lies on an induced even cycle of length at least 6, then the contraction along e is not a divisor graph; and they introduced a special type of vertex splitting that yields a divisor graph when applied to a cut vertex of a given divisor graph.

Ganesan and Uthayakumar [536] proved that $G \odot H$ is a divisor graph if and only if G is a bipartite graph and H is a divisor graph. Frayer [505] proved $K_n \times G$ is a divisor graph for each n if and only if G contains no edges and $\overline{K_n \times K_2}$ ($n \geq 3$) is a divisor graph. Vinh [1599] proved that for any $n > 1$ and $0 \leq m \leq n(n-1)/2$ there exists a divisor graph of order n and size m. She also gave a simple proof of the characterization of divisor graphs due to Chartrand, Muntean, Saenpholphat, and Zhang [364] Gera, Saenpholphat, and Zhang [561] established forbidden subgraph characterizations for all divisor graphs that contain at most three triangles. Tsao [1501] investigated the vertex-chromatic number, the clique number, the clique cover number, and the independence number of divisor graphs and their complements. In [1271] Seoud, El Sonbaty, and Mahran discuss here some necessary and sufficient conditions for a graph to be divisor graph.

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