# A Dynamic Survey of Graph Labeling

Joseph A. Gallian

Department of Mathematics and Statistics University of Minnesota Duluth Duluth, Minnesota 55812, U.S.A.

jgallian@d.umn.edu

Submitted: September 1, 1996; Accepted: November 14, 1997 Twenty-third edition, December 17, 2020 Mathematics Subject Classifications: 05C78

#### Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960s. In the intervening 50 years over 200 graph labelings techniques have been studied in over 3000 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

# Contents

1	Intr	oduction	5
<b>2</b>	Gra	ceful and Harmonious Labelings	8
	2.1	Trees	8
	2.2	Cycle-Related Graphs	13
	2.3	Product Related Graphs	19
	2.4	Complete Graphs	22
	2.5	Disconnected Graphs	24
	2.0	Joins of Graphs	27
	2.1		28
	2.8	Table 1. Summary of Gradeful Degulta	38
		Table 1: Summary of Graceful Results	39
		Table 2: Summary of Harmonious Results	42
3	Var	iations of Graceful Labelings	46
	3.1	$\alpha$ -labelings	46
	2.0	Table 3: Summary of Results on $\alpha$ -labelings	59 60
	3.2 2.2	$\gamma$ -Labelings	60 CO
	ა.ა ე_4	Gracerul-like Labellings	00
	3.4	<i>k</i> -graceful Labelings	70 71
	35	Skolom Crocoful Labolings	71 75
	3.5 3.6	Odd Craceful Labelings	70 77
	3.0	Cordial Labelings	81
	3.8	The Friendly Index-Balance Index	97
	3.9	k-equitable Labelings	02
	3.10	Hamming-graceful Labelings	.06
1	Vor	intions of Harmonious Labolings	07
4	<b>va</b> . 11	Sequential and Strongly c harmonious Labelings	07
	4.1 1 2	(k d)-arithmetic Labelings 1	13
	4.3	(k, d)-indexable Labelings 1	14
	4.0 4.4	Elegant Labelings	16
	4.5	Felicitous Labelings	18
	4.6	Odd Harmonious and Even Harmonious Labelings	20
F	Ма	ric-type Labelings 1	97
0	5 1	Magic Labelings	<b>21</b> 27
	0.1	Table 5: Summary of Magic Labelings	35
	5.2	Edge-magic Total and Super Edge-magic Total Labelings	36
		Table 6: Summary of Edge-magic Total Labelings	57
		Table 7: Summary of Super Edge-magic Labelings   1     1   1	59

	5.3	Vertex-magic Total Labelings	. 162
		Table 8: Summary of Vertex-magic Total Labelings	. 170
		Table 9: Summary of Super Vertex-magic Total Labelings	. 171
		Table 10: Summary of Totally Magic Labelings	. 172
	5.4	H-Magic Labelings	. 172
	5.5	Magic Labelings of Type $(a, b, c)$	. 177
		Table 11: Summary of Magic Labelings of Type $(a, b, c)$	. 179
	5.6	Sigma Labelings/1-vertex magic labelings/Distance Magic	. 180
	5.7	Other Types of Magic Labelings	. 184
6	Ant	imagic-type Labelings	195
	6.1	Antimagic Labelings	. 195
		Table 12: Summary of Antimagic Labelings	. 204
	6.2	(a, d)-Antimagic Labelings	. 205
		Table 13: Summary of $(a, d)$ -Antimagic Labelings	. 210
	6.3	(a, d)-Antimagic Total Labelings	. 211
		Table 14: Summary of $(a, d)$ -Vertex-Antimagic Total and Super $(a, d)$ -	
		Vertex-Antimagic Total Labelings	. 223
		Table 15: Summary of $(a, d)$ -Edge-Antimagic Total Labelings	. 224
		Table 16: Summary of $(a, d)$ -Edge-Antimagic Vertex Labelings	. 225
		Table 17: Summary of $(a, d)$ -Super-Edge-Antimagic Total Labelings	. 226
	6.4	Face Antimagic Labelings and $d$ -antimagic Labeling of Type $(1,1,1)$	. 227
		Table 18: Summary of Face Antimagic Labelings	. 231
		Table 19: Summary of <i>d</i> -antimagic Labelings of Type $(1,1,1)$	. 231
	6.5	Product Antimagic Labelings	. 232
7	Mise	cellaneous Labelings	234
	7.1	Sum Graphs	. 234
		Table 20: Summary of Sum Graph Labelings	. 242
	7.2	Prime and Vertex Prime Labelings	. 243
		Table 21: Summary of Prime Labelings	. 253
		Table 22: Summary of Vertex Prime Labelings	. 254
	7.3	Edge-graceful Labelings	. 255
		Table 23: Summary of Edge-graceful Labelings	. 264
	7.4	Line-graceful Labelings	. 266
	7.5	Radio Labelings	. 266
	7.6	Representations of Graphs modulo $n$	. 270
	7.7	Product and Divisor Cordial Labelings	. 271
	7.8	Edge Product Cordial Labelings	. 282
	7.9	Difference Cordial Labelings	. 284
	7.10	Prime Cordial Labelings	. 287
	7.11	Parity Combination Cordial Labelings	. 291
	7.12	Mean Labelings	. 291

7.13	Pair Sum and Pair Mean Graphs	12
7.14	Irregular Total Labelings	14
7.15	Geometric Labelings	24
7.16	Strongly Multiplicative Graphs	25
7.17	$k$ -sequential Labelings $\ldots \ldots 3$	26
7.18	IC-colorings	27
7.19	$Minimal k-rankings \dots \dots$	28
7.20	Set Graceful and Set Sequential Graphs	29
7.21	Vertex Equitable Graphs	31
7.22	Sequentially Additive Graphs	34
7.23	Difference Graphs	35
7.24	Square Sum Labelings and Square Difference Labelings	35
7.25	Permutation and Combination Graphs	38
7.26	Strongly *-graphs	40
7.27	Triangular Sum Graphs	41
7.28	Divisor Graphs	42
Referen	nces 3	44

# Index

536

# 1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [2136] in 1967, or one given by Graham and Sloane [925] in 1980. Rosa [2136] called a function f a  $\beta$ -valuation of a graph G with q edges if f is an injection from the vertices of G to the set  $\{0, 1, \ldots, q\}$  such that, when each edge xy is assigned the label |f(x) - f(y)|, the resulting edge labels are distinct. Golomb [902] subsequently called such labelings graceful and this is now the popular term. Alternatively, Buratti, Rinaldi, and Traetta [519] define a graph G with q edges to be graceful if there is an injection f from the vertices of G to the set  $\{0, 1, \ldots, q\}$  such that every possible difference of the vertex labels of all the edges is the set  $\{1, 2, \ldots, q\}$ . Rosa introduced  $\beta$ -valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular,  $\beta$ -valuations originated as a means of attacking the conjecture of Ringel [2117] that  $K_{2n+1}$  can be decomposed into 2n + 1 subgraphs that are all isomorphic to a given tree with n edges. In 2020 Montgomery, Pokrovskiy, and Sudakov [1783] proved Ringel's conjecture for every sufficiently large n. Although an unpublished result of Erdős says that most graphs are not graceful (see [925]), most graphs that have some sort of regularity of structure are graceful. Sheppard [2326] has shown that there are exactly q!gracefully labeled graphs with q edges. Rosa [2136] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges," (2) G "has too many edges," and (3) G "has the wrong parity." The disjoint union of trees is a case where there are too many vertices for the number of edges. An infinite class of graphs that are not graceful for the second reason is given in [462]. As an example of the third condition Rosa [2136] has shown that if every vertex has even degree and the number of edges is congruent to 1 or  $2 \pmod{4}$  then the graph is not graceful. In particular, the cycles  $C_{4n+1}$  and  $C_{4n+2}$  are not graceful. Knuth [1383] has observed the more general condition that in any graceful labeling of a graph with the number of edges congruent to 1 or 2 (mod 4), the number of vertices with an odd degree and an odd label is always odd. Knuth [1383] proved by way a computer search that all cubic graphs on 4, 6, 8, 10, 12, or 14 vertices, except  $2K_4$  and  $3K_4$ , which was proved by Kotzig, are graceful. He conjectures that that every connected cubic graph is graceful.

Acharya [22] proved that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph. Acharya, Rao, and Arumugam [42] proved: every trianglefree graph can be embedded as an induced subgraph of a triangle-free graceful graph; every planar graph can be embedded as an induced subgraph of a planar graceful graph; and every tree can be embedded as an induced subgraph of a graceful tree. Sethuraman, Ragukumar, and Slater [2288] show that every tree can be embedded in a graceful tree (see also [2287]) and pose a related open problem toward settling the Graceful Tree Conjecture. Rao and Sahoo [2097] proved that every connected graph can be embedded as an induced subgraph of an Eulerian graceful graph thereby answering a question originally posed by Rao and mentioned by Acharya and Arumugum in [28]. As a consequence they deduce that the problems on deciding whether the chromatic of a graph number is less than or equal to k, for  $k \ge 3$ , and deciding whether the clique number of a graph is greater than or equal to k, for  $k \ge 3$  are NP-complete even for Eulerian graceful graphs.

Sethuraman and Ragukumar [2286] provided an algorithm that generates a graceful tree from a given arbitrary tree by adding a sequence of new pendent edges to the given arbitrary tree thereby proving that every tree is a subtree of a graceful tree. They ask the question: If G is a graceful tree and v is any vertex of G of degree 1, is it true that G - v is graceful? If the answer is affirmative, then those additional edges of the input arbitrary tree T introduced for constructing the graceful tree T by their algorithm could be deleted in some order so that the given arbitrary tree T becomes graceful. This would imply that the Graceful Tree Conjecture is true. These results demonstrate that there is no forbidden subgraph characterization of these particular kinds of graceful graphs.

Harmonious graphs naturally arose in the study by Graham and Sloane [925] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label  $f(x) + f(y) \pmod{q}$ , the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. They proved that almost all graphs are not harmonious. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by  $2^k$  then q is divisible by  $2^{k+1}$ . Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph  $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [1607] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence  $d_1, d_2, \ldots, d_p$ then  $gcd(d_1, d_2, \ldots, d_p, q)$  divides q(q-1)/2. They have also proved that every graph is a subgraph of a harmonious graph. More generally, Sethuraman and Elumalai [2271] have shown that any given set of graphs  $G_1, G_2, \ldots, G_t$  can be embedded in a graceful or harmonious graph. Determining whether a graph has a harmonious labeling was shown to be NP-complete by Auparajita, Dulawat, and Rathore in 2001 (see [1433]).

In the early 1980s Bloom and Hsu [475], [476],[451], [477], [537] extended graceful labelings to directed graphs by defining a graceful labeling on a directed graph D(V, E)as a one-to-one map  $\theta$  from V to  $\{0, 1, 2, \ldots, |E|\}$  such that  $\theta(y) - \theta(x) \mod (|E| + 1)$ is distinct for every edge xy in E. Graceful labelings of directed graphs also arose in the characterization of finite neofields by Hsu and Keedwell [1026], [1027]. Graceful labelings of directed graphs was the subject of Marr's 2007 Ph.D. dissertation [1710]. In [1710] and [1711] Marr presents results of graceful labelings of directed paths, stars, wheels, and umbrellas. Siqinbate and Feng [2412] proved that the disjoint union of three copies of a directed cycle of fixed even length is graceful.

Over the past five decades in excess of 2800 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes of graphs have been done by several authors and in some cases the same terminology is used for different concepts. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two labeling methods, include [445], [471], [1393], [811], and [813]. The book edited by Acharya, Arumugam, and Rosa [27] includes a variety of labeling methods that we do not discuss in this survey. In 2002 Eshghi [735] wrote a 65 page paper providing an introduction to graceful graphs. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings, and neofields is discussed in [475] and [476]. The connection between graceful labelings and perfect systems of difference sets is given in [448]. The computational complexity of the gracefulness of a graph is not known, but the complexity of finding a harmonious labeling of a graph is in the NP-class [147]. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, secret sharing schemes, models for constraint programming over finite domains, [472], [473], [2552], [2040], [2430], [2431], [180], [179], [228], [2418], [1739], and network passwords-see [2768], [2534], [2767], [2769], [2191], [2588], and [2905] for details. According to Wang, B. Yao, and M. Yao [2771], graph labelings are used for incorporating redundancy in disks, designing drilling machines, creating layouts for circuit boards, and configuring resistor networks.

Terms and notation not defined below follow that used in [567] and [811].

# 2 Graceful and Harmonious Labelings

### 2.1 Trees

The Ringel-Kotzig conjecture (GTC) that all trees are graceful has been the focus of many papers. Kotzig [1031] has called the effort to prove it a "disease." Among the trees known to be graceful are: caterpillars [2136] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [1031], [2910] and [1251]; trees with diameter at most 5 [2910] and [1022]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [449], [1935], [2180]; rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity [536]; rooted trees with diameter D where every vertex has even degree except for one root and the leaves in level |D/2| [335]; rooted trees with diameter D where every vertex has even degree except for one root and the leaves, which are in level |D/2| [335]; rooted trees with diameter D where every vertex has even degree except for one root, the vertices in level |D/2| - 1, and the leaves which are in level |D/2| [335]; the graph obtained by identifying the endpoints any number of paths of a fixed length except for the case that the length has the form 4r + 1, r > 1 and the number of paths is of the form 4m with m > r [2200]; regular bamboo trees [2200] (a rooted tree consisting of branches of equal length the endpoints of which are identified with end points of stars of equal size); and olive trees [1902], [11] (a rooted tree consisting of k branches, where the *i*th branch is a path of length i); Bahls, Lake, and Wertheim [323] proved that spiders for which the lengths of every path from the center to a leaf differ by at most one are graceful. (A spider is a tree that has at most one vertex (called the *center*) of degree greater than 2.) Jampachon, Nakprasit, and Poomsa-ard [1093] provide graceful labelings for some classes of spiders. Panpa and Poomsa-ard [1891] showed that all spider graphs with at most four legs of lengths greater than one admit graceful labeling. In [1761], [1762], [1884], [1763], and [1885] Panda and Mishra and Panda, Mishra, and Dash give graceful labelings for some new classes of trees with diameter six. Pradhan and Kumar [2010] proved that all combs  $P_n \odot K_1$  with perfect matching are graceful. In [2702] Varadhan and Guruswamy give a method for combining caterpillars in a specific way such that the resulting tree is graceful. Venkatesh1 and Balasubramanian [2701] also create graceful trees by recursively merging caterpillars. In July of 2020 Gnang [895] posted a paper with a proof on arXiv of the Graceful Tree Conjecture.

Motivated by Horton's work [1020], in 2010 Fang [747] used a deterministic backtracking algorithm to prove that all trees with at most 35 vertices are graceful. In 2011 Fang [748] used a hybrid algorithm that involved probabilistic backtracking, tabu searching, and constraint programming satisfaction to verify that every tree with at most 31 vertices is harmonious. In [1690] Mahmoudzadeh and Eshghi treat graceful labelings of graphs as an optimization problem and apply an algorithm based on ant colony optimization metaheuristic to different classes of graphs and compare the results with those produced by other methods. Aldred, Širáň and Širáň [121] have proved that the number of graceful labelings of  $P_n$  grows at least as fast as  $(5/3)^n$ . They mention that this fact has an application to topological graph theory. One such application was provided by Goddyn, Richter, and and Širáň [897] who used graceful labelings of paths on 2s + 1 vertices ( $s \ge 2$ ) to obtain  $2^{2s}$  cyclic oriented triangular embeddings of the complete graph on 12s + 7 vertices. The Aldred, Širáň and Širáň bound was improved by Adamaszek [50] to  $(2.37)^n$  with the aid of a computer. Cattell [548] has shown that when finding a graceful labeling of a path one has almost complete freedom to choose a particular label *i* for any given vertex *v*. In particular, he shows that the only cases of  $P_n$  when this cannot be done are when  $n \equiv 3 \pmod{4}$  or  $n \equiv 1 \pmod{12}$ , *v* is in the smaller of the two partite sets of vertices, and i = (n-1)/2. In [2757] Wang enumerated the nonequivalent graceful trees and obtained a closed formula for the number.

Using an algorithm to run through all n! graceful graphs on n + 1 vertices Anick [173] proves that the average number of graceful labelings grows superexponentially. He provides a simple criterion to predict which trees have an exceptionally large number of graceful labelings and gives evidence that trees with an exceptionally small number of graceful labelings fall into two already known families of caterpillar graphs. Over the full set of graceful labelings for a given n, Anick shows that the distribution of vertex degrees associated with each label is very close to Poisson, with the exception of labels 0 and n. A graph is said to be k-ubiquitously graceful (also called k-rotatable) if for every vertex there is a graceful labeling which assigns that vertex the label k. He also gives two new families of trees that are not k-ubiquitously graceful and includes questions suggested by his results.

In [736] and [737] Eshghi and Azimi discuss a programming model for finding graceful labelings of large graphs. The computational results show that the models can easily solve the graceful labeling problems for large graphs. They used this method to verify that all trees with 30, 35, or 40 vertices are graceful. Stanton and Zarnke [2469] and Koh, Rogers, and Tan [1394], [1395], [1398] gave methods for combining graceful trees to yield larger graceful trees. In [2790] Wang, Yang, Hsu, and Cheng generalized the constructions of Stanton and Zarnke and Koh, Rogers, and Tan for building graceful trees from two smaller given graceful trees. Rogers in [2130] and Koh, Tan, and Rogers in [1397] provide recursive constructions to create graceful trees. Burzio and Ferrarese [520] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. and trees obtained from a graceful tree by replacing each edge with a path of fixed length is graceful.

The binomial tree  $B_0$  consists of a single vertex. The binomial tree  $B_k$  consists of two binomial trees  $B_{k-1}$  that are linked together: the root of one is the leftmost child of the root of the other. Ragukumara and Sethuraman [2049] proved that all binomial trees are graceful. Sethuraman and Murugan [2283] introduced a new method of combining graceful trees called the recursive attachment method and showed that the recursively attached tree  $T_i = T_{i-1} \oplus T^{A_{i-1}}$  is graceful for  $i \ge 1$ , where the base tree  $T_0$  is a caterpillar and the attachment tree  $T^{A_{i-1}}$  is any caterpillar. Here  $T_{i-1} \oplus T^{A_{i-1}}$  represents a tree obtained by attaching a copy of  $T^{A_{i-1}}$  at each vertex of degree at least two in  $T_{i-1}$ , for  $i \geq 1$ . Sethuraman and Murugan [2285] proved that any acyclic graph can be embedded in an unicyclic graceful graph.

It 1999 Broersma and Hoede [504] proved that an equivalent conjecture for the graceful tree conjecture is that all trees containing a perfect matching are strongly graceful (graceful with an extra condition also called an  $\alpha$ -labeling–see Section 3.1). Wang, Yang, Hsu, and Cheng [2790] showed that there exist infinitely many equivalent versions of the graceful tree conjecture (GTC). They verify these equivalent conjectures of the graceful tree conjecture are true for trees of diameter at most 7.

In 1979 Bermond [445] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a caterpillar). Morgan [1784] has shown that all lobsters with perfect matchings are graceful. Krop [1435] proved that a lobster that has a perfect matching that covers all but one vertex (i.e., that has an almost perfect matching) is graceful. Ghosh [890] used adjacency matrices to prove that three classes of lobsters are graceful. Broersma and Hoede [504] proved that if T is a tree with a perfect matching M of T such that the tree obtained from T by contracting the edges in M is caterpillar, then T is graceful. Superdock [2546] used this result to prove that all lobsters with a perfect matching are graceful. Mishra and Panda [1760] have given graceful labelings for certain lobsters. Sathiamoorthy, Natarajan, Ayyaswamy, and Janakiraman [2190] proved that the splitting graph of a caterpillar is graceful.

A Skolem sequence of order n is a sequence  $s_1, s_2, \ldots, s_{2n}$  of 2n terms such that, for each  $k \in \{1, 2, \ldots, n\}$ , there exist exactly two subscripts i(k) and j(k) with  $s_{i(k)} = s_{j(k)} = k$  and |i(k) - j(k)| = k. A Skolem sequence of order n exists if and only if  $n \equiv 0$  or 1 (mod 4). Morgan [1785] has used Skolem sequences to construct classes of graceful trees. Morgan and Rees [1786] used Skolem and Hooked-Skolem sequences to generate classes of graceful lobsters.

Mishra and Panigrahi [1764] and [1889] found classes of graceful lobsters of diameter at least five. They show other classes of lobsters are graceful in [1765] and [1766]. In [2274] Sethuraman and Jesintha [2274] explores how one can generate graceful lobsters from a graceful caterpillar while in [2278] and [2279] (see also [1115]) they show how to generate graceful trees from a graceful star. More special cases of Bermond's conjecture have been done by Ng [1850], by Wang, Jin, Lu, and Zhang [2758], Abhyanker [10], and by Mishra and Panigrahi [1765]. Renuka, Balaganesan, Selvaraju [2113] proved spider trees with n legs of even length t and odd  $n \geq 3$  and lobsters for which each vertex of the spine is adjacent to a path of length two are harmonious.

A tree in which all internal vertices have degrees r+1 except one, is called an *full r-ary* tree. A uniform full *r*-ary tree is a full *r*-ary tree in which all of its leaves are at the same level. A tree that is obtained from copies of a full *r*-ary tree by identifying each vertex of a fixed path with each vertex of the tree of degree *r* is called a *uniform-distant* tree. Suparta and Ariawan [2545] gave methods for constructing graceful classes of caterpillars, lobsters, and uniform trees that generalize results in [1795] and [2606].

Barrientos [356] defines a *y*-tree as a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point. He proves that graphs obtained from a *y*-tree T by replacing every edge  $e_i$  of T by a copy of  $K_{2,n_i}$  in such a way that the ends of  $e_i$  are merged with the two independent vertices of  $K_{2,n_i}$  after removing the edge  $e_i$  from T are graceful.

Sethuraman and Jesintha [2275], [2276], and [2277] (see also [1115]) proved that rooted trees obtained by identifying one of the end vertices adjacent to either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. They also proved that rooted trees obtained by identifying either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. In [2275], [2276], and [2277] (see also [1115] and [1127]) Sethuraman and Jesintha prove that all rooted trees in which every level contains pendent vertices and the degrees of the internal vertices in the same level are equal are graceful. Kanetkar and Sane [1322] show that trees formed by identifying one end vertex of each of six or fewer paths whose lengths determine an arithmetic progression are graceful.

Chen, Lü, and Yeh [575] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Before Sethuraman and Jesintha [2281] and [2280] (see also [1115]) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful, various kinds of bananas trees had been shown to be graceful by Bhat-Nayak and Deshmukh [457], by Murugan and Arumugam [1803], [1801] and by Vilfred [2734].

Consider a set of caterpillars, having equal diameter, in which one of the penultimate vertices has arbitrary degree and all the other internal vertices including the other penultimate vertex is of fixed even degree. Jesintha and Sethuraman [1129] call the rooted tree obtained by merging an end-vertex adjacent to the penultimate vertex of fixed even degree of each caterpillar a *arbitrarily fixed generalized banana tree*. They prove that such trees are graceful. From this it follows that all banana trees are graceful and all generalized banana trees are graceful.

Zhenbin [2913] has shown that graphs obtained by starting with any number of identical stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. He also shows that graphs obtained by starting with any two stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are graceful. In [1128] Jesintha and Sethuraman use a method of Hrnciar and Havier [1022] to generate graceful trees from a graceful star with n edges.

Aldred and McKay [120] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [925]. In a paper published in 2004 Krishnaa [1429] claims to proved that all trees have both graceful and harmonious labelings. However, her proofs were flawed.

Vietri [2728] utilized a counting technique that generalizes Rosa's graceful parity condition and provides constraints on possible graceful labelings of certain classes of trees. He expresses doubts about the validity of the graceful tree conjecture. In [2716] Vietri introduced a family of homogeneous polynomials (mod 2), one for every degree, having as many variables as the number of vertices, for any fixed graph; a so-called "graceful polynomial" that vanishes (mod 2) that may be useful for proving that the related graph is non-graceful (the degree 1 case dates back to Rosa's work). He also classified graphs whose graceful polynomials vanish for degrees 2 to 4, thereby obtaining some new non-graceful graphs.

Using a variant of the Matrix Tree Theorem, Whitty [2811] specifies an  $n \times n$  matrix of indeterminates whose determinant is a multivariate polynomial that enumerates the gracefully labeled (n + 1)-vertex trees. Whitty also gives a bijection between gracefully labelled graphs and rook placements on a chessboard on the Möbius strip. In [519] Buratti, Rinaldi, and Traetta use graceful labelings of paths to obtain a result on Hamiltonian cycle systems.

In [501] Brankovic and Wanless describe applications of graceful and graceful-like labelings of trees to several well known combinatorial problems including complete graph decompositions, the Oberwolfach problem, and coloring. They also discuss the connection between  $\alpha$ -labeling of paths and near transversals in Latin squares and show how spectral graph theory might be used to further the progress on the graceful tree conjecture.

Arkut, Arkut, and Basak [179] and Basak [228] proposed an efficient method for managing Internet Protocol (IP) networks by using graceful labelings of the nodes of the spanning caterpillars of the autonomous sub-networks to assign labels to the links in the sub-networks. Graceful labelings of trees also have been used in multi protocol label switching (MPLS) routing platforms in IP networks [180], [2415], and [2588].

Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. More specialized results about trees are contained in [445], [471], [1393], [1672], [530], [1250], and [2137]. In [706] Edwards and Howard provide a lengthy survey paper on graceful trees. Robeva [2128] provides an extensive survey of graceful labelings of trees in her 2011 undergraduate honors thesis at Stanford University. Alfalayleh, Brankovic, Giggins, and Islam [122] survey results related to the graceful tree conjecture as of 2004 and conclude with five open problems. Alfalayleh et al.: say "The faith in the [graceful tree] conjecture is so strong that if a tree without a graceful labeling were indeed found, then it probably would not be considered a tree." In his Princeton University senior thesis Superdock [2546] provided an extensive survey of results and techniques about graceful trees. He also obtained some specialized results about the gracefulness of spiders and trees with diameter 6. Arumugam and Bagga [191] discuss computational efforts aimed at verifying the graceful tree conjecture and we survey recent results on generating all graceful labelings of certain families of unicyclic graphs. Sethuraman and Murugan [2284] construct a graceful unicyclic graph G from every graceful tree T with V(G) = V(T) such that the graceful labeling of G is derived from the graceful labeling of T.

## 2.2 Cycle-Related Graphs

Cycle-related graphs have been a major focus of attention. Rosa [2136] showed that the *n*-cycle  $C_n$  is graceful if and only if  $n \equiv 0$  or 3 (mod 4) and Graham and Sloane [925] proved that  $C_n$  is harmonious if and only if n is odd. Wheels  $W_n = C_n + K_1$  are both graceful and harmonious – [795], [1018], and [925]. As a consequence we have that a subgraph of a graceful (harmonious) graph need not be graceful (harmonious). The ncone (also called the *n*-point suspension; the 1-cone is the wheel; the 2-cone is also called a double cone of  $C_m$ )  $C_m + \overline{K_n}$  has been shown to be graceful when  $m \equiv 0$  or 3 (mod 12) by Bhat-Nayak and Selvam [463]. When n is even and m is 2, 6 or 10 (mod 12)  $C_m + \overline{K_n}$ violates the parity condition for a graceful graph. Bhat-Nayak and Selvam [463] also prove that the following cones are graceful:  $C_4 + \overline{K_n}$ ,  $C_5 + \overline{K_2}$ ,  $C_7 + \overline{K_n}$ ,  $C_9 + \overline{K_2}$ ,  $C_{11} + \overline{K_n}$  and  $C_{19} + \overline{K_n}$ . The helm  $H_n$  is the graph obtained from a wheel by attaching a pendent edge at each vertex of the *n*-cycle. Helms have been shown to be graceful [212] and harmonious [894], [1618], [1619] (see also [1607], [2263], [1605], [654], and [2064]). Koh, Rogers, Teo, and Yap, [1396] define a *web* graph as one obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. They asked whether such graphs are graceful. This was proved by Kang, Liang, Gao, and Yang [1326]. Yang has extended the notion of a web by iterating the process of adding pendent points and joining them to form a cycle and then adding pendent points to the new cycle. In his notation, W(2, n) is the web graph whereas W(t, n) is the *generalized web* with t n-cycles. Yang has shown that W(3,n) and W(4,n) are graceful (see [1326]), Abhyanker and Bhat-Nayak [12] have done W(5, n) and Abhyanker [10] has done W(t,5) for  $5 \le t \le 13$ . Gnanajothi [894] has shown that webs with odd cycles are harmonious. Seoud and Youssef [2263] define a *closed helm* as the graph obtained from a helm by joining each pendent vertex to form a cycle and a *flower* as the graph obtained from a helm by joining each pendent vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A gear graph is obtained from the wheel  $W_n$  by adding a vertex between every pair of adjacent vertices of the *n*-cycle. In 1984 Ma and Feng [1675] proved all gears are graceful while in a Master's thesis in 2006 Chen [576] proved all gears are harmonious. Liu [1618] has shown that if two or more vertices are inserted between every pair of vertices of the n-cycle of the wheel  $W_n$ , the resulting graph is graceful. Sethuraman and Sankar [2291] showed that the subdivisions of wheels are graceful for even values of  $n \ge 4$ . Liu [1616] has also proved that the graph obtained from a gear graph by attaching one or more pendent edges to each vertex between the vertices of the n-cycle is graceful. Pradhan and Kumar [2010] proved that graphs obtained by adding a pendent edge to each pendent vertex of hairy cycle  $C_n \odot K_1$  are graceful if  $n \equiv 0 \pmod{4m}$ . They further provide a rule for determining the missing numbers in the graceful labeling of  $C_n \odot K_1$  and of the graph obtained by adding pendent edges to each pendent vertex of  $C_n \odot K_1$ .

Abhyanker [10] has investigated various unicyclic (that is, graphs with exactly one cycle) graphs. He proved that the unicyclic graphs obtained by identifying one vertex of  $C_4$  with the root of the olive tree with 2n branches and identifying an adjacent vertex on

 $C_4$  with the end point of the path  $P_{2n-2}$  are graceful. He showed that if one attaches any number of pendent edges to these unicyclic graphs at the vertex of  $C_4$  that is adjacent to the root of the olive tree but not adjacent to the end vertex of the attached path, the resulting graphs are graceful. Likewise, Abhyanker proved that the graph obtained by deleting the branch of length 1 from an olive tree with 2n branches and identifying the root of the edge deleted tree with a vertex of a cycle of the form  $C_{2n+3}$  is graceful. He also has a number of results similar to these. In [321] Bagga, Fotso, Max, and Arumugam investigate the gracefulness of unicyclic graphs with pendent caterpillars at two adjacent vertices of the cycle, and pendent edges at some other vertices of the cycle. In [322] Bagga and Heinz give some properties of graceful graphs obtained by adding pendent edges at each vertex of a cycle.

Delorme, Maheo, Thuillier, Koh, and Teo [658] and Ma and Feng [1674] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher, and Wegner [480], who proved various special cases. In 1985 Koh and Yap [1399] generalized this by defining a cycle with a  $P_k$ -chord to be a cycle with the path  $P_k$  joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when k = 3 and conjectured that all cycles with a  $P_k$ -chord are graceful. This was proved for k > 4 by Punnim and Pabhapote in 1987 [2041]. Chen [581] obtained the same result except for three cases which were then handled by Gao [941]. In 2005, Sethuraman and Elumalai [2270] defined a cycle with parallel  $P_k$ -chords as a graph obtained from a cycle  $C_n$   $(n \ge 6)$  with consecutive vertices  $v_0, v_1, \ldots, v_{n-1}$  by adding disjoint paths  $P_k$ ,  $(k \ge 3)$ , between each pair of nonadjacent vertices  $v_1, v_{n-1}, v_2, v_{n-2}, \ldots, v_i, v_{n-i}, \ldots, v_{\alpha}, v_{\beta}$  where  $\alpha = \lfloor n/2 \rfloor - 1$  and  $\beta = \lfloor n/2 \rfloor + 2$  if n is odd or  $\beta = \lfloor n/2 \rfloor + 1$  if n is even. They proved that every cycle  $C_n$   $(n \ge 6)$  with parallel  $P_k$ -chords is graceful for k = 3, 4, 6, 8, 6, 8and 10 and they conjecture that the cycle  $C_n$  with parallel  $P_k$ -chords is graceful for all even k. Xu [2833] proved that all cycles with a chord are harmonious except for  $C_6$  in the case where the distance in  $C_6$  between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords has also been investigated. For  $3 \leq p \leq n-r$ , let  $C_n(p,r)$  denote the *n*-cycle with consecutive vertices  $v_1, v_2, \ldots, v_n$  to which the *r* chords  $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$  have been added. Koh and Punnin [1388] and Koh, Rogers, Teo, and Yap [1396] have handled the cases r = 2, 3 and n - 3 where n is the length of the cycle. Goh and Lim [901] then proved that all remaining cases are graceful. Moreover, Ma [1677] has shown that  $C_n(p, n-p)$  is graceful when  $p \equiv 0, 3 \pmod{4}$  and Ma, Liu, and Liu [1678] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph  $C_n(3, n-3)$  any number  $k_i$  of paths of length 2 from the vertex  $v_1$  to the vertex  $v_i$  for i = 2, ..., n, the resulting graph is graceful. Chen [581] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. This generalizes the result that a cycle plus a chord is graceful. Liu [1615] has shown that the *n*-cycle with consecutive vertices  $v_1, v_2, \ldots, v_n$  to which the chords  $v_1v_k$  and  $v_1v_{k+2}$   $(2 \le k \le n-3)$  are adjointed is graceful.

For the cycle  $C_n : v_1 v_2 v_3 \cdots v_n v_1$  and a cycle with a  $C_k$  - chord Venkatesh and Sivagurunathan [2725] let  $C_{n,k}$  denote the graph obtained from  $C_n$  by adding a cycle  $C_k$  of length k between the non-adjacent vertices  $v_2$  and  $v_n$ . They define a cycle with a parallel  $C_k$  chord as the graph obtained from a cycle  $C_n$  by adding a cycle  $C_k$  of length k between every pair of non-adjacent vertices  $(v_2, v_n), (v_3, v_{n-1}), \ldots, (v_a, v_b)$  where  $a = \lfloor \frac{n}{2} \rfloor$ ,  $b = \lfloor \frac{n}{2} \rfloor + 2$ , if n is even and  $a = \lfloor \frac{n}{2} \rfloor$ ,  $b = \lfloor \frac{n}{2} \rfloor + 3$ , if n is odd. They proved that  $C_{n,4}$  and  $C_{n,4}^+$  are graceful for  $n \equiv 0 \pmod{4}$  and that  $C_{n,6}^+$  is graceful for all odd values of  $n \geq 5$ .

In [655] Deb and Limaye use the notation C(n,k) to denote the cycle  $C_n$  with k cords sharing a common endpoint called the *apex*. For certain choices of n and k there is a unique C(n,k) graph and for other choices there is more than one graph possible. They call these shell-type graphs and they call the unique graph C(n, n-3) a shell. Notice that the shell C(n, n-3) is the same as the fan  $F_{n-1} = P_{n-1} + K_1$ . Kuppusamy and Guruswamy [1446] show that the subdivision graph of  $K_{2,n}$  is graceful for  $n \ge 1$  and the subdivision graph of the shell graph C(n, n-3) is graceful for  $n \ge 4$ . Deb and Limaye define a *multiple shell* to be a collection of edge disjoint shells that have their apex in common. A multiple shell is said to be *balanced* with width w if every shell has order w or every shell has order w or w + 1. Deb and Limaye [655] have conjectured that all multiple shells are harmonious, and have shown that the conjecture is true for the balanced double shells and balanced triple shells. Yang, Xu, Xi, and Qiao [2859] proved the conjecture is true for balanced quadruple shells. Liang [1586] proved the conjecture is true when each shell has the same order and the number of copies is odd. Jeba Jesintha and Hilda [1117] define a *shell-butterfly* graph as a one-point union of two shells of any order with two pendent edges at the apex. They prove that certain shell-butterfly graphs are harmonious. Jeba Jesintha and Ezhilarasi Hilda [1116] proved butterfly graphs with one shell of order m and the other shell of order 2m + 1 are graceful and double shells in which each shell has the same order are graceful. Jeba Jesintha and Hilda [1121] define a bow graph as a double shell in which each shell has arbitrary order. A bow graph in which each shell has the same order is called a *uniform bow* graph. They prove that all uniform bow graphs are graceful. Jeba Jesintha and Ezhilarasi Hilda [1123] proved that shell-butterfly graphs are graceful.

Sethuraman and Dhavamani [2267] use H(n,t) to denote the graph obtained from the cycle  $C_n$  by adding t consecutive chords incident with a common vertex. If the common vertex is u and v is adjacent to u, then for  $k \ge 1$ ,  $n \ge 4$ , and  $1 \le t \le n-3$ , Sethuraman and Dhavamani denote by G(n,t,k) the graph obtained by taking the union of k copies of H(n,k) with the edge uv identified. They conjecture that every graph G(n,t,k) is graceful. They prove the conjecture for the case that t = n - 3.

For i = 1, 2, ..., n let  $v_{i,1}, v_{i,2}, ..., v_{i,2m}$  be the successive vertices of n copies of  $C_{2m}$ . Sekar [2200] defines a *chain of cycles*  $C_{2m,n}$  as the graph obtained by identifying  $v_{i,m}$  and  $v_{i+1,m}$  for i = 1, 2, ..., n - 1. He proves that  $C_{6,2k}$  and  $C_{8,n}$  are graceful for all k and all n. Barrientos [359] proved that all  $C_{8,n}$ ,  $C_{12,n}$ , and  $C_{6,2k}$  are graceful.

Truszczyński [2601] studied unicyclic graphs and proved several classes of such graphs are graceful. Among these are what he calls dragons. A *dragon* is formed by joining the end point of a path to a cycle (Koh, et al. [1396] call these *tadpoles*; Kim and Park [1376] call them *kites*). This work led Truszczyński to conjecture that all unicyclic graphs except  $C_n$ , where  $n \equiv 1$  or 2 (mod 4), are graceful. Guo [940] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 (mod 4). Lu [1671] uses  $C_n^{+(m,t)}$  to denote the graph obtained by identifying one vertex of  $C_n$  with one endpoint of m paths each of length t. He proves that  $C_n^{+(1,t)}$  (a tadpole) is not harmonious when a + t is odd and  $C_n^{+(2m,t)}$  is harmonious when n = 3 and when n = 2k + 1 and t = k - 1, k + 1 or 2k - 1. In his Master's thesis, Doma [684] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Guruswamy and Varadhan [942] proved that any acyclic graph can be embedded in a unicyclic graceful graph. Because of the immense diversity of unicyclic graphs, a proof of Truszczyński's conjecture seems out of reach in the near future.

Cycles that share a common edge or a vertex have received some attention. Murugan and Arumugan [1802] have shown that books with n pentagonal pages (i.e., n copies of  $C_5$  with an edge in common) are graceful when n is even and not graceful when n is odd. Lu [1671] uses  $\Theta(C_m)^n$  to denote the graph made from n copies of  $C_m$  that share an edge (an n page book with m-polygonal pages). He proves  $\Theta(C_{2m+1})^{2n+1}$  is harmonious for all m and n;  $\Theta(C_{4m+2})^{4n+1}$  and  $\Theta(C_{4m})^{4n+3}$  are not harmonious for all m and n. Xu [2833] proved that  $\Theta(C_m)^2$  is harmonious except when m = 3. ( $\Theta(C_m)^2$  is isomorphic to  $C_{2(m-1)}$ with a chord "in the middle.") Nurvazly and Sugeng [1879] proved that  $\Theta(C_3)^n$  graphs (n copies of  $C_3$  that share an edge) have graceful labelings.

A kayak paddle KP(k, m, l) is the graph obtained by joining  $C_k$  and  $C_m$  by a path of length l. Litersky [1603] proves that kayak paddles have graceful labelings in the following cases:  $k \equiv 0 \mod 4$ ,  $m \equiv 0 \text{ or } 3 \pmod{4}$ ;  $k \equiv m \equiv 2 \pmod{4}$  for  $k \ge 3$ ; and  $k \equiv 1 \pmod{4}$ ,  $m \equiv 3 \pmod{4}$ . She conjectures that KP(4k + 4, 4m + 2, l) with 2k < m is graceful when  $l \le 2m$  if l is even and when  $l \le 2m + 1$  if l is odd; and KP(10, 10, l) is graceful when  $l \ge 12$ . The cases are open: KP(4k, 4m + 1, l); KP(4k, 4m + 2, l); KP(4k + 1, 4m + 1, l); KP(4k + 1, 4m + 2, l); KP(4k + 1, 4m + 2, l); KP(4k + 2, 4m + 3, l); KP(4k + 3, 4m + 3, l).

Let  $C_n^{(t)}$  denote the one-point union of t cycles of length n. Bermond, Brouwer, and Germa [446] and Bermond, Kotzig, and Turgeon [448]) proved that  $C_3^{(t)}$  (that is, the *friendship graph* or *Dutch t-windmill*) is graceful if and only if  $t \equiv 0$  or 1 (mod 4) while Graham and Sloane [925] proved  $C_3^{(t)}$  is harmonious if and only if  $t \neq 2 \pmod{4}$ . Koh, Rogers, Lee, and Toh [1389] conjecture that  $C_n^{(t)}$  is graceful if and only if  $nt \equiv 0$  or 3 (mod 4). Yang and Lin [2851] have proved the conjecture for the case n = 5 and Yang, Xu, Xi, Li, and Haque [2857] did the case n = 7. Xu, Yang, Li and Xi [2837] did the case n = 11. Xu, Yang, Han and Li [2838] did the case n = 13. Qian [2048] verifies this conjecture for the case that t = 2 and n is even and Yang, Xu, Xi, and Li [2858] did the case n = 9. Figueroa-Centeno, Ichishima, and Muntaner-Batle [763] have shown that if  $m \equiv 0 \pmod{4}$  then the one-point union of 2, 3, or 4 copies of  $C_m$  admits a special kind of graceful labeling called an  $\alpha$ -labeling (see Section 3.1) and if  $m \equiv 2 \pmod{4}$ , then the one-point union of 2 or 4 copies of  $C_m$  admits an  $\alpha$ -labeling. Bodendiek, Schumacher, and Wegner [486] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [2320] has proved that  $C_4^{(t)}$  is graceful for all t. Seoud and Youssef [2261] have shown that the one-point union of a triangle and  $C_n$  is harmonious if and only if  $n \equiv 1 \pmod{4}$  and that if the one-point union of two cycles is harmonious then

the number of edges is divisible by 4. The question of whether this latter condition is sufficient is open. Figueroa-Centeno, Ichishima, and Muntaner-Batle [763] have shown that if G is harmonious then the one-point union of an odd number of copies of G using the vertex labeled 0 as the shared point is harmonious. Sethuraman and Selvaraju [2297] have shown that for a variety of choices of points, the one-point union of any number of non-isomorphic complete bipartite graphs is graceful. They raise the question of whether this is true for all choices of the common point.

Another class of cycle-related graphs is that of triangular cacti. The block-cutpoint graph of a graph G is a bipartite graph in which one partite set consists of the cut vertices of G, and the other has a vertex  $b_i$  for each block  $B_i$  of G. A block of a graph is a maximal connected subgraph that has no cut-vertex. A triangular cactus is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path  $v_1, v_2, \ldots, v_n$  by joining  $v_i$  and  $v_{i+1}$  to a new vertex  $w_i$  for  $i = 1, 2, \ldots, n-1$ ). Rosa [2138] conjectured that all triangular cacti with  $t \equiv 0$  or 1 (mod 4) blocks are graceful. (The cases where  $t \equiv 2$  or 3 (mod 4) fail to be graceful because of the parity condition.) Moulton [1793] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular snakes with an odd number of triangles are harmonious whereas triangular snakes with  $n \equiv 2 \pmod{4}$  triangular snakes are harmonious. Xu [2834] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to 2 (mod 4).

A double triangular snake consists of two triangular snakes that have a common path. That is, a double triangular snake is obtained from a path  $v_1, v_2, \ldots, v_n$  by joining  $v_i$  and  $v_{i+1}$  to a new vertex  $w_i$  for  $i = 1, 2, \ldots, n-1$  and to a new vertex  $u_i$  for  $i = 1, 2, \ldots, n-1$ . Xi, Yang, and Wang [2830] proved that all double triangular snakes are harmonious.

A hexagonal snake is obtained from a path  $p_1, p_2, p_3, \ldots, p_n$  by joining  $p_i, p_{i+1}$  to new vertices  $x_i$  and  $y_i$  respectively and adding edges  $x_iy_i$  for  $i = 1, 2, \ldots, n-1$  and replacing every edge with a 6-cycle; an alternate hexagonal snake is obtained from a path  $p_1, p_2, p_3, \ldots, p_n$  by joining  $p_i, p_{i+1}$  to new vertices  $x_i$  and  $y_i$  (alternatively) and adding edges  $x_iy_i$ , where  $1 \le i \le n-1$  for even n and  $1 \le i \le n-2$  for odd n and replacing each alternate edge with a 6-cycle; a double hexagonal snake is obtained from two hexagonal snakes that share the n-path; a double alternate hexagonal snake is obtained from two alternative hexagonal snakes that share the n-path. Pattabiraman, Loganathan, and Rao [1910] provided graceful labelings for double hexagonal snakes, alternate hexagonal snakes, odd alternate hexagonal snakes, and double alternate hexagonal snakes.

For any graph G defining G-snake analogous to triangular snakes, Sekar [2200] has shown that  $C_n$ -snakes are graceful when  $n \equiv 0 \pmod{4}$   $(n \geq 8)$  and when  $n \equiv 2 \pmod{4}$ 4) and the number of  $C_n$  is even. Gnanajothi [894, pp. 31-34] had earlier shown that quadrilateral snakes are graceful. Grace [923] has proved that  $K_4$ -snakes are harmonious. Rosa [2138] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases. Yu, Lee, and Chin [2891] showed that  $Q_2$ -snakes and  $Q_3$ -snakes are graceful and, when the number of blocks is greater than 1,  $Q_2$ -snakes,  $Q_3$ -snakes and  $Q_4$ -snakes are harmonious.

Barrientos [350] calls a graph a  $kC_n$ -snake if it is a connected graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to  $C_n$ . (When n > 3and k > 3 there is more than one  $kC_n$ -snake.) If a  $kC_n$ -snake where the path of minimum length that contains all the cut-vertices of the graph has the property that the distance between any two consecutive cut-vertices is  $\lfloor n/2 \rfloor$  it is called *linear*. Barrientos proves that  $kC_4$ -snakes are graceful and that the linear  $kC_6$ -snakes are graceful when k is even. He further proves that  $kC_8$ -snakes and  $kC_{12}$ -snakes are graceful in the cases where the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph are all even and that certain cases of  $kC_{4n}$ -snakes and  $kC_{5n}$ snakes are graceful (depending on the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph).

Badr [215] defines a *linear cyclic snake*  $(m, k)C_n$  as the graph consisting of k copies of  $C_n$  with two non-adjecent vertices in common where every copy has m copies of  $C_n$  and the block-cutpoint graph is not a path. He proves that the linear cyclic snakes  $(m, k)C_4$ -snake and  $(m, k)C_8$ -snake are graceful and conjectures that all the linear cyclic snakes  $(m, k)C_n$ -snakes are graceful for  $n \equiv 0 \pmod{4}$  or  $n \equiv 3 \pmod{4}$ .

Several people have studied cycles with pendent edges attached. Frucht [795] proved that any cycle with a pendent edge attached at each vertex (i.e., a *crown*) is graceful (see also [1028]). If G has order n, the *corona of* G with H,  $G \odot H$  is the graph obtained by taking one copy of G and n copies of H and joining the *i*th vertex of G with an edge to every vertex in the *i*th copy of H. Barrientos [355] also proved: if G is a graceful graph of order m and size m - 1, then  $G \odot nK_1$  and  $G + nK_1$  are graceful; if G is a graceful graph of order p and size q with q > p, then  $(G \cup (q + 1 - p)K_1) \odot nK_1$  is graceful; and all unicyclic graphs, other than a cycle, for which the deletion of any edge from the cycle results in a caterpillar are graceful.

For a given cycle  $C_n$  with  $n \equiv 0$  or 3 (mod 4) and a family of trees  $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ , let  $u_i$  and  $v_i, 1 \leq i \leq n$ , be fixed vertices of  $C_n$  and  $T_i$ , respectively. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [768] provide two construction methods that generate a graceful labeling of the unicyclic graphs obtained from  $C_n$  and  $\mathcal{T}$  by amalgamating them at each  $u_i$  and  $v_i$ . Their results encompass all previously known results for unicyclic graphs whose cycle length is 0 or 3 (mod 4) and considerably extend the known classes of graceful unicyclic graphs. Khairunnisa and Sugeng [1361] let  $A_{(m,n)}$  denote the graph obtained from  $C_m$  by connecting each two adjacent vertices with  $P_{n+1}$ . They prove that the graphs  $A_{(3,1)} \odot \overline{K_r}$  are graceful.

In [352] Barrientos proved that helms (graphs obtained from a wheel by attaching one pendent edge to each vertex) are graceful. Grace [922] showed that an odd cycle with one or more pendent edges at each vertex is harmonious and conjectured that  $C_{2n} \odot K_1$ , an even cycle with one pendent edge attached at each vertex, is harmonious. This conjecture has been proved by Liu and Zhang [1606], Liu [1618] and [1619], Hegde [980], Huang [1030], and Bu [507]. Sekar [2200] has shown that the graph  $C_m \odot P_n$  obtained by attaching the path  $P_n$  to each vertex of  $C_m$  is graceful. For any  $n \ge 3$  and any t with  $1 \le t \le n$ , let  $C_n^{+t}$  denote the class of graphs formed by adding a single pendent edge to t vertices of a cycle of length n. Ropp [2135] proved that for every n and t the class  $C_n^{+t}$  contains a graceful graph. Gallian and Ropp [811] conjectured that for all n and t, all members of  $C_n^{+t}$  are graceful. This was proved by Qian [2048] and by Kang, Liang, Gao, and Yang [1326]. Of course, such graphs are just a special case of the aforementioned conjecture of Truszczyński that all unicyclic graphs except  $C_n$  for  $n \equiv 1$  or 2 (mod 4) are graceful. Sekar [2200] proved that the graph obtained by identifying an endpoint of a star with a vertex of a cycle is graceful. Lu [1671] shows that the graph obtained by identifying each vertex of an odd cycle with a vertex disjoint copy of  $C_{2m+1}$  is harmonious if and only if m is odd. Sudha [2482] proved that the graphs obtained by starting with two or more copies of  $C_4$  and identifying a vertex of the  $i^{th}$  copy with a vertex of the  $i + 1^{th}$  copy are graceful. Sudha and Kanniga [2489] proved that the graphs obtained by identifying any vertex of  $C_m$  with any vertex of degree 1 of  $S_n$  where  $n = \lceil (m-1)/2 \rceil$  are graceful.

For a given cycle  $C_n$  with  $n \equiv 0$  or 3 (mod 4) and a family of trees  $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ , let  $u_i$  and  $v_i, 1 \leq i \leq n$ , be fixed vertices of  $C_n$  and  $T_i$ , respectively. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [768] provide two construction methods that generate a graceful labeling of the unicyclic graphs obtained from  $C_n$  and  $\mathcal{T}$  by amalgamating them at each  $u_i$  and  $v_i$ . Their results encompass all previously known results for unicyclic graphs whose cycle length is 0 or 3 (mod 4) and considerably extend the known classes of graceful unicyclic graphs.

Solairaju and Chithra [2440] defined three classes of graphs obtained by connecting copies of  $C_4$  in various ways. Denote the four consecutive vertices of *i*th copy of  $C_4$  by  $v_{i,1}, v_{i,2}, v_{i,3}, v_{i_4}$ . They show that the graphs obtained by identifying  $v_{i,4}$  with  $v_{i+1,2}$  for  $i = 1, 2, \ldots, n-1$  is graceful; the graphs obtained by joining  $v_{i,4}$  with  $v_{i+1,2}$  for  $i = 1, 2, \ldots, n-1$  by an edge is graceful; and the graphs obtained by joining  $v_{i,4}$  with  $v_{i+1,2}$  for  $i = 1, 2, \ldots, n-1$  with a path of length 2 is graceful.

Venkatesh [2721] showed that for positive integers m and n divisible by 4 the graphs obtained by appending a copy of  $C_n$  to each vertex of  $C_m$  by identifying one vertex of  $C_n$  with each vertex of  $C_m$  is graceful.

## 2.3 Product Related Graphs

Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids,  $P_m \times P_n$   $(m, n \ge 2)$ , (some authors use  $G \square H$  to denote the Cartesian product of G and H) are graceful was proved by Acharya and Gill [36] in 1978 although the much simpler labeling scheme given by Maheo [1687] in 1980 for  $P_m \times P_2$ readily extends to all grids. Liu, T. Zou, Y. Lu [1613] proved  $P_m \times P_n \times P_2$  is graceful. In 1980 Graham and Sloane [925] proved ladders,  $P_m \times P_2$ , are harmonious when m > 2 and in 1992 Jungreis and Reid [1266] showed that the grids  $P_m \times P_n$  are harmonious when  $(m, n) \neq (2, 2)$ . A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [1344] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful [1346]. Barrientos and Minion [386] showed that a graceful graph is obtained when every step of a ladder is subdivided an even number of times. In addition, they proved that when each edge of a ladder is subdivided exactly once, the resulting graph is graceful.

Acharya [25] has shown that certain subgraphs of grid graphs are graceful. Lee [1487] defines a *Mongolian tent* as a graph obtained from  $P_m \times P_n$ , *n* odd, by adding one extra vertex above the grid and joining every other vertex of the top row of  $P_m \times P_n$  to the new vertex. A *Mongolian village* is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A *Young tableau* is a subgraph of  $P_m \times P_n$  obtained by retaining the first two rows of  $P_m \times P_n$  and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and Ng [1511] have proved that all Young tableaus are graceful. Lee [1487] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful. In [2439] and [2438] Solairaju and Arockiasamy prove that various families of subgraphs of grids  $P_m \times P_n$  are graceful. Sudha [2482] proved that certain subgraphs of the grid  $P_n \times P_2$  are graceful. Knuth [1383] proved that  $K_n \times P_3$  is graceful if and only if  $n \leq 6$ .

Prisms are graphs of the form  $C_m \times P_n$ . These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher, and Wegner [480] proved that  $C_m \times P_2$  is graceful when  $m \equiv 0 \pmod{4}$ . According to the survey by Bermond [445], Gangopadhyay and Rao Hebbare did the case that m is even about the same time. In a 1979 paper, Frucht [795] stated without proof that he had done all  $C_m \times P_2$ . A complete proof of all cases and some related results were given by Frucht and Gallian [798] in 1988.

In 1992 Jungreis and Reid [1266] proved that all  $C_m \times P_n$  are graceful when m and n are even or when  $m \equiv 0 \pmod{4}$ . They also investigated the existence of a stronger form of graceful labeling called an  $\alpha$ -labeling (see Section 3.1) for graphs of the form  $P_m \times P_n$ ,  $C_m \times P_n$ , and  $C_m \times C_n$  (see also [813]).

Yang and Wang have shown that the prisms  $C_{4n+2} \times P_{4m+3}$  [2856],  $C_n \times P_2$  [2854], and  $C_6 \times P_m$   $(m \ge 2)$  (see [2856]) are graceful. Singh [2394] proved that  $C_3 \times P_n$  is graceful for all n. In their 1980 paper Graham and Sloane [925] proved that  $C_m \times P_n$  is harmonious when n is odd and they used a computer to show  $C_4 \times P_2$ , the cube, is not harmonious. In 1992 Gallian, Prout, and Winters [816] proved that  $C_m \times P_2$  is harmonious when  $m \ne 4$ . In 1992, Jungreis and Reid [1266] showed that  $C_4 \times P_n$  is harmonious when  $n \ge 3$ . Huang and Skiena [1032] have shown that  $C_m \times P_n$  is graceful for all n when m is even and for all n with  $3 \le n \le 12$  when m is odd. Abhyanker [10] proved that the graphs obtained from  $C_{2m+1} \times P_5$  by adding a pendent edge to each vertex of an outer cycle is graceful.

Torus grids are graphs of the form  $C_m \times C_n$  (m > 2, n > 2). Very little success has been achieved with these graphs. The graceful parity condition is violated for  $C_m \times C_n$ when m and n are odd and the harmonious parity condition [925, Theorem 11] is violated for  $C_m \times C_n$  when  $m \equiv 1, 2, 3 \pmod{4}$  and n is odd. In 1992 Jungreis and Reid [1266] showed that  $C_m \times C_n$  is graceful when  $m \equiv 0 \pmod{4}$  and n is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout, and Winters [816] proved that all prisms  $C_m \times P_2$  with a single vertex deleted or single edge deleted are graceful and harmonious. The *Möbius ladder*  $M_n$  is the graph obtained from the ladder  $P_n \times P_2$  by joining the opposite end points of the two copies of  $P_n$ . In 1989 Gallian [810] showed that all Möbius ladders are graceful and all but  $M_3$  are harmonious. Ropp [2135] has examined two classes of prisms with pendent edges attached. He proved that all  $C_m \times P_2$  with a single pendent edge at each vertex are graceful and all  $C_m \times P_2$  with a single pendent edge at each vertex are graceful. Ramachandran and Sekar [2077] proved that the graph obtained from the ladder  $L_n$  ( $P_n \times P_2$ ) by identifying one vertex of  $L_n$  with any vertex of the star  $S_m$  other than the center of  $S_m$  is graceful.

Another class of cartesian products that has been studied is that of books and "stacked" books. The book  $B_m$  is the graph  $S_m \times P_2$  where  $S_m$  is the star with m edges. In 1980 Maheo [1687] proved that the books of the form  $B_{2m}$  are graceful and conjectured that the books  $B_{4m+1}$  were also graceful. (The books  $B_{4m+3}$  do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [657] in 1980. Maheo [1687] also proved that  $L_n \times P_2$  and  $B_{2m} \times P_2$  are graceful. Both Grace [921] and Reid (see [815]) have given harmonious labelings for  $B_{2m}$ . The books  $B_{4m+3}$  do not satisfy the harmonious parity condition [925, Theorem 11]. Gallian and Jungreis [815] conjectured that the books  $B_{4m+1}$  are harmonious. Gnanajothi [894] has verified this conjecture by showing  $B_{4m+1}$  has an even stronger form of labeling – see Section 4.1. Liang [1582] also proved the conjecture. In 1988 Gallian and Jungreis [815] defined a stacked book as a graph of the form  $S_m \times P_n$ . They proved that the stacked books of the form  $S_{2m} \times P_n$  are graceful and posed the case  $S_{2m+1} \times P_n$  as an open question. The *n*-cube  $K_2 \times K_2 \times \cdots \times K_2$  (*n* copies) was shown to be graceful by Kotzig [1416]—see also [1687]. Although Graham and Sloane [925] used a computer in 1980 to show that the 3-cube is not harmonious (see also [1890]), Ichishima and Oshima [1059] proved that the *n*-cube  $Q_n$  has a stronger form of harmonious labeling called an  $\alpha$ -labeling (see Section 3.1) for  $n \geq 4$ .

In 1986 Reid [2110] found a harmonious labeling for  $K_4 \times P_n$ . In 2003 Petrie and Smith [1920] investigated graceful labelings of graphs as an exercise in constraint programming satisfaction. They determined that  $K_n \times P_2$  is graceful for n = 3, 4 and 5;  $K_4 \times P_3$  is graceful;  $K_4 \times C_3$  is graceful;  $(C_n \cup C_n) + K_1$  (double wheel) is graceful for n = 4 and 5; and  $(C_3 \cup C_3) + K_1$  is not graceful. That  $K_3 \times K_3$  is not graceful follows from the parity condition given in the introduction. Using significantly better methods in 2010, Smith and Puget obtained the results about graceful labelings for  $K_m \times K_1$ ,  $K_m \times P_n$ , and  $K_m \times C_n$ given in Table 1. Their labeling for  $K_5 \times P_2$  and  $K_6 \times P_3$  are the unique graceful labelings for those graphs. Redl [2109] proved that  $K_4 \times P_n$  is graceful for n = 1, 2, 3, 4, and 5 using a constraint programming approach and asked if all graphs of the form  $K_4 \times P_n$  are graceful

Vaidya, Kaneria, Srivastav, and Dani [2644] proved that  $P_n \cup P_t \cup (P_r \times P_s)$  where  $t < \min\{r, s\}$  and  $P_n \cup P_t \cup K_{r,s}$  where  $t \leq \min\{r, s\}$  and  $r, s \geq 3$  are graceful. Kaneria, Vaidya, Ghodasara, and Srivastav [1318] proved  $K_{mn} \cup (P_r \times P_s)$  where m, n, r, s > 1;

 $(P_r \times P_s) \cup P_t$  where r, s > 1 and  $t \neq 2$ ; and  $K_{mn} \cup (P_r \times P_s) \cup P_t$  where m, n, r, s > 1 and  $t \neq 2$  are graceful.

The composition  $G_1[G_2]$  is the graph having vertex set  $V(G_1) \times V(G_2)$  and edge set  $\{(x_1, y_1), (x_2, y_2) | x_1 x_2 \in E(G_1) \text{ or } x_1 = x_2 \text{ and } y_1 y_2 \in E(G_2)\}$ . The symmetric product  $G_1 \oplus G_2$  of graphs  $G_1$  and  $G_2$  is the graph with vertex set  $V(G_1) \times V(G_2)$  and edge set  $\{(x_1, y_1), (x_2, y_2) | x_1 x_2 \in E(G_1) \text{ or } y_1 y_2 \in E(G_2) \text{ but not both}\}$ . Seoud and Youssef [2262] have proved that  $P_n \oplus \overline{K_2}$  is graceful when n > 1 and  $P_n[P_2]$  is harmonious for all n. They also observe that the graphs  $C_m \oplus C_n$  and  $C_m[C_n]$  violate the parity conditions for graceful and harmonious graphs when m and n are odd.

## 2.4 Complete Graphs

The questions of the gracefulness and harmoniousness of the complete graphs  $K_n$  have been answered. In each case the answer is positive if and only if  $n \leq 4$  ([902], [2390], [925], [451]). Both Rosa [2136] and Golomb [902] proved that the complete bipartite graphs  $K_{m,n}$ are graceful while Graham and Sloane [925] showed they are harmonious if and only if m or n = 1. Aravamudhan and Murugan [178] have shown that the complete tripartite graph  $K_{1,m,n}$  is both graceful and harmonious while Gnanajothi [894, pp. 25–31] has shown that  $K_{1,1,m,n}$  is both graceful and harmonious and  $K_{2,m,n}$  is graceful. Some of the same results have been obtained by Seoud and Youssef [2257] who also observed that when m, n, and p are congruent to 2 (mod 4),  $K_{m,n,p}$  violates the parity conditions for harmonious graphs. Beutner and Harborth [451] give graceful labelings for  $K_{1,m,n}, K_{2,m,n}, K_{1,1,m,n}$ and conjecture that these and  $K_{m,n}$  are the only complete multipartite graphs that are graceful. They have verified this conjecture for graphs with up to 23 vertices via computer.

Beutner and Harborth [451] also show that  $K_n - e$  ( $K_n$  with an edge deleted) is graceful only if  $n \leq 5$ ; any  $K_n - 2e$  ( $K_n$  with two edges deleted) is graceful only if  $n \leq 6$ ; and any  $K_n - 3e$  is graceful only if  $n \leq 6$ . They also determine all graceful graphs of the form  $K_n - G$  where G is  $K_{1,a}$  with  $a \leq n - 2$  and where G is a matching  $M_a$  with  $2a \leq n$ .

The windmill graph  $K_n^{(m)}$  (n > 3) consists of m copies of  $K_n$  with a vertex in common. A necessary condition for  $K_n^{(m)}$  to be graceful is that  $n \le 5$  – see [1396]. Bermond [445] has conjectured that  $K_4^{(m)}$  is graceful for all  $m \ge 4$ . The gracefulness of  $K_4^{(m)}$  is equivalent to the existence of a (12m + 1, 4, 1)-perfect difference family, which are known to exit for  $m \le 1000$  (see [1032], [5], [2797], and [863]). Bermond, Kotzig, and Turgeon [448] proved that  $K_n^{(m)}$  is not graceful when n = 4 and m = 2 or 3, and when m = 2 and n = 5. In 1982 Hsu [1025] proved that  $K_4^{(m)}$  is harmonious for all m. Graham and Sloane [925] conjectured that  $K_n^{(2)}$  is harmonious if and only if n = 4. They verified this conjecture for the cases that n is odd or n = 6. Liu [1605] has shown that  $K_n^{(2)}$  is not harmonious if  $n = 2^a p_1^{a_1} \cdots p_s^{a_s}$  where  $a, a_1, \ldots, a_s$  are positive integers and  $p_1, \ldots, p_s$  are distinct odd primes and there is a j for which  $p_j \equiv 3 \pmod{4}$  and  $3n = 4^e(8k + 7)$  or  $n \equiv 5 \pmod{8}$ . Koh, Rogers, Lee, and Toh [1389] and Rajasingh and Pushpam [2065] have shown that  $K_{m,n}^{(t)}$ , the one-point union of t copies of  $K_{m,n}$ , is graceful. Sethuraman and Selvaraju [2293] have proved that the one-point union of graphs of the form  $K_{2,m_i}$  for i = 1, 2, ..., n, where the union is taken at a vertex from the partite set with exactly 2 vertices is graceful if at most two of the  $m_i$  are equal. They conjecture that the restriction that at most two of the  $m_i$  are equal is not necessary. Sudha [2483] proved that two or more complete bipartite graphs having one bipartite vertex set in common are graceful. Mitra and Bhoumik [1768] proved that  $K_{2n,2n} \odot K_2$  is graceful.

Koh, Rogers, Lee, and Toh [1396] introduced the notation B(n, r, m) for the graph consisting of m copies of  $K_n$  with a  $K_r$  in common  $(n \ge r)$ . (We note that Guo [941] has used the notation B(n, r, m) to denote the graph obtained by joining opposite endpoints of three disjoint paths of lengths n, r and m.) Bermond [445] raised the question: "For which m, n, and r is B(n, r, m) graceful?" Of course, the case r = 1 is the same as  $K_n^{(m)}$ . For r > 1, B(n, r, m) is graceful in the following cases: n = 3, r = 2,  $m \ge 1$  [1390];  $n = 4, r = 2, m \ge 1$  [657];  $n = 4, r = 3, m \ge 1$  (see [445]), [1390]. Seoud and Youssef [2257] have proved B(3,2,m) and B(4,3,m) are harmonious. Liu [1604] has shown that if there is a prime p such that  $p \equiv 3 \pmod{4}$  and p divides both n and n-2 and the highest power of p that divides n and n-2 is odd, then B(n,2,2) is not graceful. Smith and Puget [2431] has shown that up to symmetry, B(5,2,2) has a unique graceful labeling; B(n, 3, 2) is not graceful for n = 6, 7, 8, 9, and 10; B(6, 3, 3) and B(7, 3, 3) are not graceful; and B(5,3,3) is graceful. Combining results of Bermond and Farhi [447] and Smith and Puget [2431] show that B(n, 2, 2) is not graceful for n > 5. Lu [1671] obtained the following results: B(m, 2, 3) and B(m, 3, 3) are not harmonious when  $m \equiv 1$ (mod 8); B(m, 4, 2) and B(m, 5, 2) are not harmonious when m satisfies certain special conditions; B(m, 1, n) is not harmonious when  $m \equiv 5 \pmod{8}$  and  $n \equiv 1, 2, 3 \pmod{4}$ ;  $B(2m+1, 2m, 2n+1) \cong K_{2m} + \overline{K_{2n+1}}$  is not harmonious when  $m \equiv 2 \pmod{4}$ .

More generally, Bermond and Farhi [447] have investigated the class of graphs consisting of m copies of  $K_n$  having exactly k copies of  $K_r$  in common. They proved such graphs are not graceful for n sufficiently large compared to r. Barrientos [356] proved that the graph obtained by performing the one-point union of any collection of the complete bipartite graphs  $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_t,n_t}$ , where each  $K_{m_i,n_i}$  appears at most twice and  $gcd(n_1, n_2, \ldots, n_t) = 1$ , is graceful.

Sethuraman and Elumalai [2269] have shown that  $K_{1,m,n}$  with a pendent edge attached to each vertex is graceful and Jirimutu [1256] has shown that the graph obtained by attaching a pendent edge to every vertex of  $K_{m,n}$  is graceful (see also [148]). In [2282] Sethuraman and Kishore determine the graceful graphs that are the union of n copies of  $K_4$  with i edges deleted for  $1 \le i \le 5$  and with one edge in common. The only cases that are not graceful are those graphs where the members of the union are  $C_4$  for  $n \equiv 3$  (mod 4) and where the members of the union are  $P_2$ . They conjecture that these two cases are the only instances of edge induced subgraphs of the union of n copies of  $K_4$  with one edge in common that are not graceful.

Renuka, Balaganesan, Selvaraju [2113] proved the graphs obtained by joining a vertex of  $K_{1,m}$  to a vertex of  $K_{1,n}$  by a path are harmonious. Sethuraman and Selvaraju [2299] have shown that union of any number of copies of  $K_4$  with an edge deleted and one edge in common is harmonious.

Clemens, Coulibaly, Garvens, Gonnering, Lucas, and Winters [627] investigated the gracefulness of the one-point and two-point unions of graphs. They show the following graphs are graceful: the one-point union of an end vertex of  $P_n$  and  $K_4$ ; the graph obtained by taking the one-point union of  $K_4$  with one end vertex of  $P_n$  and the one-point union of the other end vertex of  $P_n$  with the central vertex of  $K_{1,r}$ ; the graph obtained by taking the one-point union of  $K_4$  with one end vertex of  $P_n$  and the one-point union of the other end of  $P_n$  with a vertex from the partite set of order 2 of  $K_{2,r}$ ; the graph obtained from the graph just described by appending any number of edges to the other vertex of the partite set of order 2; the two-point union of the two vertices of the partite set of order 2 in  $K_{2,r}$  and two vertices from  $K_4$ ; and the graph obtained from the graph just described by appending any number of edges to one of the vertices from the partite set of order 2.

A Golomb ruler is a marked straightedge such that the distances between different pairs of marks on the straightedge are distinct. If the set of distances between marks is every positive integer up to and including the length of the ruler, then ruler is a called a *perfect Golomb ruler*. Golomb [902] proved that perfect Golomb rulers exist only for rulers with at most 4 marks. Beavers [426] examines the relationship between Golomb rulers and graceful graphs through a correspondence between rulers and complete graphs. He proves that  $K_n$  is graceful if and only if there is a perfect Golomb ruler with n marks and Golomb rulers are equivalent to complete subgraphs of graceful graphs.

### 2.5 Disconnected Graphs

There have been many papers dealing with graphs that are not connected. For any graph G the graph mG denotes the disjoint union of m copies of G. In 1975 Kotzig [1415] investigated the gracefulness of the graphs  $rC_s$ . When  $rs \equiv 1$  or 2 (mod 4), these graphs violate the gracefulness parity condition. Kotzig proved that when r = 3 and 4k > 4, then  $rC_{4k}$  has a stronger form of graceful labeling called  $\alpha$ -labeling (see §3.1) whereas when  $r \geq 2$  and s = 3 or 5,  $rC_s$  is not graceful. In 1984 Kotzig [1417] once again investigated the gracefulness of  $rC_s$  as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of  $rC_s$ . M. He [968] has shown that graphs of the form  $2C_{2m}$  and graphs obtained by connecting two copies of  $C_{2m}$  with an edge are graceful. Cahit [533] has shown that  $rC_s$  is harmonious when r and s are odd and Seoud, Abdel Maqsoud, and Sheehan [2223] noted that when ror s is even,  $rC_s$  is not harmonious. Seoud, Abdel Maqsoud, and Sheehan [2223] proved that  $C_n \cup C_{n+1}$  is harmonious if and only if  $n \ge 4$ . They conjecture that  $C_3 \cup C_{2n}$  is harmonious when  $n \geq 3$ . This conjecture was proved when Yang, Lu, and Zeng [2852] showed that all graphs of the form  $C_{2j+1} \cup C_{2n}$  are harmonious except for (n, j) = (2, 1). As a consequence of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [767] have that  $C_n \cup C_3$  is harmonious if and only if  $n \ge 6$  and n is even. Renuka, Balaganesan, Selvaraju [2113] proved that for odd  $n \ C_n \cup P_3$  and  $C_n \odot \overline{K_m} \cup P_3$  are harmonious. Youssef [2873] has shown that if G is harmonious then mG is harmonious for all odd m.

In 1978 Kotzig and Turgeon [1420] proved that  $mK_n$  is graceful if and only if m = 1and  $n \leq 4$ . Liu and Zhang [1607] have shown that  $mK_n$  is not harmonious for n odd and  $m \equiv 2 \pmod{4}$  and is harmonious for n = 3 and m odd. They conjecture that  $mK_3$  is not harmonious when  $m \equiv 0 \pmod{4}$ . Bu and Cao [508] give some sufficient conditions for the gracefulness of graphs of the form  $K_{m,n} \cup G$  and they prove that  $K_{m,n} \cup P_t$  and the disjoint union of complete bipartite graphs are graceful under some conditions.

Recall a Skolem sequence of order n is a sequence  $s_1, s_2, \ldots, s_{2n}$  of 2n terms such that, for each  $k \in \{1, 2, \ldots, n\}$ , there exist exactly two subscripts i(k) and j(k) with  $s_{i(k)} = s_{j(k)} = k$  and |i(k) - j(k)| = k. (A Skolem sequence of order n exists if and only if  $n \equiv 0$  or 1 (mod 4)). Abrham [14] has proved that any graceful 2-regular graph of order  $n \equiv 0 \pmod{4}$  in which all the component cycles are even or of order  $n \equiv 3 \pmod{4}$ , with exactly one component an odd cycle, can be used to construct a Skolem sequence of order n + 1. Also, he showed that certain special Skolem sequences of order n can be used to generate graceful labelings on certain 2-regular graphs.

The graph  $H_n$  obtained from the cycle with consecutive vertices  $u_1, u_2, \ldots, u_n$   $(n \ge 6)$ by adding the chords  $u_2u_n, u_3u_{n-1}, \ldots, u_\alpha u_\beta$ , where  $\alpha = (n-1)/2$  for all n and  $\beta = (n-1)/2 + 3$  if n is odd or  $\beta = n/2 + 2$  if n is even is called the *cycle with parallel chords*. In Elumalai and Sethuraman [712] prove the following: for odd  $n \ge 5$ ,  $H_n \cup K_{p,q}$ is graceful; for even  $n \ge 6$  and m = (n-2)/2 or m = n/2  $H_n \cup K_{1,m}$  is graceful; for  $n \ge 6$ ,  $H_n \cup P_m$  is graceful, where m = n or n-2 depending on  $n \equiv 1$  or 3 (mod 4) or  $m \equiv n-1$  or n-3 depending on  $n \equiv 0$  or 2 (mod 4). Elumali and Sethuraman [714] proved that every *n*-cycle  $(n \ge 6)$  with parallel chords is graceful and every *n*-cycle with parallel  $P_k$ -chords of increasing lengths is graceful for  $n = 2 \pmod{4}$  with  $1 \le k \le (\lfloor n/2 \rfloor - 1)$ .

In 1985 Frucht and Salinas [799] conjectured that  $C_s \cup P_n$  is graceful if and only if  $s + n \ge 6$  and proved the conjecture for the case that s = 4. The conjecture was proved by Traetta [2595] in 2012 who used his result to get a complete solution to the well known two-table Oberwolfach problem; that is, given odd number of people and two round tables when is it possible to arrange series of seatings so that each person sits next to each other person exactly once during the series. The *t*-table Oberwolfach problem  $OP(n_1, n_2, \ldots, n_t)$  asks to arrange a series of meals for an odd number  $n = \sum n_i$  of people around *t* tables of sizes  $n_1, n_2, \ldots, n_t$  so that each person sits next to each other exactly once. A solution to  $OP(n_1, n_2, \ldots, n_t)$  is a 2-factorization of  $K_n$  whose factors consists of *t* cycles of lengths  $n_1, n_2, \ldots, n_t$ . The  $\lambda$ -fold Oberwolfach problem  $OP_{\lambda}(n_1, n_2, \ldots, n_t)$  refers to the case where  $K_n$  is replaced by  $\lambda K_n$ . Traetta used his proof of the Frucht and Salinas conjecture to provide a complete solutions to both OP(2r + 1, 2s) and OP(2r + 1, s, s), except possibly for OP(3, s, s). He also gave a complete solution of the general  $\lambda$ -fold Oberwolfach problem  $OP_{\lambda}(r, s)$ .

Seoud and Youssef [2264] have shown that  $K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q}$   $(m, n, p, q \ge 2), K_{m,n} \cup K_{p,q} \cup K_{r,s}$   $(m, n, p, q, r, s \ge 2, (p, q) \ne (2, 2))$ , and  $pK_{m,n}$   $(m, n \ge 2, (m, n) \ne (2, 2))$  are graceful. They also prove that  $C_4 \cup K_{1,n}$   $(n \ne 2)$  is not graceful whereas Choudum and Kishore [602], [1381] have proved that  $C_s \cup K_{1,n}$  is graceful for  $s \ge 7$  and  $n \ge 1$ . Lee, Quach, and Wang [1527] established the gracefulness of  $P_s \cup K_{1,n}$ . Seoud and Wilson [2256] have shown that  $C_3 \cup K_4, C_3 \cup C_3 \cup K_4$ , and certain graphs of the form  $C_3 \cup P_n$ 

and  $C_3 \cup C_3 \cup P_n$  are not graceful. Abrham and Kotzig [19] proved that  $C_p \cup C_q$  is graceful if and only if  $p + q \equiv 0$  or 3 (mod 4). Zhou [2917] proved that  $K_m \cup K_n$  (n > 1, m > 1) is graceful if and only if  $\{m, n\} = \{4, 2\}$  or  $\{5, 2\}$ . Knuth [1383] used a computer to show that  $K_5 \cup K_2$  has a unique graceful labeling up to a complement. (C. Barrientos has called to my attention that  $K_1 \cup K_n$  is graceful if and only if n = 3 or 4.) Shee [2319] has shown that graphs of the form  $P_2 \cup C_{2k+1}$  (k > 1),  $P_3 \cup C_{2k+1}$ ,  $P_n \cup C_3$ , and  $S_n \cup C_{2k+1}$  all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [458] have shown that  $C_{4t} \cup K_{1,4t-1}$  and  $C_{4t+3} \cup K_{1,4t+2}$  are graceful. Section 3.1 includes numerous families of disconnected graphs that have a stronger form of graceful labelings.

For m = 2p + 3 or 2p + 4, Wang, Liu, and Li [2783] proved the following graphs are graceful:  $W_m \cup K_{n,p}$  and  $W_{m,2m+1} \cup K_{n,p}$ ; for  $n \ge m$ ,  $W_{m,2m+1} \cup K_{1,n}$ ; for m = 2n + 5,  $W_{m,2m+1} \cup (C_3 + \overline{K_n})$ . If  $G_p$  is a graceful graph with p edges, they proved  $W_{2p+3} \cup G_p$  is graceful.

In considering graceful labelings of the disjoint unions of two or three stars  $S_e$  with e edges Yang and Wang [2855] permitted the vertex labels to range from 0 to e + 1 and 0 to e + 2, respectively. With these definitions of graceful, they proved that  $S_m \cup S_n$  is graceful if and only if m or n is even and that  $S_m \cup S_n \cup S_k$  is graceful if and only if at least one of m, n, or k is even (m > 1, n > 1, k > 1).

Seoud and Youssef [2260] investigated the gracefulness of specific families of the form  $G \cup K_{m,n}$ . They obtained the following results:  $C_3 \cup K_{m,n}$  is graceful if and only if  $m \ge 2$ and  $n \ge 2$ ;  $C_4 \cup K_{m,n}$  is graceful if and only if  $(m, n) \ne (1, 1)$ ;  $C_7 \cup K_{m,n}$  and  $C_8 \cup K_{m,n}$ are graceful for all m and n;  $mK_3 \cup nK_{1,r}$  is not graceful for all m, n and r;  $K_i \cup K_{m,n}$ is graceful for  $i \le 4$  and  $m \ge 2, n \ge 2$  except for i = 2 and (m, n) = (2, 2);  $K_5 \cup K_{1,n}$ is graceful for all n;  $K_6 \cup K_{1,n}$  is graceful if and only if n is not 1 or 3. Youssef [2875] completed the characterization of the graceful graphs of the form  $C_n \cup K_{p,q}$  where  $n \equiv 0$ or 3 (mod 4) by showing that for n > 8 and  $n \equiv 0$  or 3 (mod 4),  $C_n \cup K_{p,q}$  is graceful for all p and q (see also [354]). Note that when  $n \equiv 1$  or 2 (mod 4) certain cases of  $C_n \cup K_{p,q}$ violate the parity condition for gracefulness.

For i = 1, 2, ..., m let  $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$  be a 4-cycle. Yang and Pan [2850] define  $F_{k,4}$  to be the graph obtained by identifying  $v_{i,3}$  and  $v_{i+1,1}$  for i = 1, 2, ..., k-1. They prove that  $F_{m_{1,4}} \cup F_{m_{2,4}} \cup \cdots \cup F_{m_{n,4}}$  is graceful for all n. Pan and Lu [1883] have shown that  $(P_2 + \overline{K_n}) \cup K_{1,m}$  and  $(P_2 + \overline{K_n}) \cup T_n$  are graceful.

Barrientos [354] has shown the following graphs are graceful:  $C_6 \cup K_{1,2n+1}$ ;  $\bigcup_{i=1}^t K_{m_i,n_i}$  for  $2 \leq m_i < n_i$ ; and  $C_m \cup \bigcup_{i=1}^t K_{m_i,n_i}$  for  $2 \leq m_i < n_i, m \equiv 0$  or  $3 \pmod{4}, m \geq 11$ . In [1302] Kaneria, Makadia, and Viradia proved that the union of three grid graphs,  $\bigcup_{l=1}^3 (P_{m_l} \times P_{n_l})$ , is graceful, the union of finitely many copies of  $P_m \times P_n$  is graceful, and provided two new graceful labeling for  $P_m \times P_n$ .

Wang and Li [2781] use St(n) to denote the star  $K_{n,1}$ ,  $F_n$  to denote the fan  $P_n \odot K_1$ , and  $F_{m,n}$  to denote the graph obtained by identifying the vertex of  $F_m$  with degree mand the vertex of  $F_n$  with degree n. They showed: for all positive integers n and p and  $m \ge 2p + 2$ ,  $F_m \cup K_{n,p}$  and  $F_{m,2m} \cup K_{n,p}$  are graceful;  $F_m \cup St(n)$  is graceful; and  $F_{m,2m} \cup St(n)$  and  $F_{m,2m} \cup G_r$  are graceful. In [2787] Wang, Wang, and Li gave a sufficient condition for the gracefulness of graphs of the form  $(P_3 + \overline{K_m}) \cup G$  and  $(C_3 + \overline{K_m}) \cup G$ . Wei, Wang, and Sun [2804] provided graceful labelings for the unions of some families of wheels related graphs and complete bipartite graphs. They also gave graceful labelings for some graphs of the form  $G \cup (C_3 + \overline{K_m}) \cup S_n$  where G is wheel related. In [2892] Yu, Wang, and Song proved the following graphs are graceful:  $K_{n,m} \cup (\overline{K_2} + \overline{K_n}), K_{n,m} \cup (P_3 + \overline{K_n}),$  $K_{n,m} \cup (P_1 + P_{2n+2})$ , and  $K_{n,m} \cup K_{1,2n}$ . They proved the gracefulness of such graphs for a variety of cases when G involves stars and paths. More technical results like these are given in [2789], [2788], and [539].

#### 2.6 Joins of Graphs

A number of classes of graphs that are the join of graphs have been shown to be graceful or harmonious. Koh, Rogers, and Lim [1390] proved G + H is graceful if G is a graceful tree and H is one of  $K_n$ ,  $P_n \cup K_1$ , or a star. Koh, Phoon, and Soh [1386] point out that previous versions of this survey incorrectly stated that Acharya [22] proved that if G is a connected graceful graph, then  $G + \overline{K_n}$  is graceful. Red [2109] showed that the double cone  $C_n + \overline{K_2}$  is graceful for n = 3, 4, 5, 7, 8, 9, 11. That  $C_n + \overline{K_2}$  is not graceful for  $n \equiv 2$ (mod 4) follows that Rosa's parity condition. Redl asks what other double cones are graceful. Bras, Gomes, and Selman [229] showed that double wheels  $(C_n \cup C_n) + K_1$  are graceful. Koh, Phoon, and Soh [1386] prove that  $K_3 + \overline{K_n}$  is graceful. Reid [2110] proved that  $P_n + \overline{K_t}$  is harmonious. Sethuraman and Selvaraju [2298] and [2206] have shown that  $P_n + K_2$  is harmonious. They ask whether  $S_n + P_n$  or  $P_m + P_n$  is harmonious. As stated in an earlier section, wheels are of the form  $C_n + K_1$  and are graceful and harmonious. In 2006 Chen [576] proved that multiple wheels  $nC_m + K_1$  are harmonious for all  $n \neq 0$ mod 4. She believes that the  $n \not\equiv 0 \pmod{4}$  case is also harmonious. Chen also proved that if H has at least one edge,  $H + K_1$  is harmonious, and if n is odd, then nH + K is harmonious.

For  $n \ge t+2$  and  $t \ge 1$ , Koh, Phoon, and Soh [1387] use P(n,t) to denote the graph of order n consisting of a path of length t and n - (t+1) isolated vertices. For  $n \ge 2t+1$ and  $t \ge 1$ , they use I(n,t) to denote the disjoint union of  $tK_2$  and  $\overline{K_{n-2t}}$ . They proved:  $\overline{K_p} + P(n,t)$  is graceful for all  $p \ge 1, n \ge t+2$  and  $t \ge 1$ ;  $\overline{K_p} + I(n,t)$  is graceful for all  $p \ge 1, n \ge 2t+1$  and  $t \ge 1$ ; and for  $s, t \in \{1,2\}, P(m,s) + P(n,t)$  is graceful for all  $m \ge s+2$  and  $n \ge t+2$ . In [1387] Koh, Phoon, and Soh ask "What can be said about the gracefulness of  $C_m + P(n,t)$  where  $n \ge t+2$ " and is "Is P(m,s) + P(n,t) always graceful for all  $m \ge s+2, n \ge t+2, where s \ge 3$  or  $t \ge 3$ ?" In [1386] they state as problems about graceful graphs:  $C_m + P_n$   $(m \ge 3, n \ge 3)$ ;  $C_m + C_n$   $(m \ge 3, n \ge 3)$  and  $K_{1,p} + P(n,t)$  and prove that  $C_3 + P(n,t)$  is graceful for all  $n \ge t+2$ , where  $1 \le t \le 3$ 

Shee [2319] has proved  $K_{m,n} + K_1$  is harmonious and observed that various cases of  $K_{m,n} + K_t$  violate the harmonious parity condition in [925]. Liu and Zhang [1607] have proved that  $K_2 + K_2 + \cdots + K_2$  is harmonious. Youssef [2873] has shown that if G is harmonious then  $G^m$  is harmonious for all odd m. He asks the question of whether G is harmonious implies  $G^m$  is harmonious when  $m \equiv 0 \pmod{4}$ . Yuan and Zhu [2894] proved that  $K_{m,n} + K_2$  is graceful and harmonious. Gnanajothi [894, pp. 80–127] obtained the

following:  $C_n + \overline{K_2}$  is harmonious when n is odd and not harmonious when  $n \equiv 2, 4, 6 \pmod{8}$ ;  $S_n + \overline{K_t}$  is harmonious; and  $P_n + \overline{K_t}$  is harmonious. Balakrishnan and Kumar [337] have proved that the join of  $\overline{K_n}$  and two disjoint copies of  $K_2$  is harmonious if and only if n is even. Ramírez-Alfonsín [2082] has proved that if G is graceful and |V(G)| = |E(G)| = e and either 1 or e is not a vertex label then  $G + \overline{K_t}$  is graceful for all t. Sudha and Kanniga [2486] proved that the graph  $P_m + \overline{K_n}$  is graceful.

Seoud and Youssef [2262] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and  $C_n + \overline{K_t}$  is harmonious for every t when n is odd. They also prove that if any edge is added to  $K_{m,n}$  the resulting graph is harmonious if m or n is at least 2. Deng [660] has shown certain cases of  $C_n + \overline{K_t}$ are harmonious. Seoud and Youssef [2259] proved: the graph obtained by appending any number of edges from the two vertices of degree  $n \ge 2$  in  $K_{2,n}$  is not harmonious; dragons  $D_{m,n}$  (i.e., an endpoint of  $P_m$  is appended to  $C_n$ ) are not harmonious when m + n is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has odd order. Youssef [2872] has shown that if G is a graceful graph with p vertices and q edges with p = q + 1, then  $G + S_n$  is graceful.

Sethuraman and Elumalai [2273] have proved that for every graph G with p vertices and q edges the graph  $G+K_1+\overline{K_m}$  is graceful when  $m \geq 2^p - p - 1 - q$ . As a corollary they deduce that every graph is a vertex induced subgraph of a graceful graph. Balakrishnan and Sampathkumar [338] ask for which  $m \geq 3$  is the graph  $mK_2 + \overline{K_n}$  graceful for all n. Bhat-Nayak and Gokhale [462] have proved that  $2K_2 + \overline{K_n}$  is not graceful. Youssef [2872] has shown that  $mK_2 + \overline{K_n}$  is graceful if  $m \equiv 0$  or 1 (mod 4) and that  $mK_2 + \overline{K_n}$  is not graceful if n is odd and  $m \equiv 2$  or 3 (mod 4). Ma [1676] proved that if G is a graceful tree then,  $G + K_{1,n}$  is graceful. Amutha and Kathiresan [148] proved that the graph obtained by attaching a pendent edge to each vertex of  $2K_2 + \overline{K_n}$  is graceful.

Wu [2822] proves that if G is a graceful graph with n edges and n + 1 vertices then the join of G and  $\overline{K_m}$  and the join of G and any star are graceful. Wei and Zhang [2803] proved that for  $n \ge 3$  the disjoint union of  $P_1 + P_n$  and a star, the disjoint union of  $P_1 + P_n$ and  $P_1 + P_{2n}$ , and the disjoint union of  $P_2 + \overline{K_n}$  and a graceful graph with n edges are graceful. More technical results on disjoint unions and joins are given in [2802], [2803], [2805], [2801], and [539].

#### 2.7 Miscellaneous Results

It is easy to see that  $P_n^2$  is harmonious [922] while a proof that  $P_n^2$  is graceful has been given by Kang, Liang, Gao, and Yang [1326].  $(P_n^k)$ , the *k*th power of  $P_n$ , is the graph obtained from  $P_n$  by adding edges that join all vertices *u* and *v* with d(u, v) = k.) This latter result proved a conjecture of Grace [922]. Seoud, Abdel Maqsoud, and Sheehan [2223] proved that  $P_n^3$  is harmonious and conjecture that  $P_n^k$  is not harmonious when k > 3. The same conjecture was made by Fu and Wu [802]. However, Youssef [2882] has proved that  $P_8^4$  is harmonious and  $P_n^k$  is harmonious when *k* is odd. Yuan and Zhu [2894] proved that  $P_n^{2k}$  is harmonious when  $1 \le k \le (n-1)/2$ . Selvaraju [2202] has shown that  $P_n^3$  and the graphs obtained by joining the centers of any two stars with the end vertices of the path of length n in  $P_n^3$  are harmonious.

Cahit [533] proves that the graphs obtained by joining p disjoint paths of a fixed length k to single vertex are harmonious when p is odd and when k = 2 and p is even. Gnanajothi [894, p. 50] has shown that the graph that consists of n copies of  $C_6$  that have exactly  $P_4$  in common is graceful if and only if n is even. For a fixed n, let  $v_{i1}, v_{i2}, v_{i3}$ and  $v_{i4}$   $(1 \le i \le n)$  be consecutive vertices of n 4-cycles. Gnanajothi [894, p. 35] also proves that the graph obtained by joining each  $v_{i1}$  to  $v_{i+1,3}$  is graceful for all n and the generalized Petersen graph P(n,k) is harmonious in all cases (see also [1532]). Recall P(n,k), where  $n \geq 5$  and  $1 \leq k \leq n$ , has vertex set  $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set  $\{a_i a_{i+1} \mid i = 0, 1, \dots, n-1\} \cup \{a_i b_i \mid i = 0, 1, \dots, n-1\} \cup \{b_i b_{i+k} \mid i = 0, 1, \dots, n-1\}$  $(0, 1, \ldots, n-1)$  where all subscripts are taken modulo n [2800]. The standard Petersen graph is P(5,2).) Red [2109] has used a constraint programming approach to show that P(n,k) is graceful for n = 5, 6, 7, 8, 9, and 10. In [2715] and [2726] Vietri proved that P(8t,3) and P(8t+4,3) are graceful for all t. He conjectures that the graphs P(8t,3)have a stronger form a graceful labeling called an  $\alpha$ -labeling (see §3.1). The gracefulness of the generalized Petersen graphs is an open problem. Shao, Deng, Li, and Vese [2317] provide an backtracking algorithm that finds graceful labelings for all generalized Petersen graphs P(n,k) with n < 75 within several seconds. The algorithm strongly outperforms the standard backtracking algorithm.

Rao and Sahoo [2097] prove that every connected graph can be embedded as an induced subgraph in an Eulerian graceful graph. They also show that for an integer  $k \geq 3$ , the problems of deciding whether the chromatic number is less than or equal to k and whether the clique number is greater than or equal to k are NP-complete even for Eulerian graceful graphs. Sethuraman, Ragukumar, and Slater [2289] proved that any tree with m edges can be embedded in a graceful tree with less than 4m edges and in a graceful planar graph. A conjecture in the graph theory book by Chartrand and Lesniak [567, p. 266] that graceful graphs with arbitrarily large chromatic numbers do not exist was shown to be false by Acharya, Rao, and Arumugam [42] (see also Mahmoody [1689]).

In [362] zeke Barrientos calculates the number of non-isomorphic harmoniously labeled graphs with n edges and at most n vertices. He provides harmonious labelings for certain unicyclic graphs obtained via the corona product and triangular grids obtained via edge amalgamation of copies of  $C_3$  in such a way that each copy of a cycle shares at most two edges with other copies. Moreover, he uses the edge-switching technique on  $C_{4t}$  to generate unicyclic graphs with strongly felicitous labelings (see §4.4).

Bača and Youssef [320] investigated the existence of harmonious labelings for the corona graphs of a cycle and a graph G. They proved that if  $G+K_1$  is strongly harmonious (that is, a harmonious labeling f for which the edge labels induced by f(x)+f(y) for each edge xy are  $1, \ldots, q$ . with the 0 label on the vertex of  $K_1$ , then  $C_n \odot G$  is harmonious for all odd  $n \ge 3$ . By combining this with existing results they have as corollaries that the following graphs are harmonious:  $C_n \odot C_m$  for odd  $n \ge 3$  and  $m \not\equiv 2 \pmod{3}$ ;  $C_n \odot K_{s,t}$  for odd  $n \ge 3$ ; and  $C_n \odot K_{1,s,t}$  for odd  $n \ge 3$ .

Sethuraman and Selvaraju [2292] define a graph H to be a supersubdivision of a graph G, if every edge uv of G is replaced by  $K_{2,m}$  (m may vary for each edge) by identifying

u and v with the two vertices in  $K_{2,m}$  that form the partite set with exactly two members. Sethuraman and Selvaraju prove that every supersubdivision of a path is graceful and every cycle has some supersubdivision that is graceful. They conjecture that every supersubdivision of a star is graceful and that paths and stars are the only graphs for which every supersubdivision is graceful. Barrientos [356] disproved this latter conjecture by proving that every supersubdivision of a y-trees is graceful (recall a y-tree is obtained from a path by appending an edge to a vertex of a path adjacent to an end point). Barrientos asks if paths and y-trees are the only graphs for which every supersubdivision is graceful. This seems unlikely to be the case. The conjecture that every supersubdivision of a star is graceful was proved by Kathiresan and Amutha [1348]. In [2296] Sethuraman and Selvaraju prove that every connected graph has some supersubdivision that is graceful. They pose the question as to whether this result is valid for disconnected graphs. Barrientos and Barrientos [367] answered this question by proving that any disconnected graph has a supersubdivision that admits an  $\alpha$ -labeling (see §3.1). They also proved that every supersubdivision of a connected graph admits an  $\alpha$ -labeling. Sekar and Ramachandren proved that an arbitrary supersubdivision of disconnected graph is graceful [2201] and supersubdivisions of ladders are graceful [2079]. Sethuraman and Selvaraju also asked if there is any graph other than  $K_{2,m}$  that can be used to replace an edge of a connected graph to obtain a supersubdivision that is graceful.

Sethuraman and Selvaraju [2292] call superdivision graphs of G where every edge uvof G is replaced by  $K_{2,m}$  and m is fixed an *arbitrary supersubdivision* of G. Barrientos and Barrientos [367] answered the question of Sethuraman and Selvaraju by proving that any graph obtained from  $K_{2,m}$  by attaching k pendent edges and n pendent edges to the vertices of its 2-element stable set can be used instead of  $K_{2,m}$  to produce an arbitrary supersubdivision that admits an  $\alpha$ -labeling (a *stable* set S consists of a set of vertices such that there is not an edge  $v_i v_j$  for all pairs  $v_i, v_j$  in S).

Kathiresan and Sumathi [1356] affirmatively answer the question posed by Sethuraman and Selvaraju in [2292] of whether there are graphs different from paths whose arbitrary supersubdivisions are graceful.

For a graph G Ambili and Singh [146] call the graph  $G^*$  a strong supersubdivision of G if  $G^*$  is obtained from G by replacing every edge  $e_i$  of G by a complete bipartite graph  $K_{r_i,s_i}$ . A strong supersubdivision  $G^*$  of G is said to be an arbitrary strong supersubdivision if  $G^*$  is obtained from G by replacing every edge  $e_i$  of G by a complete bipartite graph  $K_{r,s_i}$  (r is fixed and  $s_i$  may vary). They proved that arbitrary strong supersubdivisions of paths, cycles, and stars are graceful. They conjecture that every arbitrary strong supersubdivision G, an arbitrary strong supersubdivision of G is graceful?

In [2295] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of graceful labeling called an  $\alpha$ -labeling (see §3.1 for the definition).

Kathiresan [1345] uses the notation  $P_{a,b}$  to denote the graph obtained by identifying the end points of b internally disjoint paths each of length a. He conjectures that  $P_{a,b}$  is graceful except when a is odd and  $b \equiv 2 \pmod{4}$  and proves the conjecture for the case that a is even and b is odd. Liang and Zuo [1592] proved that the graph  $P_{a,b}$  is graceful when both a and b are even. Daili, Wang and Xie [650] provided an algorithm for finding a graceful labeling of  $P_{2r,2}$  and showed that a  $P_{2r,2(2k+1)}$  is graceful for all positives r and k. Sekar [2200] has shown that  $P_{a,b}$  is graceful when  $a \neq 4r + 1$ , r > 1, b = 4m, and m > r. Yang (see [2853]) proved that  $P_{a,b}$  is graceful when a = 3, 5, 7, and 9 and b is odd and when a = 2, 4, 6, and 8 and b is even (see [2853]). Yang, Rong, and Xu [2853] proved that  $P_{a,b}$  is graceful when a = 10, 12, and 14 and b is even. Yan [2843] proved  $P_{2r,2m}$  is graceful (see [2134]). Rong and Xiong [2134] showed that  $P_{2r,b}$  is graceful for all positive integers r and b. Kathiresan also shows that the graph obtained by identifying a vertex of  $K_n$  with any noncenter vertex of the star with  $2^{n-1} - n(n-1)/2$  edges is graceful.

For a family of graphs  $G_1(u_1, u_2)$ ,  $G_2(u_2, u_3)$ , ...,  $G_m(u_m, u_{m+1})$  where  $u_i$  and  $u_{i+1}$  are vertices in  $G_i$  Cheng, Yao, Chen, and Zhang [585] define a graph-block chain  $H_m$  as the graph obtained by identifying  $u_{i+1}$  of  $G_i$  with  $u_{i+1}$  of  $G_{i+1}$  for i = 1, 2, ..., m. They denote this graph by  $H_m = G_1(u_1, u_2) \oplus G_2(u_2, u_3) \oplus \cdots \oplus G_m(u_m, u_{m+1})$ . The case where each  $G_i$  has the form  $P_{a_i,b_i}$  they call a path-block chain. The vertex  $u_1$  is called the *initial* vertex of  $H_m$ . They define a generalized spider  $S_m^*$  as a graph obtained by starting with an initial vertex  $u_0$  and m path-block graphs and join  $u_0$  with each initial vertex of each of the path-block graphs. Similarly, they define a generalized caterpillar  $T_m^*$  as a graph obtained by starting with m path-block chains  $H_1, H_2, \ldots, H_m$  and a caterpillar T with m isolated vertices  $v_1, v_2, \ldots, v_m$  and join each  $v_i$  with the initial vertex of each  $H_i$ . They prove several classes of path-block chains, generalized spiders, and generalized caterpillars are graceful.

The graph  $T_n$  with 3n vertices and 6n - 3 edges is defined as follows. Start with a triangle  $T_1$  with vertices  $v_{1,1}, v_{1,2}$  and  $v_{1,3}$ . Then  $T_{i+1}$  consists of  $T_i$  together with three new vertices  $v_{i+1,1}, v_{i+1,2}, v_{i+1,3}$  and edges  $v_{i+1,1}v_{i,2}, v_{i+1,1}v_{i,3}, v_{i+1,2}v_{i,1}, v_{i+1,2}v_{i,3}, v_{i+1,3}v_{i,1}, v_{i+1,3}v_{i,2}$ . Gnanajothi [894] proved that  $T_n$  is graceful if and only if n is odd. Sekar [2200] proved  $T_n$  is graceful when n is odd and  $T_n$  with a pendent edge attached to the starting triangle is graceful when n is even.

In [430] and [2306] Begam, Palanivelrajan, Gunasekaran, and Hameed give graceful labelings for graphs constructed by combining theta graphs (that is, a collection of edge disjoint paths that have common endpoints) with paths and stars. Khatun and Abu Nayeem [1363] prove that the zero divisor graph of the commutative ring of integers modulo n is graceful if n = pq, 4p or 9p, where p and q are prime numbers.

The torch graph  $O_n$  is defined by  $V(O_n) = \{v_i \mid 1 \leq i \leq n+4\}, E(O_n) = \{v_iv_{n+1} \mid 2 \leq i \leq n-2\} \cup \{v_iv_{n+3} \mid 2 \leq i \leq n-2\} \cup \{v_1v_i \mid 2 \leq i \leq n+4\} \cup \{v_{n-1}v_n, v_nv_{n+2}, v_nv_{n+4}, v_{n+1}v_{n+3}\}$ . Manulang and Sugeng [1698] showed that the torch graph is graceful.

For a graph G, the splitting graph of G, S'(G), is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v. Sekar [2200] has shown that  $S'(P_n)$  is graceful for all n and  $S'(C_n)$  is graceful for  $n \equiv 0, 1 \pmod{4}$ . Vaidya and Shah [2667] proved that the square graph of a bistar, the splitting graph of a bistar, and the splitting graph of a star are graceful graphs. In [2487] Sudha and Kanniga proved that fans and the splitting graph of a star are graceful. Sudha and Kanniga [2488] proved that the following graphs are graceful: arbitrary supersubdivisions of wheels; combs  $(P_n \odot K_1)$ ; double fans  $(P_n \odot \overline{K_2})$ ;  $(P_m \cup P_n) \odot K_1$ ; and graphs obtained by starting with two star graphs  $S_m$  and  $S_n$  and identifying some of the pendent vertices of each. Sudha and Kanniga [2489] proved that the graphs obtained from  $P_n \odot K_1$  by identifying the center of a  $S_n$  with the endpoint of a pendent edge attached to the endpoint of  $P_n$  are graceful; and the graphs obtained from a fan  $P_n \odot K_1$  by deleting a pendent edge attached to an endpoint of  $P_n$  are graceful. Sunda [2482] provided some results on graphs obtained by connecting copies of  $K_{m,n}$  in certain ways. Sudha and Kanniga [2485] proved that the graphs obtained by joining the vertices of a path to any number isolated points are graceful. They also proved that the arbitrary supersubdivision of all the edges of helms, combs  $(P_n \odot K_1)$  and ladders  $(P_n \times P_2)$  with pendent edges at the vertices of degree 2 by a complete bipartite graphs  $K_{2,m}$  are graceful.

The duplication of an edge e = uv of a graph G is the graph G' obtained from G by adding an edge e' = u'v' such that N(u) = N(u') and N(v) = N(v'). The duplication of a vertex of a graph G is the graph G' obtained from G by adding a new vertex v' to G such that N(v') = N(v). Kaneria, Vaidya, Ghodasara, and Srivastav [1318] proved the duplication of a vertex of a cycle, the duplication of an edge of an even cycle, and the graph obtained by joining two copies of a fixed cycle by an edge are graceful.

For a graph G and a vertex v of G, a vertex switching  $G_v$  is the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Boxwala and Vashishta [497] show that the graph obtained by switching an arbitrary vertex of  $C_n$  (n > 3), the duplication of an arbitrary vertex on the rim of a wheel with an even number of vertices, and the mirror graph of a path are graceful. Jeba Jesintha and Subashini [1130] proved that the path union of vertex switching of even cycles in increasing order is graceful.

The join sum of complete bipartite graphs  $\langle K_{m_1,n_1}, \ldots, K_{m_t,n_t} \rangle$  is the graph obtained by starting with  $K_{m_1,n_1}, \ldots, K_{m_t,n_t}$  and joining a vertex of each pair  $K_{m_i,n_i}$  and  $K_{m_{i+1},n_{i+1}}$  to a new vertex  $v_i$  where  $1 \leq i \leq k-1$ . The path union of a graph G is the graph obtained by adding an edge from n copies  $G_1, G_2, \ldots, G_n$  of G from  $G_i$  to  $G_{i+1}$  for  $i = 1, \ldots, n-1$ . We denote this graph by  $P(n \cdot G)$ . Kaneria, Makadia, and Meghpara [1298] proved the following graphs are graceful: the graph obtained by joining  $C_{4m}$  and  $C_{4n}$  by a path of arbitrary length; the path union of finite many copies of  $C_{4n}$ ; and  $C_{4n}$ with twin chords. Kaneria, Makadia, Jariya, and Meghpara [1297] proved that the join sum of complete bipartite graphs, the star of complete bipartite graphs, and the path union of a complete bipartite graphs are graceful.

Given connected graphs  $G_1, G_2, \ldots, G_n$ , Kaneria, Makadia, and Jariya [1296] define a cycle of graphs  $C(G_1, G_2, \ldots, G_n)$  as the graph obtained by adding an edge joining  $G_i$  to  $G_{i+1}$  for  $i = 1, \ldots, n-1$  and an edge joining  $G_n$  to  $G_1$ . (The resulting graph can vary depending on which vertices of the  $G_i$ s are chosen.) When the *n* graphs are isomorphic to G the notation  $C(n \cdot G)$  is used. Kaneria et al. proved that  $C(2t \cdot C_{4n})$  and  $C(2t \cdot K_{n,n})$  are graceful. In [1299] and [1301] Kaneria, Makadia, and Meghpara prove that the following graphs are graceful:  $C(2t \cdot K_{m,n})$ ;  $C(C_{4n_1}, C_{4n_2}, \ldots, C_{4n_t})$  when t is even and  $\sum_{i=1}^{\frac{t}{2}} n_i =$ 

 $\sum_{i=\frac{t}{2}}^{t} n_i$ ;  $C(2t \cdot P_m \times P_n)$ ; the star of  $P_m \times P_n$ ; and the path union of t copies of  $P_m \times P_n$ . Kaneria, Viradia, Jariya, and Makadia [1319] proved the cycle graph  $C(t \cdot P_n)$  is graceful.

The star of graphs  $G_1, G_2, \ldots, G_n$ , denoted by  $S(G_1, G_2, \ldots, G_n)$ , is the graph obtained by identifying each vertex of  $K_{1,n}$ , except the center, with one vertex from each of  $G_1, G_2, \ldots, G_n$ . The case that  $G_1 = G_2 = \cdots = G_n = G$  is denoted by  $S(n \cdot G)$ . In [1310] and [1311] Kaneria, Meghpara, and Makadia proved the following graphs are graceful:  $S(t \cdot K_{m,n})$ ;  $S(t \cdot P_m \times P_n)$ ; the barycentric subdivision of  $P_m \times P_n$  (that is, the graph obtained from  $P_m \times P_n$  by inserting a new vertex in each edge); the graph obtained by replacing each edge of  $K_{1,t}$  by  $P_n$ ; the graph obtained by identifying each end point of  $K_{1,n}$  with a vertex of  $K_{m,n}$ ; and the graph obtained by identifying each end point of  $K_{1,n}$  is a graceful tree.

Kaneria and Makadia [1287] and [1288] proved the following graphs are graceful:  $(P_m \times P_n) \cup (P_r \times P_s)$ ;  $C_{2f+3} \cup (P_m \times P_n) \cup (P_r \times P_s)$ , where f = 2(mn + rs) - (m + n + r + s); the tensor product of  $P_n$  and  $P_3$ ; the tensor product of  $P_m$  and  $P_n$  for odd m and n; the star of  $C_{4n}$ ; the *t*-supersubdivision of  $P_m \times P_n$ ; and the graph obtained by joining  $C_{4n}$  and a grid graph with a path.

The graph  $P_n^t$  is obtained by identifying one end point from each of t copies of  $P_n$ . The graph  $P_n^t(G_1, G_2, \ldots, G_{tn})$  obtained by replacing each edge of  $P_n^t$ , except those adjacent to the vertex of degree t, by the graphs  $G_1, G_2, \ldots, G_{tn}$  is called the one point path union of  $G_1, G_2, \ldots, G_{tn}$ . The case where  $G_1 = G_2 = \cdots = G_{tn} = H$  is denoted by  $P_n^t(tn \cdot H)$ . In [1310] and [1311] Kaneria, Meghpara, and Makadia proved  $P_n^t$  and  $P_n^t(tn \cdot K_{m,r})$  are graceful. In [1308] Kaneria and Meghpara proved  $P_n^t(tn \cdot P_r \times P_s)$ ,  $P_n^t(tn \cdot K_{1,m})$ ,  $S(t \cdot C_{4n})$ , and  $P_n^t(tn \cdot C_{4m})$  are graceful.

Kanneria and Makadia [1289] define a step grid graph as the graph obtained by starting with paths  $P_n, P_n, P_{n-1}, \ldots, P_2$   $(n \ge 3)$  arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. In [1289] and [1290] they prove the following graphs are graceful: step grid graphs; one point union for a path of step grid graphs; cycles of step grid graphs; stars of step grid graphs; t-super subdivisions of the step grid graphs; open stars of step grid graphs; one point unions of paths of step grid graphs; and graphs obtained by joining  $C_{4m}$  and step grid graphs with a path of arbitrary length.

For *n* even [1291] Kaneria and Makadia [1291] define a double step grid graph of size *n* (denoted by  $DSt_n$ ) as the graph obtained by starting with paths  $P_n, P_n, P_{n-2}, P_{n-4}, \ldots, P_4, P_2$  arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. They prove the following graphs are graceful: double step grid graphs; path unions of copies of  $DSt_n$ ; cycles of  $r \equiv 0, 3 \pmod{4}$  copies of double step grid graphs; and stars of double step grid graphs.

In [1303] Kaneria, Makadia and Viradia prove the following graphs are graceful: open stars of double step grid graphs; one point union of paths of double step grid graphs  $P_n^t(tn \cdot DSt_m)$ ; graphs obtained by joining  $C_{4m}$  and a double step grid graph with a path of arbitrary length; and graphs obtained by starting with a cycle  $C_m^+$  ( $m \equiv 2 \mod 4$ ) with chords that form a triangle with an edge of the cycle and joining  $C_m^+$  and a double step grid graph with a path of arbitrary length.

For even n > 2 Kaneria and Makadia [1292] define a *plus graph* of size n (denoted by  $Pl_n$ ) as the graph obtained by starting with paths  $P_2, P_4, \ldots, P_{n-2}, P_n, P_n, P_{n-2}, \ldots, P_4, P_2$  arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. They prove plus graphs, path unions of copies of  $Pl_n$ , cycles of  $r \equiv 0$ , 3 (mod 4) copies of  $Pl_n$ , and stars of plus graphs are graceful. In [1293] Kaneria and Makadia prove the following graphs are graceful: open stars of plus graphs; graphs obtained by joining  $C_{4m}$  and a plus graph with a path of arbitrary length; graphs obtained from cycles  $C_m^+$  ( $m \equiv 2 \pmod{4}$ ) with twin chords that form a triangle with an edge of the cycle by joining  $C_m^+$  and a plus graph with a path of arbitrary length.

Kaneria and Makadia [1294] define a swastik graph as the graph obtained from four copies of  $C_{4n}$  (n > 1) with vertices  $V_{i,j}$  (i = 1, 2, 3, 4, j = 1, 2, ..., 4n) and identifying  $V_{1,4t}$  and  $V_{2,1}$ ,  $V_{2,4t}$  and  $V_{3,1}$ ,  $V_{3,4t}$  and  $V_{4,1}$ , and  $V_{4,4t}$  and  $V_{1,1}$ . They proved that path unions of swastik graphs of the same size, cycles of  $r \equiv 0$ , 3 (mod 4) copies of swastik graphs of the same size, and the star of swastik graphs are graceful. In [1295] Kaneria and Makadia prove the following graphs are graceful: open stars of swastik graphs; one point unions for paths of swastik graphs; graphs obtain by joining  $C_{4m}$  and a swastik graph with a path of arbitrary length; graphs obtained from cycles  $C_m$   $(m \equiv 2 \pmod{4})$  with twin chords that form a triangle with an edge by joining  $C_m \odot K_1$  and a swastik graph with a path of arbitrary length.

In [1282] and [1281] Kaneria and Jariya define a smooth graceful graph as a bipartite graph G with q edges with the property that for all positive integers l there exists a map  $g: V \longrightarrow \{0, 1, \ldots, \lfloor \frac{q-1}{2} \rfloor, \lfloor \frac{q+1}{2} \rfloor + l, \lfloor \frac{q+3}{2} \rfloor + l, \ldots, q+l\}$  such that the induced edge labeling map  $g^*: E \longrightarrow \{1+l, 2+l, \ldots, q+l\}$  defined by  $g^*(e) = |g(u) - g(v)|$  is a bijection. Note that by taking l = 0 a smooth graceful labeling is a graceful labeling. Kaneria and Jariya proved the following graphs are smooth graceful:  $P_n$ ;  $C_{4n}$ ;  $K_{2,n}$ ;  $P_m \times P_n$ ; and the graph obtained by joining a cycle  $C_{4m+2}$  with twin chords to  $C_{4n}$ . They also proved that the graph obtained by joining  $C_{4m}$  to  $W_n$  with a path is graceful. They proved that  $K_{1,n}$ is semi smooth graceful, the star of  $K_{1,n}$  is graceful, the path union of a smooth graceful tree is graceful, and the star of a smooth graceful tree is a graceful tree.

Kaneria, Makadia and Viradia [1304] proved the following: the star of a semi smooth graceful graph is graceful;  $K_{m,n}$ ,  $P(t \cdot H)$  are semi smooth graceful where H is a semi smooth graceful graph; step grid graphs; and the cycle graphs  $C(t \cdot H)$  are smooth graceful, when  $t \equiv \pmod{4}$ , H is a semi smooth;  $C^t(m \cdot C_n)$ ,  $P^t(k \cdot T)$ ,  $< C_{n_1}, P_{n_2}, C_{n_3}, \ldots, P_{n_{2t}}, C_{n_{2t+1}} > , < K_{m_1,n_1}, P_{r_1}, K_{m_2,n_2}, P_{r_2}, \ldots, P_{r_{t-1}}, K_{m_t,n_t} > ,$ 

 $< P_{n_1} \times P_{m_1}, P_{r_1}, P_{n_2} \times P_{m_2}, \dots, P_{r_{t-1}}, P_{n_t} \times P_{m_t} >$ are graceful when T is semi smooth graceful tree.

Kaneria and Meghpara [1307] prove that  $B_{m,n}$ , the splitting graphs  $S'(B_{m,n})$  and  $S'(P_n)$  are semi-smooth graceful and if graphs obtained by joining semi-smooth graceful graph and  $B_{m,n}^2$  by an arbitrary path is graceful.

A komodo dragon is formed by attaching a path to a vertex of degree 3 in a cycle with a chord and attaching star graphs to the end points of the path. A komodo dragon with many tails is formed by attaching many paths of length two to an endpoint of the path in a komodo dragon. In [2307] and [2309] Shahul Hameed, Palanivelrajan, Gunasekaran and Raziya Begam provide graceful labelings of various komodo dragon graphs and their extensions. In [2308] and [2310] Shahul Hameed et al. investigated the gracefulness of classes of graphs constructed by combining some subdivisions of certain theta graphs with stars.

For a bipartite graph G with partite sets X and Y let G' be a copy of G and X'and Y' be copies of X and Y. Lee and Liu [1504] define the *mirror graph*, M(G), of Gas the disjoint union of G and G' with additional edges joining each vertex of Y to its corresponding vertex in Y'. The case that  $G = K_{m,n}$  is more simply denoted by M(m, n). They proved that for many cases M(m, n) has a stronger form of graceful labeling (see §3.1 for details).

The total graph  $T(P_n)$  has vertex set  $V(P_n) \cup E(P_n)$  with two vertices adjacent whenever they are neighbors in  $P_n$ . Balakrishnan, Selvam, and Yegnanarayanan [339] have proved that  $T(P_n)$  is harmonious.

For any graph G with vertices  $v_1, \ldots, v_n$  and a vector  $\mathbf{m} = (m_1, \ldots, m_n)$  of positive integers the corresponding *replicated graph*,  $R_{\mathbf{m}}(G)$ , of G is defined as follows. For each  $v_i$  form a stable set  $S_i$  consisting of  $m_i$  new vertices  $i = 1, 2, \ldots, n$  (a *stable* set S consists of a set of vertices such that there is not an edge  $v_i v_j$  for all pairs  $v_i, v_j$  in S); two stable sets  $S_i, S_j, i \neq j$ , form a complete bipartite graph if each  $v_i v_j$  is an edge in G and otherwise there are no edges between  $S_i$  and  $S_j$ . Ramírez-Alfonsín [2082] has proved that  $R_{\mathbf{m}}(P_n)$  is graceful for all  $\mathbf{m}$  and all n > 1 (see §3.4 for a stronger result) and that  $R_{(m,1,\ldots,1)}(C_{4n}), R_{(2,1,\ldots,1)}(C_n)$   $(n \geq 8)$  and  $R_{(2,2,1,\ldots,1)}(C_{4n})$   $(n \geq 12)$  are graceful.

For any permutation f on  $1, \ldots, n$ , the *f*-permutation graph on a graph G, P(G, f), consists of two disjoint copies of G,  $G_1$  and  $G_2$ , each of which has vertices labeled  $v_1, v_2, \ldots, v_n$  with n edges obtained by joining each  $v_i$  in  $G_1$  to  $v_{f(i)}$  in  $G_2$ . In 1983 Lee (see [1570]) conjectured that for all n > 1 and all permutations on  $1, 2, \ldots, n$ , the permutation graph  $P(P_n, f)$  is graceful. Lee, Wang, and Kiang [1570] proved that  $P(P_{2k}, f)$  is graceful when  $f = (12)(34) \cdots (k, k+1) \cdots (2k-1, 2k)$ . They conjectured that if G is a graceful nonbipartite graph with n vertices, then for any permutation f on  $1, 2, \ldots, n$ , the permutation graph P(G, f) is graceful. Fan and Liang [746] have shown that if f is a permutation in  $S_n$  where  $n \ge 2(m-1) + 2l$  then the permutation graph  $P(P_n, f)$  is graceful if the disjoint cycle form of f is  $\prod_{k=0}^{l-1}(m+2k,m+2k+1)$ , and if  $n \ge 2(m-1)+4l$  the permutation graph  $P(P_n, f)$  is graceful the disjoint cycle form of f is  $\prod_{k=0}^{l-1} (m+4k, m+4k)$ (4k+2)(m+4k+1, m+4k+3). For any integer  $n \ge 5$  and some permutations f in S(n), Liang and Y. Miao, [1589] discuss gracefulness of the permutation graphs  $P(P_n, f)$ if f = (m, m+1, m+2, m+3, m+4), (m, m+2)(m+1, m+3), (m, m+1, m+2, m+4, m+4)3), (m, m+1, m+4, m+3, m+2), (m, m+2, m+3, m+4, m+1), (m, m+3, m+4, m+2, m+1)and (m, m+4, m+3, m+2, m+1). In [1591] Liang, Zhang, Xu, Ye, Fan, and Ge prove the permutation graphs  $P(P_n, f)$  where f is one of the permutations (12345), (2345), (234), (123456) and (23)(45) are graceful. Some families of graceful permutation graphs are given in [1497], [1584], and [949].

In [433] Bell provided methods to combine graceful bipartite graphs to create new graceful graphs. These methods unify and generalize some well-known results in the

graceful labeling literature. She also found a new class of graceful trees.

A graph (p,q)-graph G(V, E) is said to be (k, d)-hooked Skolem graceful if there exists a bijection f from V(G) to  $\{1, 2, \ldots, p-1, p+1\}$  such that the induced edge labeling  $g_f$ from E to  $\{k, k+d, \ldots, k+(n-1)d\}$  defined by  $g_f(uv) = |f(u)-f(v)|$  for all uv in E is also bijective. Such a labeling f is called a (k, d)-hooked Skolem graceful labeling of G. Note that when k = d = 1, this notion coincides with that of hooked Skolem graceful labeling of the graph G. In [1917] Pereira, Singh, and Arumugam present some preliminary results on (k, d)-hooked Skolem graceful graphs and prove that  $nK_2$  is (2, 1)-hooked Skolem graceful if and only if  $n \equiv 1$  or 2 (mod 4).

Gnanajothi [894, p. 51] calls a graph G bigraceful if both G and its line graph are graceful. She shows the following are bigraceful:  $P_m$ ;  $P_m \times P_n$ ;  $C_n$  if and only if  $n \equiv 0, 3$ (mod 4);  $S_n$ ;  $K_n$  if and only if  $n \leq 3$ ; and  $B_n$  if and only if  $n \equiv 3 \pmod{4}$ . She also shows that  $K_{m,n}$  is not bigraceful when  $n \equiv 3 \pmod{4}$ . (Gangopadhyay and Hebbare [822] used the term "bigraceful" to mean a bipartite graceful graph.) Murugan and Arumugan [1800] have shown that graphs obtained from  $C_4$  by attaching two disjoint paths of equal length to two adjacent vertices are bigraceful.

Several well-known isolated graphs have been examined. Graceful labelings have been found for the Petersen graph [795], the cube [839], the icosahedron and the dodecahedron. Graham and Sloane [925] showed that all of these except the cube are harmonious. Winters [2816] verified that the Grőtzsch graph (see [491, p. 118]), the Heawood graph (see [491, p. 236]), and the Herschel graph (see [491, p. 53]) are graceful. Graham and Sloane [925] determined all harmonious graphs with at most five vertices. Seoud and Youssef [2261] did the same for graphs with six vertices.

In 2009 Zak [2897] defined the following generalization of harmonious labelings. For a graph G(V, E) and a positive integer  $t \ge |E|$  a function h from V(G) to  $Z_t$  (the additive group of integers modulo t) is called a t-harmonious labeling of G if h is injective for  $t \ge |V|$  or surjective for t < |V|, and  $h(u) + h(v) \ne h(x) + h(y)$  for all distinct edges uv and xy. The smallest such t for which G has a t-harmonious labeling is called the harmonious order of G. Obviously, a graph G(V, E) with  $|E| \ge |V|$  is harmonious if and only if the harmonious order of G is |E|. Zak determines the harmonious order of complete graphs, complete bipartite graphs, even cycles, some cases of  $P_n^k$ , and  $2nK_3$ . He presents some results about the harmonious order of the Cartesian products of graphs, the disjoint union of copies of a given graph, and gives an upper bound for the harmonious order of trees. He conjectures that the harmonious order of a tree of order n is n+o(n). Hegde and Murthy [991] proved Zak's conjecture [2897] using the value sets of polynomials, which partially proves the cordial tree conjecture by Hovey [1021] that all trees of order less than a prime p are p-cordial. (See Section 3.7.)

A graceful labeling of  $P_n$  is said to be an (a, b; n)-graceful labeling if one endpoint is labeled a and the other labeled b. A conjecture made in Gvozdjak's PhD Thesis [944] on the Oberwolfach Problem in 2004 is: "An (a, b; n)-graceful labeling of  $P_n$  exists if and only if the integers a, b, n satisfy (1) b - a has the same parity as n(n + 1)/2; (2)  $0 < |b - a| \le (n + 1)/2$  and (3)  $n/2 \le a + b \le 3n/2$ ." In [2906] Zhang, Zhang, and Wang showed that the conjecture is true for every n whenever it is true for  $n \le 4a + 1$  and a is
a fixed value. Moreover, they proved that the conjecture is true for a = 0, 1, 2, 3, 4, 5, 6.

For a graph with e edges Vietri [2727] generalizes the notion of a graceful labeling by allowing the vertex labels to be real numbers in the interval [0, e]. For a simple graph G(V, E) he defines an injective map  $\gamma$  from V to [0, e] to be a *real-graceful* labeling of Gprovided that  $\sum 2^{\gamma(u)-\gamma(v)} + 2^{\gamma(v)-\gamma(u)} = 2^{e+1} - 2^{-e} - 1$ , where the sum is taken over all edges uv. In the case that the labels are integers, he shows that a real-graceful labeling is equivalent to a graceful labeling. In contrast to the case for graceful labelings, he shows that the cycles  $C_{4t+1}$  and  $C_{4t+2}$  have real-graceful labelings. He also shows that the nongraceful graphs  $K_5$ ,  $K_6$ , and  $K_7$  have real-graceful labelings. With one exception, his real-graceful labels are integers.

The gamma-number (or gracefulness) of a graph G, denoted by  $\gamma(G)$ , is the smallest positive integer n for which there exists an injective function  $f: V(G) \to \{0, 1, \ldots, n\}$ such that each  $uv \in E(G)$  is labeled |f(u) - f(v)| and the resulting edge labels are distinct. The strong gamma-number of a graph G, denoted by  $\gamma_s(G)$ , is defined to be the smallest positive integer n such that  $\gamma(G) = n$  with the additional property that there exists an integer  $\lambda$  so that min  $\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$  for each  $uv \in E(G)$ . The strong gamma-number is defined to be  $+\infty$ , otherwise. Ichishima and Oshima [1063] proved that if G is a bipartite graph, then  $\gamma(mG) \leq m\gamma(G) + m - 1$ for any positive integer m. They also show that  $\gamma_s(G) < +\infty$  and  $\gamma_s(G) \leq 2\gamma(G) + 1$ for any bipartite graph G. Moreover, they provide a sharp upper bound for  $\gamma(G \cup H)$ in terms of  $\gamma(G)$  and  $\gamma_s(H)$  when G and H are graphs such that H is bipartite, and give formulas for the gamma-number of certain forests. In addition to these, they present strong gamma-number analogues to the gamma-number results and determine the exact values of the gamma-number and strong gamma-number for all cycles.

A graph G with m vertices and n edges, is said to be prime graceful if there is an injection  $\phi$  from the vertices of G to  $\{1, 2, \ldots, k\}$  where  $k = \min\{2m, 2n\}$  such that  $gcd(\phi(v_i), \phi(v_j)) = 1$  and the induced injective function  $\phi^*$  from the edges of G to  $\{1, 2, \ldots, k-1\}$  defined by  $\phi^*(v_i v_j) = |\phi(v_i) - \phi(v_j)|$ , the resulting edge labels are distinct. In [2207] Selvarajan and Subramoniam proved paths, cycles, stars, friendship graphs, bistars,  $C_4 \cup P_n$ ,  $K_{m,2}$ , and  $K_{m,2} \cup P_n$  have prime graceful labelings.

A number of authors have investigated the gracefulness of the directed graphs obtained from copies of directed cycles  $\vec{C}_m$  that have a vertex in common or have an edge in common. A digraph D(V, E) is said to be graceful if there exists an injection  $f: V(G) \rightarrow$  $\{0, 1, \ldots, |E|\}$  such that the induced function  $f': E(G) \rightarrow \{1, 2, \ldots, |E|\}$  that is defined by  $f'(u, v) = (f(v) - f(u)) \pmod{|E| + 1}$  for every directed edge uv is a bijection. The notations  $n \cdot \vec{C}_m$  and  $n - \vec{C}_m$  are used to denote the digraphs obtained from n copies of  $\vec{C}_m$  with exactly one point in common and the digraphs obtained from n copies of  $\vec{C}_m$ with exactly one edge in common. Du and Sun [696] proved that a necessary condition for  $n - \vec{C}_m$  to be graceful is that mn is even and that  $n \cdot \vec{C}_m$  is graceful when m is even. They conjectured that  $n \cdot \vec{C}_m$  and Bao in [1263]. Xu, Jirimutu, Wang, and Min [2835] proved that  $n - \vec{C}_m$  is graceful for m = 4, 6, 8, 10 and even n. Feng and Jirimutu (see [2909]) conjectured that  $n - \vec{C}_m$  is graceful for even n and asked about the situation for odd *n*. The cases where m = 5, 7, 9, 11, and 13 and even *n* were proved Zhao and Jirimutu [2908]. The cases for m = 15, 17, and 19 and even *n* were proved by Zhao et al. in [2907], and [2413]. Zhao, Siqintuya, and Jirimutu [2909] proved that a necessary condition for  $n - \vec{C}_m$  to be graceful is that nm is even. Hegde and Kumudashi [988] show that the symmetric digraph on the double cycle constructed from an *m*-cycle by replacing each edge xy by a pair of arcs, (x, y) and (y, x), is graceful for all m.

In a 1985 paper Bloom and Hsu [477] say a directed graph D with e edges has a graceful labeling  $\theta$  if for each vertex v there is a vertex labeling  $\theta$  that assigns each vertex a distinct integer from 0 to e such that for each directed edge (u, v) the integers  $\theta(v) - \theta(u) \mod (e+1)$ are distinct and nonzero. They conjectured that digraphs whose underlying graphs are wheels and that have all directed edges joining the hub and the rim in the same direction and all directed edges in the same direction are graceful. This conjecture was proved in 2009 by Hegde and Shivarajkumar [1003]. Yao, Yao, and Cheng [2864] investigated the gracefulness for many orientations of undirected trees with short diameters and proved some directed trees do not have graceful labelings. Hegde and Kumudashi [989] established the gracefulness of the directed graph that is an orientation of the planar grid graph  $P_m \times P_n$  in which each cell is a unicycle of length four. A graceful difference labeling of a directed graph G with vertex set V is a bijection  $f: V \to \{1, \ldots, |V|\}$  such that, when each arc uv is assigned the difference label f(v) - f(u), the resulting arc labels are distinct. Hertz and Picouleau [1010] conjectured that all disjoint unions of circuits have a graceful difference labeling, except in two particular cases. They provided partial results that support this conjecture. A survey of results on graceful digraphs by Feng, Xu, and Jirimutu is given in [753]. Marr [1711] and [1710] summarizes previously known results on graceful directed graphs and presents some new results on directed paths, stars, wheels, and umbrellas.

# 2.8 Summary

The results and conjectures discussed above are summarized in the tables following. The letter G after a class of graphs indicates that the graphs in that class are known to be graceful; a question mark indicates that the gracefulness of the graphs in the class is an open problem; we put a question mark after a "G" if the graphs have been conjectured to be graceful. The analogous notation with the letter H is used to indicate the status of the graphs with regard to being harmonious. The tables impart at a glimpse what has been done and what needs to be done to close out a particular class of graphs. Of course, there is an unlimited number of graphs one could consider. One wishes for some general results that would handle several broad classes at once but the experience of many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture alone has withstood the efforts of scores of people over the past four decades. Analogous sweeping conjectures are probably true but appear hopelessly difficult to prove. I thank Don Knuth for his correspendence about the results of Smith and Puget [2431] in Table 1 regarding the gracefulness  $K_m \times K$ ,  $K_m \times P_n$ , and  $K_m \times C_n$ .

Graph	Graceful
trees	G if $\leq 35$ vertices [747] G if symmetrical [449] G if at most 4 end-vertices [1031] G with diameter at most 5 [1022] G? Ringel-Kotzig G caterpillars [2136] G firecrackers [575] G bananas [2281], [2280] G? lobsters [445]
cycles $C_n$	G iff $n \equiv 0, 3 \pmod{4}$ [2136]
wheels $W_n$	G [795], [1018]
helms (see $\S2.2$ )	G [212]
webs (see $\S2.2$ )	G [1326]
gears (see $\S2.2$ )	G [1675]
cycles with $P_k$ -chord (see §2.2)	G $[658]$ , $[1674]$ , $[1399]$ , $[2041]$
$C_n$ with k consec. chords (see §2.2)	G if $k = 2, 3, n - 3$ [1388], [1396]
unicyclic graphs	G? iff $G \neq C_n, n \equiv 1, 2 \pmod{4}$ [2601]
$P_n^k$	G if $k = 2$ [1326]
$C_n^{(t)}$ (see §2.2)	$n = 3 \text{ G iff } t \equiv 0, 1 \pmod{4}$ [446], [448] G? if $nt \equiv 0, 3 \pmod{4}$ [1389] G if $n = 6, t \text{ even } [1389]$ G if $n = 4, t > 1$ [2320] G if $n = 5, t > 1$ [2851] G if $n = 7 \text{ and } t \equiv 0, 3 \pmod{4}$ [2857] G if $n = 9 \text{ and } t \equiv 0, 3 \pmod{4}$ [2858] G if $t = 2 \ n \neq 1 \pmod{4}$ [2048], [486] G if $n = 11$ [2837]

Graph	Graceful
triangular snakes (see $\S2.2$ )	G iff no. blocks $\equiv 0, 1 \pmod{4}$ [1793]
$K_4$ -snakes (see §2.2)	?
quadrilateral snakes (see $\S2.2$ )	G [894], [2048]
crowns $C_n \odot K_1$	G [795]
$C_n \odot P_k$	G [2200]
grids $P_m \times P_n$	G [36]
prisms $C_m \times P_n$	G if $n = 2$ [798], [2854] G if $m$ even [1032] G if $m$ odd and $3 \le n \le 12$ [1032] G if $m = 3$ [2394] G if $m = 6$ see [2856] G if $m \equiv 2 \pmod{4}$ , $n \equiv 3 \pmod{4}$ [2856]
$K_m \times P_n$	G if $(m, n) = (4, 2), (4, 3), (4, 4), (4, 5),$ (5, 2), (5, 3), (6, 3), (4, 6), (4, 7), (4, 8) not G if $(3, 3), (m, 2)$ $m = 6, 7, 8, 9, 10,$ 11,12 not G? for $(m, 2)$ with $m > 12$ [2431]
$K_m \times C_n$	G if $(m, n) = (4, 3), (3, 4), (4, 4), (4, 5),$ (3, 6), (4, 6) not G for $(m, n) = (6, 3)$ [2431]
$K_m \odot K_1$	G if $m = 3, 4, 5, 6, 7, 8, 9$ not G if $m = 10, 11, 12, 13, 14, 15$ not G? if $m > 15$ [2431]
$K_{m,n} \odot K_1$	G [1256]
$K_m \cup K_n \ (m, n > 1)$	G iff $\{m, n\} = \{4, 2\}$ or $\{5, 2\}$
$\bigcup_{i=1}^{t} K_{m_i,n_i}$	G 2 $\leq m_i < n_i$ [354]
torus grids $C_m \times C_n$	G if $m \equiv 0 \pmod{4}$ , n even [1266]

Table 1 – Continued from previous page

Graph	Graceful		
	not G if $m, n$ odd (parity condition)		
vertex-deleted $C_m \times P_n$	G if $n = 2$ [816]		
edge-deleted $C_m \times P_n$	G if $n = 2$ [816]		
Möbius ladders $M_n$ (see §2.3)	G [810]		
stacked books $S_m \times P_n$ (see §2.3)	$n = 2$ , G iff $m \not\equiv 3 \pmod{4}$ [1687], [657], [815] G if $m$ even [815]		
<i>n</i> -cube $K_2 \times K_2 \times \cdots \times K_2$ $K_n \times P_3$	G [1416] G iff $n \le 6$ [1383]		
$K_4 \times P_n$	G if $n = 2, 3, 4, 5$ [1920]		
$K_n$	G iff $n \le 4$ [902], [2390]		
$K_{m,n}$	G [2136], [902]		
$K_{1,m,n}$	G [178]		
$K_{1,1,m,n}$	G [894]		
windmills $K_n^{(m)}(n > 3)$ (see §2.4)	G if $n = 4, m \le 1000 \ [1032], [5], [2797], [863]$ G? if $n = 4, m \ge 4 \ [445]$ not G if $n = 4, m = 2, 3 \ [445]$ not G if $(m, n) = (2, 5) \ [448]$ not G if $n > 5 \ [1396]$		
$B(n, r, m) \ r > 1 \ (\text{see §2.4})$	G if $(n, r) = (3, 2), (4, 3)$ [1390], $(4, 2)$ [657] G $(n, r, m) = (5, 2, 2)$ [2431] not G for $(n, 2, 2)$ for $n > 5$ [447], [2431]		
$ \begin{array}{c} mK_n \text{ (see §2.5)} \\ C_m \cup P_n \end{array} $	G iff $m = 1, n \le 4$ [1420] G iff $m + n \ge 6$ [2595]		
$C_m \cup C_n$	G iff $m + n \equiv 0, 3 \pmod{4}$ [19]		

Table 1 – Continued from previous page

Graph	Graceful
$C_n \cup K_{p,q}$	for $n > 8$ G iff $n \equiv 0, 3 \pmod{4}$ [2875]
	$G C_6 \times K_{1,2n+1} [354]$
	$\begin{array}{c} G \ C_3 \times K_{m,n} \ \text{iff} \ m,n \ge 2 \ [2260] \\ G \ G \ C_3 \times K_{m,n} \ \text{iff} \ m,n \ge 2 \ [2260] \end{array}$
	$\begin{array}{c} G \ C_4 \times K_{m,n} \ \text{iff} \ (m,n) \neq (1,1)[2260] \\ G \ C_4 & K_{m,n} \ \text{iff} \ (m,n) \neq (1,1)[2260] \end{array}$
	$\begin{bmatrix} G & C_7 \times K_{m,n} & [2200] \\ G & G & \times & K \end{bmatrix} $
	$G \ C_8 \times K_{m,n} \ [2200]$
$K_i \cup K_{m,n}$	G [354]
$\bigcup_{i=1}^{t} K_{m_i,n_i}$	G $2 \le m_i < n_i \ [354]$
$C \downarrow \downarrow \downarrow \downarrow^t K$	$G_{2} \leq m \leq n$
$\bigcup_{m} \bigcup_{i=1} \prod_{m_i, n_i}$	$m = 0 \text{ or } 3 \pmod{4}$ $m > 11 [354]$
$G + \overline{K_t}$	G for connected graceful $G$ [22]
double cones $C_n + \overline{K_2}$	G for $n = 3, 4, 5, 7, 8, 9, 11, 12$
	not G for $n \equiv 2 \pmod{4} [2109]$
t-point suspension $C_n + K_t$	$\begin{array}{c} G \text{ if } n \equiv 0 \text{ or} \\ 2 \left( -1 10 \right) \left[ 4 C 2 \right] \end{array}$
	$3 \pmod{12} [403]$
	$n = 2.6, 10 \pmod{12}$
	$n \equiv 2, 0, 10 \pmod{12}$ C if $n = 4, 7, 11 \text{ or } 10 [463]$
	G if $n = 5$ or 9 and $t = 2$ [463]
	[0, 1, n = 0, 01, 0, and n = 2, [100]
$P_n^2$ (see §2.7)	G [1496]
Petersen $P(n,k)$ (see §2.7)	G for $n = 5, 6, 7, 8, 9, 10$ [2109],
	(n,k) = (8t,3) [2715]

Table 1 – Continued from previous page

Table 2: Sum	mary of	<sup>*</sup> Harmonious	Results
--------------	---------	-------------------------	---------

Graph	Harmonious
trees	H if $\leq 31$ vertices [748]
	H? [925]
	H caterpillars [925]
	? lobsters

Graph	Harmonious
cycles $C_n$	H iff $n$ is odd [925]
wheels $W_n$	H [925]
helms (see $\S2.2$ )	H [894], [1619]
webs (see $\S2.2$ )	H if cycle is odd
gears (see $\S2.2$ )	H [576]
cycles with $P_k$ -chord (see §2.2)	?
$C_n$ with k consec. chords (see §2.2)	?
unicyclic graphs	?
$P_n^k$	H if $k = 2$ [922], k odd [2223], [2882] H if k is even and $k/2 \le (n-1)/2$ [2894]
$C_n^{(t)}$ (see §2.2)	$n = 3$ H iff $t \not\equiv 2 \pmod{4}$ [925] H if $n = 4, t > 1$ [2320]
triangular snakes (see $\S2.2$ )	H if number of blocks is odd [2834] not H if number of blocks $\equiv 2 \pmod{4}$ [2834]
$K_4$ -snakes (see §2.2)	Н [923]
quadrilateral snakes (see $\S2.2$ )	?
crowns $C_n \odot K_1$	H [922], [1606]
grids $P_m \times P_n$ prisms $C_m \times P_n$	H iff $(m, n) \neq (2, 2)$ [1266] H if $n = 2, m \neq 4$ [816] H if $n \text{ odd } [925]$ H if $m = 4$ and $n \geq 3$ [1266]

Table 2 – Continued from previous page

Graph	Harmonious
torus grids $C_m \times C_n$ ,	H if $m = 4, n \ge 3$ [1266] not H if $m \not\equiv 0 \pmod{4}$ , n  odd  [1266]
vertex-deleted $C_m \times P_n$	H if $n = 2$ [816]
edge-deleted $C_m \times P_n$	H if $n = 2$ [816]
Möbius ladders $M_n$ (see §2.3)	H iff $n \neq 3$ [810]
stacked books $S_m \times P_n$ (see §2.3)	n = 2, H if m even [921], [2110] not H $m \equiv 3 \pmod{4}$ , $n = 2$ , (parity condition) H if $m \equiv 1 \pmod{4}$ , $n = 2$ [894]
<i>n</i> -cube $K_2 \times K_2 \times \cdots \times K_2$	H if and only if $n \ge 4$ [1059]
$K_4 \times P_n$	H [2110]
$K_n$	H iff $n \le 4$ [925]
$K_{m,n}$	H iff $m$ or $n = 1$ [925]
$K_{1,m,n}$	H [178]
$K_{1,1,m,n}$	H [894]
windmills $K_n^{(m)}$ $(n > 3)$ (see §2.4)	H if $n = 4$ [1025] m = 2, H? iff $n = 4$ [925] not H if $m = 2, n$ odd or 6 [925] not H for some cases $m = 3$ [1605] $(n, n) = (2, 2) \cdot (4, 2) \cdot [2257]$
$B(n, r, m) \ r > 1 \ (\text{see §2.4})$	(n,r) = (3,2), (4,3) [2257]
$mK_n \text{ (see §2.5)}$	H $n = 3$ , m odd [1607] not H for n odd and $m \equiv 2 \pmod{4}$ [1607]
nG	H when $G$ is harmonious and

Table 2 – Continued from previous page

Continued on next page

Graph	Harmonious
$G^n$	n  odd  [2873] H when $G$ is harmonious and n  odd  [2873]
$C_m \cup P_n$	?
fans $F_n = P_n + K_1$	H [925]
$nC_m + K_1 \ n \not\equiv 0 \ \mathrm{mod} \ 4$	H [576]
double fans $P_n + \overline{K_2}$	H [925]
<i>t</i> -point suspension $P_n + \overline{K_t}$ of $P_n$	H [2110]
$S_m + K_1$	H [894], [558]
t-point suspension $C_n + \overline{K_t}$ of $C_n$	H if n odd and $t = 2$ [2110], [894] not H if $n \equiv 2, 4, 6 \pmod{8}$ and $t = 2$ [894]
Petersen $P(n,k)$ (see §2.7)	H [894], [1532]

Table 2 – Continued from previous page

# **3** Variations of Graceful Labelings

## **3.1** $\alpha$ -labelings

In 1966 Rosa [2136] defined an  $\alpha$ -labeling (or  $\alpha$ -valuation) as a graceful labeling with the additional property that there exists an integer k so that for each edge xy either  $f(x) \leq k < f(y)$  or  $f(y) \leq k < f(x)$ . (Other names for such labelings are balanced, interlaced, and strongly graceful.) It follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an  $\alpha$ -labeling is necessarily bipartite and therefore can not contain a cycle of odd length. Wu [2825] has shown that a necessary condition for a bipartite graph with n edges and degree sequence  $d_1, d_2, \ldots, d_p$ to have an  $\alpha$ -labeling is that the  $gcd(d_1, d_2, \ldots, d_p, n)$  divides n(n-1)/2. Barrientos and Minion [385] proved that any tree of size n and excess  $\epsilon$  is a spanning tree of a graph of size  $n + \epsilon$  that admits an  $\alpha$ -labeling.

For a path with consecutive vertices  $v_1, v_2, \ldots, v_n$  a triangular tree is the tree obtained identifying each  $v_i$  to an end vertex of the path  $P_i$ . Barrientos [361] proved that all triangular trees admit an  $\alpha$ -labeling. He also presented several ways to combine this type of trees to construct new trees and unicyclic graphs that can  $\alpha$ -labeled.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an  $\alpha$ -labeling is a typical approach. (See [558], [922], [575], and [1266].) Moreover, Jungreis and Reid [1266] showed how sequential labelings of graphs (see Section 4.1) can often be obtained by modifying  $\alpha$ -labelings of the graphs.

Graphs with  $\alpha$ -labelings have proved to be useful in the development of the theory of graph decompositions. Rosa [2136], for instance, has shown that if G is a graph with q edges and has an  $\alpha$ -labeling, then for every natural number p, the complete graph  $K_{2qp+1}$  can be decomposed into copies of G in such a way that the automorphism group of the decomposition itself contains the cyclic group of order p. In the same vein El-Zanati and Vanden Eynden [721] proved that if G has q edges and admits an  $\alpha$ -labeling then  $K_{qm,qn}$  can be partitioned into subgraphs isomorphic to G for all positive integers m and n. Although a proof of Ringel's conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have  $\alpha$ -labelings are easy to construct (one example is the subdivision graph of  $K_{1,3}$  — see [2136]). Kotzig [1414] has shown however that almost all trees have  $\alpha$ -labelings. Sethuraman and Ragukumar [2286] have proved that every tree is a subtree of a graph with an  $\alpha$ -labeling.

As to which graphs have  $\alpha$ -labelings, Rosa [2136] observed that the *n*-cycle has an  $\alpha$ -labeling if and only if  $n \equiv 0 \pmod{4}$  whereas  $P_n$  always has an  $\alpha$ -labeling. Other familiar graphs that have  $\alpha$ -labelings include caterpillars [2136], the *n*-cube [1413], Möbius ladders  $M_n$  when *n* is odd (see §2.3) for the definition) [1899],  $B_{4n+1}$  (i.e., books with 4n + 1 pages) [815],  $C_{2m} \cup C_{2m}$  and  $C_{4m} \cup C_{4m} \cup C_{4m}$  for all m > 1 [1415],  $C_{4m} \cup C_{4m} \cup C_{4n}$  for all  $(m, n) \neq 1, 1$ ) [738],  $P_n \times Q_n$  [1687],  $K_{1,2k} \times Q_n$  [1687],  $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$  (1461],  $C_{4m} \cup C_{4n+2} \cup C_{4r+2}, C_{4m} \cup C_{4n} \cup C_{4r}$  when  $m + n \leq r$  [19],  $C_{4m} \cup C_{4n} \cup C_{4r} \cup C_{4s}$  when  $m \geq n + r + s$  [15],  $C_{4m} \cup C_{4n} \cup C_{4r+2} \cup C_{4s+2} \cup C_{4n} \cup C_{4n} \cup C_{4s+2} \cup C_{4s+2} \cup C_{4n} \cup C_{4n} \cup C_{4n} \cup C_{4s+2} \cup C_{4s+2} \cup C_{4n} \cup C_{4n} \cup C_{4n} \cup C_{4s+2} \cup C_{4s+2} \cup C_{4n} \cup C_{4n} \cup C_{4n} \cup C_{4s+2} \cup C_{4s+2} \cup C_{4n} \cup C_{4n} \cup C_{4s+2} \cup C_{4s+2} \cup C_{4n} \cup C_{4n} \cup C_{4n} \cup C_{4s+2} \cup C_{4s+2}$ 

for all m [2916],  $k^2C_4$  for all k [2916], and  $(k^2 + k)C_4$  for all k [2916]. Abrham and Kotzig [17] have shown  $kC_4$  has an  $\alpha$ -labeling for  $4 \le k \le 10$  and that if  $kC_4$  has an  $\alpha$ -labeling then so does  $(4k + 1)C_4$ ,  $(5k + 1)C_4$ , and  $(9k + 1)C_4$ . Eshghi [731] proved that  $3C_{4k}$  and  $5C_{4k}$  have an  $\alpha$ -labeling for all k. In [738] Eshghi and Carter show several families of graphs of the form  $C_{4n_1} \cup C_{4n_2} \cup \cdots \cup C_{4n_k}$  have  $\alpha$ -labelings.

In [734] Eshghi provides an integer programming model and a Tabu search algorithm to generate  $\alpha$ -labelings of the quadratic graphs  $mC_{4k}$ ) where  $6 \ge m \ge 10$  and  $2 \ge k \ge 10$ . (See also [740].) The computational complexity of the gracefulness of a graph is not known, but the complexity of finding a harmonious labeling of a graph is in the NP-class [147]. Research on programming models for finding graceful labelings of graphs can be found in [730], [740], [739], [1460], [2154], [736], [2109], [2431], [1686], and [2302].

In [147] Amini and Eshghi gave a new mathematical integer programming model for the graph labeling graphs of the form  $mC_n$  (some authors use the notation Q(m, n)). The advantages of this model are linearity and the existence of an objective function. They also gave two constraint programming models and a meta-heuristics algorithm that generate feasible graceful labeling and  $\alpha$ -labeling for special classes of quadratic graphs. Their results include:  $mC_{4k}$  with  $1 \leq 11$  and less than 1000 vertices has an  $\alpha$ -labeling with the exception of  $3C_4$ ;  $12C_{4k}$  has  $\alpha$ -labeling for  $1 \leq k \leq 19$ ; and  $13C_{4k}$  has  $\alpha$ -labeling for  $1 \leq k \leq 13$ . In [739] and [2154] Eshghi and Salarrezaei proved that  $7C_{4k}$  has an  $\alpha$ -labeling for all k. Lakshmi and Vangipuram [1460] proved that  $4C_{4k}$  is graceful.

In [387], Barrientos and Minion investigated series-parallel operations with graphs that admit  $\alpha$ -labelings. They provided necessary conditions on the graphs  $G_1$  and  $G_2$  to obtain a new  $\alpha$ -labeled graph G through each of these operations. As consequence of the series operation, they proved that the one-point union of three or four copies of  $K_{n,n}$  has an  $\alpha$ -labeling, and that any tree with maximum degree four that can be decomposed into copies of the path of length eleven has an  $\alpha$ -labeling when the distance between any pair of vertices of degree four is even. They also showed that any graph of order n + 1 and size n with an  $\alpha$ -labeling is an induced subgraph of a graph of order n + 3 and size 2n + 1. Additionally, they presented an  $\alpha$ -labeling for any graph of the form  $K_{2,n} \times P_m$ .

Figueroa-Centeno, Ichishima, and Muntaner-Batle [763] have shown that if  $m \equiv 0 \pmod{4}$  then the one-point union of 2, 3, or 4 copies of  $C_m$  admits an  $\alpha$ -labeling, and if  $m \equiv 2 \pmod{4}$  then the one-point union of 2 or 4 copies of  $C_m$  admits an  $\alpha$ -labeling. They conjecture that the one-point union of n copies of  $C_m$  admits an  $\alpha$ -labeling if and only if  $mn \equiv 0 \pmod{4}$ .

Pei-Shan Lee [1486] proved that  $C_6 \times P_{2t+1}$  and gear graphs have  $\alpha$ -labelings. He raises the question of whether  $C_{4m+2} \times P_{2t+1}$  has an  $\alpha$ -labeling for all m. Brankovic, Murch, Pond, and Rosa [499] conjectured that all trees with maximum degree three and a perfect matching have an  $\alpha$ -labeling.

In his 2001 Ph. D. thesis Selvaraju [2202] investigated the one-point union of complete bipartite graphs. He proves that the one-point unions of the following forms have an  $\alpha$ -labeling:  $K_{m,n_1}$  and  $K_{m,n_2}$ ;  $K_{m_1,n_1}$ ,  $K_{m_2,n_2}$ , and  $K_{m_3,n_3}$  where  $m_1 \leq m_2 \leq m_3$  and  $n_1 < n_2 < n_3$ ;  $K_{m_1,n}$ ,  $K_{m_2,n}$ , and  $K_{m_3,n}$  where  $m_1 < m_2 < m_3 \leq 2n$ .

Zhile [2916] uses  $C_m(n)$  to denote the connected graph all of whose blocks are  $C_m$  and

whose block-cutpoint-graph is a path. He proves that for all positive integers m and n,  $C_{4m}(n)$  has an  $\alpha$ -labeling but  $C_m(n)$  does not have an  $\alpha$ -labeling when m is odd.

Abrham and Kotzig [19] have proved that  $C_m \cup C_n$  has an  $\alpha$ -labeling if and only if both m and n are even and  $m + n \equiv 0 \pmod{4}$ . Kotzig [1415] has also shown that  $C_4 \cup C_4 \cup C_4$  does not have an  $\alpha$ -labeling. He asked if n = 3 is the only integer such that the disjoint union of n copies of  $C_4$  does not have an  $\alpha$ -labeling. This was confirmed by Abrham and Kotzig in [18]. Eshghi [730] proved that every 2-regular bipartite graph with 3 components has an  $\alpha$ -labeling if and only if the number of edges is a multiple of four except for  $C_4 \cup C_4 \cup C_4$ . In [733] Eshghi gives more results on the existence of  $\alpha$ -labelings for various families of disjoint union of cycles.

Jungreis and Reid [1266] investigated the existence of  $\alpha$ -labelings for graphs of the form  $P_m \times P_n, C_m \times P_n$ , and  $C_m \times C_n$  (see also [813]). Of course, the cases involving  $C_m$  with m odd are not bipartite, so there is no  $\alpha$ -labeling. The only unresolved cases among these three families are  $C_{4m+2} \times P_{2n+1}$  and  $C_{4m+2} \times C_{4n+2}$ . All other cases result in  $\alpha$ -labelings.

Let  $v_{1,j}, v_{2,j}, \ldots, v_{m,j}$  be the consecutive vertices of the *j*th copy of  $P_m$  in  $P_m \times P_n$ . An elementary transformation of  $P_m \times P_n$  is the graph obtained by replacing the edge  $v_{i,j}v_{i+1,j}$ by the new edge  $v_{i-x,j}v_{i+1+x,j}$ . A graph is said to be a grid-like graph if it is obtained through a sequence of elementary transformations. In [388] Barrientos and Minion proved the exitence of an  $\alpha$ -labeling for any grid-like graph. As consequence of this result, they showed that the graphs  $C_{4t} \times P_n \cup P_n$  and  $C_{4t} \times P_n \cup P_{t-1} \times P_n$  admit  $\alpha$ -labelings.

Balakrishman [333] uses the notation  $Q_n(G)$  to denote the graph  $P_2 \times P_2 \times \cdots \times P_2 \times G$ where  $P_2$  occurs n-1 times. Snevily [2434] has shown that the graphs  $Q_n(C_{4m})$  and the cycles  $C_{4m}$  with the path  $P_n$  adjoined at each vertex have  $\alpha$ -labelings. He [2435] also has shown that compositions of the form  $G[\overline{K_n}]$  (see §2.3 for the definition) have an  $\alpha$ -labeling whenever G does (see §2.3 for the definition of composition). Balakrishman and Kumar [336] have shown that all graphs of the form  $Q_n(G)$  where G is  $K_{3,3}, K_{4,4}$ , or  $P_m$  have an  $\alpha$ -labeling. Balakrishman [333] poses the following two problems. For which graphs Gdoes  $Q_n(G)$  have an  $\alpha$ -labeling? For which graphs G does  $Q_n(G)$  have a graceful labeling?

Rosa [2136] has shown that  $K_{m,n}$  has an  $\alpha$ -labeling (see also [351]). In [1062] Ichishima and Oshima proved that if m, s and t are integers with  $m \geq 1, s \geq 2$ , and  $t \geq 2$ , then the graph  $mK_{s,t}$  has an  $\alpha$ -labeling if and only if  $(m, s, t) \neq (3, 2, 2)$ . Barrientos [351] has shown that for n even the graph obtained from the wheel  $W_n$  by attaching a pendent edge at each vertex has an  $\alpha$ -labeling. In [358] Barrientos shows how to construct graceful graphs that are formed from the one-point union of a tree that has an  $\alpha$ -labeling,  $P_2$ , and the cycle  $C_n$ . In some cases,  $P_2$  is not needed. Qian [2048] has proved that quadrilateral snakes have  $\alpha$ -labelings. Yu, Lee, and Chin [2891] showed that  $Q_3$ -and  $Q_3$ -snakes have  $\alpha$ -labelings. Fu and Wu [802] showed that if T is a tree that has an  $\alpha$ -labeling with partite sets  $V_1$  and  $V_2$  then the graph obtained from T by joining new vertices  $w_1, w_2, \ldots, w_k$  to every vertex of  $V_1$  has an  $\alpha$ -labeling. Similarly, they prove that the graph obtained from Tby joining new vertices  $w_1, w_2, \ldots, w_k$  to the vertices of  $V_1$  and new vertices  $u_1, u_2, \ldots, u_t$ to every vertex of  $V_2$  has an  $\alpha$ -labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of  $V_1$  or the vertices of both  $V_1$  and  $V_2$ , the resulting graphs have  $\alpha$ -labelings. Fu and Wu [802] further show that if T is a tree with an  $\alpha$ -labeling and the sizes of the two partite sets of T differ by at most 1, then  $T \times P_m$  has an  $\alpha$ -labeling. Zhao, Ma, and Yao [2911] proved that a class of super lobster trees have  $\alpha$ -labelings. Ghosh [890] uses various methods of joining graceful graphs and graphs with  $\alpha$ -labelings to obtain some classes of graceful lobsters.

Selvaraju and G. Sethurman [2206] prove that the graphs obtained from a path  $P_n$  by joining all the pairs of vertices u, v of  $P_n$  with d(u, v) = 3 and the graphs obtained by identifying one of vertices of degree 2 of such graphs with the center of a star and the other vertex the graph of degree 2 with the center of another star (the two stars needs need not have the same size) have  $\alpha$ -labelings. They conjecture that the analogous graphs where 3 is replaced with any t with  $2 \le t \le n-2$  have  $\alpha$ -labelings.

Makadia, Karavadiya, and Kanerian [1693] proved that the graph obtained by merging t consecutive vertices of two cycle  $C_{4r}$  and  $C_{4s}$  has an  $\alpha$ -labeling when  $t \leq 2\min\{r, s\}$ . They also proved that if  $G_1$  has an  $\alpha$ -labeling and  $G_2$  is graceful then there exists a graceful labeling of the graph obtained by joining  $G_1$  and  $G_2$  by any path. Moreover, if both  $G_1$  and  $G_2$  have  $\alpha$ -labelings then there exists an  $\alpha$ -labeling of the graph obtained by joining  $G_1$  and  $G_2$  by any path. Let  $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$  be a collection of cycles where each  $n_i \equiv 0 \pmod{4}$ . In [386], Barrientos and Minion say that a graph G is the *coalescence* of these cycles if for every  $2 \leq i \leq k$ , the first  $t_i$  vertices of  $C_{n_i}$  are identified with the last  $t_i$  vertices of  $P_{n_{i-1}}$ , where  $t_i \leq n_i/2$ .

Lee and Liu [1504] investigated the mirror graph M(m, n) of  $K_{m,n}$  (see §2.3 for the definition) for  $\alpha$ -labelings. They proved: M(m, n) has an  $\alpha$ -labeling when n is odd or m is even; M(1, n) has an  $\alpha$ -labeling when  $n \equiv 0 \pmod{4}$ ; M(m, n) does not have an  $\alpha$ -labeling when m is odd and  $n \equiv 2 \pmod{4}$ , or when  $m \equiv 3 \pmod{4}$  and  $n \equiv 4 \pmod{8}$ .

Barrientos and Minion [377] proved that the Cartesian product of two  $\alpha$ -trees is an  $\alpha$ -tree when both trees admit  $\alpha$ -labelings and their stable sets are balanced. (A *stable* set S consists of a set of vertices such that there is not an edge  $v_i v_j$  for all pairs  $v_i, v_j$  in S). In addition, they present a tree that has the property that when any number of pendent vertices are attached to the vertices of any subset of its smaller stable set the resulting graph is an  $\alpha$ -tree. They also prove of an  $\alpha$ -labeling of three types of graphs obtained by connecting, sequentially, any number of paths of equal size.

Barrientos [352] defines a *chain graph* as one with blocks  $B_1, B_2, \ldots, B_m$  such that for every *i*,  $B_i$  and  $B_{i+1}$  have a common vertex in such a way that the block-cutpoint graph is a path. He shows that if  $B_1, B_2, \ldots, B_m$  are blocks that have  $\alpha$ -labelings then there exists a chain graph *G* with blocks  $B_1, B_2, \ldots, B_m$  that has an  $\alpha$ -labeling. He also shows that if  $B_1, B_2, \ldots, B_m$  are complete bipartite graphs, then any chain graph *G* obtained by concatenation of these blocks has an  $\alpha$ -labeling.

The symmetric product  $G_1 \oplus G_2$  of  $G_1$  and  $G_2$  is the graph with vertex set  $V(G_1) \times V(G_2)$  and edge set  $\{(u_1, v_1)(u_2, v_2)\}$  where  $u_1u_2$  is an edge in  $G_1$  or  $v_1v_2$  is an edge in  $G_2$  but not both  $u_1u_2$  is an edge in  $G_1$  and  $v_1v_2$  is an edge in  $G_2$ . A snake of length n > 1 is a packing of n congruent geometrical objects, called *cells*, such that the first and the last cell

each has only one neighbor and all n-2 cells in between have exactly two neighbors. In [372] Barrientos and Minion define a snake polyomino as a snake with square cells. They prove that given two graphs of sizes m and n with  $\alpha$ -labelings, the graph that results from the edge amalgamation (identification of two edges) of the edges of weight 1 and n, also has an  $\alpha$ -labeling. They use that result to prove the existence of  $\alpha$ -labelings of snake polyominoes and hexagonal chains. The result about snake polyominoes partially answers the question of Acharya. In [373], they prove that the third power of a caterpillar admits an  $\alpha$ -labeling and that the symmetric product  $G \oplus 2K_1$  has an  $\alpha$ -labeling when G does. In addition they prove that  $G \cup P_m$  is graceful provided that G admits an  $\alpha$ -labeling that does not assign the integer  $\lambda + 2$  as a label, where  $\lambda$  is its boundary value. They ask if all triangular chains are graceful.

In [379] Barrientos and Minion proved that under certain conditions, the union  $C_r \cup G$ of the cycle  $C_r$  and a caterpillar G admits a graceful labeling when r is odd, and an  $\alpha$ -labeling when r is even. They also proved the existence of an  $\alpha$ -labeling for any tree obtained by connecting with a path of length two the central vertices of  $G_i$  and  $G_{i+1}$ , where  $G_i$  is a caterpillar of diameter 2d with bipartite sets  $A_i$  and  $B_i$  such that  $|A_i| = |B_i| + 1$ and  $A_i$  contains the vertices of maximum eccentricity in  $G_i$ .

Let  $T_1, T_2, \ldots, T_s$  be trees. A *chain tree* obtained by identifying, for every  $1 \le i \le s-1$ , a vertex of  $T_i$  with a vertex of  $T_{i+1}$ . In [380], Barrientos and Minion prove that if every  $T_i$ admits an  $\alpha$ -labeling, then there exists a chain tree that also admits an  $\alpha$ -labeling. Let T be a tree of size n and v be a fixed vertex of T. The tree  $T_v^{+r}$  is obtained by connecting, with a path of length r, two copies of T, by identifying the end-point of this path with the vertices v of each copy of T. They give necessary conditions for the existence of an  $\alpha$ -labeling for a tree  $T_v^{+2}$ , where v is any of the vertices labeled  $\lambda, \lambda - 1, \ldots, \lambda - \deg(v) - 1$ by an  $\alpha$ -labeling with boundary value  $\lambda$  that assigns the labels  $\lambda + 1, \lambda + 2, \dots, \lambda + \deg(v)$ to leaves of T. In addition they proved that  $T_v^{+4}$  has an  $\alpha$ -labeling if there exists an  $\alpha$ labeling f of T, with boundary value  $\lambda$ , such that  $f(v) = \lambda - 1$ . In [380], Barrientos and Minion prove the following. The tree  $\oplus(T_1, T_2, T_3, T_4)$  obtained by connecting to a new vertex w, the vertices labeled n in  $T_1$  and  $T_3$  and the vertices labeled n/2 in  $T_2$  and  $T_4$ , where  $T_i$  is an  $\alpha$ -labeled tree of even size n that has partite sets of cardinality n/2 and n/2 + 1. If G is a graph of order m and size n, with m < n, that admits an  $\alpha$ -labeling, and H is any graceful graph of size t-1, then  $tG \cup H$  is a graceful graph. For every  $m \ge n, m \ge 3, n \ge 2, \text{ and } t \ge 2, tK_{m,n} \cup L_{t-1}$  admits an  $\alpha$ -labeling where  $L_{t-1}$  is any linear forest of size t-1. If G is a graph of order m and size n, with m < n, that admits an  $\alpha$ -labeling, then  $tG \cup L_{t-1}$  also admits an  $\alpha$ -labeling when  $L_{t-1}$  is a linear forest of size t-1. As a consequence of this result they prove that  $tG \cup P_t$  admits an  $\alpha$ -labeling provided that G does.

Barrientos [384] showed that all lobsters constructed with k copies of any caterpillar of diameter four by connecting the central vertices of all pairs of consecutive copies with an edge have an  $\alpha$ -labeling. Additionally, he proved that any chain-tree formed by caterpillars and this type of lobsters admits an  $\alpha$ -labeling. Barrientos and Minion [388] say that a tree is *regular* when the cardinalities of its stable sets are equal or differ by one. They prove if S and T are regular trees that admit  $\alpha$ -labelings then  $S \times T$  also admits an  $\alpha$ -labeling. They use this result to prove that  $S \times T$  admits a sequential labeling (see Section 4.1) as well as a harmonious labeling. They define a *fence* as the tree obtained by connecting an internal vertex of  $P_{n_i}$  with an internal vertex of  $P_{n_{i+1}}$  by a path of length  $l_i$  for every  $1 \leq i \leq t$ . They prove the existence of an  $\alpha$ -labeling for any fence constructed with tcopies of  $P_n$ , where  $l_i = 2$ . They define a 2-*link fence* as the graph obtained by connecting with an edge, two vertices of the *i*th copy of  $P_n$ , with the corresponding two vertices of the (i + 1)th copy of  $P_n$ . They prove that all such graphs admit  $\alpha$ -labelings. In [384] Barrientos says that a fence is *irregular* if two consecutive copies of  $P_n$  are connected by one or two pairs of corresponding vertices. He proved that all irregular fences have an  $\alpha$ -labeling provided that all their Eulerian subgraphs have size divisible by four. In [389] Barrientos and Minion study subfamilies of 2-link fences, a subfamily of column-convex polyominoes, and a subfamily of irregular cyclic-snakes. They prove that under certain conditions, an  $\alpha$ -labelings of these graphs can be transformed into harmonious labelings.

Barrientos and Minion [390] provided new families of harmoniously labeled graphs built on  $\alpha$ -labeled tress. Among them are  $P_n^{\ k}$ , the join of G and  $tK_1$  where G has a restrictive type of harmonious labeling and its order is different of its size by at most one,  $K_{m,n} \cup K_{1,m-1}$ , and  $G \cup T$  where G is a unicyclic graph and T is a tree built with  $\alpha$ -trees. They also showed that almost all trees admit harmonious labelings.

In [383] Barrientos and Minion extend the concept of vertex amalgamation as follows. The *k*-vertex amalgamation of  $G_1$  and  $G_2$  is the graph obtained by identifying *k* independent vertices of  $G_1$  with *k* independent vertices of  $G_2$ . A *t*-fold of a graph *G* is obtained using *t*-copies of *G*, where the *i*th copy of *G* is *k*-vertex amalgamated with the (i + 1)th copy of *G*. They prove that if *G* admits an  $\alpha$ -labeling, then any *t*-fold of *G* admits an  $\alpha$ -labeling. They consider a more general version of this construction for the case where *G* is a tree. They also introduce a new family of trees that admit  $\alpha$ -labelings; in particular, they prove that any tree of diameter 2n formed by identifying the end-vertices of four caterpillars admits an  $\alpha$ -labeling.

Fronček, Kingston, and Vezina [786] generalized snake polyomino graphs by introducing straight simple polyominal caterpillars and proving that they also admit an alpha labeling. This implies that every straight simple polyominal caterpillar with n edges decomposes the complete graph  $K_{2kn+1}$  for any positive integer k. In [781] Fronček introduced a similar family of graphs called *full hexagonal caterpillars* and prove that they admit an alpha labeling. This implies that every full hexagonal caterpillar with n edges decomposes the complete graph  $K_{2kn+1}$  for any positive integer k.

Golomb [903] introduced polyominoes in 1953 in a talk to the Harvard Mathematics Club. *Polyominoes* are planar shapes made by connecting a certain number of equal-sized squares, each joined together with at least one other square along an edge.

A graph G = (V(G), E(G)) is even graceful if there exists an injection f from the set of vertices V(G) to  $\{0, 1, 2, 3, 4, \ldots, 2|E(G)|\}$  such that when each edge uv is assigned the label |f(u) - f(v)|, the resulting edge labels are  $2, 4, 6, \ldots, 2|E(G)|$ . Elsonbaty and Mohamed [716] use even graceful labelings to give a new proof for necessary and sufficient conditions for the gracefulness of cycles. They extend this technique to odd graceful and super Fibonacci graceful labelings of cycle graphs. The *polar grid* graph  $P_{m,n}$  consists of n copies of  $C_m$  numbered from the inner most cycle to the outer cycle as  $C^{(1)}_m, \ldots, C^{(n)}_m$  and m copies of paths  $P_{n+1}$  intersected at the center vertex  $v_0$  numbered as  $P^{(1)}_{n+1}, \ldots, P^{(m)}_{n+1}$  In [717] Elsonbaty and Daoud provided edge even graceful labelings for various classes of  $P_m \times C_n$ . El Dean [2898] obtained an edge even graceful labeling for Y-trees, double stars  $B_{n,m}$ ,  $\langle K_{1,2n} : K_{1,2m} \rangle$ ,  $P_{2n-1} \odot \overline{K_{2m}}, \overline{K_2} + P_n$ , the cycle  $v_1, v_2, \ldots, v_{2n}$  with a chord from  $v_1$  to  $v_n, P_2 \odot C_n$ , flags, and flowers. Zeen El Deen and Omar [2899] gave sufficient conditions for  $K_{m,n}$  to have an edge even graceful labeling. They also provided edge even graceful labelings of the join of  $K_1$  with stars, wheels, and sunflowers, and the join of  $\overline{K_2}$  with stars and wheels. For results on Fibonacci trees see [1392].

Wu ([2824] and [2826]) has given a number of methods for constructing larger graceful graphs from graceful graphs. Let  $G_1, G_2, \ldots, G_p$  be disjoint connected graphs. Let  $w_i$  be in  $G_i$  for  $1 \leq i \leq p$ . Let w be a new vertex not in any  $G_i$ . Form a new graph  $\bigoplus_w (G_1, G_2, \ldots, G_p)$  by adjoining to the graph  $G_1 \cup G_2 \cup \cdots \cup G_p$  the edges  $ww_1, ww_2, \ldots, ww_p$ . In the case where each of  $G_1, G_2, \ldots, G_p$  is isomorphic to a graph Gthat has an  $\alpha$ -labeling and each  $w_i$  is the isomorphic image of the same vertex in  $G_i$ , Wu shows that the resulting graph is graceful. If f is an  $\alpha$ -labeling of a graph, the integer k with the property that for any edge uv either  $f(u) \leq k < f(v)$  or  $f(v) \leq k < f(u)$ is called the *boundary value* or *critical number* of f. Wu [2824] has also shown that if  $G_1, G_2, \ldots, G_p$  are graphs of the same order and have  $\alpha$ -labelings where the labelings for each pair of graphs  $G_i$  and  $G_{p-i+1}$  have the same boundary value for  $1 \leq i \leq n/2$ , then  $\bigoplus_w (G_1, G_2, \ldots, G_p)$  is graceful. In [2822] Wu proves that if G has n edges and n + 1vertices and G has an  $\alpha$ -labeling with boundary value  $\lambda$ , where  $|n - 2\lambda - 1| \leq 1$ , then  $G \times P_m$  is graceful for all m.

Given graceful graphs H and G with at least one having an  $\alpha$ -labeling Wu and Lu [2827] define four graph operations on H and G that when used repeatedly or in turns provide a large number of graceful graphs. In particular, if both H and G have  $\alpha$ -labelings, then each of the graphs obtained by the four operations on H and G has an  $\alpha$ -labeling.

Ajitha, Arumugan, and Germina [133] use a construction of Koh, Tan, and Rogers [1398] to create trees with  $\alpha$ -labelings from smaller trees with graceful labelings. These in turn allows them to generate large classes of trees that have a type of called edgeantimagic labelings (see §6.1). Shiue and Lu [2372] prove that the graph obtained from  $K_{1,k}$  by replacing each edge with a path of length 3 has an  $\alpha$ -labeling if and only if  $k \leq 4$ . In [2722] Venkatesh and Bharathi recursively construct new trees starting with caterpillars that admit  $\alpha$ -labelings.

Seoud and Helmi [2238] have shown that all gear graphs have an  $\alpha$ -labeling, all dragons with a cycle of order  $n \equiv 0 \pmod{4}$  have an  $\alpha$ -labeling, and the graphs obtained by identifying an endpoint of a star  $S_m$  ( $m \geq 3$ ) with a vertex of  $C_{4n}$  has an  $\alpha$ -labeling.

Mavonicolas and Michael [1722] say that trees  $\langle T_1, \theta_1, w_1 \rangle$  and  $\langle T_2, \theta_2, w_2 \rangle$  with roots  $w_1$  and  $w_2$  and  $|V(T_1)| = |V(T_2)|$  are gracefully consistent if either they are identical or they have  $\alpha$ -labelings with the same boundary value and  $\theta_1(w_1) = \theta_2(w_2)$ . They use this concept to show that a number of known constructions of new graceful trees using several identical copies of a given graceful rooted tree can be extended to the case where

the copies are replaced by a set of pairwise gracefully consistent trees. In particular, let  $\langle T, \theta, w \rangle$  and  $\langle T_0, \theta_0, w_0 \rangle$  be gracefully labeled trees rooted at w and  $w_0$  respectively. They show that the following four constructions are adaptable to the case when a set of copies of  $\langle T, \theta, w \rangle$  is replaced by a set of pairwise gracefully consistent trees. When  $\theta(w) = |E(T)|$  the garland construction due to Koh, Rogers, and Tan [1391] gracefully labels the tree consisting of h copies of  $\langle T, w \rangle$  with their roots connected to a new vertex r. In the case when  $\theta(w) = |E(T)|$  and whenever  $uw \in E(T)$  and  $\theta(u) \neq 0$ , then  $vw \in E(T)$  where  $\theta(u) + \theta(v) = |E(T)|$ , the attachment construction of Koh, Tan and Rogers [1398] gracefully labels the tree formed by identifying the roots of h copies of  $\langle T, w \rangle$ . A construction given by Koh, Tan and Rogers [1398] gracefully labels the tree formed by merging each vertex of  $\langle T_0, w_0 \rangle$  with the root of a distinct copy of  $\langle T, w \rangle$ . When  $\theta_0(w_0) = |E(T_0)|$ , let N be the set of neighbors of  $w_0$  and let x be the vertex of T at even distance from w with  $\theta(x) = 0$  or  $\theta(x) = |E(T)|$ . Then a construction of Burzio and Ferrarese [520] gracefully labels the tree formed by merging each non-root vertex of  $T_0$ with the root of a distinct copy of  $\langle T, w \rangle$  so that for each  $v \in N$  the edge  $vw_0$  is replaced with a new edge  $xw_0$  (where x is in the corresponding copy of T).

Snevily [2435] says that a graph G eventually has an  $\alpha$ -labeling provided that there is a graph H, called a host of G, which has an  $\alpha$ -labeling and that the edge set of H can be partitioned into subgraphs isomorphic to G. He defines the  $\alpha$ -labeling number of G to be  $G_{\alpha} = \min\{t : \text{there is a host } H \text{ of } G \text{ with } |E(H)| = t|G|\}$ . Snevily proved that even cycles have  $\alpha$ -labeling number at most 2 and he conjectured that every bipartite graph has an  $\alpha$ -labeling number. This conjecture was proved by El-Zanati, Fu, and Shiue [718]. There are no known examples of a graph G with  $G_{\alpha} > 2$ . In [2435] Snevily conjectured that the  $\alpha$ -labeling number for a tree with n edges is at most n. Shiue and Fu [2370] proved that the  $\alpha$ -labeling number for a tree with n edges and radius r is at most  $\lceil r/2 \rceil n$ . They also prove that a tree with n edges and radius r decomposes  $K_t$  for some  $t \leq (r+1)n^2+1$ .

Ahmed and Snevily [103] investigated the claim that for every tree T there exists an  $\alpha$ -labeling of T, or else there exists a graph  $H_T$  with an  $\alpha$ -labeling such that  $H_T$  can be decomposed into two edge-disjoint copies of T. They proved this claim is true for the graphs  $C_{m,k}$  obtained from  $K_{1,m}$  by replacing each edge in  $K_{1,m}$  with a path of length k.

A graph G with vertex set V and edge set E is called super edge-graceful if there is a bijection f from E to  $\{0, \pm 1, \pm 2, \ldots, \pm (|E| - 1)/2\}$  when |E| is odd and from E to  $\{\pm 1, \pm 2, \ldots, \pm |E|/2\}$  when |E| is even such that the induced vertex labeling  $f^*$  defined by  $f^*(u) = \sum f(uv)$  over all edges uv is a bijection from V to  $\{0, \pm 1, \pm 2, \ldots, \pm (|V| - 1)/2\}$ when |V| is odd and from V to  $\{\pm 1, \pm 2, \ldots, \pm |V|/2\}$  when |V| is even. Clifton and Khodkar [628] proved that graphs formed by identifying the endpoint of a path  $P_n$  and a vertex of a cycle (kites) with  $n \geq 5$  vertices,  $n \neq 6$  are super edge-graceful. Khodkar, Nolen, and Perconti [1368] proved that all complete bipartite graphs except for  $K_{2,2}, K_{2,3}$ , and  $K_{1,n}$  (n odd) are super edge-graceful. Khodkar [1370] and [1369] proved that all complete tripartite graphs except  $K_{1,1,2}$  are super edge-graceful and that the union of vertex disjoint 3-cycles is super edge-graceful. Lee, Su, and Wei [1550] provide a family of trees of odd orders which are super edge-graceful.

For a tree T with m edges, the  $\alpha$ -deficit  $\alpha_{def}(T)$  equals  $m - \alpha(T)$  where  $\alpha(T)$  is defined

as the maximum number of distinct edge labels over all bipartite labelings of T. Rosa and Siran [2139] showed that for every  $m \ge 1$ ,  $\alpha_{def}(C_{m,2}) = \lfloor m/3 \rfloor$ , which implies that  $(C_{m,2})_{\alpha} \ge 2$  for  $m \ge 3$ . Ahmed and Snevily [103] define the graph  $C'_{m,j}$  as a cometlike tree with a central vertex of degree m where each neighbor of the central vertex is attached to j pendent vertices for  $1 \le j \le (m-1)$ . For  $m \ge 3$  and  $1 \le j \le (m-1)$ they prove:  $(C'_{m,j})_{\alpha} \le 2$ ;  $(C'_{2k+1,j})_{\alpha} = 2$  for  $1 \le j \le 2k$  and conjecture if  $\Delta_T = (2k+1)$ , then  $\alpha_{def}(T) \le k$ . Ahmed and Snevily [103] prove that for every comet T (that is, graphs obtained from stars by replacing each edge by a path of some fixed length) there exists an  $\alpha$ -labeling of T, or else there exists a graph  $H_T$  with an  $\alpha$ -labeling such that  $H_T$  can be decomposed into two edge-disjoint copies of T. This is particularly noteworthy since comets are known to have arbitrarily large  $\alpha$ -deficits.

Given two bipartite graphs  $G_1$  and  $G_2$  with partite sets  $H_1$  and  $L_1$  and  $H_2$  and  $L_2$ , respectively, Snevily [2434] defines their weak tensor product  $G_1 \bigotimes G_2$  as the bipartite graph with vertex set  $(H_1 \times H_2, L_1 \times L_2)$  and with edge  $(h_1, h_2)(l_1, l_2)$  if  $h_1 l_1 \in E(G_1)$ and  $h_2 l_2 \in E(G_2)$ . He proves that if  $G_1$  and  $G_2$  have  $\alpha$ -labelings then so does  $G_1 \bigotimes G_2$ . This result considerably enlarges the class of graphs known to have  $\alpha$ -labelings. In [1628] López and Muntaner-Batle gave a generalization of Snevily's weak tensor product that allows them to significantly enlarges the classes of graphs admitting  $\alpha$ -labelings, near  $\alpha$ -labelings (defined later in this section), and bigraceful graphs.

The sequential join of graphs  $G_1, G_2, \ldots, G_n$  is formed from  $G_1 \cup G_2 \cup \cdots \cup G_n$  by adding edges joining each vertex of  $G_i$  with each vertex of  $G_{i+1}$  for  $1 \le i \le n-1$ . Lee and Wang [1559] have shown that for all  $n \ge 2$  and any positive integers  $a_1, a_2, \ldots, a_n$ the sequential join of the graphs  $\overline{K}_{a_1}, \overline{K}_{a_2}, \ldots, \overline{K}_{a_n}$  has an  $\alpha$ -labeling.

In [811] Gallian and Ropp conjectured that every graph obtained by adding a single pendent edge to one or more vertices of a cycle is graceful. Qian [2048] proved this conjecture and in the case that the cycle is even he shows the graphs have an  $\alpha$ -labeling. He further proves that for n even any graph obtained from an n-cycle by adding one or more pendent edges at some vertices has an  $\alpha$ -labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

In [1901] Pasotti introduced the following generalization of a graceful labeling. Given a graph G with e = dm edges, an injective function from  $V(\Gamma)$  to the set  $\{0, 1, 2, \ldots, d(m + 1) - 1\}$  such that  $\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, 3, \ldots, d(m + 1) - 1\} - \{m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)\}$  is called a *d*-divisible graceful labeling of G. Note that for d = 1 and of d = e one obtains the classical notion of a graceful labeling and of an odd-graceful labeling (see §3.6 for the definition), respectively. A *d*-divisible graceful labeling of a bipartite graph G with the property that the maximum value on one of the two bipartite sets is less than the minimum value on the other one is called a *d*-divisible  $\alpha$ -labeling of G. Pasotti proved that these new concepts allow to obtain certain cyclic graph decompositions. In particular, if there exists a *d*-divisible graceful labeling of a graph G of size e = dm then there exists a cyclic G-decomposition of  $K_{(\frac{e}{d}+1)\times 2dn}$  for any integer  $n \geq 1$ . She also it is proved the following: paths and stars admit a *d*-divisible  $\alpha$ -labeling for any admissible *d*;  $C_{4k}$  admits

a 2-divisible  $\alpha$ -labeling and a 4-divisible  $\alpha$ -labeling for any  $k \geq 1$ ;  $C_{2k}$  admits a 2-divisible labeling for any odd integer k > 1; and the ladder graph  $L_{2k}$  has a 2-divisible  $\alpha$ -labeling if and only if k is even.

Pasotti [1901] generalized the notion of graceful labelings for graphs G with e = $d \cdot m$  edges by defining a *d*-graceful labeling as an injective function f from V(G) to  $\{0, 1, 2, \dots, d(m+1) - 1\}$  such that  $\{|f(x) - f(y)| \mid xy \in E(G)\} = \{1, 2, \dots, d(m+1) - 1\}$  $1 - \{m + 1, 2(m + 1), \dots, (d - 1)(m + 1)\}$ . The case d = 1 is a graceful labeling and the case that d = e is an odd-graceful labeling. A *d*-graceful  $\alpha$ -labeling of a bipartite graph is a d-graceful labeling with the property that the maximum value in one of the two bipartite sets is less than the minimum value on the other bipartite set. Pasotti [1901] proved that paths and stars have d-graceful  $\alpha$ -labelings for all admissible d, ladders  $P_n \times P_2$  have a 2-graceful labeling if and only if n is even, and provided partial results about cycles of even length. He showed that the existence of d-graceful labelings can be used to prove that certain complete graphs have cyclic decompositions. Benini and Pasotti [435] used d-divisible  $\alpha$ -labelings to construct an infinite class of cyclic  $\Gamma$ -decompositions of the complete multipartite graphs, where  $\Gamma$  is a caterpillar, a hairy cycle or a cycle. Such labelings imply the existence of cyclic  $\Gamma$ -decompositions of certain complete multipartite graphs. Riasat, Kanwal, and Javed [2115] give odd-graceful labelings for disjoint unions of graphs consisting of generalized combs, ladders, stars, bistars, caterpillars and paths.

In [1900], Pasotti proved the existence of d-divisible  $\alpha$ -labelings for  $C_{4k} \times P_m$  for any integers  $k \ge 1$ ,  $m \ge 2$  for d = 2m - 1, 2(2m - 1) and 4(2m - 1). Benini and Pasotti [436] proved that the generalized Petersen graph  $P_{8n,3}$  admits an  $\alpha$ -labeling for any integer  $n \ge 1$  confirming that the conjecture posed by A. Vietri in [2715] is true.

For any tree T(V, E) whose vertices are properly 2-colored Rosa and Siráň [2139] define a bipartite labeling of T as a bijection  $f: V \to \{0, 1, 2, \dots, |E|\}$  for which there is a k such that whenever f(u) < k < f(v), then u and v have different colors. They define the  $\alpha$ -size of a tree T as the maximum number of distinct values of the induced edge labels  $|f(u) - f(v)|, uv \in E$ , taken over all bipartite labelings f of T. They prove that the  $\alpha$ -size of any tree with n edges is at least 5(n+1)/7 and that there exist trees whose  $\alpha$ -size is at most (5n+9)/6. They conjectured that minimum of the  $\alpha$ -sizes over all trees with n edges is asymptotically 5n/6. This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [522]. For trees with n vertices and maximum degree 3 Brankovic, Rosa, and Širáň [500] have shown that the  $\alpha$ -size is at least  $\lfloor \frac{6n}{7} \rfloor - 1$ . In [499] Brankovic, Murch, Pond, and Rose provide a lower bound for the  $\alpha$ -size trees with maximum degree three and a perfect matching as a function of a lower bound for minimum order of such a tree that does not have an  $\alpha$ -labeling. Using a computer search they showed that all such trees on less than 30 vertices have an  $\alpha$ -labeling. This brought the lower bound for the  $\alpha$ -size to 14n/15, for such trees of order n. They conjecture that all trees with maximum degree three and a perfect matching have an  $\alpha$ -labeling. Heinrich and Hell [1007] defined the gracesize of a graph G with n vertices as the maximum, over all bijections  $f: V(G) \to \{1, 2, \dots, n\}$ , of the number of distinct values |f(u) - f(v)| over all edges uv of G. So, from Rosa and Siráň's result, the gracesize of any tree with n edges is at least 5(n+1)/7.

In [503] Brinkmann, Crevals, Mélot, Rylands, and Steffan define the parameter  $\alpha_{\text{def}}$  which measures how far a tree is from having an  $\alpha$ -labeling as it counts the minimum number of errors, that is, the minimum number of edge labels that are missing from the set of all possible labels. Trees with an  $\alpha$ -labeling have deficit 0. For a tree T = (V, E) with bipartition classes  $V_1$  and  $V_2$  and a bipartite labeling  $f : V \to \{0, \ldots, |V| - 1\}$  the edge parity of T is  $(\sum_{i=1}^{|E|} i) \mod 2 = \frac{1}{2}(|V| - 1)|V| \mod 2$ . So if f is an  $\alpha$ -labeling this is the sum of all edge labels modulo 2; it is 0 if  $|V| \equiv 0, 1 \mod 4$  and 1 if  $|V| \equiv 2, 3 \mod 4$ . The vertex parity is the parity of the number of vertices of odd degree with odd label.

Brinkmann et al. [503] proved: in a tree T with  $\alpha$ -deficit 0 the edge parity and the vertex parities are equal; and for all non-negative integers k and d and  $n \ge k^2 + k$ , the number of trees T with n vertices,  $\alpha_{\text{def}}(T) = d$  and maximum degree n - k is the same. Furthermore, they provide computer results on the  $\alpha$ -deficit of all trees with up to 26 vertices; with maximum degree 3 and up to 36 vertices, with maximum degree 4 and up to 32 vertices, and with maximum degree 5 and up to 31 vertices.

In [816] Gallian weakened the condition for an  $\alpha$ -labeling somewhat by defining a weakly  $\alpha$ -labeling as a graceful labeling for which there is an integer k so that for each edge xy either  $f(x) \leq k \leq f(y)$  or  $f(y) \leq k \leq f(x)$ . Unlike  $\alpha$ -labelings, this condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph; namely, that the vertex with the label k must be on every odd cycle. Gallian, Prout, and Winters [816] showed that the prisms  $C_n \times P_2$  with a vertex deleted have  $\alpha$ -labelings. The same paper reveals that  $C_n \times P_2$  with an edge deleted from a cycle has an  $\alpha$ -labeling when n is even and a weakly  $\alpha$ -labeling when n > 3.

In [374] and [378] Barrientos and Minion focused on the enumeration of graphs with graceful and  $\alpha$ -labelings, respectively. They used an extended version of the adjacency matrix of a graph to count the number of labeled graphs. In [374] they count the number of gracefully-labeled graphs of size n and order m, for all possible values of m. In [378] they count the number of  $\alpha$ -labeled graphs of size n and order m, for all possible values of m. In [378] they count the number of  $\alpha$ -labeled graphs of size n with boundary value  $\lambda$ . They also count the number of  $\alpha$ -labeled graphs of size n, order m, and boundary value  $\lambda$  for all possible values of m and  $\lambda$ .

A special case of  $\alpha$ -labeling called strongly graceful was introduced by Maheo [1687] in 1980. A graceful labeling f of a graph G is called *strongly graceful* if G is bipartite with two partite sets A and B of the same order s, the number of edges is 2t + s, there is an integer k with  $t - s \leq k \leq t + s - 1$  such that if  $a \in A$ ,  $f(a) \leq k$ , and if  $b \in B$ , f(b) > k, and there is an involution  $\pi$  that is an automorphism of G such that:  $\pi$  exchanges A and B and the s edges  $a\pi(a)$  where  $a \in A$  have as labels the integers between t + 1 and t + s. Maheo's main result is that if G is strongly graceful then so is  $G \times Q_n$ . In particular, she proved that  $(P_n \times Q_n) \times K_2$ ,  $B_{2n}$ , and  $B_{2n} \times Q_n$  have strongly graceful labelings.

In 1999 Broersma and Hoede [504] conjectured that every tree containing a perfect matching is strongly graceful. Yao, Cheng, Yao, and Zhao [2861] proved that this conjecture is true for every tree with diameter at most 5 and provided a method for constructing strongly graceful trees.

El-Zanati and Vanden Eynden [722] call a strongly graceful labeling a strong  $\alpha$ -labeling.

They show that if G has a strong  $\alpha$ -labeling, then  $G \times P_n$  has an  $\alpha$ -labeling. They show that  $K_{m,2} \times K_2$  has a strong  $\alpha$ -labeling and that  $K_{m,2} \times P_n$  has an  $\alpha$ -labeling. They also show that if G is a bipartite graph with one more vertex than the number of edges, and if G has an  $\alpha$ -labeling such that the cardinalities of the sets of the corresponding bipartition of the vertices differ by at most 1, then  $G \times K_2$  has a strong  $\alpha$ -labeling and  $G \times P_n$  has an  $\alpha$ -labeling. El-Zanati and Vanden Eynden [722] also note that  $K_{3,3} \times K_2$ ,  $K_{3,4} \times K_2$ ,  $K_{4,4} \times K_2$ , and  $C_{4k} \times K_2$  all have strong  $\alpha$ -labelings. El-Zanati and Vanden Eynden proved that  $K_{m,2} \times Q_n$  has a strong  $\alpha$ -labeling and that  $K_{m,2} \times P_n$  has an  $\alpha$ -labeling for all n. They also prove that if G is a connected bipartite graph with partite sets of odd order such that in each partite set each vertex has the same degree, then  $G \times K_2$  does not have a strong  $\alpha$ -labeling. As a corollary they have that  $K_{m,n} \times K_2$  does not have a strong  $\alpha$ -labeling when m and n are odd.

An  $\alpha$ -labeling f of a graph G is called *free* by El-Zanati and Vanden Eynden in [723] if the critical number k (in the definition of  $\alpha$ -labeling) is greater than 2 and if neither 1 nor k-1 is used in the labeling. Their main result is that the union of graphs with free  $\alpha$ -labelings has an  $\alpha$ -labeling. In particular, they show that  $K_{m,n}$ , m > 1, n > 2, has a free  $\alpha$ -labeling. They also show that  $Q_n$ ,  $n \ge 3$ , and  $K_{m,2} \times Q_n$ , m > 1,  $n \ge 1$ , have free  $\alpha$ -labelings. El-Zanati [personal communication] has shown that the Heawood graph has a free  $\alpha$ -labeling.

Wannasit and El-Zanati [2799] proved that if G is a cubic bipartite graph each of whose components is either a prism, a Möbius ladder, or has order at most 14, then G admits free  $\alpha$ -labeling. They conjecture that every bipartite cubic graph admits a free  $\alpha$ -labeling.

In [1694] Makadia, Karavadiya, and Kaneria call a vertex v in a graph G with a graceful labeling f a graceful center of G if f(v) = 0 or f(v) = |E(G)|. They say a graph G is a universal graceful graph if for every  $v \in V(G)$ , v is a graceful center for G with respect to some graceful labeling of G. They call G a universal  $\alpha$ -graceful graph if for every  $v \in V(G)$ , v is a graceful center for G with respect to some  $\alpha$ -graceful labeling of G. They define the ring sum of two graphs  $G_1$  and  $G_2$  denoted  $G_1 \oplus G_2$ , as the graph with vertex set  $(V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2) - (E(G_1) \cap E(G_2))$ . They proved: any graph G that admits  $\alpha$ -labeling has at least four graceful centers; if G is a graceful graph, then  $G \oplus K_{1,n}$  is graceful; if G is a universal graceful graph, then  $G \oplus K_2$  is a graceful center of  $G_1$  and  $G_2$  have  $\alpha$  labelings, then the ring sum  $G_1 \oplus G_2$  with the graceful center of  $G_1$  and the graceful center of  $G_2$  as a common vertex is a graceful; and if  $G_1$  and  $G_2$  have  $\alpha$  labelings, then the ring sum  $G_1 \oplus G_2$  with the two graceful centers of  $G_1$  and  $G_2$  as a common vertex has an  $\alpha$  labeling.

For connected bipartite graphs Grannell, Griggs, and Holroyd [926] introduced a labeling that lies between  $\alpha$ -labelings and graceful labelings. They call a vertex labeling f of a bipartite graph G with q edges and partite sets D and U gracious if f is a bijection from the vertex set of G to  $\{0, 1, \ldots, q\}$  such that the set of edge labels induced by f(u) - f(v) for every edge uv with  $u \in U$  and  $v \in D$  is  $\{1, 2, \ldots, q\}$ . Thus a gracious labeling of G with partite sets D and U is a graceful labeling in which every vertex in D has a label lower than every adjacent vertex. They verified by computer that every tree of size up to 20 has a gracious labeling. This led them to conjecture that every tree has a gracious labeling. For any k > 1 and any tree T Grannell et al. say that T has a gracious k-labeling if the vertices of T can be partitioned into sets D and U in such a way that there is a function f from the verticies of G to the integers modulo k such that the edge labels induced by f(u) - f(v) where  $u \in U$  and  $v \in D$  have the following properties: the number of edges labeled with 0 is one less than the number of verticies labeled with 0 and for each nonzero integer t the number of edges labeled with t is the same as the number of verticies labeled with t. They prove that every nontrivial tree has a k-gracious labeling for k = 2, 3, 4, and 5 and that caterpillars are k-gracious for all  $k \geq 2$ . In [434] Bell and Cummins provided new methods for combining certain families of gracefully labeled graphs to produce new gracefully labeled graphs. If the constituent graphs have a gracious labeling, then the methods presented produce a gracious labeling. They also introduce new infinite families of gracious trees and new classes of graceful trees.

The same labeling that is called gracious by Grannell, Griggs, and Holroyd is called a *near*  $\alpha$ -*labeling* by El-Zanati, Kenig, and Vanden Eynden [720]. The latter prove that if G is a graph with n edges that has a near  $\alpha$ -labeling then there exists a cyclic Gdecomposition of  $K_{2nx+1}$  for all positive integers x and a cyclic G-decomposition of  $K_{n,n}$ . They further prove that if G and H have near  $\alpha$ -labelings, then so does their weak tensor product (see earlier part of this section) with respect to the corresponding vertex partitions. They conjecture that every tree has a near  $\alpha$ -labeling.

Another kind of labelings for trees was introduced by Ringel, Llado, and Serra [2119] in an approach to proving their conjecture  $K_{n,n}$  is edge-decomposable into n copies of any given tree with n edges. If T is a tree with n edges and partite sets A and B, they define a labeling f from the set of vertices to  $\{1, 2, \ldots, n\}$  to be a *bigraceful* labeling of T if f restricted to A is injective, f restricted to B is injective, and the edge labels given by f(y) - f(x) where yx is an edge with y in B and x in A is the set  $\{0, 1, 2, \ldots, n-1\}$ . (Notice that this terminology conflicts with that given in Section 2.7 In particular, the Ringel, Llado, and Serra bigraceful does not imply the usual graceful.) Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars  $S_{k,m}$  with k spokes of paths of length m, and complete d-ary trees for d odd. They also prove that if T is a tree then there is a vertex v and a nonnegative integer m such that the addition of m leaves to v results in a bigraceful tree. They conjecture that all trees are bigraceful.

Table 3 summarizes some of the main results about  $\alpha$ -labelings.  $\alpha$  indicates that the graphs have an  $\alpha$ -labeling.

Graph	$\alpha$ -labeling
cycles $C_n$	$\alpha \text{ iff } n \equiv 0 \pmod{4} \ [2136]$
caterpillars	$\alpha$ [2136]
<i>n</i> -cube	$\alpha$ [1413]
books $B_{2n}$ , $B_{4n+1}$	$\alpha$ [1687], [815]
Möbius ladders $M_{2k+1}$	$\alpha$ [1899]
$C_m \cup C_n$	$\alpha \text{ iff } m, n \text{ are even and} $ $m + n \equiv 0 \pmod{4}[19]$
$C_{4m} \cup C_{4m} \cup C_{4m} \ (m > 1)$	$\alpha$ [1415]
$C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$	$\alpha$ [1415]
$mK_{s,t} \ (m \ge 1, s, t \ge 2)$	iff $(m, s, t) \neq (3, 2, 2)$ [1062]
$P_n \times Q_n$	$\alpha$ [1687]
$B_{2n} \times Q_n$	$\alpha$ [1687]
$K_{1,n} \times Q_n$	$\alpha$ [1687]
$K_{m,2} \times Q_n$	$\alpha$ [722]
$K_{m,2} \times P_n$	$\alpha$ [722]
$P_2 \times P_2 \times \cdots \times P_2 \times G$	$\alpha$ when $G = C_{4m}, P_m, K_{3,3}, K_{4,4}$ [2434]
$P_2 \times P_2 \times \cdots \times P_2 \times P_m$	$\alpha$ [2434]
$P_2 \times P_2 \times \cdots \times P_2 \times K_{m,m}$	$\alpha$ [2434] when $m = 3$ or 4
$G[\overline{K_n}]$	$\alpha$ when G is $\alpha$ [2435]

Table 3: Summary of Results on  $\alpha$ -labelings

## 3.2 $\gamma$ -Labelings

In 2004 Chartrand, Erwin, VanderJagt, and Zhang [559] define a  $\gamma$ -labeling of a graph G of size m as a 1-1 function f from the vertices of G to  $\{0, 1, 2, \ldots, m\}$  that induces an edge labeling f' defined by f'(uv) = |f(u) - f(v)| for each edge uv. They define the following parameters of a  $\gamma$ -labeling: val $(f) = \Sigma f'(e)$  over all edges e of G; val<sub>max</sub> $(G) = \max\{$  val(f) : f is a  $\gamma$ -labeling of  $G\}$ , val<sub>min</sub> $(G) = \min\{$  val(f) : f is a  $\gamma$  - labeling of  $G\}$ . Among their results are the following:

 $\operatorname{val}_{\min}(P_n) = \operatorname{val}_{\max}(P_n) = \lfloor (n^2 - 2)/2 \rfloor; \operatorname{val}_{\min}(C_n) = 2(n-1); \text{ for even } n \geq 4,$  $\operatorname{val}_{\max}(C_n) = n(n+2)/2; \text{ for odd } n \geq 3, \operatorname{val}_{\max}(C_n) = (n-1)(n+3)/2; \text{ for odd } n, \operatorname{val}_{\min}(K_n) = \binom{n+1}{3}; \text{ for odd } n, \operatorname{val}_{\max}(K_n) = (n^2 - 1)(3n^2 - 5n + 6)/24; \text{ for even } n, \operatorname{val}_{\max}(K_n) = n(3n^3 - 5n^2 + 6n - 4)/24; \text{ for every } n \geq 3, \operatorname{val}_{\min}(K_{1,n-1}) = \binom{\lfloor \frac{n+1}{2} \rfloor}{2} + \binom{\lfloor \frac{n+1}{2} \rfloor}{2}; \operatorname{val}_{\max}(K_{1,n-1}) = \binom{n}{2} \text{ for a connected graph of order } n \text{ and size } m, \operatorname{val}_{\min}(G) = m \text{ if and only if } G \text{ is isomorphic to } P_n; \text{ if } G \text{ is maximal outerplanar of order } n \geq 2, \operatorname{val}_{\min}(G) \geq 3n - 5 \text{ and equality occurs if and only if } G = P_n^2; \text{ if } G \text{ is a connected } r\text{-regular bipartite graph of order } n \text{ and size } m \text{ where } r \geq 2, \text{ then } \operatorname{val}_{\max}(G) = rn(2m - n + 2)/4.$ 

In another paper on  $\gamma$ -labelings of trees Chartrand, Erwin, VanderJagt, and Zhang [560] prove for  $p, q \geq 2$ ,  $\operatorname{val}_{\min}(S_{p,q})$  (that is, the graph obtained by joining the centers of  $K_{1,p}$  and  $K_{1,q}$  by an edge) =  $(\lfloor p/2 \rfloor + 1)^2 + (\lfloor q/2 \rfloor + 1)^2 - (n_p \lfloor p/2 \rfloor + 1)^2 + (n_q \lfloor (q+2)/2 \rfloor + 1)^2)$ , where  $n_i$  is 1 if *i* is even and  $n_i$  is 0 if  $n_i$  is odd;  $\operatorname{val}_{\min}(S_{p,q}) = (p^2 + q^2 + 4pq - 3p - 3q + 2)/2$ ; for a connected graph *G* of order *n* at least 4,  $\operatorname{val}_{\min}(G) = n$  if and only if *G* is a caterpillar with maximum degree 3 and has a unique vertex of degree 3; for a tree *T* of order *n* at least 4, maximum degree  $\Delta$ , and diameter *d*,  $\operatorname{val}_{\min}(T) \geq (8n + \Delta^2 - 6\Delta - 4d + \delta_{\Delta})/4$ where  $\delta_{\Delta}$  is 0 if  $\Delta$  is even and  $\delta_{\Delta}$  is 0 if  $\Delta$  is odd. They also give a characterization of all trees of order *n* at least 5 whose minimum value is n + 1.

Saduakdee and Khemmani [2173] investigated connected graphs having the unique  $\gamma$ -min labeling. They determined the minimum value of a  $\gamma$ -labeling for some classes of trees and showed that they have no unique  $\gamma$ -min labeling.

In [2172] Sanaka determined  $\operatorname{val}_{\max}(K_{m,n})$  and  $\operatorname{val}_{\min}(K_{m,n})$ . In [518] Bunge, Chantasartraaamee, El-Zanati, and Vanden Eynden generalized  $\gamma$ -labelings by introducing two labelings for tripartite graphs. Graphs G that admit either of these labelings guarantee the existence of cyclic G-decompositions of  $K_{2nx+1}$  for all positive integers x. They also proved that, except for  $C_3 \cup C_3$ , the disjoint union of two cycles of odd length admits one of these labelings.

# 3.3 Graceful-like Labelings

As a means of attacking graph decomposition problems, Rosa [2136] invented another analogue of graceful labelings by permitting the vertices of a graph with q edges to assume labels from the set  $\{0, 1, \ldots, q+1\}$ , while the edge labels induced by the absolute value of the difference of the vertex labels are  $\{1, 2, \ldots, q-1, q\}$  or  $\{1, 2, \ldots, q-1, q+1\}$ . He calls these  $\hat{\rho}$ -labelings. Frucht [797] used the term nearly graceful labeling instead of  $\hat{\rho}$ -labelings. Frucht [797] has shown that the following graphs have nearly graceful labelings with edge labels from  $\{1, 2, \ldots, q-1, q+1\}$ :  $P_m \cup P_n$ ;  $S_m \cup S_n$ ;  $S_m \cup P_n$ ;  $G \cup K_2$  where G is graceful; and  $C_3 \cup K_2 \cup S_m$  where m is even or  $m \equiv 3 \pmod{14}$ . Seoud and Elsakhawi [2232] have shown that all cycles are nearly graceful. Barrientos [350] proved that  $C_n$  is nearly graceful with edge labels  $1, 2, \ldots, n-1, n+1$  if and only if  $n \equiv 1$  or 2 (mod 4). Nurvazly and Sugeng [1879] proved that  $\Theta(C_3)^n$  graphs (n copies of  $C_3$  that share an edge) have  $\hat{\rho}$ labelings. Gao [828] shows that a variation of banana trees is odd-graceful and in some cases has a nearly graceful labeling. (A graph G with q edges is odd-graceful if there is an injection f from V(G) to  $\{0, 1, 2, \ldots, 2q-1\}$  such that, when each edge xy is assigned the label |f(x) - f(y)|, the resulting edge labels are  $\{1, 3, 5, \ldots, 2q-1\}$ .

For a graph G with p vertices and q directed edges that are assigned distinct vertex labels in  $\{0, \ldots, q\}$  and distinct edge labels in  $\{1, \ldots, p\}$  so that the label of the directed edge from u to v is  $(f(v) - f(u)) \mod(q+1)$  (this generalizes Rosa's  $\rho$ -valuations. Knuth [1383] has observed that there is a nice data structure for storing a graph or digraph in a computer. He calls this a graceful data structure labeling.

In 1988 Rosa [2138] conjectured that triangular snakes with  $t \equiv 0$  or 1 (mod 4) blocks are graceful and those with  $t \equiv 2$  or 3 (mod 4) blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [1793] proved Rosa's conjecture while introducing the slightly stronger concept of *almost graceful* by permitting the vertex labels to come from  $\{0, 1, 2, \ldots, q-1, q+1\}$  while the edge labels are  $1, 2, \ldots, q-1, q$ , or  $1, 2, \ldots, q-1, q+1$ . More generally, Rosa [2138] conjectured that all triangular cacti are either graceful or near graceful and suggested the use of Skolem sequences to label some types of triangular cacti. Dyer, Payne, Shalaby, and Wicks [703] verified the conjecture for two families of triangular cacti using Langford sequences to obtain Skolem and hooked Skolem sequences with specific subsequences.

Seoud and Elsakhawi [2232] and [2233] have shown that the following graphs are almost graceful:  $C_n$ ;  $P_n + \overline{K_m}$ ;  $P_n + K_{1,m}$ ;  $K_{m,n}$ ;  $K_{1,m,n}$ ;  $K_{2,2,m}$ ;  $K_{1,1,m,n}$ ;  $P_n \times P_3$   $(n \ge 3)$ ;  $K_5 \cup K_{1,n}$ ;  $K_6 \cup K_{1,n}$ , and ladders.

For a graph G with p vertices, q edges, and  $1 \le k \le q$ , Eshghi [732] defines a holey  $\alpha$ labeling with respect to k as an injective vertex labeling f for which  $f(v) \in \{1, 2, \ldots, q+1\}$ for all v,  $\{|f(u) - f(v)| | \text{ for all edges } uv\} = \{1, 2, \ldots, k-1, k+1, \ldots, q+1\}$ , and there exist an integer  $\gamma$  with  $0 \le \gamma \le q$  such that  $\min\{f(u), f(v)\} \le \gamma \le \max\{f(u), f(v)\}$ . He proves the following:  $P_n$  has a holey  $\alpha$ -labeling with respect to all k;  $C_n$  has a holey  $\alpha$ labeling with respect to k if and only if either  $n \equiv 2 \pmod{4}$ , k is even, and  $(n, k) \ne (10, 6)$ , or  $n \equiv 0 \pmod{4}$  and k is odd.

Recall from Section 2.2 that a  $kC_n$ -snake is a connected graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to  $C_n$ . In addition to his results on the graceful  $kC_n$ -snakes given in Section 2.2, Barrientos [354] proved that when k is odd the linear  $kC_6$ -snake is nearly graceful and that  $C_m \cup K_{1,n}$  is nearly graceful when m = 3, 4, 5, and 6.

Yet another kind of labeling introduced by Rosa in his 1967 paper [2136] is a  $\rho$ -labeling. (Sometimes called a *rosy* labeling). A  $\rho$ -labeling (or  $\rho$ -valuation) of a graph is an injection from the vertices of the graph with q edges to the set  $\{0, 1, \ldots, 2q\}$ , where if the edge labels induced by the absolute value of the difference of the vertex labels are  $a_1, a_2, \ldots, a_q$ , then  $a_i = i$  or  $a_i = 2q + 1 - i$ . Rosa [2136] proved that a cyclic decomposition of the edge set of the complete graph  $K_{2q+1}$  into subgraphs isomorphic to a given graph G with q edges exists if and only if G has a  $\rho$ -labeling. (A decomposition of  $K_n$  into copies of Gis called *cyclic* if the automorphism group of the decomposition itself contains the cyclic group of order n.) It is known that every graph with at most 11 edges has a  $\rho$ -labeling and that all lobsters have a  $\rho$ -labeling (see [546]).

In [385] Barrientos and Minion proved that a tree admits a  $\rho$ -labeling when the deletion of some of its leaves results in a graceful tree. They use this result to prove the existence of  $\rho$ -labeling for several families of trees such as lobsters and those of diameter up to seven. Similarly, they showed that if T is any graceful tree of size n and k is an integer such that  $2k \ge n + 1$ , then any tree of size n + 2k obtained attaching a path of length 2 to k distinct vertices of T has a  $\rho$ -labeling.

Donovan, El-Zanati, Vanden Eyden, and Sutinuntopas [685] prove that  $rC_m$  has a  $\rho$ -labeling (or a more restrictive labeling) when  $r \leq 4$ . They conjecture that every 2-regular graph has a  $\rho$ -labeling. Gannon and El-Zanati [823] proved that for any odd  $n \geq 7, rC_n$  admits  $\rho$ -labelings. The cases n = 3 and n = 5 were done in [681] and [719]. Aguado, El-Zanati, Hake, Stob, and Yayla [62] give a  $\rho$ -labeling of  $C_r \cup C_s \cup C_t$  for each of the cases where  $r \equiv 0, s \equiv 1, t \equiv 1 \pmod{4}; r \equiv 0, s \equiv 3, t \equiv 3 \pmod{4}; \text{ and } r \equiv 1, s \equiv 1, t \equiv 3 \pmod{4}; \text{ (iv) } r \equiv 1, s \equiv 2, t \equiv 3 \pmod{4}; \text{ (v) } r \equiv 3, s \equiv 3, t \equiv 3 \pmod{4}$ . Caro, Roditty, and Schőnheim [546] provide a construction for the adjacency matrix for every graph that has a  $\rho$ -labeling. They ask the following question: If H is a connected graph having a  $\rho$ -labeling and q edges and G is a new graph with q edges constructed by breaking H up into disconnected parts, does G also have a  $\rho$ -labeling? Kézdy [1366] defines a stunted tree as one whose edges can be labeled with  $e_1, e_2, \ldots, e_n$  so that  $e_1$  and  $e_2$  are incident and, for all  $j = 3, 4, \ldots, n$ , edge  $e_j$  is incident to at least one edge  $e_k$  satisfying  $2k \leq j - 1$ . He uses Alon's "Combinatorial Nullstellensatz" to prove that if 2n + 1 is prime, then every stunted tree with n edges has a  $\rho$ -labeling.

Jeba Jesintha and Ezhilarasi Hilda [1124] introduced a variation of Rosas  $\rho$ -labeling as follows. A  $\rho^*$ -labeling of a graph G is an injection from the vertices of the graph with q edges to the set  $\{0, 1, \ldots, 2q\}$ , where if the edge labels induced by the absolute value of the difference of the vertex labels are  $e_1, e_2, \ldots, e_q$ , then  $e_i = i$  or  $e_i = 2q - i$ . They prove that all paths and shell-butterfly graphs have a  $\rho^*$ -labeling.

In [375] Barrientos and Minion proved the existence of  $\rho$ -labelings for some types of forests that considerably reduce the number of trees that need to be studied to prove Kotzig's Conjecture that states that  $K_{2n+1}$  can be cyclically decomposed into 2n + 1subgraphs isomorphic to a given tree with n edges. Among their results are the following. If  $T_1$  and  $T_2$  admit  $\alpha$ -labelings such that one of the end-vertices of the edge of weight 1 in  $T_2$  is a leaf, then  $T_1 \cup T_2$  admits a  $\rho$ -labeling. If  $G_1, G_2, \ldots, G_k$  is a collection of graphs that admit  $\alpha$ -labelings, where  $G_k$  is a caterpillar of size at least k-2, then  $\bigcup_{i=1}^k G_i$  admits a  $\rho$ -labeling. Let  $\mathcal{R}$  denote the family that consists of all trees G such that G has a branch H, (i.e., G - H is a tree) that is a caterpillar, where the excess of G - H is at most the size of H. They prove that G admits a  $\rho$ -labeling when  $G \in \mathcal{R}$ . Recall a kayak paddle KP(k, m, l) is the graph obtained by joining  $C_k$  and  $C_m$  by a path of length l. Fronček and Tollefeson [792], [793] proved that KP(r, s, l) has a  $\rho$ -labeling for all cases. As a corollary they have that the complete graph  $K_{2n+1}$  is decomposable into kayak paddles with n edges.

In [778] Fronček generalizes the notion of an  $\alpha$ -labeling by showing that if a graph G on n edges allows a certain type of  $\rho$ -labeling), called  $\alpha_2$ -labeling, then for any positive integer k the complete graph  $K_{2nk+1}$  can be decomposed into copies of G.

In their investigation of cyclic decompositions of complete graphs El-Zanati, Vanden Eynden, and Punnim [725] introduced two kinds of labelings. They say a bipartite graph G with n edges and partite sets A and B has a  $\theta$ -labeling h if h is a one-to-one function from V(G) to  $\{0, 1, \ldots, 2n\}$  such that  $\{|h(b) - h(a)| ab \in E(G), a \in A, b \in B\} = \{1, 2, \ldots, n\}$ . They call h a  $\rho^+$ -labeling of G if h is a one-to-one function from V(G) to  $\{0, 1, \ldots, 2n\}$ and the integers h(x) - h(y) are distinct modulo 2n + 1 taken over all ordered pairs (x, y)where xy is an edge in G, and h(b) > h(a) whenever  $a \in A, b \in B$  and ab is an edge in G. Note that  $\theta$ -labelings are  $\rho^+$ -labelings and  $\rho^+$ -labelings are  $\rho$ -labelings. They prove that if G is a bipartite graph with n edges and a  $\rho^+$ -labeling, then for every positive integer x there is a cyclic G-decomposition of  $K_{2nx+1}$ . They prove the following graphs have  $\rho^+$ -labelings: trees of diameter at most 5,  $C_{2n}$ , lobsters, and comets (that is, graphs obtained from stars by replacing each edge by a path of some fixed length). They also prove that the disjoint union of graphs with  $\alpha$ -labelings have a  $\theta$ -labeling and conjecture that all forests have  $\rho$ -labelings.

A  $\sigma$ -labeling of G(V, E) is a one-to-one function f from V to  $\{0, 1, \ldots, 2|E|\}$  such that  $\{|f(u) - f(v)| | uv \in E(G)\} = \{1, 2, \ldots, |E|\}$ . Such a labeling of G yields cyclic G-decompositions of  $K_{2n+1}$  and of  $K_{2n+2} - F$ , where F is a 1-factor of  $K_{2n+2}$ . El-Zanati and Vanden Eynden (see [61]) have conjectured that every 2-regular graph with n edges has a  $\rho$ -labeling and, if  $n \equiv 0$  or 3 (mod 4), then every 2-regular graph has a  $\sigma$ -labeling. Aguado and El-Zanati [61] have proved that the latter conjecture holds when the graph has at most three components.

Given a bipartite graph G with partite sets X and Y and graphs  $H_1$  with p vertices and  $H_2$  with q vertices, Fronček and Winters [794] define the *bicomposition* of G and  $H_1$ and  $H_2$ ,  $G[H_1, H_2]$ , as the graph obtained from G by replacing each vertex of X by a copy of  $H_1$ , each vertex of Y by a copy of  $H_2$ , and every edge xy by a graph isomorphic to  $K_{p,q}$  with the partite sets corresponding to the vertices x and y. They prove that if G is a bipartite graph with n edges and G has a  $\theta$ -labeling that maps the vertex set  $V = X \cup Y$ into a subset of  $\{0, 1, 2, \ldots, 2n\}$ , then the bicomposition  $G[\overline{K_p}, \overline{K_q}]$  has a  $\theta$ -labeling for every  $p, q \ge 1$ . As corollaries they have: if a bipartite graph G with n edges and at most n+1 vertices has a gracious labeling (see §3.1), then the bicomposition graph  $G[\overline{K_p}, \overline{K_q}]$ has a gracious labeling for every  $p, q \ge 1$ , and if a bipartite graph G with n edges has a  $\theta$ -labeling, then for every  $p, q \ge 1$ , the bicomposition  $G[\overline{K_p}, \overline{K_q}]$  decomposes the complete graph  $K_{2npq+1}$ .

In a paper published in 2009 [724] El-Zanati and Vanden Eynden survey "Rosa-type" labelings. That is, labelings of a graph G that yield cyclic G-decompositions of  $K_{2n+1}$  or  $K_{2nx+1}$  for all natural numbers x. The 2009 survey by Fronček [777] includes generalizations of  $\rho$ - and  $\alpha$ -labelings that have been used for finding decompositions of complete graphs that are not covered in [724].

Blinco, El-Zanati, and Vanden Eynden [470] call a non-bipartite graph *almost-bipartite* if the removal of some edge results in a bipartite graph. For these kinds of graphs G they call a labeling f a  $\gamma$ -labeling of G if the following conditions are met: f is a  $\rho$ -labeling; G is tripartite with vertex tripartition A, B, C with  $C = \{c\}$  and  $\bar{b} \in B$  such that  $\{\bar{b}, c\}$ is the unique edge joining an element of B to c; if av is an edge of G with  $a \in A$ , then f(a) < f(v); and  $f(c) - f(\bar{b}) = n$ . (In § 3.2 the term  $\gamma$ -labeling is used for a different kind of labeling.) They prove that if an almost-bipartite graph G with n edges has a  $\gamma$ -labeling then there is a cyclic G-decomposition of  $K_{2nx+1}$  for all x. They prove that all odd cycles with more than 3 vertices have a  $\gamma$ -labeling and that  $C_3 \cup C_{4m}$  has a  $\gamma$ -labeling if and only if m > 1. In [517] Bunge, El-Zanati, and Vanden Eynden prove that every 2-regular almost bipartite graph other than  $C_3$  and  $C_3 \cup C_4$  have a  $\gamma$ -labeling.

In [470] Blinco, El-Zanati, and Vanden Eynden consider a slightly restricted  $\rho^+$ labeling for a bipartite graph with partite sets A and B by requiring that there exists a number  $\lambda$  with the property that  $\rho^+(a) \leq \lambda$  for all  $a \in A$  and  $\rho^+(b) > \lambda$  for all  $b \in B$ . They denote such a labeling by  $\rho^{++}$ . They use this kind of labeling to show that if G is a 2-regular graph of order n in which each component has even order then there is a cyclic G-decomposition of  $K_{2nx+1}$  for all x. They also conjecture that every bipartite graph has a  $\rho$ -labeling and every 2-regular graph has a  $\rho$ -labeling.

Dufour [698] and Eldergill [708] have some results on the decomposition of complete graphs using labeling methods. Balakrishnan and Sampathkumar [338] showed that for each positive integer n the graph  $\overline{K_n} + 2K_2$  admits a  $\rho$ -labeling. Balakrishnan [333] asks if it is true that  $\overline{K_n} + mK_2$  admits a  $\rho$ -labeling for all n and m. Fronček [776] and Fronček and Kubesa [789] have introduced several kinds of labelings for the purpose of proving the existence of special kinds of decompositions of complete graphs into spanning trees.

For positive integers c and d, let  $K_{c\times d}$  denote the complete multipartite graph with c parts, each containing d vertices. Let G with n edges be the union of two vertex-disjoint even cycles. In [2473] Su et al. use Rosa-type graph labelings to show that there exists a cyclic G- decomposition of  $K_{(2n + 1)} \times t$ ,  $K_{(n/2+1)\times 4t}, K_{5\times(n/2)t}$ , and of  $K_{2nt}$  for every positive integer t. If  $n \equiv 0 \pmod{4}$ , then there also exists a cyclic G-decomposition of  $K_{n+1} \times 2t$ ,  $K_{(n/4)+1} \times 8t$ ,  $K_9 \times (n/4)t$ , and of  $K_{3\times nt}$  for every positive integer t.

For (p, q)-graphs with p = q + 1, Frucht [797] has introduced a stronger version of almost graceful graphs by permitting as vertex labels  $\{0, 1, \ldots, q - 1, q + 1\}$  and as edge labels  $\{1, 2, \ldots, q\}$ . He calls such a labeling *pseudograceful*. Frucht proved that  $P_n$   $(n \geq 3)$ , combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star),  $C_3 \cup P_n$   $(n \neq 3)$ , and  $C_4 \cup P_n$   $(n \neq 1)$  are pseudograceful whereas  $K_{1,n}$   $(n \geq 3)$  is not. Kishore [1381] proved that  $C_s \cup P_n$  is pseudograceful when  $s \geq 5$  and  $n \geq (s+7)/2$ and that  $C_s \cup S_n$  is pseudograceful when s = 3, s = 4, and  $s \geq 7$ . Seoud and Youssef [2264] and [2260] extended the definition of pseudograceful to all graphs with  $p \leq q + 1$ . They proved that  $K_m$  is pseudograceful if and only if m = 1, 3, or 4 [2260];  $K_{m,n}$  is pseudograceful when  $n \geq 2$ , and  $P_m + \overline{K_n}$   $(m \geq 2)$  [2264] is pseudograceful. They also proved that if G is pseudograceful, then  $G \cup K_{m,n}$  is graceful for  $m \geq 2$  and  $n \geq 2$  and  $G \cup K_{m,n}$  is pseudograceful for  $m \ge 2, n \ge 2$  and  $(m, n) \ne (2, 2)$  [2260]. They ask if  $G \cup K_{2,2}$  is pseudograceful whenever G is. Seoud and Youssef [2260] observed that if G is a pseudograceful Eulerian graph with q edges, then  $q \equiv 0$  or 3 (mod 4). Youssef [2875] has shown that  $C_n$  is pseudograceful if and only if  $n \equiv 0$  or 3 (mod 4), and for n > 8 and  $n \equiv 0$  or 3 (mod 4),  $C_n \cup K_{p,q}$  is pseudograceful for all  $p, q \ge 2$  except (p,q) = (2,2). Youssef [2872] has shown that if H is pseudograceful and G has an  $\alpha$ -labeling with k being the smaller vertex label of the edge labeled with 1 and if either k + 2 or k - 1 is not a vertex label of G, then  $G \cup H$  is graceful. In [2876] Youssef shows that if G is (p,q) pseudograceful graph with p = q + 1, then  $G \cup S_m$  is Skolem-graceful (see Section 3.5 for the definition). As a corollary he obtains that for all  $n \ge 2$ ,  $P_n \cup S_m$  is Skolem-graceful if and only if  $n \ge 3$  or n = 2 and m is even.

In [2881] Youssef generalizes his results in [2872] and provides new families of disconnected graphs that have  $\alpha$ -labelings and pseudo  $\alpha$ -labelings. (A pseudo  $\alpha$ -labeling f is an  $\alpha$ -labeling for which there is an integer  $k_j$  with the property that for each edge xy of the graph either  $f(x) \leq k_j < f(y)$  or  $f(y) \leq k_j < f(x)$ .)

For a graph G Ichishima, Muntaner-Batle, and Oshima [1043] defined the *beta-number* of G,  $\beta(G)$ , to be either the smallest positive integer n for which there exists an injective function f from the vertices of G to  $\{1, 2, \ldots, n\}$  such that when each edge uv is labeled |f(u) - f(v)| the resulting set of edge labels is  $\{c, c+1, \ldots, c+|E(G)|-1\}$  for some positive integer c or  $+\infty$  if there exists no such integer n. They defined the strong beta-number of G to be either the smallest positive integer n for which there exists an injective function f from the vertices of G to  $\{1, 2, \ldots, n\}$  such that when each edge uv is labeled |f(u) - f(v)| the resulting set of edge labels is  $\{1, 2, \ldots, |E(G)|\}$  or  $+\infty$  if there exists no such integer n. They gave some necessary conditions for a graph to have a finite (strong) beta-number and some sufficient conditions for a graph to have a finite (strong) beta-number. They also determined formulas for the beta-numbers and strong beta-numbers of  $C_n$ ,  $2C_n$ ,  $K_n$   $(n \ge 2)$ ,  $S_m \cup S_n$ ,  $P_m \cup S_n$ , and prove that nontrivial trees and forests without isolated vertices have finite strong beta-numbers. In [1040] Ichishima, López, Muntaner-Batte, and Oshima proved that if G is a bipartite graph and m is odd, then  $\beta(mG) \leq m|E(G)| + m - 1$ . If G has the additional property that G is a graceful nontrivial tree, then  $\beta(mG) = m|V(G)| + m - 1$ . They also investigate the (strong) betanumber of forests with components that are isomorphic to either paths or stars. They propose new conjectures on the (strong) beta-number of forests. In [1058] Ichishima and Oshima determine a formula for the (strong) beta-number of the linear forests  $P_m \cup P_n$ . As a corollary they provide a partial formula for the beta-number of the disjoint union of multiple copies of the same linear forest. In [1045] Ichishima, Muntaner-Batle, Oshima provide lower and upper bounds for  $\beta(G+nK_1)$  when  $\beta(G) = |V(G)| - 1$  and formulas for  $\beta(G+nK_1)$  and  $\beta_s(G+nK_1)$  when  $\beta_s(G) = |V(G)| - 1$ . They also determine formulas for  $\beta(G + K_{1,n})$  and  $\beta_s(G + K_{1,n})$  when  $\beta_s(G) = |V(G)| - 1$ . They conclude with two problems.

For a graph G of order p and size q and every positive integer n Ichishima, Muntaner-Batle, and Oshima [1048] proved if  $\beta(G) = p - 1$ , then there exists some positive integer c such that  $q + np \leq \beta(G + nK_1) \leq c + q + np - 1$ ; if  $\beta_s(G) = p - 1$ , then  $\beta(G + nK_1) =$   $\beta_s(G+nK_1) = q+np$  and  $G+nK_1$  is graceful; and if q = p-1 and  $\beta_s(G) = p-1$ , then  $\beta(G+S_n) = \beta_s(G+S_n) = (n+2)p+n-1$ . In particular, if T is a graceful tree of order p then  $\beta(T+nK_1) = \beta_s(T+nK_1) = (n+1)p-1$ . Moreover,  $T+nK_1$  and  $T+S_n$  are graceful.

In [1051] Ichishima, Muntaner-Batle, and Oshima establish a lower bound for the strong beta-number of an arbitrary galaxy (that is, a forest whose components are stars) under certain conditions. They also determine formulas for the (strong) beta-number and gracefulness of galaxies with three and four components. As corollaries, they provide formulas for the beta-number and gracefulness of the disjoint union of multiple copies of the same galaxies if the number of copies is odd. They pose some problems and conjecture. In [1053] Ichishima and Muntaner-Batle determined formulas for the (strong) beta-number and gracefulness of galaxies with five components. In [1055] Ichishima, Muntaner-Batle, and Oshima determined formulas for the (strong) beta-number of galaxies with five components. As a corollary of these results, they provide formulas for the beta-number and gamma-number of the disjoint union of multiple copies of galaxies with five components of the same galaxies for the same galaxies for the same determined formulas for the (strong) beta-number and gamma-number of galaxies with five components. In [1055] Ichishima, Muntaner-Batle, and Oshima determined formulas for the (strong) beta-number and gamma-number of galaxies with five components. As a corollary of these results, they provide formulas for the beta-number and gamma-number of the disjoint union of multiple copies of the same galaxies if the number of copies is odd.

McTavish [1733] has investigated labelings of graphs with q edges where the vertex and edge labels are from  $\{0, \ldots, q, q+1\}$ . She calls these  $\tilde{\rho}$ -labelings. Graphs that have  $\tilde{\rho}$ -labelings include cycles and the disjoint union of  $P_n$  or  $S_n$  with any graceful graph.

Frucht [797] has made an observation about graceful labelings that yields nearly graceful analogs of  $\alpha$ -labelings and weakly  $\alpha$ -labelings in a natural way. Suppose G(V, E) is a graceful graph with the vertex labeling f. For each edge xy in E, let [f(x), f(y)] (where  $f(x) \leq f(y)$ ) denote the interval of real numbers r with  $f(x) \leq r \leq f(y)$ . Then the intersection  $\cap [f(x), f(y)]$  over all edges  $xy \in E$  is a unit interval, a single point, or empty. Indeed, if f is an  $\alpha$ -labeling of G then the intersection is a unit interval; if f is a graceful but not an  $\alpha$ -labeling, then the intersection is a point; and, if f is a graceful but not a weakly  $\alpha$ -labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes.

Let G(V, E) be a graph without isolated vertices and with q edges. The gracefulness  $\operatorname{grac}(G)$  of G is the smallest positive integer k for which there exists an injective function  $f: V \to \{0, 1, 2, \ldots, k\}$  such that the edge induced function  $g_f: E \to \{1, 2, \ldots, k\}$  defined by  $g_f(uv) = |f(u) - f(v)|$  for all edges uv is also injective. Let  $c(f) = \max\{i: 1, 2, \ldots, i\}$  are edge labels and let  $m(G) = \max_f\{c(f)\}$  where the maximum is taken over all injective functions f from V to the nonnegative integers such that  $g_f$  is also injective. The measure m(G) is called m-gracefulness of G. It determines how close G is to being graceful. Pereira, Singh, Arumugam [1915] prove that there are infinitely many nongraceful graphs with m-gracefulness q-1 and give necessary conditions for an Eulerian graph with q edges and  $K_p$  with q edges to have m-gracefulness q-1 and q-2. They prove that  $K_5$  is the only complete graph to have m-gracefulness q-1. They also give an upper bound for the highest possible vertex label of  $K_p$  if  $m(K_p) = q-2$ .

A (p,q)-graph G is said to be a *super graceful* graph if there is a bijective function  $f: V(G) \cup E(G) \longrightarrow \{1, 2, \dots, p+q\}$  such that f(uv) = |f(u) - f(v)| for every edge  $uv \in E(G)$  labeling. Perumal, Navaneethakrishnan, Nagarajan, Arockiaraj [1918] and

[1919] show that the graphs  $P_n$ ,  $C_n$ ,  $P_m \odot nK_1$ ,  $K_{m,n}$ , and  $P_n \odot K_1$  minus a pendent edge at an endpoint of  $P_n$  are super graceful graphs. Lau, Shiu, and Ng [1469] study the super gracefulness of complete graphs, the disjoint union of certain star graphs, the complete tripartite graphs  $K_{(1,1,n)}$ , and certain families of trees. They also provide four methods of constructing new super graceful graphs. They prove all trees of order at most 7 are super graceful and conjecture that all trees are super graceful. Amutha and Uma Devi [149] proved the following graphs are super graceful: fans, double fans  $DF_n = P_n + \overline{K_2}$   $(n \ge 2)$ , and for  $(m \ge 3, n \ge 2)$  the graphs obtained by identifying a central vertex of the star  $S_m$ with an end vertex of path in  $P_n + K_1$ .

In [715] Elsonbaty and Daoud introduce a new version of gracefulness called an edge even graceful labeling of graphs. A bijective function f from the edges of a (p,q)-graph Gto  $\{2, 4, \ldots, 2q\}$  is said to be an *edge even graceful labeling* of G if the induced function  $f^*$  from the vertices to  $\{0, 2, \ldots, 2q\}$  defined by  $f^*(e)$  is the sum of  $f(e) \pmod{\max(p,q)}$ is injective. They prove the following graphs have edge even graceful labelings:  $P_n$  if and only if n is odd,  $C_n$  if and only if n is odd,  $K_{1,n}$  if and only if n is even, wheels, fans, friendship graphs, and double wheels  $W_{n,n}$ . The polar grid graph  $P_{m,n}$  consists of n copies of  $C_m$ , a new vertex  $v_0$ , and m copies on  $P_{n+1}$  that share a endpoint at  $v_0$  The graph is drawn as m concentric circles with a center at a new vertex  $v_0$  and the m vertices of each cycle lie on a line with one endpoint at  $v_0$  and the other endpoint at the outermost cycle in such a way that the n vertices of the copies on  $P_{n+1}$  other the  $v_0$  intersect the vertices of cycles. Daoud [644] provided necessary and sufficient conditions for the polar grid graph to be edge even graceful.

Singh and Devaraj [2401] call a graph G with p vertices and q edges triangular graceful if there is an injection f from V(G) to  $\{0, 1, 2, \ldots, T_q\}$  where  $T_q$  is the qth triangular number and the labels induced on each edge uv by |f(u) - f(v)| are the first q triangular numbers. They prove the following graphs are triangular graceful: paths, level 2 rooted trees, olive trees (see § 2.1 for the definition), complete n-ary trees, double stars, caterpillars,  $C_{4n}$ ,  $C_{4n}$  with pendent edges, the one-point union of  $C_3$  and  $P_n$ , and unicyclic graphs that have  $C_3$  as the unique cycle. They prove that wheels, helms, flowers (see §2.2 for the definition) and  $K_n$  with  $n \geq 3$  are not triangular graceful. They conjecture that all trees are triangular graceful. In [2301] Sethuraman and Venkatesh introduced a new method for combining graceful trees to obtain trees that have  $\alpha$ -labelings.

Van Bussel [2699] considered two kinds of relaxations of graceful labelings as applied to trees. He called a labeling *range-relaxed graceful* it is meets the same conditions as a graceful labeling except the range of possible vertex labels and edge labels are not restricted to the number of edges of the graph (the edges are distinctly labeled but not necessarily labeled 1 to q where q is the number of edges). Similarly, he calls a labeling *vertex-relaxed graceful* if it satisfies the conditions of a graceful labeling while permitting repeated vertex labels. He proves that every tree T with q edges has a range-relaxed graceful labeling with the vertex labels in the range  $0, 1, \ldots, 2q-d$  where d is the diameter of T and that every tree on n vertices has a vertex-relaxed graceful labeling such that the number of distinct vertex labels is strictly greater than n/2. In 2017 Sethuraman, Ragukumar, and Slater [2289] improved the bound on the range-relaxed graceful labeling given by Van Bussel in [2699] in 2002 for a tree T.

In [371], Barrientos and Krop introduce left- and right-layered trees as trees with a specific representation and define the excess of a tree. Applying these ideas, they show a range-relaxed graceful labeling which improves the upper bound for maximum vertex label given by Van Bussel in [2699]. They also improve the bounds given by Rosa and Širáň in [2139] for the  $\alpha$ -size and gracesize of lobsters.

Sekar [2200] calls an injective function  $\phi$  from the vertices of a graph with q edges to  $\{0, 1, 3, 4, 6, 7, \ldots, 3(q-1), 3q-2\}$  one modulo three graceful if the edge labels induced by labeling each edge uv with  $|\phi(u) - \phi(v)|$  is  $\{1, 4, 7, \ldots, 3q-2\}$ . He proves that the following graphs are one modulo three graceful:  $P_m$ ;  $C_n$  if and only if  $n \equiv 0 \mod 4$ ;  $K_{m,n}$ ;  $C_{2n}^{(2)}$  (the one-point union of two copies of  $C_{2n}$ );  $C_n^{(t)}$  for n = 4 or 8 and t > 2;  $C_6^{(t)}$  and  $t \geq 4$ ; caterpillars; stars; lobsters; banana trees; rooted trees of height 2; ladders; the graphs obtained by identifying the endpoints of any number of copies of  $P_n$ ; the graph obtained by attaching pendent edges to each endpoint of two identical stars and then identifying one endpoint from each of these graphs; the graph obtained by identifying a vertex of  $C_{4k+2}$  with an endpoint of a star; n-polygonal snakes (see §2.2) for  $n \equiv 0 \pmod{4}$ ;  $n \ge 0$  (mod 4); n-polygonal snakes for  $n \equiv 2 \pmod{4}$  where the number of polygons is even; crowns  $C_n \odot K_1$  for n even;  $C_{2n} \odot P_m$  ( $C_{2n}$  with  $P_m$  attached at each vertex of the cycle) for  $m \ge 3$ ; chains of cycles (see §2.2) of the form  $C_{4,m}, C_{6,2m}$ , and  $C_{8,m}$ . He conjectures that every one modulo three graceful graph is graceful.

A subdivided shell graph is obtained by subdividing the edges in the path of the shell graph. Jeba Jesintha and Ezhilarasi Hilda [1119] proved that the subdivided uniform shell bow graphs (that is, double shells in which each shell has the same order) are one modulo three graceful. Jeba Jesintha and Ezhilarasi Hilda [1118] proved the disjoint union of two subdivided shell graphs are one modulo three graceful.

In [2074] Ramachandran and Sekar introduced the notion of one modulo N graceful as follows. For a positive integer N a graph G with q edges is said to be one modulo N graceful if there is an injective function  $\phi$  from the vertex set of G to  $\{0, 1, N, N+1, 2N, 2N+1, \dots, (q-1)N, (q-1)N+1\}$  such that  $\phi$  induces a bijection  $\phi^*$ from the edge set of G to  $\{1, N+1, 2N+1, \dots, (q-1)N+1\}$  where  $\phi^*(uv) = |\phi(u) - \phi(v)|$ . They proved the following graph are one modulo N graceful for all positive integers N: paths, caterpillars, and stars [2074]; n-polygonal snakes,  $C_n^{(t)}$ ,  $P_{a,b}$  [2087]; the splitting graphs  $S'(P_{2n})$ ,  $S'(P_{2n+1})$ ,  $S'(K_{1,n})$ , all subdivision graphs of double triangular snakes, and all subdivision graphs of 2*m*-triangular snakes [2075]; the graph  $L_n \otimes S_m$  obtained from the ladder  $L_n$   $(P_n \times P_2)$  by identifying one vertex of  $L_n$  with any vertex of the star  $S_m$  other than the center of  $S_m$  [2077]; arbitrary supersubdivisions of paths, disconnected paths, cycles, and stars [2076]; and regular bamboo trees and coconut trees [2078]. Ramachandran and Sekar [2079] proved the supersubdivisions of ladders are one modulo Ngraceful for all positive integers N. In [2080] Ramachandran and Sekar proved that the crowns, armed crowns, and chain of even cycles are one modulo N graceful for all positive integers N.

Deviating from the standard definition of Fibonacci numbers, Kathiresan and Amutha [1349] define  $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$  They call a function  $f : V(G) \rightarrow$ 

 $\{0, 1, 2, \ldots, F_q\}$  where  $F_q$  is their qth Fibonacci number, to be Fibonacci graceful labeling if the induced edge labeling  $\overline{f}(uv) = |f(u) - f(v)|$  is a bijection onto the set  $\{F_1, F_2, \ldots, F_q\}$ . If a graph admits a Fibonacci graceful labeling, it is called a *Fibonacci graceful* graph. They prove the following:  $K_n$  is Fibonacci graceful if and only if  $n \leq 3$ ; if an Eulerian graph with q edges is Fibonacci graceful then  $q \equiv 0 \pmod{3}$ ; paths are Fibonacci graceful; fans  $P_n \odot K_1$  are Fibonacci graceful; squares of paths  $P_n^2$  are Fibonacci graceful; and caterpillars are Fibonacci graceful. They define a function  $f: V(G) \to \{0, F_1, F_2, \ldots, F_q\}$ where  $F_i$  is the *i*th Fibonacci number, to be super Fibonacci graceful labeling if the induced labeling f(uv) = |f(u) - f(v)| is a bijection onto the set  $\{F_1, F_2, \ldots, F_q\}$ . They show that bistars  $B_{n,n}$  are Fibonacci graceful but not super Fibonacci graceful for  $n \ge 5$ ; cycles  $C_n$  are super Fibonacci graceful if and only if  $n \equiv 0 \pmod{3}$ ; if G is Fibonacci or super Fibonacci graceful then  $G \odot K_1$  is Fibonacci graceful; if  $G_1$  and  $G_2$  are super Fibonacci graceful in which no two adjacent vertices have the labeling 1 and 2 then  $G_1 \cup G_2$  is Fibonacci graceful; and if  $G_1, G_2, \ldots, G_n$  are super Fibonacci graceful graphs in which no two adjacent vertices are labeled with 1 and 2 then the amalgamation of  $G_1, G_2, \ldots, G_n$ obtained by identifying the vertices having labels 0 is also a super Fibonacci graceful.

Vaidya and Prajapati [2659] proved: the graphs obtained joining a vertex of  $C_{3m}$  and a vertex of  $C_{3n}$  by a path  $P_k$  are Fibonacci graceful; the graphs obtained by starting with any number of copies of  $C_{3m}$  and joining each copy with a copy of the next by identifying the end points of a path with a vertex of each successive pair of  $C_{3m}$  (the paths need not be the same length) are Fibonacci graceful; the one point union of  $C_{3m}$  and  $C_{3n}$  is Fibonacci graceful; the one point union of k cycles  $C_{3m}$  is super Fibonacci graceful; every cycle  $C_n$  with  $n \equiv 0 \pmod{3}$  or  $n \equiv 1 \pmod{3}$  is an induced subgraph of a super Fibonacci graceful graph; and every cycle  $C_n$  with  $n \equiv 2 \pmod{3}$  can be embedded as a subgraph of a Fibonacci graceful graph. Karthikeyan, Arthi, Abinaya, Swathi, Madhumathi [1342] proved that friendship graphs  $C_3^{(t)}$  and the graphs obtained by the one-point union of copies of  $K_4$  with an edge deleted are super Fibonacci graceful.

For a graph G with q edges an injective function f from the vertices of G to  $\{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ , where  $F_i$  is the *i*th Fibonacci number (as defined by Kathiresan and Amuth above), is said to be *almost super Fibonacci graceful* if the induced edge labeling f \* (uv) = |f(u) - f(v)| is a bijection onto the set  $\{F_1, F_2, \ldots, F_q\}$  or  $\{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ .

Sridevi, Navaneethakrishnan, and Nagarajan [2463] proved that paths, combs, graphs obtained by subdividing each edge of a star, and some special types of extension of cycle related graphs are almost super Fibonacci graceful labeling.

For a graph G and a vertex v of G, a vertex switching  $G_v$  is the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Vaidya and Vihol [2685] prove the following: trees are Fibonacci graceful; the graph obtained by switching of a vertex in cycle is Fibonacci graceful; wheels and helms are not Fibonacci graceful; the graph obtained by switching of a vertex in a cycle is super Fibonacci graceful except  $n \ge 6$ ; the graph obtained by switching of a vertex in cycle  $C_n$ for  $n \ge 6$  can be embedded as an induced subgraph of a super Fibonacci graceful graph; and the graph obtained by joining two copies of a fixed fan with an edge is Fibonacci graceful.

The Perrin sequence of numbers  $P_n$  is defined by the linear recurrence relation satisfying the conditions:  $P_1 = 3$ ,  $P_2 = 0$ ,  $P_3 = 2$ , and  $P_n = P_{n-2} + P_{n-3}$ , if  $n \ge 4$ . Letting  $P_i$  be the  $i^{th}$  term of the Perrin sequence and  $P_0 = 0$ , Sugumaran and Rajesh [2527] introduced the notion of Perrin graceful labeling as follows: A function f is called a Perrin graceful labeling of a graph G, if  $f : V(G) \to \{P_0, P_1, P_2, ..., P_q\}$  is injective and the induced function  $f^{(*)} : E(G) \to \{P_1, P_2, ..., P_q\}$  defined by  $f^*(uv) = |f(u) - f(v)|$  is bijective. A graph that admits Perrin graceful labeling is called a Perrin graceful graph. In [2527] Sugumaran and Rajesh proved that the following graphs are Perrin graceful graphs:  $K_{1,n}$ ,  $B_{n,n}, P_n \odot K_1, C_n \odot K_1$ , and  $\langle K_{1,n}; 4 \rangle$ .

For  $n \ge 1$  the Pell numbers are defined as  $p_0 = 0$ ,  $p_1 = 1$ , and  $p_{n+1} = 2p_n + p_{n-1}$ . For a graph G with q edges Muthuramakrishnan and Sutha [1813] introduced the concept of *Pell graceful* labeling as an injective function f from V(G) to the Pell numbers  $\{0, 1, 2, \ldots, p_q\}$  such that the induced edge labeling  $f^*(uv) = |f(u) - f(v)|$  is a bijection onto the Pell numbers  $\{p_1, p_2, \ldots, p_q\}$  They prove that paths, combs  $P_n \odot K_1$   $(n \ge 3)$ , and the graphs obtained by the one point union of paths of lengths  $1, 2, \ldots, n$   $(n \ge 3)$  are Pell graceful.

In [502] Brešar and Klavžar define a natural extension of graceful labelings of certain tree subgraphs of hypercubes. A subgraph H of a graph G is called *isometric* if for every two vertices u, v of H, there exists a shortest u-v path that lies in H. The isometric subgraphs of hypercubes are called *partial cubes*. Two edges xy, uv of G are in  $\Theta$ -relation if  $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$ . A  $\Theta$ -relation is an equivalence relation that partitions E(G) into  $\Theta$ -classes. A  $\Theta$ -graceful labeling of a partial cube G on n vertices is a bijection  $f: V(G) \rightarrow \{0, 1, \ldots, n-1\}$  such that, under the induced edge labeling, all edges in each  $\Theta$ -class of G have the same label and distinct  $\Theta$ -classes get distinct labels. They prove that several classes of partial cubes are  $\Theta$ -graceful and the Cartesian product of  $\Theta$ -graceful partial cubes is  $\Theta$ -graceful. They also show that if there exists a class of partial cubes that contains all trees and every member of the class admits a  $\Theta$ -graceful labeling then all trees are graceful.

Table 4 provides a summary results about graceful-like labelings adapted from [501]. "Y" indicates that all graphs in that class have the labeling; "N" indicates that not all graphs in that class have the labeling; "?" means unknown; "C" means conjectured.

## 3.4 k-graceful Labelings

A natural generalization of graceful graphs is the notion of k-graceful graphs introduced independently by Slater [2424] in 1982 and by Maheo and Thuillier [1688] in 1982. A graph G with q edges is k-graceful if there is labeling f from the vertices of G to  $\{0, 1, 2, \ldots, q + k - 1\}$  such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is  $\{k, k + 1, \ldots, q + k - 1\}$ . Obviously, 1-graceful is graceful and it is readily shown that any graph that has an  $\alpha$ -labeling is k-graceful for all k. Graphs that are k-graceful for all k are sometimes called *arbitrarily graceful*. The result of Barrientos and Minion [372] that all snake polyominoes are  $\alpha$ -graphs partially answers a question of Acharya [25] and supports his conjecture that if the length of every

Graph	$\alpha$ -labeling	$\beta$ -labeling	$\sigma$ -labeling	$\rho$ -labeling
Cycle $C_n, n \equiv 0 \mod 4$	Y [2136]	Y	Y	Y
Cycle $C_n, n \equiv 3 \mod 4$	N [2136]	Y [2136]	Y	Y
Wheels	N	Y [795], [1018]	Y	Y
Trees				
Yes, if order $\leq$	5	35 [747]	54	
Paths	Y [2136]	Y	Y	Y
Caterpillars	Y [2136]	Y	Y	Y
Firecrackers	Y [575]	Y	Y	Y
Lobsters	N[471]	?C [445]	Y	Y [546]
Bananas	?	Y [2281], [2280]	Y	Y
Symmetrical trees	N [471]	Y [449]	Y	Y
Olive trees	?	Y [1902], [11]	Y	Y
Diameter $< 8$	N [2790]	Y	Y	Y
< 5 end vertices	N [471]	Y [2136]	Y	Y
Max degree 3	N [2139]	С	С	C
Max degree 3 and				
perfect matching	C [499]	C	С	C

 Table 4: Summary of Results on Graceful-like labelings

cycle of a graph is a multiple of 4, then the graph is arbitrarily graceful. In [2233] Seoud and Elsakhawi show that  $P_2 \oplus \overline{K_2}$   $(n \ge 2)$  is arbitrarily graceful. Ng [1849] has shown that there are graphs that are k-graceful for all k but do not have an  $\alpha$ -labeling.

Results of Maheo and Thuillier [1688] together with those of Slater [2424] show that:  $C_n$  is k-graceful if and only if either  $n \equiv 0$  or 1 (mod 4) with k even and  $k \leq (n-1)/2$ , or  $n \equiv 3 \pmod{4}$  with k odd and  $k \leq (n^2 - 1)/2$ . Maheo and Thuillier [1688] also proved that the wheel  $W_{2k+1}$  is k-graceful and conjectured that  $W_{2k}$  is k-graceful when  $k \neq 3$  or  $k \neq 4$ . This conjecture was proved by Liang, Sun, and Xu [1590]. Kang [1324] proved that  $P_m \times C_{4n}$  is k-graceful for all k. Lee and Wang [1557] showed that the graphs obtained from a nontrivial path of even length by joining every other vertex to one isolated vertex (a *lotus*), the graphs obtained from a nontrivial path of even length by joining every other vertex to two isolated vertices (a *diamond*), and the graphs obtained by arranging vertices into a finite number of rows with i vertices in the *i*th row and in every row the *j*th vertex in that row is joined to the *j*th vertex and j + 1st vertex of the next row (a *pyramid*) are k-graceful. Liang and Liu [1578] have shown that  $K_{m,n}$  is k-graceful. Bu, Gao, and Zhang [511] have proved that  $P_n \times P_2$  and  $(P_n \times P_2) \cup (P_n \times P_2)$  are k-graceful for all k. Acharya (see [25]) has shown that a k-graceful Eulerian graph with q edges must satisfy one of the following conditions:  $q \equiv 0 \pmod{4}$ ,  $q \equiv 1 \pmod{4}$  if k is even, or  $q \equiv 3 \pmod{4}$  if k is odd. Bu, Zhang, and He [516] have shown that an even cycle with a fixed number of pendent edges adjoined to each vertex is k-graceful. Lu, Pan, and Li [1673] have proved

that  $K_{1,m} \cup K_{p,q}$  is k-graceful when k > 1, and p and q are at least 2. Jirimutu, Bao, and Kong [1257] have shown that the graphs obtained from  $K_{2,n}$   $(n \ge 2)$  and  $K_{3,n}$   $(n \ge 3)$ by attaching  $r \ge 2$  edges at each vertex is k-graceful for all  $k \ge 2$ . Seoud and Elsakhawi [2233] proved: paths and ladders are arbitrarily graceful; and for  $n \ge 3$ ,  $K_n$  is k-graceful if and only if k = 1 and n = 3 or 4. Li, Li, and Yan [1576] proved that  $K_{m,n}$  is k-graceful graph. Pradhan and Kamesh [2009] showed that the hairy cycle  $C_n \cdot rK_1$   $(n \equiv 3 \pmod{4})$ , the graph obtained by adding a pendent edge to each pendent vertex of hairy cycle  $C_n \cdot K_1$ ;  $n \equiv 0 \pmod{4}$ , double graphs of path  $P_n$ , and double graphs of combs  $P_n \cdot K_1$ are k-graceful.

Yao, Cheng, Zhongfu, and Yao [2862] have shown: a tree of order p with maximum degree at least p/2 is k-graceful for some k; if a tree T has an edge  $u_1u_2$  such that the two components  $T_1$  and  $T_2$  of  $T - u_1u_2$  have the properties that  $d_{T_1}(u_1) \ge |T_1|/2$  and  $d_{T_2}(u_2) \ge |T_2|/2$ , then T is k-graceful for some positive k; if a tree T has two edges  $u_1u_2$  and  $u_2u_3$  such that the three components  $T_1$ ,  $T_2$ , and  $T_3$  of  $T - \{u_1u_2, u_2u_3\}$  have the properties that  $d_{T_1}(u_1) \ge |T_1|/2$ ,  $d_{T_2}(u_2) \ge |T_2|/2$ , and  $d_{T_3}(u_3) \ge |T_3|/2$ , then T is k-graceful for some k > 1; and every Skolem-graceful (see 3.5 for the definition) tree is k-graceful for all  $k \ge 1$ . They conjecture that every tree is k-graceful for some k > 1.

Several authors have investigated the k-gracefulness of various classes of subgraphs of grid graphs. Acharya [23] proved that all 2-dimensional polyminoes that are convex and Eulerian are k-graceful for all k; Lee [1487] showed that Mongolian tents and Mongolian villages are k-graceful for all k (see §2.3 for the definitions); Lee and K. C. Ng [1511] proved that all Young tableaus (see §2.3 for the definitions) are k-graceful for all k. (A special case of this is  $P_n \times P_2$ .) Lee and H. K. Ng [1511] subsequently generalized these results on Young tableaus to a wider class of planar graphs.

In [382] Barrientos and Minion say that two caterpillars  $\Gamma$  and  $\Omega$  of size *n* are analogous if the stable sets of  $\Gamma$  have the same cardinalities as the stable sets of  $\Omega$ . They prove that if  $\Omega$  is an induced subgraph of a gracefully labeled graph *G*, such that the induced labeling is a bipartite *k*-labeling shifted *c* units, then the graph *G'* obtained by replacing  $\Omega$  with any other caterpillar  $\Gamma$  analogous to  $\Omega$ , is a graceful graph. This result is used to generalize several existing results that use *k*-graceful labelings of paths such as the subdivision of graceful trees [520], the  $\alpha$ -labeling of the *i*th attachment tree [2301], the  $\alpha$ -labelings of path-like trees [353], the  $\alpha$ -labelings of the graphs obtained by identifying the end-vertices of *b* paths of length *a* with two new vertices, as well as the graceful labelings of the armed crowns [2200].

Duan and Qi [697] use  $G_t(m_1, n_1; m_2, n_2; \ldots; m_s, n_s)$  to denote the graph composed of the s complete bipartite graphs  $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_s,n_s}$  that have only t

 $(1 \leq t \leq \min\{m_1, m_2, \ldots, m_s\})$  common vertices but no common edge and  $G(m_1, n_1; m_2, n_2)$  to denote the graph composed of the complete bipartite graphs  $K_{m_1,n_1}, K_{m_2,n_2}$  with exactly one common edge. They prove that these graphs are k-graceful graphs for all k.

Let  $c, m, p_1, p_2, \ldots, p_m$  be positive integers. For  $i = 1, 2, \ldots, m$ , let  $S_i$  be a set of  $p_i + 1$  integers and let  $D_i$  be the set of positive differences of the pairs of elements of  $S_i$ . If all these differences are distinct then the system  $D_1, D_2, \ldots, D_m$  is called a *perfect system of*
difference sets starting at c if the union of all the sets  $D_i$  is  $c, c+1, \ldots, c-1 + \sum_{i=1}^{m} {p_i+1 \choose 2}$ . There is a relationship between k-graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with c describes a c-graceful labeling of a graph that is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in [13].

Acharya and Hegde [39] generalized k-graceful labelings to (k, d)-graceful labelings by permitting the vertex labels to belong to  $\{0, 1, 2, \ldots, k + (q-1)d\}$  and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be  $\{k, k + d, k + 2d, \ldots, k + (q-1)d\}$ . They also introduce an analog of  $\alpha$ -labelings in the obvious way. Notice that a (1,1)-graceful labeling is a graceful labeling and a (k, 1)-graceful labeling is a k-graceful labeling. Bu and Zhang [515] have shown:  $K_{m,n}$ is (k, d)-graceful for all k and d; for  $n > 2, K_n$  is (k, d)-graceful if and only if k = dand  $n \leq 4$ ; if  $m_i, n_i \geq 2$  and  $\max\{m_i, n_i\} \geq 3$ , then  $K_{m_1,n_1} \cup K_{m_2,n_2} \cup \cdots \cup K_{m_r,n_r}$  is (k, d)-graceful for all k, d, and r; if G has an  $\alpha$ -labeling, then G is (k, d)-graceful for all k and d; a k-graceful graph is a (kd, d)-graceful graph; a (kd, d)-graceful connected graph is k-graceful; and a (k, d)-graceful graph with q edges that is not bipartite must have  $k \leq (q-2)d$ .

Let T be a tree with adjacent vertices  $u_0$  and  $v_0$  and pendent vertices u and v such that the length of the path  $u_0 - u$  is the same as the length of the path  $v_0 - v$ . Hegde and Shetty [998] call the graph obtained from T by deleting  $u_0v_0$  and joining u and van *elementary parallel transformation* of T. They say that a tree T is a  $T_p$ -tree if it can be transformed into a path by a sequence of elementary parallel transformations. They prove that every  $T_p$ -tree is (k, d)-graceful for all k and d and every graph obtained from a  $T_p$ -tree by subdividing each edge of the tree is (k, d)-graceful for all k and d.

Yao, Cheng, Zhongfu, and Yao [2862] have shown: a tree of order p with maximum degree at least p/2 is (k, d)-graceful for some k and d; if a tree T has an edge  $u_1u_2$  such that the two components  $T_1$  and  $T_2$  of  $T - u_1u_2$  have the properties that  $d_{T_1}(u_1) \ge |T_1|/2$ and  $T_2$  is a caterpillar, then T is Skolem-graceful (see 3.5 for the definition); if a tree Thas an edge  $u_1u_2$  such that the two components  $T_1$  and  $T_2$  of  $T - u_1u_2$  have the properties that  $d_{T_1}(u_1) \ge |T_1|/2$  and  $d_{T_2}(u_2) \ge |T_2|/2$ , then T is (k, d)-graceful for some k > 1 and d > 1; if a tree T has two edges  $u_1u_2$  and  $u_2u_3$  such that the three components  $T_1$ ,  $T_2$ , and  $T_3$  of  $T - \{u_1u_2, u_2u_3\}$  have the properties that  $d_{T_1}(u_1) \ge |T_1|/2$ ,  $d_{T_2}(u_2) \ge |T_2|/2$ , and  $d_{T_3}(u_3) \ge |T_3|/2$ , then T is (k, d)-graceful for some k > 1 and d > 1; and every Skolem-graceful tree is (k, d)-graceful for  $k \ge 1$  and d > 0. They conjecture that every tree is (k, d)-graceful for some k > 1 and d > 1.

Hegde [983] has proved the following: if a graph is (k, d)-graceful for odd k and even d, then the graph is bipartite; if a graph is (k, d)-graceful and contains  $C_{2j+1}$  as a subgraph, then  $k \leq jd(q-j-1)$ ;  $K_n$  is (k, d)-graceful if and only if  $n \leq 4$ ;  $C_{4t}$  is (k, d)-graceful for all k and d;  $C_{4t+1}$  is (2t, 1)-graceful;  $C_{4t+2}$  is (2t - 1, 2)-graceful; and  $C_{4t+3}$  is (2t + 1, 1)-graceful.

A semismooth graceful graph is a bipartite graph G with the property that for some fixed positive integer  $t \leq q$  and all positive integers l there is an injective map g : $V \longrightarrow \{0, 1, \ldots, t - l, t + l + 1, \ldots, q + l\}$  such that the induced edge labeling map  $g^*: E \longrightarrow \{1+l, 2+l, \ldots, q+l\}$  defined by  $g^*(e) = |g(u) - g(v)|$  is a bijection. Kaneria, Gohil, and Makadia [1279] prove every semismooth graceful graph is a (k, d)-graceful; graphs obtained by joining two semismooth graceful graphs with an arbitrary path is a semismooth graceful graph; and the notions of graceful labeling and odd-even graceful labelings are equivalent. (A graph G with q edges is odd-even graceful if there is an injection f from the vertices of G to  $\{1, 3, 5, \ldots, 2q + 1\}$  such that, when each edge uv is assigned the label |f(u) - f(v)|, the resulting edge labels are  $\{2, 4, 6, \ldots, 2q\}$ ). Kaneria, Meghpara and Khoda [1286] prove: a smooth graceful labeling for a graph is also an  $\alpha$ -labeling for the graph; a graph that has an  $\alpha$ -labeling is a semismooth graceful graph; graphs that admit an  $\alpha$ -labeling are semismooth graceful graphs; if m is even and H has an  $\alpha$ -labeling, then the path union  $P(m \cdot H)$  is a smooth graceful graph; and the path union  $P(m \cdot H)$  has an  $\alpha$ -labeling.

In [2487] Sudha and Kanniga proved that tensor product of a star and  $P_2$  is odd-even graceful. (The tensor product  $G \otimes H$  of graphs G and H, has the vertex set  $V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in  $G \otimes H$  if and only if u' is adjacent with v' and u is adjacent with v.) In [2723] Venkatesh, Mahalakshmi, and Amirthavahini use  $C_{n,k}$  to denote the dragon obtained by joining an end point of  $P_k$  with a vertex of  $C_n$  and  $C_{n,k}^t$  to denote the graph obtained by taking one-point union of t copies of  $C_{n,k}$ at the common vertex v. They proved that the graph  $C_{n,k}^t$  admits a graceful labeling, an odd graceful labeling, and odd-even graceful labeling for all values of t with n = 4, k = 1, and that  $C_{n,1}^t$  admits vertex cordial labeling for all values of n and t, except  $n \equiv 2 \mod 4$ (see Section 3.7). Nurvazly and Sugeng [1879] proved that  $\Theta(C_3)^n$  graphs (n copies of  $C_3$ that share an edge) have odd-even graceful labelings. Anitha, Selvam, and Thirusangu [174] provide k-graceful and odd-even graceful labelings for the extended duplicate graph of the kite graph.

For a graph G let  $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$  be  $n \geq 2$  copies of G. The graph obtained by joining vertices u, v of  $G^{(i)}$  with same vertices of the graph  $G^{(i+1)}$  by two edges, for all  $i = 1, 2, \ldots, n-1$  is called the *double path union* of n copies of the graph G. Such graphs can obtained in  $\frac{p(p-1)}{2}$  different ways, where p = |V(G)| and are denoted by  $D(n \cdot G)$ . Kaneria, Teraiya and Meghpara [1314] prove the double path unions of  $C_{4m}$ ,  $K_{m,n}$ , and  $P_{2m}$  have  $\alpha$ -labelings.

Hegde [981] calls a (k, d)-graceful graph (k, d)-balanced if it has a (k, d)-graceful labeling f with the property that there is some integer m such that for every edge uv either  $f(u) \leq m$  and f(v) > m, or f(u) > m and  $f(v) \leq m$ . He proves that if a graph is (1, 1)balanced then it is (k, d)-graceful for all k and d and that a graph is (1, 1)-balanced graph if and only if it is (k, k)-balanced for all k. He conjectures that all trees are (k, d)-balanced for some values of k and d.

Slater [2427] has extended the definition of k-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices, and the countably infinite Dutch windmill is k-graceful for all k.

In [1004] Hegde and Shivarajkumar extend the idea of k-graceful labeling of undirected graphs to directed graphs as follows. A simple directed graph D with n vertices and e

edges is labeled by assigning each vertex a distinct element from the set  $Z_{e+k}$  and assigning the edge xy from vertex x to vertex y the label  $\theta(x, y) = \theta(y) - \theta(x) \mod(e+k)$ , where  $\theta(y)$  and  $\theta(x)$  are the values assigned to the vertices y and x respectively. A labeling is a k-graceful labeling if all  $\theta(x, y)$  are distinct and belong to  $\{k, k+1, \ldots, k+e-1\}$ . If a digraph D admits a k-graceful labeling then D is called a k-graceful digraph. They provide some values of k for which the unidirectional cycles admit a k-graceful labeling; give a necessary and sufficient condition for the outspoken unicyclic wheel to be k-graceful; and prove that to provide a list of values of k for which the unicyclic wheel is k-graceful is NP-complete.

More specialized results on k-graceful labelings can be found in [1487], [1511], [1515], [2424], [510], [512], [511], and [573].

Graceful-type labelings methods have been used for cryptographical password construction for network data [2768], [2767], [2769], and [2534].

#### 3.5 Skolem-Graceful Labelings

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [1541]) calls a graph G with p vertices and q edges Skolem-graceful if there is an injection from the set of vertices of G to  $\{1, 2, \ldots, p\}$ such that the edge labels induced by |f(x) - f(y)| for each edge xy are  $1, 2, \ldots, q$ . A necessary condition for a graph to be Skolem-graceful is that  $p \ge q+1$ . Lee and Wui [1571] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. Yao, Cheng, Zhongfu, and Yao [2862] have shown that a tree of order p with maximum degree at least p/2 is Skolem-graceful. Although the disjoint union of trees cannot be graceful, they can be Skolem-graceful. Lee and Wui [1571] prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. In [601] Choudum and Kishore show that the disjoint union of k copies of the star  $K_{1,2p}$  is Skolem graceful if  $k \leq 4p+1$  and the disjoint union of any number of copies of  $K_{1,2}$  is Skolem graceful. For  $k \geq 2$ , let  $St(n_1, n_2, \ldots, n_k)$  denote the disjoint union of k stars with  $n_1, n_2, \ldots, n_k$  edges. Lee, Wang, and Wui [1564] showed that the 4-star  $St(n_1, n_2, n_3, n_4)$  is Skolem-graceful for some special cases and conjectured that all 4-stars are Skolem-graceful. Denham, Leu, and Liu [661] proved this conjecture. Kishore [1381] has shown that a necessary condition for  $St(n_1, n_2, \ldots, n_k)$  to be Skolem graceful is that some  $n_i$  is even or  $k \equiv 0$  or 1 (mod 4) (see also [2893]. He conjectures that each one of these conditions is sufficient. Yue, Yuan-sheng, and Xin-hong [2893] show that for k at most 5, a k-star is Skolem-graceful if at one star has even size or  $k \equiv 0$  or 1 (mod 4). Choudum and Kishore [599] proved that all 5-stars are Skolem graceful.

Lee, Quach, and Wang [1527] showed that the disjoint union of the path  $P_n$  and the star of size m is Skolem-graceful if and only if n = 2 and m is even or  $n \ge 3$  and  $m \ge 1$ . It follows from the work of Skolem [2416] that  $nP_2$ , the disjoint union of n copies of  $P_2$ , is Skolem-graceful if and only if  $n \equiv 0$  or 1 (mod 4). Harary and Hsu [961] studied Skolem-graceful graphs under the name *node-graceful*. Frucht [797] has shown that  $P_m \cup P_n$  is Skolem-graceful when  $m + n \ge 5$ . Bhat-Nayak and Deshmukh [459] have shown that

 $P_{n_1} \cup P_{n_2} \cup P_{n_3}$  is Skolem-graceful when  $n_1 < n_2 \le n_3$ ,  $n_2 = t(n_1 + 2) + 1$  and  $n_1$  is even and when  $n_1 < n_2 \le n_3$ ,  $n_2 = t(n_1 + 3) + 1$  and  $n_1$  is odd. They also prove that the graphs of the form  $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i}$  where  $i \ge 4$  are Skolem-graceful under certain conditions. In [665] Deshmukh states the following results: the sum of all the edges on any cycle in a Skolem graceful graph is even;  $C_5 \cup K_{1,n}$  if and only if n = 1 or 2;  $C_6 \cup K_{1,n}$ if and only if n = 2 or 4.

Youssef [2872] proved that if G is Skolem-graceful, then  $G + \overline{K_n}$  is graceful. In [2876] Youssef shows that that for all  $n \geq 2$ ,  $P_n \cup S_m$  is Skolem-graceful if and only if  $n \geq 3$  or n = 2 and m is even. Yao, Cheng, Zhongfu, and Yao [2862] have shown that if a tree T has an edge  $u_1u_2$  such that the two components  $T_1$  and  $T_2$  of  $T - u_1u_2$  have the properties that  $d_{T_1}(u_1) \geq |T_1|/2$  and  $T_2$  is a caterpillar or have the properties that  $d_{T_1}(u_1) \geq |T_1|/2$ and  $d_{T_2}(u_2) \geq |T_2|/2$ , then T is Skolem-graceful.

A graph G = (V, E) is said to be (k, d)-Skolem graceful if there exists a bijection f from V(G) to  $\{12, \ldots, |V|\}$  such that the induced edge labeling  $g_f$  defined by  $g_f(uv) = |f(u) - f(v)|$  is a bijection from E to  $\{k, k+d, \ldots, k+(q-1)d\}$  where k and d are positive integers. Such a labeling is called a (k, d)-Skolem graceful labeling of G. In [1916] Pereira, Singh, and Arumugam present a few basic results on (k, d)-Skolem graceful graphs and prove that  $nK_2$  is (2, 1)-Skolem graceful if and only if  $n \equiv 0$  or  $3 \pmod{4}$ , which produces the Langford sequence L(2, n).

Mendelsohn and Shalaby [1742] defined a Skolem labeled graph G(V, E) as one for which there is a positive integer d and a function  $L: V \to \{d, d+1, \ldots, d+m\}$ , satisfying (a) there are exactly two vertices in V such that L(v) = d + i,  $0 \le i \le m$ ; (b) the distance in G between any two vertices with the same label is the value of the label; and (c) if G' is a proper spanning subgraph of G, then L restricted to G' is not a Skolem labeled graph. Note that this definition is different from the Skolem-graceful labeling of Lee, Quach, and Wang. A hooked Skolem sequence of order n is a sequence  $s_1, s_2, \ldots, s_{2n+1}$  such that  $s_{2n} = 0$  and for each  $j \in \{1, 2, ..., n\}$ , there exists a unique  $i \in \{1, 2, ..., 2n - 1, 2n + 1\}$ such that  $s_i = s_{i+j} = j$ . Mendelsohn [1741] established the following: any tree can be embedded in a Skolem labeled tree with O(v) vertices; any graph can be embedded as an induced subgraph in a Skolem labeled graph on  $O(v^3)$  vertices; for d = 1, there is a Skolem labeling or the minimum hooked Skolem (with as few unlabeled vertices as possible) labeling for paths and cycles; for d = 1, there is a minimum Skolem labeled graph containing a path or a cycle of length n as induced subgraph. In [1741] Mendelsohn and Shalaby prove that the necessary conditions in [1742] are sufficient for a Skolem or minimum hooked Skolem labeling of all trees consisting of edge-disjoint paths of the same length from some fixed vertex. Graham, Pike, and Shalaby [924] obtained various Skolem labeling results for grid graphs. Among them are  $P_1 \times P_n$  and  $P_2 \times P_n$  have Skolem labelings if and only if  $n \equiv 0$  or 1 mod 4; and  $P_m \times P_n$  has a Skolem labeling for all m and n at least 3.

In [1930] Pike, Sanaei, and Shalaby introduce pseudo-Skolem sequences, which are similar to Skolem-type sequences in their structures and applications. They use known Skolem-type sequences to constructions of such sequences and discuss applications of these sequences to Skolem labelingsre graphs such that H is bipartite, and give formulas for the gamma-number of rail-siding graphs and caterpillars.

In [626] Clark and Sanaei present (hooked) vertex Skolem labelings for generalized Dutch windmills whenever such labelings exist. They present a novel technique for showing that generalized Dutch windmills with more than two cycles cannot be Skolem labelled and that those composed of two cycles of lengths m and n,  $n \ge m$  cannot be Skolem labelled if and only if  $n - m \equiv 3$  or 5 (mod 8) and m is odd.

## **3.6 Odd-Graceful Labelings**

Gnanajothi [894, p. 182] defined a graph G with q edges to be *odd-graceful* if there is an injection f from V(G) to  $\{0, 1, 2, \dots, 2q-1\}$  such that, when each edge xy is assigned the label |f(x) - f(y)|, the resulting edge labels are  $\{1, 3, 5, \ldots, 2q - 1\}$ . She proved that the class of odd-graceful graphs lies between the class of graphs with  $\alpha$ -labelings and the class of bipartite graphs by showing that every graph with an  $\alpha$ -labeling has an odd-graceful labeling and every graph with an odd cycle is not odd-graceful. She also proved the following graphs are odd-graceful:  $P_n$ ;  $C_n$  if and only if n is even;  $K_{m,n}$ ; combs  $P_n \odot K_1$ (graphs obtained by joining a single pendent edge to each vertex of  $P_n$ ); books; crowns  $C_n \odot K_1$  (graphs obtained by joining a single pendent edge to each vertex of  $C_n$ ) if and only if n is even; the disjoint union of copies of  $C_4$ ; the one-point union of copies of  $C_4$ ;  $C_n \times K_2$ if and only if n is even; caterpillars; rooted trees of height 2; the graphs obtained from  $P_n$   $(n \geq 3)$  by adding exactly two leaves at each vertex of degree 2 of  $P_n$ ; the graphs obtained from  $P_n \times P_2$  by deleting an edge that joins to end points of the  $P_n$  paths; the graphs obtained from a star by adjoining to each end vertex the path  $P_3$  or by adjoining to each end vertex the path  $P_4$ . She conjectures that all trees are odd-graceful and proves the conjecture for all trees with order up to 10. Barrientos [357] has extended this to trees of order up to 12. Eldergill [708] generalized Gnanajothi's result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd-graceful. He also proved that the one-point union of any number of copies of  $C_6$  is odd-graceful. Kathiresan [1347] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd-graceful. Barrientos [360] and [357] has proved the following graphs are odd-graceful: every forest whose components are caterpillars; every tree with diameter at most five is odd-graceful; and all disjoint unions of caterpillars. He conjectures that every bipartite graph is odd-graceful. In [1840] Neela and Selvaraj partially resolved a Barrientos's conjecture by showing that the following graphs are odd-graceful: finite unions of paths, stars, and caterpillars; finite unions of ladders; finite unions of paths, bistars and caterpillars; finite unions of graphs obtained by the one end point union of an odd number of paths of uniform length; and the coronas  $K_{m;n} \odot rK_l$ . Gao, Zhang, and Xu [838] proved that  $P_n \times P_m$  (m = 2, 3 or 4), generalized crown graphs  $C_n \odot K_{1,t}$ , and gears are odd graceful.

Seoud, Diab, and Elsakhawi [2230] have shown that a connected complete r-partite graph is odd-graceful if and only if r = 2 and that the join of any two connected graphs is not odd-graceful. Yan [2844] proved that  $P_m \times P_n$  is odd-graceful labeling. Vaidya and Shah [2667] prove that the splitting graph and the shadow graph of bistar are odd-graceful. (The shadow graph  $D_2(G)$  of a connected graph G is constructed by taking 2 copies  $G_1$ and  $G_2$  of G and joining each vertex u in  $G_1$  to the neighbors of the corresponding vertex vin  $G_2$ . Li, Li, and Yan [1576] proved that  $K_{m,n}$  is odd-graceful. Liu, Wang, and Lu [1614] that proved that a class of bicyclic graphs with a common edge is odd-graceful. Moussa and Badr [1792] proved tha ladders and subdivisions of ladders with pendent edges are odd-graceful.

Sekar [2200] has shown the following graphs are odd-graceful: the graph obtained by identifying an end point of  $P_n$  with every vertex of  $C_m$  where  $n \ge 3$  and m is even;  $P_{a,b}$ when  $a \ge 2$  and b is odd (see §2.7);  $P_{2,b}$  and  $b \ge 2$ ;  $P_{4,b}$  and  $b \ge 2$ ;  $P_{a,b}$  when a and b are even and  $a \ge 4$  and  $b \ge 4$ ;  $P_{4r+1,4r+2}$ ;  $P_{4r-1,4r}$ ; all n-polygonal snakes with n even;  $C_n^{(t)}$  (see §2.2 for the definition); graphs obtained by beginning with  $C_6$  and repeatedly forming the one-point union with additional copies of  $C_6$  in succession; graphs obtained by beginning with  $C_8$  and repeatedly forming the one-point union with additional copies of  $C_8$  in succession; graphs obtained from even cycles by identifying a vertex of the cycle with the endpoint of a star;  $C_{6,n}$  and  $C_{8,n}$  (see §2.7); the splitting graph of  $P_n$  (see §2.7) the splitting graph of  $C_n$ , n even; lobsters, banana trees, and regular bamboo trees (see §2.1).

Yao, Cheng, Zhongfu, and Yao [2862] have shown the following: if a tree T has an edge  $u_1u_2$  such that the two components  $T_1$  and  $T_2$  of  $T - u_1u_2$  have the properties that  $d_{T_1}(u_1) \geq |T_1|/2$  and  $T_2$  is a caterpillar, then T is odd-graceful; and if a tree T has a vertex of degree at least |T|/2, then T is odd-graceful. They conjecture that for trees the properties of being Skolem-graceful and odd-graceful are equivalent. Recall a banana tree is a graph obtained by starting with any number os stars and connecting one end-vertex from each to a new vertex. Zhenbin [2913] has shown that graphs obtained by starting with any number of stars, appending an edge to exactly one edge from each star, then joining the vertices at which the appended edges were attached to a new vertex are odd-graceful.

Solairaju and Chithra [2441] defined a graph G with q edges to be *edge-odd graceful* if there is an bijection f from the edges of the graph to  $\{1, 3, 5, \ldots, 2q - 1\}$  such that, when each vertex is assigned the sum of all the edges incident to it mod 2q, the resulting vertex labels are distinct. They prove they following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders  $P_n \times P_2$   $(n \ge 3)$ ; stars with an even number of edges; and crowns  $C_n \odot K_1$ . In [2442] they prove the following graphs have edge-odd graceful labelings:  $P_n$  (n > 1) with a pendent edge attached to each vertex (combs); the graph obtained by appending 2n + 1 pendent edges to each endpoint of  $P_2$  or  $P_3$ ; and the graph obtained by subdividing each edge of the star  $K_{1,2n}$ .

A subdivided shell graph is obtained by subdividing the edges in the path of the shell graph. Let  $G_1, G_2, \ldots, G_n$  be n subdivided shell graphs of any order. The graph SSG(n) is obtained by adding an edge to apexes of  $G_i$  and  $G_{i+1}, i = 1, 2, \ldots, n-1$ . Jeba Jesintha and Ezhilarasi Hilda [1126] that SSG(2) is odd graceful. In [1120] and [1125] Jeba Jesintha and Ezhilarasi Hilda proved that the subdivided uniform shell bow graphs (that is, double shells in which each shell has the same order) are odd graceful and shell butterfly graphs are edge-odd graceful. Daoud [643] provided necessary and sufficient conditions for  $C_m \times$ 

 $P_n$  and  $C_m \times C_n$  to be edge-odd graceful.

Gao [831] has proved the following graphs are odd-graceful: the union of any number of paths; the union of any number of stars; the union of any number of stars and paths;  $C_m \cup P_n$ ;  $C_m \cup C_n$ ; and the union of any number of cycles each of which has order divisible by 4.

If f is an odd-graceful labeling of a bipartite graph G with bipartition  $(V_1, V_2)$  such that  $\max\{f(u) : u \in V_1\} < \min\{f(v) : v \in V_2\}$ , Zhou, Yao, Chen, and Tao [2921] say that f is a *set-ordered odd-graceful labeling* of G. They proved that every lobster is odd-graceful and adding leaves to a connected set-ordered odd-graceful graph is an odd-graceful graph.

In [2219] Seoud and Abdel-Aal determined all odd-graceful graphs of order at most 6 and proved that if G is odd-graceful then  $G \cup K_{m,n}$  is odd-graceful. In [2238] Seoud and Helmi proved: if G has an odd-graceful labeling f with bipartition  $(V_1, V_2)$  such that  $\max\{f(x) : f(x) \text{ is even, } x \in V_1\} < \min\{f(x) : f(x) \text{ is odd, } x \in V_2\}, \text{ then } G \text{ has}$ an  $\alpha$ -labeling; if G has an  $\alpha$ -labeling, then  $G \odot K_n$  is odd-graceful; and if  $G_1$  has an  $\alpha$ -labeling and  $G_2$  is odd-graceful, then  $G_1 \cup G_2$  is odd-graceful. They also proved the following graphs have odd-graceful labelings: dragons obtained from an even cycle; graphs obtained from a gear graph by attaching a fixed number of pendent edges to each vertex of degree 2 on rim of the wheel of the graph;  $C_{2m} \odot \overline{K_n}$ ; graphs obtained from an even cycle by attaching a fixed number of pendent edges to every other vertex; graphs obtained by identifying an endpoint of a star  $S_n$   $(n \ge 3)$  with a vertex of an even cycle; the graphs consisting of two even cycles of the same order that share a common vertex with any number of pendent edges attached at the common vertex; and the graphs obtained by joining two even cycles of the same order by an edge. Seoud, El Sonbaty, and Abd El Rehim [2231] proved that the conjunction  $P_m \wedge P_n$  for all  $n, m \geq 2$  and the conjunction  $K_2 \wedge F_n$  for n even are odd-graceful. Jeba Jesintha and Ezhilarasi Hilda [1118] proved the disjoint union of two subdivided shell graphs is odd-graceful and the one vertex union of three subdivided shells are odd-graceful.

In [1789] and [1790] Moussa proved that  $C_m \cup P_n$  is odd-graceful in some cases and gave algorithms to prove that for all  $m \ge 2$  the graphs  $P_{4r-1;m}$ , r = 1, 2, 3 and  $P_{4r+1;m}$ , r = 1, 2 are odd-graceful. ( $P_{n;m}$  is the graph obtained by identifying the endpoints of mpaths each of length n). He also presented an algorithm that showed that closed spider graphs and the graphs obtained by joining one or two copies of  $P_m$  to each vertex of the path  $P_n$  are odd-graceful. Moussa and Badr [1788] proved that  $C_m \odot P_n$  is odd-graceful if and only if m is even (see also [216]). Badr, Moussa, and Kathiresan [216] proved ladders are odd graceful.

Moussa [1791] defines the *tensor product*,  $P_m \wedge P_n$ , of  $P_m$  and  $P_n$  as the graph with vertices  $v_i^j$ , i = 1, ..., n; j = 1, ..., m and edges  $v_1^j v_2^{j+1}, v_2^{j+1} v_3^j, ..., v_{n-1}^j v_n^{j+1}$  for j odd and

 $v_1^j v_2^{j-1}, v_2^{j-1} v_3^j, \ldots, v_{n-1}^j v_n^{j-1}$  for j even. He proves that  $P_m \wedge P_m$  is odd-graceful.

In [2] Abdel-Aal generalized the notions of shadow graphs and splitting graphs are follows. The *m*-shadow graph  $D_m(G)$  of a connected graph G is constructed by taking m copies of  $G_1, G_2, \ldots, G_m$  of G, and joining each vertex u in  $G_i$  to the neighbors of the corresponding vertex v in  $G_j$  for  $1 \leq i, j \leq m$ . The *m*-splitting graph  $Spl_m(G)$  of a graph G is obtained by adding to each vertex v of G m new vertices,  $v^1, v^2, \ldots, v^m$ , such that  $v^i$ ,  $1 \leq i \leq m$  is adjacent to every vertex that is adjacent to v in  $G_j$ . Thus the 2-shadow graph is the shadow graph  $D_2(G)$  and the 1-splitting graph of G is the splitting graph of G. Abdel-Aal proved the following graphs are odd-graceful:  $D_m(P_n), D_m(P_n \oplus \overline{K_2})$  (the symmetric product of  $P_n$  and  $\overline{K_2}$ ),  $D_m(K_{r,s}), Spl_m(P_n), Spl_m(K_{1,n})$ , and  $Spl_m(P_n \oplus \overline{K_2})$ .

Vaidya and Bijukumar [2624] proved the following are odd-graceful: graphs obtained by joining two copies of  $C_n$  by a path; graphs that are two copies of an even cycle that share a common edge; graphs that are the splitting graph of a star; and graphs that are the tensor product of a star and  $P_2$ .

Acharya, Germina, Princy, and Rao [35] proved that every bipartite graph G can be embedded in an odd-graceful graph H. The construction is done in such a way that if G is planar and odd-graceful, then so is H. Varkey and Sunoj [2706] investigate some new families of odd graceful graphs generated from various graph operations on the given graph.

In [570] Chawathe and Krishna extend the definition of odd-gracefulness to countably infinite graphs and show that all countably infinite bipartite graphs that are connected and locally finite have odd-graceful labelings.

Solairaju and Chithra [2441] defined a graph G with q edges to be *edge-odd graceful* if there is an bijection f from the edges of the graph to  $\{1, 3, 5, \ldots, 2q - 1\}$  such that, when each vertex is assigned the sum of all the edges incident to it mod 2q, the resulting vertex labels are distinct. They prove they following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders  $P_n \times P_2$   $(n \ge 3)$ ; stars with an even number of edges; and crowns  $C_n \odot K_1$ . In [2442] they prove the following graphs have edge-odd graceful labelings:  $P_n$  (n > 1) with a pendent edge attached to each vertex (combs); the graph obtained by appending 2n + 1 pendent edges to each endpoint of  $P_2$  or  $P_3$ ; and the graph obtained by subdividing each edge of the star  $K_{1,2n}$ .

Singhun [2405] proved the following graphs have edge-odd graceful labelings:  $W_{2n}$ ;  $W_n \odot K_1$ ; and  $W_n \odot K_m$ , when n is odd, m is even, and n divides m. Seoud and Salim [2253] present edge-odd graceful labelings for the following families of graphs:  $W_n$  for  $n \equiv 1, 2$  and 3 (mod 4);  $C_n \odot \overline{K_{2m-1}}$ ; even helms;  $P_n \odot \overline{K_{2m}}$ ; and  $K_{2,s}$ . They also provide two theorems about non edge-odd graceful graphs. Susanti, Ernanto1, and Surodjo [2548] found edge-odd graceful labelings for some classes of prism related graphs.

In [2464] Sridevi, Navaeethakrishnan, Nagarajan, and Nagarajan call a graph G with q edges *odd-even graceful* if there is an injection f from the vertices of G to  $\{1, 3, 5, \ldots, 2q + 1\}$  such that, when each edge uv is assigned the label |f(u) - f(v)|, the resulting edge labels are  $\{2, 4, 6, \ldots, 2q\}$ . They proved that  $P_n$ , combs  $P_n \odot K_1$ , stars  $K_{1,n}, K_{1,2,n}, K_{m,n}$ , and bistars  $B_{m,n}$  are odd-even graceful.

Sudha and Babu [2484] say a graph G with q edges is *even-even graceful* if there is an injection f from the edges of G to  $\{2, 4, 6, \ldots, 2q\}$  such that, the induced map  $f^+$  from V(G) to  $\{0, 2, \ldots, 2k - 2\}$  defined by  $f^*(x) = \Sigma(f(xy) \pmod{2k})$  where  $k = \max(p, q)$  is injective and each value is  $f^*(x)$  is even. They proved that dumbbells, stars,  $C_n \times P_2$ , and  $K_1 + C_n$  are even-even graceful.

Behera, Mishra, and Nayak [431] proved the following: bistars  $B_{r,r}$  are even-even graceful, combs are even-even graceful, the trees obtained by joining and even number of pendent edges to the endpoint of a path are even-even graceful, the graphs obtained by identifying the center of a star and a vertex of  $C_3$  are odd-even graceful, the graphs obtained by identifying the center of a star and a vertex of  $C_3$  and two pendent edges at the other two verticies are odd-even graceful, and the graphs obtained by identifying the center of a star with a vertex of  $C_n$  and the endpoints of the star with the opposite vertices of  $C_n$  is odd-even graceful.

In [645] Daoud zele introduced vertex odd graceful labelings as follows. Let G be a graph with q edges. A function f is called a *vertex odd graceful*abeling of G if  $f E(G) \rightarrow \{1, 2, 3, \ldots, 2q\}$  is an injection and the induced function  $f^* V(G) \rightarrow \{1, 3, \ldots, 2q - 1\}$  defined as  $f^*(u) = \sum_{uv \in E(G)} f(uv) \pmod{2q}$  is also an injection. A graph which admits a vertex odd graceful labeling is called a *vertex odd graceful* graph. Necessary and sufficient conditions for prisme, tori, wheels, fans and books to be vertex odd graceful are given.

### 3.7 Cordial Labelings

Cahit [528] has introduced a variation of both graceful and harmonious labelings. Let f be a function from the vertices of G to  $\{0,1\}$  and for each edge xy assign the label |f(x) - f(y)|. Call f a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [529] proved the following: every tree is cordial;  $K_n$  is cordial if and only if  $n \leq 3$ ;  $K_{m,n}$  is cordial for all m and n; the friendship graph  $C_3^{(t)}$  (i.e., the one-point union of t 3-cycles) is cordial if and only if  $t \neq 2$  (mod 4); all fans are cordial; the wheel  $W_n$  is cordial if and only if  $n \neq 3$  (mod 4) (see also [694]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang, and Kwong [1445] determine all m and n for which  $mK_n$  is cordial. Liu and Zhu [1623] proved that a 3-regular graph of order n is cordial if and only if  $n \neq 4$  (mod 8).

In [2016], [2017], [2018], [2023], and [2019] Prajapati and Gajjar provided results about the existence of cordial labelings of graphs obtained from paths, cycles, flower graphs, sunflower graphs, flower snarks, lotus inside a circle graphs, helms, closed helms, armed helms  $(W_n \oplus P_2)$ , and webs by the duplication of vertices and edges.

A k-angular cactus is a connected graph all of whose blocks are cycles with k vertices. In [529] Cahit proved that a k-angular cactus with t cycles is cordial if and only if  $kt \neq 2 \pmod{4}$ . This was improved by Kirchherr [1379] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [1380] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho, Lee, and Shee [1017] proved:  $P_n \times C_{4m}$  is cordial for all m and all odd n; the composition G and H is cordial if G is cordial and H is cordial and has odd order and even size (see §2.3 for definition of composition); for  $n \geq 4$  the composition  $C_n[K_2]$  is cordial if and only if  $n \neq 2 \pmod{4}$ ; the Cartesian product of two cordial graphs of even size is cordial. Ho, Lee, and Shee [1016] showed that a unicyclic graph is cordial unless it is  $C_{4k+2}$  and that the generalized Petersen graph (see §2.7 for the definition) P(n,k) is cordial if and only if  $n \neq 2 \pmod{4}$ . Khan [1362] proved that a graph that consisting of a finite number of cycles of finite length joined at a common cut vertex is cordial if and only if the number of edges is not congruent to 2 mod 4.

Du [694] determines the maximal number of edges in a cordial graph of order n and gives a necessary condition for a k-regular graph to be cordial. Riskin [2120] proved that Möbius ladders  $M_n$  (see §2.3 for the definition) are cordial if and only if  $n \ge 3$  and  $n \ne 2$ (mod 4). (See also [2233].) Diab and Nada [677] show that  $P_n \odot P_m$  is cordial; except for n and m both equal to 2 (mod 4),  $C_n \odot C_m$  is cordial; and when  $n \equiv 2 \pmod{4}$  and mis odd,  $C_n \odot C_m$  is not cordial. In [2166] Salehi, Mukhin, and Saputro showed that  $Q_n$  is cordial for all n > 1.

Seoud and Abdel Maqusoud [2221] proved that if G is a graph with n vertices and m edges and every vertex has odd degree, then G is not cordial when  $m + n \equiv 2 \pmod{4}$ . They also prove the following: for  $m \geq 2$ ,  $C_n \times P_m$  is cordial except for the case  $C_{4k+2} \times P_2$ ;  $P_n^2$  is cordial for all n;  $P_n^3$  is cordial if and only if  $n \neq 4$ ; and  $P_n^4$  is cordial if and only if  $n \neq 4, 5, \text{ or } 6$ . Seoud, Diab, and Elsakhawi [2230] have proved the following graphs are cordial:  $P_n + P_m$  for all m and n except (m, n) = (2, 2);  $C_m + C_n$  if  $m \not\equiv 0 \pmod{4}$  and  $n \neq 2 \pmod{4}$ ;  $C_n + K_{1,m}$  for  $n \not\equiv 3 \pmod{4}$  and odd m except (n, m) = (3, 1);  $C_n + \overline{K_m}$  when n is odd, and when n is even and m is odd;  $K_{1,m,n}$ ;  $K_{2,2,m}$ ; the n-cube; books  $B_n$  if and only if  $n \not\equiv 3 \pmod{4}$ ; B(3, 2, m) for all m; B(4, 3, m) if and only if m is even; and B(5, 3, m) if and only if  $m \not\equiv 1 \pmod{4}$  (see §2.4 for the notation B(n, r, m)). In [2437] Solairaju and Arockiasamy prove that various families of subgraphs of grids  $P_m \times P_n$  are cordial.

Diab [670], [671], and [673] proved the following graphs are cordial:  $C_m + P_n$  if and only if  $(m, n) \neq (3, 3), (3, 2), \text{ or } (3, 1); P_m + K_{1,n}$  if and only if  $(m, n) \neq (1, 2); P_m \cup K_{1,n}$ if and only if  $(m, n) \neq (1, 2); C_m \cup K_{1,n}; C_m + \overline{K_n}$  for all m and n except  $m \equiv 3 \pmod{4}$  and n odd, and  $m \equiv 2 \pmod{4}$  and n even;  $C_m \cup \overline{K_n}$  for all m and n except  $m \equiv 2 \pmod{4}; P_m + \overline{K_n}; P_m \cup \overline{K_n}; P_m^2 \cup P_n^2$  except for (m, n) = (2, 2) or  $(3, 3); P_n^2 + P_m$  except for  $(m, n) = (3, 1), (3, 2), (2, 2), (3, 3), \text{ and } (4, 2); P_n^2 \cup P_m$  except for (n, m) = (2, 2), (3, 3),and  $(4, 2); P_n^2 + C_m$  if and only if  $(n, m) \neq (1, 3), (2, 3), \text{ and } (3, 3).P_n + \overline{K_m}; C_n + K_{1,m}$  for all n > 3 and all m except  $n \equiv 3 \pmod{4}; C_n + K_{1,m}$  for  $n \equiv 3 \pmod{4}$   $(n \neq 3)$  and even  $m \geq 2;$  and  $C_m \times C_n$  if and only if 2mn is not congruent to 2 (mod 4).

In [672] Diab proved the graphs  $W_n + W_m$  are cordial if and only if one of the following conditions is not satisfied: (i) (n, m) = (3, 3), (ii) n = 3 and  $m \equiv 1 \pmod{4}$ , (iii)  $n \equiv 1 \pmod{4}$  and  $m \equiv 3 \pmod{4}$ ; the graphs  $W_n \cup W_m$  are cordial if and only if one of the following conditions is not satisfied: (i) n = 3 and  $m \equiv 1 \pmod{4}$ , (ii)  $n \equiv 1 \pmod{4}$ and  $m \equiv 3 \pmod{4}$ ; the graphs  $W_n + P_m$  are cordial if and only if one of the following conditions is not satisfied: (i) (n, m) = (3, 1), (3, 2) and (3, 3), (ii)  $n \equiv 3 \pmod{4}$  and m = 1. They also prove that  $W_n \cup P_m$  and  $W_n \cup C_m$  are cordial for all m and n and  $W_n + C_m$  is cordial if and only if  $(m, n) \neq (3, 3)$  and (3, 4). In [674] Diab showed that the second power of  $C_n$  is cordial if and only if n = 3 or n is even and greater than 4. He also investigated the cordiality of the join and union of pairs of second power of cycles and graphs consisting of one second power of cycle with one cycle and one path.

In [1816] Nada, Diab, Elrokh, and Sabra proved that  $P_n \odot C_m$  is cordial if and only if  $gcd(n,m) \neq 1$  or 3 (mod 4); in [1815] they proved  $C_n \odot P_m$  is cordial for all  $n \geq 3$  and  $m \geq 1$ . Nada, Elrokh, and Elshafey [1817] provided necessary and sufficient conditions for  $F_n^2 = K_1 + P_n^2$ ,  $F_n^2 + F_m^2$ , and  $F_n^2 + F_m^2$  to be cordial.

The generalized Jahangir graph  $J_{m,n}$  m > 3, n > 1 is a graph on mn + 1 vertices, consisting of a cycle  $C_{mn}$  with one additional vertex that is adjacent to n vertices of  $C_{mn}$ at distance m to each other on  $C_{mn}$ . Gajjar and Des [809] proved  $J_{m,n}$  is cordial for all m > 3 and n > 1, except for  $J_{1,4n-1}$ .

Youssef [2878] has proved the following: If G and H are cordial and one has even size, then  $G \cup H$  is cordial; if G and H are cordial and both have even size, then G + His cordial; if G and H are cordial and one has even size and either one has even order, then G + H is cordial;  $C_m \cup C_n$  is cordial if and only if  $m + n \not\equiv 2 \pmod{4}$ ;  $mC_n$  is cordial if and only if  $mn \not\equiv 2 \pmod{4}$ ;  $C_m + C_n$  is cordial if and only if  $(m, n) \neq (3, 3)$ and  $\{m \pmod{4}, n \pmod{4}\} \neq \{0, 2\}$ ; and if  $P_n^k$  is cordial, then  $n \ge k + 1 + \sqrt{k-2}$ . He conjectures that this latter condition is also sufficient. He confirms the conjecture for k = 5, 6, 7, 8, and 9.

Lee and Liu [1505] have shown that the complete *n*-partite graph is cordial if and only if at most three of its partite sets have odd cardinality (see also [694]). Lee, Lee, and Chang [1480] prove the following graphs are cordial: the Cartesian product of an arbitrary number of paths; the Cartesian product of two cycles if and only if at least one of them is even; and the Cartesian product of an arbitrary number of cycles if at least one of them has length a multiple of 4 or at least two of them are even. Ali Al-Shamiri, Elrokh, El-Mashtawye, and Tallah [132] showed that the Cartesian product of a path and a cycle is cordial under some conditions and that the Cartesian product of two paths is cordial.

Shee and Ho [2321] have investigated the cordiality of the one-point union of n copies of various graphs. For  $C_m^{(n)}$ , the one-point union of n copies of  $C_m$ , they prove:

(i) If  $m \equiv 0 \pmod{4}$ , then  $C_m^{(n)}$  is cordial for all n;

(ii) If  $m \equiv 1$  or 3 (mod 4), then  $C_m^{(n)}$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ ;

(iii) If  $m \equiv 2 \pmod{4}$ , then  $C_m^{(n)}$  is cordial if and only if n is even.

For  $K_m^{(n)}$ , the one-point union of *n* copies of  $K_m$ , Shee and Ho [2321] prove:

(i) If  $m \equiv 0 \pmod{8}$ , then  $K_m^{(n)}$  is not cordial for  $n \equiv 3 \pmod{4}$ ;

(ii) If  $m \equiv 4 \pmod{8}$ , then  $K_m^{(n)}$  is not cordial for  $n \equiv 1 \pmod{4}$ ;

(iii) If  $m \equiv 5 \pmod{8}$ , then  $K_m^{(n)}$  is not cordial for all odd n;

(iv)  $K_4^{(n)}$  is cordial if and only if  $n \not\equiv 1 \pmod{4}$ ; (v)  $K_5^{(n)}$  is cordial if and only if n is even; (vi)  $K_6^{(n)}$  is cordial if and only if n > 2;

(vii)  $K_7^{(n)}$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ ;

(viii)  $K_n^{(2)}$  is cordial if and only if n has the form  $p^2$  or  $p^2 + 1$ .

For  $W_m^{(n)}$ , the one-point union of n copies of the wheel  $W_m$  with the common vertex being the center, Shee and Ho [2321] show:

- (i) If  $m \equiv 0$  or 2 (mod 4), then  $W_m^{(n)}$  is cordial for all n;
- (ii) If  $m \equiv 3 \pmod{4}$ , then  $W_m^{(n)}$  is cordial if  $n \not\equiv 1 \pmod{4}$ ;

(iii) If  $m \equiv 1 \pmod{4}$ , then  $W_m^{(n)}$  is cordial if  $n \not\equiv 3 \pmod{4}$ . For all n and all m > 1Shee and Ho [2321] prove  $F_m^{(n)}$ , the one-point union of n copies of the fan  $F_m = P_m + K_1$ with the common point of the fans being the center, is cordial (see also [1595]). The flag  $Fl_m$  is obtained by joining one vertex of  $C_m$  to an extra vertex called the *root*. Shee and Ho [2321] show all  $Fl_m^{(n)}$ , the one-point union of n copies of  $Fl_m$  with the common point being the root, are cordial. In his 2001 Ph. D. thesis Selvaraju [2202] proves that the one-point union of any number of copies of a complete bipartite graph is cordial. Benson and Lee [438] have investigated the regular windmill graphs  $K_m^{(n)}$  and determined precisely which ones are cordial for m < 14.

Diab and Mohammedm [676] proved the following: the join of two fans  $F_n + F_m$  is cordial if and only if  $n, m \ge 4$ ;  $F_n \cup F_m$  is cordial if and only if  $(n, m) \ne (1, 1)$  or (2, 2);  $F_n + P_m$  is cordial if and only if  $(n, m) \ne (1, 2)$ , (2, 1), (2, 2), (2, 3), or (3, 2);  $F_n \cup P_m$  is cordial if and only if  $(n, m) \ne (1, 2)$ ;  $F_n + C_m$  is cordial if and only if  $(n, m) \ne (1, 3)$ , (2, 3)or (3, 3); and  $F_n \cup C_m$  is cordial if and only if  $(n, m) \ne (2, 3)$ . Hefnawy, Elsid, and Euat Tallah [976] gave necessary and sufficient conditions for a cordial labeling of thesum of the second power of the path  $P_n^2 + K_{1,m}$  and  $P_n^2 \cup K_{1,m}$ .

Andar, Boxwala, and Limaye [153], [154], and [157] have proved the following graphs are cordial: helms; closed helms; generalized helms obtained by taking a web (see 2.2 for the definitions) and attaching pendent vertices to all the vertices of the outermost cycle in the case that the number cycles is even; flowers (graphs obtained by joining the vertices of degree one of a helm to the central vertex); sunflower graphs (that is, graphs obtained by taking a wheel with the central vertex  $v_0$  and the *n*-cycle  $v_1, v_2, \ldots, v_n$  and additional vertices  $w_1, w_2, \ldots, w_n$  where  $w_i$  is joined by edges to  $v_i, v_{i+1}$ , where i + 1 is taken modulo n); multiple shells (see §2.2); and the one point unions of helms, closed helms, flowers, gears, and sunflower graphs, where in each case the central vertex is the common vertex.

Du [695] proved that the disjoint union of  $n \ge 2$  wheels is cordial if and only if n is even or n is odd and the number of vertices of in each cycle is not 0 (mod 4) or n is odd and the number of vertices of in each cycle is not 3 (mod 4). Prajapati and Gajjar [2015] prove  $\overline{W_n}$  is not cordial if  $n \not\equiv 4, 7 \pmod{8}$  and  $\overline{C_n}$  is not cordial if  $n \not\equiv 4, 7 \pmod{8}$ .

Let  $\mathcal{O}$  be the family of all cordial graphs of odd order and odd size for which there is no cordial labeling g such that  $e_g(0) - e_g(1) = 1$ . Barrientos and Minion [376] proved that if G is a cordial graph such that  $G \notin \mathcal{O}$ , then the corona  $K_1 \odot G$  is cordial. They use this result to prove that  $H \odot G$  is cordial when G and H are cordial and G has even order and even size or  $G \notin \mathcal{O}$ . In addition,  $H \odot G$  is cordial when G is a cordial graph of odd order and even size and H is any graph of order m and size  $n \in \{m-1, m, m+1\}$ . If H is bipartite such that the difference of the cardinalities of its partite sets is at most one, and G is a cordial graph of even order and odd size that admits a cordial labeling g such that  $e_g(0) - e_g(1) = 1$ , then the corona  $H \odot G$  is cordial. Barrientos and Minion proved the cordiality of certain circulant graphs; they also proved that for every positive integer k, the k-splitting of a cordial graph of even size, results in a cordial graph. They provide sufficient conditions to prove that any super subdivision of a graph G is cordial. They study the cordiality of the join of two cordial graphs, proving that G + H is cordial when G and H have even order and even size, or both have odd order and even size, or both graphs have odd order, odd size, and the dominating weight in both graphs is not 1, or G has even order, odd size, and the dominating weight on both graphs is not the same, or both G and H have odd order, but only one has odd size, and the dominating weight is 0. They also prove that when G is a cordial graph of odd order and even size, the one-point union of t copies of G is cordial.

In [376] Barrientos and Minion provide necessary conditions for the cordiality of coronas of cordial graphs, prove the cordiality of a family of circulant graphs, prove that any splitting graph of a cordial graph of even order and even size is cordial, determine a condition that a graph must satisfy in order that any super subdivision of it is cordial, prove the cordiality of the joint of two cordial graphs, and determine when a one-point union of a cordial graph is cordial.

For positive integers m and n divisible by 4 Venkatesh [2721] constructs graphs obtained by appending a copy of  $C_n$  to each vertex of  $C_m$  by identifying one vertex of  $C_n$ with each vertex of  $C_m$  and iterating by appending a copy of  $C_n$  to each vertex of degree 2 in the previous step. He proves that the graphs obtained by successive iterations are cordial.

Elumalai and Sethurman [711] proved: cycles with parallel cords are cordial and *n*cycles with parallel  $P_k$ -chords (see §2.2 for the definition) are cordial for any odd positive integer k at least 3 and any  $n \not\equiv 2 \pmod{4}$  of length at least 4. They call a graph H an *even-multiple subdivision graph* of a graph G if it is obtained from G by replacing every edge uv of G by a pair of paths of even length starting at u and ending at v. They prove that every even-multiple subdivision graph is cordial and that every graph is a subgraph of a cordial graph. In [2807] Wen proves that generalized wheels  $C_n + mK_1$  are cordial when m is even and  $n \not\equiv 2 \pmod{4}$  and when m is odd and  $n \not\equiv 3 \pmod{4}$ . Kuppusamy and Guruswamy [1446] show that the subdivision graph of  $K_{2,n}$  is graceful for  $n \geq 1$  and the subdivision graph of the shell graph C(n, n - 3) is graceful for  $n \geq 4$ .

Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical graphs by a path. They prove: graphs obtained by joining two copies of the same cycle by a path are cordial [2635]; graphs obtained by joining two copies of the same cycle that has two chords with a common vertex with opposite ends of the chords joining two consecutive vertices of the cycle by a path are cordial [2635]; graphs obtained by joining two rim verticies of two copies of the same wheel by a path are cordial [2637]; and graphs obtained by joining two copies of the same Petersen graph by a path are cordial [2637]. They also prove that graphs obtained by replacing one vertex of a star by a fixed wheel or by replacing each vertex of a star by a fixed Petersen graph are cordial [2637]. In [2676] Vaidya, Ghodasara, Srivastav, and Kaneria investigated graphs obtained by joining two identical cycles that have a chord are cordial and the graphs obtained by starting with copies  $G_1, G_2, \ldots, G_n$  of a fixed cycle with a chord that forms a triangle with two consecutive edges of the cycle and joining each  $G_i$  to  $G_{i+1}$  ( $i = 1, 2, \ldots, n-1$ ) by an edge that is incident with the endpoints of the chords in  $G_i$  and  $G_{i+1}$  are cordial. Vaidya, Dani, Kanani, and Vihol [2630] proved that the graphs obtained by starting with copies  $G_1, G_2, \ldots, G_n$  of a fixed star and joining each center of  $G_i$  to the center of  $G_{i+1}$   $(i = 1, 2, \ldots, n-1)$  by an edge are cordial.

Ghodasara, Rokad, and Jadav [882] prove that the path union of  $P_n \times P_n$  is cordial. They also prove that the graph obtained by joining two copies of  $P_n \times P_n$  by a path is cordial. Ghodasara and Jadav [876] prove: the graph obtained by joining a finite number of copies of  $P_n \times P_n$  by path is cordial; the star of  $P_n \times P_n$  is cordial; and the path union of the star of  $P_n \times P_n$  is cordial. Rokad and Patadiya [2133] proved that the shadow graph, splitting graph, and the degree splitting graph of a star are cordial graphs. They also showed that the jewel graph and the jellyfish graph are cordial.

Ghodasara and Rokad prove [883] the star of  $K_{n,n}$   $(n \ge 2)$  is cordial, the path union of  $K_{n,n}$   $(n \ge 2)$  is cordial, and the graph obtained by joining two copies of  $K_{n,n}$   $(n \ge 2)$ by a path is cordial [883]. In [884] the same authors prove that a vertex switching of any non-apex vertex of a wheel graph, a vertex switching of any internal vertex of a flower graph, a vertex switching of any non-apex vertex of a gear graph, and a vertex switching of any non-apex vertex of a shell graph are cordial graphs. In [885] they proved that a barycentric subdivision of a shell graph, a barycentric subdivision of  $K_{n,n}$ , and a barycentric subdivision of a wheel are cordial. Ghodasara and Sonchhatra [886] prove that the graph obtained by joining two copies of the same fan by a path is cordial. They also prove that the star of a fan is cordial and the graph obtained by joining two copies of the star of the same fan by a path is cordial [886].

Vaidya, Kanani, Srivastav, and Ghodasara [2645] proved: graphs obtained by subdividing every edge of a cycle with exactly two extra edges that are chords with a common endpoint and whose other end points are joined by an edge of the cycle are cordial; graphs obtained by subdividing every edge of the graph obtained by starting with  $C_n$  and adding exactly three chords that result in two 3-cycles and a cycle of length n-3 are cordial; graphs obtained by subdividing every edge of a Petersen graph are cordial. Sankar and Sethuramam zske [2181] showed that the subdivision graph  $S(K_2, n)$  is graceful and cordial, for  $n \ge 1$  and the shell graph S(C(n, n-3)) is graceful and cordial for  $n \ge 4$ .

Recall the shell C(n, n-3) is the cycle  $C_n$  with n-3 cords sharing a common endpoint. Vaidya, Dani, Kanani, and Vihol [2631] proved that the graphs obtained by starting with copies  $G_1, G_2, \ldots, G_n$  of a fixed shell and joining common endpoint of the chords of  $G_i$ to the common endpoint of the chords of  $G_{i+1}$   $(i = 1, 2, \ldots, n-1)$  by an edge are cordial. Vaidya, Dani, Kanani, and Vihol [2646] define  $C_n(C_n)$  as the graph obtained by subdividing each edge of  $C_n$  and connecting the new n vertices to form a copy of  $C_n$ inscribed the original  $C_n$ . They prove that  $C_n(C_n)$  is cordial if  $n \neq 2 \pmod{4}$ ; the graphs obtained by starting with copies  $G_1, G_2, \ldots, G_k$  of  $C_n(C_n)$  the graph obtained by joining a vertex of degree 2 in  $G_i$  to a vertex of degree 2 in  $G_{i+1}$   $(i = 1, 2, \ldots, n-1)$  by an edge are cordial; and the graphs obtained by joining vertex of degree 2 from one copy of  $C_n(C_n)$ to a vertex of degree 2 to another copy of  $C_n(C_n)$  by any finite path are cordial. Vaidya and Shah [2672] and [2673] proved that following graphs are cordial: the shadow graph of the bistar  $B_{n,n}$ , the splitting graph of  $B_{n,n}$ , the degree splitting graph of  $B_{n,n}$ , alternate triangular snakes, alternate quadrilateral snakes. In [2675] Vaidya and Shah give cordial labelings of the degree splitting graph of paths, shells, helms, and gears.

A graph C(2n, n-2) is called an *alternate shell* if C(2n, n-2) is obtained from the cycle  $C_{2n}$   $(v_0, v_1, v_2, \ldots, v_{2n-1})$  by adding n-2 chords between the vertex  $v_0$  and the vertices  $v_{2i+1}$ , for  $1 \leq i \leq n-2$ . Sethuraman and Sankar [2290] proved that some graphs obtained by merging alternate shells and joining certain vertices by a path have  $\alpha$ -labelings.

Vaidya, Srivastav, Kaneria, and Ghodasara [2677] proved that a cycle with two chords that share a common vertex and the opposite ends of which join two consecutive vertices of the cycle is cordial. For a graph G Vaidya, Ghodasara, Srivastav, and Kaneria [2636] introduced the graph  $G^*$  called the star of G as the graph obtained by replacing each vertex of the star  $K_{1,n}$  by a copy of G and prove that  $C_n^*$  admits cordial labeling. Vaidya and Dani [2626] proved that the graphs obtained by starting with n copies  $G_1, G_2, \ldots, G_n$ of a fixed star and joining each center of  $G_i$  to the center of  $G_{i+1}$  by an edge as well as each of the centers to a new vertex  $x_i$   $(1 \le i \le n-1)$  by an edge admit cordial labelings. An arbitrary supersubdivison H of a graph G is the graph obtained from G by replacing every edge of G by  $K_{2,m}$ , where m may vary for each edge arbitrarily. Vaidya and Kanani [2638] proved that arbitrary supersubdivisions of paths and stars admit cordial labelings. Vaidya and Dani [2627] prove that arbitrary supersubdivisions of trees,  $K_{m,n}$ , and  $P_m \times P_n$ are cordial. They also prove that an arbitrary supersubdivision of the graph obtained by identifying an end vertex of a path with every vertex of a cycle  $C_n$  is cordial except when n is odd,  $m_i$   $(1 \le i \le n)$  are odd, and  $m_i$   $(n+1 \le i \le mn)$  of the  $K_{2,m_i}$  are even. Recall for a graph G and a vertex v of G Vaidya, Srivastav, Kaneria, and Kanani [2678] define a vertex switching  $G_v$  as the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. They proved that the graphs obtained by the switching of a vertex in  $C_n$  admit cordial labelings. They also show that the graphs obtained by the switching of any arbitrary vertex of cycle  $C_n$ with one chord that forms a triangle with two consecutive edges of the cycle are cordial. Moreover they prove that the graphs obtained by the switching of any arbitrary vertex in cycle with two chords that share a common vertex the opposite ends of which join two consecutive vertices of the cycle are cordial.

The middle graph M(G) of a graph G is the graph whose vertex set is  $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of Gor one is a vertex of G and the other is an edge incident with it. Vaidya and Vihol [2680] prove that the middle graph M(G) of an Eulerian graph is Eulerian with  $|E(M(G))| = \sum_{i=1}^{n} (d(v_i)^2 + 2e)/2$ . They prove that middle graphs of paths, crowns  $C_n \odot K_1$ , stars, and tadpoles (that is, graphs obtained by appending a path to a cycle) admit cordial labelings.

Vaidya and Dani [2629] define the duplication of an edge e = uv of a graph G by a new vertex w as the graph G' obtained from G by adding a new vertex w and the edges wv and wu. They prove that the graphs obtained by duplication of an arbitrary edge of a cycle and a wheel admit a cordial labeling. Starting with k copies of fixed wheel  $W_n, W_n^{(1)}, W_n^{(2)}, \ldots, W_n^{(k)}, Vaidya, Dani, Kanani, and Vihol [2633] define <math>G = \langle W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} \rangle$  as the graph obtained by joining the center vertices of each of  $W_n^{(i)}$  and  $W_n^{(i+1)}$  to a new vertex  $x_i$  where  $1 \leq i \leq k-1$ . They prove that  $\langle W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} \rangle$ 

are cordial graphs. Kaneria and Vaidya [1317] define the *index of cordiality* of G as n if the disjoint union of n copies of G is cordial but the disjoint union of fewer than n copies of G is not cordial. They obtain several results on index of cordiality of  $K_n$ . In the same paper they investigate cordial labelings of graphs obtained by replacing each vertex of  $K_{1,n}$  by a graph G. Kaneria, Jariya, and Karavadiya [1280] proved that the index of cordiality for  $K_n$  is at most 6 for n at most 105; the index of cordiality for  $K_n$  is at most 4, when n can be expressed as sum of square of two integers; and it is at most 8 when a particular different condition on the edge labels are met. See also [1094].

In [157] Andar et al. define a *t-ply graph*  $P_t(u, v)$  as a graph consisting of *t* internally disjoint paths joining vertices *u* and *v*. They prove that  $P_t(u, v)$  is cordial except when it is Eulerian and the number of edges is congruent to 2 (mod 4). In [158] Andar, Boxwala, and Limaye prove that the one-point union of any number of plys with an endpoint as the common vertex is cordial if and only if it is not Eulerian and the number of edges is congruent to 2 (mod 4). They further prove that the path union of shells obtained by joining any point of one shell to any point of the next shell is cordial; graphs obtained by attaching a pendent edge to the common vertex of the cords of a shell are cordial; and cycles with one pendent edge are cordial.

For a graph G and a positive integer t, Andar, Boxwala, and Limaye [155] define the t-uniform homeomorph  $P_t(G)$  of G as the graph obtained from G by replacing every edge of G by vertex disjoint paths of length t. They prove that if G is cordial and t is odd, then  $P_t(G)$  is cordial; if  $t \equiv 2 \pmod{4}$  a cordial labeling of G can be extended to a cordial labeling of  $P_t(G)$  if and only if the number of edges labeled 0 in G is even; and when  $t \equiv 0 \pmod{4}$  a cordial labeling of G can be extended to a cordial labeling of  $P_t(G)$  if and only if the number of edges labeled 0 in G is even; and when  $t \equiv 0 \pmod{4}$  a cordial labeling of G can be extended to a cordial labeling of  $P_t(G)$  if and only if the number of edges labeled 1 in G is even. In [156] Ander et al. prove that  $P_t(K_{2n})$  is cordial for all  $t \geq 2$  and that  $P_t(K_{2n+1})$  is cordial if and only if  $t \equiv 0 \pmod{4}$  or t is odd and  $n \not\equiv 2 \pmod{4}$ , or  $t \equiv 2 \pmod{4}$  and n is even.

In [158] Andar, Boxwala, and Limaya show that a cordial labeling of G can be extended to a cordial labeling of the graph obtained from G by attaching 2m pendent edges at each vertex of G. For a binary labeling q of the vertices of a graph G and the induced edge labels given by g(e) = |g(u) - g(v)| let  $v_q(j)$  denote the number of vertices labeled with j and  $e_q(j)$  denote the number edges labeled with j. Let  $i(G) = \min\{|e_q(0) - e_q(1)|\}$  taken over all binary labelings g of G with  $|v_q(0) - v_q(1)| \leq 1$ . And ar et al. also prove that a cordial labeling g of a graph G with p vertices can be extended to a cordial labeling of the graph obtained from G by attaching 2m + 1 pendent edges at each vertex of G if and only if G does not satisfy either of the conditions: (1) G has an even number of edges and  $p \equiv 2 \pmod{4}$ ; (2) G has an odd number of edges and either  $p \equiv 1 \pmod{4}$ with  $e_g(1) = e_g(0) + i(G)$  or  $n \equiv 3 \pmod{4}$  and  $e_g(0) = e_g(1) + i(G)$ . And ar, Boxwala, and Limaye [159] also prove: if g is a binary labeling of the n vertices of graph G with induced edge labels given by g(e) = |g(u) - g(v)| then g can be extended to a cordial labeling of  $G \odot \overline{K_{2m}}$  if and only if n is odd and  $i(G) \equiv 2 \pmod{4}$ ;  $K_n \odot \overline{K_{2m}}$  is cordial if and only if  $n \neq 4 \pmod{8}$ ;  $K_n \odot \overline{K_{2m+1}}$  is cordial if and only if  $n \neq 7 \pmod{8}$ ; if g is a binary labeling of the n vertices of graph G with induced edge labels given by g(e) = |g(u) - g(v)| then g can be extended to a cordial labeling of  $G \odot C_t$  if  $t \neq 3 \mod d$ 

4, n is odd and  $e_g(0) = e_g(1)$ . For any binary labeling g of a graph G with induced edge labels given by g(e) = |g(u) - g(v)| they also characterize in terms of i(G) when g can be extended to graphs of the form  $G \odot \overline{K_{2m+1}}$ .

For graphs  $G_1, G_2, \ldots, G_n$   $(n \ge 2)$  that are all copies of a fixed graph G, Shee and Ho [2322] call a graph obtained by adding an edge from  $G_i$  to  $G_{i+1}$  for  $i = 1, \ldots, n-1$  a path union of G (the resulting graph may depend on how the edges are chosen). Among their results they show the following graphs are cordial: path-unions of cycles; path-unions of any number of copies of  $K_m$  when m = 4, 6, or 7; path-unions of three or more copies of  $K_5$ ; and path-unions of two copies of  $K_m$  if and only if m - 2, m, or m + 2 is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees, and various compositions.

Lee and Liu [1505] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let H be a graph with an even number of edges and a cordial labeling such that the vertices of H can be divided into t parts  $H_1, H_2, \ldots, H_t$  each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let G be any graph and  $G_1, G_2, \ldots, G_t$  be any t subsets of the vertices of G. Let (G, H) be the graph that is the disjoint union of G and H augmented by edges joining every vertex in  $G_i$  to every vertex in  $H_i$  for all i. Then G is cordial if and only if (G, H) is. From this it follows that: all generalized fans  $F_{m,n} = \overline{K_m} + P_n$  are cordial; the generalized bundle  $B_{m,n}$  is cordial if and only if m is even or  $n \neq 2 \pmod{4}$  ( $B_{m,n}$  consists of 2n vertices  $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$  with an edge from  $v_i$  to  $u_i$  and 2m vertices  $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$  with  $x_i$  joined to  $v_i$  and  $y_i$  joined to  $u_i$ ); if m is odd the generalized wheel  $W_{m,n} = \overline{K_m} + C_n$  is cordial if and only if  $n \neq 3 \pmod{4}$ . If m is even,  $W_{m,n}$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ ; a complete k-partite graph is cordial if and only if  $n \neq 3$ .

Sethuraman and Selvaraju [2299] have shown that certain cases of the union of any number of copies of  $K_4$  with one or more edges deleted and one edge in common are cordial. Youssef [2882] has shown that the kth power of  $C_n$  is cordial for all n when  $k \equiv 2 \pmod{4}$  and for all even n when  $k \equiv 0 \pmod{4}$ . Ramanjaneyulu, Venkaiah, and Kothapalli [2081] give cordial labelings for a family of planar graphs for which each face is a 3-cycle and a family for which each face is a 4-cycle. Acharya, Germina, Princy, and Rao [35] prove that every graph G can be embedded in a cordial graph H. The construction is done in such a way that if G is planar or connected, then so is H.

Recall from §2.7 that a graph H is a supersubdivision of a graph G, if every edge uv of G is replaced by  $K_{2,m}$  (m may vary for each edge) by identifying u and v with the two vertices in  $K_{2,m}$  that form the partite set with exactly two members. Vaidya and Kanani [2638] prove that supersubdivisions of paths and stars are cordial. They also prove that supersubdivisions of  $C_n$  are cordial provided that n and the various values for m are odd.

Raj and Koilraj [2061] proved that the splitting graphs of  $P_n, C_n, K_{m,n}, W_n, nK_2$ , and the graphs obtained by starting with k copies of stars  $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(k)}$  and joining the central vertex of  $K_{1,n}^{(p-1)}$  and  $K_{1,n}^{(p)}$  to a new vertex  $x_{p-1}$  for each  $2 \le p \le k$  are cordial.

Seoud, El Sonbaty, and Abd El Rehim [2231] proved the following graphs are cordial:  $K_{1,l,m,n}$  when mn is even;  $P_m + K_{1,n}$  if n is even or n is odd and  $(m \neq 2)$ ; the conjunction

graph  $P_4 \wedge C_n$  is cordial if n is even; and the join of the one-point union of two copies of  $C_n$  and  $K_1$ .

Recall  $\langle K_{1,n_1}, \ldots, K_{1,n_t} \rangle$  is the graph obtained by starting with the stars  $K_{1,n_1}, \ldots, K_{1,n_t}$  and joining the center vertices of  $K_{1,n_i}$  and  $K_{1,n_{i+1}}$  to a new vertex  $v_i$  where  $1 \leq i \leq k-1$ . Kaneria, Jariya, and Meghpara [1284] proved that  $\langle K_{1,n_1}, \ldots, K_{1,n_t} \rangle$  is cordial and every graceful graph with  $|v_f(\text{odd}) - v_f(\text{even})| \leq 1$  is cordial. Kaneria, Meghpara, and Makadia [1312] proved that the cycle of complete graphs  $C(t \cdot K_{m,n})$  and the cycle of wheels  $C(t \cdot W_n)$  are cordial. Kaneria, Makadia, and Meghpara [1299] proved that the cycle of cycles  $C(t \cdot C_n)$  is cordial for  $t \geq 3$ . Kaneria, Makadia, and Meghpara [1300] proved that a star of  $K_n$  and a cycle of n copies of  $K_n$  are cordial. Kaneria, Viradia, Jariya, and Makadia [1319] proved that the cycle of paths  $C(t \cdot P_n)$  is cordial, product cordial (see Section 7.7), and total edge product cordial.

Cahit [534] calls a graph *H*-cordial if it is possible to label the edges with the numbers from the set  $\{1, -1\}$  in such a way that, for some k, at each vertex v the sum of the labels on the edges incident with v is either k or -k and the inequalities  $|v(k) - v(-k)| \leq 1$  and  $|e(1) - e(-1)| \leq 1$  are also satisfied, where v(i) and e(j) are, respectively, the number of vertices labeled with i and the number of edges labeled with j. He calls a graph  $H_n$ -cordial if it is possible to label the edges with the numbers from the set  $\{\pm 1, \pm 2, \ldots, \pm n\}$  in such a way that, at each vertex v the sum of the labels on the edges incident with v is in the set  $\{\pm 1, \pm 2, \dots, \pm n\}$  and the inequalities  $|v(i) - v(-i)| \leq 1$  and  $|e(i) - e(-i)| \leq 1$  are also satisfied for each i with  $1 \leq i \leq n$ . Among Cahit's results are:  $K_{n,n}$  is H-cordial if and only if n > 2 and n is even; and  $K_{m,n}, m \neq n$ , is H-cordial if and only if  $n \equiv 0$ (mod 4), m is even and m > 2, n > 2. Unfortunately, Ghebleh and Khoeilar [875] have shown that other statements in Cahit's paper are incorrect. In particular, Cahit states that  $K_n$  is *H*-cordial if and only if  $n \equiv 0 \pmod{4}$ ;  $W_n$  is *H*-cordial if and only if  $n \equiv 1$ (mod 4); and  $K_n$  is  $H_2$ -cordial if and only if  $n \equiv 0 \pmod{4}$  whereas Ghebleh and Khoeilar instead prove that  $K_n$  is H-cordial if and only if  $n \equiv 0$  or 3 (mod 4) and  $n \neq 3$ ;  $W_n$  is *H*-cordial if and only if n is odd;  $K_n$  is  $H_2$ -cordial if  $n \equiv 0$  or 3 (mod 4); and  $K_n$  is not  $H_2$ -cordial if  $n \equiv 1 \pmod{4}$ . Ghebleh and Khoeilar also prove every wheel has an  $H_2$ -cordial labeling. In [772] Freeda and Chellathurai prove that the following graphs are  $H_2$ -cordial: the join of two paths, the join of two cycles, ladders, and the tensor product  $P_n \otimes P_2$ . They also prove that the join of  $W_n$  and  $W_m$  where  $n + m \equiv 0 \pmod{4}$  is *H*-cordial. Cahit generalizes the notion of *H*-cordial labelings in [534].

A graph G(V, E) is called  $H_k$ - cordial if it has an H-cordial labeling f such that for each edge e and each vertex v of G have the label  $1 \leq |f(e)| \leq k, 1 \leq |f(v)| \leq k$  and  $|v_f(i) - v_f(-i))| \leq 1, |e_f(i)) - e_f(-i)| \leq 1$  for each i with  $1 \leq i \leq k$ . Ratilal and Parmar [2107] investigated  $H_k$ -cordial labelings of triangular snakes, double triangular snakes, triple triangular snakes, alternate triangular snakes, double alternate triangular snakes, irregular triangular snakes, quadrilateral snakes, double quadrilateral snakes, alternate quadrilateral snakes, and irregular quadrilateral snakes. Joshi and Parmar [1895], [1261] investigated the H-,  $H_2$ - and  $H_3$ -cordiality of the following snakes: triangular, double triangular, triple trianglar, quadrilateral, double quadrilateral, alternate trigular, double alternate trigular, irregular triangular, quadrilateral, double quadrilateral, alternate quadrilateral, and irregular quadrilateral. Joshi and Pamar [1261] investigated  $H_k$ -cordial labeling of *p*-triangular, *m*-polygonal snakes, double *m*-polygonal snakes, alternate *m*polygonal snakes, double alternate *m*-polygonal snakes, irregular *m*-polygonal snakes, and double irregular *m*-polygonal snakes.

Cahit and Yilmaz [538] call a graph  $E_k$ -cordial if it is possible to label the edges with the numbers from the set  $\{0, 1, 2, ..., k - 1\}$  in such a way that, at each vertex v, the sum of the labels on the edges incident with v modulo k satisfies the inequalities  $|v(i) - v(j)| \leq 1$  and  $|e(i) - e(j)| \leq 1$ , where v(s) and e(t) are, respectively, the number of vertices labeled with s and the number of edges labeled with t. Cahit and Yilmaz prove the following graphs are  $E_3$ -cordial:  $P_n$   $(n \geq 3)$ ; stars  $S_n$  if and only if  $n \not\equiv 1 \pmod{3}$ ;  $K_n$   $(n \geq 3)$ ;  $C_n$   $(n \geq 3)$ ; friendship graphs; and fans  $F_n$   $(n \geq 3)$ . They also prove that  $S_n$   $(n \geq 2)$  is  $E_k$ -cordial if and only if  $n \not\equiv 1 \pmod{k}$  when k is odd or  $n \not\equiv 1 \pmod{2k}$ when k is even and  $k \neq 2$ . Ni, Liu, and Lu [1842] demonstrate the  $E_3$ -cordiality of  $W_n$ ,  $P_m \times P_n$ ,  $K_{m,n}$ , and trees.

Bapat and Limaye [347] provide  $E_3$ -cordial labelings for:  $K_n$   $(n \ge 3)$ ; snakes whose blocks are all isomorphic to  $K_n$  where  $n \equiv 0$  or 2 (mod 3); the one-point union of any number of copies of  $K_n$  where  $n \equiv 0$  or 2 (mod 3); graphs obtained by attaching a copy of  $K_n$  where  $n \equiv 0$  or 3 (mod 3) at each vertex of a path; and  $K_m \odot K_n$ . Rani and Sridharan [2095] proved: for odd n > 1 and  $k \ge 2$ ,  $P_n \odot K_1$  is  $E_k$ -cordial; for n even and  $n \neq k/2$ ,  $P_n \odot K_1$  is  $E_k$ -cordial; and certain cases of fans are  $E_k$ -cordial. Youssef [2879] gives a necessary condition for a graph to be  $E_k$ -cordial for certain k. He also gives some new families of  $E_k$ -cordial graphs and proves Lee's [1537] conjecture about the edgegracefulness of the disjoint union of two cycles. Venkatesh, Salah, and Sethuraman [2724] proved that  $C_{2n+1}$  snakes and  $C_{2n+1}^{2t}$  are  $E_2$ -cordial. Liu, Liu, and Wu [1622] provide two necessary conditions for a graph G to be  $E_k$ -cordial and prove that every  $P_n$   $(n \ge 3)$  is  $E_p$ -cordial if p is odd. They also discuss the  $E_2$ -cordiality of a graph G under the condition that some subgraph of G has a 1-factor. Liu and Liu [1621] proved that a graph with no isolated vertex is  $E_2$ -cordial if and only if it does not have order 4n + 2. Bapat and Limaye [348] prove that helms, one point unions of helms, and path unions of helms are  $E_3$ -cordial. Jinnah and Beena [1252] prove the graphs  $P_n$   $(n \ge 3), C_n$  where  $n \ne 4 \mod 2$ 8, and  $K_n$   $(n \geq 3)$  are  $E_4$ -cordial graphs. They also prove that every graph of order at least 3 is a subgraph of an  $E_4$ -cordial graph.

Hovey [1021] has introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group A (under addition) and graph G(V, E) he defines G to be A-cordial if there is a labeling of V with elements of A such that for all a and b in Awhen the edge ab is labeled with f(a) + f(b), the number of vertices labeled with a and the number of vertices labeled b differ by at most one and the number of edges labeled with a and the number labeled with b differ by at most one. In the case where A is the cyclic group of order k, the labeling is called k-cordial. With this definition we have: if G(V, E) is a graph with  $|E| \geq |V| - 1$  then G(V, E) is harmonious if and only if G is |E|-cordial; G is cordial if and only if G is 2-cordial.

Hovey has obtained the following: caterpillars are k-cordial for all k; all trees are k-cordial for k = 3, 4, and 5; odd cycles with pendent edges attached are k-cordial for all

k; cycles are k-cordial for all odd k; for k even,  $C_{2mk+j}$  is k-cordial when  $0 \le j \le \frac{k}{2} + 2$ and when k < j < 2k;  $C_{(2m+1)k}$  is not k-cordial;  $K_m$  is 3-cordial; and, for k even,  $K_{mk}$  is k-cordial if and only if m = 1.

Hovey advances the following conjectures: all trees are k-cordial for all k; all connected graphs are 3-cordial; and  $C_{2mk+j}$  is k-cordial if and only if  $j \neq k$ , where k and j are even and  $0 \leq j < 2k$ . The last conjecture was verified by Tao [2571]. Tao's result combined with those of Hovey show that for all positive integers k the n-cycle is k-cordial with the exception that k is even and n = 2mk + k. Tao also proved that the crown with 2mk + jvertices is k-cordial unless j = k is even, and for  $4 \leq n \leq k$  the wheel  $W_n$  is k-cordial unless  $k \equiv 5 \pmod{8}$  and n = (k + 1)/2. In [2603] Tuczyński, Wenus, and Wesek proved a conjecture of Cichacz, Görlich, and Tuza [624] that all hypertrees are 2-cordial. They also proved that all hypertree are 3-cordial.

In [2885] Youssef and Al-Kuleab proved the following: if G is a  $(p_1, q_1)$  k-cordial graph and G is a  $(p_2, q_2)$  k-cordial graph with  $p_1$  or  $p_2 \equiv 0 \pmod{k}$  and  $q_1$  or  $q_2 \equiv 0 \pmod{k}$ , then G + H is k-cordial; if G is a  $(p_1, q_1)$  4-cordial graph and G is a  $(p_2, q_2)$  4-cordial graph with  $p_1$  or  $p_2 \not\equiv 2 \pmod{4}$  and  $q_1$  or  $q_2 \equiv 0 \pmod{k}$ , then G + H is 4-cordial; and  $K_{m,n,p}$  is 4-cordial if and only if  $(m, n, p) \mod 4 \not\equiv (0, 2, 2)$  or (2, 2, 2).

In [2877] Youssef obtained the following results:  $C_{2k}$  with one pendent edge is not (2k+1)-cordial for k > 1;  $K_n$  is 4-cordial if and only if  $n \leq 6$ ;  $C_n^2$  is 4-cordial if and only if  $n \not\equiv 2 \pmod{4}$ ; and  $K_{m,n}$  is 4-cordial if and only if  $n \not\equiv 2 \pmod{4}$ ; He also provides some necessary conditions for a graph to be k-cordial. Driscol zele [690] proved that all trees are 7-cordial.

Modha and Kanani [1776] prove that following graphs have a 5-cordial labeling: the shadow graph of a path and a cycle, graphs obtained by one point duplication and duplication of an edge by a vertex in cycle, and the graph obtained by the barycentric subdivision of wheel. In [1769] Modha and Kanani proved prisms, webs, flowers, and closed helms admit 5-cordial labelings. In [1770] they proved that fans are k-cordial for all k and double fans are k-cordial for all odd k and n = (k+1)/2. In [1772] they proved that the following graphs are k-cordial:  $W_n$  for odd  $k, n = mk + j, m \ge 0, 1 \le j \le k - 1$  except for j = (k-1)/2; the total graphs of paths (recall  $T(P_n)$  has vertex set  $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in  $P_n$ ; the square  $C_n^2$  for odd  $k \leq n$ ; the path union of n copies of  $C_k$  where k is odd; and  $C_n$  with one pendent edge for odd  $k \leq n$ . Rathod and Kanani [2103] proved  $P_n^2$  is k-cordial for all k and cycles with a single pendent edge are k-cordial for all even k. In [2100] Rathod and Kanani proved the middle graph, total graph, and splitting graph of a path are 4-cordial and  $P_n^2$  and triangular snakes are 4-cordial. Modha and Kanani [1773] proved:  $W_n$  is k-cordial for all odd k and for all n = mk + j,  $m \ge 0$ ,  $1 \le j \le k - 1$  except for j = k - 1; the path union of copies of  $C_k$  is k-cordial for odd k; the total graph of  $P_n$  is k-cordial for all k; the square  $C_n^2$  is k-cordial for odd k odd and  $n \ge k$ ; and the graphs obtained by appending an edge to  $C_n$  is k-cordial for odd k and  $n \ge k$ . Modha and Kanani [1775] prove the following graphs are k-cordial:  $P_m \times C_k$ ,  $P_m \times C_{k+1}$ ,  $P_m \times C_{k+3}$  for all odd k and  $m \ge 2$ , and  $P_m \times C_{2k-1}$  for all odd  $k, m \ge 2$  and  $m \ne tk$ . Rathod and Kanani [2103] [2105] prove that following graphs are 4-cordial: the splitting graph of  $K_{1,n}$ ; triangular books; and

the one point union any number of copies of the fan  $f_3$ ; braid graphs; triangular ladders; and irregular quadrilateral snakes obtained from the path  $P_n$  with consecutive vertices  $u_1, u_2, \ldots, u_n$  and new vertices  $v_1, v_2, \ldots, v_{n-2}, w_1, w_2$ , and edges  $u_i v_i, w_i u_{i+2}, v_i w_i$  for all  $1 \leq i \leq n-2$ . Rathod and Kanani [2104] prove wheels, fans, friendship graphs, double fans, and helms are 5-cordial. Driscoll, Krop, and Nguyen [683] proved that all trees are 6-cordial. In [1275], [1276], and [1771] Kanani and Modha prove that fans, friendship graphs, ladders, double fans, double wheels, wheels, helms, closed helms, and webs are 7-cordial graphs and wheels, fans and friendship graphs, gears, double fans, and helms are 4-cordial graphs.

Cichacz, Görlich and Tuza [624] extended the definition of k-cordial labeling for hypergraphs. They presented various sufficient conditions on a hypertree H (a connected hypergraph without cycles) to be k-cordial. From their theorems it follows that every k-uniform hypertree is k-cordial, and every hypertree with odd order or size is 2-cordial. Modha and Kanani [1774] prove the following graphs are k-cordial for all k: bistars, restricted square graphs  $B_{n,n}^2$ , the one-point union of  $C_3$  and  $K_{1,n}$ , and  $P_n \odot K_1$ .

In [2295] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph G and successively form supersubdivisions (see §2.7 for the definition) that are cordial in the case that every edge in G is replaced by  $K_{2,m}$ where m is even. Sethuraman and Selvaraju [2294] also show that the one-vertex union of any number of copies of  $K_{m,n}$  is cordial and that the one-edge union of k copies of shell graphs C(n, n-3) (see §2.2) is cordial for all  $n \ge 4$  and all k. They conjectured that the one-point union of any number of copies of graphs of the form  $C(n_i, n_i - 3)$  for various  $n_i \geq 4$  is cordial. This was proved by Yue, Yuansheng, and Liping in [2896]. Riskin [2122] claimed that  $K_n$  is  $(Z_2 \times Z_2)$ -cordial if and only if n is at most 3 and  $K_{m,n}$  is  $(Z_2 \times Z_2)$ -cordial if and only if  $(m, n) \neq (2, 2)$ . (Many authors use  $V_4$  to denote  $Z_2 \times Z_2$ .) However, Pechenik and Wise [1911] report that the correct statement for  $K_{m,n}$  is  $K_{m,n}$ is  $(Z_2 \times Z_2)$ -cordial if and only if m and n are not both congruent to 2 mod 4. Seoud and Salim [2249] gave an upper bound on the number of edges of a graph that admits a  $(Z_2 \oplus Z_2)$ -cordial labeling in terms the number of vertices. Rathod and Kanani [2102] prove the following graphs are  $(Z_2 \times Z_2)$ -cordial for all n and m:  $C_n \odot mK_1$ ,  $C_n \odot K_2$ , and graphs obtained by appending a single edge to one vertex of  $C_n$ . In Rathod and Kanani [2106] and [2101] proved the following graphs are  $(Z_2 \times Z_2)$ -cordial: alternate triangular snakes, alternate double triangular snakes, alternate triple triangular snakes, quadrilateral snakes, alternate quadrilateral snakes, double quadrilateral snakes, and double alternate quadrilateral snakes.

In [1911] Pechenik and Wise investigate  $Z_2 \times Z_2$ -cordiality of complete bipartite graphs, paths, cycles, ladders, prisms, and hypercubes. They proved that all complete bipartite graphs are  $Z_2 \times Z_2$ -cordial except  $K_{m,n}$  where  $m, n \equiv 2 \mod 4$ ; all paths are  $Z_2 \times Z_2$ -cordial except  $P_4$  and  $P_5$ ; all cycles are  $Z_2 \times Z_2$ -cordial except  $C_4, C_5, C_k$ , where  $k \equiv 2 \mod 4$ ; and all ladders  $P_2 \times P_k$  are  $Z_2 \times Z_2$ -cordial except  $C_4$ . They also introduce a generalization of A-cordiality involving digraphs and quasigroups, and show that there are infinitely many Q-cordial digraphs for every quasigroup Q. Jinnah and Nair [1253] proved that all trees except  $P_4$  and  $P_5$  are  $Z_2 \times Z_2$ -cordial and the graphs obtained by subdividing the pendent edges of  $C_n \odot K_1$  are  $Z_2 \times Z_2$  -cordial for all n.

Cairnie and Edwards [541] have determined the computational complexity of cordial and k-cordial labelings. They prove the conjecture of Kirchherr [1380] that deciding whether a graph admits a cordial labeling is NP-complete. As a corollary, this result implies that the same problem for k-cordial labelings is NP-complete. They remark that even the restricted problem of deciding whether connected graphs of diameter 2 have a cordial labeling is also NP-complete.

For a (p,q) graph G and a bijection f from V(G) to  $\{1, 2, \ldots, p\}$  Ponraj, Annathurai, and Kala [1948] introduced a new graph labeling as follows. For each edge uv assign the remainder when f(u) is divided by f(v) or when f(v) is divided by f(u) depending on whether  $f(u) \ge f(v)$  or  $f(v) \ge f(u)$ . The function f is called a remainder cordial labeling of G if  $|\eta_e - \eta_o| \leq 1$  where  $\eta_e$  and  $\eta_o$  respectively denote the number of edges labeled with even integers and the number of edges labeled with odd integers. A graph G with a remainder cordial labeling is called a *remainder cordial* graph. In [1948] and [1953] they proved that the following graphs are remainder cordial: paths, cycles, stars, bistars, crowns, combs,  $K_{2,n}$ ,  $S(K_{1,n})$ ,  $S(B_{n,n})$ ,  $P_n^2$ , wheels, subdivisions of wheels,  $K_{2,2n}$ , and the graph obtained by subdividing the pendent edges of the bistar  $B_{n,n}$ . They also proved the following star related graphs are remainder cordial:  $K_{1,n} \cup B_{n,n}$ ,  $P_n \cup K_{1,n}$ ,  $P_n \cup B_{n,n}, K_{1,n} \cup S(K_{1,n}), K_{1,n} \cup S(B_{n,n}), P_n^2 \cup K_{1,n}, P_n^2 \cup B_{n,n}, \text{ and } S(K_{1,n}) \cup S(B_{n,n}).$ They conjecture that  $K_n$  is remainder cordial if and only if  $n \leq 3$ . Ponraj, Annathurai, and Kala [1949] generalize remainder cordial labelings as follows. Let f be a function from V(G) to  $\{1, 2, \ldots, k\}$  where  $2 < k \leq |V(G)|$ . For each edge uv assign the remainder when f(u) is divided by f(v) or when f(v) is divided by f(u) depending on whether  $f(u) \ge f(v)$  or  $f(v) \ge f(u)$ . The function f is called a k-remainder cordial labeling of G if  $|v_f(i) - v_f(j)| \leq 1$ , for  $i, j \in \{1, \ldots, k\}$  where  $v_f(x)$  denote the number of vertices labeled with x and  $|\eta_e - \eta_o| \leq 1$  where  $\eta_e$  and  $\eta_o$  respectively denote the number of edges labeled with even integers and the number of edges labeled with odd integers. A graph that admits a k-remainder cordial labeling is called a k-remainder cordial graph. In [1949], [175], [176], and [1954] they proved the following. Every graph is a subgraph of a connected k-remainder cordial graph for  $k \geq 4$ . Note that when k = 2, the number of edges with label 0 is q so there does not exists a 2-remainder cordial labeling. They further investigate the 3-remainder cordial labeling behavior of paths, cycles, stars, combs, crowns, wheels, fans, squares of paths, subdivisions of wheels, subdivisions of stars, subdivisions of combs, armed crowns, and  $K_{1,n} \odot K_2$ . They further proved that  $W_n$  is 3-remainder cordial if and only if  $n \equiv 1 \pmod{3}$ ,  $K_{1,n}$  is 3-remainder cordial if and only if  $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$ , and  $K_n$  is 3-remainder cordial if and only if  $n \leq 3$ . In [1950], [1951], and [1952] Ponraj, Annathurai, and Kala proved the following graphs are 4-remainder cordial: complete graphs, paths, cycles, crowns, stars, bistars, books, subdivisions of stars, subdivisions of bistars, subdivisions of jelly fish, flowers, sunflowers, lotuses inside a circle, friendship graphs, webs, triangular snakes, durer graphs, planar grids, mongolian tents, prisms, dragon graphs  $C_m @P_n$  (the graph obtained by identifying an endpoint of  $P_n$  with one vertex of  $C_m$ , crossed prisms  $CP_{2n}$ , and  $K_2 + mK_1$  ( $m \equiv 0, 1, 3 \pmod{4}$ ). They also investigate the 4-remainder cordial labeling of  $L_n \odot mK_1$ ,  $L_n \odot K_2$ ,  $L_n \odot mK_1$ ,  $P_n \odot K_1$ ,

 $P_n \odot 2K_1, C_n \odot K_1$ , and  $S(P_n \odot K_1)$ .

In Bapat [223] introduces the following new labeling. A graph G(V, E) has a L-cordial labeling if there is a bijection f from E(G) to  $\{1, 2, \ldots, |E|\}$  that assigns 0 to a vertex v if the largest label on the edges incident to v is even and assigns 1 to v otherwise and this assignment satisfies the condition that the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1. A graph that admits an L-cordial labeling is called as L-cordial graph. He shows that stars, path, cycles, and triangular snakes are L-cordial graphs

In [566] Chartrand, Lee, and Zhang introduced the notion of uniform cordiality as follows. Let f be a labeling from V(G) to  $\{0,1\}$  and for each edge xy define  $f^*(xy) = |f(x) - f(y)|$ . For i = 0 and 1, let  $v_i(f)$  denote the number of vertices v with f(v) = iand  $e_i(f)$  denote the number of edges e with  $f^*(e) = i$ . They call a such a labeling ffriendly if  $|v_0(f) - v_1(f)| \leq 1$ . A graph G for which every friendly labeling is cordial is called uniformly cordial. They prove that a connected graph of order  $n \geq 2$  is uniformly cordial if and only if n = 3 and  $G = K_3$ , or n is even and  $G = K_{1,n-1}$ .

In [2120] Riskin introduced two measures of the noncordiality of a graph. He defines the cordial edge deficiency of a graph G as the minimum number of edges, taken over all friendly labelings of G, needed to be added to G such that the resulting graph is cordial. If a graph G has a vertex labeling f using 0 and 1 such that the edge labeling  $f_e$  given by  $f_e(xy) = |f(x) - f(y)|$  has the property that the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1, the cordial vertex deficiency defined as  $\infty$ . Riskin proved: the cordial edge deficiency of  $K_n$  (n > 1) is  $\lfloor \frac{n}{2} \rfloor - 1$ ; the cordial vertex deficiency of  $K_n$  is j - 1 if  $n = j^2 + \delta$ , when  $\delta$  is -2, 0 or 2, and  $\infty$  otherwise. In [2120] Riskin determines the cordial edge deficiency and cordial vertex deficiency for the cases when the Möbius ladders and wheels are not cordial. In [2121] Riskin determines the cordial edge deficiencies.

Recall a graph G the graph  $G^*$ , called the *star of* G, is the graph obtained by replacing each vertex G with the star  $K_{1,n}$ . In [1313] Kaneria, Patadiya and Teraiya introduced a balanced cordial labeling for a graph by saying that a cordial labeling f is a *vertex balanced cordial* if it satisfies the condition $v_f(0) = v_f(1)$ ; f is a *balanced cordial* if it satisfies the conditions  $e_f(0) = e_f(1)$  and  $v_f(0) = v_f(1)$ . Kaneria, Teraiya, and Patadiya [1316] proved the path union  $P(t \cdot C_{4n})$  is a balanced cordial if t is odd and it is vertex balanced cordial if t is even;  $C(t \cdot C_{4n})$  is a balanced cordial if  $t \equiv 0 \pmod{4}$  and it is a vertex balanced cordial if  $t \equiv 1, 3 \pmod{4}$ ; and  $C_{4n}^*$  is balanced cordial. They proved  $P_n \times C_{4t}$  is balanced cordial;  $C_{2n} \times C_{4t}$  is balanced cordial; and  $G_1 \odot G_2$  is cordial when  $G_1$  is cordial and  $G_2$ is a balanced cordial. Kaneria and Teraiya [1315] prove if G is a balanced cordial, then so is  $G^*$ ; if G is a balanced cordial, then so is  $P_{2n+1} \times G$ ; and if G is a balanced cordial, then so is  $\overline{G}^*$ .

If f is a binary vertex labeling of a graph G Lee, Liu, and Tan [1506] defined a partial edge labeling of the edges of G by  $f^*(uv) = 0$  if f(u) = f(v) = 0 and  $f^*(uv) = 1$  if f(u) = f(v) = 1. They let  $e_0(G)$  denote the number of edges uv for which  $f^*(uv) = 0$ and  $e_1(G)$  denote the number of edges uv for which  $f^*(uv) = 1$ . They say G is balanced if it has a friendly labeling f such that if  $|e_0(f) - e_1(f)| \leq 1$ . In the case that the number of vertices labeled 0 and the number of vertices labeled 1 are equal and the number of edges labeled 0 and the number of edges labeled 1 are equal they say the labeling is strongly balanced. They prove:  $P_n$  is balanced for all n and is strongly balanced if n is even;  $K_{m,n}$  is balanced if and only if m and n are even, m and n are odd and differ by at most 2, or exactly one of m or n is even (say n = 2t) and  $t \equiv -1, 0, 1 \pmod{|m - n|}$ ; a k-regular graph with p vertices is strongly balanced if and only if p is even and is balanced if and only if p is odd and k = 2; and if G is any graph and H is strongly balanced, the composition G[H] (see §2.3 for the definition) is strongly balanced. In [1411] Kong, Lee, Seah, and Tang show:  $C_m \times P_n$  is balanced if m and n are odd and is strongly balanced if either m or n is even; and  $C_m \odot K_1$  is balanced for all  $m \ge 3$  and strongly balanced if m is even. They also provide necessary and sufficient conditions for a graph to be balanced or strongly balanced. Lee, Lee, and Ng [1477] show that stars are balanced if and only if the number of edges of the star is at most 4. Kwong, Lee, Lo, and Wang [1452] define a graph G to be uniformly balanced if  $|e_0(f) - e_1(f)| \leq 1$  for every vertex labeling f that satisfies if  $|v_0(f) - v_1(f)| \leq 1$ . They present several ways to construct families of uniformly balanced graphs. Kim, Lee, and Ng [1374] prove the following: for any graph G, mG is balanced for all m; for any graph G, mG is strongly balanced for all even m; if G is strongly balanced and H is balanced, then  $G \cup H$  is balanced;  $mK_n$  is balanced for all m and strongly balanced if and only if n = 3 or mn is even; if H is balanced and G is any graph, the  $G \times H$  is strongly balanced; if one of m or n is even, then  $P_m[P_n]$ is balanced; if both m and n are even, then  $P_m[P_n]$  is balanced; and if G is any graph and H is strongly balanced, then the tensor product  $G \otimes H$  is strongly balanced. (The tensor product  $G \otimes H$  of graphs G and H, has the vertex set  $V(G) \times V(H)$  and any two vertices (u, u') and (v, v') are adjacent in  $G \otimes H$  if and only if u' is adjacent with v' and u is adjacent with v.)

A graph G is k-balanced if there is a function f from the vertices of G to  $\{0, 1, 2, \ldots, k-1\}$  such that for the induced function  $f^*$  from the edges of G to  $\{0, 1, 2, \ldots, k-1\}$  defined by  $f^*(uv) = |f(u) - f(v)|$  the number of vertices labeled i and the number of edges labeled j differ by at most 1 for each i and j. Seoud, El Sonbaty, and Abd El Rehim [2231] proved the following: if  $|E| \ge 2k + 1$  and  $|V| \le k$  then G(V, E) is not k-balanced; if  $|E| \ge 3k + 1$ ,  $(k \ge 2)$  and  $3k - 1 \ge |V| \ge 2k + 1$  then G(V, E) is not k-balanced; r-regular graphs with  $3 \le r \le n - 1$  are not r-balanced; if  $G_1$  has m vertices and  $G_2$ has n vertices then  $G_1 + G_2$  is not (m + n)-balanced for  $m, n \ge 5$ ;  $P_3 \times P_n$  with edge set E is 3n-balanced and |E|-balanced;  $L_n \times P_2$   $(L_n = P_n \times P_2)$  with vertex set V and edge set E is |V|-balanced and k-balanced for  $k \ge |E|$  but not n-balanced for  $n \ge 2$ ; the one-point union of two copies of  $K_{2,n}$  is 2n-balanced, |V|-balanced, and |E|-balanced not is 3-balanced when  $n \ge 4$ . They also proved that the composition graph  $P_n[P_2]$  is not n-balanced for  $n \ge 3$ , is not 2n-balanced for  $n \ge 5$ , and is not |E|-balanced.

A graph whose edges are labeled with 0 and 1 so that the absolute difference in the number of edges labeled 1 and 0 is no more than one is called *edge-friendly*. We say an edge-friendly labeling induces a *partial vertex labeling* if vertices which are incident to more edges labeled 1 than 0, are labeled 1, and vertices which are incident to more edges

labeled 0 than 1, are labeled 0. Vertices that are incident to an equal number of edges of both labels are called *unlabeled*. Call a procedure on a labeled graph a label switching algorithm if it consists of pairwise switches of labels. Krop, Lee, and Raridan [1436] prove that given an *edge-friendly labeling* of  $K_n$ , we show a label switching algorithm producing an edge-friendly relabeling of  $K_n$  such that all the vertices are labeled.

In 2017 [226] Bapat introduced a new labeling as follows. A function f from the vertices of a graph G(E, V) to  $\{0, 1, 2, \ldots, |V| - 1\}$  is called an *extended vertex edge additive cordial* labeling if the induced function  $f^*$  from the edges of G to  $\{0, 1\}$  defined by  $f^*(uv) = f(u) + f(v) \pmod{2}$  for all edges uv of G has the property that the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. Bapat [226] proved paths, stars,  $K_{2,n}, K_{3,n}, K_{4,n}, P_n \odot C_3$ , and  $P_n \odot C_4$  admit extended vertex edge additive cordial labeling.

Let G(p,q) a simple finite connected graph. Given a bijective function f from E(G) to  $\{0, 1, \ldots, q-1\}$  Bapat [227] calls a bijective function  $f^*$  from E(G) to  $\{0, 1, 2, \ldots, q-1\}$  an extended edge vertex cordial (eevc) labeling if the induced function  $f^*$  from V(G) to  $\{0, 1\}$  defined by  $f^*(u) = \Sigma f(uv) \mod 2$  where the sum is taken over all edges incident to u has the property that the number of vertices labeled with 0 differs from the number labeled with 1 by at most 1. He shows that  $P_n$   $(n \neq 2 \mod 4)$ ,  $C_n$   $(n \neq 2 \mod 4)$ ,  $K_{1,n}$   $(n \neq 1 \mod 4)$ , graphs obtained by joining the centers of two copies of  $K_{1,2n+1}$  by an edge, and triangular snakes have eevc labelings.

#### 3.8 The Friendly Index–Balance Index

Recall a function f from V(G) to  $\{0,1\}$  where for each edge xy,  $f^*(xy) = |f(x) - f(x)| = |f(x)| = |f(x)$  $|f(y)|, v_i(f)$  is the number of vertices v with f(v) = i, and  $e_i(f)$  is the number of edges e with  $f^*(e) = i$  is called *friendly* if  $|v_0(f) - v_1(f)| \leq 1$ . Lee and Ng [1514] define the friendly index set of a graph G as  $FI(G) = \{|e_0(f) - e_0(f)|\}$  $e_1(f)$  where f runs over all friendly labelings f of G. They proved: for any graph G with q edges  $FI(G) \subseteq \{0, 2, 4, \dots, q\}$  if q is even and  $FI(G) \subseteq \{1, 3, \dots, q\}$  if q is odd; for  $1 \leq m \leq n$ ,  $FI(K_{m,n}) = \{(m-2i)^2 \mid 0 \leq i \leq \lfloor m/2 \rfloor\}$  if m+n is even; and  $FI(K_{m,n}) = \{i(i+1) \mid 0 \leq i \leq m\}$  if m+n is odd. In [1517] Lee and Ng prove the following:  $FI(C_{2n}) = \{0, 4, 8, \dots, 2n\}$  when n is even;  $FI(C_{2n}) = \{2, 6, 10, \dots, 2n\}$  when n is odd; and  $FI(C_{2n+1}) = \{1, 3, 5, \dots, 2n-1\}$ . Elumalai [710] defines a cycle with a full set of chords as the graph  $PC_n$  obtained from  $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$  by adding the cords  $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$  when n is even and  $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$ when n is odd. Lee and Ng [1516] prove:  $FI(PC_{2m+1}) = \{3m - 2, 3m - 4, 3m - 6, \dots, 0\}$ when m is even and  $FI(PC_{2m+1}) = \{3m - 2, 3m - 4, 3m - 6, \dots, 1\}$  when m is odd;  $FI(PC_4) = \{1,3\}; \text{ for } m \ge 3, FI(PC_{2m}) = \{3m - 5, 3m - 7, 3m - 9, \dots, 1\} \text{ when } m \text{ is }$ even;  $FI(PC_{2m}) = \{3m - 5, 3m - 7, 3m - 9, \dots, 0\}$  when m is odd.

Salehi and Lee [2161] determined the friendly index for various classes of trees. Among their results are: for a tree with q edges that has a perfect matching, the friendly index is the odd integers from 1 to q and for  $n \ge 2$ ,  $FI(P_n) = \{n - 1 - 2i | 0 \le i \lfloor (n - 1)/2 \rfloor$ . Law [1474] determined the full friendly index sets of spiders and disproved a conjecture by

Salehi and Lee [2161] that the friendly index set of a tree forms an arithmetic progression. In [1520] Lee, Ng, and Lau determine the friendly index sets of several classes of spiders. Gao, Sun, and Lee [836] determined the full friendly index of  $P_m \times P_n$  with the extra mn + 1 - m - n edges  $u_{ij} - u_{(i+1)(j+1)}$ . Sun, Gao, and Lee [2533] determined the full friendly index and friendly index for the twisted product of Mőbius ladders. Sinha and Kaur [2392] determined the full edge friendly index of stars, wheels, 2-regular graphs, and  $mP_n$ . In [2334] Shiu determined the full edge-friendly index sets of complete bipartite graphs. Salehi and McGinn [2164] obtained partial results about the friendly index set of  $Q_n$  and strenghten a conjecture about the friendly index set of  $Q_n$  made in [2166]. Teffilial and Devaraj [2576] found the friendly index set of the graphs obtained by identifying the central vertex of a fan with the endpoint of a path (*umbrella*), the graphs obtained by identifying the central vertex of a star with the endpoint of a path, the graphs obtained by identifying the endpoints of copies of  $P_2$  (globe), the splitting graph of a star, and  $P_2 + mK_1$ . Lee, Low, Ng, and Wang [1508] determined the friendly index sets for various classes of disjoint unions of stars. Gao, Ruo-Yuan, Lee, Ren, and Sun [835] determined FFI(G), FI(G) and FPCI(G) for a class of cubic graphs G.

Lee and Ng [1516] define PC(n,p) as the graph obtained from the cycle  $C_n$  with consecutive vertices  $v_0, v_1, v_2, \ldots, v_{n-1}$  by adding the p cords joining  $v_i$  to  $v_{n-i}$  for  $1 \leq p\lfloor n/2 \rfloor - 1$ . They prove  $FI(PC(2m+1,p)) = \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 1\}$ if p is even and  $FI(PC(2m+1,p)) = \{2m+p-1, 2m+p-3, 2m+p-5, \ldots, 0\}$  if p is odd;  $FI(PC(2m,1)) = \{2m-1, 2m-3, 2m-5, \ldots, 1\}$ ; for  $m \geq 3$ , and  $p \geq 2$ ,  $FI(PC(2m,p)) = \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 0\}$  when p is even, and  $FI(PC(2m,p)) = \{2m+p-4, 2m+p-6, 2m+p-8, \ldots, 1\}$  when p is odd. More generally, they show that the integers in the friendly index of a cycle with an arbitrary nonempty set of parallel chords form an arithmetic progression with a common difference 2. Shiu and Kwong [2338] determine the friendly index of the grids  $P_n \times P_2$ . The maximum and minimum friendly indices for  $C_m \times P_n$  were given by Shiu and Wong in [2367].

In [1518] Lee and Ng prove: for  $n \ge 2$ ,  $FI(C_{2n} \times P_2) = \{0, 4, 8, \dots, 6n-8, 6n\}$  if n is even and  $FI(C_{2n} \times P_2) = \{2, 6, 10, \dots, 6n - 8, 6n\}$  if n is odd;  $FI(C_3 \times P_2) = \{1, 3, 5\}$ ; for  $n \ge 2$ ,  $FI(C_{2m+1} \times P_2) = \{6n-1\} \cup \{6n-5-2k | \text{ where } k \ge 0 \text{ and } 6n-5-2k \ge 0\}; FI(M_{4n}) \text{ (here } k \ge 0 \text{ and } 6n-5-2k \ge 0\};$  $M_{4n}$  is the Möbius ladder with 4n steps) =  $\{6n-4-4k \mid \text{where } k \ge 0 \text{ and } 6n-4-4k \ge 0\};$  $FI(M_{4n+2}) = \{6n+3\} \cup \{6n-5-2k | \text{ where } k \ge 0 \text{ and } 6n-5-2k > 0\}.$  In [1453] Kwong, Lee, and Ng completely determine the friendly index of all 2-regular graphs. As a corollary, they show that  $C_m \cup C_n$  is cordial if and only if m + n = 0, 1 or 3 (mod 4). Ho, Lee, and Ng [1014] determine the friendly index sets of stars and various regular windmills. In [2807] Wen determines the friendly index of generalized wheels  $C_n + mK_1$  for all m > 1. In [2160] Salehi and De determine the friendly index sets of certain caterpillars of diameter 4 and disprove a conjecture of Lee and Ng [1517] that the friendly index sets of trees form an arithmetic progression. The maximum and minimum friendly indices for for  $C_m \times P_n$ were given by Shiu and Wong in [2367]. Salehi and Bayot [2157] have determined the friendly index set of  $P_m \times P_n$ . In [1518] Lee and Ng determine the friendly index sets for two classes of cubic graphs, prisms d Möbius ladders. Sinha and Kaur [2392] investigate the full region index sets of friendly labelings of cycles, wheels fans, and  $P_2 \times P_n$ .

For positive integers  $a \leq b \leq c$ , Lee, Ng, and Tong [1523] define the broken wheel W(a, b, c) with three spokes as the graph obtained from  $K_4$  with vertices  $u_1, u_2, u_3, c$  by inserting vertices  $x_{1,1}, x_{1,2}, \ldots, x_{1,a-1}$  along the edge  $u_1u_2, x_{2,1}, x_{2,2}, \ldots, x_{2,b-1}$  along the edge  $u_2u_3, x_{3,1}, x_{3,2}, \ldots, x_{3,c-1}$  along the edge  $u_3u_1$ . They determine the friendly index set for broken wheels with three spokes.

Lee and Ng [1516] define a parallel chord of  $C_n$  as an edge of the form  $v_i v_{n-i}$ (i < n - 1) that is not an edge of  $C_n$ . For  $n \ge 6$ , they call the cycle  $C_n$  with consecutive vertices  $v_1, v_2, \ldots, v_n$  and the edges  $v_1 v_{n-1}, v_2 v_{n-2}, \ldots, v_{(n-2)/2} v_{(n+2)/2}$  for n even and  $v_2 v_{n-1}, v_3 v_{n-2}, \ldots, v_{(n-1)/2} v_{(n+3)/2}$  for n odd,  $C_n$  with a full set of parallel chords. They determine the friendly index of these graphs and show that for any cycle with an arbitrary non-empty set of parallel chords the numbers in its friendly index set form an arithmetic progression with common difference 2.

For a graph G(V, E) and a graph H rooted at one of its vertices v, Ho, Lee, and Ng [1013] define a *root-union of* (H, v) by G as the graph obtained from G by replacing each vertex of G with a copy of the root vertex v of H to which is appended the rest of the structure of H. They investigate the friendly index set of the root-union of stars by cycles.

For a graph G(V, E), the total graph T(G) of G, is the graph with vertex set  $V \cup E$ and edge set  $E \cup \{(v, uv) | v \in V, uv \in E\}$ . Note that the total graph of the *n*-star is the friendship graph and the total graph of  $P_n$  is a triangular snake. Lee and Ng [1513] use  $SP(1^n, m)$  to denote the spider with one central vertex joining *n* isolated vertices and a path of length *m*. They show:  $FI(K_1 + 2nK_2)$  (friendship graph with 2ntriangles) =  $\{2n, 2n - 4, 2n - 8, \ldots, 0\}$  if *n* is even;  $\{2n, 2n - 4, 2n - 8, \ldots, 2\}$  if *n* is odd;  $FI(K_1 + (2n+1)K_2) = \{2n+1, 2n-1, 2n-3, \ldots, 1\}$ ; for *n* odd,  $FI(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, n+1\} \cup \{n-1, n-3, n-5, \ldots, 1\}$ ; for *m* even,  $FI(T(P_n)) = \{3n-7, 3n-11, 3n-15, \ldots, n+1\} \cup \{n-1, n-3, n-5, \ldots, 1\}$ ; for  $m \leq n-1$ and m+n even,  $FI(T(SP(1^n, m))) = \{3(m+n)-4, 3(m+n)-8, 3(m+n)-12, \ldots, (m+n)$ (mod 4)}; for m + n odd,  $FI(T(SP(1^n, m))) = \{3(m+n)-4, \ldots, 1\}$ ; for  $n \geq m$  and m + n even,  $FI(T(SP(1^n, m))) = \{|4k-3(m+n)| |(n-m+2)/2 \leq k \leq m+n\}$ ; for  $n \geq m$  and m + nodd,  $FI(T(SP(1^n, m))) = \{|4k-3(m+n)| |(n-m+3)/2 \leq k \leq m+n\}$ .

Kwong and Lee [1449] determine the friendly index any number of copies of  $C_3$  that share an edge in common and the friendly index any number of copies of  $C_4$  that share an edge in common. Lau, Gao, Lee, and Sun determine the friendly index sets and the cordiality of the edge-gluing of a complete graph  $K_n$  and n copies of cycles  $C_3$ .

For a planar graph G(V, E) Sinha and Kaur [2410] extended the notion of an index set of a friendly labeling to regions of a planar graph and determined the full region index sets of friendly labeling of cycles, wheels fans, and grids  $P_n \times P_2$ .

An edge-friendly labeling f of a graph G induces a function  $f^*$  from V(G) to  $\{0,1\}$ defined as the sum of all edge labels mod 2. The *edge-friendly index set*,  $I_f(G)$ , of fis the number of vertices of f labeled 1 minus the number of vertices labeled 0. The *edge-friendly index set* of a graph G, EFI(G), is  $\{|I_f(G)|\}$  taken over all edge-friendly labelings f of G. The *full edge-friendly index set* of a graph G, FEFI(G), is  $\{I_f(G)\}$ taken over all edge-friendly labelings f of G. Sinha and Kaur [2409] determined the full edge-friendly index sets of stars, 2-regular graphs, wheels, and  $mP_n$ . In [2411] Sinha and Kaur extended the notion of index set of an edge-friendly labeling to regions of a planar graph and determined the full region index set of edge-friendly labelings of cycles, wheels, fans  $P_n + K_1$ , double fans  $P_n + \overline{K_2}$ , and grids  $P_m \times P_n$  ( $m \ge 2, n \ge 3$ ). Sinha and Kaur [2391] investigate the full edge-friendly index sets of double stars, fans generalized fans, and  $P_n \times P_2$ . In [2333] Shiu determined the extreme values of edge-friendly indices of complete bipartite graphs.

In [1375] Kim, Lee, and Ng define the balance index set of a graph G as  $\{|e_0(f)-e_1(f)|\}$ where f runs over all friendly labelings f of G. Zhang, Lee, and Wen [1477] investigate the balance index sets for the disjoint union of up to four stars and Zhang, Ho, Lee, and Wen [2900] investigate the balance index sets for trees with diameter at most four. Kwong, Lee, and Sarvate [1457] determine the balance index sets for cycles with one pendent edge, flowers, and regular windmills. Lee, Ng, and Tong [1522] determine the balance index set of certain graphs obtained by starting with copies of a given cycle and successively identifying one particular vertex of one copy with a particular vertex of the next. For graphs G and H and a bijection  $\pi$  from G to H, Lee and Su [1543] define Perm(G,  $\pi$ , H) as the graph obtaining from the disjoint union of G and H by joining each v in G to  $\pi(v)$  with an edge. They determine the balanced index sets of the disjoint union of cycles and the balanced index sets for graphs of the form Perm(G,  $\pi$ , H) where G and H are regular graphs, stars, paths, and cycles with a chord. They conjecture that the balanced index set for every graph of the form Perm(G,  $\pi$ , H) is an arithmetic progression. Lee, Ho, and Su [1493] investigated the balance index sets of k-level wheel graphs.

Wen [2806] determines the balance index set of the graph that is constructed by identifying the center of a star with one vertex from each of two copies of  $C_n$  and provides a necessary and sufficient for such graphs to be balanced. In [1546] Lee, Su, and Wang determine the balance index sets of the disjoint union of a variety of regular graphs of the same order. Kwong [1447] determines the balanced index sets of rooted trees of height at most 2, thereby settling the problem for trees with diameter at most 4. His method can be used to determine the balance index set of any tree. The homeomorph Hom(G, p) of a graph G is the collection of graphs obtained from G by adding  $p \ (p \ge 0)$  additional degree 2 vertices to its edges. For any regular graph G, Kong, Lee, and Lee [1404] studied the changes of the balance index sets of Hom(G, p) with respect to the parameter p. They derived explicit formulas for their balance index sets provided new examples of uniformly balanced graphs. In [494] Bouchard, Clark, Lee, Lo, and Su investigate the balance index sets of generalized books and ear expansion graphs. In [2140] Rose and Su provided an algorithm to calculate the balance index sets of a graph. Hua and Raridan [1029] determine the balanced index sets of all complete bipartite graphs with a larger part of odd cardinality and a smaller part of even cardinality.

In[2339] Shiu and Kwong made a major advance by introducing an easier approach to find the balance index sets of a large number of families of graphs in a unified and uniform manner. They use this method to determine the balance index sets for r-regular graphs, amalgamations of r-regular graphs, complete bipartite graphs, wheels, one point unions of regular graphs, sun graphs, generalized theta graphs, m-ary trees, spiders, grids  $P_m \times P_n$ , and cylinders  $C_m \times P_n$ . They provide a formula that enables one to determine the balance index sets of many biregular graphs (that is, graphs with the property that there exist two distinct positive integers r and s such that every vertex has degree r or s).

A labeling f from the vertices of a graph G to  $\{0, 1\}$  is said to be *vertex-friendly* if the number of vertices labeled with 0 and the number labeled with 1 differ by at most 1. The *vertex balance index set* of G is  $|e_0(f) - e_1(f)|$  taken over all vertex-friendly labelings f. Adiga, Subbaraya, Shrikanth and Sriraj [54] completely determined the vertex balance index set of  $K_n$ ,  $K_{m,n}$ ,  $C_n \times P_2$ , and complete binary trees.

In [2338] Shiu and Kwong define the *full friendly index set* of a graph G as  $\{e_0(f) - e_1(f)\}$  where f runs over all friendly labelings of G. The full friendly index for  $P_2 \times P_n$  is given by Shiu and Kwong in [2338]. The full friendly index of  $C_m \times C_n$  is given by Shiu and Ling in [2354]. In [2407] and [2408] Sinha and Kaur investigated the full friendly index sets complete graphs, cycles, fans, double fans, wheels, double stars,  $P_3 \times P_n$ , and the tensor product of  $P_2$  and  $P_n$ . Shiu and Ho [2336] investigated the full friendly index sets of cylinder graphs  $C_m \times P_2$   $(m \ge 3), C_m \times P_3$   $(m \ge 4)$ , and  $C_3 \times P_n$   $(n \ge 4)$ . These results, together with previously proven ones, completely determine the full friendly index of all cylinder graphs. Shiu and Ho [2337] study the full friendly index set and the full product-cordial index set of odd twisted cylinders and two permutation Petersen graphs. Gao [825] determined the full friendly index set of  $P_m \times P_n$ , but he used the terms "edge difference set" instead of "full friendly index set" and "direct product" instead of "Cartesian product." The *twisted cylinder* graph is the permutation graph on 4n  $(n \ge 2)$  vertices,  $P(2n; \sigma)$ , where  $\sigma = (1, 2)(3, 4) \cdots (2n - 1, 2n)$  (the product of n transpositions). Shiu and Lee [2352] determined the full friendly index sets of twisted cylinders.

In [595] and [1450] Chopra, Lee, and Su and Kwong and Lee introduce a dual of balance index sets as follows. For an edge labeling f using 0 and 1 they define a partial vertex labeling  $f^*$  by assigning 0 or 1 to  $f^*(v)$  depending on whether there are more 0-edges or 1-edges incident to v and leaving  $f^*(v)$  undefined otherwise. For i = 0 or 1 and a graph G(V, E), let  $e_f(i) = |\{uv \in E : f(uv) = i\}|$  and  $v_f(i) = |\{v \in V :$  $f^*(v) = i\}|$ . They define the edge-balance index of G as  $\text{EBI}(G) = \{|v_f(0) - v_f(1)| :$ the edge labeling f satisfies  $|e_f(0) - e_f(1)| \leq 1\}$ . Among the graphs whose edge-balance index sets have been investigated by Lee and his colleagues are: fans and wheels [595]; generalized theta graphs [1450]; flower graphs [1451] and [1451]; stars, paths, spiders, and double stars [1554]; (p, p + 1)-graphs [1548]; prisms and Möbius ladders [2780]; 2regular graphs, complete graphs [2779]; and the envelope graphs of stars, paths, and cycles [605]. (The envelope graph of G(V, E) is the graph with vertex set  $V(G) \cup E(G)$ and set  $E(G) \cup \{(u, (u, v)) : U \in V, (u, v) \in E)\}$ ).

Lee, Kong, Wang, and Lee [1405] found the  $\text{EBI}(K_{m,n})$  for m = 1, 2, 3, 4, 5 and m = n. Krop, Minion, Patel, and Raridan [1438] did the case for complete bipartite graphs with both parts of odd cardinality. Dao, Hua, Ngo, and Raridan [642] determined the edgebalanced index sets for complete even bipartite graphs. Krop and Sikes [1440] determined  $\text{EBI}(K_{m,m-2a})$  for  $1 \leq a \leq (m-3)/4$  and m odd.

For a graph G and a connected graph H with a distinguished vertex s, the L-product

of G and (H, s),  $G \times_L (H, s)$ , is the graph obtained by taking |V(G)| copies of (H, s)and identifying each vertex of G with s of a single copy of H. In [597] and [496] Chou, Galiardi, Kong, Lee, Perry, Bouchard, Clark, and Su investigated the edge-balance index sets of L-product of cycles with stars. Bouchard, Clark, and Su [495] gave the exact values of the edge-balance index sets of L-product of cycles with cycles.

Chopra, Lee, and Su [598] prove that the edge-balance index of the fan  $P_3 + K_1$ is  $\{0, 1, 2\}$  and edge-balance index of the fan  $P_n + K_1$ ,  $n \ge 4$ , is  $\{0, 1, 2, \ldots, n - 2\}$ . They define the broken fan graphs BF(a, b) as the graph with  $V(BF(a, b)) = \{c\} \cup$  $\{v_1, \ldots, v_a\} \cup \{u_1, \ldots, u_b\}$  and  $E(BF(a, b)) = \{(c, v_i) | i = 1, \ldots, a\} \cup \{(c, u_i) | 1, \ldots, b\} \cup$  $E(P_a) \cup E(P_b)$   $(a \ge 2 \text{ and } b \ge 2)$ . They prove the edge-balance index set of BF(a, b) is  $\{0, 1, 2, \ldots, a + b - 4\}$ . In [1544] Lee, Su, and Todt give the edge-balance index sets of broken wheels. See also [2474] and [2592]. In [1478] Lee, Lee, and Su present a technique that determines the balance index sets of a graph from its degree sequence. In addition, they give an explicit formula giving the exact values of the balance indices of generalized friendship graphs, envelope graphs of cycles, and envelope graphs of cubic trees.

### 3.9 k-equitable Labelings

In 1990 Cahit [530] proposed the idea of distributing the vertex and edge labels among  $\{0, 1, \ldots, k-1\}$  as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph G(V, E) and any positive integer k, assign vertex labels from  $\{0, 1, \ldots, k-1\}$  so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. Calit has called a graph with such an assignment of labels k-equitable. Note that G(V, E) is graceful if and only if it is |E| + 1-equitable and G(V, E) is cordial if and only if it is 2-equitable. Calif. (529) has shown the following:  $C_n$  is 3-equitable if and only if  $n \not\equiv 3 \pmod{6}$ ; the triangular snake with n blocks is 3-equitable if and only if n is even; the friendship graph  $C_3^{(n)}$  is 3-equitable if and only if n is even; an Eulerian graph with  $q \equiv 3 \pmod{6}$  edges is not 3-equitable; and all caterpillars are 3-equitable [529]. Cahit [529] claimed to prove that  $W_n$  is 3-equitable if and only if  $n \not\equiv 3 \pmod{6}$  but Youssef [2874] proved that  $W_n$  is 3-equitable for all  $n \geq 4$ . Youssef [2872] also proved that if G is a k-equitable Eulerian graph with q edges and  $k \equiv 2$  or 3 (mod 4) then  $q \not\equiv k \pmod{2k}$ . Calit conjectures [529] that a triangular cactus with n blocks is 3-equitable if and only if n is even. In [530] Cahit proves that every tree with fewer than five end vertices has a 3-equitable labeling. He conjectures that all trees are k-equitable [531]. In 1999 Speyer and Szaniszló [2462] proved Cahit's conjecture for k = 3. Coles, Huszar, Miller, and Szaniszlo [629] proved caterpillars, symmetric generalized *n*-stars (or symmetric spiders), and complete *n*-ary trees are 4-equitable. Vaidya and Shah [2666] proved that the splitting graphs of  $K_{1,n}$ and the bistar  $B_{n,n}$  and the shadow graph of  $B_{n,n}$  are 3-equitable. Rokad [2131] found 3-equitable labelings of the ring sum of different graphs.

Vaidya, Dani, Kanani, and Vihol [2630] proved that the graphs obtained by starting

with copies  $G_1, G_2, \ldots, G_n$  of a fixed star and joining each center of  $G_i$  to the center of  $G_{i+1}$   $(i = 1, 2, \ldots, n-1)$  by an edge are 3-equitable. Recall the shell C(n, n-3) is the cycle  $C_n$  with n-3 cords sharing a common endpoint called the *apex*. Vaidya, Dani, Kanani, and Vihol [2631] proved that the graphs obtained by starting with copies  $G_1, G_2, \ldots, G_n$  of a fixed shell and joining each apex of  $G_i$  to the apex of  $G_{i+1}$   $(i = 1, 2, \ldots, n-1)$  by an edge are 3-equitable. For a graph G and vertex v of G, Vaidya, Dani, Kanani, and Vihol [2632] prove that the graphs obtained from the wheel  $W_n, n \ge 5$ , by duplicating (see 3.7 for the definition) any rim vertex is 3-equitable and the graphs obtained from the wheel  $W_n$  by duplicating the center is 3-equitable when n is even and not 3-equitable when n is odd and at least 5. They also show that the graphs obtained from the wheel  $W_n, n \ne 5$ , by duplicating every vertex is 3-equitable.

Vaidya, Srivastav, Kaneria, and Ghodasara [2677] prove that cycle with two chords that share a common vertex with opposite ends that are incident to two consecutive vertices of the cycle is 3-equitable. Vaidya, Ghodasara, Srivastav, and Kaneria [2636] prove that star of cycle  $C_n^*$  is 3-equitable for all n. Vaidya and Dani [2626] proved that the graphs obtained by starting with n copies  $G_1, G_2, \ldots, G_n$  of a fixed star and joining the center of  $G_i$  to the center of  $G_{i+1}$  by an edge and each center to a new vertex  $x_i$   $(1 \le i \le n - 1)$  by an edge have 3-equitable labeling. Vaidya and Dani [2629] prove that the graphs obtained by duplication of an arbitrary edge of a cycle or a wheel have 3-equitable labelings.

3-equitable labelings. Recall  $G = \langle W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} \rangle$  1s the graph obtained by joining the center vertices of each of  $W_n^{(i)}$  and  $W_n^{(i+1)}$  to a new vertex  $x_i$  where  $1 \leq i \leq k-1$ . Vaidya, Dani, Kanani, and Vihol [2633] prove that  $\langle W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} \rangle$  is 3-equitable. Vaidya and Vihol [2681] prove that any graph G can be embedded as an induced subgraph of a 3-equitable graph thereby ruling out any possibility of obtaining any forbidden subgraph characterization for 3-equitable graphs.

The shadow graph  $D_2(G)$  of a connected graph G is constructed by taking two copies of G, G' and G'' and joining each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''. Vaidya, Vihol, and Barasara [2684] prove that the shadow graph of  $C_n$  is 3-equitable except for n = 3 and 5 while the shadow graph of  $P_n$  is 3-equitable except for n = 3. They also prove that the middle graph of  $P_n$  is 3-equitable and the middle graph of  $C_n$  is 3-equitable for n even and not 3-equitable for n odd.

Bhut-Nayak and Telang have shown that crowns  $C_n \odot K_1$ , are k-equitable for  $k = n, \ldots, 2n-1$  [464] and  $C_n \odot K_1$  is k-equitable for all n when k = 2, 3, 4, 5, and 6 [465].

In [2220] Seoud and Abdel Maqsoud prove: a graph with n vertices and q edges in which every vertex has odd degree is not 3-equitable if  $n \equiv 0 \pmod{3}$  and  $q \equiv 3 \pmod{6}$ ; all fans except  $P_2 + \overline{K_1}$  are 3-equitable; all double fans  $P_n + \overline{K_2}$  except  $P_4 + \overline{K_2}$  are 3-equitable;  $P_n^2$  is 3-equitable for all n except 3;  $K_{1,1,n}$  is 3-equitable if and only if  $n \equiv 0$ or 2 (mod 3);  $K_{1,2,n}$ ,  $n \geq 2$ , is 3-equitable if and only if  $n \equiv 2 \pmod{3}$ ;  $K_{m,n}$ ,  $3 \leq m \leq n$ , is 3-equitable if and only if (m, n) = (4, 4); and  $K_{1,m,n}$ ,  $3 \leq m \leq n$ , is 3-equitable if and only if (m, n) = (3, 4). They conjectured that  $C_n^2$  is not 3-equitable for all  $n \geq 3$ . However, Youssef [2880] proved that  $C_n^2$  is 3-equitable if and only if n is at least 8. Youssef [2880] also proved that  $C_n + \overline{K_2}$  is 3-equitable if and only if n is even and at least 6 and determined the maximum number of edges in a 3-equitable graph as a function of the number of its vertices. For a graph with n vertices to admit a k-equitable labeling, Seoud and Salim [2249] proved that the number of edges is at most  $k \lceil (n/k) \rceil^2 + k - 1$ .

Bapat and Limaye [345] have shown the following graphs are 3-equitable: helms  $H_n, n \ge 4$ ; flowers (see §2.2 for the definition); the one-point union of any number of helms; the one-point union of any number of copies of  $K_4$ ;  $K_4$ -snakes (see §2.2 for the definition);  $C_t$ -snakes where t = 4 or 6;  $C_5$ -snakes where the number of blocks is not congruent to 3 modulo 6. A multiple shell  $MS\{n_1^{t_1}, \ldots, n_r^{t_r}\}$  is a graph formed by  $t_i$  shells each of order  $n_i, 1 \le i \le r$ , that have a common apex. Bapat and Limaye [346] show that every multiple shell is 3-equitable and Chitre and Limaye [587] show that every multiple shell is 5-equitable. In [588] Chitre and Limaye define the *H*-union of a family of graphs  $G_1, G_2, \ldots, G_t$ , each having a graph H as an induced subgraph, as the graph obtained by starting with  $G_1 \cup G_2 \cup \cdots \cup G_t$  and identifying all the corresponding vertices and edges of H in each of  $G_1, \ldots, G_t$ . In [588] and [589] they proved that the  $\overline{K_n}$ -union of gears and helms  $H_n$   $(n \ge 6)$  are edge-3-equitable.

Szaniszló [2567] has proved the following:  $P_n$  is k-equitable for all k;  $K_n$  is 2-equitable if and only if n = 1, 2, or 3;  $K_n$  is not k-equitable for  $3 \le k < n$ ;  $S_n$  is k-equitable for all k;  $K_{2,n}$  is k-equitable if and only if  $n \equiv k - 1 \pmod{k}$ , or  $n \equiv 0, 1, 2, \ldots, \lfloor k/2 \rfloor - 1 \pmod{k}$ , or  $n = \lfloor k/2 \rfloor$  and k is odd. She also proves that  $C_n$  is k-equitable if and only if k meets all of the following conditions:  $n \ne k$ ; if  $k \equiv 2, 3 \pmod{4}$ , then  $n \ne k - 1$  and  $n \ne k \pmod{2k}$ . Coles, Huszar, Miller, and Szaniszló [629] proved that all caterpillars, symmetric generalized n-stars (or symmetric spiders), and complete n-ary trees for all are 4-equitable.

Vickrey [2714] has determined the k-equitability of complete multipartite graphs. He shows that for  $m \geq 3$  and  $k \geq 3$ ,  $K_{m,n}$  is k-equitable if and only if  $K_{m,n}$  is one of the following graphs:  $K_{4,4}$  for k = 3;  $K_{3,k-1}$  for all k; or  $K_{m,n}$  for k > mn. He also shows that when k is less than or equal to the number of edges in the graph and at least 3, the only complete multipartite graphs that are k-equitable are  $K_{kn+k-1,2,1}$  and  $K_{kn+k-1,1,1}$ . Partial results on the k-equitability of  $K_{m,n}$  were obtained by Krussel [1441].

In [2887] Youssef and Al-Kuleab proved the following:  $C_n^3$  is 3-equitable if and only if n is even and  $n \ge 12$ ; gear graphs are k-equitable for k = 3, 4, 5, 6; ladders  $P_n \times P_2$ are 3-equitable for all  $n \ge 2$ ;  $C_n \times P_2$  is 3-equitable if and only if  $n \not\equiv \pmod{6}$ ; Möbius ladders  $M_n$  are 3-equitable if and only if  $n \not\equiv \pmod{6}$ ; and the graphs obtained from  $P_n \times P_2$   $(n \ge 2)$  where by adding the edges  $u_i v_{i+1}$   $(1 \le i \le n-1)$  to the path vertices  $u_1, u_2, \ldots, u_n$  and  $v_1, v_2, \ldots, v_n$ .

In [1635] López, Muntaner-Batle, and Rius-Font prove that if n is an odd integer and F is optimal k-equitable for all proper divisors k of |E(F)|, then nF is optimal k-equitable for all proper divisors k of |E(F)|. They also prove that if m-1 and n are odd, then then  $nC_m$  is optimal k-equitable for all proper divisors k of |E(F)|.

As a corollary of the result of Cairnie and Edwards [541] on the computational complexity of cordially labeling graphs it follows that the problem of finding k-equitable labelings of graphs is NP-complete as well.

Seoud and Abdel Maqsoud [2221] call a graph k-balanced if the vertices can be labeled

from  $\{0, 1, \ldots, k-1\}$  so that the number of edges labeled *i* and the number of edges labeled *j* induced by the absolute value of the differences of the vertex labels differ by at most 1. They prove that  $P_n^2$  is 3-balanced if and only if n = 2, 3, 4, or 6; for  $k \ge 4$ ,  $P_n^2$ is not *k*-balanced if  $k \le n-2$  or  $n+1 \le k \le 2n-3$ ; for  $k \ge 4$ ,  $P_n^2$  is *k*-balanced if  $k \ge 2n-2$ ; for  $k, m, n \ge 3$ ,  $K_{m,n}$  is *k*-balanced if and only if  $k \ge mn$ ; for  $m \le n$ ,  $K_{1,m,n}$ is *k*-balanced if and only if (*i*) m = 1, n = 1 or 2, and k = 3; (*ii*) m = 1 and k = n+1or n+2; or (*iii*)  $k \ge (m+1)(n+1)$ .

In [2880] Youssef gave some necessary conditions for a graph to be k-balanced and some relations between k-equitable labelings and k-balanced labelings. Among his results are:  $C_n$  is 3-balanced for all  $n \ge 3$ ;  $K_n$  is 3-balanced if and only if  $n \le 3$ ; and all trees are 2-balanced and 3-balanced. He conjectures that all trees are k-balanced  $(k \ge 2)$ .

Bloom has used the term k-equitable to describe another kind of labeling (see [2817]) and [2818]). He calls a graph k-equitable if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label occurs exactly k times. Bloom calls a graph of order n minimally k-equitable if the vertex labels are 1,  $2, \ldots, n$  and it is k-equitable. Both Bloom and Wojciechowski [2817], [2818] proved that  $C_n$  is minimally k-equitable if and only if k is a proper divisor of n. Barrientos and Hevia [369] proved that if G is k-equitable of size q = kw (in the sense of Bloom), then  $\delta(G) \leq w$ and  $\Delta(G) \leq 2w$ . Barrientos, Dejter, and Hevia [368] have shown that forests of even size are 2-equitable. They also prove that for k = 3 or k = 4 a forest of size kw is k-equitable if and only if its maximum degree is at most 2w and that if 3 divides mn + 1, then the double star  $S_{m,n}$  is 3-equitable if and only if  $q/3 \le m \le \lfloor (q-1)/2 \rfloor$ .  $(S_{m,n}$  is  $P_2$  with m pendent edges attached at one end and n pendent edges attached at the other end.) They discuss the k-equitability of forests for  $k \geq 5$  and characterize all caterpillars of diameter 2 that are k-equitable for all possible values of k. Acharya and Bhat-Nayak [45] have shown that coronas of the form  $C_{2n} \odot K_1$  are minimally 4-equitable. In [349] Barrientos proves that the one-point union of a cycle and a path (dragon) and the disjoint union of a cycle and a path are k-equitable for all k that divide the size of the graph. Barrientos and Havia [369] have shown the following:  $C_n \times K_2$  is 2-equitable when n is even; books  $B_n$   $(n \ge 3)$ are 2-equitable when n is odd; the vertex union of k-equitable graphs is k-equitable; and wheels  $W_n$  are 2-equitable when  $n \not\equiv 3 \pmod{4}$ . They conjecture that  $W_n$  is 2-equitable when  $n \equiv 3 \pmod{4}$  except when n = 3. Their 2-equitable labelings of  $C_n \times K_2$  and the *n*-cube utilized graceful labelings of those graphs.

M. Acharya and Bhat-Nayak [46] have proved the following: the crowns  $C_{2n} \odot K_1$ are minimally 2-equitable, minimally 2*n*-equitable, minimally 4-equitable, and minimally *n*-equitable; the crowns  $C_{3n} \odot K_1$  are minimally 3-equitable, minimally 3*n*-equitable, minimally *n*-equitable, and minimally 6-equitable; the crowns  $C_{5n} \odot K_1$  are minimally 5-equitable, minimally 5*n*-equitable, minimally *n*-equitable, and minimally 10-equitable; the crowns  $C_{2n+1} \odot K_1$  are minimally (2n + 1)-equitable; and the graphs  $P_{kn+1}$  are *k*equitable.

In [351] Barrientos calls a k-equitable labeling *optimal* if the vertex labels are consecutive integers and *complete* if the induced edge labels are  $1, 2, \ldots, w$  where w is the number of distinct edge labels. Note that a graceful labeling is a complete 1-equitable

labeling. Barrientos proves that  $C_m \odot nK_1$  (that is, an *m*-cycle with *n* pendent edges attached at each vertex) is optimal 2-equitable when *m* is even;  $C_3 \odot nK_1$  is complete 2-equitable when *n* is odd; and that  $C_3 \odot nK_1$  is complete 3-equitable for all *n*. He also shows that  $C_n \odot K_1$  is *k*-equitable for every proper divisor *k* of the size 2*n*. Barrientos and Havia [369] have shown that the *n*-cube  $(n \ge 2)$  has a complete 2-equitable labeling and that  $K_{m,n}$  has a complete 2-equitable labeling when *m* or *n* is even. They conjecture that every tree of even size has an optimal 2-equitable labeling.

## 3.10 Hamming-graceful Labelings

Mollard, Payan, and Shixin [1781] introduced a generalization of graceful graphs called Hamming-graceful. A graph G = (V, E) is called Hamming-graceful if there exists an injective labeling g from V to the set of binary |E|-tuples such that  $\{d(g(v), g(u))| uv \in E\} = \{1, 2, \ldots, |E|\}$  where d is the Hamming distance. Shixin and Yu [2373] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful;  $C_n$  is Hamming-graceful if and only if  $n \equiv 0$  or 3 (mod 4); if  $K_n$  is Hamming-graceful, then n has the form  $k^2$  or  $k^2 + 2$ ; and  $K_n$  is Hamming-graceful for n = 2, 3, 4, 6, 9, 11, 16, and 18. They conjecture that  $K_n$  is Hamming-graceful for n of the forms  $k^2$  and  $k^2 + 2$  for  $k \geq 5$ .

# 4 Variations of Harmonious Labelings

### 4.1 Sequential and Strongly *c*-harmonious Labelings

Chang, Hsu, and Rogers [558] and Grace [921], [922] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling f of a graph G with q vertices to be strongly c-harmonious if the vertex labels are from  $\{0, 1, \ldots, q-1\}$  and the edge labels induced by f(x) + f(y) for each edge xy are  $c, \ldots, c + q - 1$ . Strongly 1-harmoinious labelings are more simply called *strongly harmonious*. Grace called such a labeling *sequential*. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices whereas Grace allows the vertex labels to range from 0 to q with no vertex label being used twice. For graphs other than trees, we use the term c-sequential labelings interchangeably with strongly c-harmonious labelings. By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [922] proved that caterpillars, caterpillars with a pendent edge, odd cycles with zero or more pendent edges, trees with  $\alpha$ -labelings, wheels  $W_{2n+1}$ , and  $P_n^2$  are sequential. Liu and Zhang [1606] finished off the crowns  $C_{2n} \odot K_1$ . (The case  $C_{2n+1} \odot K_1$  was a special case of Grace's results. Liu 1618 proved crowns are harmonious.)

Bača and Youssef [320] investigated the existence of harmonious labelings for the corona graphs of a cycle and a graph G. They proved that if  $G+K_1$  is strongly harmonious with the 0 label on the vertex of  $K_1$ , then  $C_n \odot G$  is harmonious for all odd  $n \ge 3$ . By combining this with existing results they have as corollaries that the following graphs are harmonious:  $C_n \odot C_m$  for odd  $n \ge 3$  and  $m \not\equiv 2 \pmod{3}$ ;  $C_n \odot K_{s,t}$  for odd  $n \ge 3$ ; and  $C_n \odot K_{1,s,t}$  for odd  $n \ge 3$ .

Bu [507] also proved that crowns are sequential as are all even cycles with m pendent edges attached at each vertex. Figueroa-Centeno, Ichishima, and Muntaner-Batle [762] proved that all cycles with m pendent edges attached at each vertex are sequential. Wu [2823] has shown that caterpillars with m pendent edges attached at each vertex are sequential. exactly one path of fixed length to each vertex of some path is sequential.

Singh has proved the following:  $C_n \odot K_2$  is sequential for all odd n > 1 [2396];  $C_n \odot P_3$ is sequential for all odd n [2397];  $K_2 \odot C_n$  (each vertex of the cycle is joined by edges to the end points of a copy of  $K_2$ ) is sequential for all odd n [2397]; helms  $H_n$  are sequential when n is even [2397]; and  $K_{1,n} + K_2$ ,  $K_{1,n} + \overline{K}_2$ , and ladders are sequential [2399]. Santhosh [2182] has shown that  $C_n \odot P_4$  is sequential for all odd  $n \ge 3$ . Both Grace [921] and Reid (see [815]) have found sequential labelings for the books  $B_{2n}$ . Jungreis and Reid [1266] have shown the following graphs are sequential:  $P_m \times P_n$   $(m, n) \neq (2, 2)$ ;  $C_{4m} \times P_n$   $(m, n) \neq (1, 2)$ ;  $C_{4m+2} \times P_{2n}$ ;  $C_{2m+1} \times P_n$ ; and  $C_4 \times C_{2n}$  (n > 1). The graphs  $C_{4m+2} \times C_{2n+1}$  and  $C_{2m+1} \times C_{2n+1}$  fail to satisfy a necessary parity condition given by Graham and Sloane [925]. The remaining cases of  $C_m \times P_n$  and  $C_m \times C_n$  are open. Gallian, Prout, and Winters [816] proved that all graphs  $C_n \times P_2$  with a vertex or an edge deleted are sequential. Zhu and Liu [2922] give necessary and sufficient conditions for sequential graphs, provide a characterization of non-tree sequential graphs by way of by vertex closure, and obtain characterizations of sequential trees.

Gnanajothi [894] [pp. 68-78] has shown the following graphs are sequential:  $K_{1,m,n}$ ;  $mC_n$ , the disjoint union of m copies of  $C_n$  if and only if m and n are odd; books with triangular pages or pentagonal pages; and books of the form  $B_{4n+1}$ , thereby answering a question and proving a conjecture of Gallian and Jungreis [815]. Sun [2529] has also proved that  $B_n$  is sequential if and only if  $n \not\equiv 3 \pmod{4}$ . Ichishima and Oshima [1062] pose determining whether or not  $mK_{s,t}$  is sequential as a problem.

Yuan and Zhu [2894] have shown that  $mC_n$  is sequential when m and n are odd. Although Graham and Sloane [925] proved that the Möbius ladder  $M_3$  is not harmonious, Gallian [810] established that all other Möbius ladders are sequential (see §2.3 for the definition of Möbius ladder). Chung, Hsu, and Rogers [558] have shown that  $K_{m,n} + K_1$ , which includes  $S_m + K_1$ , is sequential. Second and Youssef [2259] proved that if G is sequential and has the same number of edges as vertices, then  $G + \overline{K_n}$  is sequential for all n. Recall that  $\Theta(C_m)^n$  denotes the book with n m-polygonal pages. Lu [1671] proved that  $\Theta(C_{2m+1})^{2n}$  is 2mn-sequential for all n and m = 1, 2, 3, 4, and  $\Theta(C_m)^2$  is (m-2)-sequential if  $m \geq 3$  and  $m \equiv 2, 3, 4, 7 \pmod{8}$ .

Zhou and Yuan [2919] have shown that for every c-sequential graph G with p vertices and q edges and any positive integer m the graph  $(G + \overline{K_m}) + \overline{K_n}$  is also k-sequential when  $q - p + 1 \leq m \leq q - p + c$ . Zhou [2918] has shown that the analogous results hold for strongly c-harmonious graphs. Zhou and Yuan [2919] have shown that for every c-sequential graph G with p vertices and q edges and any positive integer m the graph  $(G + \overline{K_m}) + \overline{K_n}$  is c-sequential when  $q - p + 1 \leq m \leq q - p + c$ .

Shee [1532] proved that every graph is a subgraph of a sequential graph. Acharya, Germina, Princy, and Rao [35] prove that every connected graph can be embedded in a strongly c-harmonious graph for some c. Miao and Liang [1743] use  $C_n(d; i, j; P_k)$  to denote a cycle  $C_n$  with path  $P_k$  joining two nonconsecutive vertices  $x_i$  and  $x_j$  of the cycle, where d is the distance between  $x_i$  and  $x_j$  on  $C_n$ . They proved that the graph  $C_n(d; i, j; P_k)$  is strongly c-harmonious when k = 2, 3 and integer  $n \ge 6$ . Lu [1670] provides three techniques for constructing larger sequential graphs from some smaller one: an attaching construction, an adjoining construction, and the join of two graphs. Using these, he obtains various families of sequential or strongly c-indexable graphs.

For  $1 \leq s \leq n_3$ , let  $C_n(i : i_1, i_2, \ldots, i_s)$  denote an *n*-cycle with consecutive vertices  $x_1, x_2, \ldots, x_n$  to which the *s* chords  $x_i x_{i_1}, x_i x_{i_2}, \ldots, x_i x_{i_s}$  have been added. Liang [1586] proved a variety of graphs of the form  $C_n(i : i_1, i_2, \ldots, i_s)$  are strongly *c*-harmonious.

Youssef [2877] observed that a strongly c-harmonious graph with q edges is c-cordial for all  $c \ge q$  and a strongly k-indexable graph is k-cordial for every k. The converse of this latter result is not true.

In [1059] Ichishima and Oshima show that the hypercube  $Q_n$   $(n \ge 2)$  is sequential if and only if  $n \ge 4$ . They also introduce a special kind of sequential labeling of a graph G with size 2t + s by defining a sequential labeling f to be a *partitional labeling* if G is bipartite with partite sets X and Y of the same cardinality s such that  $f(x) \le t + s - 1$ for all  $x \in X$  and  $f(y) \ge t - s$  for all  $y \in Y$ , and there is a positive integer m such that
the induced edge labels are partitioned into three sets [m, m+t-1], [m+t, m+t+s-1],and [m+t+s, m+2t+s-1] with the properties that there is an involution  $\pi$ , which is an automorphism of G such that  $\pi$  exchanges X and  $Y, x\pi(x) \in E(G)$  for all  $x \in X$ , and  $\{f(x) + f(\pi(x)) | x \in X\} = [m+t, m+t+s-1]$ . They prove if G has a partitional labeling, then  $G \times Q_n$  has a partitional labeling for every nonnegative integer n. Using this together with existing results and the fact that every graph that has a partitional labeling is sequential, harmonious, and felicitous (see §4.5) they show that the following graphs are partitional, sequential, harmonious, and felicitous: for  $n \ge 4$ , hypercubes  $Q_n$ ; generalized books  $S_{2m} \times Q_n$ ; and generalized ladders  $P_{2m+1} \times Q_n$ .

In [1060] Ichishma and Oshima proved the following: if G is a partitional graph, then  $G \times K_2$  is partitional, sequential, harmonious and felicitous; if G is a connected bipartite graph with partite sets of distinct odd order such that in each partite set each vertex has the same degree, then  $G \times K_2$  is not partitional; for every positive integer m, the book  $B_m$  is partitional if and only if m is even; the graph  $B_{2m} \times Q_n$  is partitional if and only if  $(m, n) \neq (1, 1)$ ; the graph  $K_{m,2} \times Q_n$  is partitional if and only if  $(m, n) \neq (2, 1)$ ; for every positive integer n, the graph  $K_{m,3} \times Q_n$  is partitional when m = 4, 8, 12, or 16. As open problems they ask which m and n is  $K_{m,n} \times K_2$  partitional and for which l, m and n is  $K_{l,m} \times Q_n$  partitional?

Ichishma and Oshima [1060] also investigated the relationship between partitional graphs and strongly graceful graphs (see  $\S3.1$  for the definition) and partitional graphs and strongly felicitous graphs (see  $\S4.5$ ) for the definition). They proved the following. If G is a partitional graph, then  $G \times K_2$  is partitional, sequential, harmonious and felicitous. Assume that G is a partitional graph of size 2t + s with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that  $\lambda_1 = \max\{f(x) : x \in X\}$ and  $\lambda_2 = \max\{f(y) : y \in Y\}$ . If  $\lambda_1 + 1 = m + 2t + s - \lambda_2$ , where  $m = \min\{f(x) + f(y) : x \in Y\}$ .  $xy \in E(G)$  = min{ $f(y) : y \in Y$ }, then G has a strong  $\alpha$ -valuation. Assume that G is a partitional graph of size 2t + s with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that  $\lambda_1 = \max\{f(x) : x \in X\}$  and  $\lambda_2 = \max\{f(y) : y \in Y\}$ . If  $\lambda_1 + 1 = m + 2t + s - \lambda_2$ , where  $m = \min\{f(x) + f(y) : x \in Y\}$ .  $xy \in E(G)$  = min{ $f(y) : y \in Y$ }, then G is strongly felicitous. Assume that G is a partitional graph of size 2t + s with partite sets X and Y of the same cardinality s, and let f be a partitional labeling of G such that  $\mu_1 = f(x_1) = \min\{f(x) : x \in X\}$ and  $\mu_2 = f(y_1) = \min\{f(y) : y \in Y\}$ . If t + s = m + 1 and  $\mu_1 + \mu_2 = m$ , where  $m = \min\{f(x) + f(y) : xy \in E(G)\}$  and  $x_1y_1 \in E(G)$ , then G has a strong  $\alpha$ -valuation and strongly felicitous labeling.

Vaidya and Lekha [2655] proved the following graphs are odd sequential:  $P_n$ ,  $C_n$  for  $n \equiv 0 \pmod{4}$ , crowns  $C_n \odot K_1$  for even n, the graph obtained by duplication of arbitrary vertex in even cycles, path unions of stars, arbitrary super subdivisions in  $P_n$ , and shadows of stars. They also introduced the concept of a *bi-odd sequential* labeling of a graph G as one for which both G and its line graph L(G) admit odd sequential labeling. They proved  $P_n$  and  $C_n$  for  $n \equiv \pmod{4}$  are bi-odd sequential graphs and trees are bi-odd sequential if and only if they are paths. They also prove that  $P_4$  is the only graph with the property that it and its complement are odd sequential.

Arockiaraj, Mahalakshmi, and Namasivayam [184] proved that the subdivision graphs of the following graphs have odd sequential labelings (they call them *odd sum* labelings): triangular snakes; quadrilateral snakes; slanting ladders  $SL_n$  (n > 1) (the graphs obtained from two paths  $u_1u_2\ldots u_n$  and  $v_1v_2\ldots v_n$  by joining each  $u_i$  with  $v_{i+1}$ ;  $C_p \odot K_1$ ,  $H_n \odot$  $K_1, C_m @C_n$  (the graph obtained by attaching paths  $P_n$  to  $C_m$  by identifying the endpoints of the paths with each successive pairs of vertices of  $C_m$ ;  $P_m \times P_n$ ; and graphs obtained by the duplication of a vertex of a path and the duplication of a vertex of a cycle. Arockiaraj, Mahalakshmi, and Namasivayam [186] investigate the odd sum labeling behavior of paths, combs, cycles, crowns, and ladders under duplication of an edge. In [187] they investigated the odd sum property of shadow graphs, edge duplication graphs and vertex identification graphs. In [913] Gopi proved the following graphs are odd sum graphs: graphs  $H_n$  obtained from two copies of  $P_n$   $(n \ge 3)$  with vertices  $v_1, v_2, \ldots, v_n$  and  $u_1, u_2, \ldots, u_n$  by joining  $v_{(n+1)/2}$  and  $u_{(n+1)/2}$  if n is odd and  $v_{n/2}$  and  $u_{(n+2)/2}$  if n is even; graphs obtained from  $H_n$  by attaching a fixed number of pendent edges at each vertex, graphs obtained from  $P_n \ (n \ge 4)$  by attaching a two pendent edges at each interior vertex; and graphs obtained from  $P_m$   $(m \ge 4)$  by identifying an endpoint of the star  $S_n$   $(n \ge 2)$  with each vertex of  $P_m$ . In [917] Gopi and Irudaya Mary proved that slanting ladders, shadow graphs of stars and bistars and mirror graphs and duplicate vertex graphs of paths with at least four verticies are odd sum graphs. In [912] Gopi proved that alternative quadrilateral snakes  $A(D(Q_n))$   $(n \ge 4)$  are odd sum graphs.

Arockiaraj and Mahalakshmi [183] proved the following graphs have odd sequential labelings (odd sum labelings):  $P_n$  (n > 1),  $C_n$  if and only if  $n \equiv 0 \pmod{4}$ ;  $C_{2n} \odot K_1$ ;  $P_n \times P_2$  (n > 1);  $P_m \odot K_1$  if m is even or m is odd and n = 1 or 2; the balloon graph  $P_m(C_n)$  obtained by identifying an end point of  $P_m$  with a vertex of  $C_n$  if either  $n \equiv 0 \pmod{4}$  or  $n \equiv 2 \pmod{4}$  and  $m \not\equiv 1 \pmod{3}$ ; quadrilateral snakes  $Q_n$ ;  $P_m \odot C_n$ if m > 1 and  $n \equiv 0 \pmod{4}$ ;  $P_m \odot Q_3$ ; bistars;  $C_{2n} \times P_2$ ; the trees  $T_p^n$  obtained from ncopies of  $T_p$  by joining an edge uu' between every pair of consecutive paths where u is a vertex in *i*th copy of the path and u' is the corresponding vertex in the (i + 1)th copy of the path;  $H_n$ -graphs obtained by starting with two copies of  $P_n$  with vertices  $v_1, v_2, \ldots, v_n$ and  $u_1, u_2, \ldots, u_n$  and joining the vertices  $v_{(n+1)/2}$  and  $u_{(n+1)/2}$  if n is odd and the vertices  $v_{n/2+1}$  and  $u_{n/2}$  if n; and  $H_n \odot mK_1$ .

Arockiaraj and Mahalakshmi [185] proved the splitting graphs of following graphs have odd sequential labelings (odd sum labelings):  $P_n$ ;  $C_n$  if and only if  $n \equiv 0 \pmod{4}$ ;  $P_n \odot K_1$ ;  $C_{2n} \odot K_1$ ;  $K_{1,n}$  if and only if  $n \leq 2$ ;  $P_n \times P_2$  (n > 1); slanting ladders  $SL_n$  (n > 1); the quadrilateral snake  $Q_n$ ; and  $H_n$ -graphs.

Among the strongly 1-harmonious (also called *strongly harmonious*) graphs are: fans  $F_n$  with  $n \ge 2$  [558]; wheels  $W_n$  with  $n \not\equiv 2 \pmod{3}$  [558];  $K_{m,n} + K_1$  [558]; French windmills  $K_4^{(t)}$  [1025], [1327]; the friendship graphs  $C_3^{(n)}$  if and only if  $n \equiv 0$  or 1 (mod 4) [1025], [1327], [2842];  $C_{4k}^{(t)}$  [2530]; and helms [2064].

Seoud, Diab, and Elsakhawi [2230] have shown that the following graphs are strongly harmonious:  $K_{m,n}$  with an edge joining two vertices in the same partite set;  $K_{1,m,n}$ ; the composition  $P_n[P_2]$  (see §2.3 for the definition); B(3,2,m) and B(4,3,m) for all m (see §2.4 for the notation);  $P_n^2$   $(n \ge 3)$ ; and  $P_n^3$   $(n \ge 3)$ . Seoud et al. [2230] have also proved:  $B_{2n}$  is strongly 2*n*-harmonious;  $P_n$  is strongly  $\lfloor n/2 \rfloor$ -harmonious; ladders  $L_{2k+1}$  are strongly (k+1)-harmonious; and that if G is strongly c-harmonious and has an equal number of vertices and edges, then  $G + \overline{K_n}$  is also strongly c-harmonious.

Bača and Youssef [320] investigated the existence of harmonious labelings for the corona graphs of a cycle and a graph G, and for the corona graph of  $K_2$  and a tree. They prove: if join of a graph G of order p and  $K_1$ ,  $G + K_1$ , is strongly harmonious with the 0 label on the vertex of  $K_1$ , then the corona of  $C_n$  with G,  $C_n \odot G$ , is harmonious for all odd  $n \ge 3$ ; if T is a strongly c-harmonious tree of odd size q and  $c = \frac{q+1}{2}$  then the corona of  $K_2$  with T,  $K_2 \odot T$ , is also strongly c-harmonious; if a unicyclic graph G of odd size q is a strongly c-harmonious and  $c = \frac{q-1}{2}$  then the corona of  $K_2$  with G,  $K_2 \odot G$ , is also strongly c-harmonious.

Seenivasan and Lourdusamy [2199] define an absolutely harmonious labeling f as an injection from the vertex set of a graph G with q edges to the set  $\{0, 1, 2, \ldots, q-1\}$ , if when each edge uv is assigned f(u) + f(v), the resulting edge labels can be arranged as  $a_0, a_1, a_2, \ldots, a_{q-1}$  where  $a_i = q - i$  or q + i for  $0 \le i \le q - 1$ . When G is a tree one of the vertex labels may be assigned to exactly two vertices. A graph that admits absolutely harmonious labeling is called an *absolutely harmonious* graph. Observe that a strongly harmonious graph is an absolutely harmonious graph. They prove the following graphs are absolutely harmonicus:  $P_n$   $(n \geq 3)$ ,  $P_n \odot K_m$ ,  $C_n \odot K_m$ , the banana tree obtained by joining a vertex of degree 1 of each of any number of copies of  $K_{1,n}$  to an isolated vertex, ladders, triangular snakes, quadrilateral snakes,  $mK_4$ ,  $K_n$  if and only if n = 3or 4. They also prove that if G is an absolutely harmonious graph, then there exists a partition  $(V_1, V_2)$  of the vertex set V(G), such that the number of edges connecting the vertices of  $V_1$  to the vertices of  $V_2$  is exactly  $\lceil q/2 \rceil$  snd that if every vertex of an absolutely harmonious graph with q edges is even then  $q \equiv 1$  or 2. As corollaries of the latter condition, they have that  $C_n$  when  $n \equiv 1$  or 2 (mod 4),  $C_m \times C_n$  when m and n are odd, and  $mK_3$ ,  $m \ge 2$  are not absolutely harmonious.

Sethuraman and Selvaraju [2298] have proved that the graph obtained by joining two complete bipartite graphs at one edge is graceful and strongly harmonious. They ask whether these results extend to any number of complete bipartite graphs.

For a graph G(V, E) Gayathri and Hemalatha [859] define an even sequential harmonious labeling f of G as an injection from V to  $\{0, 1, 2, \ldots, 2|E|\}$  with the property that the induced mapping  $f^+$  from E to  $\{2, 4, 6, \ldots, 2|E|\}$  defined by  $f^+(uv) = f(u) + f(v)$ when f(u) + f(v) is even, and  $f^+(uv) = f(u) + f(v) + 1$  when f(u) + f(v) is odd, is an injection. They prove the following have even sequential harmonious labelings (all cases are the nontrivial ones):  $P_n, P_n^+, C_n(n \ge 3)$ , triangular snakes, quadrilateral snakes, Möbius ladders,  $P_m \times P_n$  ( $m \ge 2, n \ge 2$ ),  $K_{m,n}$ ; crowns  $C_m \odot K_1$ , graphs obtained by joining the centers of two copies of  $K_{1,n}$  by a path; banana trees (see §2.1),  $P_n^2$ , closed helms (see §2.2),  $C_3 \odot nK_1$  ( $n \ge 2$ );  $D \odot K_{1,n}$  where D is a dragon (see §2.2);  $\langle K_{1,n} : m \rangle$  ( $m, n \ge 2$ ) (see §4.5); the wreath product  $P_n * \overline{K_2}$  ( $n \ge 2$ ) (see §4.5); combs  $P_n \odot K_1$ ; the one-point union of the end point of a path to a vertex of a cycle (tadpole); the one-point union of the end point of a tadpole and the center of a star; the graphs  $PC_n$  obtained from  $C_n = v_0, v_1, v_2, \ldots, v_{n-1}$  by adding the cords  $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-2)/2}, v_{(n+2)/2}$  when n is even and  $v_1v_{n-1}, v_2v_{n-2}, \ldots, v_{(n-3)/2}, v_{(n+3)/2}$  when n is odd (that is, cycles with a full set of cords);  $P_m \odot nK_1$ ; the one-point union of a vertex of a cycle and the center of a star; graphs obtained by joining the centers of two stars with an edge; graphs obtained by joining two disjoint cycles with an edge (dumbbells); graphs consisting of two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex (butterflies).

In [1046] Ichishima, Muntaner-Batle, and Oshima define the harmonious number,  $\eta(G)$ , of a graph G with q edges as the smallest positive integer n for which there exists an injective function f from V(G) to  $Z_{n+1}$  such that each uv of G is labeled f(u) + f(v)(mod q) and the resulting edge labels are distinct, or  $+\infty$  if there exists no such integer n. If such functions exist, they are called harmonious numberings. The strong harmonious number,  $\eta_s(G)$ , of a graph G is defined to be either the smallest positive integer n such that  $n = \eta(G)$  with the additional property that there exists an integer  $\lambda$  such that  $\min\{f(u), f(v)\} \leq \lambda \leq \max\{f(u), f(v)\}$  for each edge in G or  $+\infty$  if there exists no such integer n. They provide a necessary condition for a graph to have a finite harmonious number and sufficient conditions for a graph to have an infinite (strong) harmonious number. In addition, they examine the relations between harmonious numbers, gamma-numbers, alpha-numbers, and super edge magic deficiencies (see §5.2). They determine the formulas for the (strong) harmonious numbers of some 2-regular graphs and all complete bipartite graphs.

In her PhD thesis [1812] (see also [860]) Muthuramakrishnan defined a labeling fof a graph G(V, E) to be k-even sequential harmonious if f is an injection from V to  $\{k-1,k,k+1,\ldots,k+2q-1\}$  such that the induced mapping  $f^+$  from E to  $\{2k,2k+1\}$  $2, 2k + 4, \dots, 2k + 2q - 2$  defined by  $f^+(uv) = f(u) + f(v)$  if f(u) + f(v) is even and  $f^+(uv) = f(u) + f(v) + 1$  if f(u) + f(v) is odd are distinct. A graph G is called a k-even sequential harmonious graph if it admits a k-even sequential harmonious labeling. Among the numerous graphs that she proved to be k-even sequential harmonious are: paths, cycles,  $K_{m,n}$ ,  $P_n^2$   $(n \ge 3)$ , crowns  $C_m \odot K_1$ ,  $C_m @P_n$  (the graph obtained by identifying an endpoint of  $P_n$  with one vertex of  $C_m$ ), double triangular snakes, double quadrilateral snakes, bistars, grids  $P_m \times P_n$   $(m, n \ge 2)$ ,  $P_n[P_2]$ ,  $C_3 \odot nK_1$   $(n \ge 2)$ , flags  $Fl_m$  (the cycle  $C_m$ with one pendent edge), dumbbell graphs (two disjoint cycles joined by an edge) butterfly graphs  $B_n$  (two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex),  $K_2 + nK_1$ ,  $\overline{K_n} + 2K_2$ , banana trees, sparklers  $P_m@K_{1,n}$   $(m, n \ge 2)$ , (graphs obtained by identifying an endpoint of  $P_m$ with the center of a star), twigs (graphs obtained from  $P_n$   $(n \ge 3)$  by attaching exactly two pendent edges at each internal vertex of  $P_n$ ), festoon graphs  $P_m \odot nK_1$   $(m \ge 2)$ , the graphs  $T_{m,n,t}$  obtained from a path  $P_t$  by appending m edges at one endpoint of  $P_t$  and n edges at the other endpoint of  $P_t$ ,  $L_n \odot K_1$  ( $L_n$  is the ladder  $P_n \times P_2$ ), shadow graphs of paths, stars and bistars, and split graphs of paths and stars. Muthuramakrishnan also defines k-odd sequential harmonious labeling of graphs in the natural way and obtains a handful of results.

### **4.2** (k, d)-arithmetic Labelings

Acharya and Hegde [39] have generalized sequential labelings as follows. Let G be a graph with q edges and let k and d be positive integers. A labeling f of G is said to be (k, d)-arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by f(x) + f(y) for each edge xy are  $k, k + d, k + 2d, \dots, k + (q-1)d$ . They obtained a number of necessary conditions for various kinds of graphs to have a (k, d)-arithmetic labeling. The case where k = 1 and d = 1 was called *additively graceful* by Hegde [977]. Hegde [977] showed:  $K_n$  is additively graceful if and only if n = 2, 3, or 4; every additively graceful graph except  $K_2$  or  $K_{1,2}$  contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendent edge attached. Jinnah and Singh [1254] noted that  $P_n^2$  is additively graceful. Hegde [978] proved that if G is strongly k-indexable, then G and  $G + \overline{K_n}$  are (kd, d)-arithmetic. Acharya and Hegde [41] proved that  $K_n$  is (k, d)-arithmetic if and only if  $n \geq 5$  (see also [513]). They also proved that a graph with an  $\alpha$ -labeling is a (k, d)-arithmetic for all k and d. Bu and Shi [513] proved that  $K_{m,n}$  is (k, d)-arithmetic when k is not of the form id for  $1 \leq i \leq n-1$ . For all  $d \geq 1$  and all  $r \geq 0$ , Acharya and Hegde [39] showed the following:  $K_{m,n,1}$  is (d+2r, d)-arithmetic;  $C_{4t+1}$  is (2dt+2r, d)-arithmetic;  $C_{4t+2}$  is not (k, d)-arithmetic for any values of k and d;  $C_{4t+3}$  is ((2t+1)d+2r, d)-arithmetic;  $W_{4t+2}$ is (2dt + 2r, d)-arithmetic; and  $W_{4t}$  is ((2t + 1)d + 2r, d)-arithmetic. They conjecture that  $C_{4t+1}$  is (2dt + 2r, d)-arithmetic for some r and that  $C_{4t+3}$  is (2dt + d + 2r, d)-arithmetic for some r. Hegde and Shetty [996] proved the following: the generalized web W(t,n)(see §2.2 for the definition) is ((n-1)d/2, d)-arithmetic and ((3n-1)d/2, d)-arithmetic for odd n; the join of the generalized web W(t, n) with the center removed and  $\overline{K_p}$  where n is odd is ((n-1)d/2, d)-arithmetic; every  $T_p$ -tree (see §3.2 for the definition) with q edges and every tree obtained by subdividing every edge of a  $T_p$ -tree exactly once is (k+(q-1)d,d)-arithmetic for all k and d. Lu, Pan, and Li [1673] proved that  $K_{1,m} \cup K_{p,q}$ is (k, d)-arithmetic when k > (q - 1)d + 1 and d > 1.

Yu [2889] proved that a necessary condition for  $C_{4t+1}$  to be (k, d)-arithmetic is that k = 2dt + r for some  $r \ge 0$  and a necessary condition for  $C_{4t+3}$  to be (k, d)-arithmetic is that k = (2t + 1)d + 2r for some  $r \ge 0$ . These conditions were conjectured by Acharya and Hegde [39]. Singh proved that the graph obtained by subdividing every edge of the ladder  $L_n$  is (5, 2)-arithmetic [2395] and that the ladder  $L_n$  is (n, 1)-arithmetic [2398]. He also proves that  $P_m \times C_n$  is ((n - 1)/2, 1)-arithmetic when n is odd [2398]. Acharya, Germina, and Anandavally [33] proved that the subdivision graph of the ladder  $L_n$  is (k, d)-arithmetic if either d does not divide k or k = rd for some  $r \ge 2n$  and that  $P_m \times P_n$  and the subdivision graph of the ladder  $L_n$  are (k, k)-arithmetic if and only if k is at least 3. Lu, Pan, and Li [1673] proved that  $S_m \cup K_{p,q}$  is (k, d)-arithmetic when k > (q - 1)d + 1 and d > 1.

A graph is called *arithmetic* if it is (k, d)-arithmetic for some k and d. Singh and Vilfred [2403] showed that various classes of trees are arithmetic. Singh [2398] has proved that the union of an arithmetic graph and an arithmetic bipartite graph is arithmetic. He conjectures that the union of arithmetic graphs is arithmetic. He provides an example to show that the converse is not true.

Germina and Anandavally [869] investigated embedding of graphs in arithmetic graphs. They proved: every graph can be embedded as an induced subgraph of an arithmetic graph; every bipartite graph can be embedded in a (k, d)-arithmetic graph for all k and d such that d does not divide k; and any graph containing an odd cycle cannot be embedded as an induced subgraph of a connected (k, d)-arithmetic with k < d.

In [2863] Yao, Liu, and Yao give necessary and sufficient conditions for a tree to have the following mutually equivalent labelings: set-ordered odd-graceful, (k, d)-graceful, super edge-magic total, odd-elegant (see §4.4), harmonious, (k, d)-arithmetic, and edge-antimagic (see §6.1).

## **4.3** (k, d)-indexable Labelings

Acharya and Hegde [39] call a graph with p vertices and q edges (k, d)-indexable if there is an injective function from V to  $\{0, 1, 2, \ldots, p-1\}$  such that the set of edge labels induced by adding the vertex labels is a subset of  $\{k, k+d, k+2d, \ldots, k+q(d-1)\}$ . When the set of edges is  $\{k, k+d, k+2d, \ldots, k+q(d-1)\}$  the graph is said to be strongly (k, d)-indexable. A (k, 1)-graph is more simply called k-indexable and strongly 1-indexable graphs are simply called strongly indexable. Notice that strongly indexable graphs are a stronger form of sequential graphs and for trees and unicyclic graphs the notions of sequential labelings and strongly k-indexable labelings coincide. Hegde and Shetty [1001] have shown that the notions of (1, 1)-strongly indexable graphs and super edge-magic total labelings (see §5.2) are equivalent.

Zhou [2918] has shown that for every k-indexable graph G with p vertices and q edges the graph  $(G + K_{q-p+k}) + K_1$  is strongly k-indexable. Acharaya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are  $K_2, K_3$ , and  $K_2 \times K_3$ , and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [39] call a graph with p vertices indexable if there is an injective labeling of the vertices with labels from  $\{0, 1, 2, \ldots, p-1\}$  such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [193] who also proved that all trees are indexable. Bu and Shi [514] also proved that all trees are indexable and that all unicyclic graphs with the cycle  $C_3$  are indexable. Hegde [978] has shown the following: every graph can be embedded as an induced subgraph of an indexable graph; if a connected graph with p vertices and q edges  $(q \ge 2)$  is (k, d)-indexable, then  $d \leq 2$ ;  $P_m \times P_n$  is indexable for all m and n; if G is a connected (1,2)-indexable graph, then G is a tree; the minimum degree of any (k,1)-indexable graph with at least two vertices is at most 3; a caterpillar with partite sets of orders a and bis strongly (1, 2)-indexable if and only if  $|a - b| \leq 1$ ; in a connected strongly k-indexable graph with p vertices and q edges,  $k \leq p-1$ ; and if a graph with p vertices and q edges is (k, d)-indexable, then  $q \leq (2p - 3 - k + d)/d$ . As a corollary of the latter, it follows that  $K_n$   $(n \ge 4)$  and wheels are not (k, d)-indexable.

Lee and Lee [1476] provide a way to construct a (k, d)-strongly indexable graph from

two given (k, d)-strongly indexable graphs. Lee and Lo [1507] show that every given (1,2)strongly indexable spider can extend to an (1,2)-strongly indexable spider with arbitrarily many legs.

Seoud, Abd El Hamid, and Abo Shady [2218] proved the following graphs are indexable:  $P_m \times P_n$   $(m, n \ge 2)$ ; the graphs obtained from  $P_n + K_1$  by inserting one vertex between every two consecutive vertices of  $P_n$ ; the one-point union of any number of copies of  $K_{2,n}$ ; and the graphs obtained by identifying a vertex of a cycle with the center of a star. They showed  $P_n$  is strongly  $\lceil n/2 \rceil$ -indexable; odd cycles  $C_n$  are strongly  $\lceil n/2 \rceil$ -indexable;  $K_{(m,n)}$  (m, n > 2) is indexable if and only if m or n is at most 2. For a simple indexable graph G(V, E) they proved  $|E| \le 2|V| - 3$ . Also, they determine all indexable graphs of order at most 6.

Hegde and Shetty [1000] also prove that if G is strongly k-indexable Eulerian graph with q edges then  $q \equiv 0, 3 \pmod{4}$  if k is even and  $q \equiv 0, 1 \pmod{4}$  if k is odd. They further showed how strongly k-indexable graphs can be used to construct polygons of equal internal angles with sides of different lengths.

Germina [866] has proved the following: fans  $P_n + K_1$  are strongly indexable if and only if n = 1, 2, 3, 4, 5, 6;  $P_n + K_2$  is strongly indexable if and only if  $n \leq 2$ ; the only strongly indexable complete *m*-partite graphs are  $K_{1,n}$  and  $K_{1,1,n}$ ; ladders  $P_n \times P_2$  are  $\lceil \frac{n}{2} \rceil$ -strongly indexable, if *n* is odd;  $K_n \times P_k$  is a strongly indexable if and only if n = 3;  $C_m \times P_n$  is 2-strongly indexable if *m* is odd and  $n \geq 2$ ;  $K_{1,n} + K_i$  is not strongly indexable for  $n \geq 2$ ; for  $G_i \cong K_{1,n}, 1 \leq i \leq n$ , the sequential join  $G \cong (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$ is strongly indexable if and only if, either i = n = 1 or i = 2 and n = 1 or i = 1, n = 3;  $P_1 \cup P_n$  is strongly indexable if and only if  $n \leq 3$ ;  $P_2 \cup P_n$  is not strongly indexable;  $P_2 \cup P_n$ is  $\lceil \frac{n+3}{2} \rceil$ -strongly indexable;  $mC_n$  is k-strongly indexable if and only if *m* and *n* are odd;  $K_{1,n} \cup K_{1,n+1}$  is strongly indexable; and  $mK_{1,n}$  is  $\lceil \frac{3m-1}{2} \rceil$ -strongly indexable when *m* is odd.

Acharya and Germina [28] proved that every graph can be embedded in a strongly indexable graph and gave an algorithmic characterization of strongly indexable unicyclic graphs. In [30] they provide necessary conditions for an Eulerian graph to be strongly k-indexable and investigate strongly indexable (p, q)-graphs for which q = 2p - 3.

Hegde and Shetty [996] proved that for n odd the generalized web graph W(t, n) with the center removed is strongly (n-1)/2-indexable. Hegde and Shetty [1001] define a *level joined planar grid* as follows. Let u be a vertex of  $P_m \times P_n$  of degree 2. For every pair of distinct vertices v and w that do not have degree 4, introduce an edge between v and w provided that the distance from u to v equals the distance from u to w. They prove that every level joined planar grid is strongly indexable. For any sequence of positive integers  $(a_1, a_2, \ldots, a_n)$  Lee and Lee [1475] show how to associate a strongly indexable (1, 1)-graph. As a corollary, they obtain the aforementioned result Hegde and Shetty on level joined planar grids.

Section 5.2 of this survey includes a discussion of a labeling method called super edgemagic. In 2002 Hegde and Shetty [1001] showed that a graph has a strongly k-indexable labeling if and only if it has a super edge-magic labeling.

#### 4.4 Elegant Labelings

In 1981 Chang, Hsu, and Rogers [558] defined an *elegant labeling* f of a graph G with q edges as an injective function from the vertices of G to the set  $\{0, 1, \ldots, q\}$  such that when each edge xy is assigned the label  $f(x) + f(y) \pmod{(q+1)}$  the resulting edge labels are distinct and nonzero. An injective labeling f of a graph G with q vertices is called strongly k-elegant if the vertex labels are from  $\{0, 1, \ldots, q\}$  and the edge labels induced by  $f(x) + f(y) \pmod{(q+1)}$  for each edge xy are  $k, \ldots, k+q-1$ . Note that in contrast to the definition of a harmonious labeling, for an elegant labeling it is not necessary to make an exception for trees.

Whereas the cycle  $C_n$  is harmonious if and only if n is odd, Chang et al. [558] proved that  $C_n$  is elegant when  $n \equiv 0$  or 3 (mod 4) and not elegant when  $n \equiv 1 \pmod{4}$ . Chang et al. further showed that all fans are elegant and the paths  $P_n$  are elegant for  $n \neq 0 \pmod{n}$ 4). Cahit [527] then showed that  $P_4$  is the only path that is not elegant. Balakrishnan, Selvam, and Yegnanarayanan [340] have proved numerous graphs are elegant. Among them are  $K_{m,n}$  and the *m*th-subdivision graph of  $K_{1,2n}$  for all *m*. They prove that the bistar  $B_{n,n}$  ( $K_2$  with n pendent edges at each endpoint) is elegant if and only if n is even. They also prove that every simple graph is a subgraph of an elegant graph and that several families of graphs are not elegant. Deb and Limaye [653] have shown that triangular snakes (see §2.2 for the definition) are elegant if and only if the number of triangles is not equal to  $3 \pmod{4}$ . In the case where the number of triangles is 3(mod 4) they show the triangular snakes satisfy a weaker condition they call *semi-elegant* whereby the edge label 0 is permitted. In [654] Deb and Limaye define a graph G with q edges to be *near-elegant* if there is an injective function f from the vertices of G to the set  $\{0, 1, \ldots, q\}$  such that when each edge xy is assigned the label  $f(x) + f(y) \pmod{1}$ (q+1)) the resulting edge labels are distinct and not equal to q. Thus, in a near-elegant labeling, instead of 0 being the missing value in the edge labels, q is the missing value. Deb and Limaye show that triangular snakes where the number of triangles is 3 (mod 4) are near-elegant. For any positive integers  $\alpha \leq \beta \leq \gamma$  where  $\beta$  is at least 2, the theta graph  $\theta_{\alpha,\beta,\gamma}$  consists of three edge disjoint paths of lengths  $\alpha,\beta$ , and  $\gamma$  having the same end points. Deb and Limaye [654] provide elegant and near-elegant labelings for some theta graphs where  $\alpha = 1, 2$ , or 3. Seoud and Elsakhawi [2232] have proved that the following graphs are elegant:  $K_{1,m,n}$ ;  $K_{1,1,m,n}$ ;  $K_2 + \overline{K_m}$ ;  $K_3 + \overline{K_m}$ ; and  $K_{m,n}$  with an edge joining two vertices of the same partite set. Elumalai and Sethuraman [713] proved  $P_2^n$ ,  $P_m^2 + \overline{K_n}$ ,  $S_m + S_n$ ,  $S_m + \overline{K_m}$ ,  $C_3 \times P_m$ , and even cycles  $C_{2n}$  with vertices  $a_0, a_1, \ldots, a_{2n-1}, a_0$  and 2n-3 chords  $a_0a_2, a_0a_3, \ldots, a_0a_{2n-2}$   $(n \ge 2)$  are elegant. Zhou [2918] has shown that for every strongly k-elegant graph G with p vertices and q edges and any positive integer m the graph  $(G + K_m) + K_n$  is also strongly k-elegant when  $q - p + 1 \le m \le q - p + k.$ 

Sethuraman and Elumalai [2273] proved that every graph is a vertex induced subgraph of a elegant graph and present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a strong form of elegant labeling. Acharya, Germina, Princy, and Rao [35] prove that every (p, q)- graph G can be embedded in a connected elegant graph H. The construction is done in such a way that if G is planar and elegant (harmonious), then so is H.

In [2272] Sethuraman and Elumalai define a graph H to be a  $K_{1,m}$ -star extension of a graph G with p vertices and q edges at a vertex v of G where m > p - 1 - deg(v) if H is obtained from G by merging the center of the star  $K_{1,m}$  with v and merging p - 1 - deg(v)pendent vertices of  $K_{1,m}$  with the p - 1 - deg(v) nonadjacent vertices of v in G. They prove that for every graph G with p vertices and q edges and for every vertex v of G and every  $m \ge 2^{p-1} - 1 - q$ , there is a  $K_{1,m}$ -star extension of G that is both graceful and harmonious. In the case where  $m \ge 2^{p-1} - q$ , they show that G has a  $K_{1,m}$ -star extension that is elegant. Sethuraman and Selvaraju [2299] have shown that certain cases of the union of any number of copies of  $K_4$  with one or more edges deleted and one edge in common are elegant.

In [728] Ephremnath and Elumlai say a graph G is a cycle with a chord Hamiltonian path if G is obtained from the cycle  $v_0, v_1, \ldots, v_{n-1}, v_0$   $(n \ge 6)$  by adding the chords  $v_1v_{n-1}, v_vv_{n-2}, \ldots, v_\alpha v_\beta$  where  $\alpha = \beta = (n-2)/2$  if n is even and  $\alpha = (n+3)/2$ ,  $\beta = (n-1)/2$  if n is odd. They proved that  $C_n$   $(n \ge 6)$  with a chord Hamiltonian path is harmonious and elegant.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let G be a graph with q edges and H a finite Abelian group (under addition) of order q. Define G to be H-harmonious if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label f(x) + f(y) the resulting edge labels are distinct. When G is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley, and Jungreis [422] have shown that if H is a finite Abelian group of order n > 1 then  $C_n$  is H-harmonious if and only if H has a non-cyclic or trivial Sylow 2-subgroup and H is not of the form  $Z_2 \times Z_2 \times \cdots \times Z_2$ . Thus, for example,  $C_{12}$  is not  $Z_{12}$ -harmonious but is  $(Z_2 \times Z_2 \times Z_3)$ -harmonious. In [707] Ehard, Glock, and Joos apply rainbow colorings to graph decompositions and harmonious labeling of graphs.

Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let G be a graph with q edges and H a finite Abelian group (under addition) with q + 1 elements. We say G is *H*-elegant if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label f(x) + f(y) the resulting set of edge labels is the non-identity elements of H. Beals et al. [422] proved that if H is a finite Abelian group of order n with  $n \neq 1$  and  $n \neq 3$ , then  $C_{n-1}$  is H-elegant using only the non-identity elements of H as vertex labels if and only if H has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu, and Rogers [558] by showing that  $C_n$  is elegant when  $n \equiv 2 \pmod{4}$ . Mollard and Payan [1780] also proved that  $C_n$  is elegant when  $n \equiv 2 \pmod{4}$  and gave another proof that  $P_n$  is elegant when  $n \neq 4$ . In 2014 Ollis [1880] used harmonious labelings for  $Z_m$  given by Beals, Gallian, Headley, and Jungreis in [422] to construct new Latin squares of odd order.

A function f is said to be an *odd-elegant* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to 2q - 1 such that the induced mapping  $f^*(uv) = f(u) + f(v) \pmod{2q}$  from the edges of G to the odd integers between

1 to 2q - 1 is a bijection. Zhou, Yao, and Chen [2920] proved that every lobster is oddelegant. In [2766] Wang, Xu, Ma, and Zhang gave a new type of graphical passwords based on odd-elegant labeled graphs. See also[2767] and [2905].

For a graph G(V, E) and an Abelian group H Valentin [2698] defines a *polychrome* labeling of G by H to be a bijection f from V to H such that the edge labels induced by f(uv) = f(v) + f(u) are distinct. Valentin investigates the existence of polychrome labelings for paths and cycles for various Abelian groups.

#### 4.5 Felicitous Labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function f from the vertices of a graph G with q edges to the set  $\{0, 1, \ldots, q\}$  is called *felicitous* if the edge labels induced by  $f(x) + f(y) \pmod{q}$  for each edge xy are distinct. (Recall a harmonious labeling only allows the vertex labels  $0, 1, \ldots, q-1$ .) This definition first appeared in a paper by Lee, Schmeichel, and Shee in [1532] and is attributed to E. Choo. labeling of the graph. Balakrishnan and Kumar [337] proved the conjecture of Lee, Schmeichel, and Shee [1532] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are:  $C_n$  except when  $n \equiv 2 \pmod{4}$  [1532];  $K_{m,n}$  when m, n > 1 $[1532]; P_2 \cup C_{2n+1} [1532]; P_2 \cup C_{2n} [2580]; P_3 \cup C_{2n+1} [1532]; S_m \cup C_{2n+1} [1532]; K_n$  if and only if  $n \leq 4$  [2272];  $P_n + \overline{K_m}$  [2272]; the friendship graph  $C_3^{(n)}$  for n odd [1532];  $P_n \cup C_3$  [2323];  $P_n \cup C_{n+3}$  [2580]; and the one-point union of an odd cycle and a caterpillar [2323]. Shee [2319] conjectured that  $P_m \cup C_n$  is felicitous when n > 2 and m > 3. Lee, Schmeichel, and Shee [1532] ask for which m and n is the one-point union of n copies of  $C_m$  felicitous. They showed that in the case where mn is twice an odd integer the graph is not felicitous. In contrast to the situation for felicitous labelings, we remark that  $C_{4k}$  and  $K_{m,n}$  where m, n > 1 are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel, and Shee [1532] conjectured that the n-cube is felicitous. This conjecture was proved by Figueroa-Centeno and Ichishima in 2001 [757].

Balakrishnan, Selvam, and Yegnanarayanan [339] obtained numerous results on felicitous labelings. The wreath product, G \* H, of graphs G and H has vertex set  $V(G) \times V(H)$  and  $(g_1, h_1)$  is adjacent to  $(g_2, h_2)$  whenever  $g_1g_2 \in E(G)$  or  $g_1 = g_2$ and  $h_1h_2 \in E(H)$ . They define  $H_{n,n}$  as the graph with vertex set  $\{u_1, \ldots, u_n; v_1, \ldots, v_n\}$ and edge set  $\{u_iv_j | 1 \leq i \leq j \leq n\}$ . They let  $\langle K_{1,n} : m \rangle$  denote the graph obtained by taking m disjoint copies of  $K_{1,n}$ , and joining a new vertex to the centers of the m copies of  $K_{1,n}$ . They prove the following are felicitous:  $H_{n,n}$ ;  $P_n * \overline{K_2}$ ;  $\langle K_{1,m} : m \rangle$ ;  $\langle K_{1,2} : m \rangle$ when  $m \neq 0 \pmod{3}$ , or  $m \equiv 3 \pmod{6}$ , or  $m \equiv 6 \pmod{12}$ ;  $\langle K_{1,2n} : m \rangle$  for all m and  $n \geq 2$ ;  $\langle K_{1,2t+1} : 2n+1 \rangle$  when  $n \geq t$ ;  $P_n^k$  when k = n-1 and  $n \neq 2 \pmod{4}$ , or k = 2t and  $n \geq 3$  and k < n - 1; the join of a star and  $\overline{K_n}$ ; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [2865] conjectures that the graphs obtained from an even cycle by attaching n new vertices to each vertex of the cycle is felicitous. This conjecture was verified by Figueroa-Centeno, Ichishima, and Muntaner-Batle in [762]. In [2295] Sethuraman and Selvaraju [2299] have shown that certain cases of the union of any number of copies of  $K_4$  with 3 edges deleted and one edge in common are felicitous. Sethuraman and Selvaraju [2295] present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a felicitous labeling. Krisha and Dulawat [1432] give algorithms for finding graceful, harmonious, sequential, felicitous, and antimagic (see §5.7) labelings of paths. A *linear cactus*  $P_m(K_n)$  is a connected graph in which all the blocks are isomorphic to a complete graph  $K_n$  and block-cutpoint is a path  $P_{2m-1}$ . Gomathi [910] proved the follow graphs are felicitous:  $P_m(K_4)$ , splitting graphs of  $(B_{n,n})$ , planar graphs  $\text{Pl}_{m,n}$ , and  $C_{2k+1} \odot S_m$ . Gomathi and Nagarajan [905] proved the following graphs are felicitous: a vertex switching of  $C_n$   $(n \ge 4)$ , a vertex switching of  $C_n$   $(n \ge 4)$ with one chord, a vertex duplication of  $C_n$ , and the square of the book  $B_{n,n}$   $(n \ge 2)$ . Ezhilarasi Hilda and Jeba Jesintha [1011] proved that all shell flower graphs are felicitous.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [763] define a felicitous graph to be strongly felicitous if there exists an integer k so that for every edge uv, min $\{f(u), f(v)\}$   $\leq k < \max\{f(u), f(v)\}$ . For a graph with p vertices and q edges with  $q \geq p-1$  they show that G is strongly felicitous if and only if G has an  $\alpha$ -labeling (see §3.1). They also show that for graphs  $G_1$  and  $G_2$  with strongly felicitous labelings  $f_1$  and  $f_2$  the graph obtained from  $G_1$  and  $G_2$  by identifying the vertices u and v such that  $f_1(u) = 0 = f_2(v)$ is strongly felicitous and that the one-point union of two copies of  $C_m$  where  $m \geq 4$  and m is even is strongly felicitous. As a corollary they have that the one-point union of n copies of  $C_m$  where m is even and at least 4 and  $n \equiv 2 \pmod{4}$  is felicitous. They conjecture that the one-point union of n copies of  $C_m$  is felicitous if and only if  $mn \equiv 0, 1, \text{ or } 3$ (mod 4). In [767] Figueroa-Centeno, Ichishima, and Muntaner-Batle prove that  $2C_n$  is strongly felicitous if and only if n is even and at least 4. They conjecture [767] that  $mC_n$ is felicitous if and only if  $mn \not\equiv 2 \pmod{4}$  and that  $C_m \cup C_n$  is felicitous if and only if  $m + n \not\equiv 2 \pmod{4}$ .

As consequences of their results about super edge-magic labelings (see §5.2) Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [767] have the following corollaries: if m and n are odd with  $m \ge 1$  and  $n \ge 3$ , then  $mC_n$  is felicitous;  $3C_n$  is felicitous if and only if  $n \not\equiv 2 \pmod{4}$ ; and  $C_5 \cup P_n$  is felicitous for all n.

For a graph G with q edges Shainy and Balaji [2325] call a one-to-one function f from V(G) to  $\{0, 1, 2, \ldots, 2q-1\}$  a *even felicitous* if the edge labels generated by  $(f(r) + f(s)) \mod(2q-1)$  for each edge are even and distinct. They proved that stars, bistars, the union two stars, and the union of three stars are even felicitous graph.

In [1697] Manickam, Marudai, and Kala prove the following graphs are felicitous: the one-point union of m copies of  $C_n$  if  $mn \equiv 1, 3 \mod 4$ ; the one-point union of m copies of  $C_4$ ;  $mC_n$  if  $mn \equiv 1, 3 \pmod{4}$ ; and  $mC_4$ . These results partially answer questions raised by Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima in [763] and [767].

Chang, Hsu, and Rogers [558] have given a sequential counterpart to felicitous labelings. They call a graph with q edges strongly *c*-elegant if the vertex labels are from  $\{0, 1, \ldots, q\}$  and the edge labels induced by addition are  $\{c, c+1, \ldots, c+q-1\}$ . (A strongly 1-elegant labeling has also been called a *consecutive* labeling.) Notice that every strongly

*c*-elegant graph is felicitous and that strongly *c*-elegant is the same as (c, 1)-arithmetic in the case where the vertex labels are from  $\{0, 1, \ldots, q\}$ . Chang et al. [558] have shown:  $K_n$  is strongly 1-elegant if and only if n = 2, 3, 4;  $C_n$  is strongly 1-elegant if and only if n = 3; and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [2320] has proved that  $K_{m,n}$  is strongly *c*-elegant for a particular value of *c* and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

Seoud and Elsakhawi [2234] have shown:  $K_{m,n}$   $(m \leq n)$  with an edge joining two vertices of the same partite set is strongly *c*-elegant for  $c = 1, 3, 5, \ldots, 2n + 2$ ;  $K_{1,m,n}$  is strongly *c*-elegant for  $c = 1, 3, 5, \ldots, m + n + 1$  when  $m \neq n$ ;  $K_{1,1,m,m}$  is strongly *c*-elegant for  $c = 1, 3, 5, \ldots, 2m + 1$ ;  $P_n + \overline{K_m}$  is strongly  $\lfloor n/2 \rfloor$ -elegant;  $C_m + \overline{K_n}$  is strongly *c*-elegant for odd *m* and all *n* for  $c = (m-1)/2, (m-1)/2 + 2, \ldots, 2m$  when (m-1)/2 is even and for  $c = (m-1)/2, (m-1)/2 + 2, \ldots, 2m - (m-1)/2$  when (m-1)/2 is odd; ladders  $L_{2k+1}$  (k > 1) are strongly (k+1)-elegant; and B(3, 2, m) and B(4, 3, m) (see §2.4 for notation) are strongly 1-elegant and strongly 3-elegant for all *m*; the composition  $P_n[P_2]$  (see §2.3 for the definition) is strongly *c*-elegant for  $c = 1, 3, 5, \ldots, 5n - 6$  when *n* is odd and for  $c = 1, 3, 5, \ldots, 5n - 5$  when *n* is even;  $P_n$  is strongly  $\lfloor n/2 \rfloor$ -elegant;  $P_n^2$  is strongly *c*-elegant for  $c = 1, 3, 5, \ldots, 6k - 1$  when  $n = 4k; c = 1, 3, 5, \ldots, 6k + 1$  when  $n = 4k + 1; c = 1, 3, 5, \ldots, 6k + 3$  when  $n = 4k + 2; c = 1, 3, 5, \ldots, 6k + 5$  when n = 4k + 3.

In [381] Barrientos and Minion study a technique to transform an  $\alpha$ -labeling of some snakes whose cells are squares into a felicitous labeling and the felicitous labeling into a harmonious labeling. They prove that all quadrilateral snakes, all snake polyominoes, and all hybrid quadrilateral snakes are both, felicitous (see §4.5) and harmonious. A hybrid quadrilateral snake is a snake obtained with n copies of  $C_4$  where the *i*th copy of  $C_4$  is attached to the (i+1)th copy via vertex amalgamation or edge amalgamation. Barrientos and Minion [381] prove that all hybrid quadrilateral snakes admit  $\alpha$ -labelings.

### 4.6 Odd Harmonious and Even Harmonious Labelings

Liang and Bai [1588] introduced odd harmonious labelings by defining a function f to be an odd harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to 2q-1 such that the induced mapping  $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to 2q-1 is a bijection. A function f is said to be a strongly odd harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to q such that the induced mapping  $f^*(uv) = f(u) + f(v)$  from the edges of G to the odd integers between 1 to 2q-1 is a bijection. Liang and Bai [1588] have shown the following: odd harmonious graphs are bipartite; if a (p,q)-graph is odd harmonious, then  $2\sqrt{q} \le p \le 2q-1$ ; if a (p,q)-graph with degree sequence  $(d_1, d_2, \ldots, d_p)$  is odd harmonious, then  $gcd(d_1, d_2, \ldots, d_p)$  divides  $q^2$ ;  $P_n$  (n > 1) is odd harmonious and strongly odd harmonious;  $C_n$  is odd harmonious if and only if  $n \equiv 0 \mod 4$ ;  $K_n$  is odd harmonious if and only if n = 2;  $K_{n_1,n_2,\ldots,n_k}$  is odd harmonious if and only if k = 2;  $K_n^t$  is odd harmonious if and only if n = 2;  $P_m \times P_n$  is odd harmonious; the tadpole graph obtained by identifying the endpoint of a path with a vertex of an *n*-cycle is odd harmonious if  $n \equiv 0 \mod 4$ ; the graph obtained by appending two or more pendent edges to each vertex of  $C_{4n}$  is odd harmonious; the graph obtained by subdividing every edge of the cycle of a wheel (gear graphs) is odd harmonious; the graph obtained by appending an edge to each vertex of a strongly odd harmonious graph is odd harmonious; and caterpillars and lobsters are odd harmonious. They conjecture that every tree is odd harmonious.

Liang and Bai [1588] also showned that the  $kC_4$ -snake graph is an odd harmonious graph. Abdel-Aal [3] generalize this result by showing that the  $kC_n$ -snake with string  $1, 1, \ldots, 1$  for  $n \equiv 0 \pmod{4}$  are odd harmonious. He also showed that the  $kC_4$  snake with m pendent edges is odd harmonius and that all subdivisions of 2m-triangular snakes are odd harmonious.

Abdel-Aal [3] proved that a necessary condition for odd harmonious Eulerian graphs with q edges is  $q \equiv 0 \pmod{4}$  and that the following graphs are odd harmonious:  $C_m \times P_n \ (n \geq 2, m \equiv 0 \pmod{4}; C_{4m} \odot C_4; S_n \odot \overline{K_m};$  two copies of an even *n*-cycle sharing a common edge is an odd harmonious graph when  $n \equiv 0 \pmod{4}$ ; two copies of an even *n*-cycle sharing a common vertex is odd harmonious when  $n \equiv 0 \pmod{4}$ ; and graphs obtained from  $K_{2,n} \ (n \geq 2)$  by adding r pendent edges to one of the two vertices of degree n and s pendent edges to the other vertex of degree n.

Vaidya and Shah [2663] prove that the shadow graphs (see §3.8 for the definition) of path  $P_n$  and star  $K_{1n}$  are odd harmonious. They also show that the splitting graphs (see §2.7 for the definition) ) of path  $P_n$  and star  $K_{1,n}$  are odd harmonious. In [2664] Vaidya and Shah proved the following graphs are odd harmonious: the shadow graph and the splitting graph of bistar  $B_{n,n}$ ; the arbitrary supersubdivision of paths; graphs obtained by joining two copies of cycle  $C_n$  for  $n \equiv 0 \pmod{4}$  by an edge; and the graphs  $H_{n,n}$ , where  $V(H_{n,n}) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, v_n\}$  and  $E(H_{n,n}) = \{v_i u_j : 1 \le i \le n, n-i+1 \le i \le n\}$  $j \leq n$ . In [2844] Yan proves that  $P_m \times P_n$  is strongly odd harmonious. Koppendrayer [1406] has proved that every graph with an  $\alpha$ -labeling is odd harmonious. Li, Li, and Yan [1576] proved that  $K_{m,n}$  is odd strongly harmonious. Saputri, Sugeng, and Fronček [2188] proved that the graph obtained by joining  $C_n$  to  $C_k$  by an edge (dumbbell graph  $D_{n,k,2}$ ) is odd harmonious for  $n \equiv k \equiv 0 \pmod{4}$  and  $n \equiv k \equiv 2 \pmod{4}$ , and  $C_n \times P_m$  is odd harmonious if and only if  $n \equiv 0 \pmod{4}$ . They also observe that  $C_n \odot K_1$  with  $n \equiv 0 \pmod{4}$ 4) is odd harmonious. Firmansah and Tasri [770] obtained a new class odd harmonious graphs constructed from the edge amalgamation of double quadrilateral graphs that have odd harmonious labelings.

Jeyanthi [1193] proved that the shadow and splitting graphs of  $K_{2,n}$ ,  $C_{4n}$ , the double quadrilateral snakes DQ(n)  $(n \ge 2)$ , and the graph  $H_{n,n}$  with vertex set  $V(H_{n,n}) =$  $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$  and the edge set  $E(H_{n,n}) = \{v_i u_j : 1 \le i \le n, n - i + 1 \le j \le n\}$  are odd harmonious. Jeyanthi and Philo [1193] proved that the shadow graphs  $D_2(K_{2,n})$  and  $D_2(H_{n,n})$  are odd harmonious and the splitting of graphs of  $K_{2,n}$  and  $H_{n,n}$ are odd harmonious. They also showed that the shadow graph  $D_2(C_n)$  is odd harmonious if  $n \equiv 0 \pmod{4}$ , the splitting of  $C_n$  is odd harmonious if  $n \equiv 0 \pmod{4}$ , and the double quadrilateral snake DQ(n) is odd harmonious for  $n \ge 2$ . In [1197] Jeyanthi and Philo prove that super subdivision of cycles, ladders,  $C_{4n} \oplus K_{1,m}$ , and uniform fire crackers are odd harmonious graphs. Jeyanthi and Philo [1202] proved that the graph  $P_{n-1}(1, 2, 3, \ldots, n)$  obtained from a path of n vertices  $v_1, v_2, \ldots, v_{n-1}$  by appending a path of length n - i at each  $v_i$  and certain one point unions of the end points of paths are odd harmonious.

For a  $T_p$ -tree T with vertices  $v_1, v_2, \ldots, v_n$ , the graph  $T \circ P_m$  is obtained from T and n copies of  $P_m$  by identifying a pendant vertex of *i*th copy of  $P_m$  with vertex  $v_i$  of T. For  $C_n$  with consecutive vertices  $v_1, v_2, \ldots, v_n$ , the graph  $C_n \circ P_m$  is obtained from  $C_n$  and n copies of  $P_m$  by identifying a pendant vertex of *i*th copy of  $P_m$  with vertex  $v_i$  of  $C_n$ . Jeyanthi and Philo [1204] proved that  $T_p$ -trees,  $T \circ P_m$ ,  $T \circ 2P_m$ , regular bamboo trees,  $C_n \circ P_m$ ,  $C_n \circ 2P_m$ , and subdivided grid graphs are odd harmonious.

Recall a subdivided shell graph is obtained by subdividing the edges in the path of the shell graph. Let  $G_1, G_2, \ldots, G_n$  be n subdivided shell graphs of any order. The graph SSG(n) is obtained by adding an edge to apexes of  $G_i$  and  $G_{i+1}, i = 1, 2, \ldots, (n-1)$ . Jeba Jesintha and Ezhilarasi Hilda [1126] proved that the subdivided shell graph and SSG(2) are odd harmonious.

The following definitions are taken from |1203|. The *m*-shadow graph  $D_m(G)$  of a connected graph G is constructed by taking m-copies of G,  $G_1, G_2, G_3, \ldots, G_m$ , and joining each vertex u in  $G_i$  to the neighbors of the corresponding vertex v in  $G_j$ ,  $1 \leq j \leq m$ . The *m*-splitting graph  $Spl_m(G)$  of a graph G is obtained by adding to each vertex v of G m new vertices,  $v^1, v^2, \ldots, v^m$  such that  $v^i, 1 \leq i \leq m$ , is adjacent to every vertex that is adjacent to v in G. Note that the 2-shadow graph is the shadow graph  $D_2(G)$ and the 1-splitting graph is splitting graph. The *m*-mirror graph  $M_m(G)$  is defined as the disjoint union of m copies of  $G, G_1, G_2, \ldots, G_m$ , together with additional edges joining each vertex of  $G_i$  to its corresponding vertex in  $G_{i+1}$ ,  $1 \leq i \leq m-1$ . The graph  $W_{m,n}$ is obtained from the gear graph arising from the wheel  $W_n$  as follows: Join the vertices  $v_i$  and  $v_{i+2}$  with the new vertices  $v_{i+1}^j$  for  $1 \leq j \leq m$  and  $2 \leq i \leq n-2$  and join  $v_n$ and  $v_2$  with  $v_{2i-1}$ . The graph  $K_{2,n}(r,s)$  is obtained from  $K_{2,n}$   $(n \ge 2)$  by adding r and s pendent edges to the two vertices of degree n. The graph  $G = \langle C_n : K_{2,m} : C_r \rangle$  is obtained from  $K_{2,m}$  with the partite set  $\{u, v\}$  by identifying the vertex u with a vertex of  $C_n$  and the vertex v with a vertex of  $C_r$ . Let  $P_n$  be a path on n vertices denoted by  $(1,1),(1,2),\ldots,(1,n)$  and with n-1 edges denoted by  $e_1,e_2,\ldots,e_{n-1}$  where  $e_i$  is the edge joining the vertices (1, i) and (1, i + 1). The step ladder graph  $S(T_n)$  has  $(n^2 + 3n - 2)/2$ vertices denoted by  $(1, 1), (1, 2), \dots, (1, n), (2, 1), (2, 2), \dots, (2, n), (3, 1), (3, 2), \dots, (3, n - 2)$ 1),  $(4, 1), \ldots, (4, n-2), \ldots, (n, 1), (n, 2)$  and  $n^2 + n + 2$  edges. In any ordered pair (i, j), i denotes the row (counted from bottom to top) and j denotes the column(from left to right) in which the vertex occurs.

The cocktail party graph,  $H_{m,n}$   $(m, n \geq 2)$ , is the graph with a vertex set  $V = \{v_1, v_2, \ldots, v_{mn}\}$  partitioned into n independent sets  $V = \{I_1, I_2, \ldots, I_n\}$  each of size m such that  $v_i v_j \in E$  for all  $i, j \in \{1, 2, \ldots, mn\}$  where  $i \in I_p, j \in I_q, p \neq q$ .

Jeyanthi and Philo [1196] proved that following graphs are odd harmonious:  $D_m(P_n)$ for all  $m, n \geq 2$ ;  $Spl_m(P_n)$  for  $m, n \geq 2$ ;  $D_m(H_{n,n})$  for all  $m \geq 2$  and  $n \geq 1$ ;  $Spl_m(H_{n,n})$ for all  $m \geq 2$  and  $n \geq 1$ ;  $D_m(K_{r,s})$  for all  $r, s \geq 1$ ;  $Spl_m(K_{r,s})$  for all  $m \geq 2$  and  $r, s \geq 1$ ;  $D_m(P_n \oplus \overline{K_2})$  for all  $m, n \ge 2$ ;  $Spl_m(P_n \oplus \overline{K_2}), m, n \ge 2$ ; and  $Spl_m(C_n)$  if and only if  $n \equiv 0 \pmod{4}$ .

Jeyanthi and Philo [1203] proved that following graphs are odd harmonious:  $\overline{W}_{m,n}$  for  $n \equiv 0 \pmod{4}$ ,  $m \geq 1$ ;  $D_m(P_n \odot K_1)$  (the authors use the notion  $C_{bn}$  for the comb  $P_n \odot K_1$ ) for all  $m \geq 2$  and  $n \geq 1$ ;  $Spl_m(K_{2,n}(r,s))$ ;  $\langle C_n : K_{2,m} : C_r \rangle$  for  $n, r \equiv 0 \pmod{4}$  and  $m \geq 2$ ; and the graphs obtained by arranging vertices into a finite number of rows with *i* vertices in the *i*th row and in every row the *j*th vertex in that row is joined to the *j*th vertex and j + 1st vertex of the next row (a *pyramid*) for  $n \geq 2$ . They also prove that if *G* is a strongly odd harmonious tree, then  $M_m(G)$  is odd harmonious.

Let  $P_{2n}$  be a path of length 2n-1 with 2n vertices, denoted by  $(1,1), (1,2), \ldots, (1,2n)$ and with 2n-1 edges, denoted by  $e_1, e_2, \ldots, e_{2n-1}$  where  $e_i$  is the edge joining the vertices (1, i) and (1, i+1). On each edge  $e_i$  for  $i = n+1, n+2, \ldots, 2n-1$ , we erect a ladder with 2n+1-i steps including the edge  $e_i$ . The double sided step ladder graph  $2S(T_{2\times n})$  has vertices denoted by  $(1, 1), (1, 2), \ldots, (1, 2n), (2, 1), (2, 2), \ldots, (2, 2n), (3, 2), (3, 3), \ldots, (3, 2n - 2n), (3, 2n), (3,$  $1),(4,3),(4,4),\ldots,(4,2n-2),\ldots,(n+1,n),(n+1,n+1)$ . In any ordered pair (i,j),i denotes the row (counted from bottom to top) and j denotes the column (from left to right) in which the vertex occurs. Jeyanthi and Philo [1201] proved that the path union of t copies of  $S(T_n)$ , the double sided step ladder  $2S(T_{2\times n})$ , the path union of t copies of  $2S(T_{2\times n})$ ,  $S(t.C_{bn})$ ,  $S(t.C_4)$ ,  $C_4^t$ ,  $C_6^t$ , and  $C_8^t$  are odd harmonious graphs. Jeyanthi and Philo [1198] proved that path union of r copies of  $K_{m,n}$ , the path union of r copies of  $K_{m_i,n_i}$ ,  $1 \leq i \leq r$ ,  $K_{m,n}^t$ ,  $K_{(m_1,n_1),(m_2,n_2),\dots,(m_t,n_t)}^t$ , the join sum of graph  $\langle K_{m,n}; K_{m,n}; \ldots, K_{m,n} | (t \text{ copies }), \langle K_{m_1,n_1}; K_{m_2,n_2}; \ldots, K_{m_t,n_t} \rangle$ , the circle formation of r copies of  $K_{m,n}$  when  $r \equiv 0 \pmod{4}$ ,  $S(t,K_{m,n})$  and  $P_n^t(t,n,K_{p,q})$  are odd harmonious graphs. Jeyanthi and Philo [1200] proved that the subdivided shell graphs, disjoint union of two subdivided shell graphs, subdivided shell flower graphs, and subdivided uniform shell bow graphs are odd harmonious. Jeyanthi, Philo, and Youssef [1205] proved that the path union of t copies of  $P_m \times P_n$ , the path union of t copies of  $P_{m_i} \times P_{n_i}$  where  $1 \leq i \leq t$ , the vertex union of t copies of  $P_m \times P_n$ , the vertex union of t different copies of  $P_{m_i} \times P_{n_i}$  where  $1 \leq i \leq t$ , the one point union of path of  $P_n^t(t.n.P_m \times P_m)$ , and the super subdivision of grid graph  $P_m \times P_n$  are odd harmonious graphs.

Recall from Section 2.7 that for even n > 2 a plus graph of size n (denoted by  $Pl_n$ ) is the graph obtained by starting with paths  $P_2, P_4, \ldots, P_{n-2}, P_n, P_n, P_{n-2}, \ldots, P_4, P_2$  arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. Jeyanthi [1195] proved that following graphs are odd harmonious:  $Pl_n$  where  $n \equiv 0 \pmod{2}$ ,  $n \neq 2$ ; path unions of finitely many copies of  $Pl_n$  where  $n \equiv 0 \pmod{2}$ ,  $n \neq 2$ ; open stars of plus graphs  $S(t.Pl_n)$  where  $n \equiv 0 \pmod{2}$ ,  $n \neq 2$  and todd; graphs obtained by joining  $C_m$ ,  $m \equiv 0 \pmod{4}$  and a plus graph  $Pl_n$ ,  $n \equiv 0 \pmod{2}$ ,  $n \neq 2$  with a path of arbitrary length; the graph obtained by replacing all vertices of  $K_{1,t}$ , except the apex vertex, by the path union of n copies of the graph  $Pl_m$ .

In [1197] Jeyanthi and Philo prove that super subdivision of cycles, ladders,  $C_{4n} \oplus K_{1,m}$ , and uniform fire crackers are odd harmonious graphs. They also proved the (m, n)firecracker graph obtained by the concatenation of m n-stars by linking one leaf from each is odd harmonius; the arbitrary super subdivision of cycles  $C_m$  are odd harmonious; and the super subdivision of ladders are odd harmonious. Jeyanthi and Philo [1203] proved that the *m*-mirror graph  $M_m(G)$   $(m \ge 2)$ , *m*-splitting graph of  $K_{2,n}(r,s)$  (obtained from  $K_{2,n}$ , (n > 2) by adding *r* and *s* (r, s > 1) pendent edges to the two vertices of degree *n*),  $\overline{W_{(m,4n)}}$  obtained from the gear graph of  $W_n$  by joining the vertices  $v_i$  and  $v_{i+2}$  with the new vertices  $v_{i+1}^j$  for  $1 \le j \le m$  and  $2 \le i \le n-2$  and joining  $v_n$  and  $v_2$  with  $v_1^j$  for  $1 \le j \le m$ ,  $\langle C_{4n} : K_{2,m} : C_{4r} \rangle$  obtained from  $K_{2,m}$  with one partite set  $V_1 = \{u, v\}$  and  $C_r$  by identifying the vertex *u* of  $V_1$  with a vertex of  $C_n$  and the other vertex *v* of  $V_1$  with a vertex of  $C_r$ , and the pyramid graph  $PY_n(n \ge 2)$  are odd harmonious graphs. They also proved that *G* is a strongly odd harmonious tree, then  $M_m(G)$  is an odd harmonious.

In [1197] Jeyanthi and Philo modified the notion of odd harmonious by defining an odd harmonious labelings as a function f to be an odd harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to 2q - 1 such that the induced mapping  $f^*(uv) = f(u) + f(v) \mod (2q)$  from the edges of G to the odd integers between 1 to 2q - 1 is a bijection. Using this definition they proved that an m-cycle and an n-cycle sharing a common vertex is an odd harmonious if and only if either both  $m, n \equiv 0 \pmod{4}$  or both  $m, n \equiv 2 \pmod{4}$  and the same holds for an m-cycle and an n-cycle sharing a common edge. They also proved that any two even cycles sharing a common vertex and a common edge are odd harmonious graphs.

Sarasija and Binthiya [2189] say a function f is an even harmonious labeling of a graph G with q edges if  $f: V \to \{0, 1, \ldots, 2q\}$  is injective and the induced function  $f^*: E \to \{0, 2, \ldots, 2(q-1)\}$  defined as  $f^*(uv) = f(u) + f(v) \pmod{2q}$  is bijective. Notice that for an even harmonious labeling of a connected graph all the vertex labels must have the same parity. Moveover, in the case of even harmonious labelings for connected graphs there is no loss of generality to assume that all the vertex labels are even integers and the duplicate vertex is 0. They proved the following graphs are even harmonious: non-trivial paths; complete bipartite graphs; odd cycles; bistars  $B_{m,n}$ ;  $K_2 + \overline{K_n}$ ;  $P_n^2$ ; and the friendship graphs  $F_{2n+1}$ . López, Muntaner-Batle and Rious-Font [1634] proved that every super edge-magic graph (see Section 5.2 for the definition of super edge-magic) with p vertices and q edges where  $q \ge p - 1$  has an even harmonious labeling. In [2884] Youssef provided a necessary condition for some regular graphs to be even harmonious, showed that the disjoint union of two k-sequential graphs is even 2k-sequential under some conditions, and showed that in some cases G is k-sequential implies mG is even 2k-sequential for all positive integer m.

Because 0 and 2q are equal modulo 2q the following retricted form of even harmonious labelings is of interest. A function f is said to be a *properly even harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to 2q-1 and the induced function  $f^*$  from the edges of G to  $\{0, 2, \ldots, 2q-2\}$  defined by  $f^*(xy) = f(x) + f(y) \pmod{2q}$  is bijective. In their definition of properly even harmonious in [817] Gallian and Schoenhard incorrectly required that the vertex labels should be the even integers from 0 to 2q - 2. For connected graphs the two definitions are equivalent but for disconnected graph they are not. They used vertex labels from 0 to 2q - 1 for their results on disconnected graphs.

A graph with a properly even harmonious labeling is said to be *properly even harmo*-

*nious.* Gallian and Schoenhard [817] say a properly even harmonious labeling of a graph with q edges is *strongly even harmonious* if it satisfies the additional condition that for any two adjacent vertices with labels u and v,  $0 < u + v \leq 2q$ .

Jared Bass [419] has observed that for connected graphs any harmonious labeling of a graph with q edges yields an even harmonious labeling by simply multiplying each vertex label by 2 and adding the vertex labels modulo 2q. Thus we know that every connected harmonious graph is an even harmonious graph and every connected graph that is not a tree that has a harmonious labeling also has a properly even harmonious labeling. Conversely, a properly even harmonious labeling of a connected graph with qedges (assuming that the vertex labels are even) yields a harmonious labeling of the graph by dividing each vertex label by 2 and adding the vertex labels modulo q.

Gallian and Schoenhard [817] proved the following: wheels  $W_n$  and helms  $H_n$  are properly even harmonious when n is odd;  $nP_2$  is even harmonious for n odd;  $nP_2$  is properly even harmonious if and only if n is even;  $K_n$  is even harmonious if and only if  $n \leq 4$ ;  $C_{2n}$  is not even harmonious when n is odd;  $C_n \cup P_3$  is properly even harmonious when odd  $n \ge 3$ ;  $C_4 \cup P_n$  is even harmonious when  $n \ge 2$ ;  $C_4 \cup F_n$  is even harmonious when  $n \geq 2$ ;  $S_m \cup P_n$  is even harmonious when  $n \geq 2$ ;  $K_4 \cup S_n$  is properly even harmonious;  $P_m \cup P_n$  is properly even harmonious for all  $m \ge 2$  and  $n \ge 2$ ;  $C_3 \cup P_n^2$  is even harmonious when  $n \ge 2$ ;  $C_4 \cup P_n^2$  is even harmonious when  $n \ge 2$ ; the disjoint union of two or three stars where each star has at least two edges and one has at least three edges is properly even harmonious;  $P_m^2 \cup P_n$  is even harmonious for  $m \ge 2$  and  $2 \le n < 4m-5$ ; the one-point union of two complete graphs each with at least 3 vertices is not even harmonious;  $S_m \cup P_n$ is strongly even harmonious if  $n \ge 2$ ; and  $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$  is strongly even harmonious for  $n_1 \ge n_2 \ge \cdots \ge n_t$  and  $t < \frac{n_1}{2} + 2$ . They conjecture that  $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$  is strongly even harmonious if at least one star has more than 2 edges. They also note that  $C_4, C_8, C_{12}, C_{16}, C_{20}, C_{24}$  are even harmonious and conjecture that  $C_{4n}$  is even harmonious for all n. This conjecture was proved by Youssef [2882] who also proved that if a connected even harmonious graph with q edges where q is even and each vertex has degree divisible by  $2^k$   $(k \ge 1)$ , then q is divisible by  $2^{k+1}$ . As corollary of the latter he gets that  $C_{4n+2}^2$  is not even harmonious. Hall, Hillesheim, Kocina, and Schmit [948] proved that  $nC_{2m+1}$  is properly even harmonious for all n and m.

In [818] and [819] Gallian and Stewart investigated properly even harmonious labelings of unions of graphs. They use  $P_m^{+t}$  to denote the graph obtained from the path  $P_m$  by appending t edges to an endpoint;  $Cat_m^{+t}$  to denote a caterpillar of path length m with t pendent edges; and  $C_m^{+t}$  to denote an m-cycle with t pendent edges. They proved the following graphs are properly even harmonious:  $nP_m$  if n is even and  $m \ge 2$ ;  $P_n \cup K_{m,2}$  for n odd and n > 1, m > 1;  $P_n \cup S_{m_1} \cup S_{m_2}$  for n > 2 and  $m_1 + m_2$  is odd;  $C_n \cup S_{m_1} \cup S_{m_2}$  for n odd and  $m_1$ ,  $m_2 > 3$ ;  $P_m^{+t} \cup P_n^{+s}$ ; the union of any number of caterpillars;  $C_m \cup Cat_n^{+t}$ for m > 1 odd, n > 1;  $C_4 \cup Cat_m^{+t}$ ; the union of  $C_4$  and a hairy cycle;  $K_4 \cup C_m^{+n}$  for some cases;  $W_4 \cup C_m^{+n}$  for some cases;  $C_4 \cup (P_n + \overline{K_2})$  for n > 1;  $K_4 \cup (P_n + \overline{K_m})$ for  $n \equiv 1, 2 \pmod{4}$ ;  $C_3 \cup (P_n + \overline{K_m})$  for  $n \equiv 1, 2 \pmod{4}$ ;  $W_4 \cup (P_n + \overline{K_m})$  for  $n \equiv$  $1, 2 \pmod{4}$ ;  $W_4 \cup P_n$  for  $n \equiv 1, 2 \pmod{4}$ ;  $K_4 \cup P_m^2 \cup \cdots \cup P_{m_n}^2$  for  $m_i > 2$ ,  $n \ge 1$ ;  $K_4 \cup P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$  for  $m_i > 2$ ,  $n \ge 1$ ;  $W_4 \cup P_m^2 \cup \cdots \cup P_{m_n}^2$  for  $m_i > 2$ ,  $n \ge 1$ ;  $C_m \cup P_n^2$  for  $m \equiv 3 \pmod{4}$  and n > 1; and  $2P_m \cup 2P_n$ . They also prove that  $nP_3$  is even harmonious if n > 1 is odd and  $P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$  is strongly even harmonious for  $m > 2, n \ge 1$ .

Gallian and Stewart [820] call an injective labeling f of a graph G with q edges even 2a-sequential if the vertex labels are from  $\{0, 1, \ldots, 2q - 1\}$  and the edge labels induced by f(u) + f(v) for each edge uv are  $2a, 2a + 2, \ldots, 2a + 2q - 2$ . When G is a tree, the allowable vertex labels are  $0, 1, \ldots, 2q$ . For connected a-sequential graphs, a connected 2a-sequential graph can be obtained by multiplying all the vertex labels by 2. Notice that the vertex labels in resulting graph belong to  $\{0, 2, \ldots, 2q - 2\}$  (or  $\{0, 2, \ldots, 2q\}$  for trees) and the edges labels are from 2a to 2a + 2q - 2. Moreover, a connected a-sequential graph can be obtained from a connected even 2a-sequential graph with even vertex labels by dividing all the vertex labels by 2. Likewise, a 2a-sequential labeling of a connected graph with odd vertex labels induces an a-sequential labeling of the graph by subtracting 1 from each vertex label and dividing by 2. Thus for connected graphs, a-sequential is equivalent to 2a-sequential. They prove that if G is even 2a-sequential the following graphs are properly even harmonious:  $G \cup P_m^2$  for m > 2,  $G \cup P_n$  for n > 1,  $n \equiv 1, 2 \pmod{4}$ ,  $G \cup C_m^{+t}$  for some cases,  $G \cup Cat_m^{+n}$  for m > 1, and  $G \cup W_{2n+1}$ .

For n and k odd and m, n, k, t > 1, Mbianda and Gallian (see [1724]) proved the following graphs have properly even harmonious labelings:  $mP_3$  for even m;  $2P_m \cup 2P_n \cup S_t$ ;  $2P_m \cup 2P_n \cup P_k$ ;  $2P_m \cup 2P_n \cup C_k$ ;  $2P_m \cup 2P_n \cup C_4$ ;  $2P_m \cup 2P_n \cup 2K_4$ ;  $2P_m \cup 2P_n \cup 2K_4$ ;  $P_m \cup 2P_n \cup 2K_4$ ;  $P_n \cup 2P_n \cup 2K_4$ ;  $F_n \cup W_4$ ;  $F_n \cup 2W_4$ ;  $W_n \cup K_4$ ;  $W_n \cup 2K_4$ ;  $W_n \cup W_4$ ;  $W_n \cup 2W_4$ ;  $(C_n + K_1) \cup K_4$  ( $(C_n + K_1)$ ) is the *n*-cone);  $(C_n + K_1) \cup W_4$ ;  $(C_n + K_1) \cup 2K_4$ ;  $(C_n + K_1) \cup 2W_4$ ; and  $(C_n + \overline{K_2}) \cup K_4$  ( $(C_n + \overline{K_2})$ ) is the double cone). Gallian [814] proved the following graphs have properly even harmonious labelings (in all cases m, n > 1):  $mP_n$  for m even;  $2P_m \cup 2P_n \cup 2C_3$ ;  $2P_m \cup 2P_n \cup 2C_4$ ;  $2P_m \cup 2P_n \cup C_3 \cup C_4$ ;  $F_n \cup P_4$ ;  $F_n \cup 2P_4$ ;  $F_n \cup C_4$ ; and  $F_n \cup 2C_4$ .

Binthiya and Sarasija [469] prove the following graphs are even harmonious:  $C_n \odot mK_1$  (n odd),  $P_n \odot mK_1$  (n > 1 odd),  $C_{2n} @K_2$ ,  $P_n$  (n even) with n-1 copies of  $m\overline{K_1}$ , the shadow graph  $D_2(K_{1,n})$ , the splitting graph  $spl(K_{1,n})$ , and the graph obtained from the  $P_n$  (n even) with n-1 copies of  $\overline{K_m}$  incident with first n-1 vertices of  $P_n$ . Vargheese and Arun [2705] prove that the triangular books, the disjoint union of two triangular book graphs, total graphs  $T(P_n)$ , the disjoint union of  $T(P_n)$  and a triangular book, and the graph obtained by joining the centers of two disjoint copies of  $K_{1,n}$  to an isolated vertex are even harmonious.

# 5 Magic-type Labelings

### 5.1 Magic Labelings

Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedláček [2197] in 1963.<sup>1</sup> Responding to a problem raised by Sedláček, Stewart [2471] and [2472] studied various ways to label the edges of a graph in the mid 1960s. Stewart calls a connected graph *semi-magic* if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. (Berge [440] used the term "regularisable" for this notion.) A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic* labeling. Stewart calls a magic labeling *supermagic* if the set of edge labels consists of consecutive positive integers. The classic concept of an  $n \times n$  magic square in number theory corresponds to a supermagic labeling of  $K_{n,n}$ . Stewart [2471] proved the following:  $K_n$  is magic for n = 2 and all  $n \ge 5$ ;  $K_{n,n}$  is magic for all  $n \ge 3$ ; fans  $F_n$  are magic if and only if n is odd and  $n \geq 3$ ; wheels  $W_n$  are magic for  $n \geq 4$ ; and  $W_n$  with one spoke deleted is magic for n = 4 and for  $n \ge 6$ . Stewart [2471] also proved that  $K_{m,n}$  is semi-magic if and only if m = n. In [2472] Stewart proved that  $K_n$  is supermagic for  $n \ge 5$  if and only if n > 5 and  $n \not\equiv 0 \pmod{4}$ . Sedláček [2198] showed that Möbius ladders  $M_n$  (see §2.3) for the definition) are supermagic when  $n \geq 3$  and n is odd and that  $C_n \times P_2$  is magic, but not supermagic, when  $n \ge 4$  and n is even. Shiu, Lam, and Lee [2345] have proved: the composition of  $C_m$  and  $\overline{K}_n$  (see §2.3 for the definition) is supermagic when  $m \geq 3$ and  $n \ge 2$ ; the complete *m*-partite graph  $K_{n,n,\dots,n}$  is supermagic when  $n \ge 3$ , m > 5 and  $m \not\equiv 0 \pmod{4}$ ; and if G is an r-regular supermagic graph, then so is the composition of G and  $\overline{K}_n$  for  $n \geq 3$ . Ho and Lee [1012] showed that the composition of  $K_m$  and  $\overline{K}_n$  is supermagic for m = 3 or 5 and n = 2 or n odd. Bača, Holländer, and Lih [274] have found two families of 4-regular supermagic graphs. Shiu, Lam, and Cheng [2342] proved that for  $n \ge 2$ ,  $mK_{n,n}$  is supermagic if and only if n is even or both m and n are odd. Ivančo [1078] gave a characterization of all supermagic regular complete multipartite graphs. He proved that  $Q_n$  is supermagic if and only if n = 1 or n is even and greater than 2 and that  $C_n \times C_n$  and  $C_{2m} \times C_{2n}$  are supermagic. He conjectures that  $C_m \times C_n$  is supermagic for all m and n. Trenklér [2597] has proved that a connected magic graph with p vertices and q edges other than  $P_2$  exits if and only if  $5p/4 < q \le p(p-1)/2$ . In [2531] Sun, Guan, and Lee give an efficient algorithm for finding a magic labeling of a graph. In [2810] Wen, Lee, and Sun show how to construct a supermagic multigraph from a given graph G by adding extra edges to G.

In [1423] Kovář provides a general technique for constructing supermagic labelings of copies of certain kinds of regular supermagic graphs. In particular, he proves: if G is a supermagic r-regular graph  $(r \ge 3)$  with a proper edge r coloring, then nG is supermagic when r is even and supermagic when r and n are odd; if G is a supermagic r-regular graph with m vertices and has a proper edge r coloring and H is a supermagic s-regular

<sup>&</sup>lt;sup>1</sup>A comprehensive expository treatment of magic labelings is given by Bača, Miller, Ryan, and Semaničová-Feňovčíkováin [304].

graph with n vertices and has a proper edge s coloring, then  $G \times H$  is supermagic when r is even or n is odd and is supermagic when s or m is odd.

Kovář, Kravčenko, Krbeček, and Silber [1425] affirmatively answered a question by Madaras about existence of supermagic graphs with arbitrarily many different degrees. Their construction provided graphs with all degrees even. They asked if there exists a supermagic graph with d different odd degrees for any positive integer d. This question was answered affirmatively by Fronček and Qiu with a construction based on the use of 3-dimensional magic rectangles.

In [689] Drajnová, Ivančo, and Semaničová proved that the maximal number of edges in a supermagic graph of order n is 8 for n = 5 and  $\frac{n(n-1)}{2}$  for  $6 \le n \not\equiv 0 \pmod{4}$ , and  $\frac{n(n-1)}{2}-1$  for  $8 \le n \equiv 0 \pmod{4}$ . They also establish some bounds for the minimal number of edges in a supermagic graph of order n. Ivančo, and Semaničová [1088] proved that every 3-regular triangle-free supermagic graph has an edge such that the graph obtained by contracting that edge is also supermagic and the graph obtained by contracting one of the edges joining the two n-cycles of  $C_n \times K_2$   $(n \ge 3)$  is supermagic.

Ivančo [1080] proved: the complement of a *d*-regular bipartite graph of order 8k is supermagic if and only if *d* is odd; the complement of a *d*-regular bipartite graph of order 2n where *n* is odd and *d* is even is supermagic if and only if  $(n, d) \neq (3, 2)$ ; if  $G_1$  and  $G_2$ are disjoint *d*-regular Hamiltonian graphs of odd order and  $d \geq 4$  and even, then the join  $G_1 \oplus G_2$  is supermagic; and if  $G_1$  is *d*-regular Hamiltonian graph of odd order *n*,  $G_2$  is d-2-regular Hamiltonian graph of order *n* and  $4 \leq d \equiv 0 \pmod{4}$ , then the join  $G_1 \oplus G_2$ is supermagic.

For  $k \geq 2$  and graphs G and H, the graph  $G \odot^k H$  defined as  $(G \odot^{k-1} H) \odot H$  (where  $G \odot^1 H = G \odot H$ ) is called the *k*-multilevel corona of G with H. Marbun and Salman [1699] proved  $(W_n \odot^{k-1}) \odot C_n$  is  $W_n$ -edge magic.

In [452] Bezegová and Ivančo [454] extended the notion of supermagic regular graphs by defining a graph to be *degree-magic* if the edges can be labeled with  $\{1, 2, \ldots, |E(G)|\}$ such that the sum of the labels of the edges incident with any vertex v is equal to  $(1 + |E(G)|)/\deg(v)$ . They used this notion to give some constructions of supermagic graphs and proved that for any graph G there is a supermagic regular graph which contains an induced subgraph isomorphic to G. In [454] they gave a characterization of complete tripartite degree-magic graphs and in [455] they provided some bounds on the number of edges in degree-magic graphs. They say a graph G is *conservative* if it admits an orientation and a labeling of the edges by  $\{1, 2, \ldots, |E(G)|\}$  such that at each vertex the sum of the labels on the incoming edges is equal to the sum of the labels on the outgoing edges. In [453] Bezegová and Ivančo introduced some constructions of degree-magic labelings for a large family of graphs using conservative graphs. Using a connection between degree-magic labelings and supermagic labelings they also constructed supermagic labelings for the disjoint union of some regular non-isomorphic graphs. Among their results are: If G is a  $\delta$ -regular graph where  $\delta$  is even and at least 6, and each component of G is a complete multipartite graph of even size, then G is a supermagic graph; for any  $\delta$ -regular supermagic graph H, the union of disjoint graphs H and G is supermagic; if G is a  $\delta$ -regular graph with  $\delta \equiv 0 \pmod{8}$  and each component is a circulant graph,

then G is a supermagic graph; for any  $\delta$ -regular supermagic graph H, the union of disjoint graphs H and G is a supermagic graph; and that the complement of the union of disjoint cycles  $C_{n_1}, \ldots, C_{n_k}$  is supermagic when  $k \equiv 1 \pmod{4}$  and  $11 \leq n_i \equiv 3 \pmod{8}$ for all  $i = 1, \ldots, k$ . In [1075] Inpoonjal gave necessary and sufficient conditions for the existence of degree-magic labelings of graphs obtained by taking the join and composition of complete tripartite graphs.

Let G' be a copy of a simple graph G and for each vertex  $v_i$  of G let  $u_i$  be the vertex of G' corresponding with  $v_i$ . The *double graph* has vertex set  $V(G) \cup V(G')$  and edge set  $E(G) \cup E(G') \cup \{u_i v_j \mid u_i \in V(G); v_j \in V(G') \text{ and } u_i u_j \in E(G)\}$ . Ivančo [1081] establishes sufficient conditions for generalized double graphs to be degree-magic and constructs supermagic labelings of some graphs generalizing double graphs.

Sedláček [2198] proved that graphs obtained from an odd cycle with consecutive vertices  $u_1, u_2, \ldots, u_m, u_{m+1}, v_m, \ldots, v_1 \ (m \ge 2)$  by joining each  $u_i$  to  $v_i$  and  $v_{i+1}$  and  $u_1$  to  $v_{m+1}, u_m$  to  $v_1$  and  $v_1$  to  $v_{m+1}$  are magic. Trenklér and Vetchý [2600] have shown that if G has order at least 5, then  $G^n$  is magic for all  $n \ge 3$  and  $G^2$  is magic if and only if Gis not  $P_5$  and G does not have a 1-factor whose every edge is incident with an end-vertex of G. Avadayappan, Jeyanthi, and Vasuki [209] have shown that k-sequential trees are magic (see §4.1 for the definition).

Seoud and Abdel Maqsoud [2220] proved that  $K_{1,m,n}$  is magic for all m and n and that  $P_n^2$  is magic for all n. However, Serverino has reported that  $P_n^2$  is not magic for n = 2, 3, and 5 [873]. Jeurissan [1114] characterized magic connected bipartite graphs. Ivančo [1079] proved that bipartite graphs with p > 8 vertices, equal sized partite sets, and minimum degree greater than p are magic. Bača [235] characterizes the structure of magic graphs that are formed by adding edges to a bipartite graph and proves that a regular connected magic graph of degree at least 3 remains magic if an arbitrary edge is deleted. In [2437] Solairaju and Arockiasamy prove that various families of subgraphs of grids  $P_m \times P_n$  are magic. Dayanand and Ahmed [648] investigate super magic properties of several classes of connected and disconnected graphs. They show that there can be arbitrarily large gaps among the possible valences for certain super magic graphs. They also prove that the disjoint union of multiple copies of a super magic linear forest is super magic if the number of copies is odd and that the super magic labeling is complementary edge antimagic as well. The broom  $B_{n,t}$  is a graph obtained by attaching n-t pendent edges to an end point vertex of the path  $P_t$ . Marimuthu, Raja, and Raja Durga [1707] prove that  $B_{n,n-1}$  is E-super vertex magic if and only if  $n \geq 3$  is odd and  $B_{n,t}$  is not *E*-super vertex magic for  $n-2 \ge 2$  and  $t \ge 3$ .

Ponnappan, Nagaraj, and Prabakaran [1937] say a vertex magic labeling f of a graph G(V, E) is an odd vertex magic if f maps V to  $\{1, 3, 5, \ldots, 2|V| - 1\}$  and E to  $\{1, 2, 3, 4, \ldots, |V| + |E|\} - \{1, 3, 5, \ldots, 2|V| - 1\}$  if  $|E| \ge |V| - 1$ ) and otherwise f maps E to  $\{2, 4, 6, \ldots, 2|E|\}$  and V to  $\{1, 2, 3, 4, \ldots, |V| + |E|\} - \{2, 4, 6, \ldots, 2|E|\}$ . They prove that  $P_n$   $(n \ge 3)$ ,  $C_n$  and  $mC_3$  are odd vertex magic if and only if n is odd, (3, t)-kites are vertex magic if and only if t is even, and  $C_n \odot K_1$  are not odd vertex for all n.

A triplet  $[H, \phi, t]$  is called a *supermagic frame* of G if  $\phi$  is a homomorphism of H onto G and  $t: E(H) \to \{1, 2, \ldots, |E(H)|\}$  is an injective mapping such that the sum of t(uw)

over all  $u \in \phi^{-1}(v)$  is independent of the vertex  $v \in V(G)$ . In 2000, Ivančo proved that if there is a supermagic frame of a graph G, then G is supermagic. Singhun, Boonklurb, and Charnsamorn [2406] construct a supermagic frame of  $m \ge 2$  copies of the Cartesian product of cycles and show that m copies of the Cartesian product of cycles is supermagic.

A prime-magic labeling is a magic labeling for which every label is a prime. Sedláček [2198] proved that the smallest magic constant for prime-magic labeling of  $K_{3,3}$  is 53 while Bača and Holländer [270] showed that the smallest magic constant for a prime-magic labeling of  $K_{4,4}$  is 114. Letting  $\sigma_n$  be the smallest natural number such that  $n\sigma_n$  is equal to the sum of  $n^2$  distinct prime numbers we have that the smallest magic constant for a prime-magic labeling of  $K_{n,n}$  is  $\sigma_n$ . Bača and Hollaänder [270] conjecture that for  $n \geq 5$ ,  $K_{n,n}$  has a prime-magic labeling with magic constant  $\sigma_n$ . They proved the conjecture for  $5 \leq n \leq 17$  and confirmed the conjecture for n = 5, 6 and 7.

Characterizations of regular magic graphs were given by Doob [688] and necessary and sufficient conditions for a graph to be magic were given in [1114], [1248], and [664]. Some sufficient conditions for a graph to be magic are given in [686], [2596], and [1794]. Bertault, Miller, Pé-Rosés, Feria-Puron, and Vaezpour [450] provided a heuristic algorithm for finding magic labelings for specific families of graphs. The notion of magic graphs was generalized in [687] and [2171].

Let  $m, n, a_1, a_2, \ldots, a_m$  be positive integers where  $1 \leq a_i \leq \lfloor n/2 \rfloor$  and the  $a_i$  are distinct. The circulant graph  $C_n(a_1, a_2, \ldots, a_m)$  is the graph with vertex set  $\{v_1, v_2, \ldots, v_m\}$ and edge set  $\{v_i v_{i+a_j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$  where addition of indices is done modulo n. In [2210] Semaničová characterizes magic circulant graphs and 3-regular supermagic circulant graphs. In particular, if  $G = C_n(a_1, a_2, \ldots, a_m)$  has degree r at least 3 and  $d = \gcd(a_1, n/2)$  then G is magic if and only if r = 3 and  $n/d \equiv 2 \pmod{4}, a_1/d \equiv 1 \pmod{2}$ , or  $r \geq 4$  (a necessary condition for  $C_n(a_1, a_2, \ldots, a_m)$  to be 3-regular is that nis even). In the 3-regular case,  $C_n(a_1, n/2)$  is supermagic if and only  $n/d \equiv 2 \pmod{4}$ ,  $a_1/d \equiv 1 \pmod{2}$  and  $d \equiv 1 \pmod{2}$ . Semaničová also notes that a bipartite graph that is decomposable into an even number of Hamilton cycles is supermagic. As a corollary she obtains that  $C_n(a_1, a_2, \ldots, a_{2k})$  is supermagic in the case that n is even, every  $a_i$  is odd, and  $\gcd(a_{2j-1}, a_{2j}, n) = 1$  for  $i = 1, 2, \ldots, 2k$  and  $j = 1, 2, \ldots, k$ .

Ivančo, Kovář, and Semaničová-Feňovčková [1084] characterize all pairs n and r for which an r-regular supermagic graph of order n exists. They prove that for positive integers r and n with  $n \ge r + 1$  there exists an r-regular supermagic graph of order n if and only if one of the following statements holds: r = 1 and n = 2;  $3 \le r \equiv 1 \pmod{2}$  and  $n \equiv 2 \pmod{4}$ ; and  $4 \le r \equiv 0 \pmod{2}$  and n > 5. The proof of the main result is based on finding supermagic labelings of circulant graphs. The authors construct supermagic labelings of several circulant graphs.

In [1078] Ivančo completely determines the supermagic graphs that are the disjoint unions of complete k-partite graphs where every partite set has the same order.

Trenklér [2598] extended the definition of supermagic graphs to include hypergraphs and proved that the complete k-uniform n-partite hypergraph is supermagic if  $n \neq 2$  or 6 and  $k \geq 2$  (see also [2599]). In [2503] Sugiyama gave a generalized definition of magic graphs, for which any number of digits can be used to label a vertex and edge, and described the construction of such magic graphs and their properties. He determined the minimum and maximum magic sums for regular graphs, including polygons and polyhedrons, and provided techniques for transforming and synthesizing magic graphs using an affine transform.

For connected graphs of size at least 5, Ivančo, Lastivkova, and Semaničová [1085] provide a forbidden subgraph characterization of the line graphs that can be magic. As a corollary they obtain that the line graph of every connected graph with minimum degree at least 3 is magic. They also prove that the line graph of every bipartite regular graph of degree at least 3 is supermagic.

For any non-trivial abelian group A under addition, a graph G is said to be *strong* A-magic if there exists a labeling f of the edges of G with non-zero elements of A such that the vertex labeling  $f^+$  defined as  $f^+(v) = \sum f(uv)$  taken over all edges uv incident at v is a constant, and the constant is same for all possible values of |V(G)|. Stella Arputha Mary, Navaneethakrishnan, and Nagarajan [2470] provide strong  $Z_4$ -magic labelings for various graphs and strong  $Z_{4p}$ -magic labelings for those graphs.

In [2108] Razzaq, Rizvi, and Ali introduce the concept of an *H*-groupmagic total labeling of a graph *G* over a finite Abelian group *A* as a bijection  $\lambda : V(G) \cup E(G) \to A$ such that for any subgraph H'(V', E') of *G* isomorphic to *H*, the sum  $\sum_{v \in V'} \lambda(v) + \sum_{e \in E'} \lambda(e)$  is equal to magic constant k'. A graph is called *H*-groupmagic if it admits an *H*-groupmagic total labeling. They determine the *H*-groupmagic total labelings of fan graphs over the finite Abelian group  $A \cong \mathbb{Z}_3 \times \mathbb{Z}_t$ , where  $t \geq 3$  and show that disjoint union of isomorphic as well as non-isomorphic copies of fan graphs are *H*-groupmagic over  $A \cong \mathbb{Z}_3 \times \mathbb{Z}_t$ .

For a natural number h, Salehi [2155] defines a graph G to be h-magic if there is a labeling  $\alpha$  from the edges of G to the nonzero integers in  $Z_h$  such that for each vertex v in G the sum of all  $\alpha$  values of edges incident to v is a constant (called the *magic* sum index) that is independent of the choice of v. If the constant is 0, G is called a zero-sum h-magic graph. The null set of graph G is the set of all natural numbers h for which G admits a zero-sum h-magic labeling. In [2155] Salehi determines the null sets for  $K_n$ ,  $K_{m,n}$ ,  $C_n$ , books, and cycles with a  $P_k$  chord. Lin and Wang [1596] determine the null sets of generalized wheels and generalized fans, and construct infinitely many examples of  $Z_h$ -magic graphs with magic sum zero and present some open problems.

In 1976 Sedláček [2198] defined a connected graph with at least two edges to be *pseudo-magic* if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when  $n \ge 4$  and n is even, the Möbius ladder  $M_n$  is not pseudo-magic and when  $m \ge 3$  and m is odd,  $C_m \times P_2$  is not pseudo-magic.

A vertex magic total labeling of a graph with p vertices and q edges is a bijection from the union of the vertex set and edge set to the consecutive integers  $1, 2, \ldots, p + q$ with the property that for every vertex u, the sum of the label of u and the labels of the edges incident with u is a constant k. A vertex magic total labeling is said to be *a*-vertex multiple magic if the set of the labels of the vertices is  $\{a, 2a, \ldots, na\}$  and is *b*-edge multiple magic b-edge multiple magic if the set of labels of the edges is  $\{b, 2b, \ldots, mb\}$ . Nagaraj, Ponnappan, and Prabakaran [1826] provide properties of *a*-vertex multiple magic graphs and *b*-edge multiple magic graphs. In [2903] Zhang and Wang verify the existence of *E*super vertex magic total labeling for odd regular graphs containing a particular 3-factor.

Kong, Lee, and Sun [1412] used the term "magic labeling" for a labeling of the edges with nonnegative integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. In particular, the edge labels need not be distinct. They let M(G) denote the set of all such labelings of G. For any L in M(G), they let  $s(L) = \max\{L(e) \mid e \in E\}$  and define the magic strength of G as  $m(G) = \min\{s(L) \mid L \in C\}$ M(G). To distinguish these notions from others with the same names and notation. which we will introduced in the next section for labelings from the set of vertices and edges, we call the Kong, Lee, and Sun version the edge magic strength and use em(G)for min $\{s(L): L \text{ in } M(G)\}$  instead of m(G). Kong, Lee, and Sun [1412] use DS(k) to denote the graph obtained by taking two copies of  $K_{1,k}$  and connecting the k pairs of corresponding leafs. They show: for k > 1, em(DS(k)) = k - 1;  $em(P_k + K_1) = 1$  for k = 1 or 2,  $em(P_k + K_1) = k$  if k is even and greater than 2, and 0 if k is odd and greater than 1; for  $k \ge 3$ , em(W(k)) = k/2 if k is even and em(W(k)) = (k-1)/2if k is odd;  $em(P_2 \times P_2) = 1$ ,  $em(P_2 \times P_n) = 2$  if n > 3,  $em(P_m \times P_n) = 3$  if m or n is even and greater than 2;  $em(C_3^{(n)}) = 1$  if n = 1 (Dutch windmill, - see §2.4), and  $em(C_3^{(n)}) = 2n - 1$  if n > 1. They also prove that if G and H are magic graphs then  $G \times H$  is magic and  $em(G \times H) = \max\{em(G), em(H)\}$  and that every connected graph is an induced subgraph of a magic graph (see also [726] and [760]). They conjecture that almost all connected graphs are not magic. In [1529] Lee, Saba, and Sun show that the edge magic strength of  $P_n^k$  is 0 when k and n are both odd.

Recall a *lexicographic product* of two graphs  $G_1$  and  $G_2$ , denoted by  $G_1[G_2]$ , is a graph that arises from  $G_1$  by replacing each vertex of  $G_1$  by a copy of the  $G_2$  and each edge of  $G_1$  with  $K_{n,n}$  where n is the order of  $G_2$ . Sun and Lee [2532] show that the Cartesian, conjunctive, normal, lexicographic, and disjunctive products of two magic graphs are magic and the sum of two magic graphs is magic. They also determine the edge magic strengths of the products and sums in terms of the edge magic strengths of the components graphs.

In [111] Akka and Warad define the super magic strength of a graph G, sms(G) as the minimum of all magic constants c(f) where the minimum is taken over all super magic labeling f of G if there exist at least one such super magic labeling. They determine the super magic strength of paths, cycles, wheels, stars, bistars,  $P_n^2$ ,  $\langle K_{1,n} : 2 \rangle$  (the graph obtained by joining the centers of two copies of  $K_{1,n}$  by a path of length 2), and  $(2n+1)P_2$ .

In [1338] Kanwal, Riasat, Imtiaz, Iftikhar, Javed, and Ashraf define a fork as the graph obtained by starting with three paths of length t with vertices  $x_{1,j}, x_{2,j}, x_{3,j}, 1 \leq j \leq t$ , a single new edge  $x_{2,0}$  adjacent to  $x_{2,1}$ , an edge joining  $x_{1,1}$  and  $x_{2,1}$  and an edge joining  $x_{2,1}$  and  $x_{3,1}$ . They gave super edge-magic total labelings and deficiencies of forks, the disjoint union of a fork with a star, a bistar, and a path, and of trees obtained by starting with two copies of  $P_{2t+1}$  and adding an edge joining the middle vertex of each path. The super edge-magic total labeling strengths of forks and the trees are also determined.

A Halin graph ia a planar 3-connected graphs that consist of a tree and a cycle connecting the end vertices of the tree. Let G be a (p,q)-graph in which the edges are labeled  $k, k + 1, \ldots, k + q - 1$ , where  $k \ge 0$ . In [1547] Lee, Su, and Wang define a graph with p vertices to be k-edge-magic for every vertex v the sum of the labels of the incident edges at v are constant modulo p. They investigate some classes of Halin graphs that are k-edge-magic. Lee, Su, and Wang [1549] investigated some classes of cubic graphs that are k-edge-magic andb provided a counterexample to a conjecture that any cubic graph of order  $p \equiv 2 \pmod{4}$  is k-edge-magic for all k. Shiu and Lau [2349] gave some necessary conditions for families of wheels with certain spokes missing to admit k-edge-magic labelings.

Lau, Alikhani, Lee, and Kocay [1464] (see also [131]) show that maximal outerplanar graphs of orders p = 4, 5, 7 are k-edge magic if and only if  $k \equiv 2 \pmod{p}$  and determined all maximal outerplanar graphs that are k-edge magic for k = 3 and 4. They also characterize all (p, p - h)-graphs that are k-edge magic for  $h \ge 0$  and conjecture that a maximal outerplanar graph of prime order p is k-edge magic if and only if  $k \equiv 2 \pmod{p}$ .

S. M. Lee and colleagues [1568] and [1500] call a graph G k-magic if there is a labeling from the edges of G to the set  $\{1, 2, \ldots, k-1\}$  such that for each vertex v of G the sum of all edges incident with v is a constant independent of v. The set of all k for which Gis k-magic is denoted by IM(G) and called the *integer-magic spectrum* of G. In [1568] Lee and Wong investigate the integer-magic spectrum of powers of paths. They prove:  $IM(P_4^2)$  is  $\{4, 6, 8, 10, \ldots\}$ ; for n > 5,  $IM(P_n^2)$  is the set of all positive integers except 2; for all odd d > 1,  $IM(P_{2d}^d)$  is the set of all positive integers except 1;  $IM(P_4^3)$  is the set of all positive integers; for all odd  $n \ge 5$ ,  $IM(P_n^3)$  is the set of all positive integers except 1 and 2; and for all even  $n \ge 6$ ,  $IM(P_n^3)$  is the set of all positive integers except 2. For k > 3 they conjecture:  $IM(P_n^k)$  is the set of all positive integers when n = k+1; the set of all positive integers except 1 and 2 when n and k are odd and  $n \ge k$ ; the set of all positive integers except 1 and 2 when n and k are even and  $k \ge n/2$ ; the set of all positive integers except 2 when n is even and k is odd and  $n \ge k$ ; and the set of all positive integers except 2 when n and k are even and  $k \leq n/2$ . In [1545] Lee, Su, and Wang showed that besides the natural numbers there are two types of the integer-magic spectra of honeycomb graphs. Fu, Jhuang and Lin [800] determine the integer-magic spectra of graphs obtained from attaching a path of length at least 2 to the end vertices of each edge of a cycle.

In [1500] Lee, Lee, Sun, and Wen investigated the integer-magic spectrum of various graphs such as stars, double stars (trees obtained by joining the centers of two disjoint stars  $K_{1,m}$  and  $K_{1,n}$  with an edge), wheels, and fans. In [2158] Salehi and Bennett report that a number of the results of Lee et al. are incorrect and provide a detailed accounting of these errors as well as determine the integer-magic spectra of caterpillars. Shiu and Low [2365] determined the integer-magic spectra and null sets of the Cartesian product of two trees.

Lee, Lee, Sun, and Wen [1500] use the notation  $C_m@C_n$  to denote the graph obtained by starting with  $C_m$  and attaching paths  $P_n$  to  $C_m$  by identifying the endpoints of the paths with each successive pairs of vertices of  $C_m$ . They prove that  $IM(C_m@C_n)$  is the set of all positive integers if m or n is even and  $IM(C_m@C_n)$  is the set of all even positive integers if m and n are odd.

Lee, Valdés, and Ho [1556] investigate the integer magic spectrum for special kinds of trees. For a given tree T they define the *double tree* DT of T as the graph obtained by creating a second copy  $T^*$  of T and joining each end vertex of T to its corresponding vertex in  $T^*$ . They prove that for any tree T, IM(DT) contains every positive integer with the possible exception of 2 and IM(DT) contains all positive integers if and only if the degree of every vertex that is not an end vertex is even. For a given tree T they define ADT, the *abbreviated double tree of* T, as the the graph obtained from DT by identifying the end vertices of T and  $T^*$ . They prove that for every tree T, IM(ADT) contains every positive integer with the possible exceptions of 1 and 2 and IM(ADT) contains all positive integers if and only if T is a path.

Lee, Salehi, and Sun [1531] have investigated the integer-magic spectra of trees with diameter at most four. Among their findings are: if  $n \geq 3$  and the prime power factorization of  $n-1 = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ , then  $\mathrm{IM}(K_{1,n}) = p_1 \mathbb{N} \cup p_2 \mathbb{N} \cup \cdots \cup p_k \mathbb{N}$  (here  $p_i \mathbb{N}$ means all positive integer multiples of  $p_i$ ); for  $m, n \geq 3$ , the double star  $\mathrm{IM}(DS(m,m))$ (that is, stars  $K_{m,1}$  and  $K_{n,1}$  that have an edge in common) is the set of all natural numbers excluding all divisors of m-2 greater than 1; if the prime power factorization of  $m-n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$  and the prime power factorization of  $n-2 = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ , (the exponents are permitted to be 0) then  $\mathrm{IM}(DS(m,n)) = A_1 \cup A_2 \cup \cdots \cup A_k$  where  $A_i = p_i^{1+s_i} \mathbb{N}$ if  $r_i > s_i \geq 0$  and  $A_i = \emptyset$  if  $s_i \geq r_i \geq 0$ ; for  $m, n \geq 3$ ,  $\mathrm{IM}(DS(m,n)) = \emptyset$  if and only if m-n divides n-2; if  $m, n \geq 3$  and |m-n| = 1, then DS(m,n) is not magic. Lee and Salehi [1530] give formulas for the integer-magic spectra of trees of diameter four but they are too complicated to include here.

For a graph G(V, E) and a function f from the V to the positive integers, Salehi and Lee [2162] define the functional extension of G by f, as the graph H with  $V(H) = \bigcup \{u_i | u \in V(G) \text{ and } i = 1, 2, ..., f(u)\}$  and  $E(H) = \bigcup \{u_i u_j | uv \in E(G), i = 1, 2, ..., f(u); j = 1, 2, ..., f(v)\}$ . They determine the integer-magic spectra for  $P_2, P_3$ , and  $P_4$ .

More specialized results about the integer-magic spectra of amalgamations of stars and cycles are given by Lee and Salehi in [1530].

Table 5 summarizes the state of knowledge about magic-type labelings. In the table, **SM** means semi-magic, **M** means magic, and **SPM** means supermagic. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Graph	Types	Notes
$K_n$	M	if $n = 2, n \ge 5$ [2471]
	SPM	for $n \ge 5$ iff $n > 5$ $n \ne 0$ (m and 4) [2472]
		$n \neq 0 \pmod{4} \lfloor 2472 \rfloor$
$K_{m,n}$	SM	if $n \ge 3$ [2471]
$K_{n,n}$	М	if $n \ge 3$ [2471]
fans $f_n$	M not SM	iff <i>n</i> is odd, $n \ge 3$ [2471] if $n \ge 2$ [873]
wheels $W_n$	M SM	if $n \ge 4$ [2471] if $n = 5$ or 6 [873]
wheels with one spoke deleted	М	if $n = 4, n \ge 6$ [2471]
null graph with $n$ vertices		
Möbius ladders $M_n$	SPM	if $n \ge 3$ , $n$ is odd [2198]
$C_n \times P_2$	not SPM	for $n \ge 4, n$ even [2198]
$C_m[\overline{K}_n]$	SPM	if $m \ge 3, n \ge 2$ [2345]
$K_{\underline{n,n,\ldots,n}}$	SPM	$n \ge 3, p > 5$ and
P		$p \not\equiv 0 \pmod{4} \ [2345]$
composition of $r$ -regular SPM graph and $\overline{K}_n$	SPM	if $n \ge 3$ [2345]
$K_k[\overline{K}_n]$	SPM	if $k = 3$ or 5, $n = 2$ or $n$ odd [1012]
$mK_{n,n}$	SPM	for $n \ge 2$ iff <i>n</i> is even or both <i>n</i> and <i>m</i> are odd [2342]
$Q_n$	SPM	iff $n = 1$ or $n > 2$ even [1078]

Table 5: Summary of Magic Labelings

Continued on next page

Graph	Types	Notes
$C_m \times C_n$	SPM	m = n or $m$ and $n$ are even [1078]
$C_m \times C_n$	SPM?	for all $m$ and $n$ [1078]
connected $(p,q)$ -graph other than $P_2$	М	iff $5p/4 < q \le p(p-1)/2$ [2597]
$G^i$	M	$ G  \ge 5, \ i \ge 3 \ [2600]$
$G^2$	М	$G \neq P_5$ and G does not have a 1-factor whose every edge is incident with an end-vertex of G [2600]
$K_{1,m,n}$	М	for all $m, n$ [2220]
$P_n^2$	M	for all $n$ except 2, 3, 5 [2220], [873]
$G \times H$	M	iff $G$ and $H$ are magic [1412]

Table 5 – Continued from previous page

### 5.2 Edge-magic Total and Super Edge-magic Total Labelings

In 1970 Kotzig and Rosa [1418] defined a magic valuation of a graph G(V, E) as a bijection f from  $V \cup E$  to  $\{1, 2, \dots, |V \cup E|\}$  such that for all edges xy, f(x) + f(y) + f(xy) is constant (called the *magic constant*). This notion was rediscovered by Ringel and Lladó [2118] in 1996 who called this labeling *edge-magic*. To distinguish between this usage from that of other kinds of labelings that use the word magic we will use the term *edge-magic* total labeling as introduced by Wallis [2752] in 2001. (We note that for 2-regular graphs a vertex-magic total labeling is an edge-magic total labeling and vice versa.) Kotzig and Rosa proved:  $K_{m,n}$  has an edge-magic total labeling for all m and n;  $C_n$  has an edgemagic total labeling for all  $n \geq 3$  (see also [896], [2129], [443], and [726]); and the disjoint union of n copies of  $P_2$  has an edge-magic total labeling if and only if n is odd. They further state that  $K_n$  has an edge-magic total labeling if and only if n = 1, 2, 3, 5, or 6 (see [1419], [632], and [726]) and ask whether all trees have edge-magic total labelings. Wallis, Baskoro, Miller, and Slamin [2756] enumerate every edge-magic total labeling of complete graphs. They also prove that the following graphs are edge-magic total: paths, crowns, complete bipartite graphs, and cycles with a single edge attached to one vertex. Enomoto, Llado, Nakamigana, and Ringel [726] prove that all complete bipartite graphs are edge-magic total. They also show that wheels  $W_n$  are not edge-magic total when  $n \equiv 3$  (mod 4) and conjectured that all other wheels are edge-magic total. This conjecture was proved when  $n \equiv 0, 1 \pmod{4}$  by Phillips, Rees, and Wallis [1926] and when  $n \equiv 6 \pmod{8}$  by Slamin, Bača, Lin, Miller, and Simanjuntak [2419]. Fukuchi [806] verified all cases of the conjecture independently of the work of others. Slamin et al. further show that all fans are edge-magic total. In 2002 Lee and Kong [1496] conjectured that odd star forests are super edge magic. In 2019 Cerioli, Fernandes, Lee, Lintzmayer, Mota, and da Silva [545] proved this conjecture for odd symmetric star forests and proved that odd uniform forests of catterpillars are edge-magic. Afzal, Ather, Baig, and Maheshwari [58] analyzed pyramidion ladders and  $C_n$ -books for their super edge-magicness and gave some methods for finding new super edge-magic graphs from existing ones.

Inspired by Kotzig-Rosa's notion, Enomoto, Lladó, Nakamigawa, and Ringel [726] called a graph G(V, E) with an edge-magic total labeling that has the additional property that the vertex labels are 1 to |V| a super edge-magic total labeling (SEMT). Kanwal and Kanwal [1337] determined super edge-magic total labelings and deficiencies for forests formed by two sided generalized combs, stars, combs, and banana trees. A two-sided generalized comb  $Cb_{a,b}^2$ , where b is odd, is obtained from a path  $P_{a+1}$  by attaching two paths  $P_{(b+1)/2}$  to each of the vertices of degree two and one vertex of degree one of  $P_{a+1}$ . In [1328] Kanwal, Azam, and Iftikhar investigate the SEMT strength of generalized comb and the SEMT labeling and deficiency of forests composed of two components, where one of the components for each forest is a generalized comb and other component is star, bistar, comb, or path. In [1330] Kanwal, Imtiaz, Iftikhar, Ashraf, Arshad, Irfan, and Sumbal z studied the super edge-magic deficiency of paths, caterpillars, and the disjoint union of a 2-sided generalized comb with a bistar. They also provide the super edge-magic total strength for a 2-sided generalized comb. Javed, Riasat, and Kanwal [1106] study super edge-magic total labeling and deficiencies of forests consisting of combs, generalized combs, and stars. Their results provide the evidence to support a conjecture proposed by Figueroa-Centeno, Ichishima, and Muntaner-Bartle [765]. Cerioli, Fernandes, Lee, Lintzmayer [550] proved certain forests of stars admit a super edge-magic labeling and that certain forests of caterpillars admit an edge-magic labeling.

Baskoro, Sudarsana, and Cholily [418] provided some constructions of new super edgemagic graphs from some old ones by attaching 1, 2, or 3 pendent vertices and edges. In [1377] Kim introduces a new construction of new super edge-magic graphs by attaching any number pendent vertices and edges under some conditions.

Ringel and Llado [2118] prove that a graph with p vertices and q edges is not edgemagic total if q is even and  $p + q \equiv 2 \pmod{4}$  and each vertex has odd degree. Ringel and Llado conjecture that trees are edge-magic total. In [409] Baskar Babujee and Rao show that the path with n vertices has an edge-magic total labeling with magic constant (5n + 2)/2 when n is even and (5n + 1)/2 when n is odd. For stars with n vertices they provide an edge-magic total labeling with magic constant 3n. In [737] Eshghi and Azimi discuss a zero-one integer programming model for finding edge-magic total labelings of large graphs.

Santhosh [2185] proved that for n odd and at least 3, the crown  $C_n \odot P_2$  has an edge-magic total labeling with magic constant (27n + 3)/2 and for n odd and at least 3,

 $C_n \odot P_3$  has an edge-magic total labeling with magic constant (39n + 3)/2. Ngurah and Adiwijaya [1852] investigated whether various classes of chain graphs formed from ladders, triangular ladders, diagonal ladders,  $C_4$ , and  $K_4$  have an edge-magic or super edge-magic labelings. Baig and Afzal [217] investigated the super edge-magicness of special classes of graphs having maximum magic constant k = 3p.

Ahmad, Baig, and Imran [86] define a zig-zag triangle as the graph obtained from the path  $x_1, x_2, \ldots, x_n$  by adding n new vertices  $y_1, y_2, \ldots, y_n$  and new edges  $y_1x_1, y_nx_{n-1}$ ;  $x_iy_i$  for  $1 \leq i \leq n$ ;  $y_ix_{i-1}y_ix_{i+1}$  for  $2 \leq i \leq n-1$ . They define a graph  $Cb_n$  as one obtained from the path  $x_1, x_2, \ldots, x_n$  adding n-1 new vertices  $y_1, y_2, \ldots, y_{n-1}$  and new edges  $y_ix_{i+1}$  for  $1 \leq i \leq n-1$ . The graph  $Cb_n^*$  is obtained from the  $Cb_n$  by joining a new edge  $x_1y_1$ . They prove that zig-zag triangles, graphs that are the disjoint union of a star and a banana tree, certain disjoint unions of stars, and for  $n \geq 4$ ,  $Cb_n^* \cup Cb_{n-1}$  are super edge-magic total. Baig, Afzal, Imran, and Javaid [218] investigate the existence of super edge-magic labeling of volvox and pancyclic graphs.

The super edge-magic deficiency of a graph G, denoted by  $\mu_s(G)$ , is either the minimum nonnegative integer n such that  $G \cup nK_1$  is super edge-magic or  $+\infty$  if there exists no such n. Krisnawati, Ngurah, Hidayat, and Alghofari [1434] investigated the super edge-magic deficiency of forests whose components are subdivided stars or paths. Imran, Afzal, and Baig investigate the super edge-magic deficiency of volvox and dumbbell type graphs in [1065]. Kanwal, Iftikhar, and Azam [1329] found super edge magic total labelings and deficiencies of forests consisting of two components, where one of the components for each forest is a generalized comb and the other component is a star, bistar, comb, or path. They also investigated the super edge magic total strength of generalized combs.

Let G be a graph with p vertices with  $V(G) = \{v_1, v_2, \ldots, v_p\}$  and let  $S_m$  be the star with m leaves. If in G, every vertex  $v_i$  is identified to the center vertex of  $S_{m_i}$ , for some  $m_i \ge 0, 1 \le i \le n$ , where  $S_0 = K_1$ , then the graph obtained is denoted by  $G_{(m_1,m_2,\ldots,m_p)}$ . Let  $M(G) = \{(m_1, m_2, \ldots, m_p) \mid G_{(m_1,m_2,\ldots,m_p)} \text{ is a super edge-magic graph}\}$ . The star super edge-magic deficiency  $S\mu^*(G)$  is defined as

$$S\mu^*(G) = \begin{cases} \min_{(m_1, m_2, \dots, m_p)} (m_1 + m_2 + \dots + m_p) & \text{if } M(G) \neq \emptyset, \\ +\infty, \text{ if } M(G) = \emptyset. \end{cases}$$

In [1355] Kathiresan and Sabarimalai Madha determine the star super edge-magic deficiency of certain classes of graphs.

Beardon [424] extended the notion of edge-magic total to countable infinite graphs G(V, E) (that is,  $V \cup E$  is countable). His main result is that a countably infinite tree that processes an infinite simple path has a bijective edge-magic total labeling using the integers as labels. He asks whether all countably infinite trees have an edge-magic total labeling with the integers as labels and whether the graph with the integers as vertices and an edge joining every two distinct vertices has a bijective edge-magic total labeling using the integers.

Cavenagh, Combe, and Nelson [549] investigate edge-magic total labelings of countably infinite graphs with labels from a countable Abelian group A. Their main result is that

if G is a countable graph that has an infinite set of mutually disjoint edges and A is isomorphic to a countable subgroup of the real numbers under addition then for any k in A there is an edge-magic labeling of G with elements from A that has magic constant k.

Balakrishnan and Kumar [337] proved that the join of  $K_n$  and two disjoint copies of  $K_2$ is edge-magic total if and only if n = 3. Yegnanarayanan [2866] has proved the following graphs have edge-magic total labelings:  $nP_3$  where n is odd;  $P_n + K_1$ ;  $P_n \times C_3$   $(n \ge 2)$ ; the crown  $C_n \odot K_1$ ; and  $P_m \times C_3$  with n pendent vertices attached to each vertex of the outermost  $C_3$ . He conjectures that for all n,  $C_n \odot \overline{K_n}$ , the n-cycle with n pendent vertices attached at each vertex of the cycle, and  $nP_3$  have edge-magic total labelings. In fact, Figueroa-Centeno, Ichishima, and Muntaner-Batle, [767] have proved the stronger statement that for all  $n \ge 3$ , the corona  $C_n \odot \overline{K_m}$  admits an edge-magic labeling where the set of vertex labels is  $\{1, 2, \ldots, |V|\}$ . (See also [1696].)

Yegnanarayanan [2866] also introduces several variations of edge-magic labelings and provides some results about them. Kotzig [2754] provides some necessary conditions for graphs with an even number of edges in which every vertex has odd degree to have an edge-magic total labeling. Craft and Tesar [632] proved that an *r*-regular graph with *r* odd and  $p \equiv 4 \pmod{8}$  vertices can not be edge-magic total. Wallis [2752] proved that if *G* is an edge-magic total *r*-regular graph with *p* vertices and *q* edges where  $r = 2^t s + 1$  (t > 0) and *q* is even, then  $2^{t+2}$  divides *p*.

Figueroa-Centeno, Ichishima, and Muntaner-Batle [761] have proved the following graphs are edge-magic total:  $P_4 \cup nK_2$  for n odd;  $P_3 \cup nK_2$ ;  $P_5 \cup nK_2$ ;  $nP_i$  for n odd and i = 3, 4, 5;  $2P_n$ ;  $P_1 \cup P_2 \cup \cdots \cup P_n$ ;  $mK_{1,n}$ ;  $C_m \odot nK_1$ ;  $K_1 \odot nK_2$  for n even;  $W_{2n}$ ;  $K_2 \times \overline{K}_n$ ,  $nK_3$  for n odd (the case  $nK_3$  for n even and larger than 2 is done in [1727]); binary trees, generalized Petersen graphs (see also [1854]), ladders (see also [2813]), books, fans, and odd cycles with pendent edges attached to one vertex.

In [767] Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima, investigate super edge-magic total labelings of graphs with two components. Among their results are:  $C_3 \cup C_n$  is super edge-magic total if and only if  $n \ge 6$  and n is even;  $C_4 \cup C_n$  is super edge-magic total if and only if  $n \ge 5$  and n is odd;  $C_5 \cup C_n$  is super edge-magic total if and only if  $n \ge 4$  and n is even; if m is even with  $m \ge 4$  and n is odd with  $n \ge m/2 + 2$ , then  $C_m \cup C_n$  is super edge-magic total; for m = 6, 8, or 10,  $C_m \cup C_n$  is super edge-magic total if and only if  $n \ge 3$  and n is odd;  $2C_n$  is strongly felicitous if and only if  $n \ge 4$  and nis even (the converse was proved by Lee, Schmeichel, and Shee in [1532]);  $C_3 \cup P_n$  is super edge-magic total for  $n \ge 6$ ;  $C_4 \cup P_n$  is super edge-magic total if and only if  $n \ne 3$ ;  $C_5 \cup P_n$ is super edge-magic total;  $m \cup P_n$  is super edge-magic total if and only if  $n \ne 3$ ;  $C_5 \cup P_n$ is super edge-magic total;  $P_m \cup P_n$  is super edge-magic total if and only  $(m, n) \ne (2, 2)$ or (3,3); and  $P_m \cup P_n$  is edge-magic total if and only  $(m, n) \ne (2, 2)$ . In [2125] Rizvi, Ali, Iqbal, and Gulraze give super edge-magic total labelings of forests whose components are caterpillars and stars, forests whose components are stars and banana trees, and a new families of trees.

Enomoto, Llado, Nakamigawa, and Ringel [726] conjecture that if G is a graph of order n + m that contains  $K_n$ , then G is not edge-magic total for  $n \gg m$ . Wijaya and Baskoro [2813] proved that  $P_m \times C_n$  is edge-magic total for odd n at least 3. Ngurah and Baskoro

[1854] state that  $P_2 \times C_n$  is not edge-magic total. Hegde and Shetty [992] have shown that every  $T_p$ -tree (see §4.4 for the definition) is edge-magic total. Ngurah, Simanjuntak, and Baskoro [1864] show that certain subdivisions of the star  $K_{1,3}$  have edge-magic total labelings. Ali, Hussain, Shaker, and Javaid [128] provide super edge-magic total labelings of subdivisions of stars  $K_{1,p}$  for  $p \geq 5$ . In [1859] Ngurah, Baskoro, Tomescu gave methods for construction new (super) edge-magic total graphs from old ones by adding some new pendent edges. They also proved that  $K_{1,m} \cup P_n^m$  is super edge-magic total. Wallis [2752] proves that a cycle with one pendent edge is edge-magic total. In [2752] Wallis poses a large number of research problems about edge-magic total graphs.

For  $n \geq 3$ , López, Muntaner-Batle, and Rius-Font [1635] (see [1636] for (corrigendum) let  $S_n$  denote the set of all super edge-magic total 1-regular labeled digraphs of order n where each vertex takes the name of the label that has been assigned to it. For  $\pi \in S_n$ . they define a generalization of generalized Petersen graphs that they denote by  $GGP(n;\pi)$ , which consists of an outer n-cycle  $x_0, x_1, \ldots, x_{n-1}, x_0$ , a set of n-spokes  $x_i y_i$ ,  $0 \leq i \leq n-1$ , and n inner edges defined by  $y_i y_{\pi(i)}$ ,  $i = 0, \ldots, n-1$ . Notice that, for the permutation  $\pi$  defined by  $\pi(i) = i + k \pmod{n}$  we have  $GGP(n;\pi) = P(n;k)$ . They define a second generalization of generalized Petersen graphs,  $GGP(n;\pi_2,\ldots,\pi_m)$ , as the graphs with vertex sets  $\bigcup_{j=1}^{m} \{x_i^j : i = 0, \ldots, n-1\}$ , an outer n-cycle  $x_0^1, x_1^1, \ldots, x_{n-1}^1, x_0^1$ , and inner edges  $x_i^{j-1} x_i^j$  and  $x_i^j x_{\pi_j(i)}^j$ , for  $j = 2, \ldots, m$ , and  $i = 0, \ldots, n-1$ . Notice that,  $GGP(n; \pi_2, \ldots, \pi_m) = P_m \times C_n$ , when  $\pi_j(i) = i + 1 \pmod{n}$  for every  $j = 2, \ldots, m$ . Among their results are the Petersen graphs are super edge-magic total; for each m with  $1 < l \leq m$  and  $1 \leq k \leq 2$ , the graph  $GGP(5; \pi_2, \ldots, \pi_m)$ , where  $\pi_i = \sigma_1$  for  $i \neq l$  and  $\pi_l = \sigma_k$ , is super edge-magic total; for each  $1 \leq k \leq 2$ , the graph P(5n; k + 5r) where ris the smallest integer such that  $k + 5r = 1 \pmod{n}$  is super edge-magic total.

A w-graph, W(n), has vertices  $\{(c_1, c_2, b, w, d) \cup (x^1, x^2, \dots, x^n) \cup (y^1, y^2, \dots, y^n)\}$  and edges  $\{(c_1x^1, c_1x^2, \dots, c_1x^n) \cup (c_2y^1, c_2y^2, \dots, c_2y^n) \cup (c_1b, c_1w) \cup (c_2w, c_2d)\}$ . A w-tree, WT(n, k), is a tree obtained by taking k copies of a w-graph W(n) and a new vertex **a** and joining **a** with in each copy d where  $n \ge 2$  and  $k \ge 3$ . An extended w-tree Ewt(n, k, r)is a tree obtained by taking k copies of an extended w-graph Ew(n, r) and a new vertex **a** and joining **a** with the vertex d in each of the k copies for  $n \ge 2$ ,  $k \ge 3$  and  $r \ge 2$ . Super edge-magic total labelings for w-trees, extended w-trees, and disjoint unions of extended w-trees are given in [1103], [1100], and [127]. Javaid, Hussain, Ali, and Shaker [1104] provided super edge-magic total labelings of subdivisions of  $K_{1,4}$  and w-trees. Shaker, Rana, Zobair, and Hussain [2311] gave a super edge-magic total labeling for a subdivided star with a center of degree at least 4.

In 1988 Godbod and Slater [896] made the following conjecture. If n is odd,  $n \neq 5$ ,  $C_n$  has an edge magic labeling with valence k, when  $(5n+3)/2 \leq k \leq (7n+3)/2$ . If n is even,  $C_n$  has an edge-magic labeling with valence k when  $5n/2 + 2 \leq k \leq 7n/2 + 1$ . Except for small values of n, very few valences for edge-magic labelings of  $C_n$  are known. In [1640] López, Muntaner-Batle, and Rius-Font use the  $\otimes_h$ -product in order to prove the following two results. Let  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$  be the unique prime factorization of an odd number n. Then  $C_n$  admits at least  $1 + \sum_{i=1}^k \alpha_i$  edge-magic labelings with at least  $1 + \sum_{i=1}^k \alpha_i$  mutually different valences. Let  $n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$  be the unique prime factorization of an even number n, with  $p_1 > p_2 > \cdots > p_k$ . Then  $C_n$  admits at least  $\sum_{i=1}^k \alpha_i$  edge-magic labelings with at least  $\sum_{i=1}^k \alpha_i$  mutually different valences. If  $\alpha \geq 2$  this lower bound can be improved to  $1 + \sum_{i=1}^k \alpha_i$ . In [1630] López, Muntaner-Batle, and Prabu introduce a new  $\otimes_h$  labeling construction that has a wider range of applications and applies it to the magic valences of cycles and crowns.

In [2565] Swita, Rafflesia, Henni Ms, Adji, and Simanihuruk use  $B[(C_a, m), (C_b, n), P_t]$ to denote the graph that consists of m cycles  $C_a$  and n cycles  $C_b$  with a common path  $P_t$ . They proved that  $B[(C_7, 1), (C_3, n), P_2]$  admits an edge-magic total labeling,  $B[(C_a, 1), (C_3, n), P_2]$  admits a super edge-magic total labeling for all  $a \equiv 3 \mod 4$  (a > 3), and  $B[(C_7, 2), (C_3, n), P_2]$  admits a super edge-magic total labeling.

In 1996 Erdős asked for M(n), the maximum number of edges that an edge-magic total graph of order n can have (see [632]). In 1999 Craft and Tesar [632] gave the bound  $\lfloor n^2/4 \rfloor \leq M(n) \leq \lfloor n(n-1)/2 \rfloor$ . For large n this was improved by Pikhurko [1932] in 2006 to  $2n^2/7 + O(n) \leq M(n) \leq (0.489 + \cdots + o(1)n^2)$ .

Enomoto, Lladó, Nakamigawa, and Muntaner-Batle [726] proved that a super edgemagic total graph G(V, E) with  $|V| \ge 4$  and with girth at least 4 has at most 2|V| - 5edges. They prove this bound is tight for graphs with girth 4 and 5 in [726] and [1057].

In his Ph.D. thesis, Barrientos [353] introduced the following notion. Let  $L_1, L_2, \ldots, L_h$ be ordered paths in the grid  $P_r \times P_t$  that are maximal straight segments such that the end vertex of  $L_i$  is the beginning vertex of  $L_{i+1}$  for  $i = 1, 2, \ldots, h-1$ . Suppose for some i with 1 < i < h we have  $V(L_i) = \{u_0, v_0\}$  where  $u_0$  is the end vertex of  $L_{i-1}$  and the beginning vertex of  $L_i$  and  $v_0$  is the end vertex of  $L_1$  and the beginning vertex of  $L_{i+1}$ . Let  $u \in V(L_{i-1}) - \{u_0\}$  and  $v \in V(L_{i+1}) - \{v_0\}$ . The replacement of the edge  $u_0v_0$  by a new edge uv is called an *elementary transformation* of the path  $P_n$ . A tree is called a *path-like* tree if it can be obtained from  $P_n$  by a sequence of elementary transformations on an embedding of  $P_n$  in a 2-dimensional grid. In [291] Bača, Lin, and Muntaner-Batle proved that if  $T_1, T_2, \ldots, T_m$  are path-like trees each of order  $n \ge 4$  where m is odd and at least 3, then  $T_1 \cup T_2, \cup \cdots \cup T_m$  has a super edge-magic labeling. In [290] Bača, Lin, Muntaner-Batle and Rius-Font proved that the number of such trees grows at least exponentially with m. As an open problem Bača, Lin, Muntaner-Batle and Rius-Font ask if graphs of the form  $T_1 \cup T_2 \cup \cdots \cup T_m$  where  $T_1, T_2, \ldots, T_m$  are path-like trees each of order  $n \geq 2$  and m is even have a super edge-magic labeling. In [353] Barrientos proved that all path-like trees admit an  $\alpha$ -valuation. Using Barrientos's result, it is very easy to obtain that all path-like trees are a special kind of super edge-magic by using a super edge-magic labeling of the path  $P_n$ , and hence they are also super edge-magic. Furthermore, in [760] Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that if a tree is super edge-magic, then it is also harmonious. Therefore all path-like trees are also harmonious. In [1632] López, Muntaner-Batle, and Rius-Font also use a variation of the Kronecker product of matrices in order to obtain lower bounds for the number of non isomorphic super edge-magic labeling of some types of path-like trees. As a corollary they obtain lower bounds for the number of harmonious labelings of the same type of trees. López, Muntaner-Batle, and Rius-Font [1641] proved that if  $m \ge 4$  is an even integer and  $n \ge 3$  is an odd divisor of m, then  $C_m \cup C_n$  is super edge-magic. Lee and Kong conjecture that if n is an odd, then

 $St(a_1, a_2, \ldots, a_n)$  is super edge-magic, and they proved that the following graphs are super edge-magic: St(m, n)  $(n \equiv 0 \mod(m+1)), St(1, k, n)(k = 1, 2 \text{ or } n), St(2, k, n)$  (k = 2, 3), St(1, 1, k, n) (k = 2, 3), St(k, 2, 2, n) (k = 1, 2). Zhenbin and Chongjin [2915] proved that St(1, m, n), St(3, m, m+1), St(n, n+1, n+2) are super edge-magic, and under some conditions  $St(a_1, a_2, \ldots, a_{2n+1}), St(a_1, a_2, \ldots, a_{4n+1}), St(a_1, a_2, \ldots, a_{4n+3})$  are also super edge-magic.

For a simple graph H we say that G(V, E) admits an H-covering if every edge in E(G) belongs to a subgraph of G that is isomorphic to H. In [1643] López, Muntaner-Batle, Rius-Font study a relationship existing among (super) magic coverings and the Kronecker product of matrices. (For a simple graph H, G(V, E) admits an H-covering if every edge in E(G) belongs to a subgraph of G that is isomorphic to H.) Their results can be applied to construct S-magic partitions. For m copies of a graph G and a fixed subgraph H of each copy the graph I(G, H, m) is formed by taking of all the  $G_i$ 's and identifying their subgraph H. Liang [1587] determines which I(G, H, m) and which mG have G supermagic coverings.

Bača, Lin and Muntaner-Batle in [289] using a generalization of the Kronecker product of matrices prove that the number of non-isomorphic super edge-magic labelings of the disjoint union of m copies of the path  $P_n$ ,  $m \equiv 2 \pmod{4}$ ,  $m \geq 2$ ,  $n \geq 4$ , is at least  $(m/2)^{(2n-2)}$ .

In [1634] López, Muntaner-Batle and Rius-Font proved that every super edge-magic graph with p vertices and q edges where  $q \ge p - 1$  has an even harmonious labeling (See Section 4.6.) In [1639] they stated some open problems concerning relationships among super edge-magic labelings and graceful and harmonious labelings. A Langford sequence of order m and defect d is a sequence  $(t_1, t_2, \ldots, t_{2m})$  of 2m numbers such that (i) for every  $k \in [d, d + m - 1]$  there exist exactly two subscripts  $i, j \in [1, 2m]$  with ti = tj = k and (ii) the subscripts i and j satisfy the condition |i - j| = k. López and Muntaner-Batle [1629] provided new lower bounds on the number of distinct Langford sequences with certain properties in terms of the number of 1-regular super edge-magic labeled digraphs of a particular order.

Lee and Lee [1499] prove the following graphs are super edge-magic:  $P_{2n} + \overline{K_m}$ ,  $(P_2 \cup nK_1) + \overline{K_2}$ , graphs obtained by appending a path to the apex of a fan with at least 4 vertices (*umbrella*), and *jelly fish* graphs J(m, n) obtained from a 4-cycle  $v_1, v_2, v_3, v_4$  by joining  $v_1$  and  $v_3$  with an edge and appending m pendent edges to  $v_2$  and n pendent edges to  $v_4$ .

In [57] Afzel introduces two new familes of graphs called carrom and jukebox graphs and proves they admit super edge-magic labelings. Carroms are generalizations of  $C_n \times P_2$ .

Marimuthu and Balakrishnan [1701] define a graph G(p,q) to be edge magic graceful if there exists a bijection f from  $V(G) \cup E(G)$  to  $\{1, 2, \ldots, p+q\}$  such |f(u) + f(v) - f(uv)|is a constant for all edges uv of G. An edge magic graceful graph is said to be super edge magic graceful if  $V(G) = \{1, 2, \ldots, p\}$ . They present some properties of super edge magic graceful graphs, prove some classes of graphs are super edge magic graceful, and prove that every super edge magic graceful graph with either f(uv) > f(u) + f(v) for all edges uv or f(uv) < f(u) + f(v) for all edges uv is sequential, harmonious, super edge magic and not graceful. Marimuthu, Kavitha, and Balakrishnan [1702] proved that the generalized Petersen graphs P(n, 1) and P(n, (n - 1)/2) are super edge magic graceful when n is odd.

Let G = (V, E) be a (p, q)-linear forest. In [290] Bača, Lin, Muntaner-Batle, and Rius-Font call a labeling f a strong super edge-magic labeling of G and G a strong super edgemagic graph if  $f: V \cup E \to \{1, 2, \ldots, p+q\}$  with the extra property that if  $uv \in E, u', v' \in V(G)$  and  $d_G(u, u') = d_G(v, v') < +\infty$ , then we have that f(u) + f(v) = f(u') + f(v'). In [94] Ahmad, López, Muntaner-Batle, and Rius-Font define the concept of strong super edge-magic labeling of a graph with respect to a linear forest as follows. Let G = (V, E)be a (p, q)-graph and let F be any linear forest contained in G. A strong super edgemagic labeling of G with respect to F is a super edge-magic labeling f of G with the extra property with if  $uv \in E(F), u', v' \in V(F)$  and  $d_F(u, u') = d_F(v, v') < +\infty$  then we have that f(u) + f(v) = f(u') + f(v'). If a graph G admits a strong super edge-magic labeling with respect to some linear forest F, they say that G is a strong super edge-magic graph with respect to F. They prove that if m is odd and G is an acyclic graph which is strong super edge-magic with respect to a linear forest F, then mG is strong super edge-magic with respect to  $F_1 \cup F_2 \cup \cdots \cup F_m$ , where  $F_i \simeq F$  for  $i = 1, 2, \ldots, m$  and every regular caterpillar is strong super edge-magic with respect to its spine.

Noting that for a super edge-magic labeling f of a graph G with p vertices and q edges, the magic constant k is given by the formula:  $k = (\sum_{u \in V} \deg(u) f(u) + \sum_{i=p+1}^{p+q} i)/q$ , López, Muntaner-Batle and Rius-Font [1633] define the set

 $S_G = \left\{ \left( \sum_{u \in V} \deg(u)g(u) + \sum_{i=p+1}^{p+q} i \right)/q : \text{ the function } g: V \to \{i\}_{i=1}^p \text{ is bijective} \right\}.$  If  $\left[\min S_G\right] \leq \left[\max S_G\right] \text{ then the super edge-magic interval of } G \text{ is the set } I_G = \left[\left[\min S_G\right], \left[\max S_G\right]\right] \cap \mathbb{N}.$  The super edge-magic set of G is  $\sigma_G = \{k \in I_G : \text{ there exists a super edge-magic labeling of } G \text{ with valence } k\}.$  López et al. call a graph G perfect super edge-magic graphs with  $|I_{P_n}| = 1$  if n is even and  $|I_{P_n}| = 2$  if n is odd and raise the question of whether there is an infinite family  $F_1, F_2, \ldots$  of graphs such that each member of the family is perfect super edge-magic and  $\lim_{i\to+\infty} |I_{F_i}| = +\infty$ . They show that graphs  $G \cong C_{p^k} \odot \overline{K_n}$  where p > 2 is a prime is such a family.

In [1634] López et al. define the *irregular crown*  $C(n; j_1, j_2, \ldots, j_n) = (V, E)$ , where n > 2 and  $j_i \ge 0$  for all  $i \in \{1, 2, \ldots, n\}$  as follows:  $V = \{v_i\}_{i=1}^n \cup V_1 \cup V_2 \cup \cdots \cup V_n$ , where  $V_k = \{v_k^1, v_k^2, \ldots, v_k^{j_k}\}$ , if  $j_k \ne 0$  and  $V_k = \emptyset$  if  $j_k = 0$ , for each  $k \in \{1, 2, \ldots, n\}$  and  $E = \{v_i v_{i+1}\}_{i=1}^{n-1} \cup \{v_1 v_n\} \cup (\bigcup_{k=1, j_k \ne 0}^n \{v_k v_k^l\}_{l=1}^{j_k})$ . In particular, they denote  $C_m^n \cong C(m; j_1, j_2, \ldots, j_m)$ , where  $j_{2i-1} = n$ , for each i with  $1 \le i \le (m+1)/2$ , and  $j_{2i} = 0$ , for each  $i, 1 \le i \le (m-1)/2$ . They prove that the graphs  $C_3^n$  and  $C_5^n$  are perfect edge-magic for all n > 1.

López et al. [1637] define  $\mathfrak{F}^k$ -family and  $\mathfrak{E}^k$ -family of graphs as follows. The infinite family of graphs  $(F_1, F_2, \ldots)$  is an  $\mathfrak{F}^k$ -family if each element  $F_n$  admits exactly k different valences for super edge-magic labelings, and  $\lim_{n\to+\infty} |I(F_n)| = +\infty$ . The infinite family of graphs  $(F_1, F_2, \ldots)$  is an  $\mathfrak{E}^k$ -family if each element  $F_n$  admits exactly k different valences for edge-magic labelings, and  $\lim_{n\to+\infty} |J(F_n)| = +\infty$ .

An easy observation is that  $(K_{1,2}, K_{1,3}, \dots)$  is an  $\mathfrak{F}^2$ -family and an  $\mathfrak{E}^3$ -family. They

pose the two problems: for which positive integers k is it possible to find  $\mathfrak{F}^k$ -families and  $\mathfrak{E}^k$ -families? Their main results in [1637] are that an  $\mathfrak{F}^k$ -family exits for each k = 1, 2, 3; and an  $\mathfrak{E}^k$ -family exits for each k = 3, 4 and 7.

McSorley and Trono [1732] define a relaxed version of edge-magic total labelings of a graph as follows. An *edge-magic injection*  $\mu$  of a graph G is an injection  $\mu$  from the set of vertices and edges of G to the natural numbers such that for every edge uv the sum  $\mu(u) + \mu(v) + \mu(uv)$  is some constant  $k_{\mu}$ . They investigate  $\kappa(G)$ , the smallest  $k_{\mu}$ among all edge-magic injections of a graph G. They determine  $\kappa(G)$  in the cases that G is  $K_2, K_3, K_5, K_6$  (recall that these are the only complete graphs that have edge-magic total labelings), a path, a cycle, or certain types of trees. They also show that every graph has an edge-magic injection and give bounds for  $\kappa(K_n)$ .

Avadayappan, Vasuki, and Jeyanthi [210] define the *edge-magic total strength* of a graph G as the minimum of all constants over all edge-magic total labelings of G. We denote this by emt(G). They use the notation  $\langle K_{1,n} : 2 \rangle$  for the tree obtained from the bistar  $B_{n,n}$  (the graph obtained by joining the center vertices of two copies of  $K_{1,n}$ with an edge) by subdividing the edge joining the two stars. They prove:  $emt(P_{2n}) =$ 5n+1;  $emt(P_{2n+1}) = 5n+3$ ;  $emt(\langle K_{1,n} : 2 \rangle) = 4n+9$ ;  $emt(B_{n,n}) = 5n+6$ ; emt((2n+1)) = 5n+6;  $(1)P_2) = 9n+6; emt(C_{2n+1}) = 5n+4; emt(C_{2n}) = 5n+2; emt(K_{1,n}) = 2n+4; emt(P_n^2) = 2n+4; emt(P_n^$ 3n; and  $emt(K_{n,m}) \leq (m+2)(n+1)$  where  $n \leq m$ . Using an analogous definition for super edge-magic total strength, Swaninathan and Jeyanthi [2559], [2559], [2560] provide results about the super edge-magic strength of trees, fire crackers, unicyclic graphs, and generalized theta graphs. Ngurah, Simanjuntak, and Baskoro [1864] show that certain subdivisions of the star  $K_{1,3}$  have super edge-magic total labelings. In [726] Enomoto, Lladó, Nakamigawa and Ringel conjectured that all trees have a super edge-magic total labeling. Ichishima, Muntaner-Batle, and Rius-Font [1056] have shown that any tree of order p is contained in a tree of order at most 2p-3 that has a super edge-magic total labeling.

In [290] Bača, Lin, Muntaner-Batle, and Rius-Font use a generalization of the Kronecker product of matrices introduced by Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [769] to obtain an exponential lower bound for the number of nonisomorphic strong super edge-magic labelings of the graph  $mP_n$ , for m odd and any n, starting from the strong super edge-magic labeling of  $P_n$ . They prove that the number of non-isomorphic strong super edge-magic labelings of the graph  $mP_n$ ,  $n \ge 4$ , is at least  $\frac{5}{2}2^{\lfloor\frac{m}{2}\rfloor} + 1$  where  $m \ge 3$  is an odd positive integer. This result allows them to generate an exponential number of non-isomorphic super edge-magic labelings of the forest  $F \cong \bigcup_{i=1}^{m} T_i$ , where each  $T_j$  is a path-like tree of order n and m is an odd integer.

López, Muntaner-Batle, and Rius-Font [1631] introduced a generalization of super edge-magic graphs called *super edge-magic models* and prove some results about them.

Yegnanarayanan and Vaidhyanathan [2867] use the term *nice* (1, 1) *edge-magic labeling* for a super edge-magic total labeling. They prove: a super edge-magic total labeling f of a (p,q)-graph G satisfies  $2\sum_{v \in V(G)} f(v) deg(v) \equiv 0 \mod q$ ; if G is (p,q) *r*-regular graph (r > 1) with a super edge-magic total labeling then q is odd and the magic constant is (4p+q+3)/2; every super edge-magic total labeling has at least two vertices of degree
less than 4; fans  $P_n + K_1$  are edge-magic total for all n and super edge-magic total if and only if n is at most 6; books  $B_n$  are edge-magic total for all n; a super edge-magic total (p,q)-graph with  $q \ge p$  is sequential; a super edge-magic total tree is sequential; and a super edge-magic total tree is cordial. These last three results had been proved earlier by Figueroa-Centenoa, Ichishima, and Muntaner-Batle [760].

In [2866] Yegnanarayanan conjectured that the disjoint union of 2t copies of  $P_3$  has a (1,1) edge-magic labeling and posed the problem of determining the values of m and n such that  $mP_n$  has a (1,1) edge-magic labeling. Manickam and Marudai [1696] prove the conjecture and partially settle the open problem.

Hegde and Shetty [998] (see also [997]) define the maximum magic strength of a graph Gas the maximum magic constant over all edge-magic total labelings of G. We use eMt(G)to denote the maximum magic strength of G. Hegde and Shetty call a graph G with pvertices strong magic if eMt(G) = emt(G); ideal magic if  $1 \le eMt(G) - emt(G) \le p$ ; and weak magic if eMt(G) - emt(G) > p. They prove that for an edge-magic total graph Gwith p vertices and q edges, eMt(G) = 3(p + q + 1) - emt(G). Using this result they obtain:  $P_n$  is ideal magic for n > 2;  $K_{1,1}$  is strong magic;  $K_{1,2}$  and  $K_{1,3}$  are ideal magic; and  $K_{1,n}$  is weak magic for n > 3;  $B_{n,n}$  is ideal magic;  $(2n + 1)P_2$  is strong magic; cycles are ideal magic; and the generalized web W(t,3) (see §2.2 for the definition) with the central vertex deleted is weak magic.

Santhosh [2185] has shown that for n odd and at least 3,  $eMt(C_n \odot P_2) = (27n+3)/2$ and for n odd and at least 3,  $(39n+3)/2 \le eMt(C_n \odot P_2) \le (40n+3)/2$ . Moreover, he proved that for n odd and at least 3 both  $C_n \odot P_2$  and  $C_n \odot P_3$  are weak magic. In [592] Chopra and Lee provide an number of families of super edge-magic graphs that are weak magic.

In [1798] Murugan introduces the notions of almost-magic labeling, relaxed-magic labeling, almost-magic strength, and relaxed-magic strength of a graph. He determines the magic strength of Huffman trees and twigs of odd order and the almost-magic strength of  $nP_2$  (n is even) and twigs of even order. Also, he obtains a bound on the magic strength of the path-union  $P_n(m)$  and on the relaxed-magic strength of  $kS_n$  and  $kP_n$ .

Enomoto, Llado, Nakamigawa, and Ringel [726] call an edge-magic total labeling super edge-magic if the set of vertex labels is  $\{1, 2, \ldots, |V|\}$  (Wallis [2752] calls these labelings strongly edge-magic). They prove the following:  $C_n$  is super edge-magic if and only if n = 1is odd; caterpillars are super edge-magic;  $K_{m,n}$  is super edge-magic if and only if m = 1or n = 1; and  $K_n$  is super edge-magic if and only if n = 1, 2, or 3. They also prove that if a graph with p vertices and q edges is super edge-magic then,  $q \leq 2p - 3$ . In [1683] MacDougall and Wallis study super edge-magic (p, q)-graphs where q = 2p - 3. Enomoto et al. [726] conjecture that every tree is super edge-magic. Lee and Shan [1540] have verified this conjecture for trees with up to 17 vertices with a computer. Fukuchi, and Oshima, [808] have shown that if T is a tree of order  $n \geq 2$  such that T has diameter greater than or equal to n - 5, then T has a super edge-magic labeling.

Various classes of banana trees that have super edge-magic total labelings have been found by Swaminathan and Jeyanthi [2559] and Hussain, Baskoro, and Slamin [1037]. In [72] Ahmad, Ali, and Baskoro [72] investigate the existence of super edge-magic labelings of subdivisions of banana trees and disjoint unions of banana trees. They pose three open problems.

Kotzig and Rosa's ([1418] and [1419]) proof that  $nK_2$  is edge-magic total when n is odd actually shows that it is super edge-magic. Kotzig and Rosa also prove that every caterpillar is super-edge magic. Figueroa-Centeno, Ichishima, and Muntaner-Batle prove the following: if G is a bipartite or tripartite (super) edge-magic graph, then nG is (super) edge-magic when n is odd [764]; if m is a multiple of n+1, then  $K_{1,m} \cup K_{1,n}$  is super edgemagic [764];  $K_{1,2} \cup K_{1,n}$  is super edge-magic if and only if n is a multiple of 3;  $K_{1,m} \cup K_{1,n}$ is edge-magic if and only if mn is even [764];  $K_{1,3} \cup K_{1,n}$  is super edge-magic if and only if n is a multiple of 4 [764];  $P_m \cup K_{1,n}$  is super edge-magic when  $m \ge 4$  [764];  $2P_n$  is super edge-magic if and only if n is not 2 or 3;  $K_{1,m} \cup 2nK_2$  is super edge-magic for all m and n [764];  $C_3 \cup C_n$  is super edge-magic if and only if  $n \ge 6$  and n is even [767] (see also [927]);  $C_4 \cup C_n$  is super edge-magic if and only if  $n \ge 5$  and n is odd [767] (see also [927]);  $C_5 \cup C_n$ is super edge-magic if and only if  $n \ge 4$  and n is even [767]; if m is even and at least 6 and n is odd and satisfies  $n \ge m/2 + 2$ , then  $C_m \cup C_n$  is super edge-magic [767];  $C_4 \cup P_n$ is super edge-magic if and only if  $n \neq 3$  [767];  $C_5 \cup P_n$  is super edge-magic if  $n \geq 4$  [767]; if m is even and at least 6 and  $n \ge m/2 + 2$ , then  $C_m \cup P_n$  is super edge-magic [767]; and  $P_m \cup P_n$  is super edge-magic if and only if  $(m, n) \neq (2, 2)$  or (3, 3) [767]. They [764] conjecture that  $K_{1,m} \cup K_{1,n}$  is super edge-magic only when m is a multiple of n+1 and they prove that if G is a super edge-magic graph with p vertices and q edges with  $p \geq 4$ and  $q \geq 2p-4$ , then G contains triangles. In [767] Figueroa-Centeno et al. conjecture that  $C_m \cup C_n$  is super edge-magic if and only if  $m + n \ge 9$  and m + n is odd.

Singgih [2393] gave super edge magic total labelings for unions of books mB(n) for odd m;  $m(P_2 \times P_n)$  for m and n odd;  $r(P_m \times P_n)$  for odd r and  $(m, n) \neq (2, 2)$  or (3, 3);  $r(P_3 \times mP_n)$  for odd r;  $mP_n$  for  $m \equiv 2 \pmod{4}$ ,  $n \neq 2, 3$ ; and  $mP_{4n}$  for  $m \equiv 2 \pmod{4}$ , n > 1.

In [807] Fukuchi and Oshima describe a construction of super-edge-magic labelings of some families of trees with diameter 4. Salman, Ngurah, and Izzati [2168] use  $S_n^m$   $(n \ge 3)$  to denote the graph obtained by inserting m vertices in every edge of the star  $S_n$ . They prove that  $S_n^m$  is super edge-magic when m = 1 or 2.

In [1642] López, Muntaner-Batle, and Ruis-Font introduce a new construction for super edge-magic labelings of 2-regular graphs which allows loops and is related to the knight jump in the game of chess. They also study the super edge-magic properties of cycles with cords.

Muntaner-Batle calls a bipartite graph with partite sets  $V_1$  and  $V_2$  special super edgemagic if is has a super edge-magic total labeling f with the property that  $f(V_1) = \{1, 2, \ldots, |V_1|\}$ . He proves that a tree has a special super edge-magic labeling if and only if it has an  $\alpha$ -labeling (see §3.1 for the definition). Figueroa-Centeno, Ichishima, Muntaner-Batle, and Rius-Font [769] use matrices to generate edge-magic total labeling and define the concept of super edge-magic total labelings for digraphs. They prove that if G is a graph with a super edge-magic total labeling then for every natural number dthere exists a natural number k such that G has a (k, d)-arithmetic labeling (see §4.2 for the definition). In [1476] Lee and Lee prove that a graph is super edge-magic if and only if it is (k, 1)-strongly indexable (see §4.3 for the definition of (k, d)-strongly indexable graphs). They also provide a way to construct (k, d)-strongly indexable graphs from two given (k, d)-strongly indexable graphs. This allows them to obtain several existing results about super edge-magic graphs as special cases of their constructions. Acharya and Germina [28] proved that the class of strongly indexable graphs is a proper subclass of super edge-magic graphs.

In [1041] Ichishima, López, Muntaner-Batle and Rius-Font show how one can use the product  $\otimes_h$  of super edge-magic 1-regular labeled digraphs and digraphs with harmonious, or sequential labelings to create new undirected graphs that have harmonious, sequential labelings or partitional labelings (see §4.1 for the definition). They define the product  $\otimes_h$  as follows. Let  $\overrightarrow{D} = (V, E)$  be a digraph with adjacency matrix  $A(\overrightarrow{D}) = (a_{ij})$  and let  $\Gamma = \{F_i\}_{i=1}^m$  be a family of *m* digraphs all with the same set of vertices V'. Assume that  $h: E \longrightarrow \Gamma$  is any function that assigns elements of  $\Gamma$  to the arcs of D. Then the digraph  $\overrightarrow{D} \otimes_h \Gamma$  is defined by  $V(D \otimes_h \Gamma) = V \times V'$  and  $((a_1, b_1), (a_2, b_2)) \in E(D \otimes_h \Gamma) \iff$  $[(a_1, a_2) \in E(D) \land (b_1, b_2) \in E(h(a_1, a_2))]$ . An alternative way of defining the same product is through adjacency matrices, since one can obtain the adjacency matrix of  $\overrightarrow{D} \otimes_h \Gamma$  as follows: if  $a_{ij} = 0$  then  $a_{ij}$  is multiplied by the  $p' \times p'$  0-square matrix, where p' = |V'|. If  $a_{ij} = 1$  then  $a_{ij}$  is multiplied by A(h(i, j)) where A(h(i, j)) is the adjacency matrix of the digraph h(i, j). They prove the following. Let  $\overrightarrow{D} = (V, E)$  be a harmonious (p, q)digraph with  $p \leq q$  and let h be any function from E to the set of all super edge-magic 1-regular labeled digraphs of order n, which we denote by  $S_n$ . Then the undirected graph  $und(\overrightarrow{D}\otimes_h S_n)$  is harmonious. Let  $\overrightarrow{D} = (V, E)$  be a sequential digraph and let  $h: E \longrightarrow S_n$ be any function. Then  $und(\overrightarrow{D} \otimes_h S_n)$  is sequential. Let D be a partitional graph and let  $h: E \longrightarrow S_n$  be any function, where  $\overrightarrow{D} = (V, E)$  is the digraph obtained by orienting all edges from one stable set to the other one. Then  $und(\overrightarrow{D} \otimes_h S_n)$  is partitional.

Marr, Ochel, and Perez [1713] say a digraph D with v vertices and e directed edges has an *in-magic total labeling* if there exists a bijective function  $\lambda$  from  $V(D) \cup E(D)$  to  $\{1, 2, \ldots, v+e\}$  such that for every vertex x we have  $\lambda(x) + \sum \lambda(y, x) = k$  for some integer k, where the sum is taken over all directed edges (y, x). They provide such labelings for trees and cycles and discuss some relationships between this labeling and other digraph labelings.

In [1638] López, Muntaner-Batle and Rius-Font introduce the concept of  $\{H_i\}_{i\in I}$ super edge-magic decomposable as follows: Let G = (V, E) be any graph and let  $\{H_i\}_{i\in I}$ be a set of graphs such that  $G = \bigoplus_{i\in I} H_i$  (that is, G decomposes into the graphs in the set  $\{H_i\}_{i\in I}$ ). Then we say that G is  $\{H_i\}_{i\in I}$ -super edge-magic decomposable if there is a bijection  $\beta : V \to [1, |V|]$  such that for each  $i \in I$  the subgraph  $H_i$  meets the following two requirements: (i)  $\beta(V(H_i)) = [1, |V(H_i)|]$  and (ii)  $\{\beta(a) + \beta(b) : ab \in E(H_i)\}$  is a set of consecutive integers. Such function  $\beta$  is called an  $\{H_i\}_{i\in I}$ -super edge-magic labeling of G. When  $H_i = H$  for every  $i \in I$  we just use the notation H-super edge-magic decomposable labeling. Among their results are the following. Let G = (V, E) be a (p, q)-graph which is  $\{H_1, H_2\}$ -super edge-magic decomposable for a pair of graphs  $H_1$  and  $H_2$ . Then G is super edge-bimagic; Let n be an even integer. Then the cycle  $C_n$  is  $(n/2)K_2$ -super edge-magic decomposable if and only if  $n \equiv 2 \pmod{4}$ . Let n be odd. Then for any super edge-magic tree T there exists a bipartite connected graph G = G(T, n) such that G is (nT)-super edge-magic decomposable. Let G be a  $\{H_i\}_{i\in I}$ -super edge magic decomposable graph, where  $H_i$  is an acyclic digraph for each  $i \in I$ . Assume that  $\overrightarrow{G}$  is any orientation of Gand  $h : E(\overrightarrow{G}) \to S_p$  is any function. Then  $\operatorname{und}(\overrightarrow{G} \otimes_h S_p)$  is  $\{pH_i\}_{i\in I}$ -super edge magic decomposable.

As a corollary of the last result they have that if G is a 2-regular, (1-factor)-super edge-magic decomposable graph and  $\overrightarrow{G}$  is any orientation of G and  $h : E(\overrightarrow{G}) \to S_p$  is any function, then  $\operatorname{und}(\overrightarrow{G} \otimes_h S_p)$  is a 2-regular, (1-factor)-super edge-magic decomposable graph. Moreover, if we denote the 1-factor of G by F then pF is the 1-factor of  $\operatorname{und}(\overrightarrow{G} \otimes_h S_p)$ .

They pose the following two open questions: Fix  $p \in \mathbb{N}$ . Find the maximum  $r \in \mathbb{N}$  such that there is a r-regular graph of order p which is  $(p/2)K_2$ -super edge-magic decomposable: and characterize the set of 2-regular graphs of order  $n, n \equiv 2 \pmod{4}$ , such that each component has even order and admits an  $(n/2)K_2$ -super edge-magic decomposition. In connection to open question 1 they prove: For all  $r \in \mathbb{N}$ , there is  $n \in \mathbb{N}$  such that there exists a k-regular bipartite graph B(n), with k > r and  $|V(B(n))| = 2 \cdot 3^n$ , such that B(n) is  $(3^n K_2)$ -super edge-magic decomposable.

Hendy, Sugeng, Salman, and Ayunda [1009] provided a sufficient condition for  $\overline{C_n}[\overline{K_m}]$  to have a  $P_t[\overline{K_m}]$ -magic decompositions, where n > 3, m > 1, and t = 3, 4, n - 2.

An *H*-magic labeling in an *H*-decomposable graph *G* is a bijection  $f V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$  such that, for every copy *H* in the decomposition,  $\sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$  is constant. The function *f* is said to be an *H*-*V*-super magic labeling if  $f(V(G)) = \{1, 2, \ldots, p\}$ . In [1804] Murugan and Chandra Kumar find the magic constant for *H*-factorable graphs that are *H*-*V*-super magic. Also, they give a necessary and sufficient condition for an *H*-factorable graph to be *H*-*V*-super magic and characterize the even regular graphs with a 2-factor-*V*-super magic labeling.

A bipartite graph G with partite sets  $X_1$  and  $X_2$  is called *consecutively super edge-magic* if there exists a bijective function  $f: V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\}$  such that  $f(X_1) = \{1, 2, ..., |X_1|\}, f(X_2) = \{|X_1| + 1, |X_1| + 2, ..., |V(G)|\}$  and f(u) + f(v) + f(uv) is a constant for each  $uv \in E(G)$ . In [1044] Ichishima, Muntaner-Batle, and Oshima investigated for which bipartite graphs is it possible to add a finite number of isolated vertices so that the resulting graph is consecutively super edge-magic. If it is possible for a bipartite graph G, then they say that the minimum such number  $\mu_c(G)$  of isolated vertices is the *consecutively super edge-magic deficiency* of G; otherwise, it is  $+\infty$ . Thus, the consecutively super edge-magic deficiency of a graph G is a measure of how close G is to being consecutively super edge-magic. They also include a detailed discussion of other concepts that are closely related to the consecutively super edge-magic deficiency.

In [1047] Ichishima, Muntaner-Batle, and Oshima prove that  $\alpha(G) = \mu_c(G) + |V(G)| + 1$ . Thus a tree has a consecutively super edge-magic if and only if it has an  $\alpha$ -valuation. They explore the relation between super edge-magic labelings and graceful labelings of

trees.

In [1064] Ichishima, Oshima, and Takamashi introduce the notion of strength sum of a non-empty graph as follows. The strength sum  $\operatorname{strs}_f(G)$  of a numbering  $f: V(G) \to \{1, 2, \ldots, |V(G)|\}$  is defined by  $\operatorname{strs}(G) = \min\{\operatorname{strs}_f(G) \mid f \text{ is a numbering of } G\}$ , where  $\operatorname{strs}_f(G) = \sum_{uv \in E(G)} (f(u) + f(v))$ . A numbering f of a graph G for which  $\operatorname{strs}_f(G) = \operatorname{strs}(G)$  is called a strength sum labeling of G. They also discuss relations among invariants on super edge-magic graphs and their its strength sums.

Avadayappan, Jeyanthi, and Vasuki [209] define the super magic strength of a graph G as  $sm(G) = \min\{s(L)\}$  where L runs over all super edge-magic labelings of G. They use the notation  $\langle K_{1,n} : 2 \rangle$  for the tree obtained from the bistar  $B_{n,n}$  (the graph obtained by joining the center vertices of two copies of  $K_{1,n}$  with an edge) by subdividing the edge joining the two stars. They prove:  $sm(P_{2n}) = 5n + 1$ ;  $sm(P_{2n+1}) = 5n + 3$ ;

 $sm(\langle K_{1,n}:2 \rangle) = 4n + 9; \ sm(B_{n,n}) = 5n + 6; \ sm((2n+1)P_2) = 9n + 6; \ sm(C_{2n+1}) = 5n + 4; \ emt(C_{2n}) = 5n + 2; \ sm(K_{1,n}) = 2n + 4; \ and \ sm(P_n^2) = 3n.$  Note that in each case the super magic strength of the graph is the same as its magic strength.

Santhosh and Singh [2184] proved that  $C_n \odot P_2$  and  $C_n \odot P_3$  are super edge-magic for all odd  $n \ge 3$  and prove for odd  $n \ge 3$ ,  $sm(C_n \odot P_2) = (15n+3)/2$  and  $(20n+3) \le sm(C_n \odot P_3) \le (21n+3)/2$ .

Gray [928] proves that  $C_3 \cup C_n$  is super edge-magic if and only if  $n \ge 6$  and  $C_4 \cup C_n$  is super edge-magic if and only if  $n \ge 5$ . His computer search shows that  $C_5 \cup 2C_3$  does not have a super edge-magic labeling.

In [2752] Wallis posed the problem of investigating the edge-magic properties of  $C_n$  with the path of length t attached to one vertex. Kim and Park [1376] call such a graph an (n, t)-kite. They prove that an (n, 1)-kite is super edge-magic if and only if n is odd and at (n, 3)-kite is super edge-magic if and only if n is odd and at least 5. Park, Choi, and Bae [1894] show that (n, 2)-kite is super edge-magic if and only if n is even. Wallis [2752] also posed the problem of determining when  $K_2 \cup C_n$  is super edge-magic. In [1894] and [1376] Park et al. prove that  $K_2 \cup C_n$  is super edge-magic if and only if n is even. Kim and Park [1376] show that the graph obtained by attaching a pendent edge to a vertex of degree one of a star is super-edge magic and that a super edge-magic graph with edge magic constant k and q edges satisfies  $q \leq 2k/3 - 3$ .

Lee and Kong [1496] use  $\operatorname{St}(a_1, a_2, \ldots, a_n)$  to denote the disjoint union of the *n* stars  $\operatorname{St}(a_1)$ ,  $\operatorname{St}(a_2)$ ,  $\ldots$ ,  $\operatorname{St}(a_n)$ . They prove the following graphs are super edge-magic:  $\operatorname{St}(m, n)$  where  $n \equiv 0 \mod(m+1)$ ;  $\operatorname{St}(1, 1, n)$ ;  $\operatorname{St}(1, 2, n)$ ;  $\operatorname{St}(1, n, n)$ ;  $\operatorname{St}(2, 2, n)$ ;  $\operatorname{St}(2, 3, n)$ ;  $\operatorname{St}(1, 1, 2, n) \pmod{(m \ge 2)}$ ;  $\operatorname{St}(1, 1, 3, n)$ ;  $\operatorname{St}(1, 2, 2, n)$ ; and  $\operatorname{St}(2, 2, 2, n)$ . They conjecture that  $\operatorname{St}(a_1, a_2, \ldots, a_n)$  is super edge-magic when n > 1 is odd. Gao and Fan [833] proved that  $\operatorname{St}(1, m, n)$ ;  $\operatorname{St}(3, m, m+1)$ ; and  $\operatorname{St}(n, n+1, n+2)$  are super edge-magic, and under certain conditions  $\operatorname{St}(a_1, a_2, \ldots, a_{2n+1})$ ,  $\operatorname{St}(a_1, a_2, \ldots, a_{4n+1})$ , and  $\operatorname{St}(a_1, a_2, \ldots, a_{4n+3})$  are also super edge magic.

In [1682] MacDougall and Wallis investigate the existence of super edge-magic labelings of cycles with a chord. They use  $C_v^t$  to denote the graph obtained from  $C_v$  by joining two vertices that are distance t apart in  $C_v$ . They prove:  $C_{4m+1}^t$   $(m \ge 3)$  has a super edge-magic labeling for every t except 4m - 4 and 4m - 8;  $C_{4m}^t$   $(m \ge 3)$  has a super edge-magic labeling when  $t \equiv 2 \mod 4$ ; and that  $C_{4m+2}^t$  (m > 1) has a super edge-magic labeling for all odd t other than 5, and for t = 2 and 6. They pose the problem of what values of t does  $C_{2n}^t$  have a super edge-magic labeling.

Enomoto, Masuda, and Nakamigawa [727] have proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph. Slamin, Bača, Lin, Miller, Simanjuntak [2419] proved that the friendship graph consisting of n triangles is super edge-magic if and only if n is 3, 4, 5, or 7. Fukuchi proved [805] the generalized Petersen graph P(n, 2) (see §2.7 and at least 5. Baskoro and Ngurah [417] showed that  $nP_3$  is super edge-magic for  $n \ge 4$  and n even.

Hegde and Shetty [1001] showed that a graph is super edge-magic if and only if it is strongly k-indexable (see §4.1 for the definition). Figueroa-Centeno, Ichishima, and Muntaner-Batle [760] proved that a graph is super edge-magic if and only if it is strongly 1-harmonious and that every super edge-magic graph is cordial. They also proved that  $P_n^2$  and  $K_2 \times C_{2n+1}$  are super edge-magic. In [761] Figueroa-Centeno et al. show that the following graphs are super edge-magic:  $P_3 \cup kP_2$  for all k;  $kP_n$  when k is odd;  $k(P_2 \cup P_n)$ when k is odd and n = 3 or n = 4; and fans  $F_n$  if and only if  $n \leq 6$ . They conjecture that  $kP_2$  is not super edge-magic when k is even. This conjecture has been proved by Z. Chen [579] who showed that  $kP_2$  is super edge-magic if and only if k is odd. Figueroa-Centeno et al. proved that the book  $B_n$  is not super edge-magic when  $n \equiv 1, 3, 7 \pmod{8}$  and when n = 4. They proved that  $B_n$  is super edge-magic for n = 2 and 5 and conjectured that for every  $n \ge 5$ ,  $B_n$  is super edge-magic if and only if n is even or  $n \equiv 5 \pmod{8}$ . Yuansheng, Yue, Xirong, and Xinhong [2895] proved this conjecture for the case that nis even. They prove that every tree with an  $\alpha$ -labeling is super edge-magic. Yokomura (see [726]) has shown that  $P_{2m+1} \times P_2$  and  $C_{2m+1} \times P_m$  are super edge-magic (see also [760]). In [762], Figueroa-Centeno et al. proved that if G is a (super) edge-magic 2-regular graph, then  $G \odot K_n$  is (super) edge-magic and that  $C_m \odot K_n$  is super edge-magic. Fukuchi [804] shows how to recursively create super edge-magic trees from certain kinds of existing super edge-magic trees. Ngurah, Baskoro, and Simanjuntak [1858] provide a method for constructing new (super) edge-magic graphs from existing ones. One of their results is that if G has an edge-magic total labeling and G has order p and size p or p-1, then  $G \odot nK_1$  has an edge-magic total labeling.

Ichishima, Muntaner-Batle, Oshima [1042] enlarged the classes of super edge-magic 2-regular graphs by presenting some constructions that generate large classes of super edge-magic 2-regular graphs from previously known super edge-magic 2-regular graphs or pseudo super edge-magic graphs. By virtue of known relationships among other classes of labelings the 2-regular graphs obtained from their constructions are also harmonious, sequential, felicitous and equitable. Their results add credence to the conjecture of Holden et al. [1019] that all 2-regular graphs of odd order with the exceptions of  $C_3 \cup C_4$ ,  $3C_3 \cup C_4$ , and  $2C_3 \cup C_5$  possess a strong vertex-magic total labeling, which is equivalent to super edge-magic labelings for 2-regular graphs. For a 2-regular graph G with 2m + 1 vertices that has a strong vertex-magic total labeling McQuillan and McQuillan [1728] proved that  $G \cup 2mC_3$ ,  $G \cup (2m+2)C_3$ ,  $G \cup mC_8$  and  $G \cup (m+1)C_8$  also have a strong vertex-magic total labeling. Lee and Lee [1498] investigate the existence of total edge-magic labelings and super edge-magic labelings of unicylic graphs. They obtain a variety of positive and negative results and conjecture that all unicyclic are edge-magic total.

Shiu and Lee [2351] investigated edge labelings of multigraphs. Given a multigraph G with q edges they call a bijection from the set of edges of G to  $\{1, 2, \ldots, q\}$  with the property that for each vertex v the sum of all edge labels incident to v is a constant independent of v a supermagic labeling of G. They use  $K_2[n]$  to denote the multigraph consisting of n edges joining 2 vertices and  $mK_2[n]$  to denote the disjoint union of m copies of  $K_2[n]$ . They prove that for m and n at least 2,  $mK_2[n]$  is supermagic if and only if n is even or if both m and n are odd.

In 1970 Kotzig and Rosa [1418] defined the *edge-magic deficiency*,  $\mu(G)$ , of a graph G as the minimum n such that  $G \cup nK_1$  is edge-magic total. If no such n exists they define  $\mu(G) = \infty$ . In 1999 Figueroa-Centeno, Ichishima, and Muntaner-Batle [766] extended this notion to super edge-magic deficiency,  $\mu_s(G)$ , is the analogous way. They prove the following:  $\mu_s(nK_2) = \mu(nK_2) = n - 1 \pmod{2}$ ;  $\mu_s(C_n) = 0$  if n is odd;  $\mu_s(C_n) = 1$  if  $n \equiv 0 \pmod{4}$ ;  $\mu_s(C_n) = \infty$  if  $n \equiv 2 \pmod{4}$ ;  $\mu_s(K_n) = \infty$  if and only if  $n \geq 5$ ;  $\mu_s(K_{m,n}) \leq (m-1)(n-1)$ ;  $\mu_s(K_{2,n}) = n-1$ ; and  $\mu_s(F)$  is finite for all forests F. They also prove that if a graph G has q edges with q/2 odd, and every vertex is even, then  $\mu_s(G) = \infty$  and conjecture that  $\mu_s(K_{m,n}) \leq (m-1)(n-1)$ . This conjecture was proved for m = 3, 4, and 5 by Hegde, Shetty, and Shankaran [1002] using the notion of strongly k-indexable labelings. Baig, Baskoro, and Semaničová-Feňovčíková [219] investigated the super edge-magic deficiency of a forest consisting of stars. Ngurah investigates the (super) edge-magic deficiency of chain graphs in [1853] and Ngurah and Adiwijaya does the same in [1852].

For an (n, t)-kite graph (a path of length t attached to a vertex of an n-cycle) GAhmad, Siddiqui, Nadeem, and Imran [100] proved the following: for odd  $n \ge 5$  and even  $t \ge 4, \mu_s(G) = 1$ ; for odd  $n \ge 5, t \ge 5, t \ne 11$ , and  $t \equiv 3, 7 \pmod{8}, \mu_s(G) \le 1$ ; for  $n \ge 10, n \equiv 2 \pmod{4}$  and  $t = 4, \mu_s(G) \le 1$ ; and for  $t = 5, \mu_s(G) = 1$ .

In [324] Baig, Ahmad, Baskoro, and Simanjuntak provide an upper bound for the super edge-magic deficiency of a forest formed by paths, stars, combs, banana trees, and subdivisions of  $K_{1,3}$ . Baig, Baskoro, and Semaničová-Feňovčíková [325] investigate the super edge-magic deficiency of forests consisting of stars. Among their results are: a forest consisting of  $k \geq 3$  stars has super edge-magic deficiency at most k - 2; for every positive integer n a forest consisting of 4 stars with exactly 1, n, n, and n + 2 leaves has a super edge-magic total labeling; for every positive integer n a forest consisting of 4 stars a super edge-magic total labeling; for every positive integer n a forest consisting of 4 stars a super edge-magic total labeling; for every positive integer n a forest consisting of 4 stars a super edge-magic total labeling; and for every positive integers n and k a forest consisting of k identical stars has super edge-magic deficiency at most 1 when k is even and deficiency 0 when k is odd. In [93] Ahmad, Javaid, Nadeem, and Hasni investigate the super edge-magic deficiency of some families of graphs related to ladder graphs. Kanwal, Javed, and Riasat [1331] give super edge-magic total labelings and the deficiency for forests consisting of extended w-trees, combs, stars and paths. In [96] Ahmad, Nadeem, and Gupta provided bounds for the super edge-magic deficiency of some Toeplitz graphs.

The generalized Jahangir graph  $J_{n,m}$  for  $m \geq 3$  is a graph on nm+1 vertices, consisting of a cycle  $C_{nm}$  with one additional vertex that is adjacent to m vertices of  $C_{nm}$  at distance n to each other on  $C_{nm}$ . In [326] Baig, Imran, Javaid, and Semaničová-Feňovčiková study the super edge-magic deficiencies of the web graph  $Wb_{n,m}$ , the generalized Jahangir graph  $J_{2,n}$ , crown products  $L_n \odot K_1$ ,  $K_4 \odot nK_1$ , and gave the exact value of super edge-magic deficiency for one class of lobsters.

In [765] Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that  $\mu_s(P_m \cup K_{1,n}) = 1$  if m = 2 and n is odd, or m = 3 and n is not congruent to 0 mod 3, whereas in all other cases  $\mu_s(P_m \cup K_{1,n}) = 0$ . They also proved that  $\mu_s(2K_{1,n}) = 1$  when n is odd and  $\mu_s(2K_{1,n}) \leq 1$  when n is even. They conjecture that  $\mu_s(2K_{1,n}) = 1$  in all cases. Other results in [765] are:  $\mu_s(P_m \cup P_n) = 1$  when (m, n) = (2, 2) or (3, 3) and  $\mu_s(P_m \cup P_n) = 0$  in all other cases;  $\mu_s(K_{1,m} \cup K_{1,n}) = 0$  when mn is even and  $\mu_s(K_{1,m} \cup K_{1,n}) = 1$  when mn is odd;  $\mu(P_m \cup K_{1,n}) = 1$  when m = 2 and n is odd and  $\mu(P_m \cup K_{1,n}) = 0$  in all other cases;  $\mu_s(2C_n) = 1$  when m = 2 and n is odd and  $\mu(P_m \cup K_{1,n}) = 0$  in all other cases;  $\mu_s(2C_n) = 1$  when n is even and  $\infty$  when n = 2 (mod 4);  $\mu_s(3C_n) = 0$  when n is odd;  $\mu_s(3C_n) = 1$  when  $n \equiv 0 \pmod{4}$ . They conjecture the following:  $\mu_s(mC_n) = 0$  when  $mn \equiv 2 \pmod{4}$ ;  $\mu_s(mC_n) = 1$  when  $mn \equiv 0 \pmod{4}$ ;  $\mu_s(mC_n) = \infty$  when  $n \equiv 2 \pmod{4}$ ;  $\mu_s(2K_{1,n}) = 1$ ; and if F is a forest with two components, then  $\mu(F) \leq 1$  and  $\mu_s(F) \leq 1$ . Santhosh and Singh [2183] proved: for n odd at least 3,  $\mu_s(K_2 \odot C_n) \leq (n-3)/2$ ; for n > 1,  $1 \leq \mu_s(P_n[P_2]) = \lceil (n-1)/2 \rceil$ ; and for  $n \geq 1$ ,  $1 \leq \mu_s(P_n \times K_4) \leq n$ .

Ichishima and Oshima [1062] prove the following: if a graph G(V, E) has an  $\alpha$ -labeling and no isolated vertices, then  $\mu_s(G) \leq |E| - |V| + 1$ ; if a graph G(V, E) has an  $\alpha$ -labeling, is not sequential, and has no isolated vertices, then  $\mu_s(G) = |E| - |V| + 1$ ; and, if m is even, then  $\mu_s(mK_{1,n}) \leq 1$ . As corollaries of the last result they have:  $\mu_s(2K_{1,n}) = 1$ ; when  $m \equiv 2 \pmod{4}$  and n is odd,  $\mu_s(mK_{1,n}) = 1$ ;  $\mu_s(mK_{1,3}) = 0$  when  $m \equiv 4 \pmod{8}$  or mis odd;  $\mu_s(mK_{1,3}) = 1$  when  $m \equiv 2 \pmod{4}$ ;  $\mu_s(mK_{2,2}) = 1$ ; for  $n \geq 4$ ,  $(n-4)2^{n-2}+3 \leq$  $\mu_s(Q_n) \leq (n-2)2^{n-1}-4$ ; and for  $s \geq 2$  and  $t \geq 2$ ,  $\mu_s(mK_{s,t}) \leq m(st-s-t)+1$ . They conjecture that for  $s \geq 2$  and  $t \geq 2$ ,  $\mu_s(mK_{s,t}) = m(st-s-t)+1$  and pose as a problem determining the exact value of  $\mu_s(Q_n)$ .

Ichishima and Oshima [1060] determined the super edge-magic deficiency of graphs of the form  $C_m \cup C_n$  for m and n even and for arbitrary n when m = 3, 4, 5, and 7. They state a conjecture for the super edge-magic deficiency of  $C_m \cup C_n$  in the general case. Afzal and Aslam [59] investigate the super edge-magic deficiency of various disjoint unions of  $K_{2,n}$  with stars, paths and disjoint union of paths. The *join product* of two graphs is their graph union with additional edges that connect all vertices of the first graph to each vertex of the second graph. In [1862] Ngurah and Simanjuntak investigate the super edge-magic deficiencies of a wheel minus an edge and join products of a path, a star, and a cycle with isolated vertices. They also show that the join product of a super edge-magic graph with isolated vertices has finite super edge-magic deficiency.

A *block* of a graph is a maximal subgraph with no cut-vertex. The *block-cut-vertex* graph of a graph G is a graph H whose vertices are the blocks and cut-vertices in G; two vertices are adjacent in H if and only if one vertex is a block in G and the other is a cut-

vertex in G belonging to the block. A chain graph is a graph with blocks  $B_1, B_2, B_3, \ldots, B_k$ such that for every  $i, B_i$  and  $B_{i+1}$  have a common vertex in such a way that the blockcut-vertex graph is a path. The chain graph with k blocks where each block is identical and isomorphic to the complete graph  $K_n$  is called the  $kK_n$ -path.

Ngurah, Baskoro, and Simanjuntak [1857] investigate the exact values of  $\mu_s(kK_n\text{-path})$ when n = 2 or 4 for all values of k and when n = 3 for  $k \equiv 0, 1, 2 \pmod{4}$ , and give an upper bound for  $k \equiv 3 \pmod{4}$ . They determine the exact super edge-magic deficiencies for fans, double fans, wheels of small order and provide upper and lower bounds for the general case as well as bounds for some complete partite graphs. They also include some open problems. Lee and Wang [1560] show that various chain graphs with blocks that are complete graphs are super edge-magic. In [92] investigate the super edge-magic deficiency of some kites and  $C_n \cup K_2$ .

Figueroa-Centeno and Ichishima [758] introduce the notion of the sequential number  $\sigma(G)$  of a graph G without isolated vertices to be either the smallest positive integer n for which it is possible to label the vertices of G with distinct elements from the set  $\{0, 1, \ldots, n\}$  in such a way that each  $uv \in E(G)$  is labeled f(u) + f(v) and the resulting edge labels are |E(G)| consecutive integers or  $+\infty$  if there exists no such integer n. They prove that  $\sigma(G) = \mu_s(G) + |V(G)| - 1$  for any graph G without isolated vertices, and  $\sigma(K_{m,n}) = mn$ , which settles the conjecture of Figueroa-Centeno, Ichishima, and Muntaner-Batle [766] that  $\mu_s(K_{m,n}) = (m-1)(n-1)$ .

In [1052] Ichishima and Muntaner-Batle define the strong sequential number  $\sigma_s(G)$  of G as the smallest positive integer n for which there exists an injective function from the vertices of G to [0, n] such that when each edge uv is labeled f(u) + f(v), the resulting set of edge labels is [c, c+q-1] for some positive integer c and there exists an integer  $\lambda$  so that  $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$  for all edges uv. Note that for G to have finite  $\sigma_s(G)$ , it must be bipartite. They prove for a graph G of order p,  $\sigma(G) = \mu_s(G) + p - 1$ . From this it follows that the problems of determining the sequential number and super edge-magic deficiency are equivalent and that for any graph G,  $\sigma(G)$  is finite if and only if  $\mu_s(G)$  is finite. They also introduced the following parameter as a measure of how close a graph G is to having an  $\alpha$ -labeling. The *alpha-number*  $\alpha(G)$  of a graph G with q edges is the smallest positive integer n for which there exists an injective function  $f: V(G) \to [0, n]$  such that when each edge uv is labeled |f(u) - f(v)| the resulting set of edge labels is [c, c+q-1] for some positive integer c, and there exists an integer  $\lambda$  so that  $\min \{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$  for each  $uv \in E(G)$ . If no such n exists the alpha-number of G is defined to be  $+\infty$ . Since a graph that admits an  $\alpha$ -labeling is necessarily bipartite, graphs with finite  $\alpha(G)$  are bipartite.

Ichishima and Muntaner-Batle [1052] prove: if every vertex of graph G has even degree and  $|E(G)| \equiv 2 \pmod{4}$ , then  $\sigma(G) = \sigma_s(G) = +\infty$ ; for every graph G of order  $p, \sigma_s(G) = \mu_c(G) + p - 1$ ; and if G is a super edge-magic graph with at least one edge, then the graph  $G + nK_1$  is sequential for every positive integer n. As corollaries they have: for every graph  $\sigma_s(G) = \alpha(G)$ ; a graph G has an  $\alpha$ -labeling if and only if  $\sigma_s(G) = |E(G)|$ ; and if a graph G of order p and size  $q \ge 1$  has a super edge-magic labeling f with  $s = \min\{f(u) + f(v) : uv \in E(G)\}$ , then  $\sigma(G + nK_1) \le s + q + (n-1)p - 2$ ; if G is a graph of order p and size  $q \ge 1$  and G has a super edge-magic labeling f with  $s = \min\{f(u) + f(v) : uv \in E(G)\}$ , then  $\mu_s(G + nK_1) \le s + q + (n-2)(p-1) - 3$ ; and if G is a super edge-magic graph with at least one edge, then the graph  $G + nK_1$  is harmonious and felicitous for any positive integer n.

For a graph G order p and size q Ichishima, Muntaner-Batle, and Oshima [1054] prove the following: if q = p - 1 and  $\beta_s(G) = p - 1$ , then  $\beta(G \odot nK_1) = \beta_s(G \odot nK_1) = (n+1)p - 1$  for every positive integer n; if q > p - 1 and  $\beta_s(G) = q$ , then there exists a supergraph H of G such that  $\beta(H \odot nK_1) = \beta_s(H \odot nK_1) = (n+1)(q+1) - 1$  for every positive integer n; if G has a subgraph H such that  $\beta_s(H) = q , then$  $<math>\beta(H \odot nK_1) = \beta_s(H \odot nK_1) = (n+1)(q+1) - 1$  for every positive integer n; and if G has a subgraph H such that  $\beta_s(H) = q , where H' is a subgraph of H$  $without isolated vertices, then <math>\beta(H \odot nK_1) = \beta_s(H \odot nK_1) = (n+1)(q+1) - 1$  for every positive integer n.

As the concept of super magic strength is effectively defined only for super edge-magic graphs, Ichishima, Muntaner-Batle, and Oshima [1049] generalize it for any nonempty graph as follows. A numbering f of a graph G of order p is a labeling that assigns distinct elements of the set [1, p] to the vertices of G, where each edge uv of G is labeled f(u)+f(v). The strength,  $\operatorname{str}_f(G)$ , of a numbering  $f: V(G) \to [1, p]$  of G is defined by  $\operatorname{str}_f(G) = \max\{f(u) + f(v) | uv \in E(G)\}$ , that is,  $\operatorname{str}_f(G)$  is the maximum edge label of G, and the strength,  $\operatorname{str}(G)$ , of a graph G itself is  $\operatorname{str}(G) = \min\{\operatorname{str}_f(G) | f$  is a numbering of  $G\}$ . A numbering f of a graph G for which  $\operatorname{str}_f(G) = \operatorname{str}(G)$  is called a strength labeling of G. If G is an empty graph, then  $\operatorname{str}(G)$  is undefined. For a graph G of order p they prove the following: if G has order at least 3 and contains a path of order k ( $k \in [2, p-1]$ ) as an induced subgraph, then  $\operatorname{str}(G) \leq 2p - (k-1)$ ; if  $\Delta(G) + 2 \leq \operatorname{str}(G) \leq 2p - 1$ ; and if  $p+m+\min\{p, \delta(G)+m\} \leq \operatorname{str}(G+mK_1) \leq \operatorname{str}(G) + 2m$  for every positive integer m. They determine the exact strength for many basic families of graphs such as paths, cycles complete graphs, ladders, books, and hypercubes. They conclude with six problems and a conjecture.

In [1050] Ichishima, Muntaner-Batle, and Oshima determined the strength of caterpillars and complete *n*-ary *k*-level trees. The strength str (*G*) is also given for graphs *G* obtained by taking the corona of certain graphs and arbitrary number of isolated vertices. They further proved if *G* is a graph of order *p* with  $\delta(G) \geq 1$  and str (*G*) =  $p + \delta(G)$ , then str ( $G \odot nK_1$ ) = (n + 1) p + 1. for every positive integer *n*.

The following result established in [1044] shows the connection between the alphanumber of a graph and its consecutively super edge-magic deficiency. For every graph Gof order p,  $\alpha(G) = \mu_c(G) + p - 1$ . This result shows that the problems of determining the alpha-number and consecutively super edge-magic deficiency are equivalent.

In [1863] Ngurah and Simanjuntak proved that if G is a cycle-free graph with minimum degree one and  $\mu_s(G + K_1) = 0$  then G is either a tree or a forest. They also prove: the join product of some classes of trees and forests with an isolated vertex has zero super edge-magic deficiency; for all but one tree of order at most 6, their join product with an isolated vertex has zero super edge-magic deficiency. For trees T of order at least 7 they proved that if  $\mu_s(T + K_1) = 0$ , then either  $2K_{1,3}$  or  $K_3 \cup K_{1,3}$  is a subgraph of  $T + K_1$ . For the super edge-magic deficiency of the join product of a tree T of order at least 2 with  $m \geq 2$  isolated vertices, they showed that  $\mu_s(T + mK_1) = 0$  if and only if  $T = P_2$ . For a tree  $T \neq P_2$ , they proved  $\mu_s(T + mK_1) \geq \left\lfloor \frac{(m-1)(|V(T)-2|+1)}{2} \right\rfloor$ . They also present results for the super edge-magic deficiency of some chain graphs.

Z. Chen [579] has proved: the join of  $K_1$  with any subgraph of a star is super edgemagic; the join of two nontrivial graphs is super edge-magic if and only if at least one of them has exactly two vertices and their union has exactly one edge; and if a k-regular graph is super edge-magic, then  $k \leq 3$ . Chen also obtained the following: there is a connected super edge-magic graph with p vertices and q edges if and only if  $p - 1 \leq q \leq 2p - 3$ ; there is a connected 3-regular super edge-magic graph with p vertices if and only if  $p \equiv 2 \pmod{4}$ ; and if G is a k-regular edge-magic total graph with p vertices and q edges then  $(p+q)(1+p+q) \equiv 0 \pmod{2d}$  where  $d = \gcd(k-1,q)$ . As a corollary of the last result, Chen observes that  $nK_2 + nK_2$  is not edge-magic total.

Another labeling that has been called "edge-magic" was introduced by Lee, Seah, and Tan in 1992 [1538]. They defined a graph G = (V, E) to be *edge-magic* if there exists a bijection  $f: E \to \{1, 2, \ldots, |E|\}$  such that the induced mapping  $f^+: V \to N$  defined by  $f^+(u) = \sum_{(u,v)\in E} f(u,v) \pmod{|V|}$  is a constant map. Lee (see [1526]) conjectured that a cubic graph with p vertices is edge-magic if and only if  $p \equiv 2 \pmod{4}$ . Lee, Pigg, and Cox [1526] verified this conjecture for prisms and several other classes of cubic graphs. They also show that  $C_n \times K_2$  is edge-magic if and only if n is odd. Shiu and Lee [2351] showed that the conjecture is not true for multigraphs and disconnected graphs. In [2351] Lee's conjecture was modified by restricting it to simple connected cubic graphs. A computer search by Lee, Wang, and Wen [1563] showed that the new conjecture was false for a graph of order 10. Using different methods, Shiu [2331] and Lee, Su, and Wang [1549] gave proofs that it is was false.

Lee, Seah, and Tan [1538] establish that a necessary condition for a multigraph with p vertices and q edges to be edge-magic is that p divides q(q+1) and they exhibit several new classes of cubic edge-magic graphs. They also proved:  $K_{n,n}$   $(n \ge 3)$  is edge-magic and  $K_n$  is edge-magic for  $n \equiv 1, 2 \pmod{4}$  and for  $n \equiv 3 \pmod{4}$   $(n \ge 7)$ . Lee, Seah, and Tan further proved that following graphs are not edge-magic: all trees except  $P_2$ ; all unicyclic graphs; and  $K_n$  where  $n \equiv 0 \pmod{4}$ . Schaffer and Lee [2192] have proved that  $C_m \times C_n$  is always edge-magic. Lee, Tong, and Seah [1555] have conjectured that the total graph of a (p, p)-graph is edge-magic if and only if p is odd. They prove this conjecture for cycles. Lee, Kitagaki, Young, and Kocay [1495] proved that a maximal outerplanar graph with p vertices is edge-magic if and only if p = 6. Shiu [2330] used matrices with special properties to prove that the composition of  $P_n$  with  $\overline{K_n}$  where kn is odd and n is at least 3 have edge-magic labelings. Boonklurb, Narissayaporn, and Singhun [492] show that under some conditions the m-node k-uniform hyperpaths and m-node k-uniform hypercycles are super edge-magic.

For a (p,q)-graph a bijection f from  $V(G) \cup E(G)$  to  $\{1, 2, \ldots, p+q\}$  such that for each edge  $xy \in E(G)$  the value of f(x) + f(xy) + f(y) is either  $k_1, k_2$  or  $k_3$  is said to be an *edge trimagic total labeling*. Regees and Jayasekaran [2112] prove that  $C_m \times P_n$ , the generalized web graph, and the generalized web graph without a center are super edge trimagic total graphs. In [2111] proved that the star type graphs  $P_3 \odot \overline{K_n}$ ,  $B_{m,n}$ ,  $\langle B_{m,n} : 2 \rangle$  and  $\langle K_{1,n} 3 \rangle$  admits edge trimagic total labelings and super edge trimagic total labelings.

Amuthavalli and Sugapriya [151] defined a reverse edge-trimagic labeling on a graph G(V, E) with p vertices and q edges as a one-to-one map that takes the vertices and edges onto the integers  $1, 2, \ldots, p + q$  with the property that for every edge e, when the sum of all vertex labels incident to e is subtracted from edge label f(e), the result is one of three constants. A reverse edge-trimagic labeling is said to be a reverse super edge-trimagic labeling if  $f(V) = \{1, 2, \ldots, p\}$  and  $f(E) = \{p + 1, p + 2, \ldots, p + q\}$ . They investigated the reverse super edge-trimagic labeling of barycentric subdivision of bistars, degree splitting graphs of  $K_{1,n} + K_{1,n}$  and  $K_{1,n} \cup K_{1,n}$ , and the splitting graphs of stars.

Chopra, Dios, and Lee [591] investigated the edge-magicness of joins of graphs. Among their results are:  $K_{2,m}$  is edge-magic if and only if m = 4 or 10; the only possible edgemagic graphs of the form  $K_{3,m}$  are those with m = 3, 5, 6, 15, 33, and 69; for any fixed mthere are only finitely many n such that  $K_{m,n}$  is edge-magic; for any fixed m there are only finitely many trees T such that  $T + \overline{K_m}$  is edge-magic; and wheels are not edge-magic.

Lee, Ho, Tan, and Su [1494] define the *edge-magic index* of a graph G to be the smallest positive integer k such that the graph kG is edge-magic. They completely determined the edge-magic indices of graphs which are stars. In [2347] Shiu, Lam, and Lee give the edge-magic index set of the second power of a path.

For any graph G and any positive integer k the graph G[k], called the k-fold G, is the hypergraph obtained from G by replacing each edge of G with k parallel edges. Lee, Seah, and Tan [1538] proved that for any graph G with p vertices, G[2p] is edge-magic and, if p is odd, G[p] is edge-magic. Shiu, Lam, and Lee [2346] show that if G is an (n + 1, n)-multigraph, then G is edge-magic if and only if n is odd and G is isomorphic to the disjoint union of  $K_2$  and (n - 1)/2 copies of  $K_2[2]$ . They also prove that if G is a (2m + 1, 2m)-multigraph and  $k \ge 2$ , then G[k] is edge-magic if and only if 2m + 1 divides k(k - 1). For a (2m, 2m - 1)-multigraph G and k at least 2, they show that G[k] is edgemagic if 4m divides (2m - 1)k((2m - 1)k + 1) or if 4m divides (2m + k - 1)k. In [2344] Shiu, Lam, and Lee characterize the (p, p)-multigraphs that are edge-magic as  $mK_2[2]$ or the disjoint union of  $mK_2[2]$  and two particular multigraphs or the disjoint union of  $K_2$ ,  $mK_2[2]$ , and four particular multigraphs. They also show for every (2m + 1, 2m + 1)multigraph G, G[k] is edge-magic for all k at least 2. Lee, Seah, and Tan [1538] prove that the multigraph  $C_n[k]$  is edge-magic for  $k \ge 2$ .

Tables 6 and 7 summarize what is known about edge-magic total labelings and super edge-magic total labelings. We use **SEMT** to indicate the graphs have super edge-magic total labelings and **EMT** to indicate the graphs have edge-magic total labelings. A question mark following SEMT or EMT indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová.

Graph	Types	Notes
$P_n$	EMT	[2756]
trees	EMT?	[1419], [2118]
$C_n$	EMT	for $n \ge 3$ [1418], [896], [2129], [443]
$K_n$	EMT	iff $n = 1, 2, 3, 4, 5$ , or 6 [1419], [632], [726] enumeration of all EMT of $K_n$ [2756]
$K_{m,n}$	EMT	[2756], [1418]
crowns $C_n \odot K_1$	EMT	[2866], [2756]
$C_n$ with a single edge attached to one vertex	EMT	[2756]
wheels $W_n$	EMT	iff $n \not\equiv 3 \pmod{4}$ [726], [806]
fans	EMT	[2419], [760], [761]
(p,q)-graph	not EMT	if q even, $p + q \equiv 2 \pmod{4}$ [2118]
$nP_2$	EMT	iff $n$ odd [1418]
$P_n + K_1$	EMT	[2866]
r-regular graph	not EMT	$r \text{ odd and } p \equiv 4 \pmod{8} [632]$
$P_3 \cup nK_2$ and $P_5 \cup nK_2$	EMT	[760], [761]
$P_4 \cup nK_2$	EMT	n  odd  [760], [761]
$nP_i$	EMT	n  odd, i = 3, 4, 5 [2866], [760], [761]
$nP_3$	EMT?	[2866]
$2P_n$	EMT	[760], [761]

Table 6: Summary of Edge-magic Total Labelings

Continued on next page

Graph	Types	Notes
$P_1 \cup P_2 \cup \cdots \cup P_n$	EMT	[760], [761]
$mK_{1,n}$	EMT	[760], [761]
unicylic graphs	EMT?	[1498]
$K_1 \odot nK_2$	EMT	n even [760], [761]
$K_2 \times \overline{K}_n$	EMT	[760], [761]
$nK_3$	EMT	iff $n \neq 2$ odd [760], [761], [1727]
binary trees	EMT	[760], [761]
P(m,n) (generalized Petersen graph see §2.7)	EMT	[760], [761], [1854]
ladders	EMT	[760], [761]
books	EMT	[760], [761]
odd cycle with pendent edges attached to one vertex	EMT	[760], [761]
$P_m \times C_n$	EMT	$n \text{ odd } n \ge 3 \ [2813]$
$P_m \times P_2$	EMT	$m \text{ odd } m \ge 3 \ [2813]$
$K_{1,m} \cup K_{1,n}$	EMT	iff $mn$ is even [764]
$G\odot\overline{K}_n$	EMT	if $G$ is EMT 2-regular [762]

Table 6 – Continued from previous page

Graph	Types	Notes
$C_n$	SEMT	iff $n$ is odd [726]
caterpillars	SEMT	[726], [1418], [1419]
$K_{m,n}$	SEMT	iff $m = 1$ or $n = 1$ [726]
$K_n$	SEMT	iff $n = 1, 2$ or $3 [726]$
trees	SEMT?	[726]
$nK_2$	SEMT	iff $n \text{ odd } [579]$
nG	SEMT	if $G$ is a bipartite or tripartite SEM graph and $n$ odd [764]
mB(n)	SEMT	if $m$ is odd [2393]
$m(P_2 \times P_n)$	SEMT	if $m$ , $n$ are odd [2393]
$r(P_m \times P_n)$	SEMT	if r is odd, $(m, n) \neq (2, 2)$ or $(3,3)$ [2393]
$r(P_3 \times mP_n)$	SEMT	if $r$ is odd [2393]
$K_{1,m} \cup K_{1,n}$	SEMT	if $m$ is a multiple of $n+1$ [764]
$K_{1,m} \cup K_{1,n}$	SEMT?	iff $m$ is a multiple of $n + 1$ [764]
$K_{1,2} \cup K_{1,n}$	SEMT	iff $n$ is a multiple of 3 [764]
$K_{1,3} \cup K_{1,n}$	SEMT	iff $n$ is a multiple of 4 [764]
$P_m \cup K_{1,n}$	SEMT	if $m \ge 4$ is even [764]
$2P_n$	SEMT	iff $n$ is not 2 or 3 [764]
$2P_{4n}$	SEMT	for all $n$ [764]
$mP_n$	SEMT	if $m \equiv 2 \pmod{4}, n \neq 2, 3 \pmod{2393}$

 Table 7: Summary of Super Edge-magic Labelings

Continued on next page

Graph	Types	Notes
$mP_{4n}$	SEMT	if $m \equiv 2 \pmod{4}, n > 1 \pmod{2393}$
$K_{1,m} \cup 2nK_{1,2}$	SEMT	for all $m$ and $n$ [764]
$C_3 \cup C_n$	SEMT	iff $n \ge 6$ even [767], [927]
$C_4 \cup C_n$	SEMT	iff $n \ge 5$ odd [767], [927]
$C_5 \cup C_n$	SEMT	iff $n \ge 4$ even [767]
$C_m \cup C_n$	SEMT	if $m \ge 6$ even, $n \text{ odd } n \ge m/2 + 2 \text{ [767]}$
$C_m \cup C_n$	SEMT?	iff $m + n \ge 9$ and $m + n$ odd [767]
$C_4 \cup P_n$	SEMT	iff $n \neq 3$ [767]
$C_5 \cup P_n$	SEMT	if $n \neq 4$ [767]
$C_m \cup P_n$	SEMT	if $m \ge 6$ even, $n \ge m/2 + 2$ [767]
$P_m \cup P_n$	SEMT	iff $(m, n) \neq (2, 2)$ or $(3, 3)$ [767]
corona $C_n \odot \overline{K}_m$	SEMT	$n \ge 3$ [767]
St(m,n)	SEMT	$n \equiv 0 \pmod{m+1} [1496]$
St(1,k,n)	SEMT	$k = 1, 2 \text{ or } n \ [1496]$
St(2,k,n)	SEMT	k = 2,3 [1496]
St(1,1,k,n)	SEMT	$k = 2,3 \ [1496]$
St(k,2,2,n)	SEMT	k = 1, 2 [1496]
$St(a_1,\ldots,a_n)$	SEMT?	for $n > 1$ odd [1496]
$C_{4m}^t$	SEMT	[1682]

Table 7 – Continued from previous page

Continued on next page

Graph	Types	Notes
$C_{4m+1}^t$	SEMT	[1682]
friendship graph of $n$ triangles	SEMT	iff $n = 3, 4, 5$ , or 7 [2419]
generalized Petersen graph $P(n, 2)$ (see §2.7)	SEMT	if $n \ge 3$ odd [804]
$nP_3$	SEMT	if $n \ge 4$ even [417]
$P_n^2$	SEMT	[760]
$K_2 \times C_{2n+1}$	SEMT	[760]
$P_3 \cup kP_2$	SEMT	for all $k$ [761]
$kP_n$	SEMT	if $k$ is odd [761]
$k(P_2 \cup P_n)$	SEMT	if k is odd and $n = 3, 4$ [761]
fans $F_n$	SEMT	iff $n \le 6$ [761]
books $B_n$	SEMT	if $n$ even [2895]
books $B_n$	SEMT?	if $n \equiv 5 \pmod{8}$ [761]
trees with $\alpha$ -labelings	SEMT	[761]
$P_{2m+1} \times P_2$	SEMT	[726], [760]
$C_{2m+1} \times P_m$	SEMT	[760]
$G\odot\overline{K}_n$	SEMT	if $G$ is SEM 2-regular graph [762]
$C_m \odot \overline{K}_n$	SEMT	[762]
join of $K_1$ with any subgraph of a star	SEMT	[579]
	on next page	then $k \le 3$ [579]

Table 7 – Continued from previous page

Graph	Types	Notes
graph		
G is connected $(p,q)$ -graph	SEMT	<i>G</i> exists iff $p - 1 \le q \le 2p - 3$ [579]
G is connected 3-regular graph on $p$ vertices	SEMT	iff $p \equiv 2 \pmod{4}$ [579]
$nK_2 + nK_2$	not SEMT	[579]

#### 5.3 Vertex-magic Total Labelings

MacDougall, Miller, Slamin, and Wallis [1679] introduced the notion of a vertex-magic total labeling in 1999. For a graph G(V, E) an injective mapping f from  $V \cup E$  to the set  $\{1, 2, \dots, |V| + |E|\}$  is a vertex-magic total labeling if there is a constant k, called the magic constant, such that for every vertex v,  $f(v) + \sum f(vu) = k$  where the sum is over all vertices u adjacent to v (some authors use the term "vertex-magic" for this concept). They prove that the following graphs have vertex-magic total labelings:  $C_n$ ;  $P_n$  (n > 1)2);  $K_{m,m}$  (m > 1);  $K_{m,m} - e$  (m > 2); and  $K_n$  for n odd. They also prove that when n > m + 1,  $K_{m,n}$  does not have a vertex-magic total labeling. They conjectured that  $K_{m,m+1}$  has a vertex-magic total labeling for all m and that  $K_n$  has vertex-magic total labeling for all  $n \geq 3$ . The latter conjecture was proved by Lin and Miller [1598] for the case that n is divisible by 4 while the remaining cases were done by MacDougall, Miller, Slamin, and Wallis [1679]. McQuillan [1726] provided many vertex-magic total labelings for cycles  $C_{nk}$  for  $k \geq 3$  and odd  $n \geq 3$  using given vertex-magic labelings for  $C_k$ . Gray, MacDougall, and Wallis [937] then gave a simpler proof that all complete graphs are vertex-magic total. Krishnappa, Kothapalli, and Venkaiah [1410] gave another proof that all complete graphs are vertex-magic total. Senthil Amutha and Murugesan [2217] characterized connected vertex magic total labeling graphs through their ideals in topological spaces. Among other results, Wang and Zhang [2793] settle a 2006 conjecture raised by Slamin et al., which claims the existence of the vertex magic total labeling of disjoint union of multiple copies of  $C_n \odot K_1$ . Vimal Kumar and Vijayalakshmi [2720] investigated vertex magic total labelings of the middle and total graphs of cycles.

In [1679] MacDougall, Miller, Slamin, and Wallis conjectured that for  $n \ge 5$ ,  $K_n$  has a vertex-magic total labeling with magic constant h if and only if h is an integer satisfying  $n^3 + 3n \le 4h \le n^3 + 2n^2 + n$ . In [1729] McQuillan and Smith proved that this conjecture is true when n is odd. Armstrong and McQuillan [181] proved that if  $n \equiv 2 \pmod{4}$   $(n \ge 6)$  then  $K_n$  has a vertex-magic total labeling with magic constant h for each integer h satisfying  $n^3 + 6n \le 4h \le n^3 + 2n^2 - 2n$ . If, in addition,  $n \equiv 2 \pmod{8}$ , then  $K_n$  has a vertex-magic total labeling with magic constant h for each integer h satisfying  $n^3 + 6n \le 4h \le n^3 + 2n^2 - 2n$ . If, in addition,  $n \equiv 2 \pmod{8}$ , then  $K_n$  has a vertex-magic total labeling with magic constant h for each integer h satisfying  $n^3 + 6n \le 4h \le n^3 + 2n^2 - 2n$ .

satisfying  $n^3 + 4n \leq 4h \leq n^3 + 2n^2$ . They further showed that for each odd integer  $n \geq 5$ ,  $2K_n$  has a vertex-magic total labeling with magic constant h for each integer h such that  $n^3 + 5n \leq 2h \leq n^3 + 2n^2 - 3n$ . If, in addition,  $n \equiv 1 \pmod{4}$ , then  $2K_n$  has a vertex-magic total labeling with magic constant h for each integer h such that  $n^3 + 3n \leq 2h \leq n^3 + 2n^2 - n$ .

In [1727] McQuillan and McQuillan investigate the existence of vertex-magic labelings of  $nC_3$ . They prove: for every even integer  $n \ge 4$ ,  $nC_3$  is vertex-magic (and therefore also edge-magic); for each even integer  $n \ge 6$ ,  $nC_3$  has vertex-magic total labelings with at least 2n - 2 different magic constants; if  $n \equiv 2 \mod 4$ , two extra vertex-magic total labelings with the highest possible and lowest possible magic constants exist; if  $n = 2 \cdot 3^k$ , k > 1,  $nC_3$  has a vertex-magic total labeling with magic constant k if and only if  $(1/2)(15n + 4) \le k \le (1/2)(21n + 2)$ ; if n is odd, there are vertex-magic total labelings for  $nC_3$  with n + 1 different magic constants. In [1725] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if m is an odd positive integer,  $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$  has a vertex-magic total labeling, and Jis any subset of  $I = \{1, 2, \ldots, k\}$  then  $(\bigcup_{i \in J} mC_{n_i}) \cup (\bigcup_{i \in I-J} mC_{n_i})$  has a vertex-magic total labeling.

In [620] Cichacz, Fronček and Singgih introduced a new method to expand some known vertex magic total labelings of 2-regular graphs. The also proved that for odd values of m, if  $(2r + 1) \not\equiv 0 \pmod{3}$  and  $n \not\equiv 0 \pmod{(2r + 1)}$ , then  $2mC_{rn} \cup mC_n$  has a vetex magic total labeling.

Lin and Miller [1598] have shown that  $K_{m,m}$  is vertex-magic total for all m > 1 and that  $K_n$  is vertex-magic total for all  $n \equiv 0 \pmod{4}$ . Phillips, Rees, and Wallis [1927] generalized the Lin and Miller result by proving that  $K_{m,n}$  is vertex-magic total if and only if m and n differ by at most 1. Cattell [547] has shown that a necessary condition for a graph of the form  $H + \overline{K_n}$  to be vertex-magic total is that the number of vertices of H is at least n - 1. As a corollary he gets that a necessary condition for  $K_{m_1,m_2,\dots,m_r,n}$  where nis the largest size of any partite set to be vertex-magic total is that  $m_1 + m_2 + \cdots + m_r \ge n$ . He poses as an open question whether graphs that meet the conditions of the theorem are vertex-magic total. Cattell also proves that  $K_{1,n,n}$  has a vertex-magic total labeling when n is odd and  $K_{2,n,n}$  has a vertex-magic total labeling when  $n \equiv 3 \pmod{4}$ . In [2057] Rahim and Slamin proved the disjoint union of coronas  $C_{t_1} \odot K_1 \cup C_{t_2} \odot K_1 \cup \cdots \cup C_{t_n} \odot K_1$ has a vertex-magic total labeling with magic constant  $6 \sum_{k=1}^n t_k + 1$ .

Miller, Bača, and MacDougall [1748] have proved that the generalized Petersen graphs P(n,k) (see §2.7) for the definition) are vertex-magic total when n is even and  $k \leq n/2-1$ . They conjecture that all P(n,k) are vertex-magic total when  $k \leq (n-1)/2$  and all prisms  $C_n \times P_2$  are vertex-magic total. Bača, Miller, and Slamin [306] proved the first of these conjectures (see also [2421] for partial results) while Slamin and Miller prove the second. Slamin, Prihandoko, Setiawan, Rosita and Shaleh [2422] constructed vertex-magic total labelings for the disjoint union of two copies of P(n,k) and Silaban, Parestu, Herawati, Sugeng, and Slamin [2384] extended this to any number of copies of P(n,k). More generally, they proved that for  $n_j \geq 3$  and  $1 \leq k_j \leq \lfloor (n_j - 1)/2 \rfloor$ , the union  $P(n_1, k_1) \cup P(n_2, k_2) \cup \cdots \cup P(n_t, k_t)$  has a vertex-magic total labeling with vertex magic

constant  $10(n_1 + n_2 + \dots + n_t) + 2$ . In the same article Silaban et al. define the union of t special circulant graphs  $\cup_{j=1}^t C_n(1, m_j)$  as the graph with vertex set  $\{v_i^j \mid 0 \le i \le n-1, 1 \le j \le t\}$  and edge set  $\{v_i^j v_{i+1}^j \mid 0 \le i \le n-1, 1 \le j \le t\} \cup \{v_i^j v_{i+m_j}^j \mid 0 \le i \le n-1, 1 \le j \le t\}$ . They prove that for odd n at least 5 and  $m_j \in \{2, 3, \dots, (n-1)/2\}$ , the disjoint union  $\cup_{j=1}^t C_n(1, m_j)$  has a vertex-magic total labeling with constant 8tn + (n-10/2+3).

MacDougall et al. ([1679], [1681] and [935]) have shown:  $W_n$  has a vertex-magic total labeling if and only if  $n \leq 11$ ; fans  $F_n$  have a vertex-magic total labelings if and only if  $n \leq 10$ ; friendship graphs have vertex-magic total labelings if and only if the number of triangles is at most 3;  $K_{m,n}$  (m > 1) has a vertex-magic total labeling if and only if mand n differ by at most 1. Wallis [2752] proved: if G and H have the same order and  $G \cup H$  is vertex-magic total then so is G + H; if the disjoint union of stars is vertex-magic total, then the average size of the stars is less than 3; if a tree has n internal vertices and more than 2n leaves then it does not have a vertex-magic total labeling. Wallis [2753] has shown that if G is a regular graph of even degree that has a vertex-magic total labeling then the graph consisting of an odd number of copies of G is vertex-magic total. He also proved that if G is a regular graph of odd degree (not  $K_1$ ) that has a vertex-magic total labeling then the graph consisting of any number of copies of G is vertex-magic total.

Gray, MacDougall, McSorley, and Wallis [936] investigated vertex-magic total labelings of forests. They provide sufficient conditions for the nonexistence of a vertex-magic total labeling of forests based on the maximum degree and the number of internal vertices, and leaves or the number of components. They also use Skolem sequences to prove a star forest with each component a  $K_{1,2}$  has a vertex-magic total labeling.

Recall a helm  $H_n$  is obtained from a wheel  $W_n$  by attaching a pendent edge at each vertex of the *n*-cycle of the wheel. A generalized helm H(n,t) is a graph obtained from a wheel  $W_n$  by attaching a path on t vertices at each vertex of the *n*-cycle. A generalized web W(n,t) is a graph obtained from a generalized helm H(n,t) by joining the corresponding vertices of each path to form an *n*-cycle. Thus W(n,t) has (t+1)n+1 vertices and 2(t+1)n edges. A generalized Jahangir graph  $J_{k,s}$  is a graph on ks+1 vertices consisting of a cycle  $C_{ks}$  and one additional vertex that is adjacent to k vertices of  $C_{ks}$  at distance sto each other on  $C_{ks}$ . Rahim, Tomescu, and Slamin [2058] prove:  $H_n$  has no vertex-magic total labeling for any  $n \ge 3$ ; W(n,t) has a vertex-magic total labeling for n = 3 or n = 4and t = 1, but it is not vertex-magic total for  $n \ge 17t + 12$  and  $t \ge 0$ ; and  $J_{n,t+1}$  is vertex-magic total for n = 3 and t = 1, but it does not have this property for  $n \ge 7t + 11$ and  $t \ge 1$ . Recall a flower is the graph obtained from a helm by joining each pendent vertex to the central vertex of the helm. Ahmad and Tomescu [101] proved that flower graph is vertex-magic if and only if the underlying cycle is  $C_3$ .

Fronček, Kovář, and Kovářová [787] proved that  $C_n \times C_{2m+1}$  and  $K_5 \times C_{2n+1}$  are vertexmagic total. Kovář [1421] furthermore proved some general results about products of certain regular vertex-magic total graphs. In particular, if G is a (2r + 1)-regular vertexmagic total graph that can be factored into an (r + 1)-regular graph and an r-regular graph, then  $G \times K_5$  and  $G \times C_n$  for n even are vertex-magic total. He also proved that if G an r-regular vertex-magic total graph and H is a 2s-regular supermagic graph that can be factored into two s-regular factors, then their Cartesian product  $G \times H$  is vertex-magic total if either r is odd, or r is even and |H| is odd.

Ivančo and Polláková [1087] consider supermagic graphs having a saturated vertex (i.e., a vertex that is adjacent to every other vertex). They characterize supermagic graphs  $G + K_1$ , where G is a regular graph, using a connection to vertex-magic total graphs. They prove that if G is a d-regular graph of order n then the join  $G + K_1$  is supermagic if and only if G has a VMT labeling with constant h such that (n - d - 1) is a divisor of the non-negative integer (n + 1)h - n((d + 2)/2)(n(d + 2)/2) + 1). They also prove  $K_{1,n,n}$  is supermagic if and only if  $n \ge 2$ ;  $K_{1,2,2,\dots,2}$  is supermagic except for  $K_{1,2}$ ; and the graph obtained from  $K_{n,n}$   $(n \ge 5)$  by removing all edges in a Hamilton cycle is supermagic. They also consider circulant graphs and prove that the complement of the circulant graph  $C_{2n}(1, n), n \ge 4$ , is supermagic.

MacDougall, Miller, and Sugeng [1680] define a super vertex-magic total labeling of a graph G(V, E) as a vertex-magic total labeling f of G with the additional property that  $f(V) = \{1, 2, \dots, |V|\}$  and  $f(E) = \{|V|+1, |V|+2, \dots, |V|+|E|\}$  (some authors use the term "super vertex-magic" for this concept). They show that a (p,q)-graph that has a super vertex-magic total labeling with magic constant k satisfies the following conditions:  $k = (p+q)(p+q+1)/v - (v+1)/2; k \ge (41p+21)/18;$  if G is connected,  $k \ge (7p-5)/2;$ p divides q(q+1) if p is odd, and p divides 2q(q+1) if p is even; if G has even order either  $p \equiv 0 \pmod{8}$  and  $q \equiv 0$  or 3 (mod 4) or  $p \equiv 4 \pmod{8}$  and  $q \equiv 1$  or 2 (mod 4); if G is r-regular and p and r have opposite parity then  $p \equiv 0 \pmod{8}$  implies  $q \equiv 0 \pmod{4}$ and  $p \equiv 4 \pmod{8}$  implies  $q \equiv 2 \pmod{4}$ . They also show:  $C_n$  has a super vertex-magic total labeling if and only if n is odd; and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjecture that no tree has a super vertex-magic total labeling and that  $K_{4n}$  has a super vertex-magic total labeling when n > 1. The latter conjecture was proved by Gómez in [906]. In [907] Gómez proved that if G is a d-regular graph that has a vertex-magic total labeling and k is a positive integer such that (k-1)(d+1) is even, then kG has a super vertex-magic total labeling. As a corollary, we have that if n and k are odd or if  $n \equiv 0 \pmod{4}$  and n > 4, then  $kK_n$  has a super vertex-magic total labeling. Gómez also shows how graphs with super vertex-magic total labeling can be constructed from a given graph G with super vertex-magic total labeling by adding edges to G in various ways.

Gray and MacDougall [934] establish the existence of vertex-magic total labelings for several infinite classes of regular graphs. Their method enables them to begin with any even-regular graph and from it construct a cubic graph possessing a vertex-magic total labeling. A feature of the construction is that it produces strong vertex-magic total labelings many even order regular graphs. The construction also extends to certain families of non-regular graphs. MacDougall has conjectured (see [1422]) that every rregular (r > 1) graph with the exception of  $2K_3$  has a vertex-magic total labeling. As a corollary of a general result Kovář [1422] has shown that every 2r-regular graph with an odd number of vertices and a Hamiltonian cycle has a vertex-magic total labeling.

Gómez and Kovář [908] proved that a super vertex-magic total labeling of  $kK_n$  exists for n odd and any k, for  $4 < n \equiv 0 \pmod{4}$  and any k, and for n = 4 and k even. They also showed  $kK_{4t+2}$  does not admit a super vertex-magic total labeling for k odd and provide a large number of super vertex-magic total labelings of  $kK_{4t+2}$  for any k based on a super vertex-magic total labeling of  $kK_{4t+1}$ .

Beardon [423] has shown that a necessary condition for a graph with c components, p vertices, q edges and a vertex of degree d to be vertex-magic total is  $(d+2)^2 \leq (7q^2 + (6c+5)q+c^2+3c)/p$ . When the graph is connected this reduces to  $(d+2)^2 \leq (7q^2+11q+4)/p$ . As a corollary, the following are not vertex-magic total: wheels  $W_n$  when  $n \geq 12$ ; fans  $F_n$  when  $n \geq 11$ ; and friendship graphs  $C_3^{(n)}$  when  $n \geq 4$ .

Beardon [425] has investigated how vertices of small degree effect vertex-magic total labelings. Let G(p,q) be a graph with a vertex-magic total labeling with magic constant k and let  $d_0$  be the minimum degree of any vertex. He proves  $k \leq (1 + d_0)(p + q - d_0/2)$ and  $q < (1 + d_0)q$ . He also shows that if G(p,q) is a vertex-magic graph with a vertex of degree one and t is the number of vertices of degree at least two, then  $t > q/3 \geq (p-1)/3$ . Beardon [425] has shown that the graph obtained by attaching a pendent edge to  $K_n$  is vertex-magic total if and only if n = 2, 3, or 4.

Meissner and Zwierzyński [1739] used finding vertex-magic total labelings of graphs as a way to compare the efficiency of parallel execution of a program versus sequential processing.

Swaminathan and Jeyanthi [2557] prove the following graphs are super vertex-magic total:  $P_n$  if and only if n is odd and  $n \ge 3$ ;  $C_n$  if and only if n is odd; the star graph if and only if it is  $P_2$ ; and  $mC_n$  if and only if m and n are odd. In [2558] they prove the following: no super vertex-magic total graph has two or more isolated vertices or an isolated edge; a tree with n internal edges and tn leaves is not super vertex-magic total if t > (n + 1)/n; if  $\Delta$  is the largest degree of any vertex in a tree T with p vertices and  $\Delta > (-3 + \sqrt{1 + 16p})/2$ , then T is not super vertex-magic total; the graph obtained from a comb by appending a pendent edge to each vertex of degree 2 is super vertex-magic total; the graph obtained by attaching a path with t edges to a vertex of an n-cycle is super vertex-magic total if and only if n + t is odd. Ali, Bača, and Bashir [123] proved that  $mP_3$  and  $mP_4$  have no super vertex-magic total labeling

For n > 1 and distinct odd integers x, y and z in [1, n - 1] Javaid, Ismail, and Salman [1095] define the *chordal ring* of order  $n CR_n(x, y, z)$ , as the graph with vertex set  $Z_n$ , the additive group of integers modulo n, and edges (i, i + x), (i, i + y), (i, i + z) for all even i. They prove that  $CR_n(1, 3, n - 1)$  has a super vertex-magic total labeling when  $n \equiv 0$  mod 4 and  $n \geq 8$  and conjecture that for an odd integer  $\Delta$ ,  $3 \leq \Delta \leq n - 3, n \equiv 0 \mod 4$ ,  $CR_n(1, \Delta, n - 1)$  has a super vertex-magic total labeling with magic constant 23n/4 + 2.

The Knödel graphs  $W_{\Delta,n}$  with n even and degree  $\Delta$ , where  $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$  have vertices pairs (i, j) with i = 1, 2 and  $0 \leq j \leq n/2 - 1$  where for every  $0 \leq j \leq n/2 - 1$ and there is an edge between vertex (1, j) and every vertex  $(2, (j + 2^k - 1) \mod n/2)$ , for  $k = 0, 1, \ldots, \Delta - 1$ . Xi, Yang, Mominul, and Wong [2829] have shown that  $W_{3,n}$  is super vertex-magic total when  $n \equiv 0 \mod 4$ .

A vertex magic total labeling of G(V, E) is said to be *E*-super if  $f(E(G)) = \{1, 2, 3, \ldots, |E(G)|\}$ . The cocktail party graph,  $H_{m,n}$   $(m, n \ge 2)$ , is the graph with a vertex set  $V = \{v_1, v_2, \ldots, v_{mn}\}$  partitioned into *n* independent sets  $V = \{I_1, I_2, \ldots, I_n\}$  each of size *m* such that  $v_i v_j \in E$  for all  $i, j \in \{1, 2, \ldots, mn\}$  where  $i \in I_p$ ,  $j \in I_q$ ,  $p \ne q$ .

(The graph  $H_{n,n}$  is the complement of the ladder graph and the dual graph of the *n*cube.) Marimuthu and Balakrishnan [1700] gave some basic properties of such labelings and proved that  $H_{m,n}$  is *E*-super vertex magic. Wang and Zhang [2791] show the following: Hamiltonian even regular graphs of odd order are *E*-super magic; even-regular graphs of odd order that contains a 2-factor consisting of an odd number of odd cycles with the same size are *E*-super vertex magic; graphs that can be decomposed into the sum of two spanning graphs where one is *E*-super magic and one is regular of even degree are *E*-supermagic; even-regular graphs of odd order that contain a 2-factor consisting of an odd number of odd cycles with the same size are *E*-super vertex magic; and circulant graphs with odd order are *E*-super vertex magic. Swaminathan and Jeyanthi [2557] proved that  $mC_n$  is *E*-super magic if and only if both *m* and *n* are odd.

In [1704] Marimuthu and Kumar investigate *E*-super vertex magic labelings of disconnected graphs. They prove: if a graph with p vertices and q edges and even order has an *E*-super vertex magic labeling, then either (i)  $p \equiv 0 \pmod{8}$  and  $q \equiv 0$  or 3 (mod 4), or (ii)  $p \equiv 4 \pmod{8}$  and  $q \equiv 1$  or 2 (mod 4); if an *r*-regular graph *G* of order p has an *E*-super vertex magic labeling, then p and r have opposite parity and (i) if  $p \equiv 0 \pmod{8}$ , then  $q \equiv 0 \pmod{4}$  (ii) if  $p \equiv 4 \pmod{8}$ , then  $q \equiv 2 \pmod{4}$ ;  $mC_n$  is *E*-super vertex magic if and only if  $P_n \cup (m-1)C_n$  is *E*-super vertex magic;  $P_m \cup K_{1,m}$  is not *E*-super vertex magic;  $C_m \cup P_n$  is not *E*-super vertex magic if both m and n have the same parity; the disjoint union of two non-isomorphic suns is not *E*-super vertex magic; and  $mP_3$  is not *E*-super vertex magic for any integer m > 1. They conjecture that  $K_m \cup P_m$  is *E*-super vertex magic if m = 8t + 2.

In [1808] Mutharasu and Kumar generalized the notion of super vertex-magic total labelings as follows. Let G(V, E) be a graph and k be an integer with  $1 \le k \le \operatorname{diam}(G)$ . For  $e \in E(G)$ , let  $E_k(e)$  be the set of all vertices that are at a distance at most k from e and let  $E_k(v)$  be the set of all edges that are at a distance at most k from v (u and v are at distance 1 from the edge uv). A graph G is said to be  $E_k$ -regular with regularity r if, for all edges e,  $|E_k(e)| = r$  for some positive integer r. Note that all nontrivial graphs are  $E_1$ -regular. Let G be a simple graph with p vertices and q edges. A Vsuper vertex magic labeling is a bijection  $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$  such that  $f(V(G)) = \{1, 2, \dots, p\}$  and for each vertex  $v \in V(G), f(v) + \sum_{u \in N(v)} f(uv) = M$ for some positive integer M. A  $V_k$ -super vertex magic labeling ( $V_k$ -SVML) is a bijection  $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$  with the property that  $f(V(G)) = \{1, 2, \dots, p\}$  and for each  $v \in V(G)$ ,  $f(v) + \sum_{e \in E_k(v)} f(e) = M$  for some positive integer M. A graph that admits a  $V_k$ -SVML is called  $V_k$ -super vertex magic. Mutharasu and Kumar gave a necessary and sufficient condition for the existence of  $V_k$ -SVML in graphs, determined the magic constant for  $E_k$ -regular graphs, and obtained results about  $V_2$ -SVML labelings for cycles, complement of cycles, prisms, and a family of circulant graphs.

Balbuena, Barker, Das, Lin, Miller, Ryan, and Slamin [329] call a vertex-magic total labeling of G(V, E) a strongly vertex-magic total labeling if the vertex labels are  $\{1, 2, ..., |V|\}$ . They prove: the minimum degree of a strongly vertex-magic total graph is at least 2; for a strongly vertex-magic total graph G with n vertices and e edges, if  $2e \geq \sqrt{10n^2 - 6n + 1}$  then the minimum degree of G is at least 3; and for a strongly vertex-magic total graph G with n vertices and e edges if  $2e < \sqrt{10n^2 - 6n + 1}$  then the minimum degree of G is at most 6. They also provide strongly vertex-magic total labelings for certain families of circulant graphs. In [1725] McQuillan provides a technique for constructing vertex-magic total labelings of 2-regular graphs. In particular, if m is an odd positive integer,  $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_k}$  has a strongly vertex-magic total labeling, and J is any subset of  $I = \{1, 2, \ldots, k\}$  then  $(\bigcup_{i \in J} mC_{n_i}) \cup (\bigcup_{i \in I-J} mC_{n_i})$  has a strongly vertex-magic total labeling.

Gray [928] proved that if G is a graph with a spanning subgraph H that possesses a strongly vertex-magic total labeling and G - E(H) is even regular, then G also possesses a strongly vertex-magic total labeling. As a corollary one has that regular Hamiltonian graphs of odd order have a strongly vertex-magic total labelings.

In a series of papers Gray and MacDougall expand on McQuillan's technique to obtain a variety of results. In [931] Gray and MacDougall show that for any  $r \ge 4$ , every *r*-regular graph of odd order at most 17 has a strong vertex-magic total labeling. They also show that several large classes of *r*-regular graphs of even order, including some Hamiltonian graphs, have vertex-magic total labelings. They conjecture that every 2-regular graph of odd order possesses a strong vertex-magic total labeling if and only if it is not of the form  $(2t-1)C_3 \cup C_4$  or  $2tC_3 \cup C_5$ . They include five open problems.

In [933] Gray and MacDougall introduce a procedure called a *mutation* that transforms one vertex-magic totaling labeling into another one by swapping sets of edges among vertices that may result in different labeling of the same graph or a labeling of a different graph. Among their results are: a description of all possible mutations of a labeling of the path and the cycle; for all  $n \ge 2$  and all *i* from 1 to n - 1 the graphs obtained by identifying an end points of paths of lengths i, i + 1, and 2n - 2i - 1 have a vertex-magic total labeling; for odd *n*, the graph obtained by attaching a path of length n - m to an *m* cycle, (such graphs are called (m; n - m)-kites) have strong vertex-magic total labelings for  $m = 3, \ldots, n - 2$ ;  $C_{2n+1} \cup C_{4n+4}$  and  $3C_{2n+1}$  have a strong vertex-magic total labeling; and for  $n \ge 2$ ,  $C_{4n} \cup C_{6n-1}$  has a strong vertex-magic total labeling. They conclude with three open problems.

Kimberley and MacDougall [1378] studied mutations that involve labelings of regular graphs into labelings of other regular graphs. They present results of extensive computations which confirm how prolific this procedure is. These computations add weight to MacDougall's conjecture that all nontrivial regular graphs are vertex-magic.

Gray and MacDougall [932] show how to construct vertex-magic total labelings for several families of non-regular graphs, including the disjoint union of two other graphs already possessing vertex-magic total labelings. They prove that if G is a d-regular graph of order v and H a t-regular graph of order u with each having a strong vertex magic total labeling and  $vd^2 + 2d + 2v + 2u = 2tvd + 2t + ut^2$  then  $G \cup H$  possesses a strong vertex-magic total labeling. They also provide bounds on the minimum degree of a graph with a vertex-magic total labeling.

In [934] Gray and MacDougall establish the existence of vertex-magic total labelings for several infinite classes of regular graphs. Their method enables them to begin with any even-regular graph and construct a cubic graph possessing a vertex-magic total labeling that produces strong vertex-magic total labelings for many even order regular graphs. The construction also extends to certain families of non-regular graphs.

In [1820] Nagaraj, Ponnappan, and Prabakaran define a vertex-magic total labeling of G to be an even vertex magic total labeling if the set of vertex labels is  $\{2, 4, 6, \ldots, 2|V(G)|\}$ . They prove the following:  $C_n$  is even vertex magic total if and only if n is odd;  $rC_s$  is even vertex magic total if and only if r and s are odd;  $C_n \odot K_1$  is even vertex magic total; wheels are not even vertex magic total; fans (excluding  $C_3$ ) are not even vertex magic total; kites are not even vertex magic total; and  $K_{4n}$  is not even vertex magic total. In [1823] they prove that  $C_3 \cup C_{2t}$  (t > 2) and  $C_4 \cup C_{2t+1}$  ( $t \ge 2$ ) have even vertex magic total labelings. In [1822] Nagaraj, Ponnappan, and Prabakan prove that the union of any finite numbers of graphs of the form  $C_n \odot K_1$  (the sizes may vary) has an even vertex magic total labeling.

Rahim and Slamin [2056] give the bounds for the number of vertices for Jahangir graphs, helms, webs, flower graphs and sunflower graphs when the graphs considered are not vertex-magic total. Thirusangu, Nagar, and Rajeswari [2590] show that certain Cayley digraphs of cyclic groups have vertex-magic total labelings.

Balbuena, Barker, Lin, Miller, and Sugeng [334] call vertex-magic total labeling an *a-vertex consecutive magic* labeling if the vertex labels are  $\{a, a + 1, \ldots, a + |V|\}$ . For an *a*-vertex consecutive magic labeling of a graph G with p vertices and q edges they prove: if G has one isolated vertex, then a = q and  $(p - 1)^2 + p^2 = (2q + 1)^2$ ; if q = p - 1, then p is odd and a = p - 1; if p = q, then p is odd and if G has minimum degree 1, then a = (p + 1)/2 or a = p; if G is 2-regular, then p is odd and a = 0 or p; and if G is r-regular, then p and r have opposite parities. They also define an *b-edge consecutive* magic labeling analogously and state some results for these labelings.

Wood [2820] generalizes vertex-magic total and edge-magic total labelings by requiring only that the labels be positive integers rather than consecutive positive integers. He gives upper bounds for the minimum values of the magic constant and the largest label for complete graphs, forests, and arbitrary graphs.

Exoo, Ling, McSorley, Phillips, and Wallis [744] call a function  $\lambda$  a totally magic labeling of a graph G if  $\lambda$  is both an edge-magic total and a vertex-magic total labeling of G. A graph with such a labeling is called totally magic. Among their results are:  $P_3$ is the only connected totally magic graph that has a vertex of degree 1; the only totally magic graphs with a component  $K_1$  are  $K_1$  and  $K_1 \cup P_3$ ; the only totally magic complete graphs are  $K_1$  and  $K_3$ ; the only totally magic complete bipartite graph is  $K_{1,2}$ ;  $nK_3$  is totally magic if and only if n is odd;  $P_3 \cup nK_3$  is totally magic? That question was answered by Calhoun, Ferland, Lister, and Polhill [540] who proved that if  $K_{1,m} \cup nK_3$  is totally magic then m = 2 and  $K_{1,2} \cup nK_3$  is totally magic if and only if n is even.

McSorley and Wallis [1731] examine the possible totally magic labelings of a union of an odd number of triangles and determine the spectrum of possible values for the sum of the label on a vertex and the labels on its incident edges and the sum of an edge label and the labels of the endpoints of the edge for all known totally magic graphs. Gray and MacDougall [929] define an order n sparse semi-magic square to be an  $n \times n$  array containing the entries  $1, 2, \ldots, m$  once (for some  $m < n^2$ ), has its remaining entries equal to 0, and whose rows and columns have a constant sum of k. They prove some basic properties of such squares and provide constructions for several infinite families of squares, including squares of all orders  $n \ge 3$ . Moreover, they show how such arrays can be used to construct vertex-magic total labelings for certain families of graphs.

In Tables 8, 9 and 10, **VMT** means vertex-magic total labeling, **SVMT** means super vertex magic total, and **TM** means totally magic labeling. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2007.

Graph	Types	Notes
$C_n$	VMT	[1679]
$P_n$	VMT	n > 2 [1679]
$K_{m,m} - e$	VMT	m > 2 [1679]
$K_{m,n}$	VMT	iff $ m - n  \le 1$ [1927], [1679], [1681]
$K_n$	VMT	for $n \text{ odd } [1679]$ for $n = 2 \pmod{4}$ $n > 2 [1598]$
$nK_3$	VMT	iff $n \neq 2$ [760], [761], [1727]
$mK_n$	VMT	$m \ge 1, \ n \ge 4 \ [1730]$
Petersen $P(n,k)$	VMT	[306]
prisms $C_n \times P_2$	VMT	[2421]
$W_n$	VMT	iff $n \le 11$ [1679], [1681]
$F_n$	VMT	iff $n \le 10$ [1679], [1681]
friendship graphs	VMT	iff # of triangles $\leq 3$ [1679], [1681]
G + H	VMT	V(G)  =  V(H)  and $G \cup H$ is VMT [2752]

Table 8: Summary of Vertex-magic Total Labelings

Continued on next page

Graph	Types	Notes
unions of stars	VMT	[2752]
tree with $n$ internal vertices	not VMT	[2752]
and more than $2n$ leaves		
nG	VMT	n  odd, G  regular of even
		degree, VMT [2753]
		G is regular of odd
		degree, VMT, but not $K_1$ [2753]
$C_n \times C_{2m+1}$	VMT	[787]
		[]
$K_5 \times C_{2n+1}$	VMT	[787]
<i>a a</i>		$C_{2} \rightarrow 1$ $\ldots$ $L_{2}$ VMT [1491]
$G \times U_{2n}$	V IVI I	G 2r + 1-regular VM1 [1421]
$C \times K_{\tau}$	VMT	$C 2r \pm 1$ -regular VMT [1/21]
$O \land H_5$	V IVI I	
$G \times H$	VMT	G r-regular VMT, r odd
		or $r$ even and $ H $ odd,
		H 2s-regular supermagic [1421]

Table 8 – Continued from previous page

# Table 9:Summary of Super Vertex-magic TotalLabelings

Graph	Types	Notes
$P_n$	SVMT	iff $n > 1$ is odd [2557]
$C_n$	SVMT	iff $n$ is odd [2557] and [1680]
$K_{1,n}$	SVMT	iff $n = 1$ [2557]
$mC_n$	SVMT	iff $m$ and $n$ are odd [2557]
W <sub>n</sub>	not SVMT	[1680]
ladders	not SVMT	[1680]
friendship graphs	not SVMT	[1680]

Continued on next page

Table 9 – Continued from previous page

Graph	Types	Notes
$K_{m,n}$	not SVMT	[1680]
dragons (see $\S2.2$ )	SVMT	iff order is even [2558], [2558]
Knödel graphs $W_{3,n}$	SVMT	$n \equiv 0 \pmod{4} \ [2829]$
graphs with min. deg. 1	not SVMT	[1680]
$K_{4n}$	SVMT	n > 1 [906]

 Table 10: Summary of Totally Magic Labelings

Graph	Types	Notes
$P_3$	ТМ	the only connected
		TM graph with vertex
		of deg 1 $[744]$
$K_n$	ТМ	iff $n = 1, 3$ [744]
$K_{m,n}$	TM	iff $K_{m,n} = K_{1,2}$ [744]
$nK_3$	TM	iff $n$ is odd [744]
$P_3 \cup nK_3$	1'M	iff $n$ is even [744]
$K_{1,m} \cup nK_3$	TM	iff $m = 2$ and $n$ is even [540]

### 5.4 *H*-Magic Labelings

In 2005 Gutiérrez and Lladó [943] introduced the notion of an *H*-magic labeling of a graph, which generalizes the concept of a magic valuation. Let *H* and G = (V, E) be finite simple graphs with the property that every edge of *G* belongs to at least one subgraph isomorphic to *H*. A bijection  $f: V \cup E \to \{1, \ldots, |V| + |E|\}$  is an *H*-magic labeling of *G* if there exists a positive integer m(f), called the magic sum, such that for any subgraph H'(V', E') of *G* isomorphic to *H*, the sum  $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$  is equal to the magic sum, m(f). A graph is *H*-magic if it admits an *H*-magic labeling. If, in addition, the *H*-magic labeling *f* has the property that  $\{f(v)\}_{v \in V} = \{1, \ldots, |V|\}$ , then the graph is *H*-supermagic. A  $K_2$ -magic labeling is also known as an edge-magic total labeling. Gutiérrez and Lladó investigate the cases where  $G = K_n$  or  $G = K_{m,n}$  and H is a star or a path. Among their results are: a *d*-regular graph is not  $K_{1,h}$  for any 1 < h < d;  $K_{n,n}$  is  $K_{1,n}$ -magic for all n;  $K_{n,n}$  is not  $K_{1,n}$ -supermagic for n > 1; for any integers 1 < r < s,  $K_{r,s}$  is  $K_{1,h}$ -supermagic if and only if h = s;  $P_n$  is  $P_h$ -supermagic for all  $2 \le h \le n$ ;  $K_n$  is not  $P_h$ -magic for any  $2 < h \le n$ ;  $C_n$  is  $P_h$ -magic for any  $2 \le h < n$  such that gcd(n, h(h - 1)) = 1. They also show that by uniformly gluing copies of H along edges of another graph G, one can construct connected H-magic graphs from a given 2-connected graph H and an H-free supermagic graph G.

Lladó and Moragas [1624] studied cycle-magic graphs. They proved: wheels  $W_n$  are  $C_3$ -magic for odd n at least 5; for  $r \ge 3$  and  $k \ge 2$  the windmill graphs  $C_r^{(k)}$  (the one-point union of k copies of  $C_r$ ) are  $C_r$ -supermagic; and if G is  $C_4$ -free supermagic graph of odd size, then  $G \times K_2$  is  $C_4$ -supermagic. As corollaries of the latter result, they have that for n odd, prisms  $C_n \times K_2$  and books  $K_{1,n} \times K_2$  are  $C_4$ -magic. They define a subdivided wheel  $W_n(r,k)$  as the graph obtained from a wheel  $W_n$  by replacing each radial edge  $vv_i, 1 \leq i \leq n$  by a  $vv_i$ -path of size  $r \geq 1$ , and every external edge  $v_iv_{i+1}$  by a  $v_iv_{i+1}$ -path of size  $k \ge 1$ . They prove that  $W_n(r,k)$  is  $C_{2r+k}$ -magic for any odd  $n \ne 2r/k+1$  and that  $W_n(r,1)$  is  $C_{2r+1}$ -supermagic. They also prove that the graph obtained by joining the end points of any number of internally disjoint paths of length  $p \ge 2$  is  $C_{2p}$ -supermagic. Asif, Ali, Numan, and Semaničová-Feňovčíková [206] proved that if G is  $C_r$ -(super)magic, then so is nG and that  $P_m \times P_n$   $(m, n \ge 4)$  is  $C_4$ -supermagic. In [2011] Pradipta and Salman define a *calendula graph*, denoted by  $Cl_{m,n}$ , as the graph constructed from  $C_m$ and m copies of  $C_n, C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ , and grafting the *i*-th edge of  $C_m$  to an edge of  $C_{n_i}$ for each *i*. They provide some cycle-supermagic labelings of calendula graphs. Chithra, Marimuthu, and Kumar [586] provided some basic results on the magic constant of graphs, on cycle-supermagic labelings of generalized splitting graphs, and proved that  $mC_n$  is cycle -supermagic for m > 2 and n > 3.

A decomposition of a graph G into isomorphic copies of a graph H is H-magic if there is a bijection f from  $V(G) \cup E(G)$  onto  $\{0, 1, \ldots, |V(G)| + |E(G)| - 1\}$  such that the sum of labels of edges and vertices of each copy of H in the decomposition is constant. By using the results on the sumset partition problem, Inayah, Lladó, and Moragas [1067] show that  $K_{2m+1}$  admits T-magic decompositions by any graceful tree with m edges. They address analogous problems for complete bipartite graphs and for antimagic and (a, d)-antimagic decompositions.

An edge of H-magic graph G is said to be a good edge if it belongs to only one subgraph isomorphic to H. For  $s \geq 1$ , B is the collection of good edges obtained by choosing exactly s good edges from each subgraph isomorphic to H in G. A uniform subdivided graph  $\mathcal{G}$  of the graph G is obtained by subdividing all edges of B with  $k \geq 1$  vertices. A nonuniform subdivided graph is obtained by subdividing the edges of  $E(G) \setminus B$ . Rizvi, Khalid, Ali, Miller, and Ryan [2126] prove that if a graph G is a  $C_n$ -supermagic graph then its uniform subdivided graph  $\mathcal{G}$  is  $C_{n+sk}$ -(super)magic for positive integers n, s, and k. Using known results on the cycle-supermagicness they immediately obtain that uniform subdivided graphs of fans, antiprisms, triangular ladders, ladders and grids are cycle-(super)magic. They also prove that some special nonuniform subdivisions of fans and triangular ladders are cycle-supermagic.

Jeyanthi and Muthuraja [1191] established that  $P_{m,n}$  is  $C_{2m}$ -supermagic for all  $m, n \geq 2$  and the splitting graph of  $C_n$  is  $C_4$ -supermagic for  $n \neq 4$ . Nirmalasari Wijaya, Ryan, and Kalinowski [1869] show that for odd n and arbitrary k, the firecracker  $F_{k,n}$  is  $F_{2,n}$ -supermagic, the banana tree  $B_{k,n}$  is  $B_{1,n}$ -supermagic, and flower graphs are  $C_3$ supermagic. Kojima [1401] proved that for two positive integers m and t with  $m > t \ge 2$ , if  $C_m$  is  $P_t$ -supermagic, then  $C_{3m}$  is also  $P_t$ -supermagic and for t = 2, 3, 4, or 9 and  $C_n$  is  $P_t$ -supermagic if and only if n is odd with n > t. Nirmalasari Wijaya, Ryan, and Kalinowski [1847] proved that every d-dimensional grid graph (d > 2) is  $Q_d$ -supermagic where  $Q_d$  is the *d*-cube. Pu, Numan, Butt, Asif, Rafique, and Shao [2039] showed that toroidal fullerenes, Klein-bottle fullerenes, and the disjoint union of toroidal and Kleinbottle fullerenes are  $C_6$ -supermagic and the subdivision of toroidal fullerenes, Klein-bottle fullerenes, and any graph homeomorphic to a toroidal fullerene or Klein-bottle fullerene are cyclic-supermagic. Ulfatimah, Roswitha, and Kusmayadi [2607] proved that a star with one or more appended edges at each end-point admits a double star  $S_{2,2}$ -supermagic labeling and  $L_m \odot P_n$  admits supermagic labeling of the one-point union of  $C_3$  and  $C_4$  for  $m, n \ge 2.$ 

The edge corona path graph  $G_m \diamond P_n$  is the graph obtained from one copy of the gear graph  $G_m$  and 3m copies of  $P_n$ ,  $P_n^i$ , by joining two end vertices of  $e_i \in E(G_m)$  to every vertex  $v_j inV(P_n)$  in the *i*-th copy of  $G_m$  with i = 1, 2, ..., 3m and j = 1, 2, ..., n. Noviati, Martini, and Indriati [1872] provided a  $C_3 \diamond P_n$ -supermagic labeling for  $f_n \diamond P_n$  and a  $P_3 \diamond P_n$ -supermagic labeling for  $S_n \diamond P_n$  for odd  $n \geq 3$ .

Rizvi, Ali, and Hussian [2124] proved: the disjoint union of two or more copies of G is  $C_3$ -supermagic when G is a fan, triangular ladder, wheel, or a generalized antiprism; the disjoint union of two or more copies of G is  $C_3$ -supermagic when G is a ladder or a book;  $sF_{n+1} \cup kF_n$  is  $C_3$ -supermagic; and  $sL_{n+1} \cup kL_n$  is  $C_4$ -supermagic. Khalid, Rizvi, and Ali [1360] investigated whether the disjoint union of isomorphic copies of a connected cycle-supermagic graph is cycle-supermagic or not. They also study cycle-supermagic labelings for the disjoint union of isomorphic copies of fans, ladders, triangular ladders, wheels, books, and generalized antiprisms as well as disjoint unions of non-isomorphic copies of ladders and fans. Ali, Rizvi, Semaničová-Feňovčíková [129] proved that the disjoint union of an arbitrary number of isomorphic copies of prisms  $C_n \times P_m$ ,  $m \ge 2$  and  $n \ge 3$ ,  $n \ne 4$ , is  $C_4$ -supermagic. They propose an open problem to find a  $C_4$ -supermagic labeling of the graph  $t(C_4 \times P_m)$  for  $m \ge 2$  and  $t \ge 1$ .

Liang [1585] proved the following: if there exist an even integer k and  $m_i \equiv 0 \pmod{k}$ for every i in [1,n], then there exist  $K_{k,k}$ - and  $C_{2k}$ -supermagic decompositions of  $K_{m_1,\ldots,m_n}$ ; if k and  $t_n \geq k$  are even integers, then for any positive integers  $t_i \equiv 0 \pmod{k}$ , i in [1, n-1], there exists a  $C_{2k}$ -supermagic decomposition of  $K_{t_1,\ldots,t_{n-1},t_n}$ ; if there exists an even integer k and  $K_{m,n}$  is  $C_{2k}$ -decomposable, then there exists a  $C_{2k}$ -supermagic decomposition of  $K_{m,n}$ ; and if G is a graph with p vertices and p edges, H is a graph with q vertices and q edges, and there is an H-supermagic decomposition of G, then there exists an Hsupermagic decomposition of nG. In [2812] Wichianpaisarn and Mato gave necessary and sufficient conditions for the existence of  $K_{1,n-1}$ -supermagic decomposition of  $K_{n,n}$  minus a one-factor.

In [1715] Maryati, Baskoro, and Salman provided  $P_n$ -(super) magic labelings of subdivisions of stars, shrubs and banana trees. Ngurah, Salman, and Sudarsana [1860] construct  $C_n$ -(super) magic labelings for some fans and ladders. For any connected graph H, Maryati, Salman, Baskoro, and Irawati [1718] proved that the disjoint union of k isomorphic copies of a connected graph H is a H-supermagic graph if and only if |V(H)| + |E(H)|is even or k is odd. In [1716] Maryati, Baskoro, Salman, and Irawati give some necessary conditions for any  $P_n$ -magic graph and provide some  $P_n$ -supermagic labelings of a cycle with some pendent edges and its subdivisions.

The *m*-shadow of graph G,  $D_m(G)$ , is a graph obtained by taking *m* copies of G, namely,  $G_1, G_2, \ldots, G_m$ , and then joining every vertex u in  $G_i, i \in \{1, 2, \ldots, m-1\}$ , to the neighbors of the corresponding vertex v in  $G_{i+1}$ . Agustin, Susanto, Dafik, Prihandini, Alfarisi, and Sudarsana [56] studied the *H*-supermagic labelings of  $D_m(G)$  where G are paths and cycles.

Kojima [1401] proved the following. Let G be a  $C_4$ -free super edge-magic (p, q)-graph with the minimum degree at least one and  $m \ge 2$ . If q odd and m = 2 or  $|p - q| \ge 2$ , then  $P_m \times G$  is  $C_4$ -supermagic; if p is odd and m = 2 or |p - q| = 1 and  $m \le 5$ , then  $P_m \times G$  is  $C_4$ -supermagic; if  $n \ge 3$  is odd and m is even, then  $P_2 \times (C_n \odot \overline{K_m})$  is  $C_4$ -supermagic; if  $n \ge 3$  is odd and m is odd, then  $P_2 \times (C_n \odot \overline{K_m})$  is not  $C_4$ -supermagic; if G is a caterpillar, then  $P_m \times G$  is  $C_4$ -supermagic for  $m \ge 2$ ; and  $P_m \times C_n$  is  $C_4$ -supermagic for  $m \ge 2$  and  $n \ge 3$ . The latter result solved an open problem in [1861]. Kojma also proved that if a  $C_4$ -free bipartite (p, p - 1)-graph G with the minimum degree at least one and partite sets U and V has a super edge-magic labeling f of G such that  $f(U) = \{1, 2, \ldots, |U|\}$ , then  $P_m \times (2G)$  is  $C_4$ -supermagic.

Maryati, Salman, Baskoro, Ryan, and Miller [1719] define a shackle as a graph obtained from nontrivial connected graphs  $G_1, G_2, \ldots, G_k$   $(k \ge 2)$  such that  $G_s$  and  $G_t$  have no common vertex for every s and t in [1, k] with  $|s - t| \ge 2$ , and for every i in [1, k -1],  $G_i$  and  $G_{i+1}$  share exactly one common vertex that are all distinct. They prove that shackles and amalgamations constructed from copies of a connected graph H is Hsupermagic. (Recall for finite collection of graph  $G_1, G_2, \ldots, G_k$  with a fixed vertex  $v_i$ from each  $G_i$ , an amalgamation, Amal $G_i, v_i$ ), is the graph obtained by identifying the  $v_i$ .) Ashari and Salman [202] gave sufficient conditions for  $(H_1, H_2)$ -supermagic labelings for shackles involving cycles, flowers, and prisms.

Ngurah, Salman, and Susilowati [1861] proved the following: chain graphs with identical blocks each isomorphic to  $C_n$  are  $C_n$ -supermagic; fans are  $C_3$ -supermagic; ladders and books are  $C_4$ -supermagic;  $K_{1,n} + K_1$  are  $C_3$ -supermagic; grids  $P_m \times P_n$  are  $C_4$ -supermagic for  $m \ge 3$  and n = 3, 4, and 5. They pose the case that  $P_m \times P_n$  are  $C_4$ -supermagic for n > 5 as an open problem. They also have some results on  $P_t$ -(super) magic labelings of cycles.

Roswitha, Baskoro, Maryati, Kurdhi, and Susanti [2141] proved: the generalized Jahangir graph  $J_{k,s}$  is  $C_{s+2}$ -supermagic;  $K_{2,n}$  is  $C_4$ -supermagic; and  $W_n$  for n even and  $n \ge 4$  is  $C_3$ -supermagic. As an open problem they asked if  $K_{m,n}$ ,  $2 < m \le n$ , admits a  $C_{2m}$ -

supermagic labeling. Roswitha and Baskoro [2142] proved that double stars, caterpillars, firecrackers, and banana trees admit star-supermagic labelings.

Maryati, Salman, and Baskoro [1717] characterized all graphs G such that the disjoint union of copies of G is G-supermagic. They also showed: the disjoint union of any paths is  $mP_n$ -supermagic for certain values of m and n; some subgraph amalgamations of graphs G are G-supermagic; and for any subgraph H of G Amal(G, H, k) is G-supermagic. Salman and Maryati [2167] proved that Amal $(G, P_n, k)$  is G-supermagic.

Selvagopal and Jeyanthi proved: for any positive integer n, a the k-polygonal snake of length n is  $C_k$ -supermagic [2203]; for  $m \ge 2$ , n = 3, or n > 4,  $C_n \times P_m$  is  $C_4$ -supermagic [1231];  $P_2 \times P_n$  and  $P_3 \times P_n$  are  $C_4$ -supermagic for all  $n \ge 2$  [1231]; the one-point union of any number of copies of a 2-connected H is H-magic [1229]; graphs obtained by taking copies  $H_1, H_2, \ldots, H_n$  of a 2-connected graph H and two distinct edges  $e_i, e'_i$  from each  $H_i$ and identifying  $e'_i$  of  $H_i$  with  $e_{i+1}$  of  $H_{i+1}$  where  $|V(H)| \ge 4$ ,  $|E(H)| \ge 4$  and n is odd or both n and |V(H)| + |E(H)| are even are H-supermagic [1229]. For simple graphs H and G the H-supermagic strength of G is the minimum constant value of all H-magic total labelings of G for which the vertex labels are  $\{1, 2, \ldots, |V|\}$ . Jeyanthi and Selvagopal [1230] found the  $C_n$ -supermagic strength of n-polygonal snakes of any length and the H-supermagic strength of a chain of an arbitrary 2-connected simple graph.

Let  $H_1, H_2, \ldots, H_n$  be copies of a graph H. Let  $u_i$  and  $v_i$  be two distinct vertices of  $H_i$  for  $i = 1, 2, \ldots, n$ . The chain graph  $H_n$  of H of length n is the graph obtained by identifying the vertices  $u_i$  and  $v_{i+1}$  for  $i = 1, 2, \ldots, n-1$ . In [1228] Jayanthi and Selvagopal show that a chain graph of any 2-connected simple graph H is H-supermagic and if H is a 2-connected (p, q) simple graph, then  $H_n$  is H-supermagic if p + q is even or p + q + n is even.

The antiprism on 2n vertices has vertex set  $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$  and edge set  $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$  (subscripts are taken modulo n). Jeyanthi, Selvagopal, and Sundaram [1233] proved the following graphs are  $C_3$ -supermagic: antiprisms, fans, and graphs obtained from the ladders  $P_2 \times P_n$  with the two paths  $v_{1,1}, \ldots, v_{1,n}$  and  $v_{2,1}, \ldots, v_{2,n}$  by adding the edges  $v_{1,j}v_{2,j+1}$ .

Jeyanthi and Selvagopal [1232] show that for any 2-connected simple graph H the edge amalgamation of a finite number of copies of H is H-supermagic. They also show that the graph obtained by picking one endpoint  $v_i$  from each of k copies of  $K_{1,k}$  then creating a new graph by joining each  $v_i$  to a fixed new vertex v is  $K_{1,k}$ -supermagic.

An *H*-magic labeling in an *H*-decomposable of a graph *G* is a bijection  $f: V(G) \cup E(G)$ onto  $\{1, 2, \ldots, p + q\}$  such that for every copy of *H* in the decomposition, the sum of f(v) + f(e) over all v in V(H) and e in E(H) is constant. The labeling f is said to be H - V-super magic if  $f(V(G)) = \{1, 2, \ldots, p\}$ . Marimuthu and Kumar [1706] prove that  $K_{n,n}$   $(n \geq 2)$  is *H*-*V*-super magic decomposable when *H* is  $K_{1,n}$ . Marimuthu and Kumar [1705] provide a necessary and sufficient condition for the existence of *V*-super vertex-magic labeling and give *E*-super and *V*-super vertex-magic total labeling of certain families of generalized Petersen graphs. They also prove that no wheel is *E*-super vertexmagic,  $C_3$  is the only friendship graph that is *V*-super vertex-magic, and  $C_3$  is the only friendship graph that is *E*-super vertex-magic. An *H*-magic labeling f is said to be an *H*-*E*-super magic labeling if  $f(E(G)) = \{1, 2, ..., q\}$ . A graph that admits an *H*-*E*-super magic labeling is called an *H*-*E*-super magic decomposable graph. Subbiah and Pandimadevi [2477] study some elementary properties of *H*-*E*-super magic labelings with *H* an *m*-factor and provide a necessary and sufficient condition for an even regular graph to be *H*-*E*-super magic decomposable where *H* is a 2-factor.

## **5.5** Magic Labelings of Type (a, b, c)

A magic-type method for labeling the vertices, edges, and faces of a planar graph was introduced by Lih [1594] in 1983. Lih defines a magic labeling of type (1,1,0) of a planar graph G(V, E) as an injective function from  $\{1, 2, \ldots, |V|+|E|\}$  to  $V \cup E$  with the property that for each interior face the sum of the labels of the vertices and the edges surrounding that face is some fixed value. Similarly, Lih defines a magic labeling of type (1, 1, 1) of a planar graph G(V, E) with face set F as an injective function from  $\{1, 2, \ldots, |V|+|E|+|F|\}$ to  $V \cup E \cup F$  with the property that for each interior face the sum of the labels of the face and the vertices and the edges surrounding that face is some fixed value. Lih calls a labeling involving the faces of a plane graph consecutive if for every integer s the weights of all s-sided faces constitute a set of consecutive integers. Lih gave consecutive magic labelings of type (1, 1, 0) for wheels, friendship graphs, prisms, and some members of the Platonic family. In [236] Bača shows that the cylinders  $C_n \times P_m$  have magic labelings of type (1, 1, 0) when  $m \ge 2, n \ge 3, n \ne 4$ . In [246] Bača proves that the generalized Petersen graph P(n, k) (see §2.7 for the definition) has a consecutive magic labeling if and only if n is even and at least 4 and  $k \le n/2 - 1$ .

Bača gave magic labelings of type (1, 1, 1) for fans [230], ladders [230], planar bipyramids (that is, 2-point suspensions of paths) [230], grids [239], hexagonal lattices [238], Möbius ladders [233], and  $P_n \times P_3$  [234]. Kathiresan and Ganesan [1350] show that the graph  $P_{a,b}$  consisting of  $b \ge 2$  internally disjoint paths of length  $a \ge 2$  with common end points has a magic labeling of type (1, 1, 1) when b is odd, and when a = 2 and  $b \equiv 0 \pmod{4}$ . They also show that  $P_{a,b}$  has a consecutive labeling of type (1, 1, 1) when b is even and  $a \ne 2$ . Ali, Hussain, Ahmad, and Miller [126] study magic labeling of type (1, 1, 1)for wheels and subdivided wheels. They prove: wheels admits a magic labeling of type and (1, 1, 1) and (0, 1, 1), for odd n wheels  $W_n$  n admit a magic labeling of type (0, 1, 0), and subdivided wheels admit a magic labeling of type (1, 1, 0). As an open problem they ask for a magic labeling of type (1, 1, 0) for  $W_n$  and n even. Ahmad [67] proves that subdivided ladders admit magic labelings of type (1, 1, 1) and admit consecutive magic labelings of type (1, 1, 0).

Bača [232], [231], [242], [240], [234], [241] and Bača and Holländer [271] gave magic labelings of type (1, 1, 1) and type (1, 1, 0) for certain classes of convex polytopes. Kathiresan and Gokulakrishnan [1352] provided magic labelings of type (1, 1, 1) for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces, and one external infinite face. Bača [237] also provides consecutive and magic labelings of type (0, 1, 1) (that is, an injective function from  $\{1, 2, \ldots, |E| + |F|\}$  to  $E \cup F$  with the property that for each interior face the sum of the labels of the face and the edges surrounding that face is some fixed value) and a consecutive labeling of type (1,1,1) for a kind of planar graph with hexagonal faces. Tabraiz and Hussain [2568] provide a super magic labeling of type (1, 0, 0) for ladders and a super magic labeling of type (1, 0, 0) for subdivided ladders.

A magic labeling of type (1,0,0) of a planar graph G with vertex set V is an injective function from  $\{1, 2, \ldots, |V|\}$  to V with the property that for each interior face the sum of the labels of the vertices surrounding that face is some fixed value. Kathiresan, Muthuvel, and Nagasubbu [1354] define a lotus inside a circle as the graph obtained from the cycle with consecutive vertices  $a_1, a_2, \ldots, a_n$  and the star with central vertex  $b_0$  and end vertices  $b_1, b_2, \ldots, b_n$  by joining each  $b_i$  to  $a_i$  and  $a_{i+1}$  ( $a_{n+1} = a_1$ ). They prove that these graphs ( $n \geq 5$ ) and subdivisions of ladders have consecutive labelings of type (1, 0, 0). Devaraj [668] proves that graphs obtained by subdividing each edge of a ladder exactly the same number of times has a magic labeling of type (1, 0, 0).

In Table 11 we use following abbreviations

 $\mathbf{M}(a, b, c)$  magic labeling of type (a, b, c)

CM(a, b, c) consecutive magic labeling of type (a, b, c).

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová.

Graph	Labeling	Notes
W <sub>n</sub>	CM(1,1,0)	[1594]
friendship graphs	CM(1,1,0)	[1594]
prisms	CM(1,1,0)	[1594]
cylinders $C_n \times P_m$	M(1,1,0)	$m \ge 2, n \ge 3, n \ne 4$ [236]
fans $F_n$	M(1,1,1)	[230]
ladders	M(1,1,1)	[230]
planar bipyramids (see $\S5.3$ )	M(1,1,1)	[230]
grids	M(1,1,1)	[239]
hexagonal lattices	M(1,1,1)	[238]
Möbius ladders	M(1,1,1)	[233]
$P_n \times P_3$	M(1,1,1)	[234]
certain classes of convex polytopes	${f M}(1,1,1) \ {f M}(1,1,0)$	[232], [242], [240], [234] [241], [271]
certain classes of planar graphs with hexagonal faces	$\begin{array}{c} {\rm M}(0,1,1)\\ {\rm CM}(0,1,1)\\ {\rm CM}(1,1,1) \end{array}$	[237]
lotus inside a circle (see $\S5.3$ )	CM(1,0,0)	$n \ge 5 \ [1354]$
subdivisions of ladders	$ \begin{array}{ c c } M(1,0,0) \\ CM(1,0,0) \end{array} $	[668] [1354]

Table 11: Summary of Magic Labelings of Type (a, b, c)

#### 5.6 Sigma Labelings/1-vertex magic labelings/Distance Magic

In 1987 Vilfred [2736] (see also [2737]) defined a sigma-labeling of a graph G with n vertices as a bijection f from the vertices of G to  $\{1, 2, \ldots, n\}$  such that there is a constant k with the property that, at any vertex v the sum  $\sum f(u)$  taken over all neighbors u of v is k. The concept of sigma labeling was independently studied in 2003 by Miller, Rodger, and Simanjuntak in [1755] under the name 1-vertex magic. In a 2009 article Sugeng, Fronček, Miller, Ryan, and Walker [2491] used the term distance magic labeling. For convenience, we will use the term distance magic. In [2738] Vilfred and Jinnah give a number of necessary conditions for a graph to have a distance magic labeling. One of them is that if u and v are vertices of a graph with a distance labeling, then the order of the symmetric difference of N(u) and N(v) (neighborhoods of u and v) is not 1 or 2. This condition rules out a large class of graphs as having distance magic labelings. Rao, Singh, and Parameswaran [2098] have shown  $C_m \times C_n$  has a distance magic labeling if and only if  $m = n \equiv 2 \pmod{4}$  and  $K_m \times K_n$ ,  $m \geq 2, n \geq 3$  does not have a distance magic labeling. In [429] Benna gives necessary and sufficient condition for  $K_{m,n}$  to be a distance magic graph and proves that if  $G_1$  and  $G_2$  are connected graphs with minimum degree 1 and at least three vertices, then  $G_1 \times G_2$  does not have a distance magic labeling. Rao, Sighn, and Parameswaran [44] prove that every graph is an induced subgraph of a regular graph that has a distance magic labeling. As open problems, Rao [2096] asks for a characterize 4-regular graphs that have distance magic labelings and which graphs of the form  $C_m \times C_n$ ,  $m = n \equiv 2 \pmod{4}$  have distance magic labelings. Kovář, Fronček, and Kovářová [1424] classified all orders n for which a 4-regular distance magic graph exists and also showed that there exists a distance magic graph with k = 2t for every integer  $t \geq 6$ . Acharaya, Rao, Signh, and Parameswaran [43] proved  $P_m \times C_n$  does not have a distance magic labeling when m is at least 3 and provide necessary and sufficient conditions for  $K_{m,n}$  to have a distance magic labeling. Kovár and Silber [1426] proved that an (n-3)regular distance magic graph with n vertices exists if and only if  $n \equiv 3 \pmod{6}$  and that its structure is determined uniquely. Moreover, they reduce constructions of Fronček to a single construction and provide another sufficient condition for the existence a distance magic graph with an odd number of vertices. Fronček, Kovář, and Kovářová [788] provide a construction for distance magic graphs arising from arbitrary regular graphs based on an application of magic rectangles. They also solve a problem posed by Shafiq, Ali, and Simanjuntak [2303]. Godinho and Singh [899] investigate the distance magic labelings for neighborhood expansions of graphs and present a method for embedding regular graphs into distance magic graphs.

Among the results of Miller, Rodger, and Simanjuntak in [1755]: the only trees that have a distance magic labeling are  $P_1$  and  $P_3$ ;  $C_n$  has a distance magic labeling if and only if n = 4;  $K_n$  has a distance magic labeling if and only if n = 1; the wheel  $W_n = C_n + P_1$ has a distance magic labeling if and only if n = 4; the complete graph  $K_{n,n,\dots,n}$  with ppartite sets has a distance magic labeling if and only if n is even or both n and p are odd; an r-regular graph where n is odd does not have a distance magic labeling; and  $G \times \overline{K_{2n}}$ has a distance magic labeling for any regular graph G. They also give necessary and
sufficient conditions for complete tripartite graphs to have a distance magic labeling.

An orientable  $\Gamma$ -distance magic labeling of a graph, was introduced by Cichacz, Freyberg and Fronček [617] as a generalization of group distance magic labeling for oriented graphs. They showed that an even regular circulant graph of order n is orientable  $Z_n$ distance magic, the direct product  $C_n \times C_m$  is orientable  $Z_{nm}$ -distance magic. They also considered some products of circulant graphs. Moreover they proved that if G has order  $n \equiv 2 \pmod{4}$  and all vertices of odd degree, then there does not exist an orientable  $\Gamma$ -distance magic labeling of G for any Abelian group  $\Gamma$  of order n. Dyrlaga and Szopa in [704] gave necessary and sufficient conditions for lexicographic product  $K_m \circ \overline{K_n} \cong K_{m,m,\ldots,m}$  to be orientable  $\zeta_{mn}$ -distance magic. As a consequence, they

provide an infinite family of odd regular graphs possessing orientable  $\zeta_n$ -distance magic labeling. In [773] and [774] Freyberg and Keranen found orientable  $Z_n$ -distance magic labelings of the Cartesian product of cycles. In [775] they studied  $Z_n$ -distance magic labelings for the strong product of cycles.

Anholcer, Cichacz, Peterin, and Tepeh [165] proved that the direct product of two cycles  $C_m$  and  $C_n$  is distance magic if and only if m = 4 or n = 4, or  $m, n \equiv 0 \pmod{4}$ (the direct product of graphs G and H has the vertex set  $V(G) \times V(H)$  and (g, h) is adjacent to (g', h') if g is adjacent to g' in G and h is adjacent to h' in H). In [613] Cichacz gave necessary and sufficient conditions for circulant graph  $C_n(1, 2, \ldots, p)$  to be distance magic for p odd. In [618] Cichacz and Fronček characterized all distance magic circulant graphs  $C_n(1, p)$  for p odd. Cichacz, Fronček, Krop, and Raridan [619] proved that r-partite graph  $K_{n,n,\ldots,n} \times C_4$  is distance magic if and only if r > 1 and n > 2 is even. Anholcer and Cichacz [168] gave necessary and sufficient conditions for lexicographic product of an r-regular graph G and  $K_{m,n}$  to be distance magic. Cichacz and Görlich [623] gave necessary and sufficient conditions for the direct product of an r-regular graph G and  $K_{m,n}$  to be distance magic. In [615] the necessary and sufficient conditions for complete tripartite graphs to be group distance magic was given by Cichacz. In [196] Arumugam, Kamatchi, and Kovář give several results on distance magic graphs and open problems.

A finite r-regular graph G has a p-partition (resp. closed p-partition)  $(p \ge 2)$  if there exists a partition of the set V(G) into  $V_1, V_2, \ldots, V_p$  such that for every  $x \in V(G)$ , all  $V(x) \cap V_i$  (respectively,  $V[x] \cap V_1$ ) have the same size. In [625] Cichacz and Nikodem proved the following for finite r-regular graphs G. If G is distance magic (resp. closed distance magic) graph with a p-partition and p(t-1) even then tG is also distance (resp. closed distance) magic. If G has order t and H is p-regular such that tH is distance (resp. closed distance) magic, then the lexicographic product of G and H is distance (resp. closed distance) magic. If G has order t and H is such that tH is distance magic, then the lexicographic product of G and H and the direct product of G and H are distance (resp. closed distance) magic. If H is a p-regular distance magic graph with a 2-partition, then the lexicographic product of G and H and the direct product of G and H are distance magic. They further proved that if  $G = C_3$  or G is the strong product of  $C_n$  and  $C_m$  for n = 3 and m odd, or  $m, n \equiv 3 \pmod{6}$ , then tG is closed distance magic if and only if t is odd. (The strong direct product of G and H has vertex set  $V(G) \times V(H)$  and (g, h) is adjacent to (g', h') if g = g' and h is adjacent to h' in H, or h = h' and g is adjacent to g' in G.)

In [2222] Seoud, Maqsoud, and Aldiban determined whether or not the following families of graphs have a distance magic vertex labeling:  $K_n - \{e\}$ ;  $K_n - \{2e\}$ ;  $P_n^k$ ;  $C_n^2$ ;  $K_m \times C_n$ ;  $C_m + P_n$ ;  $C_m + C_n$ ;  $P_m + P_n$ ;  $K_{1,r,s}$ ;  $K_{1,r,m,n}$ ;  $K_{2,r,m,n}$ ;  $K_{m,n} + P_k$ ;  $K_{m,n} + C_k$ ;  $C_m + \overline{K_n}$ ;  $P_m + \overline{K_n}$ ;  $P_m \times P_n$ ;  $K_{m,n} \times P_k$ ;  $K_m \times P_n$ ; the splitting graph of  $K_{m,n}$ ;  $K_n + G$ ;  $K_m + \overline{K_n}$ ;  $K_m + C_n$ ;  $K_m + P_n$ ;  $K_{m,n} + K_r$ ;  $C_m \times P_n$ ;  $C_m \times K_{1,n}$ ;  $C_m \times K_{n,n}$ ;  $K_m \times K_{n,r}$ ; and  $K_m \times K_n$ . Typically, distance magic labelings exist only a few low parameter cases.

In [779] Fronček defined the notion of a  $\Gamma$ -distance magic graph as one that has a bijective labeling of vertices with elements of an Abelian group  $\Gamma$  resulting in constant sums of neighbor labels. A graph that is  $\Gamma$ -distance magic for an Abelian group  $\Gamma$  is called group distance magic. Cichacz and Fronček [618] showed that for an r-regular distance magic graph G on n vertices, where r is odd there does not exist an Abelian group  $\Gamma$ of order n having exactly one involution (i.e., an element that is its own inverse) that is  $\Gamma$ -distance magic. Fronček [779] proved that  $C_m \times C_n$  is a  $Z_{mn}$ -distance magic graph if and only if mn is even. He also showed that  $C_{2^n} \times C_{2^n}$  has a  $Z_{2^{2n}}$ -distance magic labeling. In [609] Cichacz showed some  $\Gamma$ -distance magic labelings for  $C_m \times C_n$  where  $\Gamma \not\approx Z_{mn}$  and  $\Gamma \not\approx Z_{2^{2n}}$ . Anholcer, Cichacz, Peterin, and Tepeh [167] proved that if an  $r_1$ -regular graph  $G_1$  is  $\Gamma_1$ -distance magic and an  $r_2$ -regular graph  $G_2$  is  $\Gamma_2$ -distance magic, then the direct product of graphs  $G_1$  and  $G_2$  is  $\Gamma_1 \times \Gamma_2$ -distance magic. Moreover they showed that if G is an r-regular graph of order n and m = 4 or m = 8 and r is even, then  $C_m \times G$  is group distance magic. They proved that  $C_m \times C_n$  is  $Z_{mn}$ -distance magic if and only if  $m \in \{4, 8\}$  or  $n \in \{4, 8\}$  or  $m, n \equiv 0 \pmod{4}$ . They also showed that if  $m, n \not\equiv 0 \pmod{4}$ 4) then  $C_m \times C_n$  is not  $\Gamma$ -distance magic for any Abelian group  $\Gamma$  of order mn. Cichacz [610] gave necessary and sufficient conditions for complete k-partite graphs of odd order pto be  $Z_p$ -distance magic. Moreover she showed that if  $p \equiv 2 \pmod{4}$  and k is even, then there does not exist a group  $\Gamma$  of order p that has a  $\Gamma$ -distance labeling for a k-partite complete graph of order p. She also proved that  $K_{m,n}$  is a group distance magic graph if and only if  $n + m \not\equiv 2 \pmod{4}$ . In [611] Cichacz proved that if G is an Eulerian graph, then the lexicographic product of G and  $C_4$  is group distance magic. In the same paper she also showed that if m + n is odd, then the lexicographic product of  $K_{m,n}$  and  $C_4$  is group distance magic. In [612] Cichacz gave necessary and sufficient conditions for direct product of  $K_{m,n}$  and  $C_4$  for m + n odd and for  $K_{m,n} \times C_8$  to be group distance magic. In [614] Cichacz proved that for n even and r > 1 the Cartesian product the complete r-partitie graph  $K_{n,n,\dots,n}$  and  $C_4$  is group distance magic. Godinho and Singh [898] obtain group distance magic labelings of  $C_n^r$  for certain classes of abelian groups and provide necessary conditions for existence of such labelings.

Cichacz [616] showed there exists an infinite family of odd regular graphs possessing  $\Gamma$ -distance magic labeling for groups  $\Gamma$  with more than one involution. In [613] Cichacz using a notion of a  $\Gamma$ -magic rectangle set  $MRS_{\Gamma}(a, b; c)$  showed group distance labeling for Cartesian and direct product of complete *r*-partite graphs. These results supported a conjecture in [618] that says that if *G* is a distance magic graph, then *G* is group distance

magic.

A directed  $\Gamma$ -distance magic labeling of an oriented graph  $\overrightarrow{G} = (V, A)$  of order n is a bijective mapping f from the vertex set of G to an abelian group  $\Gamma$  of order n with the property that there exists a constant  $c \in \Gamma$  such that, for every vertex  $v \in V(\overrightarrow{G})$ ,  $w(v) = \sum_{u \in N_G^{in}(v)} f(u) - \sum_{u \in N_G^{out}(v)} f(u) = c$ , where  $N_G^{in}(v)$  is the open in-neighborhood and  $N_G^{out}(v)$  is the open out-neighborhood of vertex v, that is  $N_G^{in}(v) = \{u : uv \in A\}$  and  $N_G^{out}(v) = \{u : vu \in A\}$ . If for a graph G there exists an orientation  $\overrightarrow{G}$  such that there is a directed  $\Gamma$ -distance magic labeling f for  $\overrightarrow{G}$  the graph G is called orientable  $\Gamma$ -distance magic. Freyberg and Keranen [774] proved that  $C_m \times C_n$  is orientable  $\mathbb{Z}_{mn}$ -distance magic for all  $m, n \geq 3$ .

In [166] Anholcer, Cichacz, Peterin, and Tepeh introduce the notion of balanced distance magic graphs. They say that a distance magic graph G with an even number of vertices is balanced if there exists a bijection f from V(G) to  $\{1, 2, \ldots, |V(G)|\}$  such that for every vertex w the following holds: If  $u \in N(w)$  with f(u) = i, then there exists  $v \in N(w), u \neq v$  with f(v) = |V(G)| - i + 1. They prove that a graph G is balanced distance magic if and only if G is regular and V(G) can be partitioned in pairs  $(u_i, v_i), i \in \{1, 2, \dots, |V(G)|/2\}$ , such that  $N(u_i) = N(v_i)$  for all *i*. Using this characterization, the following theorems are proved: if G is a regular graph and H is a graph not isomorphic to  $K_n$  where n is odd, then  $G \odot H$  is a balanced distance magic graph if and only if H is a balanced distance magic graph;  $G \times H$  is balanced distance magic if and only if one of G and H is balanced distance magic and the other one is regular; and  $C_m \times C_n$ is distance magic if and only if n = 4 or m = 4 or  $m, n \equiv 0 \pmod{4}$  and  $C_m \times C_n$  is balanced distance magic if and only if n = 4 or m = 4. In [169] they prove that every balanced distance magic graph is also group distance magic; the Cartesian product of a balanced distance magic graph and a regular graph is group distance magic; the direct product of  $C_4$  or  $C_8$  and a regular graph is group distance magic; and they show that  $C_8 \times G$  is also group distance magic for any even-regular graph G. They also prove that  $C_{4s} \times C_{4t}$  is  $A \times B$ -distance magic for any Abelian groups A and B of order 4s and 4t, respectively. Moreover, they conjecture that  $C_{4m} \times C_{4n}$  is a group distance magic graph. They prove that  $C_m \times C_n$  is  $Z_{mn}$ -distance magic if and only if  $m \in \{4, 8\}$  or  $n \in \{4, 8\}$  or both n and m are divisible by 4, and that  $C_m \times C_n$  with orders not divisible by 4 is not  $\Gamma$ -distance magic for any Abelian group  $\Gamma$  of order mn.

Let G = (V, E) be a graph on *n* vertices. A bijection *f* from the vertices of graph G to  $\{1, 2, \ldots, |V(G)|\}$  is called a *nearly distance magic* labeling of *G* if there exists a positive integer *k* such that  $\sum f(x)$  over all  $x \in N(v) = k$  or k+1 for all v. The constant *k* is called a *magic constant* of the graph and any graph which admits such a labeling is called a *nearly distance magic* graph. Godinho, Singh, and Arumugam [900] give several basic results on nearly distance magic graphs and compute the magic constant *k* in terms of the fractional total domination number of the graph.

A survey of results on distance magic (sigma, 1-vertex) labelings through 2009 is given in [192].

#### 5.7 Other Types of Magic Labelings

In 2004 Baskar Babujee [392] and [393] introduced the notion of *vertex-bimagic* labeling in which there exist two constants  $k_1$  and  $k_2$  such that the sums involved in a specified type of magic labeling is  $k_1$  or  $k_2$ . Thus a vertex-bimagic total labeling with bimagic constants  $k_1$  and  $k_2$  is the same as a vertex-magic total labeling except for each vertex v the sum of the label of v and all edges adjacent to v may be  $k_1$  or  $k_2$ . Murugesan and Senthil Amutha [1805] proved that the bistar  $B_{n,n}$  is vertex-bimagic total labeling for n > 2. An edge bimagic total labeling edge bimagic total of a graph G(V, E) with p vertices and q edges is a bijection f from the set of vertices and edges to such that for every edge  $uv \in E, f(u) + f(uv) + f(v)$  is one of two occustants  $k_1$  or  $k_2$ , independent of the choice of the edge. A bimagic labeling is of interest for graphs that do not have a magic labeling of a particular type. Bimagic labelings for which the number of sums equal to  $k_1$  and the number of sums equal to  $k_2$  differ by at most 1 are called *equitable*. When all sums except one are the same the labeling is called *almost magic*. Although the wheel  $W_n$  does not have an edge-magic total labeling when when  $n \equiv 3 \pmod{4}$ , Marr, Phillips and Wallis [1712] showed that these wheels have both equitable bimagic and almost magic labelings. They also show that whereas  $nK_2$  has an edge-magic total labeling if and only if n is odd,  $nK_2$  has an edge-bimagic total labeling when n is even and although even cycles do not have super edge-magic total labelings all cycles have super edge-bimagic total labelings. They conjecture that there is a constant N such that  $K_n$ has a edge-bimagic total labeling if and only if n is at most N. They show that such an N must be at least 8. They also prove that if G has an edge-magic total labeling then 2G has an edge-bimagic total equitable labeling. Amara Jothi, David, and Baskar Babujee [145] provide edge-bimagic labelings for switching of paths, cycles, stars, crowns and helms. They also examine whether operations on edge magic graphs results in edge bimagic graphs or not.

Baskar Babujee and Babitha [396] call a graph with p vertices 1-vertex bimagic if there is a bijective labeling f from the vertices to  $\{1, 2, \ldots, p\}$  such that for each vertex u the sum of all f(v) where v is adjacent to u is either a constant  $k_1$  or a constant  $k_2$  and  $k_1 \neq k_2$ . A graph with p vertices is called odd 1-vertex bimagic if there is a bijective labeling f from the vertices to  $\{1, 3, \ldots, 2p - 1\}$  such that for each vertex u the sum of all f(v) where v is adjacent to u is either a constant  $k_1$  or a constant  $k_2$  and  $k_1 \neq k_2$ . A graph with p vertices is called even 1-vertex bimagic if there is a bijective labeling f from the vertices to  $\{0, 2, \ldots, 2(p - 1)\}$  such that for each vertex u the sum of all f(v) where v is adjacent to u is either a constant  $k_1$  or a constant  $k_2$  and  $k_1 \neq k_2$ .

Baskar Babujee and Babitha [396] prove that a necessary condition for the existence of a 1-vertex bimagic vertex labeling f of a graph G is  $\sum_{x \in V(G)} d(x)f(x) = k_1p_1 + k_2p_2$ where d(x) is the degree of vertex x and  $p_1$  and  $p_2$  are the number of vertices with common count  $k_1$  and  $k_2$ , respectively. Among their results are: if G has a 1-vertex bimagic vertex labeling and  $G \neq C_4$ , then  $G+K_1$  admits a 1- vertex bimagic vertex labeling;  $C_n$  a 1-vertex bimagic if and only if n = 4;  $K_{m,n}$  is 1-vertex bimagic; graphs obtained from  $P_n$   $(n \geq 3)$ by adding edges joining every pair of vertices an odd distance apart are 1-vertex bimagic; *n*-partite graphs of the form  $K_{p,p,\dots,p}$  are 1-vertex bimagic for all p > 1 when *n* is even and 1-vertex bimagic for all even *p* when *n* is odd; a regular or biregular graph admits a 1-vertex bimagic labeling if and only if it the admits an odd 1-vertex bimagic labeling and if and only it admits an even 1-vertex bimagic labeling.

In [2215] Semenyuta, Nedilko, and Nedilko introduce the notion of the equivalence of vertex labelings on a given graph. They prove the equivalence of three bimagic labelings for regular graphs and obtain a particular solution for the problem of the existence a 1-vertex bimagic vertex labeling for graphs of isomorphic  $K_{n,n,m}$ . They prove that the sequence of biregular graphs  $K_{n(ij)} = ((K_{n-1} - M) + K_1) - (u_n u_i) - (u_n u_j)$  admits a 1-vertex bimagic vertex labeling, where  $u_i, u_j$  is any pair of nonadjacent vertices in the graph  $K_{n-1} - M$ ,  $u_n$  is the vertex of  $K_1$ , and M is the perfect matching of the complete graph, then the graph G + G has a 1-vertex bimagic vertex labeling with magic constants  $(n+1)(n+r)/2 + n^2$  and (n+1)(n+r)/2 + nr. They also define two new types of graphs that do not admit 1-vertex bimagic vertex labeling.

Baskar Babujee and Jagadesh [393], [400], [401], and [399] proved the following graphs have super edge bimagic labelings: cycles of length 3 with a nontrivial path attached;  $P_3 \odot K_{1,n} \ n \text{ even}; \ P_n + \overline{K_2} \ (n \text{ odd}); \ P_2 + mK_1 \ (m \ge 2); \ 2P_n \ (n \ge 2); \text{ the disjoint union}$ of two stars;  $3K_{1,n} \ (n \ge 2); \ P_n \cup P_{n+1} \ (n \ge 2); \ C_3 \cup K_{1,n}; \ P_n; \ K_{1,n}; \ K_{1,n,n}; \text{ the graphs}$ obtained by joining the centers of any two stars with an edge or a path of length 2; the graphs obtained by joining the centers of two copies of  $K_{1,n} \ (n \ge 3)$  with a path of length 2 then joining the center one of copies of  $K_{1,n}$  to the center of a third copy of  $K_{1,n}$  with a path of length 2; combs  $P_n \odot K_1$ ; cycles; wheels; fans; gears;  $K_n$  if and only if  $n \le 5$ .

Given positive integers k and  $\lambda$ , Yao, Chen, Yao, and Cheng [2860] say that a total labeling f of a connected graph G(V, E) from  $V \cup E$  to  $\{1, 2, \ldots, |V| + |E|\}$  such that  $f(x) \neq f(y)$  for distinct  $x, y \in V \cup E$  and  $f(u) + f(v) = k + \lambda f(uv)$  for each edge uv in E is a  $(k, \lambda)$ -magically total labeling of G. They provide necessary and sufficient conditions for graphs with  $(k, \lambda)$  )-magically total labelings to also have graceful, odd-graceful, felicitous, and (a, d)-edge antimagic total labelings (see §6.2).

In [1635] López, Muntaner-Batle, and Rius-Font give a necessary condition for a complete graph to be edge bimagic in the case that the two constants have the same parity.

In [397] Baskar Babujee, Babitha, and Vishnupriya make the following definitions. For any natural number a, a graph G(p,q) is said to be a-additive super edge bimagic if there exists a bijective function f from  $V(G) \cup E(G)$  to  $\{a+1, a+2, \ldots, a+p+q\}$  such that for every edge uv,  $f(u)+f(v)+f(uv) = k_1$  or  $k_2$ . For any natural number a, a graph G(p,q) is said to be a-multiplicative super edge bimagic if there exists a bijective f from  $V(G) \cup E(G)$ to  $\{a, 2a, \ldots, (p+q)a\}$  such that for every edge uv,  $f(u)+f(v)+f(uv) = k_1$  or  $k_2$ . A graph G(p,q) is said to be super edge-odd bimagic if there exists a bijection f from  $V(G) \cup E(G)$ to  $\{1, 3, 5, \ldots, 2(p+q)-1\}$  such that for every edge uv  $f(u) + f(v) + f(uv) = k_1$  or  $k_2$ . If f is a super edge bimagic labeling, then a function g from E(G) to  $\{0, 1\}$  with the property that for every edge uv, g(uv) = 0 if  $f(u) + f(v) + f(uv) = k_1$  and g(uv) = 1 if  $f(u) + f(v) + f(uv) = k_2$  is called a super edge bimagic cordial labeling if the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. They prove: super edge bimagic graphs are *a*-additive super edge bimagic; super edge bimagic graphs are *a*-multiplicative super edge bimagic; if G is super edge-magic, then  $G + K_1$  is super edge bimagic labeling; the union of two super edge magic graphs is super edge bimagic; and  $P_n$ ,  $C_{2n}$  and  $K_{1,n}$  are super edge bimagic cordial.

For any nontrivial Abelian group A under addition a graph G is said to be A-magic if there exists a labeling f of the edges of G with the nonzero elements of A such that the vertex labeling  $f^+$  defined by  $f^+(v) = \Sigma f(vu)$  over all edges vu is a constant. In [2467] and [2468] Stanley noted that Z-magic graphs can be viewed in the more general context of linear homogeneous diophantine equations. Shiu, Lam, and Sun [2348] have shown the following: the union of two edge-disjoint A-magic graphs with the same vertex set is A-magic; the Cartesian product of two A-magic graphs is A-magic; the lexicographic product of two A-magic connected graphs is A-magic; for an Abelian group A of even order a graph is A-magic if and only if the degrees of all of its vertices have the same parity; if G and H are connected and A-magic, G composed with H is A-magic;  $K_{m,n}$  is A-magic when  $m, n \geq 2$  and A has order at least 4;  $K_n$  with an edge deleted is A-magic when  $n \ge 4$  and A has order at least 4; all generalized theta graphs (§4.4 for the definition) are A-magic when A has order at least 4;  $C_n + \overline{K_m}$  is A-magic when  $n \ge 3, m \ge 2$  and A has order at least 2; wheels are A-magic when A has order at least 4; flower graphs  $C_m @C_n$  are A-magic when  $m, n \ge 2$  and A has order at least 4 ( $C_m @C_n$  is obtained from  $C_n$  by joining the end points of a path of length m-1 to each pair of consecutive vertices of  $C_n$ ).

When the constant sum of an A-magic graph is zero the graph is called *zero-sum* A-magic. The null set N(G) of a graph G is the set of all positive integers h such that G is zero-sum  $Z_h$ -magic. Akbari, Ghareghani, Khosrovshahi, and Zare [106] and Akbari, Kano, and Zare [107] proved that the null set N(G) of an r-regular graph G,  $r \geq 3$ , does not contain the numbers 2, 3 and 4. Akbari, Rahmati, and Zare [108] proved the following: if G is an even regular graph then G is zero-sum  $Z_h$ -magic for all h; if G is an odd r-regular graph,  $r \geq 3$  and  $r \neq 5$  then N(G) contains all positive integers except 2 and 4; if an odd regular graph is also 2-edge connected then N(G) contains all positive integers except 2; and a 2-edge connected bipartite graph is zero-sum  $Z_h$ -magic for  $h \geq 6$ . They also determine the null set of 2-edge connected bipartite graphs, describe the structure of some odd regular graphs,  $r \geq 3$ , that are not zero-sum 4-magic, and describe the structure of some 2-edge connected bipartite graphs that are not zero-sum  $Z_h$ -magic for h = 2, 3, 4. They conjecture that every 5-regular graph admits a zero-sum 3-magic labeling.

In [1528] Lee, Saba, Salehi, and Sun investigate graphs that are A-magic where  $A = V_4 \approx Z_2 \oplus Z_2$  is the Klein four-group. Many of theorems are special cases of the results of Shiu, Lam, and Sun [2348] given in the previous paragraph. They also prove the following are  $V_4$ -magic: a tree if and only if every vertex has odd degree; the star  $K_{1,n}$  if and only if n is odd;  $K_{m,n}$  for all  $m, n \geq 2$ ;  $K_n - e$  (edge deleted  $K_n$ ) when n > 3; even cycles with k pendent edges if and only if k is even; odd cycles with k pendent edges if and only if k is odd; wheels;  $C_n + \overline{K_2}$ ; generalized theta graphs; graphs that are copies of  $C_n$  that share a common edge; and  $G + \overline{K_2}$  whenever G is  $V_4$ -magic.

In [590] Choi, Georges, and Mauro explore  $Z_2^k$ -magic graphs in terms of even edge-

coverings, graph parity, factorability, and nowhere-zero 4-flows. They prove that the minimum k such that bridgeless G is zero-sum  $Z_2^k$ -magic is equal to the minimum number of even subgraphs that cover the edges of G, known to be at most 3. They also show that bridgeless G is zero-sum  $Z_2^k$ -magic for all  $k \geq 2$  if and only if G has a nowhere-zero 4-flow, and that G is zero-sum  $Z_2^k$ -magic for all  $k \geq 2$  if G is Hamiltonian, bridgeless planar, or isomorphic to a bridgeless complete multipartite graph, and establish equivalent conditions for graphs of even order with bridges to be  $Z_2^k$ -magic for all  $k \geq 4$ . In [864] Georges, Mauro, and Wang utilized well-known results on edge-colorings in order to construct infinite families that are  $V_4$ -magic but not  $Z_4$ -magic.

Baskar Babujee and Shobana [412] prove that the following graphs have  $Z_3$ -magic labelings:  $C_{2n}$ ;  $K_n$   $(n \ge 4)$ ;  $K_{m,2m}$   $(m \ge 3)$ ; ladders  $P_n \times P_2$   $(n \ge 4)$ ; bistars  $B_{3n-1,3n-1}$ ; and cyclic, dihedral and symmetric Cayley digraphs for certain generating sets. Siddiqui [2380] proved that generalized prisms, generalized antiprisms, fans and friendship graphs are  $Z_{3k}$ -magic for  $k \ge 1$ . In [596] Chou and Lee investigated  $Z_3$ -magic graphs.

Chou and Lee [596] showed that every graph is an induced subgraph of an A-magic graph for any nontrivial Abelian group A. Thus it is impossible to find a Kuratowski type characterization of A-magic graphs. Low and Lee [1663] have shown that if a graph is  $A_1$ -magic then it is  $A_2$ -magic for any subgroup  $A_2$  of  $A_1$  and for any nontrivial Abelian group A every Eulerian graph of even size is A-magic. For a connected graph G, Low and Lee define T(G) to be the graph obtained from G by adding a disjoint uv path of length 2 for every pair of adjacent vertices u and v. They prove that for every finite nontrivial Abelian group A the graphs  $T(P_{2k})$  and  $T(K_{1,2n+1})$  are A-magic. Shiu and Low [2357] show that  $K_{k_1,k_2,\ldots,k_n}(k_i \geq 2)$  is A-magic, for all A where  $|A| \geq 3$ . In [2362] Shiu and Low analyze the A-magic property for complete n-partite graphs and composition graphs with deleted edges. Lee, Salehi and Sun [1531] have shown that for  $m, n \geq 3$  the double star DS(m, n) is Z-magic if and only if m = n.

S. M. Lee [1490] calls a graph G fully magic if it is A-magic for all nontrivial abelian groups A. Low and Lee [1663] showed that if G is an Eulerian graph of even size, then Gis fully magic. In [1490] Lee gives several constructions that produce infinite families of fully magic graphs and proves that every graph is an induced subgraph of a fully magic graph.

In [1448] Kwong and Lee call the set of all k for which a graph is  $Z_k$ -magic the *integer-magic spectrum* of the graph. They investigate the integer-magic spectra of the coronas of some specific graphs including paths, cycles, complete graphs, and stars. Low and Sue [1666] have obtained some results on the integer-magic spectra of tessellation graphs. Shiu and Low [2358] provide the integer-magic spectra of sun graphs. Chopra and Lee [594] determined the integer-magic spectra of all graphs consisting of any number of pairwise disjoint paths with common end vertices (that is, generalized theta graphs). Low and Lee [1663] show that Eulerian graphs of even size are A-magic for every finite nontrivial Abelian group A whereas Wen and Lee [2808] provide two families of Eularian graphs that are not A-magic for every finite nontrivial Abelian group A and eight infinite families of Eulerian graphs of odd sizes that are A-magic for every finite nontrivial Abelian group A. Low and Lee [1663] also prove that if A is an Abelian group and G and H are A-magic,

then so are  $G \times H$  and the lexicographic product of G and H. Low and Shiu [1665] prove:  $K_{1,n} \times K_{1,n}$  has a  $Z_{n+1}$ -magic labeling with magic constant 0; if  $G \times H$  is  $Z_2$ -magic, then so are G and H; if G is  $Z_m$ -magic and H is  $Z_n$ -magic, then the integer-magic spectra of  $G \times H$  contains all common multiples of m and n; if n is even and  $k_i \geq 3$  then the integer-magic spectra of  $P_{k_1} \times P_{k_2} \times \cdots \times P_{k_n} = \{3, 4, 5, \ldots\}$ . In [2360] Shiu and Low determine all positive integers k for which fans and wheels have a  $Z_k$ -magic labeling with magic constant 0. Shiu and Low [2361] determined for which  $k \geq 2$  a connected bicyclic graph without a pendent has a  $Z_k$ -magic labeling.

Jeyanthi and Jeya Daisy [1149] prove that  $P_n^2$  (n > 4),  $C_n^2$ , the total graph of  $C_n$ , and the splitting graph of  $C_{2n}$  are  $Z_k$ -magic graphs. They also prove: the splitting graph of  $C_n$  is  $Z_k$ -magic when n is even and n is odd and k is even, the middle graph of  $C_n$  is  $Z_k$ magic when n and k are odd, the  $m\Delta_{2n}$ -snake graph is  $Z_k$ -magic when k > m, the graph obtained by joining the vertices  $u_i$  and  $u_{i+1}$  of  $C_n$  by a path of length  $m_i$  for  $1 \le i \le n-1$ , and  $u_1$  and  $u_n$  by a path of length  $m_n$  is  $Z_k$ -magic if either all  $m_1, m_2, \ldots, m_n$  are even or all are odd. In [1150] Jeyanthi and Jeya Daisy prove total graphs of the paths, flower graphs, and  $C_m \times P_n$  are  $Z_k$ -magic. They also prove closed helms are  $Z_k$ -magic when k > 4 is even, lotuses inside a circle are  $Z_{4k}$ -magic, and graphs consisting of two cycles with a common edge are  $Z_k$ -magic: two odd cycles connected by a path; the graph obtained by identifying a vertex of  $C_n$  with a pendent vertex of a star, m-splitting graphs of paths, and m-middle graphs of paths. They prove that if G is  $Z_m$ -magic with magic constant a then  $G \odot \overline{K_m}$  is  $Z_m$ -magic.

Jeyanthi and Jeya Daisy [1148] prove that the subdivision graphs of the following families of graphs are  $Z_k$ -magic: ladders, triangular ladders, the shadow graph of paths, the total graph of paths, flowers, generalized prisms  $C_m \times P_n$  for m even,  $m\Delta_n$ -snakes, lotuses inside a circle, the square graph of paths, gears of even cycles, closed helms of even cycles, and antiprisms  $A_n^m$  for m even.

Let G be a graph and let  $G_1, G_2, \ldots, G_n$  be  $n \ge 2$  copies of G. The graph obtained by replacing each endpoint vertex of  $K_{1,n}$  by the graphs  $G_1, G_2, \ldots, G_n$  is called the *open star* of G. Jeyanthi and Jeya Daisy [1152] proved that the open star graphs of shells, flowers, double wheels, cylinders, wheels, generalised Peterson graphs, lotuses inside a circle, and closed helms are  $Z_k$ -magic graphs. They also prove that the super subdivision of any graph is  $Z_k$ -magic.

Jeyanthi and Jeya Daisy [1153] proved that the path union of  $n \ge 2$  copies of the following families of graphs are  $Z_k$ -magic: odd cycles; generalised Peterson graphs P(r, m)when r is odd and  $1 \le m \le \frac{r}{2}$ ; shell graphs  $S_r$  when r > 3; wheels  $W_r$  when r > 3; closed helms  $CH_r$  when (i) r > 3 is odd and (ii) r is even and k is even; double wheels  $DW_r$ when r > 3 is odd; flowers  $Fl_r$  when r > 2;  $C_r \times P_2$  when r is odd; total graphs of paths  $T(P_r)$  for all n, r > 4; lotuses inside a circle  $LC_r$  when r > 3; and  $C_r \odot K_1$  for odd r.

Jeyanthi and Jeya Daisy [1154] proved that the following graphs are k-magic: shell graphs  $S_n$  when n is odd or n is even and k is even; generalised Jahangir graphs  $J_{n,s}$ when n and s have the same parity or n is even, s is odd, and k is even;  $(P_n + P_1) \times P_2$ when n is odd; double wheels  $2C_n + K_1$ ; mongolian tents M(m, n) when m is even; flower snark graphs; slanting ladders (that is, graphs obtained from two paths  $u_1, u_2, \ldots, u_n$  and  $v_1, v_2, \ldots, v_n$  by joining each  $u_i$  with  $v_{i+1}, 1 \leq i \leq n-1$ ) when n is even; double step grid graphs; double arrow graphs obtained from  $P_m \times P_n$  by joining a new vertex with the m vertices of the first copy of  $P_m$  and another new vertex with the m vertices of the last copy of  $P_m$  when m is even; semi Jahangir graphs (the connected graph with vertex set  $\{p, x_i, y_k : 1 \leq i \leq n+1, 1 \leq k \leq n\}$  and the edge set  $\{px_i : 1 \leq i \leq n+1\} \cup \{x_iy_i : 1 \leq i \leq n\} \cup \{y_ix_{i+1} : 1 \leq i \leq n\}$ ); graphs obtained by connecting double wheels  $DW_{n_1}$  and  $DW_{n_2}$  by a path when  $n_1$  and  $n_2$  are odd; graphs obtained by joining two copies of shell graphs by a path; and the splitting graph of a  $Z_k$  magic graph with magic constant 0.

Let G be a graph with n vertices  $\{u_1, u_2, \ldots, u_n\}$  and consider n copies of  $G, G_1, G_2, \ldots, G_n$ , with vertex sets  $V(G_i) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq n\}$ . The cycle of a graph G, denoted by C(n.G), is obtained by identifying the vertex  $u_1^j$  of  $G_j$  with  $u_i$  of G for  $1 \leq i \leq n, 1 \leq j \leq n$ . Jeyanthi and Jeya Daisy [1155] prove that the following graphs are  $Z_k$ -magic:  $C(n.C_r)$  except r is even, n is odd, and k is odd; generalised Peterson graphs C(n.P(r,m)) except r is even, n is odd, and k is odd; cycles of shell graphs; cycles of wheel graphs; cycles of closed helms; cycles of double wheels  $C(n.DW_r)$  except r is even, n is odd, and k is odd; cycles of flower graphs; and cycles of lotus inside a circle graphs. Jeyanthi and Jeya Daisy [1155] also prove that if G is  $Z_k$ -magic then C(n.G) is  $Z_k$ -magic if n or k are even.

Shiu and Low [2359] have introduced the notion of ring-magic as follows. Given a commutative ring R with unity, a graph G is called R-ring-magic if there exists a labeling f of the edges of G with the nonzero elements of R such that the vertex labeling  $f^+$  defined by  $f^+(v) = \Sigma f(vu)$  over all edges vu and vertex labeling  $f^{\times}$  defined by  $f^{\times}(v) = \Pi f(vu)$  over all edges vu are constant. They give some results about R-ring-magic graphs.

In [535] Cahit says that a graph G(p,q) is total magic cordial (TMC) provided there is a mapping f from  $V(G) \cup E(G)$  to  $\{0,1\}$  such that  $(f(a) + f(b) + f(ab)) \mod 2$  is a constant modulo 2 for all edges  $ab \in E(G)$  and  $|f(0) - f(1)| \leq 1$  where f(0) denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and f(1) denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He says a graph G is total sequential cordial (TSC) if there is a mapping f from  $V(G) \cup E(G)$  to  $\{0,1\}$  such that for each edge e = ab with f(e) = |f(a) - f(b)|it is true that  $|f(0) - f(1)| \leq 1$  where f(0) denotes the sum of the number of vertices labeled with 0 and the number of edges labeled with 0 and f(1) denotes the sum of the number of vertices labeled with 1 and the number of edges labeled with 1. He proves that the following graphs have a TMC labeling:  $K_{m,n}$  (m, n > 1), trees, cordial graphs, and  $K_n$  if and only if n = 2, 3, 5, or 6. He also proves that the following graphs have a TSC labeling: trees; cycles; complete bipartite graphs; friendship graphs; cordial graphs; cubic graphs other than  $K_4$ ; wheels  $W_n$  (n > 3);  $K_{4k+1}$  if and only if  $k \ge 1$  and  $\sqrt{k}$  is an integer;  $K_{4k+2}$  if and only if  $\sqrt{4k+1}$  is an integer;  $K_{4k}$  if and only if  $\sqrt{4k+1}$  is an integer; and  $K_{4k+3}$  if and only if  $\sqrt{k+1}$  is an integer. In [1137] Jeyanthi, Angel Benseera, and Cahit prove  $mP_2$  is TMC if  $m \not\equiv 2 \pmod{4}$ ,  $mP_n$  is TMC for all  $m \geq 1$  and  $n \geq 3$ , and obtain partial results about TMC labelings of  $mK_n$ . Neela and Selvaraj proved that the complete tripartite graphs are TMC and complete multipartite graphs are TMC when

the partite sets have even sizes

Jeyanthi and Angel Benseera [1135] investigated the existence of TMC labelings of the one-point unions of copies of cycles, complete graphs and wheels. In [1136] Jeyanthi and Angel Benseera prove that if  $G_i(p_i, q_i)$ , i = 1, 2, 3, ..., n are totally magic cordial graphs with C = 0 such that  $p_i + q_i$ , i = 1, 2, 3, ..., n are even, and  $|p_i - 2m_i| \leq 1$ , where  $m_i$  is the number of vertices labeled with 0 in  $G_i$ , i = 1, 2, ..., n, then  $G_1 + G_2 + \cdots + G_n$  is TMC; if G is an odd graph with  $p + q \equiv 2 \pmod{4}$ , then G is not TMC; fans  $F_n$  are TMC for  $n \geq 2$ ; wheels  $W_n$   $(n \geq 3)$  are TMC if and only if  $n \not\equiv 3 \pmod{4}$ ;  $mW_{4t+3}$  is TMC if and only if  $n \not\equiv 3 \pmod{4}$ ;  $C_{2n+1} \odot \overline{K}_m$  is TMC if and only if m is odd; the disjoint union of  $K_{1,m}$  and  $K_{1,n}$  is TMC if and only if m or n is even.

For a bijection  $f: V(G) \cup E(G) \to Z_k$  such that for each edge  $uv \in E(G), f(u) +$ f(v) + f(uv) is constant (mod k)  $n_f(i)$  denotes the number vertices and edges labeled by i under f. If  $|n_f(i) - n_f(j)| \leq 1$  for all  $0 \leq i < j \leq k - 1$ , f is called a k-totally magic cordial labeling of G. A graph is said to be k-totally magic cordial if it admits a k-totally magic cordial labeling. In [1138] Jeyanthi, Angel Benseera, and Lau provide some ways to construct new families of k-totally magic cordial (k-TMC) graphs from a known k-totally magic cordial graph. Let G (respectively, H) be a (p,q)-graph (respectively, an (n,m)graph) that admits a k-TMC labeling f (respectively, g) with constant C such that  $n_f(i)$ and  $v_f(i) = \frac{p}{k}$  (or  $n_g(i)$  and  $v_g(i) = \frac{n}{k}$ ) are constants for all  $0 \le i \le k-1$ , they show that G + H also admits a k-TMC labeling with constant C. They prove the following. If G is an edge magic total graph, then G is k-TMC for  $k \ge 2$ ; if G is an odd graph with  $p + q \equiv k \pmod{2k}$  and  $k \equiv 2 \pmod{4}$ , then G is not k-TMC; if  $n \equiv 7 \pmod{8}$ ,  $K_n \odot K_1$  is not 2*n*-TMC; if  $n \equiv 2 \pmod{4}$ ,  $C_n \odot C_3$  is not *n*-TMC; if  $n \equiv 1 \pmod{2}$ ,  $C_n \odot K_5$  is not 2*n*-TMC; if  $n \equiv 2 \pmod{4}$ ,  $C_n \times P_2$  is not *n*-TMC;  $K_n \ (n \geq 3)$  is *n*-TMC;  $K_n \odot K_1 \ (n \ge 3)$  is n-TMC;  $S_n$  is n-TMC for all  $n \ge 1$ ;  $K_{m,n} \ (m \ge n \ge 2)$  is both *m*-TMC and *n*-TMC;  $W_n$  is *n*-TMC for all odd  $n \ge 3$  and is 3-TMC for  $n \equiv 0 \pmod{6}$ ;  $mK_n \ (n \ge 2)$  is *n*-TMC if  $n \ge 3$  is odd;  $K_n + K_n$  is *n*-TMC if  $n \ge 3$  is odd;  $S_n + S_n$  $(n \ge 1)$  is (n+1)-TMC; and if  $m \ge 3$  and n is odd,  $C_n \times P_m$   $(n \ge 3)$  is n-TMC. In [1140] Jeyanthi, Angel Benseera, and Lau call a graph G hypo-k-TMC if  $G - \{v\}$  is k-TMC for each vertex v in V(G) and establish that some families of graphs admit and do not admit hypo-k-TMC labeling.

A graph G(V, E) where  $V = \{v_i, 1 \le i \le n\}$  and  $E = \{v_i v_{i+1}, 1 \le i \le n\}$  is 0-edge magic if there exists a bijection  $f: V(G) \to \{1, -1\}$  such that the induced edge labeling defined by  $f^*(uv) = f(u) + f(v)$  is 0 for all  $uv \in E$ . Paths, cycles, complete *n*-ary pseudo trees,  $P_m \times C_n$  where  $n \equiv 0 \pmod{2}$ ,  $Q_n$ , the graph  $C_m$  attached to  $mK_1, m \equiv 0 \pmod{2}$ , friendship graphs  $C_n^{(m)}$ , and the graph  $P_m \times P_m \times P_m$  are 0-edge magic graphs [1422], [1108], [1838]. Jayapriya [1107] proved the splitting graphs  $spl(P_n)$ ,  $spl(C_n)$ ,  $spl(K_{1,n})$ ,  $spl(B_{m,n})$ , and splitting graph of any tree admits 0-edge magic labelings. Laurejas and Pedrano [1473] determine the 0-edge magic labeling of  $P_m \times P_n$ ,  $C_m \times C_n$ , and the generalized Petersen graph. They also prove that odd cycles are not 0-edge magic.

A binary magic total labeling of a graph G is a function  $f: V(G) \cup E(G) \rightarrow \{0, 1\}$ such that  $f(a) + f(b) + f(ab) \equiv C \pmod{2}$  for all  $ab \in E(G)$ . Jeyanthi and Angel Benseera [1139] define the totally magic cordial deficiency of G as the minimum number of vertices taken over all binary magic total labeling of G that is necessary to add so that that the resulting graph is totally magic cordial. The totally magic cordial deficiency of G is denoted by  $\mu_T(G)$ . They provide  $\mu_T(K_n)$  for some cases.

Let G be a graph rooted at a vertex u and  $f_i$  be a binary magic total labeling of G and  $f_i(u) = 0$  for i = 1, 2, ..., k and  $n_{f_i}(0) = \alpha_i$ ,  $n_{f_i}(1) = \beta_i$  for i = 1, 2, ..., k. Jeyanthi and Angel Benseera [1139] determine the totally magic cordial deficiency of the one-point union  $G^{(n)}$  of n copies of G. They show that for  $n \equiv 3 \pmod{4}$  the totally magic cordial deficiency of  $W_n$ ,  $W_n^{(4t+1)}$ ,  $W_{4t+1}^{(n)}$  and  $C_n + \overline{K}_{2m+1}$  is 1; for m odd,  $\mu_T(mW_{4t+3}) = 1$ ; and for  $n \equiv 1 \pmod{4}$ ,  $\mu_T(K_4^{(n)}) = 1$ .

In 2001, Simanjuntak, Rodgers, and Miller [1755] defined a 1-vertex magic (also known as distance magic labeling vertex labeling of G(V, E) as a bijection from V to  $\{1, 2, \ldots, |V|\}$  with the property that there is a constant k such that at any vertex v the sum  $\sum f(u)$  taken over all neighbors of v is k. Among their results are:  $H \times \overline{K}_{2k}$ has a 1-vertex-magic vertex labeling for any regular graph H; the symmetric complete multipartite graph with p parts, each of which contains n vertices, has a 1-vertex-magic vertex labeling if and only if whenever n is odd, p is also odd;  $P_n$  has a 1-vertex-magic vertex labeling if and only if n = 1 or 3;  $C_n$  has a 1-vertex-magic vertex labeling if and only if n = 4;  $K_n$  has a 1-vertex-magic vertex labeling if and only if n = 1 or P<sub>3</sub>; and r-regular graphs with r odd do not have a 1-vertex-magic vertex labeling.

Miller, Rogers, and Simanjuntak [1755] the complete *p*-partite (p > 1) graph  $K_{n,n,\dots,n}$  (n > 1) has a 1-vertex-magic vertex labeling if and only if either *n* is even or *np* is odd. Shafiq, Ali, Simanjuntak [2303] proved  $mK_{n,n,\dots,n}$  has a 1-vertex-magic vertex labeling if *n* is even or *mnp* is odd and  $m \ge 1, n > 1, p > 1$ ; and  $mK_{n,n,\dots,n}$  does not have a 1-vertex-magic vertex labeling if *np* is odd,  $p \equiv 3 \pmod{4}$ , and *m* is even.

Recall if  $V(G) = \{v_1, v_2, \ldots, v_p\}$  is the vertex set of a graph G and  $H_1, H_2, \ldots, H_p$  are isomorphic copies of a graph H, then G[H] is the graph obtained from G by replacing each vertex  $v_i$  of G by  $H_i$  and joining every vertex in  $H_i$  to every neighbor of  $v_i$ . Shafiq, Ali, Simanjuntak [2303] proved if G is an r-regular graph  $(r \ge 1)$  then  $G[C_n]$  has a 1vertex-magic vertex labeling if and only if n = 4. They also prove that for  $m \ge 1$  and n > 1,  $mC_p[\overline{K_n}]$  has 1-vertex-magic vertex labeling if and only if either n is even or mnpis odd or n is odd and  $p \equiv 3 \pmod{4}$ .

For a graph G Jeyanthi and Angel Benseera [1134] define a function f from  $V(G) \cup E(G)$  to  $\{0, 1\}$  to be a totally vertex-magic cordial labeling (TVMC) with a constant C if  $f(a) + \sum_{b \in N(a)} f(ab) \equiv C \pmod{2}$  for all vertices  $a \in V(G)$  and  $|n_f(0) - n_f(1)| \leq 1$ , where N(a) is the set of vertices adjacent to the vertex a and  $n_f(i)$  is the sum of the number of vertices and edges with label i. They prove the following graphs have totally vertex-magic cordial labelings: vertex-magic total graphs; trees;  $K_n$ ;  $K_{m,n}$  whenever  $|m-n| \leq 1$ ;  $P_n + P_2$ ; friendship graphs with C = 0; and flower graphs  $Fl_n$  for  $n \geq 3$  with C = 0. They also proved that if G is TVMC with C = 1, then the graph obtained by identifying any vertex of G with any vertex of a tree is TVMC with C = 1; if G is a (p, q) graph with

 $|p-q| \leq 1$ , then G is TVMC with C = 1; and if G(p,q) is a TVMC graph with constant C = 0 where p is odd, then  $G + \overline{K_{2m}}$  is TVMC with C = 1 if m is odd and with C = 0 if m is even.

Jeyanthi, Angel Benseera, and Immaculate Mary [1133] showed that the following graphs have totally magic cordial labelings: (p,q) graphs with  $|p-q| \leq 1$ ; flower graphs  $Fl_n$  for  $n \geq 3$ ; ladders; and graphs obtained by identifying a vertex of  $C_m$  with each vertex of  $C_n$ . They also proved that if  $G_1(p_1, q_1)$  and  $G_2(p_2, q_2)$  are two disjoint totally magic cordial graphs with  $p_1 = q_1$  or  $p_2 = q_2$  then  $G_1 \cup G_2$  is totally magic cordial. In Theorem 10 in [535] Cahit stated that  $K_n$  is totally magic cordial if and only if  $n \in \{2, 3, 5, 6\}$ . Jeyanthi and Angel Benseera [1139] proved that  $K_n$  is totally magic cordial if and only if  $\sqrt{4k+1}$  has an integer value when n = 4k;  $\sqrt{k+1}$  or  $\sqrt{k}$  have an integer value when n = 4k + 1;  $\sqrt{4k+5}$  or  $\sqrt{4k+1}$  have an integer value when n = 4k + 2; or  $\sqrt{k+1}$  has an integer value when n = 4k + 3.

A graph G is said to have a *totally magic cordial* TMC labeling with constant C if there exists a mapping  $f: V(G) \cup E(G) \to \{0,1\}$  such that  $f(a) + f(b) + f(ab) \equiv C \pmod{2}$  for all  $ab \in E(G)$  and  $|n_f(0) - n_f(1)| \leq 1$ , where  $n_f(i)$  (i = 0, 1) is the sum of the number of vertices and edges with label i. In [1136] Jeyanthi and Angel Benseera prove that if  $G_i(p_i, q_i), i = 1, 2, 3, \ldots, n$  are totally magic cordial graphs with C = 0 such that  $p_i + q_i$ ,  $i = 1, 2, 3, \ldots, n$  are even, and  $|p_i - 2m_i| \leq 1$ , where  $m_i$  is the number of vertices labeled with 0 in  $G_i, i = 1, 2, \ldots, n$ , then  $G_1 + G_2 + \cdots + G_n$  is TMC. They also prove the following. If G be an odd graph with  $p + q \equiv 2 \pmod{4}$ , then G is not TMC; fan graph  $F_n$  is TMC for  $n \geq 2$ ; the wheel graph  $W_n$   $(n \geq 3)$  is TMC if and only if  $n \not\equiv 3 \pmod{4}$ ;  $mW_{4t+3}$  is TMC if and only if m is even;  $mW_n$  is TMC if  $n \not\equiv 3 \pmod{4}$  and  $m \geq 1$ ;  $C_n + \overline{K}_{2m+1}$  is TMC if and only if  $n \not\equiv 3 \pmod{4}$ ;  $C_{2n+1} \odot \overline{K}_m$  is TMC if and only if m is odd; and the disjoint union of  $K_{1,m}$  and  $K_{1,n}$  is TMC if and only if m or n is even.

Balbuena, Barker, Lin, Miller, and Sugeng [341] call a vertex-magic total labeling of a graph G(V, E) an *a-vertex consecutive magic labeling* if the vertex labels are  $\{a + 1, a + 2, \ldots, a + |V|\}$  where  $0 \le a \le |E|$ . They prove: if a tree of order n has an a-vertex consecutive magic labeling then n is odd and a = n - 1; if G has an a-vertex consecutive magic labeling with n vertices and e = n edges, then n is odd and if G has minimum degree 1, then a = (n + 1)/2 or a = n; if G has an a-vertex consecutive magic labeling with n vertices and e edges such that  $2a \le e$  and  $2e \ge \sqrt{6n} - 1$ , then the minimum degree of G is at least 2; if a 2-regular graph of order n has an a-vertex consecutive magic labeling, then n is odd and a = 0 or n; and if a r-regular graph of order n has an a-vertex consecutive magic labeling, then n and r have opposite parities.

Balbuena et al. also call a vertex-magic total labeling of a graph G(V, E) a *b-edge* consecutive magic labeling if the edge labels are  $\{b+1, b+2, \ldots, b+|E|\}$  where  $0 \le b \le |V|$ . They prove: if G has n vertices and e edges and has a b-edge consecutive magic labeling and one isolated vertex, then b = 0 and  $(n - 1)^2 + n^2 = (2e + 1)^2$ ; if a tree with odd order has a b-edge consecutive magic labeling then b = 0; if a tree with even order has a b-edge consecutive magic labeling then it is  $P_4$ ; a graph with n vertices and e edges such that  $e \ge 7n/4$  and  $b \ge n/4$  and a b-edge consecutive magic labeling has minimum degree 2; if a 2-regular graph of order n has a b-edge consecutive magic labeling, then n is odd and b = 0 or b = n; and if a r-regular graph of order n has an b-edge consecutive magic labeling, then n and r have opposite parities.

Sugeng and Miller [2494] prove: If (V, E) has an *a*-vertex consecutive edge magic labeling, where  $a \neq 0$  and  $a \neq |E|$ , then *G* is disconnected; if (V, E) has an *a*-vertex consecutive edge magic labeling, where  $a \neq 0$  and  $a \neq |E|$ , then *G* cannot be the union of three trees with more than one vertex each; for each nonnegative *a* and each positive *n*, there is an *a*-vertex consecutive edge magic labeling with *n* vertices; the union of *r* stars and a set of r-1 isolated vertices has an *s*-vertex consecutive edge magic labeling, where *s* is the minimum order of the stars; for every *b* every caterpillar has a *b*-edge consecutive edge magic labeling; if a connected graph *G* with *n* vertices has a *b*-edge consecutive edge magic labeling where  $1 \leq b \leq n-1$ , then *G* is a tree; the union of *r* stars and a set of r-1 isolated vertices has an *r*-edge consecutive edge magic labeling.

Baskar Babujee, Vishnupriya, and Jagadesh [415] introduced a labeling called *a-vertex* consecutive edge bimagic total as a graph G(V, E) for which there are two positive integers  $k_1$  and  $k_2$  and a bijection f from  $V \cup E$  to  $\{1, 2, \ldots, |V| + |E|\}$  such that f(u) + f(v) + f(v) $f(uv) = k_1$  or  $k_2$  for all edges uv and  $f(V) = \{a + 1, a + 2, \dots, a + |V|\}, 0 \le a \le |V|.$ They proved the following graphs have such labelings:  $P_n$ ,  $K_{1,n}$ , combs, bistars  $B_{m,n}$ , trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from  $P_5$  by identifying the center of a copy  $K_{1,n}$  with the two end vertices and the middle vertex. In [405] Baskar Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have *a*-vertex consecutive edge bimagic total labelings. Baskar Babujee, Jagadesh, Vishnupriya [407] study the properties of *a*-vertex consecutive edge bimagic total labeling for  $P_3 \odot K_{1,2n}$ ,  $P_n + \overline{K_2}$  (n is odd and  $n \ge 3$ ),  $(P_2 \cup mK_1) + \overline{K_2}$ ,  $(P_2 + mK_1)$  ( $m \ge 2$ ),  $C_n$ , fans  $P_n + K_1$ , double fans  $P_n + 2K_1$ , and graphs obtained by appending a path of length at least 2 to a vertex of  $C_3$ . Baskar Babujee and Jagadesh [406] prove the following graphs have a-vertex consecutive edge bimagic total labelings:  $2P_n$   $(n \ge 2)$ ,  $P_n \cup P_{n+1}$   $(n \ge 2)$ ,  $K_{2,n}, C_n \odot K_1$ , and that  $C_3 \cup K_{1,n}$  an *a*-vertex consecutive edge bimagic labeling for a = n + 3.

Vishnupriya, Manimekalai, and Baskar Babujee [2750] define a labeling f of a graph G(p,q) to be a *edge bimagic total labeling* if there exists a bijection f from  $V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$  such that for each edge  $e = (u, v) \in E(G)$  we have  $f(u) + f(e) + f(v) = k_1$  or  $k_2$ , where  $k_1$  and  $k_2$  are two constants. They provide edge bimagic total labelings for  $B_{m,n}$ ,  $K_{1,n,n}$ , and trees obtained from a path by appending an edge to one of the vertices adjacent to an endpoint of the path. An edge bimagic total labeling is G(V, E) is called an *a-vertex consecutive edge bimagic total labeling* if the vertex labels are  $\{a+1, a+2, \ldots, a+|V|\}$  where  $0 \leq a \leq |E|$ . Baskar Babujee and Jagadesh [403] prove the following graphs *a*-vertex consecutive edge-bimagic total labelings: the trees obtained from  $K_{1,n}$  by adding a new pendent edge to each of the existing n pendent vertices; the trees obtained by adding a pendent path of length 2 to each of the n pendent vertices of  $K_{1,n}$ ; the graphs obtained by joining the centers of two copies of identical stars by a path of length 2; and the trees obtained from a path by adding new pendent edges to one pendent vertex of the path. Baskar Babujee, Vishnupriya, and Jagadesh [415] proved the following graphs have

such labelings:  $P_n$ ,  $K_{1,n}$ , combs, bistars  $B_{m,n}$ , trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, trees obtained by joining the centers of two stars with a path of length 2, trees obtained from  $P_5$  by identifying the center of a copy  $K_{1,n}$  with the two end vertices and the middle vertex. In [405] Baskar Babujee and Jagadesh proved that cycles, fans, wheels, and gear graphs have *a*-vertex consecutive edge bimagic total labelings. Baskar Babujee, Jagadesh, Vishnupriya [407] study the properties of *a*-vertex consecutive edge bimagic total labeling for  $P_3 \odot K_{1,2n}$ ,  $P_n + \overline{K_2}$  (*n* is odd and  $n \ge 3$ ),  $(P_2 \cup mK_1) + \overline{K_2}$ ,  $(P_2 + mK_1)$  ( $m \ge 2$ ),  $C_n$ , fans  $P_n + K_1$ , double fans  $P_n + 2K_1$ , and graphs obtained by appending a path of length at least 2 to a vertex of  $C_3$ .

Vishnupriya, Manimekalai, and Baskar Babujee [2750] prove that bistars, trees obtained by adding a pendent edge to a vertex adjacent to the end point of a path, and trees obtained subdividing each edge of a star have edge bimagic total labelings. Prathap and Baskar Babujee [2037] obtain all possible edge magic total labelings and edge bimagic total labelings for the star  $K_{1,n}$ . Jayasekaran1 and Flower [1110] proved that the shadow graph and the splitting graph of paths stars and cycles are edge trimagic total and super edge trimagic total. Magic labelings of directed graphs are discussed in [1710] and [478].

# 6 Antimagic-type Labelings

### 6.1 Antimagic Labelings

Hartsfield and Ringel [964] introduced antimagic graphs in 1990. A graph with q edges is called *antimagic* if its edges can be labeled with  $1, 2, \ldots, q$  without repetition such that the sums of the labels of the edges incident to each vertex are distinct.<sup>2</sup> Among the graphs they prove are antimagic are:  $P_n$   $(n \ge 3)$ , cycles, wheels, and  $K_n$   $(n \ge 3)$ . T. Wang [2774] has shown that the toroidal grids  $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}$  are antimagic and, more generally, graphs of the form  $G \times C_n$  are antimagic if G is an r-regular antimagic graph with r > 1. Cheng [584] proved that all Cartesian products or two or more regular graphs of positive degree are antimagic and that if G is j-regular and H has maximum degree at most k, minimum degree at least one (G and H need not be connected), then  $G \times H$  is antimagic provided that j is odd and  $j^2 - j \ge 2k$ , or j is even and  $j^2 > 2k$ . Wang and Hsiao [2775] prove the following graphs are antimagic:  $G \times P_n$  (n > 1) where G is regular;  $G \times K_{1,n}$ where G is regular; compositions G[H] (see §2.3 for the definition) where H is d-regular with d > 1; and the Cartesian product of any double star (two stars with an edge joining their centers) and a regular graph. In [583] Cheng proved that  $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_t}$   $(t \ge 2)$ and  $C_m \times P_n$  are antimagic. In [2437] Solairaju and Arockiasamy prove that various families of subgraphs of grids  $P_m \times P_n$  are antimagic. Liang and Zhu [1580] proved that if G is k-regular  $(k \ge 2)$ , then for any graph H with  $|E(H)| \ge |V(H)| - 1 \ge 1$ , the Cartesian product  $H \times G$  is antimagic. They also showed that if  $|E(H)| \geq |V(H)| - 1$  and each connected component of H has a vertex of odd degree, or H has at least 2|V(H)| - 2 edges, then the prism of H is antimagic. Shang [2313] showed that all spiders are antimagic. Lee, Lin, and Tsai [1484] proved that  $C_n^2$  is antimagic and the vertex sums form a set of successive integers when n is odd. Shang, Lin, and Liaw [2316] show that a star forest containing no  $S_1$  and at most one  $S_2$  as components is antimagic. They also prove that if a star forest  $mS_2$  is antimagic then m = 1 and  $mS_2 \cup S_n$   $(n \ge 3)$  is antimagic if and only if  $m < \min\{2n+1, 2n-5 + \sqrt{8n^2 - 24n + 17}/2\}$ . Wang, Miao, and Li [2786] show that certain graphs with even factors are antimagic. Li [1574] gives antimagic labelings for  $C_n^k$  for k = 2, 3, and 4. In [2792] Wang and Zhang showed that certain classes of regular graphs of odd degree with particular type of perfect matchings are antimagic. As a by-product, they get that generalized Petersen graphs and a subclass of Cayley graphs of  $Z_n$  are antimagic. Deng and Li [659] proved that caterpillars with maximum degree 3 are antimagic.

For a graph G and a vertex v of G, the vertex switching graph  $G_v$  is the graph obtained from G by removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G. Vaidya and Vyas [2691] proved that the graphs obtained by the switching of a pendent vertex of a path, a vertex of a cycle, a rim vertex of a wheel, the center vertex of a helm, or a vertex of degree 2 of a fan are antimagic graphs.

Phanalasy, Miller, Rylands, and Lieby [1925] in 2011 showed that there is a relation-

 $<sup>^2\</sup>mathrm{A}$  comprehensive expository treatment of antimagic labelings is given by Bača, Miller, Ryan, and Semaničová-Feňovčíkováin [304].

ship between completely separating systems and labeling of regular graphs. Based on this relationship they proved that some regular graphs are antimagic. Phanalasy, Miller, Iliopoulos, Pissis, and Vaezpour [1923] proved the Cartesian product of regular graphs obtained from [1925] is antimagic. Ryan, Phanalasy, Miller, and Rylands introduced the generalized web and flower graphs in [2148] and proved that these families of graphs are antimagic. Rylands, Phanalasy, Ryan, and Miller extended the concept of generalized web graphs to the single apex multi-generalized web graphs and they proved these graphs to be antimagic in [2150]. Ryan, Phanalasy, Rylands and Miller extended the concept of generalized flower to the single apex multi-(complete) generalized flower graphs and constructed antimagic labeling for this family of graphs in [2149]. For more about antimagicness of generalized web and flower graphs see [1751]. Phanalasy, Ryan, Miller and Arumugam [1924] introduced the concept of generalized pyramid graphs and they constructed antimagic labeling for these graphs. Bača, Miller, Phanalasy, and Feňovčíková proved that some join graphs and incomplete join graphs are antimagic in [301]. Moreover, in [300] they proved that the complete bipartite graph  $K_{m,m}$  and complete 3-partite graph  $K_{m,m,m}$  are antimagic and if G is a k-regular (connected or disconnected) graph with p vertices and  $k \ge 2$ , then the join of G and  $(p-k)K_1$ ,  $G + (p-k)K_1$  is antimagic. Arumugam, Miller, Phanalasy, and Ryan [197] provided antimagic labelings for a family of generalized pyramid graphs. Daykin, Iliopoulas. Miller, and Phanalasy [651] show several families of graphs recursively defined from a sequence of graphs that are generalizations of corona graphs are antimagic. Lozano, Mora, Seara, and Tey [1667] proved that caterpillars are antimagic.

Let G be a k-regular graph with p vertices and q edges. The generalized sausage graph, denoted by S(G; m), is the graph obtained from  $G \times P_m$  ( $G \times P_1 = G$ ), by joining each end vertex of the  $P_m$  to a new vertex (which we call *apexes*) with an edge. In particular, when m = 1, each vertex of the graph G joins to two vertices with two edges. The mixed generalized sausage graph, denoted by MS(G;m), is the graph obtained from the generalized sausage graph  $S(G;m), m \geq 3$ , by joining each vertex of each copy of the [m/2] copies of G on the left hand side to the left hand side apex, except the nearest copy to the apex, and similarly for the right hand side apex. The *complete mixed generalized* sausage graph, denoted by CMS(G; m) is the graph obtained from the generalized sausage graph by joining each vertex of each copy of G, except the two nearest copies of G to the apexes, to each apex with an edge, and each corresponding pair of vertices of the two nearest copies of G to the apexes with an edge. The complete mixed generalized sausage graph  $CMS^{-}(G;m)$  is the graph obtained from CMS(G;m) by deleting the edge connecting each corresponding pair of vertices of the two nearest copies of G to the apexes. In [1922] Phanalasy proved a families of generalized sausage graphs, mixed generalized sausage graphs, and complete mixed generalized sausage graphs are antimagic.

A split graph is a graph that has a vertex set that can be partitioned into a clique and an independent set. Tyshkevich (see [391]) defines a canonically decomposable graph as follows. For a split graph S with a given partition of its vertex set into an independent set A and a clique B (denoted by S(A, B)), and an arbitrary graph H the composition  $S(A, B) \circ H$  is the graph obtained by taking the disjoint union of S(A, B) and H and adding to it all edges having an endpoint in each of B and V(H). If G contains nonempty induced subgraphs H and S and vertex subsets A and B such that  $G = S(A, B) \circ H$ , then G is *canonically decomposable*; otherwise G is *canonically indecomposable*. Barrus [391] proved that every connected graph on at least 3 vertices that is split or canonically decomposable is antimagic.

Hartsfield and Ringel [964] conjecture that every tree except  $P_2$  is antimagic and, moreover, every connected graph except  $P_2$  is antimagic. In 2004 Alon, Kaplan, Lev, Roditty, and Yuster [139] use probabilistic methods and analytic number theory to show that this conjecture is true for all graphs with n vertices and minimum degree  $\Omega(\log n)$ . They also prove that if G is a graph with  $n \ge 4$  vertices and  $\Delta(G) \ge n - 2$ , then G is antimagic and all complete partite graphs except  $K_2$  are antimagic. Slíva [2428] proved the conjecture for graphs with a regular dominating subgraph. In 2016 Eccles [705] improved the result of Alon et al. by proving that there exists an absolute constant  $d_0$  such that if G is a graph with average degree at least  $d_0$  and G contains no isolated edge and at most one isolated vertex, then G is antimagic.

Chawathe and Krishna [571] proved that every complete *m*-ary tree is antimagic. Yilma [2868] extended results on antimagic graphs that contain vertices of large degree by proving that a connected graph with  $\Delta(G) \ge |V(G)| - 3$ ,  $|V(G)| \ge 9$  is antimagic and that if G is a graph with  $\Delta(G) = \deg(u) = |V(G)| - k$ , where  $k \le |V(G)|/3$  and there exists a vertex v in G such that the union of neighborhoods of the vertices u and v forms the whole vertex set V(G), then G is antimagic.

Fronček [780] defines a handicap incomplete tournament of n teams with r rounds, HIT(n, r), as a tournament in which every team plays r other teams and the total strength of the opponents that team i plays is  $\vec{S}_{n,r}(i) = t - i$  for every i and some fixed constant t. (This means that the strongest team plays strongest opponents, and the lowest ranked team plays weakest opponents.) In terms of distance magic graphs this restriction corresponds to finding a distance antimagic graph with the additional property that the sequence  $w(1), w(2), \ldots, w(n)$  (where team i is again the i-th ranked team) is an increasing arithmetic progression with difference one. These graphs are called *handicap distance* antimagic graphs. A handicap distance d-antimagic labeling of a graph G(V, E) with n vertices is a bijection  $\vec{f}: V \to \{1, 2, \dots, n\}$  with the property that  $\vec{f}(x_i) = i$  and the sequence of the weights  $w(x_1), w(x_2), \ldots, w(x_n)$  forms an increasing arithmetic progression with difference d. A graph G is a handicap distance d-antimagic graph if it admits a handicap distance d-antimagic labeling, and handicap distance antimagic graph when d = 1. In [780] Fronček establishes a relationship between handicap incomplete tournaments and distance antimagic graphs and construct some new infinite classes of distance antimagic graphs and infinite classes of handicap incomplete round robin tournaments. Fronček and Shepanik [791] construct r-regular handicap distance antimagic graphs of order  $n \equiv 0$ (mod 8) for all feasible values of r. Fronček [784] proved that regular handicap distance antimagic graphs exist for every feasible odd order by proving that there exists a regular handicap graph of an odd order n if and only if n = 9 or n > 13. In [783] Fronček constructed a class of regular 2-handicap distance antimagic graphs for every order  $n \equiv 0$ (mod 16). In [785] he proved that a k-regular 2-handicap distance antimagic graph of order  $n \equiv 0 \pmod{16}$  exists if and only if  $n \ge 16$  and  $4 \le k \le n - 6$ .

Cranston [633] proved that for  $k \ge 2$ , every k-regular bipartite graph is antimagic. For non-bipartite regular graphs, Liang and Zhu [1581] proved that every cubic graph is antimagic. That result was generalized by Cranston, Liang, and Zhu [634], who proved that odd degree regular graphs are antimagic. Hartsfield and Ringel [964] proved that every 2-regular graph is antimagic. Bérczi, Bernäth, and Vizer [439] use a slight modification of an argument of Cranston et al. [634] to prove that k-regular graphs are antimagic for  $k \ge 2$ . The same was done by Chang, Liang, Pan, and Zhu [555] proved that every even degree regular graph is antimagic.

Beck and Jackanich [428] showed that every connected bipartite graph except  $P_2$  with |E| edges admits an edge labeling with labels from  $\{1, 2, \ldots, |E|\}$ , with repetition allowed, such that the sums of the labels of the edges incident to each vertex are distinct. They call such a graph *weak antimagic*.

Wang, Liu, and Li [2784] proved:  $mP_3$   $(m \ge 2)$  is not antimagic;  $P_n \cup P_n$   $(n \ge 4)$ is antimagic;  $S_n \cup P_n$  is antimagic;  $S_n \cup P_{n+1}$  is antimagic;  $C_n \cup S_m$  is antimagic for  $m \ge 2\sqrt{n} + 2$ ;  $mS_n$  is antimagic; if G and H are graphs of the same order and  $G \cup H$ is antimagic, then so is G + H; and if G and H are r-regular graphs of even order, then G+H is antimagic. In [2785] Wang, Liu, and Li proved that if G is an n-vertex graph with minimum degree at least r and H is an m-vertex graph with maximum degree at most 2r-1  $(m \ge n)$ , then G+H is antimagic. Bača, Kimáková, Semaničová-Feňovčikovă, and Umar [278] prove the disjoint union of multiple copies of a (a, 1)-(super)-tree-antimagic graph is also a (b, 1)-(super)-tree-antimagic for certain a and b.

For any given degree sequence pertaining to a tree, Miller, Phanalasy, Ryan, and Rylands [1753] gave a construction for two vertex antimagic edge trees with the given degree sequence and provided a construction to obtain an antimagic unicyclic graph with a given degree sequence pertaining to a unicyclic graph.

Kaplan, Lev, and Roditty [1340] prove that every non-trivial rooted tree for which every vertex that is a not a leaf has at least two children is antimagic (see [1579]) for a correction of a minor error in the the proof). For a graph G with m vertices and an Abelian group A they define G to be A-antimagic if there is a one-to-one mapping from the edges of G to the nonzero elements of A such that the sums of the labels of the edges incident to v, taken over all vertices v of G, are distinct. For any  $n \ge 2$  they show that a non-trivial rooted tree with n vertices for which every vertex that is a not a leaf has at least two children is  $Z_n$ -antimagic if and only if n is odd. They also show that these same trees are A-antimagic for elementary Abelian groups G with prime exponent congruent to 1 (mod 3).

In [551] Chan, Low, and Shiu use [G, A] to denote the class of distinct A-antimagic labelings of G. They prove: for a non-trivial Abelian group A that underlies some commutative ring R with unity, if d is a unit in R and  $f \in [G, A]$ , then  $df \in [G, A]$ ; if A is an Abelian group that contains a subgroup isomorphic to B and a graph G is B-antimagic, then G is A-antimagic;  $P_{4m+r}$  and  $C_{4m+r}$  are  $Z_k$ -antimagic for  $k \ge 4m + r$  and r = 0, 1, 3;  $P_{4m+2}$  is  $Z_k$ -antimagic for  $k \ge 4m + 3$ ; regular Hamiltonian graphs of order 4m + r are  $Z_k$ -antimagic for  $k \ge 4m + r$  and r = 0, 1, 3, and  $Z_k$ -antimagic for  $k \ge 4m + 3$  and r = 2; for odd n,  $S_n$  is  $Z_k$ -antimagic for  $k \ge n > 4$ ; for even n,  $S_n$  is  $Z_k$ -antimagic for  $k \ge n + 2 \ge 6$  but not  $Z_n$ -antimagic or  $Z_{n+1}$ -antimagic; trees of order n with exactly one vertex of even degree are  $Z_k$ -antimagic for  $k \ge n$ ; trees of order n with exactly two vertices of even degree are  $Z_k$ -antimagic for  $k \ge n + 1$ ; and double stars of order are  $Z_k$ -antimagic for  $k \ge n + 1$ ; and double stars of order are  $Z_k$ -antimagic for  $k \ge n + 1$ ; and  $Z_k$ -antimagic for  $k \ge n$  when  $n \not\equiv 2 \pmod{4}$ .

The integer-antimagic spectrum of a graph G is the set  $\{k \mid G \text{ is } Z_k\text{-antimagic } (k \geq 2\}$ . Shiu, Sun, and Low [2366] determine the integer-antimagic spectra of tadpoles and lollipops. Shiu and Low [2363] determine the integer-antimagic spectra of complete bipartite graphs and complete bipartite graphs with a deleted edge. Shiu [2335] determined the integer-antimagic spectra of disjoint unions of cycles.

Liang, Wong, and Zhu [1579] study trees with many degree 2 vertices with a restriction on the subgraph induced by degree 2 vertices and its complement. Denoting the set of degree 2 vertices of a tree T by  $V_2(T)$  Liang, Wong, and Zhu proved that if  $V_2(T)$  and  $V \setminus V_2(T)$  are both independent sets, or  $V_2(T)$  induces a path and every other vertex has an odd degree, then T is antimagic. In [1669] Lozano, Mora, Seara, and Tey extended this result by showing that trees whose vertices of even degree induce a path are antimagic.

In [2695] Vaidya and Vyas proved that the middle graphs, total graphs, and shadow graphs of paths and cycles are antimagic. In [1430] and [1431] Krishnaa provided some results for antimagic labelings for graphs derived from wheels and antimagic labelings of helm related graphs.

Bertault, Miller, Pé-Rosés, Feria-Puron, and Vaezpour [450] approached labeling problems as combinatorial optimization problems. They developed a general algorithm to determine whether a graph has a magic labeling, antimagic labeling, or an (a, d)-antimagic labeling (see Section 6.3). They verified that all trees with fewer than 10 vertices are super edge magic and all graphs of the form  $P_2^r \times P_3^s$  with less than 50 vertices are antimagic. In [292] Bača, MacDougall, Miller, Slamin, and Wallis survey results on antimagic, edgemagic total, and vertex-magic total labelings.

A total labeling of a graph G is a bijection f from  $V(G) \cup E(G)$  to  $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ . When  $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ , we say the total labeling is super. For a labeling f the associated edge-weight of an edge uv is defined by wtf(uv) = f(uv) + f(u) + f(v). The associated vertex-weight of a vertex v is defined by  $wt_f(v) = \sum_{u \in N(v)} f(uv) + f(v)$ , where N(v) is the set of the neighbors of v. A labeling f is called *edge-antimagic total* (vertex-antimagic total) if all edge-weights (vertex-weights) are pairwise distinct. A graph that admits an edge-antimagic total (vertex-antimagic total) graph. A labeling that is simultaneously edge-antimagic total and vertex-antimagic total is called a *totally antimagic total labeling*. A graph that admits a totally antimagic total labeling is called a *totally antimagic total graph*. A labeling g is said to be ordered (sharp ordered) if  $wt_g(u) \leq wt_g(v)$  ( $wt_g(u) < wt_g(v)$ ) holds for every pair of vertices  $u, v \in V(G)$  such that g(u) < g(v). A graph that admits a (sharp) ordered labeling is called a (sharp) ordered graph.

Miller, Phanalasy, and Ryan [1750] proved that all graphs have vertex-antimagic total labelings. Bača, Miller, Phanalasy, Ryan, Semaničová-Feňovčíková, and Abildgaard Sillasen [298] prove that  $mK_1, mK_2, P_n$   $(n \ge 2)$ , and  $C_n$  are sharp ordered super totally antimagic total. They prove if G is an ordered super edge-antimagic total graph then  $G + K_1$  is a totally antimagic total graph. As a corollary they get that stars, friendship graphs  $nK_2 + K_1$ , fans, and wheels are totally antimagic total. They also prove that if G is a regular ordered super edge-antimagic total graph then  $G \odot nK_1$  is totally antimagic total. As a corollary of this result, they have double-stars  $K_2 \odot nK_1$  and crowns  $C_m \odot nK_1$ are totally antimagic total. They show that a union of regular totally antimagic total graphs is a totally antimagic total graph.

Ahmed and Baskar Baskar [102] proved that complete bipartite graphs admit a totally antimagic total labeling. The same result was proved by Akwu and Ajayi [115] who also showed that the join of a complete bipartite graph and  $K_1$  is a totally antimagic total graph. Ahmed, Baskar Babujee, Bača, Semaničová-Feňovčíková [87] proved that complete graphs admit totally antimagic total labeling. They also considered the problem of finding total labelings for prisms and for two special classes of graphs related to paths that are simultaneously edge-magic and vertex-antimagic.

Miller, Phanalasy, Ryan, and Rylands [1752] provide a method whereby, given any degree sequence pertaining to a tree, one can construct an antimagic tree based on this sequence. By swapping the roles of edges and vertices with respect to a labeling, they provide a method to construct an edge antimagic vertex labeling for any tree. Ahmad, Semaničová-Feňovčíková, Siddiqui, and Kamran [98] construct  $\alpha$ -labelings from graceful labelings of smaller trees and transform this labeling to edge-antimagic vertex labeling of trees. Shang [2314] shows that linear forests without either of the paths  $P_2$  or  $P_3$  as components are antimagic. Shang [2315] proved that  $P_2$ ,  $P_3$ , and  $P_4$ -free linear forests are antimagic.

In [974] Hefetz, Mütze, and Schwartz investigate antimagic labelings of directed graphs. An antimagic labeling of a directed graph D with n vertices and m arcs is a bijection from the set of arcs of D to the integers  $\{1, \ldots, m\}$  such that all n oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of all edges entering that vertex minus the sum of labels of all edges leaving it. Hefetz et al. raise the questions "Is every orientation of any simple connected undirected graph antimagic? and "Given any undirected graph G, does there exist an orientation of G which is antimagic?" They call such an orientation an *antimagic orientation* of G. Regarding the first question, they state that, except for  $K_{1,2}$  and  $K_3$ , they know of no other counterexamples. They prove that there exists an absolute constant C such that for every undirected graph on n vertices with minimum degree at least  $C \log n$  every orientation is antimagic. They also show that every orientation of  $S_n$ ,  $n \neq 2$ , is antimagic; every orientation of  $W_n$  is antimagic; and every orientation of  $K_n$ ,  $n \neq 3$ , is antimagic. For the second question they prove: for odd r, every undirected r-regular graph has an antimagic orientation; for even r every connected undirected r-regular graph that admits a matching that covers all but at most one vertex has an antimagic orientation; and if G is a graph with 2n vertices that admits a perfect matching and has an independent set of size n such that every vertex in the independent set has degree at least 3, then G has an antimagic orientation. They conjecture that every connected undirected graph admits an antimagic

orientation and ask if it true that every connected directed graph with at least 4 vertices is antimagic. Motivated by the Hartsfield and Ringel on antimagic labelings of graphs, in 2010 Hefetz, Mütze, and Schwartz [974] initiated the study of antimagic orientations of graphs, and conjectured that every connected graph admits an antimagic orientation. The conjecture has been veried to be true for regular graphs (see [[974], [1575], [2849]]), and biregular bipartite graphs with minimum degree at least two by Shan and Yu [2312]. Yang, Carlson, Owens, Perry, Singgih, Song, Zhang, Zhang [2848] proved that every connected graph G on  $n \ge 9$  vertices with maximum degree at least n-5 admits an antimagic orientation. Li, Song, Wang, Yang, and Zhang [1575] proved that every 2-regular graph has an antimagic orientation and for all integers  $d \ge 2$ , every connected 2*d*-regular graph has an antimagic orientation.

Sonntag [2456] investigated antimagic labelings of hypergraphs. He shows that certain classes of cacti, cycle, and wheel hypergraphs have antimagic labelings. Javaid and Bhatti [1099] extended some of Sonntag's results to disjoint unions of hypergraphs. In [1828] Nalliah investigated the existence of antimagic labelings of some families of digraphs using hooked Skolem sequences. Marimuthu, Raja Durga, and Durga Devi [1708] investigated the existence of super vertex in-antimagic total labelings of generalized de Bruijn digraphs.

Hefetz [973] calls a graph with q edges k-antimagic if its edges can be labeled with  $1, 2, \ldots, q + k$  such that the sums of the labels of the edges incident to each vertex are distinct. In particular, antimagic is the same as 0-antimagic. More generally, given a weight function  $\omega$  from the vertices to the natural numbers Hefetz calls a graph with q edges  $(\omega, k)$ -antimagic if its edges can be labeled with  $1, 2, \ldots, q + k$  such that the sums of the labels of the edges incident to each vertex and the weight assigned to each vertex by  $\omega$  are distinct. In particular, antimagic is the same as  $(\omega, 0)$ -antimagic where  $\omega$  is the zero function. Using Alon's combinatorial nullstellensatz [138] as his main tool, Hefetz has proved the following: a graph with  $3^m$  vertices and a  $K_3$  factor is antimagic; a graph with q edges and at most one isolated vertex and no isolated edges is  $(\omega, 2q-4)$ -antimagic; a graph with p > 2 vertices that admits a 1-factor is (p-2)-antimagic; a graph with p vertices and maximum degree n-k, where  $k \geq 3$  is any function of p is (3k-7)-antimagic and, in the case that  $p \geq 6k^2$ , is (k-1)-antimagic. Hefetz, Saluz, and Tran [975] improved the first of Hefetz's results by showing that a graph with  $p^m$  vertices, where p is an odd prime and m is positive, and a  $C_p$  factor is antimagic.

A graph G = (V, E) is strongly antimagic if there is a bijective mapping  $f : E \to 1, 2, \ldots, |E|$  such that for any two vertices  $u \neq v$ , not only  $\sum_{e \in E(u)} f(e) \neq \sum_{e \in E(v)} f(e)$  and also  $\sum_{e \in E(u)} f(e) < \sum_{e \in E(v)} f(e)$  whenever  $\deg(u) < \deg(v)$ , where E(u) is the set of edges incident to u. Chang, Chin, Li, and Pan [556] proved double spiders (the trees contains exactly two vertices of degree at least 3) are strongly antimagic. They raise the following two questions. Does there exist a strongly antimagic labellings for every antimagic graph? Is there a k-antimagic graph but not (k + 1)-antimagic?

Ahmad, Bača, Lascsáková and Semaničová-Feňovčíková [80] call a labeling of a plane graph *d*-antimagic if for every positive integer s, the set of s-sided face weights is  $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$  for some positive integers as  $a_s$  and d, where  $f_s$  is the number of the s-sided faces. (They allow different sets  $W_s$  for different s). A dantimagic labeling is called *super* if the smallest possible labels appear on the vertices. In [125] they investigated the existence of super *d*-antimagic labelings of type (1, 1, 0) for disjoint union of plane graphs for several values of difference *d*. Bača, Numan, and Semaničová-Feňovčíková [308] invesitgate the existence of super *d*-antimagic labelings of generalized prisms. Hussainn and Tabraiz [1038] investigated super *d*-antimagic labeling of type (1,1,1) on the snakes  $kC_5$ ; subdivided  $kC_5$ ; and isomorphic copies of  $kC_5$  for strings  $(1,1,\ldots,1)$  and  $(2,2\ldots,2)$ .

Bača, Baskoro, Jendrol, and Miller [257] investigated various k-antimagic labelings for graphs in the shape of hexagonal honeycombs. They use  $H_n^m$  to denote the honeycomb graph with m rows, n columns, and mn 6-sided faces. They prove: for n odd  $H_n^m$ , has a 0-antimagic vertex labeling and a 2-antimagic edge labeling, and if n is odd and mn > 1,  $H_n^m$  has a 1-antimagic face labeling. In [2364] Shiu and Low show how to construct k-antimagic graphs from existing graphs G with particular labeling properties by joining G to cycles and dumbell related graphs with an edge.

Huang, Wong, and Zhu [1033] say a graph G is weighted-k-antimagic if for any vertex weight function w from the vertices of G to the natural numbers there is an injection f from the edges of G to  $\{1, 2, \ldots, |E| + k\}$  such that for any two distinct vertices u and  $v, \sum (f(e) + w(v)) \neq \sum (f(e) + w(u))$  over all edges incidence to v. They proved that if G has odd prime power order  $p^{z}$  and has total domination number 2 with the degree of one vertex in the total dominating set not a multiple of p, then G is weighted-1-antimagic, and if G has odd prime power order  $p^z$ ,  $p \neq 3$  and has maximum degree at least |V(G)| = 3, then G is weighted-1-antimagic. Wong and Zhu [2773] proved: graphs that have a vertex that is adjacent to all other vertices are weighted-2-antimagic; graphs with a prime number of vertices that have a Hamiltonian path are weighted-1-antimagic; and connected graphs  $G \neq K_2$  on n vertices are weighted  $\lfloor 3n/2 \rfloor$ -antimagic. Matamala and Zamora [1721] proved that  $K_{m,n}$ ,  $3 \le m \le n$ ,  $n \ge 3$ , is weighted-0-antimagic and described a polynomial time algorithm that computes a (w, 0)-antimagic labeling of  $K_{m,n}$ . They also prove the following. Let H be an arbitrary complete partite graph with  $n \geq 5$ vertices not isomorphic to  $K_{1,n}$ . Then, any graph containing H as a spanning subgraph is weighted-0-antimagic and given a weight function w, a (w, 0)-antimagic labeling can be computed in polynomial time. They prove that each connected graph G on  $n \geq 3$  vertices having  $K_{1,n}$  as a spanning subgraph is weighted-1-antimagic unless G is isomorphic to  $K_{1,n}$  and n is even.

A distance k-antimagic labeling of a graph G(V, E) is a bijection f from V to  $\{1, 2, \ldots, |V|\}$  with the property that there exists an ordering of the vertices of G such that the sequence of the weights  $w(x_1), w(x_2), \ldots, w(x_n)$  forms an arithmetic progression with difference k. When k = 1, then f is simply called a distance antimagic labeling. A distance k-antimagic graph is a distance k-antimagic graph that admits a distance k-antimagic labeling, and is called distance antimagic when k = 1. Cichacz, Froncek, Sugeng and Zhou in [621] gave a necessary condition for a graph with an even number of vertices to be distance antimagic with respect to an Abelian group with a unique involution. They also gave sufficient conditions for a Cayley graph on an Abelian group to be distance antimagic or magic with respect to the same group, and discussed the

consequences of these results to Cayley graphs on elementary Abelian groups. In [950] Handa, Godinho, and Singh investigate the existence of distance antimagic labelings of ladders.

For a positive integer k, define  $f_k : V(G) \longrightarrow \{1 + k, 2 + k, ..., n + k\}$  by  $f_k(x) = f(x) + k$ . If  $w_{f_k}(u) \neq w_{f_k}(v)$  for every pair of vertices  $u, v \in V$ , for any  $k \geq 0$  then f is said to be an *arbitrarily distance antimagic* labeling and the graph which admits such a labeling is said to be an *arbitrarily distance antimagic* graph. Handa, Godinho, and Singh [951] provide arbitrarily distance antimagic labelings for  $rP_n$ , the generalized Petersen graph  $P(n,k), n \geq 5$ , the Harary graph  $H_{4,n}$  for  $n \neq 6$  and prove that join of these graphs is distance antimagic.

For an arbitrary set of distances  $D \subseteq \{0, 1, \ldots, diam(G)\}$ , a *D*-weight of a vertex x in a graph G under a vertex labeling  $f : V \to \{1, 2, \ldots, v\}$  is defined as  $w_D(x) = \sum_{y \in N_D(x)} f(y)$ , where  $N_D(x) = \{y \in V \mid d(x, y) \in D\}$ . A graph G is said to be *D*-distance magic if all vertices has the same *D*-vertex-weight, it is said to be *D*-distance antimagic if all vertices have distinct *D*-vertex-weights. In [2387] Simanjuntak and Wijaya gave some necessary conditions for the existence of *D*-distance antimagic graphs and conjectured that those conditions are sufficient. They also gave  $\{1\}$ -distance antimagic labelings for cycles, suns, prisms, complete graphs, wheels, fans, and friendship graphs.

In [195] Arumugam and Kamatchi introduced the notion of (a, d)-distance antimagic graphs as follows. Let G be a graph with vertex set V and  $f: V \to \{1, 2, \ldots, |V|\}$  be a bijection. If for all v in G the set of sums  $\sum f(u)$  taken over all neighbors u of v is the arithmetic progression  $\{a, a + d, a + 2d, \dots, a + (|V| - 1)d\}, f$  is called an (a, d)-distance antimagic labeling and G is called a (a, d)-distance antimagic graph. Arumugam and Kamatchi [195] proved:  $C_n$  is (a, d)-distance antimagic if and only if n is odd and d = 1; there is no (1, d)-distance antimagic labeling for  $P_n$  when  $n \geq 3$ ; a graph G is (1, d)distance antimagic graph if and only if every component of G is  $K_2$ ;  $C_n \times K_2$  is (n+2,1)distance antimagic; and the graph obtained from  $C_{2n} = (v_1, v_2, \ldots, v_{2n})$  by adding the edges  $v_1v_{n+1}$  and  $v_iv_{2n+2-i}$  for  $i=2,3,\ldots,n$  is (2n+2,1)-distance antimagic. In [780] and [782] Froncek proved that disjoint copies of the Cartesian product of two complete graphs and its complement are (a, 2)-distance antimagic and (a, 1)-distance antimagic. He also proved that disjoint copies of the hypercube  $Q_3$  is (a, 1)-distance antimagic. Semeniuta [2213] proved that the crown  $P_n \odot P_1$  does not admit an (a, 1)-distance antimagic labeling for  $n \ge 2$  and  $a \ge 2$  and determines the values of a for which  $P_n$  can be an (a, 1)-distance antimagic graph. The circulant graph is also investigated. Semenyuta [2214] proved that  $P_n \odot P_1$  is not an (a, d)-distance antimagic graph for all a and d and that  $Q_n$  is a  $(2^n + n - 1, n - 2)$ -distance antimagic graph. He found two types of graphs that do not allow 1-vertex bimagic vertex labeling and established a relation between the distance magic labeling of a regular graph G with 1-vertex bimagic vertex labeling  $G \cup G$ .

Kamatchi, Vijayakumar, Ramalakshmi, Nilavarasi, and Arumugam [1271] prove that the hypercube is (a, d)-distance antimagic and the bistar  $K_2(n, n)$  is distance antimagic. They also show that if G is a regular distance antimagic graph, then 2G is also distance antimagic and several families of disconnected graphs are distance antimagic graphs.

A connected graph G = (V, E) with m edges is called if for each set B of m positive

integers there is an bijective function  $f: E \to B$  such that the function  $\tilde{f}: V \to \mathbb{N}$  defined at each vertex v as the sum of all labels of edges incident to v is injective. Matamala and Zamora [1720] proved that paths, cycles, split graphs, and graphs that contains the complete bipartite graph  $K_{2,n}$  as a spanning subgraph are universal antimagic.

In 2019 [305] Bača, Miller, Ryan, and Semaničová-Feňovčíková published a monograph that focuses on variations of magic and antimagic type lsbelings and includes new results, techniques, constructions, and open problems and conjectures.

In Table 12 we use the abbreviation **A** to mean antimagic. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2014.

Graph	Labeling	Notes
$P_n$	А	for $n \ge 3$ [964]
$C_n$	А	[964]
$W_n$	А	[964]
$K_n$	А	for $n \ge 3$ [964]
every tree except $K_2$	A?	[964]
caterpillars	А	[1667]
regular graphs	А	[1581], [964], [555]
every connected graph except $K_2$	A?	[964]
$n \ge 4$ vertices $\Delta(G) \ge n-2$	А	[139]
all complete partite graphs except $K_2$	А	[139]
$C_m \times P_n$	А	[583]
$P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k}$	А	[583]

Table 12: Summary of Antimagic Labelings

Continued on next page

Table 12 – Continued from previous page

Graph	Labeling	Notes
$C_{m_1} \times C_{m_2} \times \cdots \times C_{m_k}$	А	[2774]
$C_n^2$	А	[1484]
$mP_3 \ m \geq 2$	not A	[2784]

## **6.2** (a, d)-Antimagic Labelings

The concept of an (a, d)-antimagic labelings was introduced by Bodendiek and Walther [481] in 1993. A connected graph G = (V, E) is said to be (a, d)-antimagic if there exist positive integers a, d and a bijection  $f: E \to \{1, 2, \dots, |E|\}$  such that the induced mapping  $g_f: V \to N$ , defined by  $g_f(v) = \sum \{f(uv) \mid uv \in E(G)\}$ , is injective and  $g_f(V) =$  $\{a, a + d, \dots, a + (|V| - 1)d\}$ . (In [1599] Lin, Miller, Simanjuntak, and Slamim called these (a, d)-vertex-antimagic edge labelings). Bodendick and Walther ([483] and [484]) prove the Herschel graph is not (a, d)-antimagic and obtain both positive and negative results about (a, d)-antimagic labelings for various cases of graphs called *parachutes*  $P_{g,p}$ .  $(P_{g,p})$  is the graph obtained from the wheel  $W_{g+p}$  by deleting p consecutive spokes.) In [272] Bača and Holländer prove that necessary conditions for  $C_n \times P_2$  to be (a, d)-antimagic are d = 1, a = (7n+4)/2 or d = 3, a = (3n+6)/2 when n is even, and d = 2, a = (5n+5)/2or d = 4, a = (n + 7)/2 when n is odd. Bodendiek and Walther [482] conjectured that  $C_n \times P_2$   $(n \ge 3)$  is ((7n+4)/2, 1)-antimagic when n is even and is ((5n+5)/2, 2)antimagic when n is odd. These conjectures were verified by Bača and Holländer [272] who further proved that  $C_n \times P_2$   $(n \ge 3)$  is ((3n+6)/2, 3)-antimagic when n is even. Bača and Holländer [272] conjecture that  $C_n \times P_2$  is ((n+7)/2, 4)-antimagic when n is odd and at least 7. Bodendiek and Walther [482] also conjectured that  $C_n \times P_2$   $(n \ge 7)$  is ((n+7)/2, 4)-antimagic. Miller and Bača [1746] prove that the generalized Petersen graph P(n,2) is ((3n+6)/2,3)-antimagic for  $n \equiv 0 \pmod{4}$ ,  $n \geq 8$  and conjectured that P(n,k)is ((3n+6)/2, 3)-antimagic for even n and  $2 \le k \le n/2 - 1$  (see §2.7 for the definition of P(n,k)). This conjecture was proved for k=3 by Xu, Yang, Xi, and Li [2840]. Jirimutu and Wang proved that P(n,2) is ((5n+5)/2,2)-antimagic for  $n \equiv 3 \pmod{4}$  and  $n \geq 7$ . Xu, Xu, Lü, Baosheng, and Nan [2836] proved that P(n,2) is ((3n+6)/2, 2)-antimagic for  $n \equiv 2 \pmod{4}$  and  $n \geq 10$ . Xu, Yang, Xi, and Li [2840] proved that P(n,3) is ((3n+6)/2, 3)-antimagic for even  $n \ge 10$  and for  $n \equiv 0 \pmod{4}$ ,  $n \ge 8$ . In [1602] Lingqi, Linna, Yuan show that P(n,3) is (5n+5)/2, 2)-antimagic for odd  $n \ge 7$ . Feng, Hong, Yang, and Jirimutu [754] show that P(n, 5) is (3n+6)/2, 3)-antimagic for even  $n \ge 12$ . Bao, Zhao, Yang, Feng, and Jirimutu [344] proved that P(n,7) is  $(\frac{3n+6}{2},3)$ -antimagic for even  $n \geq 16$ . Ivančo [1081] investigated (a, 1)-antimagic labelings and their connection with supermagic generalized double graphs. Bodendiek and Walther [485] proved that the following graphs are not (a, d)-antimagic: even cycles; paths of even order; stars;

 $C_3^{(k)}; C_4^{(k)};$  trees of odd order at least 5 that have a vertex that is adjacent to three or more end vertices; *n*-ary trees with at least two layers when d = 1; the Petersen graph;  $K_4$  and  $K_{3,3}$ . They also prove:  $P_{2k+1}$  is (k, 1)-antimagic;  $C_{2k+1}$  is (k+2, 1)-antimagic; if a tree of odd order 2k + 1 (k > 1) is (a, d)-antimagic, then d = 1 and a = k; if  $K_{4k}$   $(k \ge 2)$ is (a, d)-antimagic, then d is odd and  $d \le 2k(4k - 3) + 1$ ; if  $K_{4k+2}$  is (a, d)-antimagic, then d is even and  $d \le (2k + 1)(4k - 1) + 1$ ; and if  $K_{2k+1}$   $(k \ge 2)$  is (a, d)-antimagic, then  $d \le (2k + 1)(k - 1)$ . Lin, Miller, Simanjuntak, and Slamin [1599] show that no wheel  $W_n$  (n > 3) has an (a, d)-antimagic labeling. In [2549] Susanto provided super (a, d)- $C_n$ -antimagic total labelings for various cases of  $mC_n$ .

In [1089] Ivančo, and Semaničová show that a 2-regular graph is super edge-magic if and only if it is (a, 1)-antimagic. As a corollary we have that each of the following graphs are (a, 1)-antimagic:  $kC_n$  for n odd and at least 3;  $k(C_3 \cup C_n)$  for n even and at least 6;  $k(C_4 \cup C_n)$  for n odd and at least 5;  $k(C_5 \cup C_n)$  for n even and at least 4;  $k(C_m \cup C_n)$  for m even and at least 6, n odd, and  $n \ge m/2 + 2$ . Extending a idea of Kovář they prove if G is  $(a_1, 1)$ -antimagic and H is obtained from G by adding an arbitrary 2k-factor then H is  $(a_2, 1)$ -antimagic for some  $a_2$ . As corollaries they observe that the following graphs are (a, 1)-antimagic: circulant graphs of odd order; 2r-regular Hamiltonian graphs of odd order; and 2r-regular graphs of odd order n < 4r. They further show that if G is an (a, 1)-antimagic r-regular graph of order n and n - r - 1 is a divisor of the non-negative integer a + n(1 + r - (n + 1)/2), then  $G \oplus K_1$  is supermagic. As a corollary of this result they have if G is (n - 3)-regular for n odd and  $n \ge 7$  or (n - 7)-regular for n odd and  $n \ge 15$ , then  $G \oplus K_1$  is supermagic.

Bertault, Miller, Feria-Purón, and Vaezpour [450] approached labeling problems as combinatorial optimization problems. They developed a general algorithm to determine whether a graph has a magic labeling, antimagic labeling, or an (a, d)-antimagic labeling. They verified that all trees with fewer than 10 vertices are super edge magic and all graphs of the form  $P_2^r \times P_3^s$  with less than 50 vertices are antimagic. Javaid, Hussain, Ali, and Dar [1103] and Javaid, Bhatti, and Hussain [1100] constructed super (a, d)-edge-antimagic total labelings for w-trees and extended w-trees (see 5.2 for the definitions) as well as super (a, d)-edge-antimagic total labelings for disjoint union of isomorphic and non-isomorphic copies of extended w-trees. In [1101] Javaid and Bhatt defined a generalized w-tree and proved that they admit a super (a, d)-edge-antimagic total labeling. In [2782] Wang, Li, and Wang proved that some classes of graphs derived from regular or regular bipartite graphs are antimagic. A subdivided star  $T(n_1, n_2, \ldots, n_r)$  is a tree obtained by inserting  $n_i \geq 1, 1 \leq i \leq r$  with  $r \geq 3$  vertices. In [2051] Raheem, Javaid, and Baig study a super (a, d)-edge-antimagic total labelings of the subdivided stars  $T(n, n + 1, n_3, \ldots, n_r)$ when n is even and  $T(n, n, n+1, n_4, \ldots, n_r)$  when n is odd for all possible values of d. In [2052] Raheem and Baig proved the super edge antimagicness of subdivided stars for all possible values of d. Bhatti, Tahir, and Javaid [467] give super (a, d)-edge antimagic total labelings of some wheel-like graphs. In [198] investigate the existence of super (a, d)-edge antimagic total labeling for friendship graphs and generalized friendship graphs.

For graphs G and F, if every edge of G belongs to a subgraph of G isomorphic to F and there exists a total labeling  $\lambda$  of G such that for every subgraph F' of G that is

isomorphic to F, the set  $\{\Sigma\lambda(F'): F' \cong F, F' \subseteq G\}$  forms an arithmetic progression starting with a with common difference d, Lee, Tsai, and Lin [1483] say that G is (a, d)-F-antimagic. Furthermore, if  $\lambda(V(G)) = \{1, 2, \ldots, |V(G)|\}$  then G is said to be super (a, d)-F-antimagic and  $\lambda$  is said to be a super (a, d)-F-antimagic labeling of G. Lee, Tsai, and Lin [1483] proved that  $P_m \times P_n$   $(m, n \geq 2)$  is super (a, 1)- $C_4$ -antimagic. In [2204] Selvagopal, Jeyanthi, Muthuraja, and Semaničová-Feňovčiková investigated the existence super (a, d)-star-antimagic labelings of a particular class of banana trees and construct a star-antimagic graph.

The edge corona path graph  $G_m \diamond P_n$  is the graph obtained from one copy of the gear graph  $G_m$  and 3m copies of  $P_n$ ,  $P_n^i$ , by joining two end vertices of  $e_i \in E(G_m)$  to every vertex  $v_j \in V(P_n)$  in the *i*-th copy of  $G_m$  with i = 1, 2, ..., 3m and j = 1, 2, ..., n. The graph  $G_m \cdot C_n$  is the graph obtained from  $G_m$  and 2m + 1 copies of  $C_n$  namely  $C_n^i$  by joined every vertex  $v_i \in G_m$  to all vertices  $v_i \in C_n$  for  $i \in \{1, 2, ..., 2m + 1\}$ . Nistyawati and Martini [1870] proved that for every odd m, the gear edge corona path graph  $G_m \cdot C_n$  is super  $C_4 \diamond P_n$ -antimagic and for every odd m, the gear corona cycle graph  $G_m \cdot C_n$  is super  $C_4 \circ C_n$ -antimagic. Roswitha, Martini, and S. A. Nugroho [2143] proved: for  $n \geq 5$  the double cone  $DC_n = C_n + \overline{K_2}$  is  $(14 + 7n + (n + 1)2, 1) \cdot C_3$ -antimagic and  $(a, 1) \cdot C_3$ -antimagic;  $DC_n$  is  $(a, d) \cdot W_n$ -antimagic;  $DC_{2n}$  is  $(a, 1) \cdot W_{2n}$ -antimagic; and  $DC_{2n+1}$  is  $(a, 2) \cdot W_{2n+1}$ -antimagic.

Yegnanarayanan [2866] introduced several variations of antimagic labelings and provides some results about them.

The antiprism on 2n vertices has vertex set  $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$  and edge set  $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$  (subscripts are taken modulo n). For  $n \geq 3$  and  $n \not\equiv 2 \pmod{4}$  Bača [244] gives (6n + 3, 2)-antimagic labelings and (4n + 4, 4)-antimagic labelings for the antiprism on 2n vertices. He conjectures that for  $n \equiv 2 \pmod{4}$ ,  $n \geq 6$ , the antiprism on 2n vertices has a (6n+3, 2)-antimagic labeling and a (4n+4, 4)-antimagic labeling.

Nicholas, Somasundaram, and Vilfred [1867] prove the following: If  $K_{m,n}$  where  $m \leq n$ is (a, d)-antimagic, then d divides ((m - n)(2a + d(m + n - 1)))/4 + dmn/2; if m + n is prime, then  $K_{m,n}$ , where n > m > 1, is not (a, d)-antimagic; if  $K_{n,n+2}$  is (a, d)-antimagic, then d is even and  $n + 1 \leq d < (n + 1)^2/2$ ; if  $K_{n,n+2}$  is (a, d)-antimagic and n is odd, then a is even and d divides a; if  $K_{n,n+2}$  is (a, d)-antimagic and n is even, then d divides 2a; if  $K_{n,n}$  is (a, d)-antimagic, then n and d are even and  $0 < d < n^2/2$ ; if G has order nand is unicylic and (a, d)-antimagic, then (a, d) = (2, 2) when n is even and (a, d) = (2, 2)or (a, d) = ((n + 3)/2, 1) when n is odd; a cycle with m pendent edges attached at each vertex is (a, d)-antimagic if and only if m = 1; the graph obtained by joining an endpoint of  $P_m$  with one vertex of the cycle  $C_n$  is (2, 2)-antimagic if m = n or m = n - 1; if m + nis even the graph obtained by joining an endpoint of  $P_m$  with one vertex of the cycle  $C_n$ is (a, d)-antimagic if and only if m = n or m = n - 1. They conjecture that for n odd and at least 3,  $K_{n,n+2}$  is  $((n + 1)(n^2 - 1)/2, n + 1)$ -antimagic and they have obtained several results about (a, d)-antimagic labelings of caterpillars.

In [1668] Lozano, Mora, and Seara prove that any caterpillar of order n is  $(\lfloor (n-1)/2, \rfloor - 2)$ -antimagic. Furthermore, if C is a caterpillar with a spine of order s, they

prove that when C has at least  $\lfloor (3s+1)/2 \rfloor$  leaves or  $\lfloor (s-1)/2 \rfloor$  consecutive vertices of degree at most 2 at one end of a longest path, then C is antimagic. As a consequence of a result by Wong and Zhu [2819], they also prove that if p is a prime number, any caterpillar with a spine of order p, p-1 or p-2 is 1-antimagic.

In [2739] Vilfred and Florida proved the following: the one-sided infinite path is (1, 2)antimagic;  $P_{2n}$  is not (a, d)-antimagic for any a and d;  $P_{2n+1}$  is (a, d)-antimagic if and only if (a, d) = (n, 1);  $C_{2n+1}$  has an (n + 2, 1)-antimagic labeling; and that a 2-regular graph G is (a, d)-antimagic if and only if |V(G)| = 2n + 1 and (a, d) = (n + 2, 1). They also prove that for a graph with an (a, d)-antimagic labeling, q edges, minimum degree  $\delta$ and maximum degree  $\Delta$ , the vertex labels lie between  $\delta(\delta + 1)/2$  and  $\Delta(2q - \Delta + 1)/2$ .

Chelvam, Rilwan, and Kalaimurugan [572] proved that Cayley digraph of any finite group admits a super vertex (a, d)-antimagic labeling depending on d and the size of the generating set. They provide algorithms for constructing the labelings.

Irfan and Semaničová-Feňovčiková [1076] provide some classes of graphs that admit a labeling that is simultaneously a super edge-magic total and a super vertex-antimagic total and give some results for fans, sun graphs, caterpillars, and prisms.

For n > 1 and distinct odd integers x, y and z in [1, n - 1] Javaid, Ismail, and Salman [1095] define the *chordal ring* of order n,  $CR_n(x, y, z)$ , as the graph with vertex set  $Z_n$ , the additive group of integers modulo n, and edges (i, i + x), (i, i + y), (i, i + z) for all even i. They prove that  $CR_n(1, 3, 7)$  and  $CR_n(1, 5, n - 1)$  have (a, d)-antimagic labelings when  $n \equiv 0 \mod 4$  and conjecture that for an odd integer  $\Delta$ ,  $3 \leq \Delta \leq n - 3, n \equiv 0 \mod 4$ ,  $CR_n((1, \Delta, n - 1) \max ((7n + 8)/4, 1)$ -antimagic labeling.

For an arbitrary set of distances  $D \subseteq \{0, 1, \dots, diam(G)\}$ , a D-weight of a vertex x in a graph G under a vertex labeling  $f: V \to \{1, 2, \dots, v\}$  is defined as  $w_D(x) =$  $\sum_{y \in N_D(x)} f(y)$ , where  $N_D(x) = \{y \in V | d(x, y) \in D\}$ . A graph G is said to be Ddistance magic if all vertices have the same D-vertex-weight, it is said to be D-distance antimagic index D-distance antimagic if all vertices have distinct D-vertex-weights, and it is called (a, d) - D-distance antimagic if the D-vertex-weights constitute an arithmetic progression with difference d and starting value a. In [2387] Simanjuntak and Wijaya gave some necessary conditions for the existence of D-distance antimagic graphs and conjectured that those conditions are sufficient. They also gave  $\{1\}$ -distance antimagic labelings for cycles, suns, prisms, complete graphs, wheels, fans, and friendship graphs. Arumugam and Kamatchi [195] characterized (a, d)-distance antimagic cycles and (a, d)distance antimagic labelings for paths and prisms. In [780] and [782] Fronček proved that disjoint copies of the Cartesian product of two complete graphs and its complement are (a, 2)-distance antimagic and (a, 1)-distance antimagic. He also proved that disjoint copies of the hypercube  $Q_3$  is (a, 1)-distance antimagic. In [950] Handa, Godinho and Singh investigate the existence of distance antimagic labeling of ladders.

In [2740] Vilfred and Florida call a graph G = (V, E) odd antimagic if there exist a bijection  $f: E \to \{1, 3, 5, \ldots, 2|E| - 1\}$  such that the induced mapping  $g_f: V \to N$ , defined by  $g_f(v) = \sum \{f(uv)| uv \in E(G)\}$ , is injective and odd (a, d)-antimagic if there exist positive integers a, d and a bijection  $f: E \to \{1, 3, 5, \ldots, 2|E| - 1\}$  such that the induced mapping  $g_f: V \to N$ , defined by  $g_f(v) = \sum \{f(uv)| uv \in E(G)\}$ , is injective and  $g_f(V) = \{a, a + d, a + 2d, \ldots, a + (|V| - 1)d\}$ . Although every (a, d)-antimagic graph is antimagic,  $C_4$  has an antimagic labeling but does not have an (a, d)-antimagic labeling. They prove:  $P_{2n+1}$  is not odd (a, d)-antimagic for any a and d;  $C_{2n+1}$  has an odd (2n+2, 2)-antimagic labeling; if a 2-regular graph G has an odd (a, d)-antimagic labeling, then |V(G)| = 2n + 1 and (a, d) = (2n + 2, 2);  $C_{2n}$  is odd magic; and an odd magic graph with at least three vertices, minimum degree  $\delta$ , maximum degree  $\Delta$ , and  $q \geq 2$  edges has all its vertex labels between  $\delta^2$  and  $\Delta(2q - \Delta)$ .

Combining the notions of 1-vertex-magic vertex labelings and antimagic labelings Swaminathan and Jeyanthi [2562] introduced a new labeling as follows. For a graph with p vertices a 1-1 mapping from the vertices to  $\{1, 2, \ldots, p\}$  is called an (a, d)-1-vertexantimagic vertex labeling if the sums of the labels of the vertices adjacent to each vertex taken over all vertices form the set  $\{a, a+d, a+2d, \ldots, a+(p-1)d\}$ . They give some basic properties of such labelings and provide some results for some classes of regular graphs.

For a graph G = (V, E), a bijection g from  $V(G) \cup E(G)$  into  $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called a (a, d)-edge-antimagic graceful labeling of G if the edge-weights w(xy) = |g(x) + g(y) - g(xy)|,  $xy \in E(G)$ , form an arithmetic progression starting from a and having a common difference d. An (a, d)-edge-antimagic graceful labeling is called super (a, d)edge-antimagic graceful if  $g(V(G)) = \{1, 2, \ldots, |V(G)|\}$ . Marimuthu and Krishnaveni [1703] proved  $mC_n$  has a super (0, 1)-edge-antimagic graceful labeling for every  $m \geq 2$ and  $n \geq 3$ ; and  $mK_n$  and  $MP_n$  have a super (a, 1)-edge-antimagic graceful labeling for every  $m \geq 2$  and  $n \geq 2$ .

For a connected graph G with q edges a bijection  $f: E \to \{1, 2, \ldots, q\}$  is called a *local* antimagic labeling if for any two adjacent vertices u and  $v, w(u) \neq w(v)$ , where  $w(u) = \sum_{e \in E(u)} f(e)$ , and E(u) is the set of edges incident to u. In [201] Arumugam, Premalatha, Bača, and Semaničová-Feňovčíková proved several basic results on this new parameter and conjectured that any connected graph other than  $K_2$  admits a local antimagic labeling. This conjecture was proved by Haslegrave [966] using the probabilistic method, proves that the local antimagic conjecture is true. Lau [1463] proved that every graph admits a local antimagic total labeling.

In Table 13 we use the abbreviation (a, d)-**A** to mean that the graph has an (a, d)antimagic labeling. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Graph	Labeling	Notes
$P_{2n}$	not $(a, d)$ -A	[485]
$P_{2n+1}$	iff $(n, 1)$ -A	[485]
$C_{2n}$	not $(a, d)$ -A	[485]
$C_{2n+1}$	(n+2,1)-A	[485]
stars	not $(a, d)$ -A	[485]
$C_3^{(k)}, C_4^{(k)}$	not $(a, d)$ -A	[485]
$K_{3,3}$	not $(a, d)$ -A	[485]
$K_4$	not $(a, d)$ -A	[485]
Petersen graph	not $(a, d)$ -A	[485]
	not $(a, d)$ -A	n > 3 [1599]
antiprism on $2n$ vertices (see §6.2)	$\begin{array}{c} (6n+3,2)\text{-A} \\ (4n+4,4)\text{-A} \\ (2n+5,6)\text{-A?} \\ (6n+3,2)\text{-A?} \\ (4n+4,4)\text{-A?} \end{array}$	$n \ge 3, n \not\equiv 2 \pmod{4} [244]$ $n \ge 3, n \not\equiv 2 \pmod{4} [244]$ $n \ge 4 [244]$ $n \ge 6, n \not\equiv 2 \pmod{4} [244]$ $n \ge 6, n \not\equiv 2 \pmod{4} [244]$
Hershel graph (see [567])	not $(a, d)$ -A	[481], [483]
parachutes $P_{g,p}$ (see §6.2)	(a,d)-A	for certain classes [481], [483]
prisms $C_n \times P_2$ generalized Petersen graph $P(n, 2)$	$ \begin{array}{c} ((7n+4)/2,1)\text{-A} \\ ((5n+5)/2,2)\text{-A} \\ ((3n+6)/2,3)\text{-A} \\ ((n+7)/2,4)\text{-A}? \\ ((3n+6)/2,3)\text{-A} \end{array} $	$n \ge 3, n \text{ even } [482], [272]$ $n \ge 3, n \text{ odd } [482], [272]$ $n \ge 3, n \text{ even } [272]$ $n \ge 7, [483], [272]$ $n \ge 8, n \equiv 0 \pmod{4} [273]$

Table 13: Summary of (a, d)-Antimagic Labelings

### **6.3** (a, d)-Antimagic Total Labelings

Bača, Bertault, MacDougall, Miller, Simanjuntak, and Slamin [262] introduced the notion of a (a, d)-vertex-antimagic total labeling in 2000. For a graph G(V, E), an injective mapping f from  $V \cup E$  to the set  $\{1, 2, \ldots, |V| + |E|\}$  is a (a, d)-vertex-antimagic total labeling if the set  $\{f(v) + \sum f(vu)\}$  where the sum is over all vertices u adjacent to v for all v in G is  $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$ . In the case where the vertex labels are 1,2,  $\ldots, |V|, (a, d)$ -vertex-antimagic total labeling is called a super (a, d)-vertex-antimagic total labeling. Among their results are: every super-magic graph has an (a, 1)-vertex-antimagic total labeling; every (a, d)-antimagic graph G(V, E) is (a+|E|+1, d+1)-vertex-antimagic total; and, for d > 1, every (a, d)-antimagic graph G(V, E) is (a+|V|+|E|, d-1)-vertexantimagic total. They also show that paths and cycles have (a, d)-vertex-antimagic total labelings for a wide variety of a and d. In [263] Bača et al. use their results in [262] to obtain numerous (a, d)-vertex-antimagic total labelings for prisms, and generalized Petersen graphs (see §2.7 for the definition). (See also [276] and [2496] for more results on generalized Petersen graphs.)

Sugeng, Miller, Lin, and Bača [2496] prove:  $C_n$  has a super (a, d)-vertex-antimagic total labeling if and only if d = 0 or 2 and n is odd, or d = 1;  $P_n$  has a super (a, d)-vertexantimagic total labeling if and only if d = 2 and  $n \ge 3$  is odd, or d = 3 and  $n \ge 3$ ; no even order tree has a super (a, 1)-vertex antimagic total labeling; no cycle with at least one tail and an even number of vertices has a super (a, 1)-vertex-antimagic labeling; and the star  $S_n$ ,  $n \ge 3$ , has no super (a, d)-super antimagic labeling. As open problems they ask whether  $K_{n,n}$  has a super (a, d)-vertex-antimagic total labeling and the generalized Petersen graph has a super (a, d)-vertex-antimagic total labeling for specific values a, d, and n. In [2050] Raheem proved that various subclasses of stars admit super (a, d)-edge antimagic total labelings for d = 1, 2, and 3. Lin, Miller, Simanjuntak, and Slamin [1599] have shown that for n > 20,  $W_n$  has no (a, d)-vertex-antimagic total labeling. Tezer and Cahit [2584] proved that neither  $P_n$  nor  $C_n$  has (a, d)-vertex-antimagic total labelings for  $a \geq 3$  and  $d \geq 6$ . Kovář [1422] has shown that every 2*r*-regular graph with *n* vertices has an (s, 1)-vertex antimagic total labeling for  $s \in \{(rn+1)(r+1) + tn \mid t = 0, 1, \dots, r\}$ . Dafik, Slamin, Romdhani, and Arianti [641] studied the super (a, d)-antimagicness of generalized flower and disk brake graphs.

Several papers have been written about vertex-antimagic total labeling of graphs that are the disjoint union of suns. The sun graph  $S_n$  is  $C_n \odot K_1$ . Rahim and Sugeng [2055] proved that  $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$  is (a, 0)-vertex-antimagic total (or vertex magic total). Parestu, Silaban, and Sugeng [1892] and [1893] proved  $S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_t}$  is (a, d)vertex-antimagic total for d = 1, 2, 3, 4, and 6 and particular values of a. In [2053] Rahim, Ali, Kashif, and Javaid provide (a, d)-vertex antimagic total labelings of disjoint unions of cycles, sun graphs, and disjoint unions of sun graphs. In [726] Enomoto et al. proposed the conjecture that every tree is a super (a, 0)-edge-antimagic total graph. Javaid [1097] gave (a, d)-edge-antimagic total labelings for certain subclasses of subdivided stars. Javaid [1098] gave a super (a, d)-edge-antimagic total labeling for the subdivided star  $T(n, n, n+4, n+4, n_5, n_6, \ldots, n_r)$  for d = 0, 1, 2, where  $n_p = 2^{p-4}(n+3)+1$ ,  $5 \le p \le r$  and  $n \geq 3$  is odd.

In [1855] Ngurah, Baskova, and Simanjuntak provide (a, d)-vertex-antimagic total labelings for the generalized Petersen graphs P(n,m) for the cases:  $n \ge 3$ ,  $1 \le m \le \lfloor (n-1)/2 \rfloor$ , (a,d) = (8n+3,2); odd  $n \ge 5$ , m = 2, (a,d) = ((15n+5)/2,1); odd  $n \ge 5$ , m = 2, (a,d) = ((15n+5)/2,1); odd  $n \ge 5$ , m = 2, (a,d) = (((15n+5)/2,1); odd  $n \ge 7$ , m = 3, (a,d) = (((21n+5)/2,1); odd  $n \ge 9$ , m = 4, (a,d) = (((15n+5)/2,1); and (a,d) = (((21n+5)/2,1). They conjecture that for n odd and  $1 \le m \le \lfloor (m-1)/2 \rfloor$ , P(n,m) has an ((21n+5)/2,1)-vertex-antimagic labeling. In [2501] Sugeng and Silaban show: the disjoint union of any number of odd cycles of orders  $n_1, n_2, \ldots, n_t$ , each at least 5, has a super  $(3(n_1 + n_2 + \cdots + n_t) + 2, 1)$ -vertex-antimagic total labeling; for any odd positive integer t, the disjoint union of t copies of the generalized Petersen graph P(n, 1) has a super  $(10t + 2)n - \lfloor n/2 \rfloor + 2, 1)$ -vertex-antimagic total labeling; and for any odd positive integers t and n  $(n \ge 3)$ , the disjoint union of t copies of the generalized Petersen graph P(n, 2) has a super (21n + 5)/2, 1)-vertex-antimagic total labeling; and for any odd positive integers t and n  $(n \ge 3)$ , the disjoint union of t copies of the generalized Petersen graph P(n, 2) has a super (21n + 5)/2, 1)-vertex-antimagic total labeling.

Ail, Bača, Lin, and Semaničová-Feňovčiková [125] investigated super-(a, d)-vertex antimagic total labelings of disjoint unions of regular graphs. Among their results are: if m and (m-1)(r+1)/2 are positive integers and G is an r-regular graph that admits a super-vertex magic total labeling, then mG has a super-(a, 2)-vertex antimagic total labeling; if G has a 2-regular super-(a, 1)-vertex antimagic total labeling, then mG has a super-(m(a-2)+2, 1), 1)-vertex antimagic total labeling;  $mC_n$  has a super-(a, d)-vertex antimagic total labeling if and only if either d is 0 or 2 and m and n are odd and at least 3 or d = 1 and  $n \geq 3$ ; and if G is an even regular Hamilton graph, then mG has a super-(a, 1)-vertex antimagic total labeling for all positive integers m.

In [315] Bača, A. Semaničová-Feňovčíková, Wang, and Zhang investigate the existence of (a, 1)-vertex-antimagic edge labelings for disconnected 3-regular graphs. As an extension of (a, d)-vertex-antimagic edge labeling they also introduce the concept of (a, d)vertex-antimagic edge deficiency for measuring how close a graph is away from being an (a, d)-antimagic graph. In [200] Arumugam and Nalliah investigate the existence of a super (a, d)-edge-antimagic total labelings of disconnected graphs.

Ahmad, Ali, Bača, Kovář and Semaničová-Feňovčíková [70] provided a technique that allows one to construct several (a, r)-vertex-antimagic edge labelings for any 2r-regular graph G of odd order provided the graph is Hamiltonian or has a 2-regular factor that has (b, 1)-vertex-antimagic edge labeling. A similar technique allows them to construct a super (a, d)-vertex-antimagic total labeling for any 2r-regular Hamiltonian graph of odd order with differences d = 1, 2, ..., r and d = 2r + 2.

For  $n \ge 2$  Dafik, Setiawani, and Azizah [680] define a *shackle* as a graph constructed from connected graphs  $G_1, G_2, \ldots, G_n$ , all isomorphic to G, such that  $G_s$  and  $G_t$  are disjoint when  $|s - t| \ge 2$  and for every  $i = 1, 2, \ldots, n - 1$ ,  $G_i$  and  $G_{i+1}$  share exactly one common vertex v. In a *generalized shackle* a common subgraph is shared by each  $G_i$  and  $G_{i+1}$ . Dafik, Setiawani, and Azizah prove that the generalized shackle of a fan of order four and five admits a super (a, d)-edge antimagic total labeling for d = 0, 1, 2.

Suggeng and Bong [2490] show how to construct super (a, d)-vertex antimagic total labelings for the circulant graphs  $C_n(1, 2, 3)$ , for d = 0, 1, 2, 3, 4, 8. Thirusangu, Nagar,

and Rajeswari [2590] show that certain Cayley digraphs of dihedral groups have (a, d)-vertex-magic total labelings.

For a simple graph H we say that G(V, E) admits an H-covering if every edge in E(G) belongs to a subgraph of G that is isomorphic to H. Inayah, Salman, and Simanjuntak [1068] define an (a, d)-H-antimagic total labeling of G as a bijective function  $\xi$ from  $V \cup E \to \{1, 2, \dots, |V| + |E|\}$  such that for all subgraphs H' isomorphic to H, the *H*-weights  $w(H') = \sum_{v \in V(H')} \xi(v) + \sum_{e \in E(H')} \xi(e)$  constitute an arithmetic progression  $a, a+d, a+2d, \ldots, a+(t-1)d$  where a and d are positive integers and t is the number of subgraphs of G isomorphic to H. Such a labeling  $\xi$  is called a super (a, d)-H-antimagic total labeling, if  $\xi(V) = \{1, 2, \dots, |V|\}$ . Inayah et al. study some basic properties of such labeling and give (a, d)-cycle-antimagic labelings of fans. Taimur, Numan, Mumtaz, and Semaničová-Feňovčíková [2569] proved that if a graph G is super cycle-antimagic then the subdivided graph of G also admits a super cycle-antimagic labeling and they showed that the subdivided wheel is super (a, d)-cycle-antimagic for wide range of values. Laurence and Kathiresan [1471] investigated super (a, d)- $P_n$ -antimagic total labeling of stars. Bača, Jeyanthi, Selvagopal, Muthu Raja, and Semaničová-Feňovčíková [214] proved the existence of super (a, d)-H-antimagic labelings of fan graphs and ladders for H isomorphic to a cycle.

In [2212] Semaničová-Feňovčíková, Bača, Lascsáková, Miller, and Ryan investigated the super (a, d)- $C_n$ -antimagic total labelings of wheels and super (a, d)- $P_n$ -antimagic total labelings of cycles and paths. Ovais, Umar, Bača, and Semaničová-Feňovčíková [1881] proved that fans admits a super (a, d)- $C_k$ -antimagic labeling for d = 1, 3, 2k - 5, 2k -1, 3k - 1, k - 7, k + 1, 3k - 9. They also prove that fans admits a super (a, d)- $C_3$ antimagic labeling for d = 0, 1, 2, 3, 4, 5, 6, 8, and a super (a, d)- $C_4$ -antimagic labeling for d = 0, 1, 2, 3, 4, 5, 6, 7, 11. They propose an open problem to find a super (a, d)- $C_k$ antimagic labeling of fans for  $d \neq 1, 3, k - 7, k + 1, 2k - 5, 2k - 1, 3k - 1, 3k - 9$ . Bača, Miller, Ryan, and Semaničová-Feňovčíková [303] study super (a, d)-H-antimagic labelings of a disjoint union of graphs for d = |E(H)| - |V(H)|.

For a vertex u of a graph G,  $G_u[S_n]$  is the graph obtained by identifying u with the center of  $S_n$ . Then for any vertex w of  $S_n$  G + e, e = uw is a subgraph of  $G_u[S_n]$ . Kathiresan and Laurence [1353] prove that the graph  $G_u[S_n]$  admits a super-(a, d)-(G+e)antimagic total labeling if and only if  $d \in \{0, 1, 2, \ldots, |V(G)| + |E(G)| + 2\}$ . Moreover, they show that a caterpillar  $S_{n_1,n_2,\ldots,n_k}$  has a super- $(a, 4n^2)$ - $S_{n,n}$ -antimagic total labeling for  $n_1 = n_2 = \cdots = n_k = n$ .

Jeyanthi, Muthuraja, Semaničová-Feňovčíková, and Dharshikha proved [1192] proved that fans, triangular ladders, and middle graphs of cycles are super (a, d)- $C_3$ -antimagic for some values of a and d. They also proved that ladder are super (a, d)- $C_4$ -antimagic for  $1 \leq d \leq 8$ . Inayah, Simanjuntak, and Salman [1069] proved that there exists a super (a, d) - H-antimagic total labelings for shackles of a connected graph H. Nadzima and Martini [1818] determined (a, d)-H-antimagic total labeling for certains cases of  $W_n \odot P_n$ with H as  $C_3 \odot P_n$  and  $W_n \odot C_n$  with H as  $C_3 \odot C_n$ .

A graph G is said to have an  $(H_1, H_2, \ldots, H_k)$ -covering if every edge in G belongs to at least one of the  $H_i$ 's. Susilowati, Sania, and Estuningsih [2551] investigated such antimagic labelings for the ladders  $P_n \times P_2$  with  $C_t$ -coverings for t = 4, 6, and 8 for some value of d.

Simanjuntak, Bertault, and Miller [2386] define an (a, d)-edge-antimagic vertex labeling for a graph G(V, E) as an injective mapping f from V onto the set  $\{1, 2, \ldots, |V|\}$  such that the set  $\{f(u) + f(v) | uv \in E\}$  is  $\{a, a+d, a+2d, \ldots, a+(|E|-1)d\}$ . (The equivalent notion of (a, d)-indexable labeling was defined by Hegde in 1989 in his Ph. D. thesis-see [978].) Similarly, Simanjuntak et al. define an (a, d)-edge-antimagic total labeling for a graph G(V, E) as an injective mapping f from  $V \cup E$  onto the set  $\{1, 2, \ldots, |V| + |E|\}$ such that the set  $\{f(v) + f(vu) + f(v) | uv \in E\}$  where v ranges over all of V is  $\{a, a + v\}$  $d, a + 2d, \ldots, a + (|V| - 1)d$ . Among their results are:  $C_{2n}$  has no (a, d)-edge-antimagic vertex labeling;  $C_{2n+1}$  has a (n+2,1)-edge-antimagic vertex labeling and a (n+3,1)edge-antimagic vertex labeling;  $P_{2n}$  has a (n+2, 1)-edge-antimagic vertex labeling;  $P_n$  has a (3,2)-edge-antimagic vertex labeling;  $C_n$  has (2n+2,1)- and (3n+2,1)-edge-antimagic total labelings;  $C_{2n}$  has (4n+2,2)- and (4n+3,2)-edge-antimagic total labelings;  $C_{2n+1}$ has (3n + 4, 3)- and (3n + 5, 3)-edge-antimagic total labelings;  $P_{2n+1}$  has (3n + 4, 2)-, (3n + 4, 3)-, (2n + 4, 4)-, (5n + 4, 2)-, (3n + 5, 2)-, and (2n + 6, 4)-edge-antimagic total labelings;  $P_{2n}$  has (6n, 1)- and (6n+2, 2)-edge-antimagic total labelings; and several parity conditions for (a, d)-edge-antimagic total labelings. They conjecture:  $C_{2n}$  has a (2n+3, 4)or a (2n+4,4)-edge-antimagic total labeling;  $C_{2n+1}$  has a (n+4,5)- or a (n+5,5)-edgeantimagic total labeling; paths have no (a, d)-edge-antimagic vertex labelings with d > 2; and cycles have no (a, d)-antimagic total labelings with d > 5. The first and last of these conjectures were proved by Zhenbin in [2914] and the last two were verified by Bača, Lin, Miller, and Simanjuntak [286] who proved that a graph with v vertices and e edges that has an (a, d)-edge-antimagic vertex labeling must satisfy  $d(e-1) \leq 2v-1-a \leq 2v-4$ . As a consequence, they obtain: for every path there is no (a, d)-edge-antimagic vertex labeling with d > 2; for every cycle there is no (a, d)-edge-antimagic vertex labeling with d > 1; for  $K_n$  (n > 1) there is no (a, d)-edge-antimagic vertex labeling (the cases for n = 2 and n = 3are handled individually);  $K_{n,n}$  (n > 3) has no (a, d)-edge-antimagic vertex labeling; for every wheel there is no (a, d)-edge-antimagic vertex labeling; for every generalized Petersen graph there is no (a, d)-edge-antimagic vertex labeling with d > 1. They also study the relationship between graphs with (a, d)-edge-antimagic labelings and magic and antimagic labelings. They conjecture that every tree has an (a, 1)-edge-antimagic total labeling.

Bača and Barrientos [248] prove that if a tree T has an  $\alpha$ -labeling and  $\{A, B\}$  is the bipartition of the vertices of T, then T also admits an (a, 1)-edge-antimagic vertex labeling and it admits a (3, 2)-edge-antimagic vertex labeling if and only if  $||A| - |B|| \leq 1$ .

In [286] Bača, Lin, Miller, and Simanjuntak prove: if  $P_n$  has an (a, d)-edge-antimagic total labeling, then  $d \leq 6$ ;  $P_n$  has (2n + 2, 1)-, (3n, 1)-, (n + 4, 3)-, and (2n + 2, 3)-edge-antimagic total labelings;  $P_{2n+1}$  has (3n + 4, 2)-, (5n + 4, 3)-, (2n + 4, 4)-, and (2n + 6, 4)-edge-antimagic total labelings; and  $P_{2n}$  has (3n+3, 2)- and (5n+1, 2)-edge-antimagic total labelings. Ngurah [1851] proved  $P_{2n+1}$  has (4n + 4, 1)-, (6n + 5, 3)-, (4n + 4, 2)-, (4n + 5, 2)-edge-antimagic total labelings and  $C_{2n+1}$  has (4n + 4, 2)- and (4n + 5, 2)-edge-antimagic total labelings. Silaban and Sugeng [2385] prove:  $P_n$  has (n + 4, 4)- and (6, 6)-edge-antimagic total labelings; if  $C_m \odot \overline{K_n}$  has an (a, d)-edge-antimagic total labeling, then

 $d \leq 5$ ;  $C_m \odot \overline{K_n}$  has (a, d)-edge-antimagic total labelings for m > 3, n > 1 and d = 2or 4; and  $C_m \odot \overline{K_n}$  has no (a, d)-edge-antimagic total labelings for m and d and  $n \equiv 1$ mod 4. They conjecture that  $P_n$   $(n \geq 3)$  has (a, 5)-edge-antimagic total labelings. In [2502] Sugeng and Xie use adjacency methods to construct super edge magic graphs from (a, d)-edge-antimagic vertex graphs. Pushpam and Saibulla [2042] determined super (a, d)-edge antimagic total labelings for graphs derived from copies of generalized ladders, fans, generalized prisms and web graphs. Ahmad, Ali, Bača, Kovar, and Semaničová-Feňovčíková, investigated the vertex-antimagicness of regular graphs and the existence of (super) (a, d)-vertex antimagic total labelings for regular graphs in general.

In [319] Bača and Youssef used parity arguments to find a large number of conditions on p, q and d for which a graph with p vertices and q edges cannot have an (a, d)-edgeantimagic total labeling or vertex-antimagic total labeling. Bača and Youssef [319] made the following connection between (a, d)-edge-antimagic vertex labelings and sequential labelings: if G is a connected graph other than a tree that has an (a, d)-edge-antimagic vertex labeling, then  $G + K_1$  has a sequential labeling.

In [2481] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every  $n \geq 2$ ,  $P_n \cup$  $P_{n+1}$  has a (6n + 1, 1)- and a (4n + 3, 3)-edge-antimagic total labeling, for every odd  $n \geq 3, P_n \cup P_{n+1}$  has a (6n, 1)- and a (5n + 1, 2)-edge-antimagic total labeling, for every  $n \geq 2$ ,  $nP_2 \cup P_n$  has a (7n, 1)- and a (6n + 1, 2)-edge-antimagic total labeling. In [2478] the same authors show that  $P_n \cup P_{n+1}$ ,  $nP_2 \cup P_n$   $(n \ge 2)$ , and  $nP_2 \cup P_{n+2}$  are super edgemagic total. They also show that under certain conditions one can construct new super edge-magic total graphs from existing ones by joining a particular vertex of the existing super edge-magic total graph to every vertex in a path or every vertex of a star and by joining one extra vertex to some vertices of the existing graph. Baskoro, Sudarsana, and Cholily 418 also provide algorithms for constructing new super edge-magic total graphs from existing ones by adding pendent vertices to the existing graph. A corollary to one of their results is that the graph obtained by attaching a fixed number of pendent edges to each vertex of a path of even length is super edge-magic. Baskoro and Cholily [416] show that the graphs obtained by attaching any numbers of pendent edges to a single vertex or a fix number of pendent edges to every vertex of the following graphs are super edge-magic total graphs: odd cycles, the generalized Petersen graphs P(n,2) (n odd and at least 5), and  $C_n \times P_m$  (n odd,  $m \ge 2$ ).

Arumugam and Nalliah [199] proved: the friendship graph  $C_3^{(n)}$  with  $n \equiv 0, 8 \pmod{12}$ has no super (a, 2)-edge-antimagic total labeling;  $C_n^{(n)}$  with  $n \equiv 2 \pmod{4}$  has no super (a, 2)-edge-antimagic total labeling; and the generalized friendship graph  $F_{2,p}$  consisting of 2 cycles of various lengths, having a common vertex, and having order p where  $p \geq 5$ , has a super (2p + 2, 1)-edge-antimagic total labeling if and only if p is odd.

An (a, d)-edge-antimagic total labeling of G(V, E) is called a super (a, d)-edgeantimagic total if the vertex labels are  $\{1, 2, \ldots, |V(G)|\}$  and the edge labels are  $\{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$ . Bača, Baskoro, Simanjuntak, and Sugeng [261] prove the following:  $C_n$  has a super (a, d)-edge-antimagic total labeling if and only if either d is 0 or 2 and n is odd, or d = 1; for odd  $n \ge 3$  and m = 1 or 2, the generalized Petersen graph P(n, m) has a super (11n + 3)/2, 0)-edge-antimagic total labeling and a

super ((5n+5)/2, 2)-edge-antimagic total labeling; for odd  $n \ge 3$ , P(n, (n-1)/2) has a super ((11n + 3)/2, 0)-edge-antimagic total labeling and a super ((5n + 5)/2, 2)-edgeantimagic total labeling. They also prove: if  $P(n,m), n \geq 3, 1 \leq m \leq \lfloor (n-1)/2 \rfloor$ is super (a,d)-edge-antimagic total, then (a,d) = (4n+2,1) if n is even, and either (a,d) = ((11n+3)/2,0), or (a,d) = (4n+2,1), or (a,d) = ((5n+5)/2,2), if n is odd; and for odd  $n \ge 3$  and m = 1, 2, or (n-1)/2, P(n,m) has an (n,0)-edge-antimagic total labeling and an (a, 2)-edge-antimagic total labeling. (In a personal communication MacDougall argues that "edge-magic" is a better term than "(a, 0)-edge-antimagic" for while the latter is technically correct, "antimagic" suggests different weights whereas "magic" emphasizes equal weights and that the edge-magic case is much more important, interesting, and fundamental rather than being just one subcase of equal value to all the others.) They conjecture that for odd  $n \ge 9$  and  $3 \le m \le (n-3)/2$ , P(n,m) has a (a,0)edge-antimagic total labeling and an (a, 2)-edge-antimagic total labeling. Ngurah and Baskoro [1854] have shown that for odd  $n \ge 3$ , P(n, 1) and P(n, 2) have ((5n + 5)/2, 2)edge-antimagic total labelings and when  $n \geq 3$  and  $1 \leq m < n/2$ , P(n,m) has a super (4n + 2, 1)-edge-antimagic total labeling. In [1855] Ngurah, Baskova, and Simanjuntak provide (a, d)-edge-antimagic total labelings for the generalized Petersen graphs P(n, m)for the cases m = 1 or 2, odd n > 3, and (a, d) = ((9n + 5)/2, 2).

In [2479] Sudarsana, Baskoro, Uttunggadewa, and Ismaimuza show how to construct new larger super (a, d)-edge-antimagic-total graphs from existing smaller ones.

In [1856] Ngurah, Baskoro, and Simanjuntak prove that  $mC_n$   $(n \ge 3)$  has an (a, d)edge-antimagic total in the following cases: (a, d) = (5mn/2 + 2, 1) where m is even; (a, d) = (2mn+2, 2); (a, d) = ((3mn+5)/2, 3) for m and n odd; and (a, d) = ((mn+3), 4)for m and n odd; and  $mC_n$  has a super (2mn+2, 1)-edge-antimagic total labeling.

Bača and Barrientos [249] have shown that  $mK_n$  has a super (a, d)-edge-antimagic total labeling if and only if  $(i) \ d \in \{0, 2\}, \ n \in \{2, 3\}$  and  $m \ge 3$  is odd, or  $(ii) \ d = 1, \ n \ge 2$ and  $m \ge 2$ , or  $(iii) \ d \in \{3, 5\}, \ n = 2$  and  $m \ge 2$ , or  $(iv) \ d = 4, \ n = 2, \ and \ m \ge 3$  is odd. In [248] Bača and Barrientos proved the following: if a graph with q edges and q + 1vertices has an  $\alpha$ -labeling, than it has an (a, 1)-edge-antimagic vertex labeling; a tree has a (3, 2)-edge-antimagic vertex labeling if and only if it has an  $\alpha$ -labeling and the number of vertices in its two partite sets differ by at most 1; if a tree with at least two vertices has a super (a, d)-edge-antimagic total labeling, then d is at most 3; if a graph has an (a, 1)-edge-antimagic vertex labeling, then it also has a super  $(a_1, 0)$ -edge-antimagic total labeling and a super  $(a_2, 2)$ -edge-antimagic total labeling.

Bača and Youssef [319] proved the following: if G is a connected (a, d)-edge-antimagic vertex graph that is not a tree, then  $G+K_1$  is sequential;  $mC_n$  has an (a, d)-edge-antimagic vertex labeling if and only if m and n are odd and d = 1; an odd degree (p, q)-graph G cannot have a (a, d)-edge-antimagic total labeling if  $p \equiv 2 \pmod{4}$  and  $q \equiv 0 \pmod{4}$ , or  $p \equiv 0 \pmod{4}$ ,  $q \equiv 2 \pmod{4}$ , and d is even; a (p, q)-graph G cannot have a super (a, d)-edge-antimagic total labeling if G has odd degree,  $p \equiv 2 \pmod{4}$ , q is even, and d is odd, or G has even degree,  $q \equiv 2 \pmod{4}$ , and d is even;  $C_n$  has a (2n + 2, 3)- and an (n+4, 3)-edge-antimagic total labeling; a (p, q)-graph is not super (a, d)-vertex-antimagic total if:  $p \equiv 2 \pmod{4}$  and d is even;  $p \equiv 0 \pmod{4}$ ,  $q \equiv 2 \pmod{4}$ , and d is odd;  $p \equiv 0$
(mod 8) and  $q \equiv 2 \pmod{4}$ .

In [2481] Sudarsana, Ismaimuza, Baskoro, and Assiyatun prove: for every  $n \ge 2$ ,  $P_n \cup P_{n+1}$  has super (n + 4, 1)- and (2n + 6, 3)-edge antimagic total labelings; for every odd  $n \ge 3$ ,  $P_n \cup P_{n+1}$  has super (4n + 5, 1)-,(3n + 6, 2)-, (4n + 3, 1)- and (3n + 4, 2)-edge antimagic total labelings; for every  $n \ge 2$ ,  $nP_2 \cup P_n$  has super (6n + 2, 1)- and (5n + 3, 2)-edge antimagic total labelings; and for every  $n \ge 1$ ,  $nP_2 \cup P_{n+2}$  has super (6n + 6, 1)- and (5n + 6, 2)-edge antimagic total labelings. They pose a number of open problems about constructing (a, d)-edge antimagic labelings and super (a, d)-edge antimagic labelings for the graphs  $P_n \cup P_{n+1}$ ,  $nP_2 \cup P_n$ , and  $nP_2 \cup P_{n+2}$  for specific values of d.

Dafik, Miller, Ryan, and Bača [637] investigated the super edge-antimagicness of the disconnected graph  $mC_n$  and  $mP_n$ . For the first case they prove that  $mC_n$ ,  $m \ge 2$ , has a super (a, d)-edge-antimagic total labeling if and only if either d is 0 or 2 and m and n are odd and at least 3, or d = 1,  $m \ge 2$ , and  $n \ge 3$ . For the case of the disjoint union of paths they determine all feasible values for m, n and d for  $mP_n$  to have a super (a, d)-edge-antimagic total labeling except when m is even and at least 2,  $n \ge 2$ , and d is 0 or 2. In [639] Dafik, Miller, Ryan, and Bača obtain a number of results about super edge-antimagicness of the disjoint union of two stars and state three open problems. Nalliah and Arumugam [1830] proved that  $K_{1,6} \cup K_{1,5}$  does not have such a labeling and prove that some special cases of  $K_{1,n+1} \cup K_{1,n}$  do have them.

Sudarsana, Hendra, Adiwijaya, and Setyawan [2480] show that the t-joint copies of wheel  $W_n$  have a super edge antimagic ((2n+2)t+2, 1)-total labeling for  $n \ge 4$  and  $t \ge 2$ .

In [281] Bača, Lascsáková, and Semaničová investigated the connection between graphs with  $\alpha$ -labelings and graphs with super (a, d)-edge-antimagic total labelings. Among their results are: If G is a graph with n vertices and n-1 edges  $(n \ge 3)$  and G has an  $\alpha$ labeling, then mG is super (a, d)-edge-antimagic total if either d is 0 or 2 and m is odd, or d = 1 and n is even; if G has an  $\alpha$ -labeling and has n vertices and n-1 edges with vertex bipartition sets  $V_1$  and  $V_2$  where  $|V_1|$  and  $|V_2|$  differ by at most 1, then mG is super (a, d)-edge-antimagic total for d = 1 and d = 3. In the same paper Bača et al. prove: caterpillars with odd order at least 3 have super (a, 1)-edge-antimagic total labeling; if G is a caterpillar of odd order at least 3 and G has a super (a, 1)-edge-antimagic total labeling, then mG has a super (b, 1)-edge-antimagic total labeling for some b that is a function of a and m.

In [636] Dafik, Miller, Ryan, and Bača investigated the existence of antimagic labelings of disjoint unions of s-partite graphs. They proved: if  $s \equiv 0$  or 1 (mod 4),  $s \geq 4, m \geq 2, n \geq 1$  or mn is even,  $m \geq 2, n \geq 1, s \geq 4$ , then the complete s-partite graph  $mK_{n,n,\dots,n}$ has no super (a, 0)-edge-antimagic total labeling; if  $m \geq 2$  and  $n \geq 1$ , then  $mK_{n,n,n,n}$  has no super (a, 2)-antimagic total labeling; and for  $m \geq 2$  and  $n \geq 1$ ,  $mK_{n,n,n,n}$  has an (8mn + 2, 1)-edge-antimagic total labeling. They conjecture that for  $m \geq 2, n \geq 1$  and  $s \geq 5$ , the complete s-partite graph  $mK_{n,n,\dots,n}$  has a super (a, 1)-antimagic total labeling.

In [307] Bača, Muntaner-Batle, Semaničová-Feňovčiková, and Shafiq investigate super (a, d)-edge-antimagic total labelings of disconnected graphs. Among their results are: If G is a (super) (a, 2)-edge-antimagic total labeling and m is odd, then mG has a (super) (a', 2)-edge-antimagic-total labeling where a' = m(a - 3) + (m + 1)/2 + 2; and if d a

positive even integer and k a positive odd integer, G is a graph with all of its vertices having odd degree, and the order and size of G have opposite parity, then 2kG has no (a, d)-edge-antimagic total labeling. Bača and Brankovic [264] have obtained a number of results about the existence of super (a, d)-edge-antimagic totaling of disjoint unions of the form  $mK_{n,n}$ . In [268] Bača, Dafik, Miller, and Ryan provide (a, d)-edge-antimagic vertex labelings and super (a, d)-edge-antimagic total labelings for a variety of disjoint unions of caterpillars. Bača and Youssef [319] proved that  $mC_n$  has an (a, d)-edge-antimagic vertex labeling if and only if m and n are odd and d = 1. Bača, Dafik, Miller, and Ryan [269] constructed super (a, d)-edge-antimagic total labeling for graphs of the form  $m(C_n \odot K_s)$  and  $mP_n \cup kC_n$  while Dafik, Miller, Ryan, and Bača [638] do the same for graphs of the form  $mK_{n,n,n}$  and  $K_{1,m} \cup 2sK_{1,n}$ . Both papers provide a number of open problems. In [291] Bača, Lin, and Muntaner-Batle provide super (a, d)-edge-antimagic total labeling of forests in which every component is a specific kind of tree. In [279] Bača. Koýar, Semaničová-Feňovčiková, and Shafiq prove that every even regular graph and every odd regular graph with a 1-factor are super (a, 1)-edge-antimagic total and provide some constructions of non-regular super (a, 1)-edge-antimagic total graphs. Bača, Lin, and Semaničová-Feňovčiková [293] show: the disjoint union of m graphs with super (a, 1)-edge antimagic total labelings have super (m(a-2)+2, 1)-edge antimagic total labelings; the disjoint union of m graphs with super (a, 3)-edge antimagic total labelings have super (m(a-3)+3,3)-edge antimagic total labelings; if G has a (a,1)-edge antimagic total labelings then mG has an (b, 1)-edge antimagic total labeling for some b; and if G has a (a, 3)-edge antimagic total labelings then mG has an (b, 3)-edge antimagic total labeling for some b.

Bača, Miller, Ryan, and Semaničová-Feňovčíková [303] prove that if G admits a (super) (a, d)-H-antimagic labeling, where d = |E(H)| - |V(H)|, then mG admits a (super) (b, d)-H-antimagic labelling. By considering special H-coverings of a given H-antimagic graph G they derive many corollaries. In [2211] Semaničová-Feňovčíková, Bača, and Lascsáková provide two constructions of (super) H-antimagic graphs obtained from smaller (super) H'-antimagic graphs. Dafik, Slamin, Tana, Semaničová-Feňovčíková, and Bača [640] show a connection between a constructions of H-antimagic labelings of graph and edge-antimagic total labelings and describe how to obtain the H-antimagic graph using smaller edge-antimagic graph. Bača, Semaničová-Feňovčíková, Umar, and Welyyanti [314] gave sufficient conditions for  $G_1 \times G_2$  to admit an H-supermagic or a super (a, d)-H-antimagic labeling but provide no examples of graphs that satisfy the given conditions.

For  $t \ge 2$  and  $n \ge 4$  the Harary graph,  $C_p^t$ , is the graph obtained by joining every two vertices of  $C_p$  that are at distance t in  $C_p$ . In [2053] Rahim, Ali, Kashif, and Javaid provide super (a, d)-edge antimagic total labelings for disjoint unions of Harary graphs and disjoint unions of cycles. In [1035] Hussain, Ali, Rahim, and Baskoro construct various (a, d)-vertex-antimagic labelings for Harary graphs and disjoint unions of identical Harary graphs. For p odd and at least 5, Balbuena, Barker, Das, Lin, Miller, Ryan, Slamin, Sugeng, and Tkac [329] give a super ((17p + 5)/2)-vertex-antimagic total labeling of  $C_p^t$ . MacDougall and Wallis [1682] have proved the following:  $C_{4m+3}^t$ ,  $m \ge 1$ , has a super (a, 0)edge-antimagic total labeling for all possible values of t with a = 10m + 9 or 10m + 10;  $C_{4m+1}^t$ ,  $m \ge 3$ , has a super (a, 0)-edge-antimagic total labeling for all possible values except t = 5, 9, 4m - 4, and 4m - 8 with a = 10m + 4 and 10m + 5;  $C_{4m+1}^t$ ,  $m \ge 1$ , has a super (10m + 4, 0)-edge-antimagic total labeling for all  $t \equiv 1 \pmod{4}$  except 4m - 3;  $C_{4m}^t$ , m > 1, has a super (10m + 2, 0)-edge-antimagic total labeling for all  $t \equiv 2 \pmod{4}$ ;  $C_{4m+2}^t$ , m > 1, has a super (10m + 7, 0)-edge-antimagic total labeling for all odd t other than 5 and for t = 2 or 6. In [1036] Hussain, Baskoro, and Ali prove the following: for any  $p \ge 4$  and for any  $t \ge 2$ ,  $C_p^t$  admits a super (2p + 2, 1)-edge-antimagic total labeling; for  $n \ge 4$ ,  $k \ge 2$  and  $t \ge 2$ ,  $C_p^t$  admits a super (2nk + 2, 1)-edge-antimagic total labeling; provided if  $p \ne 2t$ .

Bača and Murugan [312] have proved: if  $C_n^t$ ,  $n \ge 4, 2 \le t \le n-2$ , is super (a, d)-edge-antimagic total, then d = 0, 1, or 2; for  $n = 2k + 1 \ge 5$ ,  $C_n^t$  has a super (a, 0)-edge-antimagic total labeling for all possible values of t with a = 5k + 4 or 5k + 5; for  $n = 2k+1 \ge 5$ ,  $C_n^t$  has a super (a, 2)-edge-antimagic total labeling for all possible values of t with a = 3k+3 or 3k+4; for  $n \equiv 0 \pmod{4}$ ,  $C_n^t$  has a super (5n/2+2, 0)-edge-antimagic total labeling and a super (3n/2+2, 0)-edge-antimagic total labeling for all  $t \equiv 2 \pmod{4}$ ; for n = 10 and  $n \equiv 2 \pmod{4}$ ,  $n \ge 18$ ,  $C_n^t$  has a super (5n/2+2, 0)-edge-antimagic total labeling and a super (3n/2+2, 0)-edge-antimagic total labeling for all  $t \equiv 3 \pmod{4}$  and for t = 2 and 6; for odd  $n \ge 5$ ,  $C_n^t$  has a super (2n + 2, 1)-edge-antimagic total labeling for all odd  $t \ge 3$ ; and for even  $n \equiv 0 \pmod{4}$ . They conjecture that there is a super (2n + 2, 1)-edge-antimagic total labeling of  $C_n^t$  for  $n \equiv 0 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t = 0 \pmod{4}$  and for  $t = 0 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$  and for  $t \equiv 2 \pmod{4}$  and for  $t = 0 \pmod{4}$  and for  $t \equiv 0 \pmod{4}$ 

In [287] Bača, Lin, Miller, and Youssef prove: if the friendship  $C_3^{(n)}$  is super (a, d)antimagic total, then d < 3;  $C_3^{(n)}$  has an (a, 1)-edge antimagic vertex labeling if and only if n = 1, 3, 4, 5, and 7;  $C_3^{(n)}$  has a super (a, d)-edge-antimagic total labelings for d = 0 and 2;  $C_3^{(n)}$  has a super (a, 1)-edge-antimagic total labeling; if a fan  $F_n$   $(n \ge 2)$  has a super (a, d)-edge-antimagic total labeling, then d < 3;  $F_n$  has a super (a, d)-edge-antimagic total labeling if  $2 \le n \le 6$  and d = 0, 1 or 2; the wheel  $W_n$  has a super (a, d)-edge-antimagic total labeling if and only if d = 1 and  $n \not\equiv 1 \pmod{4}$ ;  $K_n$ ,  $n \ge 3$ , has a super (a, d)-edgeantimagic total labeling if and only if either d = 0 and n = 3, or d = 1 and  $n \ge 3$ , or d = 2 and n = 3; and  $K_{n,n}$  has a super (a, d)-edge antimagic total labeling if and only if d = 1 and  $n \ge 2$ .

Bača, Lin, and Muntaner-Batle [288] have shown that if a tree with at least two vertices has a super (a, d)-edge-antimagic total labeling, then d is at most three and  $P_n$ ,  $n \ge 2$ , has a super (a, d)-edge-antimagic total labeling if and only if d = 0, 1, 2, or 3. They also characterize certain path-like graphs in a grid that have  $\operatorname{super}(a, d)$ -edge-antimagic total labelings.

In [2495] Sugeng, Miller, and Bača prove that the ladder,  $P_n \times P_2$ , is super (a, d)-edgeantimagic total if n is odd and d = 0, 1, or 2 and  $P_n \times P_2$  is super (a, 1)-antimagic total if n is even. They conjecture that  $P_n \times P_2$  is super (a, 0)- and (a, 2)-edge-antimagic when n is even. Sugeng, Miller, and Bača [2495] prove that  $C_m \times P_2$  has a super (a, d)-edgeantimagic total labeling if and only if either d = 0, 1 or 2 and m is odd and at least 3, or d = 1 and m is even and at least 4. They conjecture that if m is even,  $m \ge 4$ ,  $n \ge 3$ , and d = 0 or 2, then  $C_m \times P_n$  has a super (a, d)-edge-antimagic total labeling. In [1482] M.-J. Lee studied super (a, 1)-edge-antimagic properties of  $m(P_4 \times P_n)$  for  $m, n \ge 1$  and  $m(C_n \odot \overline{K_t})$  for n even and  $m, t \ge 1$ . He also proved that for  $n \ge 2$  the graph  $P_4 \times P_n$  has a super (8n + 2, 1)-edge antimagic total labeling.

Sugeng, Miller, and Bača [2495] define a variation of a ladder,  $\mathbb{L}_n$ , as the graph obtained from  $P_n \times P_2$  by joining each vertex  $u_i$  of one path to the vertex  $v_{i+1}$  of the other path for  $i = 1, 2, \ldots, n-1$ . They prove  $\mathbb{L}_n$ ,  $n \ge 2$ , has a super (a, d)-edge-antimagic total labeling if and only if d = 0, 1, or 2.

In [635] Dafik, Miller, and Ryan investigate the existence of super (a, d)-edge-antimagic total labelings of  $mK_{n,n,n}$  and  $K_{1,m} \cup 2sK_{1,n}$ . Among their results are: for d = 0 or 2,  $mK_{n,n,n}$  has a super (a, d)-edge-antimagic total labeling if and only if n = 1 and m is odd and at least 3;  $K_{1,m} \cup 2sK_{1,n}$  has a super (a, d)-edge-antimagic labeling for (a, d) = (4n + 5)s + 2m + 4, 0), ((2n + 5)s + m + 5, 2), ((3n + 5)s + (3m + 9)/2, 1) and (5s + 7, 4).

In [252] Bača, Bashir, and Semaničová showed that for  $n \ge 4$  and d = 0, 1, 2, 3, 4, 5, and 6 the antiprism  $A_n$  has a super *d*-antimagic labeling of type (1, 1, 1). The generalized antiprism  $A_m^n$  is obtained from  $C_m \times P_n$  by inserting the edges  $\{v_{i,j+1}, v_{i+1,j}\}$  for  $1 \le i \le m$ and  $1 \le j \le n - 1$  where the subscripts are taken modulo m. Suggeng et al. prove that  $A_m^n, m \ge 3, n \ge 2$ , is super (a, d)-edge-antimagic total if and only if d = 1.

A toroidal polyhex (toroidal fullerene) is a cubic bipartite graph embedded on the torus such that each face is a hexagon. Note that the torus is a closed surface that can carry a toroidal polyhex such that all its vertices have degree 3 and all faces of the embedding are hexagons. Bača and Shabbir [316] proved the toroidal polyhex  $\mathbb{H}_m^n$  with mn hexagons,  $m, n \geq 2$ , admits a super (a, d)-edge-antimagic total labeling if and only if d = 1 and a = 4mn + 2.

Bača, Miller, Phanalasy, and A. Semaničová-Feňovčíková [299] investigated the existence of (super) 1-antimagic labelings of type (1, 1, 1) for disjoint union of plane graphs. They prove that if a plane graph G(V, E, F) has a (super) 1-antimagic labeling h of type (1, 1, 1) such that  $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$  where  $z_{ext}$  denotes the unique external face then, for every positive integer m, the graph mG also admits a (super) 1-antimagic labeling of type (1, 1, 1); and if a plane graph G(V, E, F) has 4sided inner faces and h is a (super) d-antimagic labeling of type (1, 1, 1) of G such that  $h(z_{ext}) = |V(G)| + |E(G)| + |F(G)|$  where d = 1, 3, 5, 7, 9 then, for every positive integer m, the graph mG also admits a (super) d-antimagic labeling of type (1, 1, 1). They also give a similar result about plane graphs with inner faces that are 3-sided.

Sugeng, Miller, Slamin, and Bača [2498] proved: the star  $S_n$  has a super (a, d)antimagic total labeling if and only if either d = 0, 1 or 2, or d = 3 and n = 1 or 2;
if a nontrivial caterpillar has a super (a, d)-edge-antimagic total labeling, then  $d \leq 3$ ; all
caterpillars have super (a, 0)-, (a, 1)- and (a, 2)-edge-antimagic total labelings; all caterpillars have a super (a, 1)-edge-antimagic total labeling; if m and n differ by at least 2
the double star  $S_{m,n}$  (that is, the graph obtained by joining the centers of  $K_{1,m}$  and  $K_{1,n}$ 

with an edge) has no (a, 3)-edge-antimagic total labeling.

Sugeng and Miller [2493] show how to manipulate adjacency matrices of graphs with (a, d)-edge-antimagic vertex labelings and super (a, d)-edge-antimagic total labelings to obtain new (a, d)-edge-antimagic vertex labelings and super (a, d)-edge-antimagic total labelings. Among their results are: every graph can be embedded in a connected (a, d)-edge-antimagic vertex graph; every (a, d)-edge-antimagic vertex graph has a proper (a, d)-edge-antimagic vertex subgraph; if a graph has a (a, 1)-edge-antimagic vertex labeling; every super edge magic total graph has an (a, 1)-edge-antimagic vertex labeling; and every graph can be embedded in a connected super (a, d)-edge-antimagic total graph has an (a, 1)-edge-antimagic vertex labeling; and every graph can be embedded in a connected super (a, d)-edge-antimagic total graph.

Rahmawati, Sugeng, Silaban, Miller, and Bača [2059] construct new larger (a, d)edge-antimagic vertex graphs from an existing (a, d)-edge-antimagic vertex graph using adjacency matrix for difference d = 1, 2. The results are extended for super (a, d)-edgeantimagic total graphs with differences d = 0, 1, 2, 3.

Ajitha, Arumugan, and Germina [133] show that (p, p-1) graphs with  $\alpha$ -labelings (see §3.1) and partite sets with sizes that differ by at most 1 have super (a, d)-edge antimagic total labelings for d = 0, 1, 2 and 3. They also show how to generate large classes of trees with super (a, d)-edge-antimagic total labelings from smaller graceful trees.

Bača, Lin, Miller, and Ryan [285] define a *Möbius grid*,  $M_n^m$ , as the graph with vertex set  $\{x_{i,j} | i = 1, 2, ..., m + 1, j = 1, 2, ..., n\}$  and edge set  $\{x_{i,j}x_{i,j+1} | i = 1, 2, ..., m + 1, j = 1, 2, ..., m, 1 \} \cup \{x_{i,j}x_{i+1,j} | i = 1, 2, ..., m, j = 1, 2, ..., n\} \cup \{x_{i,n}x_{m+2-i,1} | i = 1, 2, ..., m + 1\}$ . They prove that for  $n \ge 2$  and  $m \ge 4$ ,  $M_n^m$  has no *d*-antimagic vertex labeling with  $d \ge 5$  and no *d*-antimagic-edge labeling with  $d \ge 9$ .

Ali, Bača, and Bashir, [123] investigated super (a, d)-vertex-antimagic total labelings of the disjoint unions of paths. They prove:  $mP_2$  has a super (a, d)-vertex-antimagic total labeling if and only if m is odd and d = 1;  $mP_3$ , m > 1, has no super (a, 3)-vertexantimagic total labeling;  $mP_3$  has a super (a, 2)-vertex-antimagic total labeling for  $m \equiv 1$ (mod 6); and  $mP_4$  has a super (a, 2)-vertex-antimagic total labeling for  $m \equiv 3 \pmod{4}$ .

Lee, Tsai, and Lin [1485] denote the subdivision of a star  $S_n$  obtained by inserting m vertices into every edge of the star  $S_n$  by  $S_m^n$ . They proved that for  $n \ge 3$ , the graph  $kS_m^n$  is super (a, d)-edge antimagic total for certain values. In [1041] Ichishima, López, Muntaner-Batle and Rius-Font proved that if G is tripartite and has a (super) (a, d)-edge antimagic total labeling, then nG  $(n \ge 3)$  has a (super) (a, d)-edge antimagic total labeling for d = 1 and for d = 0, 2 when n is odd.

Let  $p, t_1, t_2, \ldots, t_k$  be integers such that  $1 \leq t_1 < t_2 < \cdots < t_k < p$ . A Toeplitz graph, denoted by  $T_p\langle t_1 \ldots, t_k \rangle$ , is a graph with vertex set  $\{v_1, v_2, \ldots, v_p\}$  and edge set  $\{v_i v_j : |i - j| \in \{t_1, t_2, \ldots, t_k\}\}$ . Bača, Bashir, Nadeem, and Shabbir [251] give an upper bound on the difference d when a Toeplitz graph  $T_p\langle t_1, t_2, \ldots, t_k \rangle$  is super (a, d)edge-antimagic total. They also construct a super (a, 1)-edge-antimagic total labeling for an arbitrary Toeplitz graph without isolated vertices and prove that the Toeplitz graph  $T_p\langle t_1 \rangle$  admits a super (a, 3)-edge-antimagic total labeling. Moreover, when p and  $t_1$  satisfy certain conditions  $T_p\langle t_1 \rangle$  also admits a super (a, d)-edge-antimagic total labeling for d = 0and d = 2. When k = 2 they show the existence of a super (a, 2)-edge-antimagic total labeling for the Toeplitz graph  $T_p\langle t_1, t_1 + 1 \rangle$ .

Pandimadevi and Subbiah [1886] show the existence and nonexistence of (a, d)-vertex antimagic total labeling for several class of digraphs and show how to construct labelings for generalized de Bruijn digraphs.

In [874] Getzimah and Palani define vertex antimagic total labeling, edge antimagic total labeling on  $Z_{p+q}$ , and discuss these labelings for cycles, stars, complete bipartite graphs, the subdivision graphs of ladders, and combs. They also investigate totally (a, d)-edge antimagic graphs, totally super vertex graphs, edge antimagic graphs, and determine the bounds for the vertices and the edges under total labelings.

Chang, Chen, Li, and Pan [554] investigated a weak version of antimagic labelings called k-shifted-antimagic labelings that allow the consecutive numbers to start from k+1, instead of starting from 1. They established connections among various concepts proposed in the literature of antimagic labelings and extend previous results in three ways: some classes of graphs, including trees and graphs whose vertices are of odd degrees, that have not been verified to be antimagic are shown to be k-shifted-antimagic for sufficiently large k; some graphs are proved k-shifted-antimagic for all k, whereas some are proved not for some particular k; and disconnected graphs are also considered.

The book [297] by Bača and Miller has a wealth of material and open problems on super edge-antimagic labelings. In [260] Bača, Baskoro, Miller, Ryan, Simanjuntak, and Sugeng provide detailed survey of results on edge antimagic labelings and include many conjectures and open problems. In 2015 Nalliah [1829] published a list of open problems on super (a, d)-edge antimagic total labelings of graphs. In 2017 Brankovic, Jendrol, Lin, Phanalasy, Ryan, Semaničová-Feňovčíková, Slamin, and Sugeng [253] provided a survey of recent results on face-antimagic labelings. It was dedicated to the memory of Mirka Miller, who introducted the concept of face-antimagic labeling of plane graphs in 2003.

In Tables 14, 15, 16 and 17 we use the abbreviations

- (a, d)-VAT (a, d)-vertex-antimagic total labeling
- (a, d)-SVAT super (a, d)-vertex-antimagic total labeling
- (a, d)-EAT (a, d)-edge-antimagic total labeling
- (a, d)-SEAT super (a, d)-edge-antimagic total labeling
- (a, d)-EAV (a, d)-edge-antimagic vertex labeling

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Graph	Labeling	Notes
$P_n$	(a,d)-VAT	wide variety of $a$ and $d$ [262]
$P_n$	(a, d)-SVAT	iff $d = 3$ , $d = 2, n \ge 3$ odd or $d = 3$ , $n \ge 3$ [2496]
$C_n$	(a, d)-VAT	wide variety of $a$ and $d$ [261]
$C_n$	(a, d)-SVAT	iff $d = 0, 2$ and $n$ odd or $d = 1$ [2496]
generalized Petersen graph $P(n,k)$	( <i>a</i> , <i>d</i> )-VAT ( <i>a</i> , 1)-VAT	[263] $n \ge 3, \ 1 \le k \le n/2 \ [2497]$
prisms $C_n \times P_2$	(a, d)-VAT	[263]
antiprisms	(a, d)-VAT	[263]
$S_{n_1} \cup \ldots \cup S_{n_t}$	(a, d)-VAT	d = 1, 2, 3, 4, 6 [1893], [2055]
$W_n$	not $(a, d)$ -VAT	for $n > 20$ [1599]
$K_{1,n}$	not $(a, d)$ -SVAT	$n \ge 3$ [2496]

Table 14: Summary of (a, d)-Vertex-Antimagic Total and Super (a, d)-Vertex-Antimagic Total Labelings

Graph	Labeling	Notes
trees	(a, 1)-EAT?	[286]
$P_n$	not $(a, d)$ -EAT	d > 2 [286]
$P_{2n}$	(6n, 1)-EAT (6n + 2, 2)-EAT	[2386] [2386]
$P_{2n+1}$	(3n + 4, 2)-EAT (3n + 4, 3)-EAT (2n + 4, 4)-EAT (5n + 4, 2)-EAT (3n + 5, 2)-EAT (2n + 6, 4)-EAT	[2386] [2386] [2386] [2386] [2386] [2386]
$C_n$	(2n + 2, 1)-EAT (3n + 2, 1)-EAT not $(a, d)$ -EAT	
$C_{2n}$	(4n + 2, 2)-EAT (4n + 3, 2)-EAT (2n + 3, 4)-EAT? (2n + 4, 4)-EAT?	[2386] [2386] [2386] [2386]
$C_{2n+1}$	(3n + 4, 3)-EAT (3n + 5, 3)-EAT (n + 4, 5)-EAT? (n + 5, 5)-EAT?	[2386] [2386] [2386] [2386]
$K_n$	not $(a, d)$ -EAT	d > 5 [286]
$K_{n,n}$	(a, d)-EAT	iff $d = 1, n \ge 2$ [287]
caterpillars	(a, d)-EAT	$d \le 3 \ [2498]$
$W_n$	not $(a, d)$ -EAT	d > 4 [286]
generalized Petersen	not $(a, d)$ -EAT	d > 4 [286]
graph $P(n,k)$	((5n+5)/2, 2)-EAT super $(4n+2, 1)$ -EAT	$ \begin{vmatrix} n \ge 3 \text{ odd}, \ k = 1, 2 \ [1854] \\ n \ge 3, \ 1 \le k \le n/2 \ [1854] \end{aligned} $

Table 15: Summary of (a, d)-Edge-Antimagic Total Labelings

The electronic journal of combinatorics  $\ (2019),\ \#DS6$ 

Graph	Labeling	Notes
$P_n$	(3,2)-EAV	[2386]
	not $(a, d)$ -EAV	d > 2 [2386]
$P_{2n}$	(n+2,1)-EAV	[2386]
$C_n$	not $(a, d)$ -EAV	d > 1 [286]
$C_{2n}$	not $(a, d)$ -EAV	[2386]
$C_{2n+1}$	(n+2,1)-EAV (n+2,1) EAV	[2386]
	(n+3,1)-LAV	[2000]
$K_n$	not $(a, d)$ -EAV	for $n > 1$ [286]
$K_{n,n}$	not $(a, d)$ -EAV	for $n > 3$ [286]
$W_n$	not $(a, d)$ -EAV	[286]
$C_3^{(n)}$ (friendship graph)	(a, 1)-EAV	iff $n = 1, 3, 4, 5, 7$ [287]
generalized Petersen graph $P(n,k)$	not $(a, d)$ -EAV	d > 1 [286]

Table 16: Summary of (a, d)-Edge-Antimagic Vertex Labelings

Graph	Labeling	Notes
$C_n \odot K_1$	(a, d)-SEAT	variety of cases [241], [312]
$P_n \times P_2$ (ladders)	(a, d)-SEAT (a, d)-SEAT?	$n \text{ odd}, d \le 2 \ [2495]$ $n \text{ even}, d = 1 \ [2495]$ $d = 0, 2, n \text{ even} \ [2495]$
$C_n \times P_2$	(a, d)-SEAT	iff $d \le 3 \ n \text{ odd } [2495]$ or $d = 1, \ n \ge 4 \text{ even } [2495]$
$C_m \times P_n$	(a, d)-SEAT?	$m \ge 4$ even, $n \ge 3, \ d = 0, 2 \ [2495]$
caterpillars	(a, 1)-SEAT	[2498]
$C_3^{(n)}$ (friendship graphs)	(a, d)-SEAT	d = 0, 1, 2 [287]
$F_n \ (n \ge 2) \ (\text{fans})$	(a, d) SEAT (a, d)-SEAT	only if $d < 3$ [287] $2 \le n \le 6, d = 0, 1, 2$ [287]
$W_n$	(a, d)-SEAT	iff $d = 1, n \not\equiv 1 \pmod{4}$ [287]
$K_n \ (n \ge 3)$	(a,d) SEAT	iff $d = 0, n = 3$ [287] $d = 1, n \ge 3$ [287] d = 2, n = 3 [287]
trees	(a, d)-SEAT	only if $d \leq 3$ [288]
$P_n \ (n > 1)$	(a, d)-SEAT	iff $d \le 3$ [288]
$mK_n$	(a,d)-SEAT	$ \begin{array}{l} \text{iff } d \in \{0, 2\}, n \in \{2, 3\}, \\ m \geq 3 \text{ odd } [249] \\ d = 1, m, n \geq 2 \ [249] \\ d = 3 \text{ or } 5, n = 2, m \geq 2 \ [249] \\ d = 4, n = 2, m \geq 3 \text{ odd } [249] \end{array} $
$C_n$	(a, d)-SEAT	$ \inf_{d = 1} d = 0 \text{ or } 2, n \text{ odd } [288] $ $ d = 1 \ [261] $
P(m,n)	(a, d)-SEAT	many cases [261]

Table 17: Summary of (a, d)-Super-Edge-Antimagic Total Labelings

# 6.4 Face Antimagic Labelings and *d*-antimagic Labeling of Type (1,1,1)

Bača [243] defines a connected plane graph G with edge set E and face set F to be (a, d)face antimagic if there exist positive integers a and d and a bijection  $g: E \to \{1, 2, \dots, |E|\}$ such that the induced mapping  $\psi_g \colon F \to \{a, a+d, \ldots, a+(|F(G)|-1)d\}$ , where for a face f,  $\psi_g(f)$  is the sum of all g(e) for all edges e surrounding f is also a bijection. In [245] Bača proves that for n even and at least 4, the prism  $C_n \times P_2$  is (6n + 3, 2)-face antimagic and (4n + 4, 4)-face antimagic. He also conjectures that  $C_n \times P_2$  is (2n + 5, 6)face antimagic. In |283| Bača, Lin, and Miller investigate (a, d)-face antimagic labelings of the convex polytopes  $P_{m+1} \times C_n$ . They show that if these graphs are (a, d)-face antimagic then either d = 2 and a = 3n(m+1) + 3, or d = 4 and a = 2n(m+1) + 4, or d = 6 and a = n(m+1) + 5. They also prove that if n is even,  $n \ge 4$  and  $m \equiv 1 \pmod{4}$ ,  $m \ge 3$ , then  $P_{m+1} \times C_n$  has a (3n(m+1)+3, 2)-face antimagic labeling and if n is at least 4 and even and m is at least 3 and odd, or if  $n \equiv 2 \pmod{4}$ ,  $n \geq 6$  and m is even,  $m \geq 4$ , then  $P_{m+1} \times C_n$  has a (3n(m+1)+3, 2)-face antimagic labeling and a (2n(m+1)+4, 4)face antimagic labeling. They conjecture that  $P_{m+1} \times C_n$  has (3n(m+1)+3,2)- and (2n(m+1)+4, 4)-face antimagic labelings when  $m \equiv 0 \pmod{4}$ ,  $n \geq 4$ , and for m even and  $m \ge 4$ , that  $P_{m+1} \times C_n$  has a (n(m+1)+5, 6)-face antimagic labeling when n is even and at least 4. Bača, Baskoro, Jendrol, and Miller [257] proved that graphs in the shape of hexagonal honeycombs with m rows, n columns, and mn 6-sided faces have d-antimagic labelings of type (1, 1, 1) for d = 1, 2, 3, and 4 when n odd and mn > 1.

In [295] Bača and Miller define the class  $Q_n^m$  of convex polytopes with vertex set  $\{y_{j,i}: i = 1, 2, \ldots, n; j = 1, 2, \ldots, m + 1\}$  and edge set  $\{y_{j,i}y_{j,i+1}: i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, j \cup \{y_{j,i+1}y_{j+1,i}: 1 + 1, 2, \ldots, n; j = 1, 2, \ldots, m, j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1}: i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, j \text{ even}\}$ where  $y_{j,n+1} = y_{j,1}$ . They prove that for m odd,  $m \ge 3, n \ge 3, Q_n^m$  is (7n(m+1)/2+2, 1)-face antimagic and when m and n are even,  $m \ge 4, n \ge 4, Q_n^m$  is (7n(m+1)/2+2, 1)-face antimagic. They conjecture that when n is odd,  $n \ge 3$ , and m is even, then  $Q_n^m$  is ((5n(m+1)+5)/2, 2)-face antimagic and ((n(m+1)+7)/2, 4)-face antimagic. They further conjecture that when n is even, n > 4, m > 1 or n is odd, n > 3 and m is odd, m > 1, then  $Q_n^m$  is (3n(m+1)/2+3, 3)-face antimagic. In [247] Bača proves that for the case m = 1 and  $n \ge 3$  the only possibilities for (a, d)-antimagic labelings for  $Q_n^m$  are (7n+2, 1) and (3n+3, 3). He provides the labelings for the first case and conjectures that they exist for the second case. Bača [243] and Bača and Miller [294] describe (a, d)-face antimagic labelings for a certain classes of convex polytopes.

In [256] Bača et al. provide a detailed survey of results on face antimagic labelings and include many conjectures and open problems.

For a plane graph G, Bača and Miller [296] call a bijection h from  $V(G) \cup E(G) \cup F(G)$ to  $\{1, 2, \ldots, |V(G)| + |E(G)| \cup |F(G)|\}$  a *d*-antimagic labeling of type (1, 1, 1) if for every number s the set of s-sided face weights is  $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$  for some integers  $a_s$  and d, where  $f_s$  is the number of s-sided faces ( $W_s$  varies with s). They show that the prisms  $C_n \times P_2$  ( $n \ge 3$ ) have a 1-antimagic labeling of type (1, 1, 1) and that for  $n \equiv 3 \pmod{4}$ ,  $C_n \times P_2$  have a *d*-antimagic labeling of type (1, 1, 1) for d = 2, 3, 4, and 6. They conjecture that for all  $n \geq 3$ ,  $C_n \times P_2$  has a *d*-antimagic labeling of type (1, 1, 1) for d = 2, 3, 4, 5, and 6. This conjecture has been proved for the case d = 3 and  $n \neq 4$  by Bača, Miller, and Ryan [302] (the case d = 3 and n = 4 is open). The cases for d = 2, 4, 5, and 6 were done by Lin, Slamin, Bača, and Miller [1600]. Bača, Lin, and Miller [284] prove: for m, n > 8,  $P_m \times P_n$  has no *d*-antimagic edge labeling of type (1, 1, 1)with  $d \geq 9$ ; for  $m \geq 2, n \geq 2$ , and  $(m, n) \neq (2, 2)$ ,  $P_m \times P_n$  has *d*-antimagic labelings of type (1, 1, 1) for d = 1, 2, 3, 4, and 6. They conjecture the same is true for d = 5. Butt, Numan, Shah, and Ali [521] prove that the generalized prims  $C_n \times P_m$  have *d*-antimagic face labelings of type (1, 1, 1) for  $n \geq 5$  and  $m \geq 2$ .

Bača, Miller, and Ryan [302] also prove that for  $n \ge 4$  the antiprism (see §6.1 for the definition) on 2n vertices has a *d*-antimagic labeling of type (1, 1, 1) for d = 1, 2, and 4. They conjecture the result holds for d = 3, 5, and 6 as well. Lin, Ahmad, Miller, Sugeng, and Bača [1597] did the cases that d = 7 for  $n \ge 3$  and d = 12 for  $n \ge 11$ . Sugeng, Miller, Lin, and Bača [2497] did the cases: d = 7, 8, 9, 10 for  $n \ge 5$ ; d = 15 for  $n \ge 6$ ; d = 18 for  $n \ge 7$ ; d = 12, 14, 17, 20, 21, 24, 27, 30, 36 for n odd and  $n \ge 7$ ; and d = 16, 26 for n odd and  $n \ge 9$ .

Baca, Numan, and Semaničová-Feňovčíková [309] investigated the problem of labeling the vertices, edges, and faces of a disjoint union of r copies  $C_n \times P_m$  by the consecutive integers starting from 1 in such a way that the sum of the labels of a face and the labels of vertices and edges surrounding that face for all *s*-sided faces form an arithmetic progression with common difference d.

Ali, Bača, Bashir, and Semaničová-Feňovčíková [124] investigated antimagic labelings for disjoint unions of prisms and cycles. They prove: for  $m \ge 2$  and  $n \ge 3$ ,  $m(C_n \times P_2)$ has no super *d*-antimagic labeling of type (1, 1, 1) with  $d \ge 30$ ; for  $m \ge 2$  and  $n \ge$  $3, n \ne 4, m(C_n \times P_2)$  has super *d*-antimagic labeling of type (1, 1, 1) for d = 0, 1, 2, 3, 4, and 5; and for  $m \ge 2$  and  $n \ge 3, mC_n$  has (m(n+1)+3, 3)- and (2mn+2, 2)-vertexantimagic total labeling. Bača and Bashir [250] proved that for  $m \ge 2$  and  $n \ge 3, n \ne$  $4, m(C_n \times P_2)$  has super 7-antimagic labeling of type (1, 1, 1) and for  $n \ge 3, n \ne 4$  and  $2 \le m \le 2n m(C_n \times P_2)$  has super 6-antimagic labeling of type (1, 1, 1).

Bača, Numan and Siddiqui [311] investigated the existence of the super d-antimagic labeling of type (1, 1, 1) for the disjoint union of m copies of antiprism  $mA_n$ . They proved that for  $m \ge 2$ ,  $n \ge 4$ ,  $mA_n$  has super d-antimagic labelings of type (1, 1, 1) for d = 1, 2, 3, 5, 6. Ahmad, Bača, Lascsáková, and Semaničová-Feňovčíková [80] investigated super d-antimagicness of type (1, 1, 0) for mG in a more general sense. They prove: if there exists a super 0-antimagic labeling of type (1, 1, 0) of a plane graph G then, for every positive integer m, the graph mG also admits a super 0-antimagic labeling of type (1, 1, 0); if a plane graph G with 3-sided inner faces admits a super d-antimagic labeling of type (1, 1, 0) for d = 0, 6 then, for every positive integer m, the graph mG also admits a super d-antimagic labeling of type (1, 1, 0); if a plane graph G with 3-sided inner faces is a tripartite graph with a super d-antimagic labeling of type (1, 1, 0) for d = 2, 4 then, for every positive integer m, the graph mG also admits a super d-antimagic labeling of type (1, 1, 0); if a plane graph G with 4-sided inner faces admits a super d-antimagic labeling of type (1, 1, 0); if a plane graph G with 4-sided inner faces admits a super d-antimagic labeling of type (1, 1, 0); if a plane graph G with 4-sided inner faces admits a super d-antimagic labeling of type (1, 1, 0); if a plane graph G with 4-sided inner faces admits a super d-antimagic labeling of type type (1, 1, 0) for d = 0, 4, 8 then the disjoint union of arbitrary number of copies of G also admits a super *d*-antimagic labeling of type (1, 1, 0); if a plane graph G with *k*-sided inner faces,  $k \ge 3$ , admits a super *d*-antimagic labeling of type (1, 1, 0) for d = 0, 2k then, for every positive integer m, the graph mG also admits a super *d*-antimagic labeling of type (1, 1, 0); if a plane graph G with *k*-sided inner faces admits a super *k*-antimagic labeling of type (1, 1, 0) for k even then, for every positive integer m, the graph mG also admits a super *k*-antimagic labeling of type (1, 1, 0).

Bača, Jendral, Miller, and Ryan [276] prove: for n even,  $n \ge 6$ , the generalized Petersen graph P(n,2) has a 1-antimagic labeling of type (1,1,1); for n even,  $n \ge 6$ ,  $n \ne 10$ , and d = 2 or 3, P(n,2) has a d-antimagic labeling of type (1,1,1); and for  $n \equiv 0 \pmod{4}$ ,  $n \ge 8$  and d = 6 or 9, P(n,2) has a d-antimagic labeling of type (1,1,1). They conjecture that there is an d-antimagic labeling of type (1,1,1) for P(n,2) when  $n \equiv 2 \pmod{4}$ ,  $n \ge 6$ , and d = 6 or 9.

In [266] Bača, Brankovic, and A. Semaničová-Feňovčiková provide super *d*-antimagic labelings of type (1,1,1) for friendship graphs  $F_n$   $(n \ge 2)$  and several other families of planar graphs.

Bača, Brankovic, Lascsáková, Phanalasy, and Semaničová-Feňovčíková [265] provided super *d*-antimagic labeling of type (1, 1, 0) for friendship graphs  $F_n$ ,  $n \ge 2$ , for  $d \in \{1, 3, 5, 7, 9, 11, 13\}$ . Moreover, they show that for  $n \equiv 1 \pmod{2}$  the graph  $F_n$  also admits a super *d*-antimagic labeling of type (1, 1, 0) for  $d \in \{0, 2, 4, 6, 8, 10\}$ .

Bača, Baskoro, and Miller [258] have proved that hexagonal planar honeycomb graphs with an even number of columns have 2-antimagic and 4-antimagic labelings of type (1,1,1). They conjecture that these honeycombs also have *d*-antimagic labelings of type (1,1,1) for d = 3 and 5. They pose the odd number of columns case for  $1 \le d \le 5$  as an open problem. Bača, Baskoro, and Miller [259] give *d*-antimagic labelings of a special class of plane graphs with 3-sided internal faces for d = 0, 2, and 4. Bača, Lin, Miller, and Ryan [285] prove for odd  $n \ge 3$ ,  $m \ge 1$  and d = 0, 1, 2 or 4, the Möbius grid  $M_n^m$  has an *d*-antimagic labeling of type (1,1,1). Siddiqui, Numan, and Umar [2383] examined the existence of super *d*-antimagic labelings of type (1,1,1) for Jahangir graphs for certain differences *d*.

Bača, Numan, and Shabbir [310] studied the existence of super *d*-antimagic labelings of type (1, 1, 1) for the toroidal polyhex  $\mathbb{H}_m^n$ . They labeled the edges of a 1-factor by consecutive integers and then in successive steps they labeled the edges of 2*m*-cycles (respectively 2*n*-cycles) in a 2-factor by consecutive integers. This technique allowed them to construct super *d*-antimagic labelings of type (1, 1, 1) for  $\mathbb{H}_m^n$  with d = 1, 3, 5. They suppose that such labelings exist also for d = 0, 2, 4.

Kathiresan and Ganesan [1351] define a class of plane graphs denoted by  $P_a^b$   $(a \ge 3, b \ge 2)$  as the graph obtained by starting with vertices  $v_1, v_2, \ldots, v_a$  and for each  $i = 1, 2, \ldots, a - 1$  joining  $v_i$  and  $v_{i+1}$  with b internally disjoint paths of length i + 1. They prove that  $P_a^b$  has d-antimagic labelings of type (1, 1, 1) for d = 0, 1, 2, 3, 4, and 6. Lin and Sugen [1601] prove that  $P_a^b$  has a d-antimagic labeling of type (1, 1, 1) for d = 5, 7a - 2, a + 1, a - 3, a - 7, a + 5, a - 4, a + 2, 2a - 3, 2a - 1, a - 1, 3a - 3, a + 3, 2a + 1, 2a + 3, 3a + 1, 4a - 1, 4a - 3, 5a - 3, 3a - 1, 6a - 5, 6a - 7, 7a - 7, and 5a - 5. Similarly, Bača, Baskoro,

and Cholily [255] define a class of plane graphs denoted by  $C_a^b$  as the graph obtained by starting with vertices  $v_1, v_2, \ldots, v_a$  and for each  $i = 1, 2, \ldots, a$  joining  $v_i$  and  $v_{i+1}$  with b internally disjoint paths of length i+1 (subscripts are taken modulo a). In [255] and [254] they prove that for  $a \ge 3$  and  $b \ge 2$ ,  $C_a^b$  has a d-antimagic labeling of type (1, 1, 1) for d = 0, 1, 2, 3, a + 1, a - 1, a + 2, and a - 2.

In [267] Bača, Brankovic, and Semaničová-Feňovčiková investigated the existence of super d-antimagic labelings of type (1,1,1) for plane graphs containing a special kind of Hamilton path. They proved: if there exists a Hamilton path in a plane graph Gsuch that for every face except the external face, the Hamilton path contains all but one of the edges surrounding that face, then G is super d-antimagic of type (1,1,1) for d = 0, 1, 2, 3, 5; if there exists a Hamilton path in a plane graph G such that for every face except the external face, the Hamilton path contains all but one of the edges surrounding that face and if  $2(|F(G)| - 1) \leq |V(G)|$ , then G is super d-antimagic of type (1, 1, 1) for d = 0, 1, 2, 3, 4, 5, 6; if G is a plane graph with  $M = \lfloor \frac{|V(G)|}{|F(G)|-1} \rfloor$  and a Hamilton path such that for every face, except the external face, the Hamilton path contains all but one of the edges surrounding that face, then for M = 1, G admits a super d-antimagic labeling of type (1,1,1) for d = 0, 1, 2, 3, 5; and for  $M \ge 2$ , G admits a super d-antimagic labeling of type (1,1,1) for  $d = 0, 1, 2, 3, \ldots, M + 4$ . They also proved that  $P_n \times P_2$   $(n \ge 3)$  admits a super d-antimagic labeling of type (1,1,1) for  $d \in \{0, 1, 2, \dots, 15\}$  and the graph obtained from  $P_n \times P_m$   $(n \ge 2)$  by adding a new edge in every 4-sided face such that the added edges are "parallel" admits a super d-antimagic labeling of type (1,1,1) for  $d \in \{0,1,2,\ldots,9\}$ .

In [1066] Imran, Siddiqui, and Numan examine the existence of super d-antimagic labelings of type (1,1,1) for uniform subdivision of wheel for certain differences d.

In the following tables we use the abbreviations

(a, d)-**FA** (a, d)-face antimagic labeling

d-AT(1,1,1) d-antimagic labeling of type (1,1,1).

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The tables were prepared by Petr Kovář and Tereza Kovářová and updated by J. Gallian in 2008.

Graph	Labeling	Notes
$Q_n^m$ (see §6.4)	(7n(m+1)/2+2,1)-FA	$m \ge 3, n \ge 3, m \text{ odd } [295]$
	(7n(m+1)/2+2,1)-FA	$m \ge 4, n \ge 4, m, n \text{ even } [295]$
	((5n(m+1)+5)/2,2)-FA?	$m \geq 2, n \geq 3, m$ even, $n$ odd [295]
	((n(m+1)+7)/2, 4)-FA?	$m \geq 2, n \geq 3, m$ even, $n$ odd [295]
	(3n(m+1)/2+3,3)-FA?	m > 1, n > 4, n even [295]
	(3n(m+1)/2+3,3)-FA?	m > 1, n > 3, m  odd, n  odd  [295]
$C_n \times P_2$	(6n+3,2)-FA	n > 4, n even [245]
	(4n + 4, 4)-FA	$n \ge 4, n \text{ even } [245]$
	(2n+5,6)-FA?	[245]
$P_{m+1} \times C_n$	(3n(m+1)+3,2)-FA	$n \ge 4, n$ even and [283]
		$m \ge 3, m \equiv 1 \pmod{4},$
	(3n(m+1)+3,2)-FA and	$n \ge 4, n \text{ even and } [283]$
	(2n(m+1)+4,4)-FA	$m \ge 3, m \text{ odd } [283],$
		or $n \ge 6$ , $n \equiv 2 \pmod{4}$ and
		$m \geq 4, m$ even
	(3n(m+1)+3,2)-FA?	$m \ge 4, n \ge 4, m \equiv 0 \pmod{4}$ [283]
	(2n(m+1)+4,4)-FA?	$m \ge 4, n \ge 4, m \equiv 0 \pmod{4}$ [283]
	(n(m+1)+5,6)-FA?	$n \geq 4, n \text{ even } [283]$

Table 18: Summary of Face Antimagic Labelings

Table 19: Summary of d-antimagic Labelings of Type (1,1,1)

Graph	Labeling	Notes
$P_m \times P_n$	not $d$ -AT(1,1,1)	$m, n, d \ge 9,  [284]$
$P_m \times P_n$	d-AT(1,1,1)	d = 1, 2, 3, 4, 6;
		$m, n \ge 2, \ (m, n) \ne (2, 2) \ [284]$
$P_m \times P_n$	5-AT(1,1,1)	$m, n \ge 2, \ (m, n) \ne (2, 2) \ [284]$
$C_n \times P_2$	1-AT(1,1,1)	[296]
	d-AT(1,1,1)	d = 2, 3, 4  and  6 [296]
		for $n \equiv 3 \pmod{4}$
	d-AT(1,1,1)	$d = 2, 4, 5, 6 \text{ for } n \ge 3 [1600]$
	d-AT(1,1,1)	$d = 3 \text{ for } n \ge 5 [302]$

Continued on next page

Graph	Labeling	Notes
$P_m \times P_n$	5-AT(1,1,1)?not d-AT	$\begin{bmatrix} 1600 \\ m, n > 8, \ d \ge 9 \ [1600] \end{bmatrix}$
antiprism on $2n$ vertices	d-AT(1,1,1) d-AT(1,1,1)?	$d = 1, 2 \text{ and } 4 \text{ for } n \ge 4 [302]$ d = 3, 5 and 6 for n \ge 4 [302]
$M_n^m$ (Möbius grids)	<i>d</i> -AT(1,1,1)	$n \ge 3 \text{ odd}, d = 0, 1, 2, 4 \text{ [285]}$ $d = 7, n \ge 3 \text{ [1597]}$ $d = 12, n \ge 11 \text{ [1597]}$ $d = 7, 8, 9, 10, n \ge 5 \text{ [2497]}$ $d = 15, n \ge 6 \text{ [2497]}$ $d = 18 n \ge 7 \text{ [2497]}$
P(n,2)	d-AT(1,1,1)	$d = 1; \ d = 2, 3, \ n \ge 6, n \ne 10 \ [276]$
P(4n,2)	d-AT(1,1,1)	$d = 6, 9, \ n \ge 2, \ n \ne 10 \ [276]$
P(4n+2,2)	d-AT(1,1,1)?	$d = 6, 9, \ n \ge 1, \ n \ne 10 \ [276]$
honeycomb graphs with even number of columns	d-AT(1,1,1)  d-AT(1,1,1)?	$d = 2, 4 \ [258] \\ d = 3, 5 \ [258]$
$C_n \times P_2$	d-AT(1,1,1)	d = 1, 2, 4, 5, 6 [1600], [296]
$C_n \times P_2$	3-AT(1,1,1)	$n \neq 4 \ [302]$

Table 19 – Continued from previous page

## 6.5 Product Antimagic Labelings

Figueroa-Centeno, Ichishima, and Muntaner-Batle [759] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph G of size q to be product magic if there is a labeling from E(G) onto  $\{1, 2, \ldots, q\}$  such that, at each vertex v, the product of the labels on the edges incident with v is the same. They call a graph G of size q product antimagic if there is a labeling f from E(G) onto  $\{1, 2, \ldots, q\}$  such that the products of the labels on the edges incident at each vertex v are distinct. They prove: a graph of size q is product magic if and only if  $q \leq 1$  (that is, if and only if it is  $K_2, \overline{K_n}$  or  $K_2 \cup \overline{K_n}$ );  $P_n$   $(n \geq 4)$  is product antimagic; every 2-regular graph is product antimagic; and, if G is product antimagic, then so are  $G + K_1$  and  $G \odot \overline{K}_n$ . They conjecture that a connected graph of size q is product antimagic if and only if  $q \ge 3$ . Kaplan, Lev, and Roditty [1339] proved the following graphs are product antimagic: the disjoint union of cycles and paths where each path has least three edges; connected graphs with n vertices and m edges where  $m \ge 4n \ln n$ ; graphs G = (V, E) where each component has at least two edges and the minimum degree of G is at least  $8\sqrt{\ln |E|} \ln (\ln |E|)$ ; all complete k-partite graphs except  $K_2$  and  $K_{1,2}$ ; and  $G \odot H$  where G has no isolated vertices and H is regular.

In [1934] Pikhurko characterizes all large graphs that are product antimagic graphs. More precisely, it is shown that there is an  $n_0$  such that a graph with  $n \ge n_0$  vertices is product antimagic if and only if it does not belong to any of the following four classes: graphs that have at least one isolated edge; graphs that have at least two isolated vertices; unions of vertex-disjoint of copies of  $K_{1,2}$ ; graphs consisting of one isolated vertex; and graphs obtained by subdividing some edges of the star  $K_{1,k+l}$ .

In [759] Figueroa-Centeno, Ichishima, and Muntaner-Batle also define a graph G with p vertices and q edges to be *product edge-magic* if there is a labeling f from  $V(G) \cup E(G)$  onto  $\{1, 2, \ldots, p+q\}$  such that  $f(u) \cdot f(v) \cdot f(uv)$  is a constant for all edges uv and *product edge-antimagic* if there is a labeling f from  $V(G) \cup E(G)$  onto  $\{1, 2, \ldots, p+q\}$  such that for all edges uv the products  $f(u) \cdot f(v) \cdot f(uv)$  are distinct. They prove  $K_2 \cup \overline{K_n}$  is product edge-magic, a graph of size q without isolated vertices is product edge-magic if and only if  $q \leq 1$  and every graph other than  $K_2$  and  $K_2 \cup \overline{K_n}$  is product edge-antimagic.

## 7 Miscellaneous Labelings

#### 7.1 Sum Graphs

In 1990, Harary [958] introduced the notion of a sum graph. A graph G(V, E) is called a sum graph if there is an bijection f from V to a set of positive integers S such that  $xy \in E$  if and only if  $f(x) + f(y) \in S$ . Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In 1991 Harary, Hentzel, and Jacobs [960] defined a real sum graph in an analogous way by allowing S to be any finite set of positive real numbers. However, they proved that every real sum graph is a sum graph. Bergstrand, Hodges, Jennings, Kuklinski, Wiener, and Harary [442] defined a product graph analogous to a sum graph except that 1 is not permitted to belong to S. They proved that every product graph is a sum graph and vice versa.

For a connected graph G, let  $\sigma(G)$ , the sum number of G, denote the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph (some authors use s(G) for the sum number of G). A labeling that makes G together with  $\sigma(G)$  isolated points a sum graph is called an *optimal sum graph* labeling. Ellingham [709] proved the conjecture of Harary [958] that  $\sigma(T) = 1$  for every tree  $T \neq K_1$ . Smyth [2432] proved that there is no graph G with e edges and  $\sigma(G) = 1$  when  $n^2/4 < e \leq n(n-1)/2$ . Smyth [2433] conjectures that the disjoint union of graphs with sum number 1 has sum number 1. More generally, Kratochvil, Miller, and Nguyen [1428] conjecture that  $\sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1$ . Hao [952] has shown that if  $d_1 \leq d_2 \leq \cdots \leq d_n$  is the degree sequence of a graph G, then  $\sigma(G) > \max(d_i - i)$  where the maximum is taken over all i. Bergstand et al. [441] proved that  $\sigma(K_n) = 2n - 3$ . Hartsfield and Smyth [965] claimed to have proved that  $\sigma(K_{m,n}) = \lceil 3m + n - 3 \rceil/2$  when  $n \geq m$  but Yan and Liu [2845] found counterexamples to this assertion when  $m \neq n$ . Pyatkin [2043], Liaw, Kuo, and Chang [1593], Wang, and Liu [2798], and He, Shen, Wang, Chang, Kang, and Yu [971] have shown that for  $2 \leq m \leq n$ ,  $\sigma(K_{m,n}) = \lceil \frac{n}{p} + \frac{(p+1)p(m-1)}{2} \rceil$  where  $p = \lceil \sqrt{\frac{2n}{m-1} + \frac{1}{4} - \frac{1}{2} \rceil$ is the unique integer such that  $\frac{(p-1)p(m-1)}{2} < n \leq \frac{(p+1)p(m-1)}{2}$ .

Miller, Ryan, Slamin, and Smyth [1757] proved that  $\sigma(W_n) = \frac{n}{2} + 2$  for n even and  $\sigma(W_n) = n$  for  $n \ge 5$  and n odd (see also [2555]). Miller, Ryan, and Smyth [1759] prove that the complete n-partite graph on n sets of 2 nonadjacent vertices has sum number 4n - 5 and obtain upper and lower bounds on the complete n-partite graph on n sets of m nonadjacent vertices. Fernau, Ryan, and Sugeng [756] proved that the generalized friendship graphs  $C_n^{(t)}$  (see §2.2) has sum number 2 except for  $C_4$ . Gould and Rödl [920] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [909] proved that  $\sigma(K_n - \text{edge}) \le 2n - 4$ . The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the spum. They proved spum of  $K_n$  is 4n - 6 and the spum of  $C_n$  is at most 4n - 10. Kratochvil, Miller, and Nguyen [1428] have proved that every sum graph on n vertices has a sum labeling such that every label is at most  $4^n$ . Konečný, Kučera, Novotná, Pekárek, Šimsa, and Töpfer [1403] showed that if one allows for non-

injective labelings or graphs with loops then there are sum graphs without a minimal sum labeling, which partially answers the question posed by Miller, Ryan and Smyth in [1759].

At a conference in 2000 Miller [1745] posed the following two problems: Given any graph G, does there exist an optimal sum graph labeling that uses the label 1; Find a class of graphs G that have sum number of the order  $|V(G)|^s$  for s > 1. (Such graphs were shown to exist for s = 2 by Gould and Rödl in [920]).

In [2418] Slamet, Sugeng, and Miller show how one can use sum graph labelings to distribute secret information to set of people so that only authorized subsets can reconstruct the secret.

Chang [557] generalized the notion of sum graph by permitting x = y in the definition of sum graph. He calls graphs that have this kind of labeling *strong sum graphs* and uses  $i^*(G)$  to denote the minimum positive integer m such that  $G \cup mK_1$  is a strong sum graph. Chang proves that  $i^*(K_n) = \sigma(K_n)$  for n = 2, 3, and 4 and  $i^*(K_n) > \sigma(K_n)$  for  $n \ge 5$ . He further shows that for  $n \ge 5$ ,  $3n^{\log_2 3} > i^*(K_n) \ge 12\lfloor n/5 \rfloor - 3$ .

In 1994 Harary [959] generalized sum graphs by permitting S to be any set of integers. He calls these graphs *integral sum graphs*. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [2318] has shown that  $C_n$  and  $W_n$  are integral sum graphs for all  $n \neq 4$ . Chen [578] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. In [578] and [580] Chen gives methods for constructing new connected integral sum graphs from given integral sum graphs by identifying vertices. Chen [580] has shown that every graph is an induced subgraph of a connected integral sum graph. Chen [580] calls a vertex of a graph saturated if it is adjacent to every other vertex of the graph. He proves that every integral sum graph except  $K_3$  has at most two saturated vertices and gives the exact structure of all integral sum graphs that have exactly two saturated vertices. Chen [580] also proves that a connected integral sum graph with p > 1 vertices and q edges and no saturated vertices satisfies  $q \leq p(3p-2)/8 - 2$ . Wu, Mao, and Le [2821] proved that  $mP_n$ are integral sum graphs. They also show that the conjecture of Harary [959] that the sum number of  $C_n$  equals the integral sum number of  $C_n$  if and only if  $n \neq 3$  or 5 is false and that for  $n \neq 4$  or 6 the integral sum number of  $C_n$  is at most 1. Vilfred and Nicholas [2733] prove that graphs G of order n with  $\Delta(G) = n-1$  and  $|V_{\Delta}(G)| > 2$  are not integral sum graphs, except  $K_3$ , and that integral sum graphs G of order n with  $\Delta(G) = n - 1$  and  $|V_{\Delta}(G)| = 2$  exist and are unique up to isomorphism. Chen [582] proved that if G(V, E) is an integral sum other than  $K_3$  that has vertex of degree |V| - 1, then the edge-chromatic number of G is |V| - 1.

He, Wang, Mi, Shen, and Yu [969] say that a graph has a *tail* if the graph contains a path for which each interior vertex has degree 2 and an end vertex of degree at least 3. They prove that every tree with a tail of length at least 3 is an integral sum graph.

B. Xu [2832] has shown that the following are integral sum graphs: the union of any three stars;  $T \cup K_{1,n}$  for all trees T;  $mK_3$  for all m; and the union of any number of integral sum trees. Xu also proved that if 2G and 3G are integral sum graphs, then so is mG for all m > 1. Xu poses the question as to whether all disconnected forests are integral sum graphs. Nicholas and Somasundaram [1865] prove that all banana trees (see Section 2.1 for the definition) and the union of any number of stars are integral sum graphs.

Liaw, Kuo, and Chang [1593] proved that all caterpillars are integral sum graphs (see also [2821] and [2832] for some special cases of caterpillars). This shows that the assertion by Harary in [959] that K(1,3) and S(2,2) are not integral sum graphs is incorrect. They also prove that all cycles except  $C_4$  are integral sum graphs and they conjecture that every tree is an integral sum graph. Singh and Santhosh show that the crowns  $C_n \odot K_1$ are integral sum graphs for  $n \ge 4$  [2402] and that the subdivision graphs of  $C_n \odot K_1$  are integral sum graphs for  $n \ge 3$  [2186]. Wang, Li, and Wei [2764] proved that there exists a connected integral sum graph with any minimum degree and give an upper bound for the relation between the vertex number and the edge number of a connected integral sum graph with no saturated vertex.

For graphs with n vertices, Tiwari and Tripathi [2591] show that there exist sum graphs with m edges if and only if  $m \leq \lfloor (n-1^2)/4 \rfloor$  and that there exists integral sum graphs with m edges if and only if  $m \leq \lceil 3(n-1)^2/8 \rceil + \lfloor (n-1)/2 \rfloor$ , except for  $m = \lceil 3(n-1)^2/8 \rceil + \lfloor (n-1)/2 \rfloor - 1$  when n is of the form 4k + 1. They also characterize sets of positive integers (respectively, integers) that are in bijection with sum graphs (respectively, integral sum graphs) of maximum size for a given order.

The integral sum number,  $\zeta(G)$ , of G is the minimum number of isolated vertices that must be added to G so that the resulting graph is an integral sum graph. Thus, by definition, G is a integral sum graph if and only if  $\zeta(G) = 0$ . Harary [959] conjectured that  $\zeta(K_n) = 2n - 3$  for  $n \ge 4$ . This conjecture was verified by Chen [577], by Sharary [2318], and by B. Xu [2832]. Yan and Liu proved:  $\zeta(K_n - E(K_r)) = n - 1$  when  $n \ge 6$ ,  $n \equiv 0 \pmod{3}$  and r = 2n/3 - 1 [2846];  $\zeta(K_{m,m}) = 2m - 1$  for  $m \ge 2$  [2846];  $\zeta(K_n - edge) = 2n - 4$ for  $n \ge 4$  [2846], [2832]; if  $n \ge 5$  and  $n - 3 \ge r$ , then  $\zeta(K_n \setminus E(K_r)) \ge n - 1$  [2846]; if  $\lceil 2n/3 \rceil - 1 > r \ge 2$ , then  $\zeta(K_n \setminus E(K_r)) \ge 2n - r - 2$  [2846]; and if  $2 \le m < n$ , and n = (i + 1)(im - i + 2)/2, then  $\sigma(K_{m,n}) = \zeta(K_{m,n}) = (m - 1)(i + 1) + 1$  while if (i + 1)(im - i + 2)/2 < n < (i + 2)[(i + 1)m - i + 1]/2, then  $\sigma(K_{m,n}) = \zeta(K_{m,n}) =$  $\lceil ((m - 1)(i + 1)(i + 2) - 2n - 2$  when r + 1, 2n - 3 when  $2 \le r \le n - 1$ , and 2n - 4 when r = n.

Nagamochi, Miller, and Slamin [1814] have determined upper and lower bounds on the sum number a graph. For most graphs G(V, E) they show that  $\sigma(G) = \Omega(|E|)$ . He, Yu, Mi, Sheng, and Wang [970] investigated  $\zeta(K_n \setminus E(K_r))$  where  $n \ge 5$  and  $r \ge 2$ . They proved that  $\zeta(K_n \setminus E(K_r)) = 0$  when r = n or n - 1;

 $\begin{aligned} \zeta(K_n \setminus E(K_r)) &= n-2 \text{ when } r = n-2; \zeta(K_n \setminus E(K_r)) = n-1 \text{ when } n-3 \geq r \geq \\ \lceil 2n/3 \rceil -1; \zeta(K_n \setminus E(K_r)) = 3n-2r-4 \text{ when } \lceil 2n/3 \rceil -1 > r \geq n/2; \zeta(K_n \setminus E(K_r)) = 2n-4 \\ \text{when } \lceil 2n/3 \rceil -1 \geq n/2 > r \geq 2. \end{aligned}$  Moreover, they prove that if  $n \geq 5, r \geq 2$ , and  $r \neq n-1$ , then  $\sigma(K_n \setminus E(K_r)) = \zeta(K_n \setminus E(K_r)).$ 

Dou and Gao [692] prove that for  $n \ge 3$ , the fan  $F_n = P_n + K_1$  is an integral sum graph,  $\rho(F_4) = 1$ ,  $\rho(F_n) = 2$  for  $n \ne 4$ , and  $\sigma(F_4) = 2$ ,  $\sigma(F_n) = 3$  for n = 3 or  $n \ge 6$  and n even, and  $\sigma(F_n) = 4$  for  $n \ge 6$  and n odd.

Wang and Gao [2760] and [2761] determined the sum numbers and the integral sum

numbers of the complements of paths, cycles, wheels, and fans as follows:  $0 = \zeta(\overline{P_4}) < \sigma(\overline{P_4}) = 1$ ;  $1 = \zeta(\overline{P_5}) < \sigma(\overline{P_5}) = 2$ ;  $3 = \zeta(\overline{P_6}) < \sigma(\overline{P_6}) = 4$ ;  $\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 0$ , n = 1, 2, 3;  $\zeta(\overline{P_n}) = \sigma(\overline{P_n}) = 2n - 7$ ,  $n \ge 7$ .  $\zeta(\overline{C_n}) = \sigma(\overline{C_n}) = 2n - 7$ ,  $n \ge 7$ .  $\zeta(\overline{W_n}) = \sigma(\overline{W_n}) = 2n - 8$ ,  $n \ge 7$ .  $0 = \zeta(\overline{F_5}) < \sigma(\overline{F_5}) = 1$ ;  $2 = \zeta(\overline{F_6}) < \sigma(\overline{F_6}) = 3$ ;  $\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 0$ , n = 3, 4;  $\zeta(\overline{F_n}) = \sigma(\overline{F_n}) = 2n - 8$ ,  $n \ge 7$ .

Wang, Yang and Li [2765] proved:  $\zeta(K_n \setminus E(C_{n-1})) = 0$  for  $n = 4, 5, 6, 7; \ \zeta(K_n \setminus E(C_{n-1})) = 2n - 7$  for  $n \ge 8; \ \sigma(K_4 \setminus E(C_{n-1})) = 1; \ \sigma(K_5 \setminus E(C_{n-1})) = 2; \ \sigma(K_6 \setminus E(C_{n-1})) = 5; \ \sigma(K_7 \setminus E(C_{n-1})) = 7; \ \sigma(K_n \setminus E(C_{n-1})) = 2n - 7$  for  $n \ge 8$ .

Wang and Li [2763] proved: a graph with  $n \ge 6$  vertices and degree greater than (n+1)/2 is not an integral sum graph; for  $n \ge 8$ ,  $\zeta(K_n \setminus E(2P_3)) = \sigma(K_n \setminus E(2P_3)) = \epsilon(K_n \setminus E(2P_3)) = 2n - 7$ ; for  $n \ge 7$ ,  $\zeta(K_n \setminus E(K_2)) = \sigma(K_n \setminus E(K_2)) = 2n - 4$ ; and for  $n \ge 7$  and  $1 \le r \le \lceil \frac{n}{2} \rceil$ ,  $\zeta(K_n \setminus E(rK_2)) = \sigma(K_n \setminus E(rK_2)) = 2n - 5$ .

Chen [577] has given some properties of integral sum labelings of graphs G with  $\Delta(G) < |V(G)| - 1$  whereas Nicholas, Somasundaram, and Vilfred [1867] provided some general properties of connected integral sum graphs G with  $\Delta(G) = |V(G)| - 1$ . They have shown that connected integral sum graphs G other than  $K_3$  with the property that G has exactly two vertices of maximum degree are unique and that a connected integral sum graph G other than  $K_3$  can have at most two vertices with degree |V(G)| - 1 (see also [2745]).

Vilfred and Florida [2742] have examined one-point unions of pairs of small complete graphs. They show that the one-point union of  $K_3$  and  $K_2$  and the one-point union of  $K_3$  and  $K_3$  are integral sum graphs whereas the one-point union of  $K_4$  and  $K_2$  and the one-point union of  $K_4$  and  $K_3$  are not integral sum graphs. In [2743] Vilfred and Florida defined and investigated properties of maximal integral sum graphs.

Vilfred and Nicholas [2746] have shown that the following graphs are integral sum graphs: banana trees, the union of any number of stars, fans  $P_n + K_1$   $(n \ge 2)$ , Dutch windmills  $K_3^{(m)}$ , and the graph obtained by starting with any finite number of integral sum graphs  $G_1, G_2, \ldots, G_n$  and any collections of n vertices with  $v_i \in G_i$  and creating a graph by identifying  $v_1, v_2, \ldots, v_n$ . The same authors [2747] also proved that G + v where G is a union of stars is an integral sum graph.

Melnikov and Pyatkin [1740] have shown that every 2-regular graph except  $C_4$  is an integral sum graph and that for every positive integer r there exists an r-regular integral sum graph. They also show that the cube is not an integral sum graph. For any integral sum graph G, Melnikov and Pyatkin define the *integral radius of* G as the smallest natural number r(G) that has all its vertex labels in the interval [-r(G), r(G)]. For the family of all integral sum graphs of order n they use r(n) to denote maximum integral radius among all members of the family. Two questions they raise are: Is there a constant C such that  $r(n) \leq C_n$  and for n > 2, is r(n) equal to the (n-2)th prime?

The concepts of sum number and integral sum number have been extended to hypergraphs. Sonntag and Teichert [2458] prove that every hypertree (i.e., every connected, non-trivial, cycle-free hypergraph) has sum number 1 provided that a certain cardinality condition for the number of edges is fulfilled. In [2459] the same authors prove that for  $d \geq 3$  every d-uniform hypertree is an integral sum graph and that for  $n \geq d+2$  the sum number of the complete d-uniform hypergraph on n vertices is d(n-d) + 1. They also prove that the integral sum number for the complete d-uniform hypergraph on nvertices is 0 when d = n or n - 1 and is between (d - 1)(n - d - 1) and d(n - d) + 1for  $d \leq n - 2$ . They conjecture that for  $d \leq n - 2$  the sum number and the integral sum number of the complete d-uniform hypergraph are equal. Teichert [2578] proves that hypercycles have sum number 1 when each edge has cardinality at least 3 and that hyperwheels have sum number 1 under certain restrictions for the edge cardinalities. (A hypercycle  $C_n = (\mathcal{V}_n, \mathcal{E}_n)$  has  $\mathcal{V}_n = \bigcup_{i=1}^n \{v_1^i, v_2^i, \ldots, v_{d_i-1}^i\}, \mathcal{E}_n = \{e_1, e_2, \ldots, e_n\}$  with  $e_i = \{v_1^i, \ldots, v_{d_i}^i = v_1^{i+1}\}$  where i + 1 is taken modulo n. A hyperwheel  $\mathcal{W}_n = (\mathcal{V}'_n, \mathcal{E}'_n)$  has  $\mathcal{V}'_n = \mathcal{V}_n \cup \{c\} \bigcup_{i=1}^n \{v_2^{n+i}, \ldots, v_{d_{n+i}-1}^{n+i}\}, \mathcal{E}'_n = \mathcal{E}_n \cup \{e_{n+1}, \ldots, e_{2n}\}$  with  $e_{n+i} = \{v_1^{n+i} = c, v_2^{n+i}, \ldots, v_{d_{n+i}-1}^{n+i}, v_{d_{n+i}}^{n+i} = v_1^i\}$ .)

Teichert [2577] determined an upper bound for the sum number of the *d*-partite complete hypergraph  $K_{n_1,\ldots,n_d}^d$ . In [2579] Teichert defines the strong hypercycle  $\mathcal{C}_n^d$  to be the *d*-uniform hypergraph with the same vertices as  $C_n$  where any *d* consecutive vertices of  $C_n$  form an edge of  $\mathcal{C}_n^d$ . He proves that for  $n \ge 2d + 1 \ge 5$ ,  $\sigma(\mathcal{C}_n^d) = d$  and for  $d \ge 2$ ,  $\sigma(\mathcal{C}_{d+1}^d) = d$ . He also shows that  $\sigma(\mathcal{C}_5^3) = 3$ ;  $\sigma(\mathcal{C}_6^3) = 2$ , and he conjectures that  $\sigma(\mathcal{C}_n^d) < d$  for  $d \ge 4$  and  $d+2 \le n \le 2d$ .

In [1868] Nicholas and Vilfred define the *edge reduced sum number* of a graph as the minimum number of edges whose removal from the graph results in a sum graph. They show that for  $K_n$ ,  $n \ge 3$ , this number is  $(n(n-1)/2 + \lfloor n/2 \rfloor)/2$ . They ask for a characterization of graphs for which the edge reduced sum number is the same as its sum number. They conjecture that an integral sum graph of order p and size q exists if and only if  $q \le 3(p^2 - 1)/8 - \lfloor (p - 1)/4 \rfloor$  when p is odd and  $q \le 3(3p - 2)/8$  when p is even. They also define the *edge reduced integral sum number* in an analogous way and conjecture that for  $K_n$  this number is  $(n - 1)(n - 3)/8 + \lfloor (n - 1)/4 \rfloor$  when n is odd and n(n - 2)/8 when n is even.

For certain graphs G Vilfred and Florida [2741] investigated the relationships among  $\sigma(G), \zeta(G), \chi(G), \text{ and } \chi'(G)$  where  $\chi(G)$  is the chromatic number of G and  $\chi'(G)$  is the edge chromatic number of G. They prove:  $\sigma(C_4) = \zeta(C_4) > \chi(C_4) = \chi'(C_4)$ ; for  $n \ge 3$ ,  $\zeta(C_{2n}) < \sigma(C_{2n}) = \chi(C_{2n}) = \chi'(C_{2n})$ ;  $\zeta(C_{2n+1}) < \sigma(C_{2n+1}) < \chi(C_{2n+1}) = \chi'(C_{2n+1})$ ; for  $n \ge 4$ ,  $\chi'(K_n) \le \chi(K_n) < \zeta(K_n) = \sigma(K_n)$ ; and for  $n \ge 2$ ,  $\chi(P_n \times P_2) < \chi'(P_n \times P_2) = \zeta(P_n \times P_2)$ .

Alon and Scheinermann [140] generalized sum graphs by replacing the condition  $f(x) + f(y) \in S$  with  $g(f(x), f(y)) \in S$  where g is an arbitrary symmetric polynomial. They called a graph with this property a g-graph and proved that for a given symmetric polynomial g not all graphs are g-graphs. On the other hand, for every symmetric polynomial g and every graph G there is some vertex labeling such that G together with at most |E(G)| isolated vertices is a g-graph.

Boland, Laskar, Turner, and Domke [490] investigated a modular version of sum graphs. They call a graph G(V, E) a mod sum graph (MSG) if there exists a positive integer n and an injective labeling from V to  $\{1, 2, \ldots, n-1\}$  such that  $xy \in E$  if and only if  $(f(x) + f(y)) \pmod{n} = f(z)$  for some vertex z. Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [490]

have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and  $K_{2,n}$ . They further proved that  $K_p$   $(p \ge 2)$  is not MSG (see also [893]) and that  $W_4$  is MSG. They conjecture that  $W_p$  is MSG for  $p \ge 4$ . This conjecture was refuted by Sutton, Miller, Ryan, and Slamin [2556] who proved that for  $n \ne 4$ ,  $W_n$  is not MSG (the case where n is prime had been proved in 1994 by Ghoshal, Laskar, Pillone, and Fricke [893]. In the same paper Sutton et al. also showed that for  $n \ge 3$ ,  $K_{n,n}$  is not MSG. Ghoshal, Laskar, Pillone, and Fricke [893] proved that every connected graph is an induced subgraph of a connected MSG graph and any graph with n vertices and at least two vertices of degree n - 1 is not MSG.

Sutton, Miller, Ryan, and Slamin [2556] define the mod sum number,  $\rho(G)$ , of a connected graph G to be the least integer r such that  $G \cup \overline{K_r}$  is MSG. Recall the cocktail party graph  $H_{m,n}$ ,  $m, n \geq 2$ , as the graph with a vertex set  $V = \{v_1, v_2, \ldots, v_{mn}\}$  partitioned into n independent sets  $V = \{I_1, I_2, \ldots, I_n\}$  each of size m such that  $v_i v_j \in E$  for all  $i, j \in \{1, 2, \ldots, mn\}$  where  $i \in I_p$ ,  $j \in I_q$ ,  $p \neq q$ . The graphs  $H_{m,n}$  can be used to model relational database management systems (see [2552]). Sutton and Miller [2554] prove that  $H_{m,n}$  is not MSG for  $n > m \geq 3$  and  $\rho(K_n) = n$  for  $n \geq 4$ . In [2553] Sutton, Draganova, and Miller prove that for n odd and  $n \geq 5$ ,  $\rho(W_n) = n$  and when n is even,  $\rho(W_n) = 2$ . Wang, Zhang, Yu, and Shi [2796] proved that fan  $F_n(n \geq 2)$  are not mod sum graphs and  $\rho(F_n) = 2$  for even n at least 6. They also prove that  $\rho(K_{n,n}) = n$  for  $n \geq 3$ .

Dou and Gao [693] obtained exact values for  $\rho(K_{m,n})$  and  $\rho(K_m - E(K_n))$  for some cases of m and n and bounds in the remaining cases. They call a graph G(V, E) a mod integral sum graph if there exists a positive integer n and an injective labeling from V to  $\{0, 1, 2, \ldots, n-1\}$  (note that 0 is included) such that  $xy \in E$  if and only if (f(x) + f(y))(mod n) = f(z) for some vertex z. They define the mod integral sum number,  $\psi(G)$ , of a connected graph G to be the least integer r such that  $G \cup \overline{K_r}$  is a mod integral sum graph. They prove that for  $m + n \geq 3$ ,  $\psi(K_{m,n}) = \rho(K_{m,n})$  and obtained exact values for  $\psi(K_m - E(K_n))$  for some cases of m and n and bounds in the remaining cases.

Wallace [2751] has proved that  $K_{m,n}$  is MSG when n is even and  $n \ge 2m$  or when n is odd and  $n \ge 3m - 3$  and that  $\rho(K_{m,n}) = m$  when  $3 \le m \le n < 2m$ . He also proves that the complete m-partite  $K_{n_1,n_2,\ldots,n_m}$  is not MSG when there exist  $n_i$  and  $n_j$  such that  $n_i < n_j < 2n_i$ . He poses the following conjectures:  $\rho(K_{m,n}) = n$  when  $3m - 3 > n \ge m \ge 3$ ; if  $K_{n_1,n_2,\ldots,n_m}$  where  $n_1 > n_2 > \cdots > n_m$ , is not MSG, then  $(m-1)n_m \le \rho(K_{n_1,n_2,\ldots,n_m}) \le (m-1)n_1$ ; if G has n vertices, then  $\rho(G) \le n$ ; and determining the mod sum number of a graph is NP-complete (Sutton has observed that Wallace probably meant to say 'NP-hard'). Miller [1745] has asked if it is possible for the mod sum number of a graph G be of the order  $|V(G)|^2$ .

In a sum graph G, a vertex w is called a *working vertex* if there is an edge uv in G such that w = u + v. If  $G = H \cup \overline{H_r}$  has a sum labeling such that H has no working vertex the labeling is called an *exclusive sum labeling of* H with respect G. The *exclusive sum number*,  $\epsilon(H)$ , of a graph H is the smallest integer r such that  $G \cup \overline{K_r}$  has an exclusive sum labeling. The exclusive sum number is known in the following cases (see [1749] and [1758]): for  $n \geq 3$ ,  $\epsilon(P_n) = 2$ ; for  $n \geq 3$ ,  $\epsilon(C_n) = 3$ ; for  $n \geq 3$ ,  $\epsilon(K_n) = 2n - 3$ ; for

 $n \geq 4$ ,  $\epsilon(F_n) = n$  (fan of order n + 1); for  $n \geq 4$ ,  $\epsilon(W_n) = n$ ;  $\epsilon(C_3^{(n)}) = 2n$  (friendship graph-see §2.2);  $m \geq 2$ ,  $n \geq 2$ ,  $\epsilon(K_{m,n}) = m + n - 1$ ; for  $n \geq 2$ ,  $S_n = n$  (star of order n + 1);  $\epsilon(S_{m,n}) = \max\{m, n\}$  (double star);  $H_{2,n} = 4n - 5$  (cocktail party graph); and  $\epsilon$ (caterpillar G) =  $\Delta(G)$ . Dou [691] showed that  $H_{m,n}$  is not a mod sum graph for  $m \geq 3$ and  $n \geq 3$ ;  $\rho(H_{m,3}) = m$  for  $m \geq 3$ ;  $H_{m,n} \cup \rho(H_{m,n})K_1$  is exclusive for  $m \geq 3$  and  $\geq 4$ ; and  $m(n-1) \leq \rho(H_{m,n}) \leq mn(n-1)/2$  for  $m \geq 3$  and  $n \geq 4$ . Vilfred and Florida [2744] proved that  $\epsilon(P_3 \times P_3) = 4$  and  $\epsilon(P_n \times P_2) = 3$ . In [1006] Hegde and Vasudeva provide an  $O(n^2)$  algorithm that produces an exclusive sum labeling of a graph with n vertices given its adjacency matrix.

In 2001 Kratochvil, Miller, and Nguyen proved that  $\sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1$ . In 2003 Miller, Ryan, Slamin, Sugeng, and Tuga [1754] posed the problem of finding the exclusive sum number of the disjoint union of graphs. In 2010 Wang and Li [2762] proved the following. Let  $G_1$  and  $G_2$  be graphs without isolated vertices,  $L_i$  be an exclusive sum labeling of  $G_i \cup \epsilon(G_i)K_1$ , and  $C_i$  be the isolated set of  $L_i$  for i = 1 and 2. If max $C_1$  and min $C_2$  are relatively prime, then  $\epsilon(G_1 \cup G_2) \leq \epsilon(G_1) + \epsilon(G_2) - 1$ . Wang and Li also proved the following:  $\epsilon(K_{r,s}) = s + r - 1$ ;  $\epsilon(K_{r,s} - E(K_2)) = s - 1$ ; for  $s \geq r \geq 2$ ,  $\epsilon(K_{r,s} - E(rK_2)) = s + r - 3$ . For  $n \geq 5$  they prove:  $\epsilon(K_n - E(K_n)) = 0$ ;  $\epsilon(K_n - E(K_{n-1})) = n - 1$ ; for  $2 \leq r < n/2$ ,  $\epsilon(K_n - E(K_r)) = 2n - 4$ ; for  $n/2 \leq r \leq n-2$ ,  $\epsilon(K_n - E(K_r)) = 3n - 2r - 4$ , and  $\epsilon(C_n \odot K_1)$  is 3 or 4. They show that  $\epsilon(C_3 \odot K_1) = 3$  and guess that for  $n \geq 4$ ,  $\epsilon(C_n \odot K_1) = 4$ . A survey of exclusive sum labelings of graphs is given by Ryan in [2147].

If  $\epsilon(G) = \Delta(G)$ , then G is said to be an  $\Delta$ -optimum summable graph. An exclusive sum labeling of a graph G using  $\Delta(G)$  isolates is called a  $\Delta$ -optimum exclusive sum labeling of G. Tuga, Miller, Ryan, and Ryjáček [2604] show that some families of trees that are  $\Delta$ -optimum summable and some that are not. They prove that if G is a tree that has at least one vertex that has two or more neighbors that are not leaves then  $\epsilon(G) = \Delta(G)$ .

Koh, Miller, Smyth, and Wang [1385] show the following: the graphs obtained by identifying one end of a q-path with a vertex of a p-cycle are 1-optimum summable, and that two of these graphs can be joined via a new edge to create a 2-optimum summable graph; generalized  $\theta$ -graphs are 2-optimum summable;  $\theta(p, q, r)$  which consists of a pair of vertices joined by 3 independent paths of lengths p, q and r (with a few small exceptions) are 2-optimum summable; there exists a 3-optimum summable graph of order 4l + 3 for all  $l \geq 1$ ; how to construct for all  $k \geq 4$  a k-optimum summable graph; and if G is a k-optimum summable graph of order n, then  $n \geq 2k$ .

In [1096] Javaid, Khalid, Ahmad, and Imran introduce a weaker version of sum labeling of graphs as follows. Let H = (V, E) be a simple, finite, undirected graph with |V| = p. H is a weak sum graph if there exists a labeling L (called a w-sum) of the vertices of Vby distinct positive integers such that  $(u, v) \in E$  if there exists a vertex  $w \in V$  such that L(w) = L(u) + L(v). (A sum graph also requires the "only if" condition). If H is a w-sum graph with the additional constraint that the labels L all fall in the range  $1, \ldots, p$ , then His called a super weak sumgraph (sw-sumgraph). Because sumgraphs must have isolated vertices we may write  $H = G + K_{\delta}$ , where G is connected and  $K_{\delta}$  denotes  $\delta$  isolated vertices If  $\delta$  is a minimum with respect to G, we say that the sumgraph (respectively, wsumgraph, sw-sumgraph) H is  $\delta$ -optimal and that G is  $\delta$ -optimal summable (respectively, w-summable, sw-summable). Javaid et al. prove: paths are 1-optimal sw-summable; cycles are 2-optimal sw-summable; wheels are 3-optimal sw-summable;  $K_n$  is (n-1)optimal sw-summable; and  $G = K_{n_1,n_2,\ldots,n_q}$  are t-optimal sw-summable, where t is the minimum degree of any vertex in G. They also prove that for  $n \geq 5$ , the Cayley graph  $\operatorname{Cay}(\mathbb{Z}_n, \pm 1, \pm 2)$  is 4-optimal w-summable. They conjecture that all connected graphs are  $\delta$ -optimal w-summable for some  $\delta$ . See also [1385] and [1754].

Grimaldi [938] has investigated labeling the vertices of a graph G(V, E) with *n* vertices with distinct elements of the ring  $Z_n$  so that  $xy \in E$  whenever  $(x + y)^{-1}$  exists in  $Z_n$ .

In his 2001 Ph.D. thesis Sutton [2552] introduced two methods of graph labelings with applications to storage and manipulation of relational database links specifically in mind. He calls a graph  $G = (V_p \cup V_i, E)$  a  $sum^*$  graph of  $G_p = (V_p, E_p)$  if there is an injective labeling  $\lambda$  of the vertices of G with non-negative integers with the property that  $uv \in E_p$  if and only if  $\lambda(u) + \lambda(v) = \lambda(z)$  for some vertex  $z \in G$ . The  $sum^*$  number,  $\sigma^*(G_p)$ , is the minimum cardinality of a set of new vertices  $V_i$  such that there exists a sum<sup>\*</sup> graph of  $G_p$  on the set of vertices  $V_p \cup V_i$ . A mod  $sum^*$  graph of  $G_p$  is defined in the identical fashion except the sum  $\lambda(u) + \lambda(v)$  is taken modulo n where the vertex labels of G are restricted to  $\{0, 1, 2, \ldots, n-1\}$ . The mod  $sum^*$  number,  $\rho^*(G_p)$ , of a graph  $G_p$  is defined in the analogous way. Sum<sup>\*</sup> graphs are a generalization of sum graphs and mod sum<sup>\*</sup> graphs are a generalization of mod sum graphs. Sutton shows that every graph is an induced subgraph of a connected sum<sup>\*</sup> graph. Sutton [2552] poses the following conjectures:  $\rho(H_{m,n}) \leq mn$  for  $m, n \geq 2$ ;  $\sigma^*(G_p) \leq |V_p|$ ; and  $\rho^*(G_p) \leq |V_p|$ .

The following table summarizes what is known about sum graphs, mod sum graphs, sum<sup>\*</sup> graphs, and mod sum<sup>\*</sup> graphs is reproduced from Sutton's Ph. D. thesis [2552]. It was updated by J. Gallian in 2006. A question mark indicates the value is unknown. The results on sum<sup>\*</sup> and mod sum<sup>\*</sup> graphs are found in [2552].

Graph	$\sigma(G)$	$\rho(G)$	$\sigma^*(G)$	$\rho^*(G)$
$K_2 = S_1$	1	1	0	0
stars, $S_n, n \ge 2$	1	0	0	0
trees $T_n, n \ge 3$ when $T_n \ne S_n$	1	0	1	0
$C_3$	2	1	1	0
$C_4$	3	0	2	0
$C_n, n > 4$	2	0	2	0
$W_4$	4	0	2	0
$W_n, n \ge 5, n \text{ odd}$	n	n	2	0
$W_n, n \ge 6, n$ even	$\frac{n}{2} + 2$	2	2	0
fan, $F_4$ ,	2	1	1	0
fans, $F_n$ , $n \ge 5$ , $n$ odd	?	2	1	0
fans, $F_n$ , $n \ge 6$ , $n$ even	3	2	1	0
$K_n, n \ge 4$	2n - 3	n	n-2	0
cocktail party graphs, $H_{2,n}$	4n - 5	0	?	0
$C_n^{(t)}(n,t) \neq (4,1) \text{ (see §2.2)}$	2	?	?	?
K <sub>n,n</sub>	$\left\lceil \frac{4n-3}{2} \right\rceil$	$n(n \ge 3)$	?	?
$K_{m,n}, \ 2nm \ge n \ge 3$	?	n	?	?
$K_{m,n} \ m \ge 3n-3, \ n \ge 3, \ m \text{ odd}$	?	0	?	0
$K_{m,n}, m \ge 2n, n \ge 3, m$ even	?	0	?	0
$K_{m,n}, m < n$	$\left\lceil \frac{kn-k}{2} + \frac{m}{k-1} \right\rceil$	?	?	?
$k = \lceil \sqrt{1 + (8m + n - 1)(n - 1)/2} \rceil$				
$K_{n,n} - E(nK_2), \ n \ge 6$	2n-3	n-2	?	?

Table 20: Summary of Sum Graph Labelings

### 7.2 Prime and Vertex Prime Labelings<sup>3</sup>

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla [2594]. A graph with vertex set V is said to have a prime *labeling* if its vertices are labeled with distinct integers  $1, 2, \ldots, |V|$  such that for each edge xy the labels assigned to x and y are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. Little progress was made on this conjecture until 2011 when Haxell, Pikhurko, Taraz [967] proved that all large trees are prime. Also, their method allowed them to determine the smallest size of a non-prime connected order-ngraph for all large n, proving a conjecture of Rao [2099] in this range. Among the classes of trees known to have prime labelings are: paths, stars, complete binary trees, spiders (i.e., trees with one vertex of degree at least 3 and with all other vertices with degree at most 2), olive trees (i.e., a rooted tree consisting of k branches such that the *i*th branch is a path of length i), all trees of order up to 50, palm trees (i.e., trees obtained by appending identical stars to each vertex of a path), banana trees, and binomial trees (the binomial tree  $B_0$  of order 0 consists of a single vertex; the binomial tree  $B_n$  of order n has a root vertex whose children are the roots of the binomial trees of order  $0, 1, 2, \ldots, n-1$  (see [2169], [1933], [2594], [801], and [2127]). Tout, Dabboucy, and Howalla [2594] showed t-toe caterpillars (the internal vertices on the spine are regular in degree) are prime and that all caterpillars with maximum degree at most 5 are prime.

Seoud, Sonbaty, and Mahran [2255] provide necessary and sufficient conditions for a graph to be prime. They also give a procedure to determine whether or not a graph is prime. Other graphs with prime labelings include all cycles and the disjoint union of  $C_{2k}$  and  $C_n$  [663]. The complete graph  $K_n$  does not have a prime labeling for  $n \ge 4$  and  $W_n$  is prime if and only if n is even (see [1572]). Lee, Wui, and Yeh [1572] proved that friendship graphs have prime labelings. Diefenderfer et al. [679] and [678] proved that the graph obtained by identifying a vertex of  $C_n$  with an endpoint of the star  $S_m$  where  $1 \le m \le 9$ , chains of  $C_n$  where n = 4, 6, or  $8, C_n \times P_2$  where n - 1 is prime and  $n \ge 4$ , generalized books  $S_n \times P_m$  where  $3 \le m \le 7$ , and other families of uncylic graphs have prime vertex labelings.

Seoud, Diab, and Elsakhawi [2230] have shown the following graphs are prime: fans; helms; flowers (see §2.2); stars;  $K_{2,n}$ ; and  $K_{3,n}$  unless n = 3 or 7. They also shown that  $P_n + \overline{K_m}$  ( $m \ge 3$ ) is not prime. Berliner, Dean, Hook, Marr, Mbirka, and McBee give consecutive cyclic prime labelings of certain classes of ladders. Although  $K_{n,n}$  does not have a prime labeling when n > 2, Berliner et al. give minimal prime labelings for all nvalues  $1 \le n \le 23$  and give conditions on m and n for which  $K_{m,n}$  are prime. They provide specific values of n for m up to 13. Dissanayake, Abeysekara, Dhananjaya, Perera, and Ranasinghe [682] provide necessary and sufficient conditions for  $K_{1,m,n}$  to have a prime labeling.

Tout, Dabboucy, and Howalla [2594] proved that  $C_m \odot \overline{K_n}$  is prime for all m and n. Vaidya and Prajapati [2660] proved that the graphs obtained by duplication of a vertex

 $<sup>^{3}</sup>$ I am grateful to John Asplund and N. Bradley Fox for their helpful comments on the results in this section.

by a vertex in  $P_n$  and  $K_{1,n}$  are prime graphs and the graphs obtained by duplication of a vertex by an edge, duplication of an edge by a vertex, duplication of an edge by an edge in  $P_n$ ,  $K_{1,n}$ , and  $C_n$  are prime graphs. They also proved that graph obtained by duplication of every vertex by an edge in  $P_n$ ,  $K_{1,n}$ , and  $C_n$  are not prime graphs. Ghorbani and Kamali [889] proved that ladders have prime labelings.

For m and n at least 3, Seoud and Youssef [2258] define  $S_n^{(m)}$ , the (m, n)-gon star, as the graph obtained from the cycle  $C_n$  by joining the two end vertices of the path  $P_{m-2}$  to every pair of consecutive vertices of the cycle such that each of the end vertices of the path is connected to exactly one vertex of the cycle. Seoud and Youssef [2258] have proved the following graphs have prime labelings: books;  $S_n^{(m)}$ ;  $P_n + \overline{K_2}$  if and only if n = 2 or n is odd;  $C_n \odot K_1$  with a complete binary tree of order  $2^k - 1$  ( $k \ge 2$ ) attached at each pendent vertex, and that  $C_m$ -snakes are prime (see §2.2) for the definition). They also prove that every spanning subgraph of a prime graph is prime and every graph is a subgraph of a prime graph. They conjecture that all unicycle graphs have prime labelings. Diefenderfer, Hastings, Heath, Prawzinsky, Preston, White, and Whittemore [678] proved that certain families of graphs that are special cases of Seoud and Youssef's conjecture [2258] have prime labelings. Seoud and Youssef [2258] proved the following graphs are not prime:  $C_m + C_n$ ;  $C_n^2$  for  $n \ge 4$ ;  $P_n^2$  for n = 6 and for  $n \ge 8$ ; and Möbius ladders  $M_n$  for n even (see §2.3 for the definition). They also give an exact formula for the maximum number of edges in a prime graph of order n and an upper bound for the chromatic number of a prime graph.

Youssef and Elsakhawi [2888] have shown: the union of stars  $S_m \cup S_n$ , are prime; the union of cycles and stars  $C_m \cup S_n$  are prime;  $K_m \cup P_n$  is prime if and only if m is at most 3 or if m = 4 and n is odd;  $K_n \odot K_1$  is prime if and only if  $n \leq 7$ ;  $K_n \odot \overline{K_2}$  is prime if and only if  $n \leq 16$ ;  $6K_m \cup S_n$  is prime if and only if the number of primes less than or equal to m + n + 1 is at least m; and that the complement of every prime graph with order at least 20 is not prime. Michael and Youssef [1744] determined all self-complementary graphs that have prime labelings.

Salmasian [2169] has shown that every tree with n vertices  $(n \ge 50)$  can be labeled with n integers between 1 and 4n such that every two adjacent vertices have relatively prime labels. Pikhurko [1933] has improved this by showing that for any c > 0 there is an N such that any tree of order n > N can be labeled with n integers between 1 and (1 + c)n such that labels of adjacent vertices are relatively prime.

Baskar Babujee and Vishnupriya [413] proved the following graphs have prime labelings:  $nP_2$ ,  $P_n \cup P_n \cup \cdots \cup P_n$ , bistars (that is, the graphs obtained by joining the centers of two identical stars with an edge), and the graph obtained by subdividing the edge joining edge of a bistar. Baskar Babujee [395] obtained prime labelings for the graphs:  $(P_m \cup nK_1) + \overline{K_2}$ ,  $(C_m \cup nK_1) + \overline{K_2}$ ,  $(P_m \cup C_n \cup \overline{K_r}) + \overline{K_2}$ ,  $C_n \cup C_{n+1}$ ,  $(2n-2)C_{2n}$  (n >1),  $C_n \cup mP_k$  and the graph obtained by subdividing each edge of a star once. In [404] Baskar Babujee and Jagadesh prove the following graphs have prime labelings: bistars  $B_m, n; P_3 \odot K_{1,n}$ ; the union of  $K_{1,n}$  and the graph obtained from  $K_{1,n}$  by appending a pendent edge to every pendent edge of  $K_{1,n}$ ; and the graph obtained by identifying the center of  $K_{1,n}$  with the two endpoints and the middle vertex of  $P_5$ .

In [2656] Vaidya and Prajapati prove the following graphs have prime labelings: a t-ply graph of prime order; graphs obtained by joining center vertices of wheels  $W_m$  and  $W_n$  to a new vertex w where m and n are even positive integers such that m + n + 3 = pand p and p-2 are twin primes; the disjoint union of the wheel  $W_{2n}$  and a path; the graph obtained by identifying any vertex of a wheel  $W_{2n}$  with an end vertex of a path; the graph obtained from a prime graph of order n by identifying an end vertex of a path with the vertex labeled with 1 or n; the graph obtained by identifying the center vertices of any number of fans (that is, a "multiple shell"); the graph obtained by identifying the center vertices of m wheels  $W_{n_1}, W_{n_2}, \ldots, W_{n_m}$  where each  $n_i \ge 4$  is an even integer and each  $n_i$  is relatively prime to  $2 + \sum_{k=1}^{i-1} n_k$  for each  $i \in \{2, 3, \ldots, m\}$ . Prajapati and Suther [2034] provided results about the existence of prime labelings of graphs obtained from  $K_{2,n}$  by the duplication of vertices and edges. In [2012] Prajapati and Gajjar provided conditions under which the disjoint union of two graphs admit a prime labeling. They showed that  $C_{2n+1} \times P_2$  is not prime,  $W_n$  is prime if and only if  $3 \le n \le 6$ , and, for a prime  $p \geq 3$ ,  $C_{p-1} \times P_2$  is prime and a wheel graph of odd order is switching invariant. In [2013] they proved that generalized Petersen graph P(n,k) is prime then n must be even and k must be odd found some classes of generalized Petersen graphs that admit prime labelings.

The Knödel graphs  $W_{\Delta,n}$  with n even and degree  $\Delta$ , where  $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$  have vertices pairs (i, j) with i = 1, 2 and  $0 \leq j \leq n/2 - 1$  where for every  $0 \leq j \leq n/2 - 1$ and there is an edge between vertex (1, j) and every vertex  $(2, (j + 2^k - 1) \mod n/2)$ , for  $k = 0, 1, \ldots, \Delta - 1$ . Haque, Lin, Yang, and Zhao [957] have shown that  $W_{3,n}$  is prime when  $n \leq 130$ .

In [2194] Schuchter and Wilson gave evidence for their conjecture that a generalized Petersen graph P(n, k) is prime if and only if it is bipartite, which occurs for n even and k. They show that it is true for all even n and odd k such that  $n \leq 9000$  and  $1 \leq k \leq \frac{n}{2}$ . They conjectured that all cubic bipartite graphs with at least 8 vertices are prime and verified it for all such graphs G, connected or not, satisfying  $8 \leq V(G) \leq 22$ . Schroeder [2195] proved that every bipartite graph is prime except  $K_{3,3}$ . This result establishes that the generalized Petersen graph P(n, k) is prime precisely when it is bipartite, the Knődel graph  $W_{3,n}$  is prime for all even  $n \geq 4$ , and the union of any number of even cycles is prime. He also classifies precisely when 1-regular graphs and 3-regular graphs are prime, provides evidence to support a conjectured classification for r = 2, and conjectures cycle  $C_n$  has a k-prime labeling where  $n \geq 3$  and k is odd.

Sundaram, Ponraj, and Somasundaram [2540] investigated the prime labeling behavior of all graphs of order at most 6 and established that only one graph of order 4, one graph of order 5, and 42 graphs of order 6 are not prime.

Given a collection of graphs  $G_1, \ldots, G_n$  and some fixed vertex  $v_i$  from each  $G_i$ , Lee, Wui, and Yeh [1572] define Amal $\{(G_i, v_i)\}$ , the amalgamation of  $\{(G_i, v_i) | i = 1, \ldots, n\}$ , as the graph obtained by taking the union of the  $G_i$  and identifying  $v_1, v_2, \ldots, v_n$ . They proved  $Amal\{(G_i, v_i)\}$  has a prime labeling when  $G_i$  are paths and when  $G_i$  are cycles. They also showed that the amalgamation of any number of copies of  $W_n$ , n odd, with a common vertex is not prime. They conjecture that for any tree T and any vertex v from T, the amalgamation of two or more copies of T with v in common is prime. They further conjecture that the amalgamation of two or more copies of  $W_n$  that share a common point is prime when n is even  $(n \neq 4)$ . Vilfred, Somasundaram, and Nicholas [2749] have proved this conjecture for the case that  $n \equiv 2 \pmod{4}$  where the central vertices are identified.

Vilfred, Somasundaram, and Nicholas [2749] have also proved the following graphs are prime: helms;  $P_m \times P_n$  where *n* is prime,  $m \leq 3$  and  $m \leq n$ ; double fans  $P_n + \overline{K_2}$  if and only if *n* is odd; and cycles with a  $P_k$ -chord. They conjecture that  $P_m \times P_n$  where m < nand *n* is prime is prime and ladders  $P_n \times P_2$  are prime. The conjecture about grids was proved by Sundaram, Ponraj, and Somasundaram [2538]. In the same article they also showed that  $P_n \times P_n$  is prime when *n* is prime. Kanetkar [1321] proved:  $P_6 \times P_6$  is prime;  $P_{n+1} \times P_{n+1}$  is prime when *n* is a prime with  $n \equiv 3$  or 9 (mod 10) and  $(n+1)^2 + 1$  is also prime; and  $P_n \times P_{n+2}$  is prime when *n* is an odd prime with  $n \not\equiv 2 \pmod{7}$ .

Seoud, El Sonbaty, and Abd El Rehim [2231] proved that for  $m = p_{n+t-1} - (t+n)$ where  $p_i$  is the *i*<sup>th</sup> prime number in the natural order  $K_n \cup K_{t,m}$  is prime and graphs obtained from  $K_{2,n}$ ,  $(n \ge 2)$  by adding p and q edges out from the two vertices of degree n of  $K_{2,n}$  are prime. They also proved that if G is not prime, then  $G \cup K_{1,n}$  is prime if  $\pi(n+m+1) \ge m$  where m is the order of G and  $\pi(x)$  is the number of primes less than or equal to x.

Recall that  $C_n^{(k)}$  is the graph obtained from the  $k \ge 2$  copies of the cycle  $C_n$  by identifying exactly one vertex of each of these k copies of  $C_n$ . Patel and Vasava [1909] proved the following:  $C_n^{(j)} \cup C_m^{(k)}$  is a prime graph if and only if either n is even or m is even;  $C_{2n}^{(2)} \cup C_{2m}^{(2)} \cup C_k^{(2)}$  is a prime graph for all n, m and k;  $C_{2n} \cup C_{2n} \cup C_{2n} \cup C_{2n} \cup C_{2m} \cup C_k$ is a prime graph for all n, m and k; and  $G = \left(\bigcup_{k=1}^N C_{n_k}^{(2)}\right) \cup \left(\bigcup_{j=1}^M C_{m_j}^{(2)}\right)$  is not a prime graph if  $M \le N-2$  They also provided conditions for which  $G = C_{2n}^{(2)} \cup C_{2m+1}^{(2)} \cup C_{2k+1}^{(2)}$ is a prime graph.

For any finite collection  $\{G_i, u_i v_i\}$  of graphs  $G_i$ , each with a fixed edge  $u_i v_i$ , Carlson [544] defines the edge amalgamation  $Edgeamal\{(G_i, u_i v_i)\}$  as the graph obtained by taking the union of all the  $G_i$  and identifying their fixed edges. The case where all the graphs are cycles she calls generalized books. She proves that all generalized books are prime graphs. Moreover, she shows that graphs obtained by taking the union of cycles and identifying in each cycle the path  $P_n$  are also prime.

In [394] Baskar Babujee proves that the maximum number of edges in a simple graph with n vertices that has a prime labeling is  $\sum_{k=2}^{n} \phi(k)$ . He also shows that the planar graphs having n vertices and 3(n-2) edges (i.e., the maximum number of edges for a planar graph with n vertices) obtained from  $K_n$   $(n \ge 5)$  with vertices  $v_1, v_2, \ldots, v_n$  by deleting the edges joining  $v_s$  and  $v_t$  for all s and t satisfying  $3 \le s \le n-2$  and  $s+2 \le t \le n$ has a prime labeling if and only if n is odd.

By showing that for every even  $n \leq 2.468 \times 10^9$  there exists  $1 \leq s \leq n-1$  such that both n + s and 2n + s are prime, Schluchter, Schroeder, Cokus, Ellingson, Harris, Rarity, and Wilson [2193] prove the generalized Peterson graph P(n, 1) (which is isomorphic to  $C_n \times P_2$ ) is prime for all even  $4 \leq n \leq 2.468 \times 10^9$ . For a fixed *n* they also describe a method for labeling P(n, k) that is a prime labeling for multiple values of *k*. Using this method, they prove P(n,k) is prime for all even  $n \leq 50$  and odd k < n/2.

Yao, Cheng, Zhongfu, and Yao [2862] have shown: a tree of order p with maximum degree at least p/2 is prime; a tree of order p with maximum degree at least p/2 has a vertex subdivision that is prime; if a tree T has an edge  $u_1u_2$  such that the two components  $T_1$  and  $T_2$  of  $T - u_1u_2$  have the properties that  $d_{T_1}(u_1) \ge |T_1|/2$  and  $d_{T_2}(u_2) \ge |T_2|/2$ , then T is prime when  $|T_1| + |T_2|$  is prime; if a tree T has two edges  $u_1u_2$  and  $u_2u_3$ such that the three components  $T_1$ ,  $T_2$ , and  $T_3$  of  $T - \{u_1u_2, u_2u_3\}$  have the properties that  $d_{T_1}(u_1) \ge |T_1|/2$ ,  $d_{T_2}(u_2) \ge |T_2|/2$ , and  $d_{T_3}(u_3) \ge |T_3|/2$ , then T is prime when  $|T_1| + |T_2| + |T_3|$  is prime.

Vaidya and Prajapati [2657] define a vertex switching  $G_v$  of a graph G as the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex that is not adjacent to v in G. They say a prime graph Gis switching invariant if for every vertex v of G, the graph  $G_v$  obtained by switching the vertex v in G is also a prime graph. They prove:  $P_n$  and  $K_{1,n}$  are switching invariant; the graph obtained by switching the center of a wheel is a prime graph; and the graph obtained by switching a rim vertex of  $W_n$  is a prime graph if n + 1 is a prime. They also prove that the graph obtained by switching a rim vertex in  $W_n$  is not a prime graph if n + 1 is an even integer greater than 9.

Prajapati and Gajjar [2012] prove the following graphs are prime: graphs obtained from  $P_{m+1}$  and m copies of  $C_n$  by identifying each edge of  $P_{m+1}$  with an edge of a corresponding copy of  $C_n$ ; graphs obtained from  $C_m$  and m copies of  $C_n$  by identifying each edge of  $C_m$  with an edge of corresponding copy of  $C_n$ ; for a prime  $p \ge 3$  and p-2copies of  $C_{p+1}$ , the graph obtained by identifying one vertex of each copy of  $C_{p+1}$  with corresponding pendent vertex of  $K_{1,p-2}$ ; for a prime  $p \ge 3$ ,  $C_{p-1} \times P_2$ ; and for a prime  $p \ge 3$ , the graphs obtained by joining every rim vertex of a wheel graph  $W_{p-1}$  with the corresponding vertex of  $C_{p-1}$ . They also prove that the complement of  $W_n$  is prime if and only if  $3 \le n \le 6$ ; for odd  $n \ge 3$   $C_n \times P_2$  is not prime; and  $W_{2n}$  is switching invariant.

Selvaraju and Moha [2205] proved that the one-point union of any number of cycles and the one-point union of any number of wheels at the center are prime graphs. Haque, Xiaohui, Yuansheng, and Pingzhong proved that the generalized Petersen graph P(n,k)is prime for all even  $n \leq 2500$  when k = 1 [954] and for all even  $n \leq 100$  when k = 3[956]. They show P(n,3) is not prime for odd n and conjecture that P(n,3) are prime for all even n.

In [2236] Seoud, El-Sonbaty, and Mahran discuss the primality of some corona graphs  $G \odot H$  and conjecture that  $K_n \odot \overline{K_m}$  is prime if and only if  $n \leq \pi(nm+n) + 1$ , where  $\pi(x)$  is the number of primes less than or equal to x. For  $m \leq 20$  they give the exact values of n for which  $K_n \odot \overline{K_m}$  is prime. They also show that  $K_{m,n}$  is prime if and only if  $\min\{m,n\} \leq \pi(m+n) - \pi((m+n)/2) + 1$ .

Klee, Lehmann, and Park [1382] we extended the notion of prime labeling to the Gaussian integers. They showed that paths, stars, spiders, graphs obtained by joining the centers of two stars with a path, and some firecrackers admit Gaussian prime labelings.

The Prime Ladder Conjecture states that every ladder  $P_n \times P_2$  is prime. This was proved by Dean [652] in 2017. He conjectures that every integer  $n \ge 50$  has a canonical

partition with at most three terms and he states that this conjecture was verified by computer up to 5,000,000.

Lau, Chu, Suhadak, Foo, and Ng [1465] introduced SD-prime cordial labelings as follows. Given a finite, simple graph G with n vertices and a bijection  $f: V(G) \rightarrow G$  $\{1, 2, \ldots, n\}$ , for each edge uv let S = f(u) + f(v) and D = |f(u) - f(v)|. For each edge uv define f' induced by f by assigning f'(uv) = 1 if gcd(S, D) = 1 and f'(uv) = 0otherwise. Then f' is said to be SD-prime cordial if f'(uv) = 1 for all edges uv. They provide results about paths, complete bipartite graphs, stars, double stars, wheels, fans, double fans, ladders, and grids. They conjecture that  $P_m \times P_n$  is SD-prime for all  $m \ge 2$ and  $n \geq 2$ . Lau, Shiu, Ng, and Jeyanthi [1470] give sufficient conditions for a theta graph to have an SD-prime cordial labeling, provide a way to construct new SD-prime cordial graphs from existing ones, and investigate SD-primae cordialness of some general graphs. Lourdusamy and Patrick [1652] provide a way to construct SD-prime cordial graphs from an existing graph G with an SD-prime cordial labeling by identifying a vertex of G having a particular label with a vertex of maximum degree of a star or fan or with an endpoint of a path. In [1653] Lourdusamy and Patrick investigated SD-prime cordial labelings of subdivision graphs, splitting graphs, shadow graphs of stars and bistars,  $T(P_n), T(C_n)$ , the graph obtained by duplication of each vertex of a path and a cycle by an edge,  $Q_n$ ,  $A(T_n)$ , triangular ladders,  $P_n \odot K_1$ ,  $C_n \odot K_1$ , and jewel graphs.

The following definitions appear in [1976], [1963], [1964], and [1965]. A double triangular snake  $DT_n$  consists of two triangular snakes that have a common path; a double quadrilateral snake  $DQ_n$  consists of two quadrilateral snakes that have a common path; an alternate triangular snake  $A(T_n)$  is the graph obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertex  $v_i$  (that is, every alternate edge of a path is replaced by  $C_3$ ); a double alternate triangular snake  $DA(T_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to two new vertices  $v_i$  and  $w_i$ ; an alternate quadrilateral snake  $A(Q_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertices  $v_i$  and  $w_i$  respectively and then joining  $v_i$  and  $w_i$  (that is, every alternate edge of a path is replaced by a cycle  $C_4$ ); a double alternate quadrilateral snake  $DA(Q_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$ (alternatively) to new vertices  $v_i$ ,  $x_i$  and  $w_i$  and  $y_i$  respectively and then joining  $v_i$  and  $w_i$  and  $x_i$  and  $y_i$ .

Prajapati and Vantiya [2035] proved that the following snake graphs have SD-prime cordial labelings: triangular (except for n = 3), alternate triangular, quadrilateral, alternate quadrilateral, double triangular, double alternate triangular, double quadrilateral, and double alternate quadrilateral. Lourdusamy, Wency, and Patrick [1661] proved that the union of stars and paths, subdivision of combs, subdivision of ladders, and the graph obtained by attaching a star at one end of a path are SD-prime graphs. They proved that the union of two SD-prime cordial graphs need not be SD-prime cordial graphs. Also, they proved that given a positive integer n, there is SD-prime cordial graph with n vertices.

Vaidya and Prajapati [2656] have introduced the notion of k-prime labeling. A k-prime labeling of a graph G is an injective function  $f: V(G) \to \{k+1, k+2, k+3, \ldots, k+|V(G)|-1\}$  for some positive integer k that induces a function  $f^+$  on the edges of G

defined by  $f^+(uv) = \gcd(f(u), f(v))$  such that  $\gcd(f(u), f(v)) = 1$  for all edges uv. A graph that admits a k-prime labeling is called a k-prime graph. They prove the following are prime graphs: a tadpole (that is, a graph obtained by identifying a vertex of a cycle to an end vertex of a path); the union of a prime graph of order n and a (n + 1)-prime graph; the graph obtained by identifying the vertex labeled with n in an n-prime graph with either of the vertices labeled with 1 or n in a prime graph of order n.

A dual of prime labelings has been introduced by Deretsky, Lee, and Mitchem [663]. They say a graph with edge set E has a vertex prime labeling if its edges can be labeled with distinct integers  $1, \ldots, |E|$  such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee, and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs;  $C_{2k} \cup C_n$ ;  $C_{2m} \cup C_{2n} \cup C_{2k+1}$ ;  $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$ ; and  $5C_{2m}$ . They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Let  $G = \bigcup_{i=1}^{t} C_{2n_i}$  and  $N = \sum_{i=1}^{t} n_i$ . In [493] Borosh, Hensley and Hobbs proved that there is a positive constant  $n_0$  such that the conjecture of Deretsky et al. is true for the following cases: G is the disjoint union of at most seven cycles; G is a union of cycles all of the same even length 2n where  $n \leq 150\,000$  or where  $n \ge n_0; n_i \ge (\log N)^{4\log \log \log n}$  for all  $i = 1, \ldots, t$ ; and when each  $C_{2n_i}$  is repeated at most  $n_i$  times. They end their paper with a discussion of graphs whose components are all even cycles, and of graphs with some components that are not cycles and some components that are odd cycles.

In [223] Bapat proved the following graphs have vertex prime labelings: kayak paddles KP(k, m, l); books; irregular books not necessarily with pages of the same size; triangular snakes; *m*-fold triangular snakes of length *n* obtained from a path  $v_1, v_2, \ldots, v_n, v_{n+1}$  by joining  $v_i$  and  $v_{i+1}$  to new *m* vertices  $w_1^i, w_2^i, \ldots, w_m^i$ , for  $i = 1, 2, \ldots, n$  giving edges  $v_i w_j^i$  and  $w_j^i v_{i+1}$ , for  $j = 1, \ldots, m$ ,  $i = 1, 2, \ldots, n$ ; *m*-fold petal sunflowers obtained from a cycle  $v_1, v_2, \ldots, v_n$  by joining  $v_i$  and  $v_{i+1}$  to new *m* vertices  $w_1^i, w_2^i, \ldots, w_m^i$ , for  $i = 1, 2, \ldots, n$  giving edges  $v_i w_j^i$  and  $w_j^i v_{i+1}$  for  $j = 1, \ldots, m$ ,  $i = 1, 2, \ldots, n$ ; *m*-fold petal sunflowers obtained from a cycle  $v_1, v_2, \ldots, v_n$  by joining  $v_i$  and  $v_{i+1}$  to new *m* vertices  $w_1^i, w_2^i, \ldots, w_m^i$ , for  $i = 1, 2, \ldots, n$  giving edges  $v_i w_j^i$  and  $w_j^i v_{i+1}$  for  $j = 1, \ldots, m$ ,  $i = 1, 2, \ldots, n$  ( $v_{n+1} = v_1$ ); and one-point unions of cycles not necessarily of the same length.

A bijection f from V(G to  $\{1, 2, \ldots, |V| + |E|\}$  is said to be a *total prime* if for each edge uv the labels assigned to u and v are relatively prime and for each vertex of degree at least 2, the labels on the incident edges are relatively prime. A graph that admits a total prime labeling is called a *total prime graph*. In [2022] Prajapati and Gajjar defined a *braided star* graph as follows. Let  $a_0$  be the apex vertex and  $a_1, a_2, \ldots, a_{n-1}, a_n$  be consecutive n rim vertices of  $W_n$ ,  $(n \geq 3)$ . Let  $b_1, b_2, b_3, \ldots, b_{2n-1}, b_{2n}$  be consecutive 2nvertices of  $C_{2n}$  (n > 1); and let  $c_1, c_2, c_3, \ldots, c_{2n-1}, c_{2n}$  be consecutive 2n vertices of a second copy of  $C_{2n}$ . Join each  $a_i$  to  $b_{2i-1}$  by an edge and  $b_{2i}$  to  $c_{2i}$  by an edge. For each i, join a new vertices  $d_i$  to each  $c_{2i-1}$  and  $c_{2i+1}$  by an edge taking the subscripts modulo n. They proved that braided stars are prime, total prime, and vertex prime.

Jothi [1264] calls a graph G highly vertex prime if its edges can be labeled with distinct integers  $\{1, 2, ..., |E|\}$  such that the labels assigned to any two adjacent edges are relatively prime. Such labeling is called a *highly vertex prime* labeling. He proves: if G is highly vertex prime then the line graph of G is prime; cycles are highly vertex prime; paths are highly vertex prime;  $K_n$  is highly vertex prime if and only if  $n \leq 3$ ;  $K_{1,n}$  is highly vertex prime if and only if  $n \leq 2$ ; even cycles with a chord are highly vertex prime;  $C_p \cup C_q$  is not highly vertex prime when both p and q are odd; and crowns  $C_n \odot K_1$  are highly vertex prime.

For a finite simple graph G(V, E) with n vertices and  $v \in V$  let N(v) denote the open neighborhood of v. Patel and Shrimali [1906] say a bijective function  $f :\to \{1, 2, 3, \ldots, n\}$ is a *neighborhood-prime labeling* of G, if for every vertex  $v \in V$  with deg(v) > 1,  $gcd \{f(u) : u \in N(v)\} = 1$ . A graph that admits a neighborhood-prime labeling is called a *neighborhood-prime* graph. In [1906], [1907], and [1908] they prove the following graphs have a prime-neighborhood labeling: graphs with a vertex of degree |V| - 1; paths;  $C_n$ if and only if  $n \not\equiv 2 \pmod{4}$ ; helms; closed helms; flowers; graphs obtained by the duplication of an arbitrary vertex of cycle or path;  $G_1 + G_2$  where each of  $G_1$  and  $G_2$  have at least 2 vertices;  $C_n \cup C_m$  is a neighborhood-prime graph if and only if  $n \equiv 0 \pmod{4}$ and  $m \equiv 0 \pmod{4}$ , or  $n \equiv 0 \pmod{4}$  and  $m \equiv 1 \pmod{2}$ ;  $W_m \cup W_n$ ; the union of a finite number of paths;  $P_m \times P_n$ ; and the tensor product of two paths of the same order. They also prove that if G is neighborhood-prime graph and v is a vertex in G that is not adjacent to any pendent vertices, then the graph obtained by duplicating the vertex v is neighborhood-prime [1906].

Patel [1903] showed that the generalized Petersen graph P(n,k) is neighborhoodprime when the greatest common divisor of n and k is 1, 2, or 4 and that P(n, 8) is neighborhood-prime for all n. Rozario Raj and Sheriff [2146] gave neighborhood-prime labelings for books with triangular and rectangle pages. Shrimali, Rathod, and Vihol [2376] proved the following graphs are neighborhood-prime: graphs obtained from the helm  $H_n$  by identifying each vertex of degree 1 with a vertex of  $W_n$ , graphs obtained from the helm  $H_n$  by identifying each vertex of degree 1 with a vertex of the fan  $F_n$  graphs obtained by identifying each pendent vertex of  $H_n$  with a vertex of outer cycle of closed helm of  $H_n$ , and graphs obtained by identifying each pendent vertex of  $H_n$  by vertex of outer cycle of the Petersen graph.

Let G(V, E) be a graph with p vertices and Q edges. Rajesh Kumar and Mathew Varkey [2073] call a bijection from  $V(G) \cup E(G)$  to  $\{1, 2, \ldots, p+q\}$  an total neighborhood prime labeling if for for each vertex of degree at least two, the gcd of labeling on its neighborhood vertices is 1 and for each vertex of degree at least two, the gcd of labeling on the induced edges is 1. They proved that paths, combs, and  $C_{4n+2}$  are total neighborhood prime graphs. Shrimali and Pandya [2374] proved that the following graphs have total neighborhood prime labelings: combs  $P_n \odot K_1$ ,  $P_m \cup P_n$ ,  $(P_m \odot K_1) \cup (P_n \odot K_1)$ ,  $W_m \cup W_n$ , graphs obtained from a copy of  $P_n$  and n copies  $K_{1,n}$  by joining the *i*th vertex of  $P_n$  with an edge to the center vertex of the *i*th copy of  $K_{1,m}$ ,  $C_n \odot mK_1$ , and subdivisions of bistars  $B_{m,n}$ . Shrimali, Rathod, and Vihol [2376] proved that following graphs are neighborhoodprime graphs: the graph obtained by identifying each pendent vertex of a helm  $H_n$  with a rim vertex of the wheel  $W_n$ ; the graph obtained by identifying each pendent vertex of a helm  $H_n$  with a vertex of maximum degree of the fan  $P_n + K_1$ ; and the graph obtained by identifying each pendent vertex of  $H_n$  with a vertex of outer cycle of closed helm graph  $\overline{H_n}$ .

In [2072] Rajesh Kumar and Mathew Varkey extend the neighborhood prime labeling concept to Gaussian integers. Using the spiral order on the Gaussian integers, they showed the following graphs have Gaussian neighborhood prime labelings: graphs obtained by connecting the centers of two stars with a path, combs  $P_n \odot K_1$ , spiders,  $C_n$  where  $n \neq 2$ mod 4,  $C_4$  ( $n \geq 4$ ) with a cord, graphs obtained by switching of any vertex  $C_n$ , and graphs obtained by duplicating arbitrary vertex of  $C_n$ .

In [1887] Pandya and Shrimali introduce a vertex-edge neighborhood prime labeling of graphs as follows. For a graph G an injective function  $f : V(G) \cup E(G) \rightarrow$  $\{1, 2, \ldots, |V(G)| + |E(G)|\}$  is said to be a vertex-edge neighborhood prime labeling if it has the following properties: If u has degree 1, then  $gcd\{f(w), f(uw)\} = 1$  taken over all vertices w adjacent to u; if u has degree greater than 1, then  $gcd\{f(w)\} = 1$  taken over all vertices w adjacent to u and  $gcd\{f(wu)\} = 1$  taken over all vertices w adjacent to u. A graph that admits vertex-edge neighborhood prime labeling is called a vertexedge neighborhood prime graph. They give vertex-edge neighborhood prime labelings for paths, helms,  $C_n \oplus K_1$ , bistars, the central edge subdivision of bistars, and subdivisions of edges of bistars. They observe that every vertex-edge neighborhood prime graph is a total neighborhood prime graph and that a total neighborhood prime graph that does not have a vertex of degree 1 is vertex-edge neighborhood prime.

Patel and Ghodasara [1904] proved the following graphs are neighborhood-prime: one point union  $C_n^{(k)}$   $(k \ge 2, n \ge 3)$  of k copies of cycle  $C_n$ , the barycentric subdivision of wheels and gears, the middle and total graph of crowns  $C_n \odot K_1$   $(n \ge 3)$ , the square of crowns, tadpoles T(n,l)  $(n \ge 3, l \ge 1)$ , cycles, and umbrellas. In [656] Delman, Koilraj, and Raj gave neighborhood-prime labelings of arbitrary super subdivision of helms, tadpoles, and triangular snakes.

In [2032] Prajapati and Shah introduce an odd prime labeling as follows. Let G(V, E) be a graph. A bijection f from V to  $\{1, 3, \ldots, 2|V| - 1\}$  is called an *odd prime labeling* if for each edge uv, gcd(f(u), f(v)) = 1. A graph that admits odd prime labeling is called an *odd prime graph*. They prove paths, ladders, complete bipartite graphs, wheels, gears, flowers, helms, closed helms, and generalized Petersen graphs P(n, 2) are odd prime and conjecture that generalized Petersen graphs P(n,k) and every prime graph is an odd prime graph. In [2033] they proved the following graphs are odd prime graphs: graphs obtained by duplication of a vertex of paths, stars, and wheels, and graphs obtained by duplication of an edge of cycles, stars, and wheels.

For a graph G(V, E) with p vertices and q edges Shiu, Lau, and Lee [2350] call a bijection f from E to  $\{1, 2, \ldots, q\}$  an *edge-prime labeling* if for each edge uv in E, we have  $gcd(f^+(u), f^+(v)) = 1$ , where  $f^+(u) = \Sigma f(uw)$  over all  $uw \in E$ . A graph that admits an edge-prime labeling is called an *edge-prime graph*. A bijection f from E to  $\{1, 2, \ldots, q\}$ is an *semi-edge-prime labeling* if for each edge uv in E, we have  $gcd(f^+(u), f^+(v)) = 1$  or  $f^+(u) = f^+(v)$ . They obtained a necessary and sufficient condition for the disjoint union of paths to be edge-prime, proved that all 2-regular graphs are edge-prime, proved that many bipartite and tripartite graphs are edge-prime (or not edge-prime), and showed that certain bipartite and tripartite graphs are semi-edge-prime graphs. In [1468] Lau, Lee, and Shiu proved that if G is a cubic graph and every component is of order 4, 6 or 8, then G is edge-prime if and only if  $G \not\approx K_4$  or  $nK_{3,3}$  and n = 2 or 3 (mod 4). They conjectured that a connected cubic graph G is not edge-prime if and only if  $G \approx K_4$ .

In [1092] Jagadesh and Baskar Babujee introduced an *edge vertex prime* labeling of a graph G as an injection f from  $V(G) \cup E(G)$  to  $\{1, 2, \dots, |V(G)| + |E(G)|\}$  such that for every edge uv, the labels f(u), f(v), and f(uv) are pairwise relatively prime. A graph that admits such a labeling is called an *edge vertex prime* graph. They proved that paths, cycles, and stars are edge vertex prime. In [1896] and [1897] Parmar proved that wheels, fans, friendship graphs, and  $K_{2,n}$  are edge vertex prime. Simaringa and Muthukumaran [2388] proved that following graphs have edge vertex prime labelings: triangular and rectangular books, butterfly graphs,  $\overline{K_n} \cup K_{1,m}$ ,  $K_{1,m} + K_1$ ,  $\overline{K_m} \cup \overline{K_n}$ , Jahangir graphs  $J_{n,3}$  and  $J_{n,4}$ . In [2389] Simaringa and Muthukumaran investigated the existence of edge vertex prime labelings for crowns, unions of cycles, and wheel relate graphs. Shrimali and Parmar [2375] proved that the following graphs have edge vertex prime labelings: bistars B(m, n), n-centipede trees, coconut trees obtained from the path  $P_n$  by appending m new pendent edges at an end vertex of  $P_n$ , double coconut trees (graphs obtained by attaching n > 1 pendent vertices to one end of the path  $P_r$  and m > 1 pendent vertices to the other end of path  $P_r$ ), and special classes of banana trees and fire crackers.

A {coprime labeling of vertices of a graph G with distinct labels from the set  $\{1, \ldots, m\}$  for some integer  $m \ge n$  such that adjacent labels are relatively prime. The minimum value m for which G has a coprime labeling is defined as the minimum coprime number, denoted by pr(G), and a coprime labeling of G with largest label being pr(G) is called a minimum coprime labeling of G. Obviously, if G is a prime graph of order n, then pr(G) = n. In [208] Asplund and Fox focus on the problem of determining the minimum coprime number for graphs that have been shown to not be prime. Among them are  $K_n$   $(n \ge 4)$ ,  $W_{2n+1}$ , and the union of odd cycles. C. Lee [1479] determined the minimum coprime number of coronas of complete graphs with empty graphs, the joins of two paths, and prisms. She also proved that gears are prime, double wheels  $DW_n$  are prime if and only if n is even, and the graph that obtained by attaching  $P_2$  to each vertex of  $C_n$  followed by attaching the star  $S_m$  at its center to each pendent vertex is prime.

The tables following summarize the state of knowledge about prime labelings and vertex prime labelings. In the table, **P** means prime labeling exists, and **VP** means vertex prime labeling exists. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.
Graph	Types	Notes
$P_n$	Р	[801]
stars	Р	[801]
complete binary trees	Р	[801]
spiders	Р	[801]
trees	P?	[1572]
$C_n$	Р	[663]
$C_n \cup C_{2m}$	Р	[663]
$K_n$	Р	iff $n \le 3$ [1572]
$W_n$	Р	iff $n$ is even [2594]
helms	Р	[2230]
fans	Р	[2230]
flowers	Р	[2230]
$K_{2,n}$	Р	[2230]
$K_{3,n}$	Р	$n \neq 3,7$ [2230]
$P_n + \overline{K_m}$	not P	$n \ge 3$ [2230]
$P_n + \overline{K_2}$	Р	iff $n = 2$ or $n$ is odd [2230]
books	Р	[2258]
$C_m + C_n$	not P	[2258]
$C_n^2$	not P	$n \ge 4$ [2258]
$P_n^2$	not P	$n \ge 6, \ n \ne 7 \ [2258]$

# Table 21: Summary of Prime Labelings

Continued on next page

Graph	Types	Notes
$M_n$ (Möbius ladders)	not P	n  even  [2258]
$S_m \cup S_n$	Р	[2888]
$C_m \cup S_n$	Р	[2888]
$K_m \cup S_n$	Р	iff no. of primes $\leq m + n + 1$ is at least $m$ [2888]
$K_n \odot K_1$	Р	iff $n \le 7$ [2888]
$P_m \times P_n$ (grids)	Р	$m \le 3, \ m > n, \ n \text{ prime } [2749]$
$C_n \odot \overline{K_i} $ (crowns)	Р	[2594]
$P_n \odot \overline{K_2}$	Р	iff $n \neq 2$ [2749]
$C_m$ -snakes (see §2.2)	Р	[544]
unicyclic	P?	[2230]

Table 21 – Continued from previous page

Table 22: Summary of Vertex Prime Labelings

Graph	Types	Notes
$C_m + C_n$	not P	[2258]
$C_n^2$	not P	$n \ge 4$ [2258]
$P_n$	not P	$n = 6, \ n \ge 8 \ [2258]$
$M_{2n}$ (Möbius ladders)	not P	[2258]
connected graphs	VP	[663]
forests	VP	[663]

Continued on next page

Graph	Types	Notes
$C_{2m} \cup C_n$	VP	[663]
$C_{2m} \cup C_{2n} \cup C_{2k+1}$	VP	[663]
$C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$	VP	[663]
$5C_{2m}$	VP	[663]
$G \cup H$	VP	if $G$ , $H$ are connected and one is not an odd cycle [663]
2-regular graph G	not VP	G has at least 2 odd cycles
	VP?	iff $G$ has at most 1 odd cycle [663]

Table 22 – Continued from previous page

## 7.3 Edge-graceful Labelings

In 1985, Lo [1626] introduced the notion of edge-graceful graphs. A graph G(V, E) is said to be *edge-graceful* if there exists a bijection f from E to  $\{1, 2, \ldots, |E|\}$  such that the induced mapping  $f^+$  from V to  $\{0, 1, \dots, |V| - 1\}$  given by  $f^+(x) = (\sum f(xy)) \pmod{|V|}$ taken over all edges xy is a bijection. Note that an edge-graceful graph is antimagic (see  $\S6.1$ ). A necessary condition for a graph with p vertices and q edges to be edge-graceful is that  $q(q+1) \equiv p(p+1)/2 \pmod{p}$ . Lee [1489] notes that this necessary condition extends to any multigraph with p vertices and q edges. It was conjectured by Lee [1489] that any connected simple (p,q)-graph with  $q(q+1) \equiv p(p-1)/2 \pmod{p}$  vertices is edge-graceful. Lee, Kitagaki, Young, and Kocay [1495] prove that the conjecture is true for maximal outerplanar graphs. Lee and Murthy [1481] proved that  $K_n$  is edge-graceful if and only if  $n \not\equiv 2 \pmod{4}$ . (An edge-graceful labeling given in [1626] for  $K_n$  for  $n \not\equiv 2 \pmod{4}$  is incorrect.) Lee [1489] notes that a multigraph with  $p \equiv 2 \pmod{4}$ vertices is not edge-graceful and conjectures that this condition is sufficient for the edgegracefulness of connected graphs. Lee [1488] has conjectured that all trees of odd order are edge-graceful. Small [2429] has proved that spiders for which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [1365] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low, and Mitchem [525] have shown that regular spiders of odd order are edge-graceful. For a (p,q)connected edge-graceful graph G with q = kp + r, where k is an integer and  $0 \le r < p$ . Kayathri and Amutha [1357] proved that every edge-graceful labeling f of G induces  $((k+1)!)^r (k!)^{p-r}$  edge-graceful labelings of G.

Lee and Seah [1534] have shown that  $K_{n,n,\dots,n}$  is edge-graceful if and only if n is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [1533] have also proved that  $C_n^k$  (the kth power of  $C_n$ ) is edge-graceful for  $k < \lfloor n/2 \rfloor$  if and only if n is odd and  $C_n^k$  is edge-graceful for  $k \ge \lfloor n/2 \rfloor$  if and only if  $n \not\equiv 2 \pmod{4}$  (see also [525]). Lee, Seah, and Wang [1539] gave a complete characterization of edge-graceful  $P_n^k$ graphs. Shiu, Lam, and Cheng [2343] proved that the composition of the path  $P_3$  and any null graph of odd order is edge-graceful. Uma and Mazuda Shanofer [2605] prove that  $C_{2n+1} \odot \overline{K_2}$  is edge graceful and the graphs obtained by starting with  $C_n$  and, for each edge of  $C_n$ , adjoining a copy of  $C_n$  that shares an edge with the starting copy (the flower graph  $FL_n$ ) is not edge-graceful.

Lo [1626] proved that all odd cycles are edge-graceful and Wilson and Riskin [2815] proved the Cartesian product of any number of odd cycles is edge-graceful. Lee, Ma, Valdes, and Tong [1509] investigated the edge-gracefulness of grids  $P_m \times P_n$ . The necessity condition of Lo [1626] that a (p,q) graph must satisfy  $q(q+1) \equiv 0$  or  $p/2 \pmod{p}$  severely limits the possibilities. Lee et al. prove the following:  $P_2 \times P_n$  is not edge-graceful for all n > 1;  $P_3 \times P_n$  is edge-graceful if and only if n = 1 or n = 4;  $P_4 \times P_n$  is edge-graceful if and only if n = 3 or n = 4;  $P_5 \times P_n$  is edge-graceful if and only if n = 1;  $P_{2m} \times P_{2n}$  is edge-graceful if and only if m = n = 2. They conjecture that for all  $m, n \ge 10$  of the form m = (2k+1)(4k+1), n = (2k+1)(4k+3), the grids  $P_m \times P_n$  are edge-graceful. Riskin and Weidman [2123] proved: if G is an edge-graceful 2r-regular graph with p vertices and q edges and (r, kp) = 1, then kG is edge-graceful when k is odd; when n and k are odd,  $kC_n^r$  is edge-graceful; and if G is the cartesian product of an odd number of odd cycles and k is odd, then kG is edge-graceful. They conjecture that the disjoint union of an odd number of copies of a 2r-regular edge-graceful graph is edge-graceful.

Shiu, Lee, and Schaffer [2353] investigated the edge-gracefulness of multigraphs derived from paths, combs, and spiders obtained by replacing each edge by k parallel edges. Lee, Ng, Ho, and Saba [1519] construct edge-graceful multigraphs starting with paths and spiders by adding certain edges to the original graphs. Lee and Seah [1535] have also investigated edge-gracefulness of various multigraphs.

Lee and Seah (see [1489]) define a sunflower graph SF(n) as the graph obtained by starting with an n-cycle with consecutive vertices  $v_1, v_2, \ldots, v_n$  and creating new vertices  $w_1, w_2, \ldots, w_n$  with  $w_i$  connected to  $v_i$  and  $v_{i+1}$  ( $v_{n+1}$  is  $v_1$ ). In [1536] they prove that SF(n) is edge-graceful if and only if n is even. In the same paper they prove that  $C_3$ is the only triangular snake that is edge-graceful. Lee and Seah [1533] prove that for  $k \leq n/2$ ,  $C_n^k$  is edge-graceful if and only if n is odd, and for  $k \geq n/2$ ,  $C_n^k$  is edge-graceful if and only if  $n \neq 2 \pmod{4}$ . Lee, Seah, and Lo (see [1489]) have proved that for n odd,  $C_{2n} \cup C_{2n+1}, C_n \cup C_{2n+2}$ , and  $C_n \cup C_{4n}$  are edge-graceful. They also show that for odd k and odd n,  $kC_n$  is edge-graceful. Lee and Seah (see [1489]) prove that the generalized Petersen graph P(n,k) (see Section 2.7 for the definition) is edge-graceful if and only if n is even. Schaffer and Lee [2192] proved that  $C_m \times C_n$  (m > 2, n > 2) is edge-graceful if and only if m and n are odd. They also showed that if G and H are edge-graceful regular graphs of odd order then  $G \times H$  is edge-graceful and that if G and H are edge-graceful graphs where G is c-regular of odd order m and H is d-regular of odd order n, then  $G \times H$ is edge-magic if gcd(c, n) = gcd(d, m) = 1. They further show that if H has odd order, is 2d-regular and edge-graceful with gcd(d, m) = 1, then  $C_{2m} \times H$  is edge-magic, and if G is odd-regular, edge-graceful of even order m that is not divisible by 3, and G can be partitioned into 1-factors, then  $G \times C_m$  is edge-graceful.

In 1987 Lee (see [1537]) conjectured that  $C_{2m} \cup C_{2n+1}$  is edge-graceful for all m and n except for  $C_4 \cup C_3$ . Lee, Seah, and Lo [1537] have proved this for the case that m = n and m is odd. They also prove: the disjoint union of an odd number copies of  $C_m$  is edge-graceful when m is odd;  $C_n \cup C_{2n+2}$  is edge-graceful; and  $C_n \cup C_{4n}$  is edge-graceful for n odd. Bu [506] gave necessary and sufficient conditions for graphs of the form  $mC_n \cup P_{n-1}$  to be edge-graceful.

Kendrick and Lee (see [1489]) proved that there are only finitely many n for which  $K_{m,n}$  is edge-graceful and they completely solve the problem for m = 2 and m = 3. Ho, Lee, and Seah [1015] use  $S(n; a_1, a_2, \ldots, a_k)$  where n is odd and  $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k < n/2$  to denote the (n, nk)-multigraph with vertices  $v_0, v_1, \ldots, v_{n-1}$  and edge set  $\{v_i v_j | i \neq j, i - j \equiv a_t \pmod{n}$  for  $t = 1, 2, \ldots, k\}$ . They prove that all such multigraphs are edge-graceful. Lee and Pritikin (see [1489]) prove that the Möbius ladders (see §2.2 for definition) of order 4n are edge-graceful. Lee, Tong, and Seah [1555] have conjectured that the total graph of a (p, p)-graph is edge-graceful if and only if p is even. They have proved this conjecture for cycles. In [1372] Khodkar and Vinhage proved that there exists a super edge-graceful labeling of the total graph of  $K_{1,n}$  and the total graph of  $C_n$ . Wang and Zang [2794] proved that a regular graph of odd degree is edge-graceful if it contains either a quasi-prism factor or a claw factor.

Kuang, Lee, Mitchem, and Wang [1442] have conjectured that unicyclic graphs of odd order are edge-graceful. They have verified this conjecture in the following cases: graphs obtained by identifying an endpoint of a path  $P_m$  with a vertex of  $C_n$  when m + n is even; crowns with one pendent edge deleted; graphs obtained from crowns by identifying an endpoint of  $P_m$ , m odd, with a vertex of degree 1; amalgamations of a cycle and a star obtained by identifying the center of the star with a cycle vertex where the resulting graph has odd order; graphs obtained from  $C_n$  by joining a pendent edge to n - 1 of the cycle vertices and two pendent edges to the remaining cycle vertex.

In [2795] Wang and Zhang introduced the notion called edge-graceful deficiency, which is a parameter to measure how close a graph is away from being an edge-graceful graph. The *edge-graceful deficiency* of a graph G is the minimum value of k such that the edge labeling  $f E \rightarrow \{1, 2, \ldots, q+k\}$  is edge-graceful. They proved that an odd regular graph is edge-graceful if it contains a quasi-prism factor or a claw factor and completely determine the edge-graceful deficiency of Hamiltonian regular graphs of even degree.

Gayathri and Subbiah [861] say a graph G(V, E) has a strong edge-graceful labeling if there is an injection f from the E to  $\{1, 2, 3, \ldots, \langle 3|E|/2\rangle\}$  such that the induced mapping  $f^+$  from V defined by  $f^+(u) = (\Sigma f(uv)) \pmod{2|V|}$  taken all edges uv is an injection. They proved the following graphs have strong edge graceful labelings:  $P_n(n \ge 3), C_n, K_{1,n} (n \ge 2)$ , crowns  $C_n \odot K_1$ , and fans  $P_n + K_1 (n \ge 2)$ . In his Ph.D. thesis [2476] Subbiah provided edge-graceful and strong edge-graceful labelings for a large variety of graphs. Among them are bistars, twigs, *y*-trees, spiders, flags, kites, friendship graphs, mirror of paths, flowers, sunflowers, graphs obtained by identifying a vertex of a cycle with an endpoint of a star, and  $K_2 \odot C_n$ , and various disjoint unions of path, cycles, and stars.

Hefetz [973] has shown that a graph G(V, E) of the form  $G = H \cup f_1 \cup f_2 \cup \cdots \cup f_r$ where H = (V, E') is edge-graceful and the  $f_i$ 's are 2-factors is also edge-graceful and that a regular graph of even degree that has a 2-factor consisting of k cycles each of length t where k and t are odd is edge-graceful.

Bača and Holländer [273] investigated a generalization of edge-graceful labeling called (a, b)-consecutive labelings. A connected graph G(V, E) is said to have an (a, b)-consecutive labeling where a is a nonnegative integer and b is a positive proper divisor of |V|, if there is a bijection from E to  $\{1, 2, \ldots, |E|\}$  such that if each vertex v is assigned the sum of all edges incident to v the vertex labels are distinct and they can be partitioned into |V|/b intervals

 $W_j = [w_{\min} = (j-1)b + (j-1)a, w_{\min} + jb + (j-1)a - 1]$ , where  $1 \le j \le p/b$  and  $w_{\min}$  is the minimum value of the vertices. They present necessary conditions for (a, b)-consecutive labelings and describe (a, b)-consecutive labelings of the generalized Petersen graphs for some values of a and b.

A graph with p vertices and q edges is said to be k-edge-graceful if its edges can be labeled with  $k, k + 1, \ldots, k + q - 1$  such that the sums of the edges incident to each vertex are distinct modulo p. In [1558] Lee and Wang show that for each  $k \neq 1$  there are only finitely many trees that are k-edge graceful (there are infinitely many 1-edge graceful trees). They describe completely the k-edge-graceful trees for k = 0, 2, 3, 4, and 5. Gayathri and Sarada Devi [843] obtained some necessary conditions and characterizations for k-edge-gracefulness of trees. They also proved that specific families of trees are edgegraceful and k-edge-graceful and conjecture that all odd trees are k-edge-graceful.

Gayathri and Sarada Devi [669] defined a k-even edge-graceful labeling of a (p, q) graph G(V, E) as an injection f from E to  $\{2k - 1, 2k, 2k + 1, \ldots, 2k + 2q - 2\}$  such that the induced mapping  $f^+$  of V defined by  $f^+(x) = \sum f(xy) \pmod{2s}$  taken over all edges xy, are distinct and even, where  $s = \max\{p, q\}$  and k is a positive integer. A graph G that admits a k-even-edge-graceful labeling is called a k-even-edge-graceful graph. In [669], [844], [845], and [846] Gayathri and Sarada Devi investigate the k-even edge-gracefulness of a wide variety of graphs. Among them are: paths; stars; bistars; cycles with a pendent edge; cycles with a cord; crowns  $C_n \odot K_1$ ; graphs obtained from  $P_n$  by replacing each edge by a fixed number of parallel edges; and sparklers (paths with a star appended at an endpoint of the path).

In 1991 Lee [1489] defined the *edge-graceful spectrum* of a graph G as the set of all nonnegative integers k such that G has a k-edge graceful labeling. In [1561] Lee, Wang, Ng, and Wang determine the edge-graceful spectrum of the following graphs:  $G \odot K_1$  where G is an even cycle with one chord; two even cycles of the same order joined by an

edge; and two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex (*butterfly graph*). Lee, Chen, and Wang [1492] have determined the edge-graceful spectra for various cases of cycles with a chord and for certain cases of graphs obtained by joining two disjoint cycles with an edge (i.e., *dumbbell graphs*). More generally, Shiu, Ling, and Low [2355] call a connected with p vertices and p+1 edges *bicyclic*. In particular, the family of bicyclic graphs includes the one-point union of two cycles, two cycles joined by a path and cycles with one cord. In [2356] they determine the edge-graceful spectra of bicyclic graphs that do not have pendent edges. Kang, Lee, and Wang [1325] determined the edge-graceful spectra of wheels and Wang, Hsiao, and Lee [2777] determined the edge-graceful spectra of the square of  $P_n$ for odd n. Results about the edge-graceful spectra of three types of (p, p + 1)-graphs are given by Chen, Lee, and Wang [574]. In [2778] Wang and Lee determine the edge-graceful spectra of the one-point union of two cycles, the corona product of the one-point union of two cycles with  $K_1$ , and the cycles with one chord.

Lee, Levesque, Lo, and Schaffer [1503] investigate the edge-graceful spectra of cylinders. They prove: for odd  $n \ge 3$  and  $m \equiv 2 \pmod{4}$ , the spectra of  $C_n \times P_m$  is  $\emptyset$ ; for m = 3 and  $m \equiv 0, 1$  or  $3 \pmod{4}$ , the spectra of  $C_4 \times P_m$  is  $\emptyset$ ; for even  $n \ge 4$ , the spectra of  $C_n \times P_2$  is all natural numbers; the spectra of  $C_n \times P_4$  is all odd positive integers if and only if  $n \equiv 3 \pmod{4}$ ; and  $C_n \times P_4$  is all even positive integers if and only if  $n \equiv 1$ (mod) 4. They conjecture that  $C_4 \times P_m$  is k-edge-graceful for some k if and only if  $m \equiv 2$ (mod) 4. Shiu, Ling, and Low [2356] determine the edge-graceful spectra of all connected bicyclic graphs without pendent edges.

A graph G(V, E) is called *super edge-graceful* if there is a bijection f from E to  $\{0, \pm 1, \pm 2, \ldots, \pm (|E|-1)/2\}$  when |E| is odd and from E to  $\{\pm 1, \pm 2, \ldots, \pm |E|/2\}$  when |E| is even such that the induced vertex labeling  $f^*$  defined by  $f^*(u) = \Sigma f(uv)$  over all edges uv is a bijection from V to  $\{0, \pm 1, \pm 2, \ldots, \pm (p-1)/2\}$  when p is odd and from V to  $\{\pm 1, \pm 2, \ldots, \pm p/2\}$  when p is even. Lee, Wang, Nowak, and Wei [1562] proved the following:  $K_{1,n}$  is super-edge-magic if and only if n is even; the double star DS(m, n) (that is, the graph obtained by joining the centers of  $K_{1,m}$  and  $K_{1,n}$  by an edge) is super edge-graceful if and only if m and n are both odd. They conjecture that all trees of odd orders which are super edge-graceful. In [1550] Lee, Su, and Wei exhibit a family of trees of odd orders which are super edge-graceful. Chung, Lee, Gao and Schaffer [607] posed the problems of characterizing the paths and tress of diameter 4 that are super edge-graceful.

In [606] Chung, Lee, and Gao prove various classes of caterpillars, combs, and amalgamations of combs and stars of even order are super edge-graceful. Lee, Sun, Wei, Wen, and Yiu [1551] proved that trees obtained by starting with the paths the  $P_{2n+2}$  or  $P_{2n+3}$ and identifying each internal vertex with an endpoint of a path of length 2 are super edge-graceful.

Shiu [2329] has shown that  $C_n \times P_2$  is super-edge-graceful for all  $n \ge 2$ . More generally, he defines a family of graphs that includes  $C_n \times P_2$  and generalized Petersen graphs are follows. For any permutation  $\theta$  on n symbols without a fixed point the  $\theta$ -Petersen graph  $P(n;\theta)$  is the graph with vertex set  $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$  and edge set  $\{u_i u_{i+1}, u_i w_i, w_i w_{\theta(i)} \mid 1 \le i \le n\}$  where addition of subscripts is done modulo n. (The graph  $P(n; \theta)$  need not be simple.) Shiu proves that  $P(n; \theta)$  is super-edge-graceful for all  $n \geq 2$ . He also shows that certain other families of connected cubic multigraphs are super-edge-graceful and conjectures that every connected cubic of multigraph except  $K_4$ and the graph with 2 vertices and 3 edges is super-edge-graceful.

In [2341] Shiu and Lam investigated the super-edge-gracefulness of fans and wheel-like graphs. They showed that fans  $F_{2n}$  and wheels  $W_{2n}$  are super-edge-graceful. Although  $F_3$ and  $W_3$  are not super-edge-graceful the general cases  $F_{2n+1}$  and  $W_{2n+1}$  are open. For a positive integer  $n_1$  and even positive integers  $n_2, n_3, \ldots, n_m$  they define an *m*-level wheel as follows. A wheel is a 1-level wheel and the cycle of the wheel is the 1-level cycle. An *i*-level wheel is obtained from an (i-1)-level wheel by appending  $n_i/2$  pairs of edges from any number of vertices of the i-1-level cycle to  $n_i$  new vertices that form the vertices in the i-level cycle. They prove that all m-level wheels are super-edge-graceful. They also prove that for n odd  $C_m \odot \overline{K_n}$  is super-edge-graceful, for odd  $m \geq 3$  and even  $n \geq 2$  $C_m \odot \overline{K_n}$  is edge-graceful, and for  $m \geq 3$  and  $n \geq 1$   $C_m \odot \overline{K_n}$  is super-edge-graceful. For a cycle  $C_m$  with consecutive vertices  $v_1, v_2, \ldots, v_m$  and nonnegative integers  $n_1, n_2, \ldots, n_m$ they define the graph  $A(m; n_1, n_2, \ldots, n_m)$  as the graph obtained from  $C_m$  by attaching  $n_i$  edges to the vertex  $v_i$  for  $1 \leq i \leq m$ . They prove  $A(m; n_1, n_2, \ldots, n_m)$  is super-edgegraceful if m is odd and  $A(m; n_1, n_2, \ldots, n_m)$  is super-edge-graceful if m is even and all the  $n_i$  are positive and have the same parity. Chung, Lee, Gao, and Schaffer [607] provide super edge-graceful labelings for various even order paths, spiders and disjoint unions of two stars. In [604] Chung and Lee characterize spiders of even orders that are not super-edge-graceful and exhibit some spiders of even order of diameter at most four that are super-edge-graceful. They raised the question of which paths are super edge-graceful. This was answered by Cichacz, Fronček, and Xu [622] who showed that the only paths that are not super edge-graceful are  $P_2$  and  $P_4$ . Cichacz et al. also proved that the only cycles that are not super edge-graceful are  $C_4$  and  $C_6$ . Gao and Zhang [837] proved that some cases of caterpillars are super edge-graceful.

In [607] Chung, Lee, Gao, and Schaffer asked for a characterize trees of diameter 4 that are super edge-graceful. Krop, Mutiso, and Raridan [1439] provide a super edge-graceful labelings for all caterpillars and even size lobsters of diameter 4 that permit such labelings. They also provide super edge-graceful labelings for several families of odd size lobsters of diameter 4. They were unable to find general methods that describe super edge-graceful labelings for a few families of odd size lobsters of diameter 4, although they are able to show that certain lobsters in these families are super-edge graceful. They conclude with three conjectures about rooted trees of height 2 and diameter 4.

Although it is not the case that a super edge-graceful graph is edge-graceful, Lee, Chen, Yera, and Wang [1491] proved that if G is a super edge-graceful with p vertices and q edges and  $q \equiv -1 \pmod{p}$  when q is even, or  $q \equiv 0 \pmod{p}$  when q is odd, then G is also edge-graceful. They also prove: the graph obtained from a connected super edge-graceful unicyclic graph of even order by joining any two nonadjacent vertices by an edge is super edge-graceful; the graph obtained from a super edge-graceful graph with p vertices and p + 1 edges by appending two edges to any vertex is super edge-graceful; and the one-point union of two identical cycles is super edge-graceful. Collins, Magnant, and Wang [630] present a stronger concept of "tight" super-edge-graceful labeling. Such a super-edge graceful labeling has an additional constraint on the edge and vertices with the largest and smallest labels. They use this concept to recursively construct tight super-edge graceful trees of any order.

Gayathri, Duraisamy, and Tamilselvi [848] calls a (p, q)-graph with  $q \ge p$  even edgegraceful if there is an injection f from the set of edges to  $\{1, 2, 3, \ldots, 2q\}$  such that the values of the induced mapping  $f^+$  from the vertex set to  $\{0, 1, 2, \ldots, 2q - 1\}$  given by  $f^+(x) = (\Sigma f(xy)) \pmod{2q}$  over all edges xy are distinct and even. In [848] and [847] Gayathri et al. prove the following: cycles are even edge-graceful if and only if the cycles are odd; even cycles with one pendent edge are even edge-graceful; wheels are even edgegraceful; gears (see §2.2 for the definition) are not even edge-graceful; fans  $P_n + K_1$  are even edge-graceful;  $C_4 \cup P_m$  for all m are even edge-graceful;  $C_{2n+1} \cup P_{2n+1}$  are even edgegraceful; sunflowers (see §3.7 for the definition) are even edge-graceful; triangular snakes (see §2.2 for the definition) are even edge-graceful; closed helms (see §2.2 for the definition) with the center vertex removed are even edge-graceful; graphs decomposable into two odd Hamiltonian cycles are even edge-graceful; and odd order graphs that are decomposable into three Hamiltonian cycles are even edge-graceful.

In [847] Gayathri and Duraisamy generalized the definition of even edge-graceful to include (p, q)-graphs with q < p by changing the modulus from 2q the maximum of 2q and 2p. With this version of the definition, they have shown that trees of even order are not even edge-graceful whereas, for odd order graphs, the following are even edge-graceful: banana trees (see §2.1 for the definition); graphs obtained joining the centers of two stars by a path;  $P_n \odot K_{1,m}$ ; graphs obtained by identifying an endpoint from each of any number of copies of  $P_3$  and  $P_2$ ; bistars (that is, graphs obtained by joining the centers of two stars with an edge); and graphs obtained by appending the endpoint of a path to the center of a star. They define odd edge-graceful graphs in the analogous way and provide a few results about such graphs.

Lee, Pan, and Tsai [1525] call a graph G with p vertices and q edges vertex-graceful if there exists a labeling  $f V(G) \rightarrow \{1, 2, ..., p\}$  such that the induced labeling  $f^+$  from E(G) to  $Z_q$  defined by  $f^+(uv) = f(u) + f(v) \pmod{q}$  is a bijection. Vertex-graceful graphs can be viewed the dual of edge-graceful graphs. They call a vertex-graceful graph strong vertex-graceful if the values of  $f^+(E(G))$  are consecutive. They observe that the class of vertex-graceful graphs properly contains the super edge-magic graphs and strong vertexgraceful graphs are super edge-magic. They provide vertex-graceful and strong vertexgraceful labelings for various (p, p + 1)-graphs of small order and their amalgamations.

Shiu and Wong [2368] proved the one-point union of an *m*-cycle and an *n*-cycle is vertex-graceful only if  $m+n \equiv 0 \pmod{4}$ ; for  $k \geq 2$ , C(3, 4k-3) is strong vertex-graceful; C(2n+3, 2n+1) is strong vertex-graceful for  $n \geq 1$ ; and if the one-point union of two cycles is vertex-graceful, then it is also strong vertex-graceful. In [2443] Somashekara and Veena found the number of (n, 2n-3) strong vertex graceful graphs. Gao, Zhang, and Xu [824] proved that  $C_n$ ,  $C_n \odot K_1$  and  $C_n \odot K_{1,t}$  are vertex-graceful if *n* is odd;  $C_n$  is super vertex-graceful if  $n \neq 4, 6$ ; and  $C_n \odot K_1$  is super vertex-graceful if *n* is even. They proposed two conjectures on (super)vertex-graceful labelings.

As a dual to super edge-graceful graphs Lee and Wei [1565] define a graph G(V, E) to be super vertex-graceful if there is a bijection f from V to  $\{\pm 1, \pm 2, \ldots, \pm (|V|-1)/2\}$  when |V| is odd and from V to  $\{\pm 1, \pm 2, \ldots, \pm |V|/2\}$  when |V| is even such that the induced edge labeling  $f^*$  defined by  $f^+(uv) = f(u) + f(v)$  over all edges uv is a bijection from E to  $\{0, \pm 1, \pm 2, \ldots, \pm (|E|-1)/2\}$  when |E| is odd and from E to  $\{\pm 1, \pm 2, \ldots, \pm |E|/2\}$  when |E| is even. They show: for m and  $n_1, n_2, \ldots, n_m$  each at least 3,  $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m}$ is not super vertex-graceful; for n odd, books  $K_{1,n} \times P_2$  are not super vertex-graceful; for  $n \geq 3$ ,  $P_n^2 \times P_2$  is super vertex-graceful if and only if n = 3, 4, or 5; and  $C_m \times C_n$  is not super vertex-graceful. They conjecture that  $P_n \times P_n$  is super vertex-graceful for  $n \geq 3$ .

In [1569] Lee and Wong generalize super edge-vertex graphs by defining a graph G(V,E) to be P(a)Q(1)-super vertex-graceful if there is a bijection f from V to  $\{0, \pm a, \pm (a+1), \dots, \pm (a-1+(|V|-1)/2)\}$  when |V| is odd and from V to  $\{\pm a, \pm (a+1)\}$ 1),...,  $\pm (a-1+|V|/2)$  when |V| is even such that the induced edge labeling  $f^*$  defined by  $f^+(uv) = f(u) + f(v)$  over all edges uv is a bijection from E to  $\{0, \pm 1, \pm 2, \dots, \pm (|E| - 1)\}$ 1)/2} when |E| is odd and from E to  $\{\pm 1, \pm 2, \dots, \pm |E|/2\}$  when |E| is even. They show various classes of unicyclic graphs are P(a)Q(1)-super vertex-graceful. In [1502] Lee, Leung, and Ng more simply refer to P(1)Q(1)-super vertex-graceful graphs as super *vertex-graceful* and show how to construct a variety of unicyclic graphs that are super vertex-graceful. They conjecture that every unicyclic graph is an induced subgraph of a super vertex-graceful unicyclic graph. Lee and Leung [1501] determine which trees of diameter at most 6 are super vertex-graceful graphs and propose two conjectures. Lee, Ng, and Sun [1521] found many classes of caterpillars that are super vertex-graceful. In [832] Gao shows that the generalized butterfly graph  $B_n^t$  is super vertex-graceful when t > 0 is even,  $B_n^0$  is super vertex-graceful when  $n \equiv 0$  or 3 (mod 4), and  $C_3^{(t)}$  is super vertex-graceful if and only if t = 1, 2, 3, 5, or 7.

In [593] Chopra and Lee define a graph G(V, E) to be Q(a)P(b)-super edge-graceful if there is a bijection f from E to  $\{\pm a, \pm (a+1), \ldots, \pm (a+(|E|-2)/2)\}$  when |E| is even and from E to  $\{0, \pm a, \pm (a+1), \ldots, \pm (a+(|E|-3)/2)\}$  when |E| is odd and  $f^+(u)$  is equal to the sum of f(uv) over all edges uv is a bijection from V to  $\{\pm b, \pm (b+1), \ldots, (|V|-2)/2\}$ when |V| is even and from V to  $\{0, \pm b, \pm (b+1), \ldots, \pm (|V|-3)/2\}$  when |V| is odd. They say a graph is strongly super edge-graceful if it is Q(a)P(b)-super edge-graceful for all  $a \geq 1$ . Among their results are: a star with n pendent edges is strongly super edgegraceful if and only if n is even; wheels with n spokes are strongly super edge-graceful if and only if n is even; coronas  $C_n \odot K_1$  are strongly super edge-graceful for all  $n \geq 3$ ; and double stars DS(m, n) are strongly super edge-graceful in the case that m is odd and at least 3 and n is even and at least 2 and in the case that both m and n are odd and one of them is at least 3. Lee, Song, and Valdés [1542] investigate the Q(a)P(b)-super edge-gracefulness of wheels  $W_n$  for n = 3, 4, 5, and 6.

In [1566] Lee, Wang, and Yera proved that some Eulerian graphs are super edgegraceful, but not edge-graceful, and that some are edge-graceful, but not super edgegraceful. They also showed that a Rosa-type condition for Eulerian super edge-graceful graphs does not exist and pose some conjectures, one of which was: For which n, is  $K_n$  is super edge-graceful? It was known that the complete graphs  $K_n$  for n = 3, 5, 6, 7, 8 are super edge-graceful and  $K_4$  is not super edge-graceful. Khodkar, Rasi, and Sheikholeslami, [1371] answered this question by proving that all complete graphs of order  $n \ge 3$ , except 4, are super edge-graceful.

In 1997 Yilmaz and Cahit [2869] introduced a weaker version of edge-graceful called *E*-cordial. Let *G* be a graph with vertex set *V* and edge set *E* and let *f* a function from *E* to {0,1}. Define *f* on *V* by  $f(v) = \sum \{f(uv) | uv \in E\} \pmod{2}$ . The function *f* is called an *E*-cordial labeling of *G* if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an *E*-cordial labeling is called *E*-cordial. Yilmaz and Cahit prove the following graphs are *E*-cordial: trees with *n* vertices if and only if  $n \not\equiv 2 \pmod{4}$ ;  $K_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ;  $K_{m,n}$  if and only if  $m + n \not\equiv 2 \pmod{4}$ ;  $C_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ; regular graphs of degree 1 on 2*n* vertices if and only if *n* is even; friendship graphs  $C_3^{(n)}$  for all *n* (see §2.2 for the definition); fans  $F_n$  if and only if  $n \not\equiv 1 \pmod{4}$ ; and wheels  $W_n$  if and only if  $n \not\equiv 1$ (mod 4). They observe that graphs with  $n \equiv 2 \pmod{4}$  vertices can not be *E*-cordial. They generalized *E*-cordial labelings to  $E_k$ -cordial (k > 1) labelings by replacing {0,1} by  $\{0, 1, 2, \ldots, k - 1\}$ . Of course,  $E_2$ -cordial is the same as *E*-cordial (see §3.7).

Liu, liu, and Wu [1622] provide two necessary conditions for a graph to be  $E_k$ -cordial and prove that  $P_n$   $(n \ge 3)$  is  $E_p$ -cordial for odd p. They also discuss the  $E_2$ -cordiality of graphs that have a subgraph that is a 1-factor.

In [2688] Vaidya and Vyas prove that the following graphs are E-cordial: the mirror graphs (see §2.3 for the definition) even paths, even cycles, and the hypercube are E-cordial. In [2653] they show that the middle graph, the total graph, and the splitting graph of a path are E-cordial and the composition of  $P_2n$  with  $P_2$ . (See §2.7 for the definitions of middle, total and splitting graphs.) In [2654] Vaidya and Lekha [2654] prove the following graphs are E-cordial: the graph obtained by duplication of a vertex (see §2.7 for the definition) of a cycle; the graph obtained by duplication of an edge (see §2.7 for the definition) of a cycle; the graph obtained by joining of two copies of even cycle by an edge; the splitting graph of an even cycle; and the shadow graph (see §3.8 for the definition) of a path of even order.

Vaidya and Vyas [2689] proved the following graphs have *E*-cordial labelings:  $K_{2n} \times P_2$ ;  $P_{2n} \times P_2$ ;  $W_n \times P_2$  for odd *n*; and  $K_{1,n} \times P_2$  for odd *n*. Vaidya and Vyas [2690] proved that the Möbius ladders, the middle graph of  $C_n$ , and crowns  $C_n \odot K_1$  are *E*-cordial graphs for even *n* while bistars  $B_{n,n}$  and its square graph  $B_{n,n}^2$  are *E*-cordial graphs for odd *n*. In [2692] and [2693] Vaidya and Vyas proved the following graphs are *E*-cordial: flowers, closed helms, double triangular snakes, gears, graphs obtained by switching of an arbitrary vertex in  $C_n$  except  $n \equiv 2 \pmod{4}$ , switching of rim vertex in wheel  $W_n$  except  $n \equiv 1 \pmod{4}$ , switching of an apex vertex in helms, and switching of an apex vertex in closed helms. Sugumaran and Vishnu Prakash [2518] proved that the following graphs are *E*-cordial: theta graphs, the fusion of any two vertices in theta graphs, and the open star of *n* copies of a fixed theta graph (that is, the graph obtained by replacing each endpoint vertex of  $K_{1,n}$  by copies of the theta graph).

Seoud and Jaber [2241] proved that the butterfly  $BF_{n,m}$ , helms  $H_n$ , the graph  $\langle W_n : W_m \rangle$  obtained by joining apex vertices of two wheels with a new vertex are prime cordial, and determine the prime cordial graphs of order 7. They also gave an algorithm to calculate the maximum number of edges in a 3-equitable prime cordial graph.

For a graph G(V, E) and the group  $S_3$  of all permutations of  $\{1, 2, 3\}$  Chandra and Kala [552] define a function  $g: V(G) \to S_3$  such that  $xy \in E$  if g(x) and g(y) have relatively prime orders. Let  $n_j(g)$  denote the number of vertices of G having label j under g. Then g is called a group  $S_3$  cordial prime labeling if  $|n_i(g) - n_j(g)| \leq 1$  for every  $i, j \in S_3$ . A graph that admits a group  $S_3$  cordial prime labeling is called a group  $S_3$ cordial prime graph. Chandra and Kala [552] prove that all paths, cycles, gears, ladders, and fans are group  $S_3$  cordial prime and characterize wheels that are group  $S_3$  cordial prime.

In her PhD thesis [2700] Vanitha defines a (p,q) graph G to be directed edge-graceful if there exists an orientation of G and a labeling of the arcs of G with  $\{1, 2, \ldots, q\}$  such that the induced mapping g on V defined by  $g(v) = |f^+(v) - f^-(v)| \pmod{p}$  is a bijection where,  $f^+(v)$  is the sum of the labels of all arcs with head v and  $f^-(v)$  is the sum of the labels of all arcs with tail v. She proves that a necessary condition for a graph with pvertices to be directed edge-graceful is that p is odd. Among the numerous graphs that she proved to be directed edge-graceful are: odd paths, odd cycles, fans  $F_{2n}$   $(n \ge 2)$ , wheels  $W_{2n}$ ,  $nC_3$ -snakes, butterfly graphs  $B_n$  (two even cycles of the same order sharing a common vertex with an arbitrary number of pendent edges attached at the common vertex),  $K_{1,2n}$   $(n \ge 2)$ , odd order y-trees with at least 5 vertices, flags  $Fl_{2n}$  (the cycle  $C_{2n}$  with one pendent edge), festoon graphs  $P_n \odot mK_1$ , the graphs  $T_{m,n,t}$  obtained from a path  $P_t$   $(t \ge 2)$  by appending m edges at one endpoint of  $P_t$  and n edges at the other endpoint of  $P_t$ ,  $C_3^n$ ,  $P_3 \cup K_{1,2n+1}$ ,  $P_5 \cup K_{1,2n+1}$ , and  $K_{1,2n} \cup K_{1,2m+1}$ .

Devaraj [667] has shown that M(m, n), the mirror graph of K(m, n), is *E*-cordial when m + n is even and the generalized Petersen graph P(n, k) is *E*-cordial when n is even. (Recall that P(n, 1) is  $C_n \times P_2$ .)

The table following summarizes the state of knowledge about edge-graceful labelings. In the table **EG** means edge-graceful labeling exists. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property.

Graph	Types	Notes
$K_n$	EG	$\text{iff } n \not\equiv 2 \pmod{4} \ [1481]$
odd order trees	EG?	[1488]
$K_{n,n,\dots,n}$ (k terms)	EG	iff <i>n</i> is odd or $k \not\equiv 2 \pmod{4} [1534]$

Table 23: Summary of Edge-graceful Labelings

Continued on next page

Graph	Types	Notes
$C_n^k, \ k < \lfloor n/2 \rfloor$	EG	iff $n$ is odd [1533]
$C_n^k, \ k \ge \lfloor n/2 \rfloor$	EG	iff $n \not\equiv 2 \pmod{4}$ [1533]
$P_3[K_n]$	EG	n  is odd  [1533]
$M_{4n}$ (Möbius ladders)	EG	[1489]
odd order dragons	EG	[1442]
odd order unicycilc graphs	EG?	[1442]
$P_{2m} \times P_{2n}$	EG	iff $m = n = 2$ [1509]
$C_n \cup P_2$	EG	n even [1537]
$C_{2n} \cup C_{2n+1}$	EG	n  odd  [1537]
$C_n \cup C_{2n+2}$	EG	[1537]
$C_n \cup C_{4n}$	EG	n  odd  [1537]
$C_{2m} \cup C_{2n+1}$	EG?	$(m,n) \neq (4,3) \text{ odd } [1538]$
P(n,k) generalized Petersen graph	EG	n even, $k < n/2$ [1489]
$C_m \times C_n$	EG?	$(m,n) \neq (4,3) \ [1538]$

## 7.4 Line-graceful Labelings

Gnanajothi [894] has defined a concept similar to edge-graceful. She calls a graph with nvertices *line-graceful* if it is possible to label its edges with  $0, 1, 2, \ldots, n$  such that when each vertex is assigned the sum modulo n of all the edge labels incident with that vertex the resulting vertex labels are  $0, 1, \ldots, n-1$ . A necessary condition for the line-gracefulness of a graph is that its order is not congruent to 2 (mod 4). Among line-graceful graphs are (see [pp. 132–181][894])  $P_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ;  $C_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ;  $K_{1,n}$  if and only if  $n \not\equiv 1 \pmod{4}$ ;  $P_n \odot K_1 \pmod{5}$  if and only if n is even;  $(P_n \odot K_1) \odot K_1$ if and only if  $n \not\equiv 2 \pmod{4}$ ; (in general, if G has order n,  $G \odot H$  is the graph obtained by taking one copy of G and n copies of H and joining the *i*th vertex of G with an edge to every vertex in the *i*th copy of H);  $mC_n$  when mn is odd;  $C_n \odot K_1$  (crowns) if and only if n is even;  $mC_4$  for all m; complete n-ary trees when n is even;  $K_{1,n} \cup K_{1,n}$  if and only if n is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete n-ary tree when n is even; trees for which exactly one vertex has even degree. She conjectures that all trees with  $p \not\equiv 2 \pmod{4}$  vertices are line-graceful and proved this conjecture for p < 9.

Gnanajothi [894] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from  $K_{1,4}$  by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has  $p \not\equiv 2 \pmod{4}$  vertices; the graph obtained from  $K_{1,n}$  by replacing each spoke with a path of length m (counting the center vertex) is line-graceful in the following cases: n = 2; n = 3 and  $m \not\equiv 3 \pmod{4}$ ; and m is even and  $mn + 1 \equiv 0 \pmod{4}$ .

Gnanajothi studied graphs obtained by joining disjoint graphs G and H with an edge. She proved such graphs are line-graceful in the following circumstances: G = H;  $G = P_n, H = P_m$  and  $m + n \not\equiv 0 \pmod{4}$ ; and  $G = P_n \odot K_1$ ,  $H = P_m \odot K_1$  and  $m + n \not\equiv 0 \pmod{4}$ .

In [2647] and [2648] Vaidya and Kothari proved following graphs are line graceful: fans  $F_n$  for  $n \not\equiv 1 \pmod{4}$ ;  $W_n$  for  $n \not\equiv 1 \pmod{4}$ ; bistars  $B_{n,n}$  if and only if for  $n \equiv 1, 3 \pmod{4}$ ; helms  $H_n$  for all n;  $S'(P_n)$  for  $n \equiv 0, 2 \pmod{4}$ ;  $D_2(P_n)$  for  $n \equiv 0, 2 \pmod{4}$ ;  $T(P_n)$ ,  $M(P_n)$ , alternate triangular snakes, and graphs obtained by duplication of each edge of  $P_n$  by a vertex are line graceful graphs.

# 7.5 Radio Labelings

In 2001 Chartrand, Erwin, Zhang, and Harary [563] were motivated by regulations for channel assignments of FM radio stations to introduce radio labelings of graphs. A radio labeling of a connected graph G is an injection c from the vertices of G to the natural numbers such that  $d(u, v) + |c(u) - c(v)| \ge 1 + diam(G)$  for every two distinct vertices u and v of G. The radio number of c, rn(c), is the maximum number assigned to any vertex of G. The radio number of G, rn(G), is the minimum value of rn(c) taken over

all radio labelings c of G. Chartrand et al. and Zhang [2904] gave bounds for the radio numbers of cycles. The exact values for the radio numbers for paths and cycles were reported by Liu and Zhu [1612] as follows: for odd  $n \ge 3$ ,  $rn(P_n) = (n-1)^2/2 + 2$ ; for 1(k-1)/2;  $rn(C_{4k+2}) = (k+2)(k-2)/2+1$ ; and  $rn(C_{4k+3}) = (k+2)(k-1)/2$ . However, Chartrand, Erwin, and Zhang [562] obtained different values than Liu and Zhu for  $P_4$  and  $P_5$ . Chartrand, Erwin, and Zhang [562] proved:  $rn(P_n) \leq (n-1)(n-2)/2 + n/2 + 1$  when n is even;  $rn(P_n) \leq n(n-1)/2+1$  when n is odd;  $rn(P_n) < rn(P_{n+1})$  (n > 1); for a connected graph G of diameter d,  $rn(G) \ge (d+1)^2/4 + 1$  when d is odd; and  $rn(G) \ge d(d+2)/4 + 1$ when d is even. Benson, Porter, and Tomova [437] have determined the radio numbers of all graphs of order n and diameter n-2. In [1608] Liu obtained lower bounds for the radio number of trees and the radio number of spiders (trees with at most one vertex of degree greater than 2) and characterized the graphs that achieve these bounds. Bantva, Vaidya, and Zhou [420] and [421] give a lower bound for the radio number of trees and a necessary and sufficient condition for their bound to be achieved. They determine the radio number for symmetric trees (that is, trees whose non-leaf vertices all have the same degree and whose leaf vertices all have the same eccentricity), banana trees, and firecracker trees. In [1400] Kola and Panigrahi provide the radio number for a class of caterpillars. Nazeer, Khan, Kousar, and Nazeer [1835] investigated the radio number for some families of generalized caterpillar graphs.

Chartrand, Erwin, Zhang, and Harary [563] proved:  $rn(K_{n_1,n_2,\ldots,n_k}) = n_1 + n_2 + \cdots + n_k + k - 1$ ; if G is a connected graph of order n and diameter 2, then  $n \leq rn(G) \leq 2n - 2$ ; and for every pair of integers k and n with  $n \leq k \leq 2n - 2$ , there exists a connected graph of order n and diameter 2 with rn(G) = k. They further provide a characterization of connected graphs of order n and diameter 2 with prescribed radio number.

Fernandez, Flores, Tomova, and Wyels [755] proved  $rn(K_n) = n$ ;  $rn(W_n) = n + 2$ ; and the radio number of the gear graph obtained from  $W_n$  by inserting a vertex between each vertex of the rim is 4n + 2. Morris-Rivera, Tomova, Wyels, and Yeager [1787] determine the radio number of  $C_n \times C_n$ . Martinez, Ortiz, Tomova, and Wyels [1714] define generalized prisms, denoted  $Z_{n,s}$ ,  $s \ge 1$ ,  $n \ge s$ , as the graphs with vertex set  $\{(i,j) | i = 1, 2 \text{ and } j = 1, ..., n\}$  and edge set  $\{((i,j), (i, j \pm 1))\} \cup \{((1,i), (2, i + \sigma)) | \sigma = -\lfloor \frac{s-1}{2} \rfloor \dots, 0, \dots, \lfloor \frac{s}{2} \rfloor\}$ . They determine the radio number of  $Z_{n,s}$  for s = 1, 2 and 3. In [221] and [222] Bantva determines the radio number for three families of trees obtained by taking a graph operation on a given tree or a family of trees and the radio number for the middle graph of paths. Zhang, Nazeer, Habib, Zia, and Ren [2902] determined the radio number for the generalized Petersen graphs P(4k + 2, 2) and provided a lower bound for P(4k, 2).

In [2094] Ramyal and Sooryanarayana generalized the notion of radio labeling as follows. Let M be a subset of non-negative integers and  $(M, \star)$  be a monoid with the identity e. A radio  $\star$ -labeling of graph G(V, E) is a mapping  $f : V \to M$  such that  $|f(u) - f(v)| \star d(u, v) \geq diam(G) + 1 - e$ , for all  $u, v \in V$ . The radio  $\star$ -number  $rn \star (f)$ of a radio  $\star$ -labeling f of G is the maximum label assigned to a vertex of G. The radio  $\star$ -number of G, denoted by  $rn \star (G)$ , is the minimum of  $rn \star (f)$  taken over all radio \*-labeling f of G. They completely determine  $rn \star (G)$  of some transformation graphs of paths and cycles where  $\star$  is the usual multiplication of integers.

Sooryanarayana and Ranghunath [2460] define a radio labeling f of a graph G to be a consecutive radio labeling of G if  $f(V(G)) = \{1, 2, ..., |V(G)|\}$ . They call a graph for which a consecutive radio labeling exists radio graceful. In her Ph.D. thesis [1844] Niedzialomski (see also [1845]) investigated the existence of radio graceful labelings of Cartesian products of graphs. Among her results are: for  $n \ge 3$  and  $1 \le t \le n - 1$  the Cartesian product of t copies of  $K_n$  is radio graceful; for  $2 \le p \le n_2$  the Cartesian product of  $p \cdot \lceil n/p \rceil$  copies of  $K_n$  is radio graceful; the Cartesian product  $K_{n_1} \times K_{n_2}, \ldots, \times K_{n_s}$ is radio graceful when  $n_1, n_2, \ldots, n_s$  are relatively prime; certain families of generalized Petersen graphs are radio graceful; and the Cartesian product of  $t \ge 1 + n(n^2 - 1)/6$ copies of  $K_n$  is not radio graceful. Locke and Niedzialomski [1627] proved that  $K_n \times P$  is radio graceful where P is the Peterson graph. Wyels and Tomova [1627] proved that that  $P \times P$  is radio graceful.

The generalized gear graph  $J_{t,n}$  is obtained from a wheel  $W_n$  by introducing *t*-vertices between every pair  $(v_i, v_{i+1})$  of adjacent vertices on the *n*-cycle of wheel. Ali, Rahim, Ali, and Farooq [130] gave an upper bound for the radio number of generalized gear graph, which coincided with the lower bound found in and [2054]. They proved for t < n - 1and  $n \ge 7$ ,  $\operatorname{rn}(J_{t,n}) = (nt^2 + 4nt + 3n + 4)/2$ . They pose the determination of the radio number of  $J_{t,n}$  when  $n \le 7$  and t > n - 1 as an open problem.

Saha and Panigrahi [2152] determined the radio number of the toroidal grid  $C_m \times C_n$ when at least one of m and n is an even integer and gave a lower bound for the radio number when both m and n are odd integers. Liu and Xie [1610] determined the radio numbers of squares of cycles for most values of n. In [1611] Liu and Xie proved that  $rn(P_n^2)$  is  $\lfloor n/2 \rfloor + 2$  if  $n \equiv 1 \pmod{4}$  and  $n \geq 9$  and  $rn(P_n^2)$  is  $\lfloor n/2 + 1 \rfloor$  otherwise. In [1609] Liu found a lower bound for the radio number of trees and characterizes the trees that achieve the bound. She also provides a lower bound for the radio number of spiders in terms of the lengths of their legs and characterizes the spiders that achieve this bound. Sweetly and Joseph [2566] prove that the radio number of the graph obtained from the wheel  $W_n$  by subdividing each edge of the rim exactly twice is 5n-3. Marinescu-Ghemeci [1709] determined the radio number of the caterpillar obtained from a path by attaching a new terminal vertex to each non-terminal vertex of the path and the graph obtained from a star by attaching k new terminal vertices to each terminal vertex of the star. Ahmad and Marinescu-Ghemeci [95] determined the radio numbers of Mongolian tents, diamonds, fans, and double fans.

Sooryanarayana and Raghunath [2460] determined the radio number of  $C_n^3$ , for  $n \leq 20$ and for  $n \equiv 0$  or 2 or 4 (mod 6). Sooryanarayana, Vishu Kumar, Manjula [2461] determine the radio number of  $P_n^3$ , for  $n \geq 4$ . Lo and Alegria [1625] completely determine the radio number for the fourth-power of  $P_n$  for  $n \geq 6$ , except when  $n \equiv 1 \pmod{8}$ . Saha and Panigrahi [2153] prove that for an *n*-vertex simple connected graph *G*, the difference between the upper and lower bounds of the radio number of  $G^2$  is at most  $\lfloor (n-1)/2 \rfloor$ . They also determine the radio number for square of graphs belonging to some specific class and apply this to find the radio number for square of hypercube  $Q_n^2$  ( $n \neq 0 \pmod{2}$  4)), the square of toroidal grid  $T_{m,n}^2$   $(m+n \equiv 1, 2, 3, 4, 6 \pmod{8})$ , and the square of some generalized prism graphs. Wang, Xu, Yang, Zhang, Luo, and Wang [2770] determine the radio number of ladder graphs. Jiang [1249] completely determined the radio number of the grid graph  $P_m \times P_n$  (m, n > 2). In [2686] Vaidya and Vihol determined upper bounds on radio numbers of cycles with chords and determined the exact radio numbers for the splitting graph and the middle graph of  $C_n$ . In [1577] Li, Mak, and Zhou determine the radio number of complete *m*-ary trees. Kim, Hwang, and Song [1373] determine the radio numbers of  $P_n$  with  $n \ge 4$  and  $K_m$  with  $m \ge 3$ . Bantva [220] improved the lower bound for the radio number of graphs given by Das *et al.* in [647] and gave necessary and sufficient condition to achieve the lower bound. He also determined the radio number for cartesian product of paths  $P_n$  and the Peterson graph P and provided a short proof for the radio number of cartesian product of paths  $P_n$  and complete graphs  $K_m$  given by Kim [1373]. In [1836] Nazeer, Kousar, and Nazeer give radio and radio antipodal labelings for certain circulant graphs. In [1837] Nazeer, Kousar, and Munir determined the radio number and radio antipodal number of non-bipartite cubic graphs of order  $2^k$ . Shen, Dong, Zheng, and Guo [2324] use C(m,t) to denote the caterpiller consisting of a path  $x_1x_2\cdots x_m$  with t pendent edges at each inner vertex. They determine the exact value of the radio number of C(m,t) for all integers  $m \ge 4$  and  $t \ge 2$ , and explicitly construct an optimal radio labeling. They also show that the radio number and the construction of optimal radio labelings of paths are the special cases of C(m,t) with t=2. An edge-joint graph G is a 1-edge connected graph having an edge uv such that eccentricity of u equuls the eccentricity of v and deletion of uv disconnects G. In [1834] Naseem, Shabbir, and Shaker gave a lower bound for the radio number of edge-joint graphs. Adefokun and Ajayi [51] proved that for  $m \ge 4$  and n even  $rn(S_m \times P_n) = mn^2/2 + n - 1$  and that for n even  $rn(S_3 \times P_n) = 3n^2/2 + n.$ 

In [543] Canales, Tomova, and Wyels investigated the question of which radio numbers of graphs of order n are achievable. They proved that the achievable radio numbers of graphs of order n must lie in the interval  $[n, rn(P_n)]$ , and that these bounds are the best possible. They also show that for odd n, the integer  $rn(P_n) - 1 = \frac{(n-1)^2}{2} + 2$  is an unachievable radio number for any graph of order n. In [2436] Sokolowsky settled the question of exactly which radio numbers are achievable for a graph of order n.

For any connected graph G and positive integer k Chartrand, Erwin, and Zhang, [561] define a radio k-coloring as an injection f from the vertices of G to the natural numbers such that  $d(u, v) + |f(u) - f(v)| \ge 1 + k$  for every two distinct vertices u and v of G. Using  $rc_k(f)$  to denote the maximum number assigned to any vertex of G by f, the radio k-chromatic number of G,  $rc_k(G)$ , is the minimum value of  $rc_k(f)$  taken over all radio k-colorings of G. Note that  $rc_1(G)$  is  $\chi(G)$ , the chromatic number of G, and when k = diam(G),  $rc_k(G)$  is rn(G), the radio number of G. Chartrand, Nebesky, and Zang [569] gave upper and lower bounds for  $rc_k(P_n)$  for  $1 \le k \le n - 1$ . Kchikech, Khennoufa, and Togni [1358] improved Chartrand et al.'s lower bound for  $rc_k(P_n)$  and Kola and Panigrahi [1402] improved the upper bound for certain special cases of n. The exact value of  $rc_{n-2}(P_n)$  for  $n \ge 5$  was given by Khennoufa and Togni in [1367] and the exact value of  $rc_{n-3}(P_n)$  for  $n \ge 8$  was given by Kola and Panigrahi in [1402]. Kola and Panigrahi [1402] gave the exact value of  $rc_{n-4}(P_n)$  when n is odd and  $n \ge 11$  and an upper bound for  $rc_{n-4}(P_n)$  when n is even and  $n \ge 12$ . In [2151] Saha and Panigrahi provided an upper and a lower bound for  $rc_k(C_n^r)$  for all possible values of n, k and r and showed that these bounds are sharp for antipodal number of  $C_n^r$  for several values of n and r. Kchikech, Khennoufa, and Togni [1359] gave upper and lower bounds for  $rc_k(G \times H)$ and  $rc_k(Q_n)$ . In [1358] the same authors proved that  $rc_k(K_{1,n}) = n(k-1) + 2$  and for any tree T and  $k \ge 2$ ,  $rc_k(T) \le (n-1)(k-1)$ . Karst, Langowitz, Oehrlein, and Troxell [1341] provide general lower bounds for  $rc_k(C_n)$  for all cycles  $C_n$  when  $k \ge \text{diam}(C_n)$  and show that these bounds are exact values when  $k = \text{diam}(C_n) + 1$ .

A radio k-coloring of G when  $k = \operatorname{diam}(G) - 1$  is called a radio antipodal labeling. The minimum span of a radio antipodal labeling of G is called the radio antipodal number of G and is denoted by an(G). Khennoufa and Togni [1364] determined the radio number and the radio antipodal number of the hypercube by using a generalization of binary Gray codes. They proved that  $rn(Q_n) = (2^{n-1}-1)\lceil \frac{n+3}{2}\rceil + 1$  and  $an(Q_n) = (2^{n-1}-1)\lceil \frac{n}{2}\rceil + \varepsilon(n)$ , with  $\varepsilon(n) = 1$  if  $n \equiv 0 \mod 4$ , and  $\varepsilon(n) = 0$  otherwise.

Sooryanarayana and Raghunath [2460] say a graph with *n* vertices is *radio graceful* if rn(G) = n. They determine the values of *n* for which  $C_n^3$  is radio graceful.

The survey article by Panigrahi [1888] includes background information and further results about radio k-colorings.

#### 7.6 Representations of Graphs modulo n

In 1989 Erdős and Evans [729] defined a representation modulo n of a graph G with vertices  $v_1, v_2, \ldots, v_r$  as a set  $\{a_1, \ldots, a_r\}$  of distinct, nonnegative integers each less than n satisfying  $gcd(a_i - a_i, n) = 1$  if and only if  $v_i$  is adjacent to  $v_i$ . They proved that every finite graph can be represented modulo some positive integer. The representation number,  $\operatorname{Rep}(G)$ , is smallest such integer. Obviously the representation number of a graph is prime if and only if a graph is complete. Evans, Fricke, Maneri, McKee, and Perkel [742] have shown that a graph is representable modulo a product of a pair of distinct primes if and only if the graph does not contain an induced subgraph isomorphic to  $K_2 \cup 2K_1, K_3 \cup K_1$ , or the complement of a chordless cycle of length at least five. Nešetřil and Pultr [1841] showed that every graph can be represented modulo a product of some set of distinct primes. Evans et al. [742] proved that if G is representable modulo n and p is a prime divisor of n, then  $p > \chi(G)$ . Evans, Isaak, and Narayan [743] determined representation numbers for specific families as follows (here we use  $q_i$  to denote the *i*th prime and for any prime  $p_i$  we use  $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$  to denote the next k primes larger than  $p_i$ ): Rep $(P_n) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil}$ ; Rep $(C_4) = 4$  and for  $n \ge 3$ , Rep $(C_{2n}) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil + 1}$ ; Rep $(C_5) = 3 \cdot 5 \cdot 7 = 105$  and for  $n \ge 4$  and not a power of 2,  $\operatorname{Rep}(C_{2n+1}) = 3 \cdot 5 \cdot \cdots \cdot q_{\lceil \log_2 n \rceil + 1}$ ; if  $m \ge n \ge 3$ , then  $\operatorname{Rep}(K_m - P_n) = p_i p_{i+1}$  where  $p_i$ is the smallest prime greater than or equal to  $m - n + \lfloor n/2 \rfloor$ ; if  $m \ge n \ge 4$ , and  $p_i$  is the smallest prime greater than or equal to  $m - n + \lceil n/2 \rceil$ , then  $\operatorname{Rep}(K_m - C_n) = q_i q_{i+1}$  if n is even and  $\operatorname{Rep}(K_m - C_n) = q_i q_{i+1} q_{i+2}$  if n is odd; if  $n \leq m-1$ , then  $\operatorname{Rep}(K_m - K_{1,n}) =$  $p_s p_{s+1} \cdots p_{s+n-1}$  where  $p_s$  is the smallest prime greater than or equal to m-1; Rep $(K_m)$ 

is the smallest prime greater than or equal to m;  $\operatorname{Rep}(nK_2) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2 n \rceil + 1}$ ; if  $n, m \geq 2$ , then  $\operatorname{Rep}(nK_m) = p_i p_{i+1} \cdots p_{i+m-1}$ , where  $p_i$  is the smallest prime satisfying  $p_i \geq m$ , if and only if there exists a set of n-1 mutually orthogonal Latin squares of order m;  $\operatorname{Rep}(mK_1) = 2m$ ; and if  $t \leq (m-1)!$ , then  $\operatorname{Rep}(K_m + tK_1) = p_s p_{s+1} \cdots p_{s+m-1}$  where  $p_s$  is the smallest prime greater than or equal to m. Narayan [1832] proved that for  $r \geq 3$  the maximum value for  $\operatorname{Rep}(G)$  over all graphs of order r is  $p_s p_{s+1} \cdots p_{s+r-2}$ , where  $p_s$  is the smallest prime that is greater than or equal to r-1. Agarwal and Lopez [60] determined the representation numbers for complete graphs minus a set of stars.

Evans [741] used matrices over the additive group of a finite field to obtain various bounds for the representation number of graphs of the form  $nK_m$ . Among them are  $\operatorname{Rep}(4K_3) = 3 \cdot 5 \cdot 7 \cdot 11; \operatorname{Rep}(7K_5) = 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23;$  and  $\operatorname{Rep}((3q-1)/2)K_q) \leq p_q p_{q+1} \cdots p_{(3q-1)/2)}$  where q is a prime power with  $q \equiv 3 \pmod{4}$ ,  $p_q$  is the smallest prime greater than or equal to q, and the remaining terms are the next consecutive (3q-3)/2primes;  $\operatorname{Rep}(2q-2)K_q) \leq p_q p_{q+1} \cdots p_{(3q-3)/2)}$  where q is a prime power with  $q \equiv 3 \mod 4$ , and  $p_q$  is the smallest prime greater than or equal to q;  $\operatorname{Rep}((2q-2)K_q) \leq p_q p_{q+1} \cdots p_{2q-3}$ .

In [1831] Narayan asked for the values of  $\operatorname{Rep}(C_{2^{k}+1})$  when  $k \geq 3$  and  $\operatorname{Rep}(G)$  when G is a complete multipartite graph or a disjoint union of complete graphs. He also asked about the behavior of the representation number for random graphs. Yahyaei and Katre [2841] gave upper and lower bounds for the representation number of a caterpillar and exact values in some cases.

Akhtar, Evans, and Pritikin [112] characterized the representation number of  $K_{1,n}$ using Euler's phi function, and conjectured that this representation number is always of the form  $2^a$  or  $2^a p$ , where  $a \ge 1$  and p is a prime. They proved this conjecture for "small" n and proved that for sufficiently large n, the representation number of  $K_{1,n}$  is of the form  $2^a$ ,  $2^a p$ , or  $2^a pq$ , where  $a \ge 1$  and p and q are primes. In [113] they showed that for sufficiently large  $n \ge m$ , rep $(K_{m,n}) = 2^a, 3^a, 2^a p^b$ , or  $2^a pq$ , where  $a, b \ge 1$  and p and q are primes; and for sufficiently large order, rep $(K_{n_1,n_2,...,n_t} = p^a, p^a q^b)$ , or  $p^a q^b u$ , where p, q, uare primes with p, q < u. Akhtar [114] determined the representation number of graphs of the form  $K_2 \cup nK_1$  (he uses the notation  $K_2 + nK_1$ ) and studies their prime decompositions. Using relations between representation modulo r and product representations, he determined representation number of binary trees and gave an improved lower bound for hypercubes.

#### 7.7 Product and Divisor Cordial Labelings

Sundaram, Ponraj, and Somasundaram [2535] introduced the notion of product cordial labelings. A product cordial labeling of a graph G with vertex set V is a function ffrom V to  $\{0,1\}$  such that if each edge uv is assigned the label f(u)f(v), the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a *product cordial* graph. In [2535] and [2544] Sundaram, Ponraj, and Somasundaram prove the following graphs are product cordial: trees; unicyclic graphs of odd order; triangular snakes; dragons; helms;  $P_m \cup P_n$ ;  $C_m \cup P_n$ ;  $P_m \cup K_{1,n}$ ;  $W_m \cup F_n$  ( $F_n$  is the fan  $P_n + K_1$ );  $K_{1,m} \cup K_{1,n}$ ;  $W_m \cup K_{1,n}$ ;  $W_m \cup P_n$ ;  $W_m \cup C_n$ ; the total graph of  $P_n$  (the total graph of  $P_n$  has vertex set  $V(P_n) \cup E(P_n)$  with two vertices adjacent whenever they are neighbors in  $P_n$ );  $C_n$  if and only if n is odd;  $C_n^{(t)}$ , the one-point union of t copies of  $C_n$ , provided t is even or both t and n are even;  $K_2 + mK_1$  if and only if m is odd;  $C_m \cup P_n$  if and only if m + n is odd;  $K_{m,n} \cup P_s$  if s > mn;  $C_{n+2} \cup K_{1,n}$ ;  $K_n \cup K_{n,(n-1)/2}$  when n is odd;  $K_n \cup K_{n-1,n/2}$  when n is even; and  $P_n^2$  if and only if n is odd. They also prove that  $K_{m,n}$  (m, n > 2),  $P_m \times P_n$  (m, n > 2) and wheels are not product cordial and if a (p,q)-graph is product cordial graph, then  $q \leq (p-1)(p+1)/4 + 1$ .

In [2237] Seoud and Helmi obtained the following results:  $K_n$  is not product cordial for all  $n \ge 4$ ;  $C_m$  is product cordial if and only if m is odd; the gear graph  $G_m$  is product cordial if and only if m is odd; all web graphs are product cordial; the corona of a triangular snake with at least two triangles is product cordial; the  $C_4$ -snake is product cordial if and only if the number of 4-cycles is odd;  $C_m \odot \overline{K_n}$  is product cordial; and they determine all graphs of order less than 7 that are not product cordial. Seoud and Helmi define the conjunction  $G_1 G_2$  of graphs  $G_1$  and  $G_2$  as the graph with vertex set  $V(G_1) \times V(G_2)$  and edge set  $\{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$ . They prove:  $P_m P_n (m, n \ge 2)$  and  $P_m S_n (m, n \ge 2)$  are product cordial. Nada, Diab, Elrokh, and Sabra [1816] proved that  $P_n \odot C_m$  is product cordial if and only if  $(n, m) \ne (1, 3) \pmod{4}$ . Gao, Lau, and Lee [834] investigated the friendly index and product-cordial index sets of a family of Möbius-like cubic graphs. Rokad [2132] proved the following graphs are product cordial: double wheels  $DW_n = 2C_n + K_1$ , path unions of finite number of copies of double wheels, the graphs obtained by joining two copies of double wheels by a path of arbitrary length,  $DW_n \oplus K_{1,n}$ , and  $DF_n \oplus K_{1,n} (DF_n = P_n + \overline{K}_2)$ .

Vaidya and Kanani [2639] prove the following graphs are product cordial: the path union of k copies of  $C_n$  except when k is odd and n is even; the graph obtained by joining two copies of a cycle by path; the path union of an odd number copies of the shadow of a cycle (see §3.8 for the definition); and the graph obtained by joining two copies of the shadow of a cycle by a path of arbitrary length. In [2642] Vaidya and Kanani prove the following graphs are product cordial: the path union of an even number of copies of  $C_n(C_n)$ ; the graph obtained by joining two copies of  $C_n(C_n)$  by a path of arbitrary length; the path union of any number of copies of the Petersen graph; and the graph obtained by joining two copies of the Petersen graph by a path of arbitrary length.

Vaidya and Barasara [2608] prove that the following graphs are product cordial: friendship graphs; the middle graph of a path; odd cycles with one chord except when the chord joins the vertices at a diameter distance apart; and odd cycles with two chords that share a common vertex and form a triangle with an edge of the cycle and neither chord joins vertices at a diameter apart. In [2623] Vaidya and Barasara investigated the product cordial labeling of the line graph of the middle graphs of paths, triangular snakes, armed crowns, the square of paths, the splitting graphs of paths, and the total graph of paths.

In [2628] Vaidya and Dani prove the following graphs are product cordial:  $\langle S_n^{(1)}: S_n^{(2)}: \ldots: S_n^{(k)} \rangle$  except when k odd and n even;  $\langle K_{1,n}^{(1)}: K_{1,n}^{(2)}: \ldots: K_{1,n}^{(k)} \rangle$ ; and  $\langle W_n^{(1)}: W_n^{(2)}: \ldots: W_n^{(k)} \rangle$  if and only if k is even or k is odd and n is even with k > n. (See  $\S3.7$  for the definitions.)

Vaidya and Barasara [2609] proved the following graphs are product cordial: closed helms, web graphs, flower graphs, double triangular snakes obtained from the path  $P_n$  if and only if n is odd, and gear graphs obtained from the wheel  $W_n$  if and only if n is odd. Vaidya and Barasara [2610] proved that the graphs obtained by the duplication of an edge of a cycle, the mutual duplication of pair of edges of a cycle, and mutual duplication of pair of vertices between two copies of  $C_n$  admit product cordial labelings. Moreover, if G and G' are the graphs such that their orders or sizes differ at most by 1 then the new graph obtained by joining G and G' by a path  $P_k$  of arbitrary length admits product cordial labeling.

Vaidya and Barasara [2611] define the duplication of a vertex v of a graph G by a new edge u'v' as the graph G' obtained from G by adding the edges u'v', vu' and vv' to G. They define the duplication of an edge uv of a graph G by a new vertex v' as the graph G' obtained from G by adding the edges uv' and vv' to G. They proved the following graphs have product cordial labelings: the graph obtained by duplication of an arbitrary vertex by a new edge in  $C_n$  or  $P_n$  (n > 2); the graph obtained by duplication of an arbitrary edge by a new vertex in  $C_n$  (n > 3) or  $P_n$  (n > 3); and the graph obtained by duplicating all the vertices by edges in path  $P_n$ . They also proved that the graph obtained by duplicating all the vertices by edges in  $C_n$  (n > 3) and the graph obtained by duplicating all the edges by vertices in  $C_n$  are not product cordial.

Recall (see [1976], [1963], [1964], [1965]) a double triangular snake  $DT_n$  consists of two triangular snakes that have a common path; a double quadrilateral snake  $DQ_n$  consists of two quadrilateral snakes that have a common path; an alternate triangular snake  $A(T_n)$  is the graph obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertex  $v_i$  (that is, every alternate edge of a path is replaced by  $C_3$ ); a double alternate triangular snake  $DA(T_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$ (alternatively) to two new vertices  $v_i$  and  $w_i$ ; an alternate quadrilateral snake  $A(Q_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertices  $v_i$  and  $w_i$  respectively and then joining  $v_i$  and  $w_i$  (that is, every alternate edge of a path is replaced by a cycle  $C_4$ ); a double alternate quadrilateral snake  $DA(Q_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertices  $v_i$  and  $w_i$  respectively and then joining  $v_i$  and  $w_i$  (that is, every alternate edge of a path is replaced by a cycle  $C_4$ ); a double alternate quadrilateral snake  $DA(Q_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertices  $v_i, x_i$  and  $w_i$ and  $y_i$  respectively and then joining  $v_i$  and  $w_i$  and  $x_i$  and  $y_i$ .

Vaidya and Barasara [2613] prove that the shell graph  $S_n$  is product cordial for odd n and not product cordial for even n. They also show that  $D_2(C_n)$ ;  $D_2(P_n)$ ;  $C_n^2$ ;  $M(C_n)$ ;  $S'(C_n)$ ; circular ladder  $CL_n$ ; Möbius ladder  $M_n$ ; step ladder  $S(T_n)$  and  $H_{n,n}$  does not admit product cordial labeling.

Vaidya and Vyas [2696] prove the following graphs are product cordial: alternate triangular snakes  $A(T_n)$  except  $n \equiv 3 \pmod{4}$ ; alternate quadrilateral snakes  $A(QS_n)$ except except  $n \equiv 2 \pmod{4}$ ; double alternate triangular snakes  $DA(T_n)$  and double alternate quadrilateral snakes  $DA(QS_n)$ . Vaidya and Vyas [2697] prove the following graphs are product cordial: the splitting graph of bistar  $S'(B_{n,n})$ ; duplicating each edge by a vertex in bistar  $B_{n,n}$  and duplicating each vertex by an edge in bistar  $B_{n,n}$ . They also proved that  $D_2(B_{n,n})$  is not product cordial. Ghodasara and Vaghasiya [887] prove the following graphs admit product cordial labelings: the path union of an odd number of copies of  $C_n$  with a chord except for n = 4, the path union of an odd number of copies of  $C_n$  with twin chords except when n = 6, the path union of  $C_n$  (n > 6) with three cords that form two triangles and a cycle of length n - 3, the graph obtained by joining two copies of the same cycle that has one chord by a path, the graph obtained by joining two copies of same cycle that has twin chords by a path, and the graph obtained by joining two copies of  $C_n$   $(n \ge 7)$  with three cords that form two triangles and a cycle of length n - 3 by a path. Ghodasara and Vaghasiya [888] prove the following graphs are product cordial: the path union of helms, the path union of closed helms, the path union of gear graphs  $G_n$  for odd n, the graph obtained by joining two copies of the same helm by a path, the graph obtained by joining two copies of the same closed helm by a path, and the graph obtained by joining two copies of same gear graph by a path.

In [225] Bapat proves the following graphs are product cordial: graphs obtained by identifying an endpoint of  $P_n$  with each vertex of  $C_3$ , graphs obtained by identifying an endpoint of  $P_n$  with each vertex of  $C_4$ , graphs obtained by identifying the degree m vertex of  $K_{1,m}$  with each vertex of  $C_3$ , and graphs obtained by identifying the degree m vertex of  $K_{1,m}$  with each vertex of the shell  $C_{n,n-3}$  ( $C_n$  with n-3 chords that share a common endpoint) if and only n is even or n is odd and m is even. In [224] Bapat proves  $K_5 \odot C_n$ and kayak paddles are product cordial, the one-point union of n copies of  $K_m$  is product cordial if and only if n is even, and graphs obtained by identifying one edge of  $K_5$  with each edge of  $P_n$  is product cordial if and only if n is even.

Prajapati and Raval [2028] investigated product cordial labelings of the graphs obtained by duplication of verticies and edges of gears and graphs obtained by the vertex switching operation of gears. In [2029] Prajapati and Raval proved that the book  $B_{m,n}$  is a product cordial graph if and only if m and n both are odd and  $m \geq 3$ . They showed that graphs obtained from books by duplicating or deleting vertices or edges are product cordial. For graphs with an even number of vertices they proved that the duplication of each of the vertices of a product cordial graph with an edge is a product cordial graph and that for graphs that have an odd number of vertices and even number of edges the duplication of each of the vertices of a product cordial graph with an edge is a product cordial graph.

Kwong, Lee, and Ng [1456] determine the product-cordial index sets of Möbius ladders and the graphs obtained by subdividing an edge of  $K_4$  and an edge of a Möbius ladder that is not a rung and joining the two new vertices by an edge. They show that no Möbius ladder is product cordial. Gao, Sun, Zhang, Meng, and Lau [830] provide sufficient conditions for a graph to admit (or not admit) a product cordial labeling. Gao, Lau, and Lee [829] investigated the friendly index and product-cordial index sets of a family of cubic graphs known as Möbius-like graphs. Prajapati and Raval [2027] proved that windmills, barbells, the one point union at the apex of copies of a wheel (generalized wheel), and the one point union of copies of a wheel connected at one common rim vertex of the wheel are product cordial graphs. They also showed that duplicating all rim edges with a vertex and duplicating all the vertices with an edge of generalized wheels, and the graphs obtained by switching an apex vertex in a generalized wheel are product cordial graphs. Patel, Prajapati, and Kansagara [1905] proved that graphs obtained from the barbell graph by duplicating all vertices by edges and duplicating all edges by vertices in the path joining complete graphs are product cordial and the graphs obtained by switching a vertex of path in the barbell graph are product cordial.

In [2156] Salehi called the set  $\{|e_f(0) - e_f(1)| : f \text{ is a friendly labeling of } G\}$  the product-cordial set of G. He determines the product-cordial sets for paths, cycles, wheels, complete graphs, bipartite complete graphs, double stars, and complete graphs with an edge deleted. Salehi and Mukhin [2165] say a graph G of size q is fully product-cordial if its product cordial set is  $\{q - 2k : 0 \le k \le \lfloor q/2 \rfloor\}$ . They proved:  $P_n$   $(n \ge 2)$  is fully product-cordial; trees with a perfect matching are fully product-cordial; and  $P_2 \times P_n$  is not fully product-cordial. They determine the product-cordial set is the multiplicative version of the friendly index set, Kwong, Lee, and Ng [1454] called it the product-cordial index set of  $C_m$  and  $C_m \times P_n$  and that  $P_m \times P_n$  has the maximum product cordial-index 2mn - m - n. In [1455] Kwong, Lee, and Ng determined the friendly index sets of any number of wheels. In [2159] z Salehi, Churchman, Hill, and Jordan determine the product-cordial index sets of certain classes of trees.

In [2340] Shiu and Kwong define the *full product-cordial index* of G under f as FPCI(G) =  $\{i_f^*(G) \mid f \text{ is a friendly labeling of } G\}$ . They provide a relation between the friendly index and the product-cordial index of a regular graph. As applications, they determine the full product-cordial index sets of  $C_m$  and  $C_m \times C_n$ , which was asked by Kwong, Lee, and Ng in [1454]. Shiu [2332] determined the product-cordial index sets of grids  $P_m \times P_n$ . Recall the *twisted cylinder* graph is the permutation graph on 4n ( $n \ge 2$ ) vertices,  $P(2n; \sigma)$ , where  $\sigma = (1, 2)(3, 4) \cdots (2n - 1, 2n)$  (the product of n transpositions). Shiu and Lee [2352] determined the full friendly index sets and the full product-cordial index.

Jeyanthi and Maheswari define a mapping  $f : V(G) \to \{0, 1, 2\}$  to be a 3-product cordial labeling if  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(i) - e_f(j)| \leq 1$  for any  $i, j \in \{0, 1, 2\}$ , where  $v_f(i)$  denotes the number of vertices labeled with  $i, e_f(i)$  denotes the number of edges xy with  $f(x)f(y) \equiv i \pmod{3}$ . A graph with a 3-product cordial labeling is called a 3-product cordial graph. In [1162] they prove that for a (p,q) 3-product cordial graph:  $p \equiv 0 \pmod{3}$  implies  $q \leq \frac{p^2 - 3p + 6}{3}$ ;  $p \equiv 1 \pmod{3}$  implies  $q \leq \frac{p^2 - 2p + 7}{3}$ ; and  $p \equiv 2 \pmod{3}$ implies  $q \leq \frac{p^2 - p + 4}{3}$ . They prove the following graphs are 3-product cordial: paths; stars;  $C_n$  if and only if  $n \equiv 1, 2 \pmod{3}$ ;  $C_n \cup P_n$ ,  $C_m \odot \overline{K_n}$ ;  $P_m \odot \overline{K_n}$  for  $m \geq 3$  and  $n \geq 1$ ;  $W_n$  when  $n \equiv 1 \pmod{3}$ ; and the graph obtained by joining the centers of two identical stars to a new vertex. They also prove that  $K_n$  is not 3-product cordial for  $n \geq 3$  and if  $G_1$  is a 3-product cordial graph, with 3m vertices and 3n edges and  $G_2$  is any 3-product cordial graph, then  $G_1 \cup G_2$  is a 3-product cordial graph. In [1163] they prove that ladders,  $\langle W_n^{(1)} : W_n^{(2)} : \ldots : W_n^{(k)} \rangle$  (see §3.7 for the definition), graphs obtained by duplicating an arbitrary edge of a wheel, graphs obtained by duplicating an

arbitrary vertex of a cycle or a wheel are 3-product cordial. They also prove that the graphs obtained by from the ladders  $L_n = P_n \times P_2$   $(n \ge 2)$  by adding the edges  $u_i v_{i+1}$ for  $1 \leq i \leq n-1$ , where the consecutive vertices of two copies of  $P_n$  are  $u_1, u_2, \ldots, u_n$ and  $v_1, v_2, \ldots, v_n$  and the edges are  $u_i v_i$ . They call these graphs triangular ladders. The graph  $B_{n,n}^*$  is obtained from the bistar  $B_{n,n}$  with  $V(B_{n,n}) = \{u, v, u_i, v_i \mid 1 \le i \le n\}$ and  $E(B_{n,n}) = \{uv, uu_i, vv_i, vu_i, uv_i \mid 1 \leq i \leq n\}$  by joining u with  $v_i$  and v with  $u_i$ for  $1 \leq i \leq 4$ . Jeyanthi and Maheswari [1170] proved: the splitting graphs  $S'(K_{1,n})$ and  $S'(B_{n,n})$  are 3-product cordial graphs;  $B_{n,n}^*$  is a 3-product cordial graph if and only if  $n \equiv 0, 1 \pmod{3}$ ; and the shadow graph  $D_2(B_{n,n})$  is a 3-product cordial graph if and only if  $n \equiv 0, 1 \mod 3$ . Jeyanthi, Maheswari, and Vijaya Laksmi [1185] prove the following: graphs obtained by switching an apex vertex in a closed helm are 3-product cordial;  $W_n$  are 3-product cordial if and only if  $n \equiv 2 \pmod{3}$ ; double fans are 3-product cordial if and only if  $n \equiv 0 \pmod{3}$ ; books are 3-product cordial; and permutation graphs  $P(K_2 + mK_1; T)$  are 3-product cordial if and only if  $m \equiv 2 \pmod{3}$ . In [1189] Jeyanthi, Maheswari, and Vijayalaksmi investigated the 3-product cordial behavior of alternate triangular snakes, double alternate triangular snakes, and triangular snakes.

Sundaram and Somasundaram [2539] also have introduced the notion of total product cordial labelings. A total product cordial labeling of a graph G with vertex set V is a function f from V to  $\{0,1\}$  such that if each edge uv is assigned the label f(u)f(v) the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a total product cordial graph. In [2539] and [2537] Sundaram, Ponraj, and Somasundaram prove the following graphs are total product cordial: every product cordial graph of even order or odd order and even size; trees; all cycles except  $C_4$ ;  $K_{n,2n-1}$ ;  $C_n$  with m edges appended at each vertex; fans; double fans; wheels; helms;  $C_n \times P_2$ ;  $K_{2,n}$  if and only if  $n \equiv 2 \pmod{4}$ ;  $P_m \times P_n$  if and only if  $(m, n) \neq (2, 2)$ ;  $C_n + 2K_1$  if and only if n is even or  $n \equiv 1 \pmod{3}$ ;  $K_n \times 2K_2$  if n is odd, or  $n \equiv 0$  or 2 (mod 6), or  $n \equiv 2 \pmod{8}$ . Y.-L. Lai, the reviewer for MathSciNet [1458], called attention to some errors in [2537]. Pedrano and Rulete [1912] determined the total product cordial labeling of  $P_m \times C_n$ ,  $C_m \times C_n$  and the generalized Petersen graph P(m, n). In [1913] Pedrano and Rulete determined the total product cordial labeling of  $P_m \odot C_n$ ,  $P_m \odot P_n$ ,  $C_m \odot P_n$ ,  $P_m \odot F_n$ ,  $P_m \odot W_n$ , and  $P_m \odot K_n$ . Villar [2719] proved  $P_n \odot C_m$   $(n \ge 2, m \ge 3)$  is product cordial,  $P_n \odot P_m$   $(n, m \ge 2)$  is product cordial except when n and m are both even, and  $P_{2n+1} \odot K_m$   $(n \ge 1, m \ge 4)$ is not product cordial. Gao, Sun, Zhang, Meng, and Lau [830] proved that  $P_{n+1}^m$  is total product cordial. Ramanjaneyulu, Venkaiah, and Kothapalli [2081] give total product cordial labeling for a family of planar graphs for which each face is a 4-cycle.

Vaidya and Vihol [2679] prove the following graphs have total product cordial labelings: a split graph; the total graph of  $C_n$ ; the star of  $C_n$  (recall that the *star of a graph* G is the graph obtained from G by replacing each vertex of star  $K_{1,n}$  by a graph G); the friendship graph  $F_n$ ; the one point union of k copies of a cycle; and the graph obtained by the switching of an arbitrary vertex in  $C_n$ .

Sundaram, Ponraj, and Somasundaram [2542] introduced the notion of EP-cordial labeling (extended product cordial) labeling of a graph G as a function f from the

verticies of a graph to  $\{-1, 0, 1\}$  such that if each edge uv is assigned the label f(u)f(v), then  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(i) - e_f(j)| \leq 1$  where  $i, j \in \{-1, 0, 1\}$  and  $v_f(k)$  and  $e_f(k)$  denote the number of vertices and edges respectively labeled with k. An EPcordial graph is one that admits an EP-cordial labeling. In [2542] Sundaram, Ponraj, and Somasundaram prove the following: every graph is an induced subgraph of an EP-cordial graph,  $K_n$  is EP-cordial if and only if  $n \leq 3$ ;  $C_n$  is EP-cordial if and only if  $n \equiv 1, 2 \pmod{3}$ ,  $W_n$  is EP-cordial if and only if  $n \equiv 1 \pmod{3}$ ; and caterpillars are EP-cordial. They prove that all  $K_{2,n}$ , paths, stars and the graphs obtained by subdividing each edge of of a star exactly once are EP-cordial. They also prove that if a (p,q) graph is EP-cordial, then  $q \leq 1 + p/3 + p^2/3$ . They conjecture that every tree is EP-cordial.

Ponraj, Sivakumar, and Sundaram [1998] introduced the notion of k-product cordial labeling of graphs. Let f be a map from V(G) to  $\{0, 1, 2, \dots, k-1\}$ , where  $2 \le k \le |V|$ . For each edge uv assign the label  $f(u)f(v) \pmod{k}$ . f is called a k-product cordial labeling if  $|v_f(i) - v_f(j)| \le 1$  and  $|e_f(i) - e_f(j)| \le 1$ ,  $i, j \in \{0, 1, 2, \dots, k-1\}$ , where  $v_f(x)$ and  $e_f(x)$  denote the number of vertices and edges labeled with x. A graph with a kproduct cordial labeling is called a *k*-product cordial graph. Observe that 2-product cordial labeling is simply a product cordial labeling and 3-product cordial labeling is an EP-cordial labeling. In [1998] and [1999] Ponraj et al. prove the following are 4-product cordial:  $P_n$  if and only  $n \leq 11$ ,  $C_n$  if and only if n = 5, 6, 7, 8, 9, or 10,  $K_n$  if and only if  $n \leq 2$ ,  $P_n \odot K_1$ ,  $P_n \odot 2K_1, K_{2,n}$  if and only if  $n \equiv 0, 3 \pmod{4}, W_n$  if and only if n = 5 or  $9, \overline{K_n} + 2K_2$  iff  $n \leq 2$ , and the subdivision graph of  $K_{1,n}$ . Sivakumar [2414] proved the following coronas are 4-total product cordial:  $P_n \odot K_1$ ,  $P_n \odot 2K_1$ ,  $S(P_n \odot K_1)$ ,  $S(P_n \odot 2K_1)$ ,  $S(C_n \odot K_1)$  and  $S(C_n \odot 2K_1)$ . Jeyanthi, Maheswari, and Vijayalakshmi [1184] investigated the 3-product cordial behavior of alternate triangular snakes, double alternate triangular snakes, and triangular snake graphs. In [1186] they establish that vertex switching graphs of wheels, gears, and degree splitting of bistars are 3-product cordial graphs.

Let f be a map from V(G) to  $\{0, 1, 2, \dots, k-1\}$  where  $2 \le k \le |V|$ . For each edge uv assign the label  $f(u)f(v) \pmod{k}$ . Ponraj, Sivakumar, and Sundaram [2000] define f to be a k-total product cordial labeling if  $|f(i) - f(j)| \leq 1, i, j \in \{0, 1, 2, \dots, k-1\},\$ where f(x) denote the number of vertices and edges labeled with x. A graph with a ktotal product cordial labeling is called a k-total product cordial graph. A 2-total product cordial labeling is simply a total product cordial labeling. In [2000], [2001], [2002], [2003] and [2004], Ponraj et al. proved the following graphs are 3-total product cordial:  $P_n$ ,  $C_n$ if and only if  $n \neq 3$  or 6,  $K_{1,n}$  if and only if  $n \equiv 0, 2 \pmod{3}$ ,  $P_n \odot K_1$ ,  $P_n \odot 2K_1$ ,  $K_2 + mK_1$  if and only if  $m \equiv 2 \pmod{3}$ , helms, wheels,  $C_n \odot 2K_1$ ,  $C_n \odot K_2$ , dragons  $C_m @P_n$  (obtained by identitying an endpoint of  $P_n$  with a vertex of  $C_m$ ),  $C_n \odot K_1$ , bistars  $B_{m,n}$ , and the subdivision graphs of  $K_{1,n}$ ,  $C_n \odot K_1$ ,  $K_{2,n}$ ,  $P_n \odot K_1$ ,  $P_n \odot 2K_1$ ,  $C_n \odot K_2$ , wheels and helms. They also proved that every graph is a subgraph of a connected k-total product cordial graph,  $B_{m,n}$  is (n+2)-total product cordial, and  $K_{m,n}$  is (n+2)-total product cordial. Sharon Philomena and Thirusangu [1928] proved that the flower graph is 3-total product cordial. Ahmada, Bača, Naseemc, and Semaničová-Feňovčíková [82] described a method for obtaining a 3-total edge product cordial labeling of the hexagonal grid from a smaller hexagonal grid. In [66] Ahmad proved that the generalized Petersen

graphs P(n, m) are 3-total edge product cordial. In [213] Azaizeh, Hasni, Lau, and Ahmad proved that complete graphs, bipartite graphs and generalised friendship graphs have 3total edge product cordial labelings. Ahmad, Ali, Bilal, Zafar, and Zahid [69] prove that webs, helms, gears, and  $L_n \odot 2K_1$  ( $L_n$  is the laddar with 2n vertices) have 3-total edge product cordial labelings. Ivančo [1082] characterized graphs admitting a 2-total edge product cordial labeling and proved that dense graphs and regular graphs of degree 2(k-1) admit a k-total edge product cordial labeling. Javed and Jamil [1105] proved that the rhombic grid graphs  $R_1^m$ ,  $R_2^m$  and  $R_3^m$  are 3-total edge product cordial for  $m \ge 1$ and that the rhombic grid graph  $R_n^m$  is 3-total edge product cordial for  $m, n \ge 1$ .

In [1444] Kumari and Mehra call a vertex labeling f of a graph G with 0 and 1 with the induced edge labeling f given by f(uv) = f(u)f(v) a vertex product cordial labeling if the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. They prove the following graphs have vertex product cordial labelings:  $P_n^2$  if and only if n is odd, the path unions of k copies of  $P_n^2$ ,  $P_n \odot K_1$ , helms, gears  $G_n$ for odd n, graphs obtained from  $C_n$  after switching of a vertex,  $C_n \odot K_1$ ,  $C_n \odot \bar{K}_m$ , and certain banana trees.

For a graph G Sundaram, Ponraj, and Somasundaram [2543] defined the *index of* product cordiality,  $i_p(G)$ , of G as the minimum of  $\{|e_f(0) - e_f(1)|\}$  taken over all the 0-1 binary labelings f of G with  $|v_f(i) - v_f(j)| \leq 1$  and f(uv) = f(u)f(v), where  $e_f(k)$ and  $v_f(k)$  denote the number of edges and the number of vertices labeled with k. They established that  $i_p(K_n) = \lfloor n/2 \rfloor^2$ ;  $i_p(C_n) = 2$  if n is even;  $i_p(W_n) = 2$  or 4 according as n is even or odd;  $i_p(K_{2,n}) = 4$  or 2 according as n is even or odd;  $i_p(K_2 + nK_1) = 3$  if n is even;  $i_p(G \times P_2) \leq 2i_p(G)$ ;  $i_p(G_1 \cup G_2) \leq i_p(G_1) + i_p(G_2) + 2\min\{\Delta(G_1), \Delta(G_2)\}$  where  $G_1$  and  $G_2$  are graphs of odd order; and  $i_p(G_1 \odot G_2) \leq i_p(G_1) + i_p(G_2) + 2\delta(G_2) + 3$  where  $G_1$  and  $G_2$  have odd order.

In [2581] Tenguria and Verma called a mapping f from V(G) to  $\{0, 1, 2\}$  such that each edge uv is labeled  $(f(u)+f(v)) \mod 3$  a 3-total super sum cordial labeling if  $|f(i)-f(j)| \leq 1$ for  $i, j \in \{0, 1, 2\}$ , where f(x) denotes the total number of vertices and edges labeled with x and for each edge  $uv, |f(u) - f(v)| \leq 1$ . A graph that has a 3-total super sum cordial labeling is called 3-total super sum cordial graph. They proved  $P_m \cup P_n, C_m \cup C_n$ , and  $K_{1,m} \cup K_{1,n}$  are 3-total super sum cordial graphs. (These results also appeared in [2582] and [2583]).

Vaidya and Vyas [2687] define the tensor product  $G_1(T_p)G_2$  of graphs  $G_1$  and  $G_2$  as the graph with vertex set  $V(G_1) \times V(G_2)$  and edge set

 $\{(u_1, v_1)(u_2, v_2)| u_1u_2 \in E(G_1), v_1v_2 \in V(G_2)\}$ . They proved the following graphs are product cordial:  $P_m(T_p)P_n$ ;  $C_{2m}(T_p)P_{2n}$ ;  $C_{2m}(T_p)C_{2n}$ ; the graph obtained by joining two components of  $P_m(T_p)P_n$  an by arbitrary path; the graph obtained by joining two components of  $C_{2m}(T_p)P_{2n}$  by an arbitrary path; and the graph obtained by joining two components of  $C_{2m}(T_p)P_{2n}$  by an arbitrary path.

In [1938] Ponraj introduced the notion of an  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial labeling of a graph. Let  $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$  be a finite set of distinct integers and f be a function from a vertex set V(G) to S. For each edge uv of G assign the label f(u)f(v). He calls f an  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial labeling of G if  $|v_f(\alpha_i) - v_f(\alpha_j)| \leq 1$  for all  $i, j \in \{1, 2, \ldots, k\}$  and  $|e_f(\alpha_i \alpha_j) - e_f(\alpha_r \alpha_s)| \leq 1$  for all  $i, j, r, s \in \{1, 2, \ldots, k\}$ , where  $v_f(t)$  and  $e_f(t)$  denote the number of vertices labeled with t and the number of edges labeled with t, respectively. A graph that admits an  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial labeling is called an  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial graph. Note that an  $(-\alpha, \alpha)$ -cordial graph is simply a cordial graph and a  $(0, \alpha)$ -cordial graph is a product cordial graph. Ponraj proved that  $K_{1,n}$  is  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial if and only if  $n \leq k$  and for  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ ,  $\alpha_1 + \alpha_2 \neq 0$  proved the following:  $K_n$  is  $(\alpha_1, \alpha_2)$ -cordial if and only if  $n \leq 2$ ;  $P_n$  is  $(\alpha_1, \alpha_2)$ -cordial;  $C_n$  is  $(\alpha_1, \alpha_2)$ -cordial if and only if  $n \geq 3$ ;  $K_{m,n}$  (m, n > 2) is not  $(\alpha_1, \alpha_2)$ -cordial; the bistar  $B_{n,n+1}$  is  $(\alpha_1, \alpha_2)$ -cordial;  $B_{n+2,n}$  is  $(\alpha_1, \alpha_2)$ -cordial if and only if  $n \equiv 1, 2 \pmod{3}$ ;  $B_{n+3,n}$  is  $(\alpha_1, \alpha_2)$ -cordial graph with p vertices and q edges, then  $q \leq 3p^2/8 - p/2 + 9/8$ . In [1938] Ponraj proved that combs  $P_n \odot K_1$  are  $(\alpha_1, \alpha_2)$ -cordial; coronas  $C_n \odot K_1$  are  $(\alpha_1, \alpha_2)$ -cordial for  $n \equiv 0, 2, 4, 5 \pmod{6}$ ;  $C_3^{(t)}$  is not  $(\alpha_1, \alpha_2)$ -cordial;  $W_n$  is not  $(\alpha_1, \alpha_2)$ -cordial; and only if n = 2.

In [2703] Varatharajan, Navanaeethakrishnan Nagarajan define a divisor cordial labeling of a graph G with vertex set V as a bijection f from V to  $\{1, 2, \ldots, |V|\}$  such that an edge uv is assigned the label 1 if one f(u) or f(v) divides the other and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If graph that has a divisor cordial labeling, it is called a divisor cordial graph. They proved the standard graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial. They also proved that complete graphs are not divisor cordial. In [2704] they proved dragons, coronas, wheels, and complete binary trees are divisor cordial. For t copies  $S_1, S_2, \ldots, S_t$  of an n-star  $K_{1,n}$  they define  $\langle S_1, S_2, \ldots, S_t \rangle$  as the graph obtained by starting with  $S_1, S_2, \ldots, S_t$  and joining the central vertices of  $S_{k-1}$ and  $S_k$  to a new vertex  $x_{k-1}$ . They prove that  $\langle S_1, S_2 \rangle$  and  $\langle S_1, S_2, S_3 \rangle$  are divisor cordial.

Vaidya and Shah [2668] proved that the splitting graphs of stars and bistars are divisor cordial and the shadow graphs and the squares of bistars are divisor cordial. In [2670] they proved that helms, flower graphs, and gears are divisor cordial graphs. They also proved that graphs obtained by switching of a vertex in a cycle, switching of a rim vertex in a wheel, and switching of an apex vertex in a helm admit divisor cordial labelings. Raj and Valli [2062] proved the following graphs divisor cordial: the duplication of a vertex of a cycle; graphs obtained by joining two wheels of the same size by a path of length at least 3;  $G_v \odot K_1$ , where  $G_v$  is a graph obtained by switching any vertex of a cycle of size at least 4; graphs obtained by joining the apex vertices of two shells of the same size to an isolated vertex; graphs obtained by joining the centers of two wheels of the same size to an isolated vertex; and a class of graphs obtained by removing certain edges from complete graphs. Bosmia and Kanani [523] proved that the graphs of the form  $G \odot K_1$ where G any of the following admits a divisor cordial labeling:  $K_{1,n}$ ,  $K_{2,n}$ ,  $K_{3,n}$ , a wheel, a helm, a flower, a fan, a double fan, and a barycentric subdivision of a star. Bosmia and Kanani [524] prove that the following graphs admit divisor cordial labelings: bistars, the splitting graph of bistars, the degree splitting graph of bistars, the shadow graph of bistars, the restricted square graph of bistars, the barycentric subdivision of bistars, and

the corona product of a bistar with  $K_1$ . Thirusangu and Madhu [2589] proved that the extended duplicate graph of star and bistar graphs are divisor cordial graphs.

In [1777] Sugumaran and Mohan proved that the following graphs are divisor cordial graphs: degree splitting graph of  $K_{1,n,n}$ ), the splitting graph of the graph obtained from two isolated vertices are joined by n paths of length 2, the W- graph (the graph obtained from two copies of  $K_{1,n}$  and identifying the last pendent vertex of the first copy with the first pendent edge of the second copy),  $B(n) \odot_{u_m} K_1$ , where  $B(n) = 2P_n + K_1$  (bow graph, the Herschel graph  $H_s$  and switching of an apex vertex in the Herschel graph (see [491, p. 53]). In [2508] Sugumaran and Suresh proved that the following graphs are divisor cordial graphs: fans, Petersen graphs,  $C_m \odot K_1$ , friendship graphs  $F_n$ , and the switching of a end vertex in path  $P_n$ , switching of any one of the inner vertices of Petersen graph  $P_e$ .

Motivated by the concept of divisor cordial labeling, Lourdusamy and Patrick [1649] introduced a new concept of divisor cordial labeling called sum divisor cordial labeling. Let G = (V(G), E(G)) be a simple graph and f be a bijection from V(G) to  $\{1, 2, \ldots, |V(G)|\}$ . For each edge uv, assign the label 1 if 2 divides f(u) + f(v) and the label 0 otherwise. The function f is called a *sum divisor cordial* labeling if the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph which admits a sum divisor cordial labeling is called a {it sum divisor cordial. They prove that paths, combs, stars, complete bipartite,  $K_2 + mK_1$ , bistars, jewels, crowns, flowers, gears, subdivisions of stars, the graph obtained from  $K_{1,3}$  by attaching the root of  $K_{1,n}$  at each pendent vertex of  $K_{1,3}$ , and the square  $B_{n,n}$  are sum divisor cordial graphs.

In [2520], [2521], [2522], and [2523] Sugumaran and Rajesh proved that the following graphs are sum divisor cordial: swastiks, path unions of finite number of copies of swastiks, cycles of k copies of swastiks, when k is odd, jelly fish, Petersen graphs, theta graphs, the fusion of any two vertices in the cycle of swastiks, duplication of any vertex in the cycle of swastiks, the switchings of a central vertex in swastiks, the path unions of two copies of a swastik, the star graph of the theta graphs, the Herschel graph, the fusion of any two adjacent vertices of degree 3 in Herschel graphs, the duplication of any vertex of degree 3 in the Herschel graph, the switching of central vertex in Herschel graph, the path union of two copies of the Herschel graph, H-graph  $H_n$ , when n is odd,  $C_3@K_{1,n}$ (obtained by identifying the center of  $K_{1,n}$  with a vertex of  $C_3$ ),  $\langle F_n^1 \Delta F_n^2 \rangle$  (the graph obtained by joining the apex vertices of  $F_n^1$  and  $F_n^2$  by an edge and by joining the two apex vertices to a new vertex v') and open star of swastik graphs  $S(t.Sw_n)$ , when t is odd. In [2524], [2525] and [2526] Sugumaran and Rajesh proved that the following graphs are sum divisor cordial graphs: H-graph  $H_n$ , when n is even, duplication of all edges of the H-graph  $H_n$ , when n is even,  $H_n \odot K_1$ ,  $P(r.H_n)$ ,  $C(r.H_n)$ , plus graphs, umbrella graphs, path unions of odd cycles, kites, complete binary trees, drums graph (two copies of  $C_n$ that share exactly one vertex v and two copies of  $P_n$  that have an end point at v, twigs (graphs obtained from a aath by attaching exactly two pendent edges to each internal vertices of the path), fire crackers of the form  $P_n \odot S_n$ , where n is even, and the double arrow graph  $DA_m^n$ , where  $|m-n| \leq 1$  and n is even (obtained from  $P_m \times P_n$  by adding two new vertices u and v such that each of the top row vertices of  $P_m \times P_n$  are connected

to u by an edge and the bottom row vertices of  $P_m \times P_n$  are connected to v by an edge). Sugumaran and Rajesh [2507] proved that the following graphs are sum divisor cordial:  $P_n + P_n$  (n is odd),  $P_n@K_{1,m}$  (obtained by identifying an endpoint of  $P_n$  with the center of  $K_{1,n}$ ),  $C_n@K_{1,m}$  (n is odd), the graph obtained from  $W_{2n}$  by attaching the apex vertex of a copy of  $K_{1,m}$  to each rim vertex, the graph obtained by joining the central vertices of two copies of  $K_{1,n,n}$  by an edge and to a new vertex, the graph obtained by starting with  $C_n$  and, for each edge of  $C_n$ , adjoining a copy of  $C_n$  that shares an edge with the starting copy (the flower graph  $FL_n$ ). In [49] Adalja and Ghodasara provide sum divisor cordial labelings for the graphs resulting from the duplication of graph elements in stars, cycles, and paths.

Murugesan [1806] introduced a square divisor cordial labeling. Let G be a simple graph and  $f :\to \{1, 2, \ldots, |V(G)|\}$  a bijection. For each edge uv, assign the label 1 if either  $(f(u))^2$  divides f(v) or  $(f(v))^2$  divides f(u) and the label 0 otherwise. Call f a square divisor cordial labeling if  $|e_f(0) - e_f(1)| \leq 1$ . A graph with a square divisor cordial labeling is called a square divisor cordial graph. Murugesan proved that the following are square divisor cordial graphs:  $P_n$   $(n \leq 12)$ ,  $C_n$   $(3 \leq n \leq 11)$ , wheels, some stars, some complete bipartite graphs, and some complete graphs. Vaidya and Shah [2674] proved that the following are square divisor cordial graphs: flowers, bistars, shadow graphs of stars, splitting graphs of stars and bistars, degree splitting graphs of paths and bistars.

Kanani and Bosmia [1272] define a cube divisor cordial labeling f of a simple graph Gas a bijection from V(G) to  $\{1, 2, ..., |V(G)|\}$  such that, when each edge uv is assigned the label 1 if  $(f(u))^3$  divides f(v) or  $(f(v))^3$  divides f(u) and the label 0 otherwise, it holds that  $|e_f(0) - e_f(1)| \leq 1$ . A graph with a cube divisor cordial labeling is called a *cube* divisor cordial graph. They proved that the following graphs admit cube divisor cordial labelings:  $K_n$  if and only if n = 1, 2, 3;  $K_{1,n}$  if and only if n = 1, 2, 3;  $K_{2,n}$  for all n;  $K_{3,n}$ if and only if n = 1, 2; bistars  $B_{n,n}$  for all n; and the graph obtained by joining leaves of one star of a bistar with the center of the opposite star of the bistar. Kanani and Bosmia [1272] prove: the edge deleted graph of a cube divisor cordial graph is also a cube divisor cordial graph;  $P_n$  is a cube divisor cordial graph if and only if n = 1, 2, 3, 4, 5, 6, 8;  $C_n$  is a cube divisor cordial graph if and only if n = 3, 4, 5; and wheels, flowers and fans are cube divisor cordial,

The Lucas sequence of numbers is a linear recurrence relation satisfying the conditions:  $l_1 = 1, l_2 = 3$  and  $l_n = l_{n-1} + l_{n-2}, n \ge 3$ . Let G = (V, E) be a simple graph and  $f: V(G) \to \{l_1, l_2, \ldots, l_{|V(G)|}\}$  be a bijection such that each edge uv, assign the label 1 if either f(u) divides f(v) or f(v) divides f(u) and label 0 otherwise. In [2528] Sugumaran and Rajesh call such an f a Lucas divisor cordial labeling if  $|e_f(0) - e_f(1)| \le 1$ . A graph with a Lucas divisor cordial labeling is called a Lucas divisor cordial graph. In [2528] Sugumaran and Rajesh proved that the following graphs are Lucas divisor cordial graphs: bistars, jelly fish, square graphs of bistars, switching of a vertex in cycles, and switching of a pendent vertex in paths.

A variation of divisor cordial labeling called vertex odd divisor cordial labeling was introduced by Muthaiyan and Pugalenthi (see [1807]) as follows. Let G be a graph with p vertices and a bijection f from V(G) to  $\{1, 3, 5, \ldots, 2p - 1\}$  such that if each edge uv is

assigned the label 1 if f(u) divides f(v) or f(v) divides f(u), and the label 0 otherwise. The function f is called a vertex odd divisor cordial labeling if  $|e_f(0) - e_f(1)| \leq 1$ . A graph with vertex odd divisor cordial labeling is called a vertex odd divisor cordial graph. Muthaiyan and Pugalenthi (see [1807]) proved paths, cycles,  $K_{2,n}, K_{1,n} \cup K_{1,m}$ , helms, flowers,  $\langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \rangle$ , the switching of the apex vertex in helms, and the splitting graph of stars are vertex odd divisor cordial graphs under some conditions. In [1807] Muthaiyan and Pugalenthi proved the following graphs have vertex odd divisor cordial labelings: wheels, the switching of a pendent vertex in paths and cycles, bistars  $B_{n,n}$ , the subdivision graph of  $K_{1,n}, B_{n,n}^2, DS(B_{n,n})$ , the splitting graph of  $B_{n,n}$ , and  $\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)} \rangle$ . Let  $G_1$  and  $G_2$  be two copies of any graph G that has an apex vertex. The graph

Let  $G_1$  and  $G_2$  be two copies of any graph G that has an apex vertex. The graph obtained by joining the apex vertices of  $G_1$  and  $G_2$  by an edge and bThe graph obtained by joining the apex vertices of  $G_1$  and  $G_2$  by an edge and by joining the two apex vertices to a new vertex v', is denoted  $G_1 \triangle G_2$ . y joining the two apex vertices to a new vertex v', is denoted  $G_1 \triangle G_2$ . For any vertex u of  $K_{m,n}$  the graph obtained by joining u to a new pendent vertex is denoted by  $K_{m,n} \odot u(K_1)$ . In [2512] Sugumaran and Suresh proved that the following graphs are vertex odd divisor cordial graphs: the shadow graph of  $K_{1,n}$ ,  $K_{2,n} \odot u(K_1)$ ,  $K_{1,n} \triangle K_{1,n}$ , the subdivision of the edge between the apex vertices of  $B_{n,n}$ , and the graph  $K_{1,n} * P_{n+2}$  (the graph obtained by identifying an end vertex of  $P_{n+2}$  with the apex vertex of  $K_{1,n}$ ). In [2511] they showed that the graphs  $F_n \triangle F_n$ ,  $K_{1,n} \triangle K_{1,n} \triangle K_{1,n}$ ,  $K_{1,n} \triangle K_{1,n} \triangle K_{1,n}$ , theta graphs, and switching of a vertex in a Petersen graph are vertex in  $S(K_{1,n})$ ,  $P_2 + mK_1$ , C(n, n-3), and C(n, n-4) are vertex odd divisor cordial graphs. In [2509] they showed that the globe Gl(n), jewels,  $G * W_n$  (appending the central vertex of wheel  $W_n$  with any one of the vertices of G), G \* C(n, n-3), and wheels are vertex odd divisor cordial graphs.

Let  $f: V(G) \to \{1, 2, ..., |V(G)|\}$  be an injective map. For each edge uv assign the label r where r is the remainder when f(u) is divided by f(v) or f(v) is divided by f(u)according as  $f(u) \ge f(v)$  or  $f(v) \ge f(u)$ . The function f is called a *remainder cordial* labeling of G if  $|e_f(0) - e_f(1)| \le 1$  where  $e_f(0)$  and  $e_f(1)$  respectively denote the number of edges labelled with even integers and number of edges labelled with odd integers. A graph G with admits a remainder cordial labeling is called a *remainder cordial* graph. In [1953] investigated the remainder cordial behavior of  $S(K_{1,n}), S(B_{n,n}), S(W_n)$  and union of some star related graphs.

## 7.8 Edge Product Cordial Labelings

Vaidya and Barasara [2614] introduced the concept of edge product cordial labeling as edge analogue of product cordial labeling. An *edge product cordial labeling* of graph Gis an edge labeling function  $f : E(G) \to \{0, 1\}$  that induces a vertex labeling function  $f^* : V(G) \to \{0, 1\}$  defined as  $f^*(u) = \prod\{f(uv) \mid uv \in E(G)\}$  such that the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1 and the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1. A graph with an edge product cordial labeling is called an *edge product cordial*  graph.

In [2614], [2616], [2617], [2618], and [2621] Vaidya and Barasara proved the following graphs are edge product cordial:  $C_n$  for n odd; trees with order greater than 2; unicyclic graphs of odd order;  $C_n^{(t)}$ , the one point union of t copies of  $C_n$  for t even or t and n both odd;  $C_n \odot K_1$ ; armed crowns  $C_m \odot P_n$ ; helms; closed helms; webs; flowers; gears; shells  $S_n$  for odd n; tadpoles  $C_n @P_m$  for m + n even or m + n odd and m > n while not edge product cordial for m + n odd and m < n; triangular snakes; for odd n, double triangular snakes  $DT_n$ , quadrilateral snakes  $Q_n$  and double quadrilateral snakes  $DQ_n$ ;  $P_n^2$  for odd n;  $M(P_n)$ ,  $T(P_n)$ ;  $S'(P_n)$  for even n; the tensor product of  $P_m$  and  $P_n$ ; and the tensor product of  $C_n$  and  $C_m$  if m and n are even. In [2622] Vaidya and Barasara investigate product and edge product cordial labelings of the degree splitting graphs of paths, shells, bistars, and gear graphs.

Vaidya and Barasara proved the following graphs are not edge product cordial:  $C_n$  for n even;  $K_n$  for  $n \ge 4$ ;  $K_{m,n}$  for  $m, n \ge 2$ ; wheels; the one point union of t copies of  $C_n$  for t odd and n even; shells  $S_n$  for even n; tadpoles  $C_n@P_m$  for m + n odd and m < n; for n even double triangular snake  $DT_n$ , quadrilateral snake  $Q_n$  and double quadrilateral snake  $DQ_n$ ; double fans;  $C_n^2$  for n > 3;  $P_n^2$  for even n;  $D_2(C_n)$ ,  $D_2(P_n)$ ;  $M(C_n)$ ;  $T(C_n)$ ;  $S'(C_n)$ ;  $S'(P_n)$  for odd n;  $P_m \times P_n$  and  $C_m \times C_n$ ; the tensor product of  $C_n$  and  $C_m$  if m or n odd; and  $P_n[P_2]$  and  $C_n[P_2]$ .

Prajapati and Shah [2031] proved the following graphs are edge product cordial: graphs obtained from a crown by duplication of a vertex, duplication of a vertex by an edge, or duplication of an edge by a vertex; graphs obtained from a gear graph by duplication of each of the vertices of degree three by an edge; and the graph obtained from a helm by duplication of each of the pendent vertices by a new vertex. In [2025] Prajapati and Patel provided results about the existence of edge product cordial labelings closed webs, lotus inside a circle, and sunflower graphs.

Vaidya and Barasara [2619] introduced the concept of a total edge product cordial labeling as edge analogue of total product cordial labeling. An *total edge product cordial labeling* of graph G is an edge labeling function  $f : E(G) \to \{0, 1\}$  that induces a vertex labeling function  $f^* : V(G) \to \{0, 1\}$  defined as  $f^*(u) = \prod\{f(uv) \mid uv \in E(G)\}$  such that the number of edges and vertices labeled with 0 and the number of edges and vertices labeled with 1 differ by at most 1. A graph with total edge product cordial labeling is called a *total edge product cordial graph*.

In [2619] and [2620] Vaidya and Barasara proved the following graphs are total edge product cordial:  $C_n$  for  $n \neq 4$ ;  $K_n$  for n > 2;  $W_n$ ;  $K_{m,n}$  except  $K_{1,1}$  and  $K_{2,2}$ ; gears;  $C_n^{(t)}$ , the one point union of t copies of  $C_n$ ; fans; double fans;  $C_n^2$ ;  $M(C_n)$ ;  $D_2(C_n)$ ;  $T(C_n)$ ;  $S'(C_n)$ ;  $P_n^2$  for n > 2;  $M(C_n)$ ;  $D_2(C_n)$  for n > 2;  $T(C_n)$ ;  $S'(C_n)$ . Moreover, they prove that every edge product cordial graph of either even order or even size admits total edge product cordial labeling. Bača, Irfan, Javad, and Semaničová-Feňočová [275] investigated the existence of total edge product cordial labeling of toroidal fullerenes and for Kleinbottle fullerenes. Prajapati and Patel [2026] proved that the one point union of t copies of a wheel with a rim vertex in common is edge product cordial if and only is t is even; all pentagonal snakes (obtained from the path by replacing every edge of a path by  $C_5$ ) are edge product cordial; and a double pentagonal snakes (two pentagonal snakes that have a common path) is edge product cordial if and only is t is odd.

#### 7.9 Difference Cordial Labelings

Ponraj, Sathish Narayanan, and Kala [1975] introduced the notion of difference cordial labelings. A difference cordial labeling of a graph G is an injective function f from V(G) to  $\{1, \ldots, |V(G)|\}$  such that if each edge uv is assigned the label |f(u) - f(v)|, the number of edges labeled with 1 and the number of edges not labeled with 1 differ by at most 1. A graph with a difference cordial labeling is called a difference cordial graph.

The following definitions appear in [1976], [1963], [1964], and [1965]. A double triangular snake  $DT_n$  consists of two triangular snakes that have a common path; a double quadrilateral snake  $DQ_n$  consists of two quadrilateral snakes that have a common path; an alternate triangular snake  $A(T_n)$  is the graph obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertex  $v_i$  (that is, every alternate edge of a path is replaced by  $C_3$ ); a double alternate triangular snake  $DA(T_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to two new vertices  $v_i$  and  $w_i$ ; an alternate quadrilateral snake  $A(Q_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to new vertices  $v_i$  and  $w_i$  respectively and then joining  $v_i$  and  $w_i$  (that is, every alternate edge of a path is replaced by a cycle  $C_4$ ); a double alternate quadrilateral snake  $DA(Q_n)$  is obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$ (alternatively) to new vertices  $v_i$ ,  $x_i$  and  $w_i$  and  $y_i$  respectively and then joining  $v_i$  and  $w_i$  and  $x_i$  and  $y_i$ .

In [1964] and [1965] Ponraj and Sathish Narayanan define the *irregular triangular snake*  $IT_n$  as the graph obtained from the path  $P_n : u_1, u_2, \ldots, u_n$  with vertex set  $V(IT_n) = V(P_n) \cup \{v_i : 1 \le i \le n \le 2\}$  and the edge set  $E(IT_n) = E(P_n) \cup \{u_iv_i, v_iu_{i+2} : 1 \le i \le n-2\}$ . The *irregular quadrilateral snake*  $IQ_n$  is obtained from the path  $P_n : u_1, u_2, \ldots, u_n$  with vertex set  $V(IQ_n) = V(P_n) \cup \{v_i, w_i : 1 \le i \le n-2\}$  and edge set  $E(IQ_n) = E(P_n) \cup \{u_iv_i, w_iu_{i+2}, v_iw_i : 1 \le i \le n-2\}$ . They proved the following graphs are difference cordial: triangular snakes  $T_n$ , quadrilateral snakes, irregular quadrilateral snakes, irregular quadrilateral snakes, irregular snakes, ir

In [1975], [1962], [1976], and [1963] Ponraj, Sathish Narayanan, and Kala proved the following graphs have difference cordial labelings: paths; cycles; wheels; fans; gears; helms;  $K_{1,n}$  if and only if  $n \leq 5$ ;  $K_n$  if and only if  $n \leq 4$ ;  $K_{2,n}$  if and only if  $n \leq 4$ ;  $K_{3,n}$  if and only if  $n \leq 5$ ;  $K_n$  if and only if  $n \leq 5$ ;  $B_{2,n}$  if and only if  $n \leq 6$ ;  $B_{3,n}$  if and only if  $n \leq 5$ ;  $DT_n \odot K_1$ ;  $DT_n \odot 2K_1$ ;  $DT_n \odot K_2$ ;  $DQ_n \odot K_1$ ;  $DQ_n \odot 2K_1$ ;  $DQ_n \odot K_2$ ;  $DA(T_n) \odot K_1$ ;  $DA(T_n) \odot K_2$ ;  $DA(Q_n) \odot K_1$ ;  $DA(Q_n) \odot 2K_1$ ; and  $DA(Q_n) \odot K_2$ . They also proved: if G is a (p,q) difference cordial graph, then  $q \leq 2p - 1$ ; if G is a r-regular graph with  $r \geq 4$ , then G is not difference cordial; if  $m \geq 4$  and  $n \geq 4$ , then  $K_{m,n}$  is not difference cordial; if m + n > 8 then the bistar  $B_{m,n}$  is not difference cordial; and every graph is a subgraph of a connected difference cordial graph. If G is a book, sunflower,

lotus inside a circle, or square of a path, they prove that  $G \odot mK_1$  (m = 1, 2) and  $G \odot K_2$  is difference cordial.

In [1977], [1979], and [1978] Ponraj, Sathish Narayanan, and Kala proved that the following graphs are difference cordial: crowns  $C_n \odot K_1$ ; combs  $P_n \odot K_1$ ;  $P_n \odot C_m$ ;  $C_n \odot C_m$ ;  $W_n \odot K_2$ ;  $W_n \odot 2K_1$ ;  $G_n \odot K_1$  where  $G_n$  is the gear graph;  $G_n \odot 2K_1$ ;  $G_n \odot K_2$ ;  $(C_n \times P_2) \odot K_1$ ;  $(C_n \times P_2) \odot 2K_1$ ;  $(C_n \times P_2) \odot K_2$ ;  $L_n \odot K_1$ ;  $L_n \odot 2K_1$ ; and  $L_n \odot K_2$ . Ponraj, Sathish Narayanan and Kala proved that the following subdivision graphs are difference cordial:  $S(T_n)$ ;  $S(Q_n)$ ;  $S(DT_n)$ ;  $S(DQ_n)$ ;  $S(A(T_n))$ ;  $S(DA(T_n))$ ;  $S(AQ_n)$ ;  $S(DAQ_n)$ ;  $S(K_{1,n})$ ;  $S(K_{2,n})$ ;  $S(W_n)$ ;  $S(P_n \odot K_1)$ ;  $S(P_n \odot 2K_1)$ ;  $S(LC_n)$ ;  $S(P_n^2)$ ;  $S(K_2 + mK_1)$ ; subdivision graphs of sunflowers  $S(SF_n)$ ; subdivisions graphs flowers  $S(Fl_n)$ ;  $S(M_n n)$ ; S(W(t, n)); subdivisions of Young tableaus  $S(Y_{n,n})$ ; and if S(G) is difference cordial, then  $S(G \odot mK_1)$  is difference cordial. For graphs G that are a tree, a unicycle, or when |E(G)| = |V(G)| + 1, they proved that  $G \odot P_n$  and  $G \odot mK_1$  (m = 1, 2, 3) are difference cordial.

In [2020] Prajapati and Gajjar define a *holiday star* as follows. Let  $v_1, v_2, \ldots, v_{4n-1}, v_{4n}$ be the consecutive 4n vertices of  $C_{4n}$   $(n \geq 3)$ . Let  $u_0$  be the central vertex and  $u_1, u_2, \ldots, u_{2n-1}, u_{2n}$  be end vertices of  $K_{1,2n}$ . Join  $u_0$  to  $v_{4i-3}$  by an edge; for each i from 1 to n. In [2021] they define a Kusadama flower graph as follows. Let  $v_0$  be the apex vertex and  $v_1, v_2, v_3, \ldots, v_{2n-1}, v_{2n}$  be 2n consecutive rim vertices of the wheel  $W_{2n}$   $(n \ge 3)$ . Subdivide the spoke edge  $v_0v_{2i-1}$  by a vertex  $w_i$  and at each  $w_i$  join two copies of path of length 2;  $P_2^{\ell} = v_0, u_{2i-1}, w_i$  and  $P_2^r = v_0, u_{2i}, w_i$ , for each  $i \in [n]$ . In [2020] and [2021] Prajapati and Gajjar proved that the holiday star graph and the Kusudama flower graph admit cordial, E-cordial, difference cordial, prime, vertex prime, and total prime labelings. In [2022] Prajapati and Gajjar define a braided star graph as follows: Let  $a_0$  be the apex vertex and  $a_1, a_2, \ldots, a_{n-1}, a_n$  be consecutive n rim vertices of  $W_n$   $(n \ge 3)$ ; let  $b_1, b_2, b_3, \ldots, b_{2n-1}, b_{2n}$  be 2n consecutive vertices of the cycle  $C_{2n}$ ; let  $c_1, c_2, \ldots, c_{2n-1}, c_{2n}$ be consecutive 2n vertices of  $C_{2n}$ . Join each  $a_i$  to  $b_{2i-1}$  by an edge and  $b_{2i}$  to  $c_{2i}$  by an edge. Take a new vertex  $d_i$  and join each  $d_i$  to  $c_{2i-1}$  and  $c_{2i+1}$  by an edge for each  $i \in [n]$ where subscripts are taken modulo n. Prajapati and Gajjar [2022] proved that braided star graph are cordial, E-cordial and difference cordial.

Recall the splitting graph of G, S'(G), is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v and the shadow graph  $D_2(G)$  of a connected graph G is constructed by taking two copies of G, G' and G'', and joining each vertex u' in G' to the neighbors of the corresponding vertex u' in G''.

Ponraj and Sathish Narayanan [1964], [1965] proved the following graphs are difference cordial:  $S'(P_n)$ ;  $S'(C_n)$ ;  $S'(P_n \odot K_1)$ ; and  $S'(K_{1,n})$  if and only if  $n \leq 3$ . They proved following are not difference cordial:  $S'(W_n)$ ;  $S'(K_n)$ ;  $S'(C_n \times P_2)$ ; the splitting graph of a flower graph;  $DS(SF_n)$ ;  $DS(LC_n)$ ;  $DS(Fl_n)$ ;  $D_2(G)$  where G is a (p,q) graph with  $q \geq p$ ; and  $DS(B_{m,n})$   $(m \neq n)$  with m + n > 8.

Let G(V, E) be a graph with  $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$  where each  $S_i$  is a set of vertices having at least two vertices and having the same degree. Panraj and Sathish

Narayanan [1964], [1965] define the *degree splitting graph* of G denoted by DS(G) as the graph obtained from G by adding vertices  $w_1, w_2, \ldots, w_t$  and joining  $w_i$  to each vertex of  $S_i$   $(1 \le i \le t)$ . They proved the following graphs are difference cordial:  $DS(P_n)$ ;  $W_n$ ;  $DS(C_n)$ ;  $DS(K_n)$  if and only if  $n \le 3$ ;  $DS(K_{1,n})$  if and only if  $n \le 4$ ;  $DS(W_n)$  if and only if n = 3;  $DS(K_n^c + 2K_2)$  if and only if n = 1;  $DS(K_2 + mK_1)$  if and only if  $n \le 3$ ;  $DS(K_{n,n})$  if and only if  $n \le 2$ ;  $DS(T_n)$  if and only if  $n \le 5$ ;  $DS(Q_n)$  if and only if  $n \le 5$ ;  $DS(L_n)$  if and only if  $n \le 5$ ;  $DS(B_{n,n})$  if and only if  $n \le 2$ ;  $DS(B_{1,n})$  if and only  $n \le 4$ ;  $D_2(P_n)$ ;  $D_2(K_n)$  if and only if  $n \le 2$ ; and  $D_2(K_{1,m})$  if and only if  $m \le 2$ .

In [1966], Ponraj and Sathish Narayanan proved the following graphs are difference cordial:  $T_n \odot K_1, T_n \odot 2K_1, T_n \odot K_2, A(T_n) \odot K_1, A(T_n) \odot 2K_1$  and  $A(T_n) \odot K_2$  where  $T_n$  and  $A(T_n)$  are triangular snake and alternate triangular snake respectively. In [1980, 1981] Ponraj, Sathish Narayanan, and Kala proved the following graphs are difference cordial:  $C_n \times P_2$ ; Möbius ladders; the *n*-cube; sunflower graphs; lotuses inside a circle; pyramids; books with n pentagonal pages; mongolian tents; graphs obtained from a ladder by subdividing each step exactly once; permutation graphs  $P(P_{2k}, f)$  where f = $(12)(34)\cdots(kk+1)\cdots(2k-12k)$ ; and  $P(P_n, I)$ ,  $P(C_n, I)$ ,  $P(P_n \odot K_1, I)$ ,  $P(P_n \odot 2K_1, I)$ where I is the identity permutation. Ponraj, Sathish Narayanan, and Kala [1980] [1981] proved the following graphs are not difference cordial:  $G_1(p_1, q_1) \times G_2(p_2, q_2)$  with  $q_1 \ge p_1$ and  $q_2 \ge p_2$ ;  $C_m \times C_n$ ;  $G \times K_n$  where G connected graph and  $n \ge 5$ ,  $G + K_1$  where  $|E(G)\rangle > |V(G) + 1; G_1 + G_2$  where  $G_1$  and  $G_2$  are connected and  $|E(G_1)| > 1$  and  $|E(G_2)| > 3$ ; permutation graphs  $P(G \times K_2, f)$  where  $|E(G)| \geq |V(G)|$  and f is any permutation;  $P(W_n, f)$  for any permutation f; P(S'(G), f) where S'(G) is the splitting graph of G,  $|E(G)| \geq |V(G)|$ , and f is any permutation; and  $P(Fl_n, f)$  where  $Fl_n$  is a flower graph and f is any permutation. They also obtained the following necessary and sufficient conditions for difference cordiality:  $K_m \times P_2$  if and only if  $m \leq 3$ ; for a connected graph G,  $G \times W_n$  if and only if  $G = K_1$ ; books  $B_m$  if and only if  $m \leq 6$ ; G + G if and only if  $|V(G)| \leq 3$  and  $|E(G)| \leq 1$ ;  $K_2 + mK_1$  if and only if  $m \leq 4$ ;  $\overline{K_n} + 2K_2$  if and only if  $n \leq 2$ ; the double fan  $DF_n$  if and only if  $n \leq 4$ ; the t-fold wheel  $W_n + \overline{K_t}$  if and only if  $t \leq 2$  and n = 3; cocktail party graphs  $H_{n,n}$  if and only  $n \leq 6$ ;  $P(K_n, I)$  if and only if  $n \leq 3$ ;  $P(K_2 + mK_1, I)$  if and only if  $m \leq 3$ ; and  $P(K_{m,n}, I)$  (m, n > 1) if and only if m = n = 2 and n = 3, 4, 5.

In [1946], Ponraj, Maria Adaickalam, and Kala introduced a new graph labeling called a k-difference cordial labeling. Let G be a (p,q)-graph and  $2 \leq k \leq |V(G)|$ . Let  $f: V(G) \to \{1, 2, \ldots, k\}$  be a map. For each edge uv, assign the label |f(u) - f(v)|. They say f is a k-difference cordial labeling of G if  $|v_f(i) - v_f(j)| \leq 1$  and  $|e_f(0) - e_f(1)| \leq 1$ , where  $v_f(x)$  denotes the number of vertices labeled with x,  $e_f(1)$  denotes the number of edges labeled with 1, and  $e_f(0)$  denotes the number of edges that are not labeled with 1. A graph witha k-difference cordial labeling is called a k-difference cordial graph. They proved the following: every graph is a subgraph of a connected k-difference cordial graph; if k is even, then k-copies of  $K_{1,p}$  is k-difference cordial; and if  $n \equiv 0 \pmod{k}$  and  $k \geq 6$ , then  $K_{1,n}$  is not k-difference cordial. They further prove the following are 3-difference cordial graphs: paths;  $C_n$  where  $n \equiv 0, 3 \pmod{4}$ ;  $K_{m,n}$   $(m \leq n)$  and m is even; combs; double combs; quadrilateral snakes; bistars; subdivisions of a star; subdivisions of a bistar;  $C_4^{(t)}$ ;  $K_n$  if and only if  $n \in \{1, 2, 3, 4, 6, 7, 9, 10\}$ ; and  $K_{1,n}$  if and only if  $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$ .

In [1941], [1942], [1943], Ponraj and Maria Adaickalam proved the following are 3difference cordial graphs:  $K_{1,n} \odot K_2$ ,  $P_n \odot 3K_1$ ,  $C_n \odot K_2$ ,  $mC_4$ , splitting graph of a star, fan, double fan,  $W_n$  where  $n \equiv 0, 1 \pmod{3}$ , helms, flower, sunflower graph, lotus inside a circle, closed helm, double wheel  $DW_n$  where  $V(DW_n) = V(W_n) \cup \{v_i : 1 \le i \le n\}$ and edge set  $E(DW_n) = E(W_n) \cup \{uv_i : 1 \le i \le n\} \cup \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_1 v_n\},\$ degree splitting graph of a bistar,  $spl(K_{1,n}) \cup K_{1,n}$ ,  $spl(K_{1,n}) \cup P_n$ ,  $K_{3,n} \cup spl(K_{1,n})$ ,  $DF_n \cup spl(K_{1,n}), S(K_{1,n}) \cup S(B_{n,n}), K_{2,n} \cup S(K_{1,n}), F_n \cup S(K_{1,n}), W_n \cup S(K_{1,n}), B_{n,n} \cup S(K_{1,n})$  $S(B_{n,n}), K_{2,n} \cup B_{n,n}, (C_n \odot K_1) \cup (P_n \odot K_1), F_n \cup F_n$ , jelly fish,  $P_n \cup K_{1,n}, K_{1,n} \cup K_{2,n}$  $K_{1,n} \cup S(K_{1,n})$ , are Let  $C_n$  be the cycle  $u_1 u_2 \ldots u_n u_1$ . If G is (p,q) 3-difference cordial graph with  $p \equiv 0 \pmod{2}$  and  $q \equiv 0 \pmod{3}$ , then  $G \cup G$  also 3-difference cordial. Let G be the graph obtained from  $C_n$  with  $V(G) = V(C_n) \cup \{v_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$  and  $E(G) = \{u_i v_i, u_{i+1} v_i : 1 \le i \le n\}$ . Then G is 3-difference cordial. The graph  $G_n$  with the vertex set  $V(G_n) = \{u_i, v_i, w_i : 1 \le i \le n\}$  and  $E(G_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \le i \le n\}$ n-1  $\cup$  { $u_nu_1, v_1u_1$ }  $\cup$  { $u_iv_i, v_iw_i : 1 \le i \le n$ } is 3-difference cordial. Let  $C_3$  be the cycle  $u_1u_2u_3u_1$ . Let G be a graph obtained from  $C_3$  with  $V(G) = V(C_3) \cup \{v_i, w_i, z_i : 1 \le i \le n\}$ and  $E(G) = E(C_3) = \{u_1v_i, u_2w_i, u_3z_i : 1 \leq i \leq n\}$ . Then G is 3-difference cordial if  $n \equiv 0, 2, 3 \pmod{4}$ . If  $n \equiv 0, 1 \pmod{3}$ , then  $K_{1,n} \cup K_{1,n}$  is 3-difference cordial. Ponraj, Adaickalam, and Kala [1947] proved the following graphs have 3-difference cordial labelings:  $DA(T_n) \odot K_1, DA(T_n) \odot 2K_1, DA(T_n) \odot K_2, DA(Q_n) \odot K_1$ , and  $DA(Q_n) \odot 2K_1$  $(T_n \text{ is a triangular snake.})$  In [1944] Ponraj, Adaickalam, Maria Adaickalam, and Kala investigated the 3-difference cordial labeling behavior of ladders, books, dumbbell graphs, and umbrella graphs.

For graphs G and H and a vertex v of G the graph  $G \odot_v H$  is obtained by joining any particular vertex of H to vertex v. In [1778] Sugumaran and Mohan proved that the following graphs are difference cordial graphs: the path union of r copies of  $P_n^2$ (that is,  $P(r.P_n^2)$ )-see Section 2.7 for the definition), the cycle union of r copies of  $C_n^2$ , (that is,  $C(r.C_n^2)$ ), the open star of r copies the square graph  $P_n^2$  (that is,  $S(r.P_n^2)$ ), the graph  $C_n^2 \odot_{v_n} P_k$ , and the graph  $C_n^2 \odot_{v_n} P_k^2$ . In [1779] they proved that the plus graph  $Pl_n$ , the path union of plus graph  $P(r.Pl_n)$ , the cycle union of plus graph  $C(r.Pl_n)$ , the barycentric subdivision of  $Pl_n$ , the hanging pyramid  $HP_{y_n}$  graph, and the path union of hanging pyramid  $P(r.HP_{y_n})$ . In [2506] they proved that switching of a pendent vertex in path  $P_n$ , switching of an apex vertex in  $CH_n$ , the graph obtained by duplication of each vertex of path  $P_n$  by an edge, the barycentric subdivision of  $C_n \odot K_1$ , the path union of r copies of fan  $P(r.F_n)$ , the cycle union of r copies of fan  $C(r.F_n)$ , and the open star of rcopies of fan  $S(r.F_n)$  are difference cordial graphs.

## 7.10 Prime Cordial Labelings

Sundaram, Ponraj, and Somasundaram [2536] have introduced the notion of prime cordial labelings. A prime cordial labeling of a graph G with vertex set V is a bijection f from V to  $\{1, 2, \ldots, |V|\}$  such that if each edge uv is assigned the label 1 if gcd(f(u), f(v)) = 1

and 0 if gcd(f(u), f(v)) > 1, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In [2536] Sundaram, Ponraj, and Somasundram prove the following graphs are prime cordial:  $C_n$  if and only if  $n \ge 6$ ;  $P_n$  if and only if  $n \neq 3$  or 5;  $K_{1,n}$  (n odd); the graph obtained by subdividing each edge of  $K_{1,n}$  if and only if  $n \geq 3$ ; bistars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders;  $K_{1,n}$  if n is even and there exists a prime p such that 2p < n+1 < 3p;  $K_{2,n}$  if n is even and if there exists a prime p such that 3p < n+2 < 4p; and  $K_{3,n}$  if n is odd and if there exists a prime p such that 5p < n + 3 < 6p. They also prove that if G is a prime cordial graph of even size, then the graph obtained by identifying the central vertex of  $K_{1,n}$  with the vertex of G labeled with 2 is prime cordial, and if G is a prime cordial graph of odd size, then the graph obtained by identifying the central vertex of  $K_{1,2n}$  with the vertex of G labeled with 2 is prime cordial. They further prove that  $K_{m,n}$  is not prime cordial for a number of special cases of m and n. Sundaram and Somasundaram [2539] and Youssef [2882] observed that for  $n \geq 3$ ,  $K_n$  is not prime cordial provided that the inequality  $\phi(2) + \phi(3) + \cdots + \phi(n) \ge n(n-1)/4 + 1$  is valid for  $n \geq 3$  ( $\phi$  is the Euler phi-function). This inequality was proved by Yufei Zhao [2912]. Haque, Lin, Yang, and Zhao [955] show that with the exception of P(4, 1), all generalized Petersen graphs are prime cordial. Haque, Lin, Yang, and Zhang [953] show that the flower snark and related graphs are prime cordial. In [891] Ghosh, Mohanty, and Pal gave an algorithmic approach to find cordial labelings of Cartesian product of two balanced bipartite graphs. The algorithm works for signed product cordial lablings, total signed product cordial labelings, and prime cordial labelings of such graphs.

Seoud and Salim [2247] give an upper bound for the number of edges of a graph with a prime cordial labeling as a function of the number of vertices. For bipartite graphs they give a stronger bound. They prove that  $K_n$  does not have a prime cordial labeling for 2 < n < 500 and conjecture that  $K_n$  is not prime cordial for all n > 2. They determine all prime cordial graphs of order at most 6. For a graph with n vertices to admit a prime cordial labeling, Seoud and Salim [2249] proved that the number of edges must be less than  $n(n-1) - 6n^2/\pi^2 + 3$ . As a corollary they get that  $K_n$  (n > 2) is not prime cordial thereby proving their earlier conjecture.

In [877] Ghodasara and Jena prove that the following graphs are prime cordial:  $C_n$  with one chord,  $C_n$  with twin chords (that is, two cords that form a triangle with an edge of the cycle),  $C_n$  with three cords that form two triangles and a cycle of length n-3 ( $n \ge 7$ ), the graph obtained by joining two copies of  $C_n$  with one chord by a path, and the graph obtained by joining two copies of the same cycle with twin chords by a path is prime cordial.

In [410] Baskar Babujee and Shobana proved sun graphs  $C_n \odot K_1$ ;  $C_n$  with a path of length n-3 attached to a vertex; and  $P_n$  ( $n \ge 6$ ) with n-3 pendent edges attached to a pendent vertex of  $P_n$  have prime cordial labelings. Additional results on prime cordial labelings are given in [411].

In [2682] and [2683] Vaidya and Vihol prove following graphs are prime cordial: the total graph of  $P_n$  and the total graph of  $C_n$  for  $n \ge 5$  (see §2.7 for the definition);  $P_2[P_m]$  for all  $m \ge 5$ ; the graph obtained by joining two copies of a fixed cycle by a path; and the
graph obtained by switching of a vertex of  $C_n$  except for n = 5 (see §3.6 for the definition); the graph obtained by duplicating each edge by a vertex in  $C_n$  except for n = 4 (see §2.7 for the definition); the graph obtained by duplicating a vertex by an edge in cycle  $C_n$  (see §2.7 for the definition); the path union of any number of copies of a fixed cycle (see §3.7 for the definition); and the friendship graph  $F_n$  for  $n \ge 3$ . Vaidya and Shah [2662] prove following results:  $P_n^2$  is prime cordial for n = 6 and  $n \ge 8$ ;  $C_n^2$  is prime cordial for  $n \ge 10$ ; the shadow graphs of  $K_{1,n}$  (see §3.8 for the definition) for  $n \ge 4$  and the bistar  $B_{n,n}$  are prime cordial graphs.

Let  $G_n$  be a simple nontrival connected cubic graph with vertex set  $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \le i \le n-1\}$ , and edge set  $E(G_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i a_i, d_i b_i, d_i c_i : 0 \le i \le n-1\}$ , where the edge labels are taken modulo n. Let  $H_n$  be a graph obtained from  $G_n$  by replacing the edges  $b_{n-1}b_0$  and  $c_{n-1}c_0$  with  $b_{n-1}c_0$  and  $c_{n-1}b_0$  respectively. For odd  $n \ge 5$ ,  $H_n$  is called a *flower snark* whereas  $G_n$ ,  $H_3$  and all  $H_n$  with even  $n \ge 4$ , are called the *related graphs* of a flower snark. Mominul Haque, Lin, Yang, and Zhang [1782] proved that flower snarks and related graphs are prime cordial for all  $n \ge 3$ .

In [2665] Vaidya and Shah prove that the following graphs are prime cordial: split graphs of  $K_{1,n}$  and  $B_{n,n}$ ; the square graph of  $B_{n,n}$ ; the middle graph of  $P_n$  for  $n \ge 4$ ; and  $W_n$  if and only if  $n \ge 8$ . Vaidya and Shah [2665] prove following graphs are prime cordial: the splitting graphs of  $K_{1,n}$  and  $B_{n,n}$ ; the square of  $B_{n,n}$ ; the middle graph of  $P_n$  for  $n \ge 4$ ; and wheels  $W_n$  for  $n \ge 8$ .

In [2669] [2671] Vaidya and Shah proved following graphs are prime cordial: gear graphs  $G_n$  for  $n \ge 4$ ; helms; closed helms  $CH_n$  for  $n \ge 5$ ; flower graphs  $Fl_n$  for  $n \ge 4$ ; degree splitting graphs of  $P_n$  and the bistar  $B_{n,n}$ ; double fans  $Df_n$  for n = 8 and  $n \ge 10$ ; the graphs obtained by duplication of an arbitrary rim edge by an edge in  $W_n$  where  $n \ge 6$ ; and the graphs obtained by duplication of an arbitrary spoke edge by an edge in wheel  $W_n$  where n = 7 and  $n \ge 9$ .

Let G(p,q) with  $p \ge 4$  be a prime cordial graph and  $K_{2,n}$  be a bipartite graph with bipartition  $V = V_1 \cup V_2$  with  $V_1 = \{v_1, v_2\}$  and  $V_2 = \{u_1, u_2, \ldots, u_n\}$ . If  $G_1$  is the graph obtained by identifying the vertices  $v_1$  and  $v_2$  of  $K_{2,n}$  with the vertices of G having labels 2 and 4 respectively, Vaidya and Prajapati [2660] proved that  $G_1$  admits a prime cordial labeling if n is even; if n, p, q are odd and with  $e_f(0) = \lfloor q/2 \rfloor$ ; and if n is odd, p is even and q is odd with  $e_f(0) = \lceil q/2 \rceil$ . Prajapati and Gajjar [2024] proved the following graphs are prime cordial:  $C_n \times P_2$  except for n = 1, 2 and  $4, C_n \times P_4$   $(n \ge 3), C_3 \times P_n$  (n > 1), $C_5 \times P_n$   $(n > 1), C_6 \times P_n$   $(n > 1), C_{2p} \times P_n$  where p is an odd prime and n > 1, and  $C_4 \times P_n$  (n > 2).

In [2505] Sugumaran and Mohan proved the following graphs are prime cordial: the cycle butterfly graph  $B_{n,m}$  (two copies of  $C_n$  that share a common vertex with m pendent vertices attached to the common vertex), W- graph (obtained by starting with the two copies of  $K_{1,n}$  and merging the last pendent vertex in the first copy of  $K_{1,n}$  with the initial pendent vertex in the second copy of  $K_{1,n}$ ),  $H_n$  graph (the graph obtained from two paths  $u_1, u_2, \ldots, u_n$  and  $v_1, v_2, \ldots, v_n$  by joining the vertices  $u_{(n+1)/2}$  and  $v_{(n+1)/2}$  if n is odd and joining  $u_{n/2}$  and  $v_{n/2+1}$  if n is even), and duplication of all edges of an  $H_n$  graph. In [2504] Sugumaran and Mohan proved that the following graphs are prime cordial:  $H_n \odot K_1$ , the

path union of r copies of  $H_n$ , the cycle union of r copies of an  $H_n$ , the open star of r copies of an  $H_n$ -graph (obtained by replacing each pendent vertex of  $K_{1,n}$  by a copy of  $H_n$ .

In [2510] Sugumaran and Suresh proved that the following graphs are prime cordial graphs: the duplication of each vertex by an edge of paths, stars, jelly fish, bistars, and  $C_n \odot K_1$ . Sugumaran and Vishnu Prakash [2515] proved that the following graphs are prime cordial graphs: duplication of any vertex of degree 3 in theta graph, switching of any vertex of degree 3 in theta graph, fusion of any two vertices in theta graph, the path union of two copies of theta graph, and two copies of theta graph joined by a path of any length. They further proved that the theta graph is not a prime cordial labeling. In [2516] they showed that one point union of path graph  $P_n^t(tn.T_\alpha)$ , the open star of theta graph, and any path union of even number of theta graphs are prime cordial graphs. Also they proved [2514] the subdivision of bistar  $B_{n,n}$ ,  $P_n \bigcirc K_{1,n-1}$ , (that is, each  $i^{th}$  vertex of path  $P_n$  is append with the apex vertex of  $i^{th}$  copy of  $K_{(1, n - 1)}$ ), the disconnected graph  $P_n \bigcup P_m$  are prime cordial graphs.

Vaidya and Prajapati [2658] call a graph strongly prime cordial if for any vertex v there is a prime labeling f of G such that f(v) = 1. They prove the following: the graphs obtained by identifying any two vertices of  $K_{1,n}$  are prime cordial; the graphs obtained by identifying any two vertices of  $P_n$  are prime cordial;  $C_n, P_n$ , and  $K_{1,n}$  are strongly prime cordial; and  $W_n$  is a strongly prime cordial for every even integer  $n \ge 4$ . Prajapati and Gajjar [2014] proved that generalized prism graphs  $Y_{n,2}$  is prime cordial except for n = 1, 2 and 4;  $Y_{n,4}$  is prime cordial for  $n \ge 3$ ;  $Y_{3,n}, Y_{5,n}, Y_{6,n}$  and  $Y_{2p,n}$  (for odd prime p) are prime cordial for n > 1; and  $Y_{4,n}$  is prime cordial for n > 2. They also proved the following graphs are prime cordial:  $C_n \times P_2$  except for n = 1, 2 and 4,  $C_n \times P_4$   $(n \ge 3), C_3 \times P_n$   $(n > 1), C_5 \times P_n$   $(n > 1), C_6 \times P_n$   $(n > 1), C_{2p} \times P_n$  where p is an odd prime and n > 1, and  $C_4 \times P_n$  (n > 2).

In [1992] Ponraj, Singh, Kala, and Sathish Narayanan introduced a new graph labeling called k-prime cordial labeling. Let G be a (p,q)-graph and  $2 \le p \le k$  and let  $f: V(G) \rightarrow Q$  $\{1, 2, \ldots, k\}$  be a map. For each edge uv, assign the label gcd (f(u), f(v)). They say that f is a k-prime cordial labeling of G if  $|v_f(i) - v_f(j)| \leq 1$  for  $i, j \in \{1, 2, \dots, k\}$  and  $|e_f(0) - e_f(1)| \leq 1$ , where  $v_f(x)$  denotes the number of vertices labeled with x, and  $e_f(1)$ and  $e_f(0)$ , respectively, denote the number of edges labeled with 1 and not labeled with 1. A graph with a k-prime cordial labeling is a k-prime cordial graph. They proved that every graph is a subgraph of a connected k-prime cordial graph; if k is even, then  $P_n$ ,  $n \neq 3$ , is k-prime cordial;  $C_n$ ,  $n \neq 3$ , is k-prime cordial when k is even; and the bistar  $B_{n,n}$ is k-prime cordial for all even k. They studied 3-prime cordiality of paths, cycles, and olive trees. They also proved that if T is a 3-prime cordial tree, then  $T \odot K_1$  is 3-prime cordial;  $K_{1,n}$  is 3-prime cordial if and only if  $n \leq 3$ ;  $K_n$  is 3-prime cordial if and only if n < 3; combs  $P_n \odot K_1$  are 3-prime cordial; and  $C_n \odot K_1$  is 3-prime cordial if and only if  $n \neq 3$ . They proved that  $K_2 + mK_1$ ,  $K_{2,n}$ , and wheels are not 3-prime cordial graphs. In [1993] Ponraj, Singh, and Sathish Narayana proved if G is 3-prime cordial, then  $G \cup P_n$  is a 3-prime cordial for n > 12, the splitting graph of a star is not a 3-prime cordial graph, and the jelly fish J(m, n) is 3-prime cordial if  $10m \ge n+2$ .

For a 4-prime cordial graph G Ponraj and Singh [1991] proved  $G \cup P_n$   $(n \ge 5)$ ,  $G \cup 2mK_{n,n}$ , and  $G \cup 2mK_{1,n}$  are 4-prime cordial. For a (4t, q) 4-prime cordial graph G they prove that  $G + K_1$  and  $G + 2K_1$  are 4-prime cordial. Ponraj, Singh, and Kala [1994] prove that  $P_m \times P_n$  and subdivisions of wheels and helms are 4-prime cordial. They also show that if G is bipartite then  $G \cup G$  is 4-prime cordial; and if G is 4-prime cordial then  $G \odot K_1$  is 4-prime cordial. Ponraj, Singh, and Kala [1995] proved the following graphs are 4-prime cordial:  $2m(K_{n,n})$ ,  $2m(P_n \times P_2)$ ,  $m(C_n \oplus K_1)$ ,  $mB_{n,n}$ , and  $2W_{2n+1}$ .

Murugesan, Jayaraman, and Shiama (see [2517]) defined a 3-equitable prime cordial labeling of a graph G as a bijection f from V(G) to  $\{1, 2, \ldots, |V(G)|\}$  such that if an edge uv is assigned the label 1 when gcd(f(u), f(v)) = 1 and gcd(f(u) + f(v), f(u) - f(v)) = 1, the label 2 when gcd(f(u), f(v)) = 1 and gcd(f(u) + f(v), f(u) - f(v)) = 2, and the label 0 otherwise, then the number of edges labeled with i and the number of edges labeled with j differ by at most 1 for  $0 \le i \le 2$  and  $0 \le j \le 2$ . A graph that has a 3-equitable prime cordial labeling is called a 3-equitable prime cordial graph. Sugumaran and Vishnu Prakash [2517] proved the following graphs are 3-equitable prime cordial graphs: bistars, combs, ladders, kites, and slanting ladders. In [2519] they showed that theta graphs, the duplication of any vertex in theta graphs, path unions of two copies of theta graphs, open star graphs of copies of a fixed theta graph are 3-equitable prime cordial graphs.

## 7.11 Parity Combination Cordial Labelings

In [1990] Ponraj, Sathish Narayanan, and Ramasamy introduced a new graph labeling called parity combination cordial labeling. Let G be a (p,q)-graph. Let f be an injective map from V(G) to  $\{1, 2, \ldots, p\}$ . For each edge xy, assign the label  $\binom{x}{y}$  or  $\binom{y}{x}$  according as x > y or y > x. Call f a parity combination cordial labeling if f is a one to one map and  $|e_f(0) - e_f(1)| \leq 1$ , where  $e_f(0)$  and  $e_f(1)$  denote the number of edges labeled with an even number and odd number, respectively. A graph with a parity combination cordial labeling is called a parity combination cordial graph. They proved that the following are parity combination cordial graphs: paths, cycles, stars, triangular snakes, alternate triangular snakes, olive trees, combs, crowns, fans, umbrellas,  $P_n^2$ , helms, dragons, bistars, butterfly graphs, and graphs obtained from  $C_n$  and  $K_{1,m}$  by unifying a vertex of  $C_n$  and a pendent vertex of  $K_{1,m}$ . They also proved that  $W_n$  admits a parity combination cordial labeling if and only if  $n \geq 4$  and conjectured that for  $n \geq 4$ ,  $K_n$  is not a parity combination cordial graph. In [1996], Ponraj, Rajpal Singh, and Sathish Narayanan proved that if Gis a parity combination cordial graph, then  $G \cup P_n$  is also parity combination cordial if  $n \neq 2, 4$ .

### 7.12 Mean Labelings

Somasundaram and Ponraj [2446] have introduced the notion of mean labelings of graphs. A graph G with p vertices and q edges is called a *mean graph* if there is an injective function f from the vertices of G to  $\{0, 1, 2, \ldots, q\}$  such that when each edge uv is labeled with (f(u) + f(v))/2 if f(u) + f(v) is even, and (f(u) + f(v) + 1)/2 if f(u) + f(v) is odd, then the resulting edge labels are distinct.

In [2446], [2447], [2448], [2449], [2006], and [2007] they prove the following graphs are mean graphs:  $P_n$ ,  $C_n$ ,  $K_{2,n}$ ,  $K_2 + mK_1$ ,  $\overline{K_n} + 2K_2$ ,  $C_m \cup P_n$ ,  $P_m \times P_n$ ,  $P_m \times C_n$ ,  $C_m \odot K_1$ ,  $P_m \odot K_1$ , triangular snakes, quadrilateral snakes,  $K_n$  if and only if n < 3,  $K_{1,n}$  if and only if n < 3, bistars  $B_{m,n}$  (m > n) if and only if m < n + 2, the subdivision graph of the star  $K_{1,n}$  if and only if n < 4, the friendship graph  $C_3^{(t)}$  if and only if t < 2, the one point union of two copies a fixed cycle, dragons (the one point union of  $C_m$  and  $P_n$ , where the chosen vertex of the path is an end vertex), the one point union of a cycle and  $K_{1,n}$ for small values of n, and the arbitrary super subdivision of a path, which is obtained by replacing each edge of a path by  $K_{2,m}$ . They also prove that  $W_n$  is not a mean graph for n > 3 and enumerate all mean graphs of order less than 5.

Gayathri and Gopi [856] prove the following are mean graphs: double triangular snakes; double quadrilateral snakes; generalized antiprisms; graphs obtained by joining the 2 vertices of  $K_{2,n}$  of degree n with an edge; and graphs obtained from  $C_n$  with consecutive vertices  $v_1, v_2, \ldots, v_n$  by adding the chords joining  $v_i$  and  $v_{n-i+2}$  for  $2 \le i \le \lfloor n/2 \rfloor$ . In [854] Gayathri and Gopi gave various necessary conditions for mean labelings.

Lourdusamy and Seenivasan [1657] prove that  $kC_n$ -snakes are means graphs and every cycle has a super subdivision that is a mean graph. They define a generalized  $kC_n$ -snake in the same way as a  $C_n$ -snake except that the sizes of the cycle blocks can vary (see Section 2.2). They prove that generalized  $kC_n$ -snakes are mean graphs. Recall that  $P_{a,b}$ denotes the graph obtained by identifying the endpoints of b internally disjoint paths each of length a. Vasuki and Nagarajan [2708] proved that the following graphs admit mean labelings:  $P_{r,2m+1}$  for all r and m;  $P_{r,2m}$  for all m and  $2 \le r \le 6$ ;  $P_r^{2m+1}$  for all r and m; and  $P_r^{2m}$  for all m and  $2 \le r \le 6$ . Anusa, Sandhya, and Somasundaram [152] proved that triangular ladders, triangular snakes, double triangular snakes, quadrilateral snakes, and double quadrilateral snakes are mean graphs.

Lourdusamy and Seenivasan [1658] define an *edge linked cyclic snake*,  $EL(kC_n)$ , as the connected graph obtained from k copies of  $C_n$   $(n \ge 4)$  by identifying an edge of the  $(i + 1)^{th}$  copy to an edge of the  $i^{th}$  copy for i = 1, 2, ..., k - 1 in such a way that the consecutive edges so chosen are not adjacent. They proved that all  $EL(kC_{2n})$  are mean graphs and some cases of  $EL(C_{2n-1})$  are mean graphs. They also define a *generalized edge linked cyclic snake* in the same way but allow the cycle lengths (at least 4) to vary. They prove that certain cases of generalized edge linked cyclic snakes are mean graphs.

Barrientos and Krop [370] proved that there exist n! graphs of size n that admit mean labelings. They give two necessary conditions for the existence of a mean labeling of a graph G with m vertices and n edges: if G is a mean graph, then  $n + 1 \ge m$ ; if G is a mean graph with n edges and maximum degree  $\Delta(G)$ , then  $\Delta(G) \le \frac{n+3}{2}$  when n is odd and  $\Delta(G) \le \frac{n+2}{2}$  when n is even. They proved that the disjoint union of n copies of  $C_3$  is a mean graph and if a mean r-regular graph has n vertices, then r < n - 2. They established a connection between  $\alpha$ -labelings and mean labelings by proving that every tree that admits an  $\alpha$ -labeling is a mean graph when the size of its stable sets differ by at most one. When the tree is a caterpillar, this difference can be up to two. Barrientos and Krop call a mean labeling of a bipartite graph an  $\alpha$ -mean labeling if the labels assigned to vertices of the same color have the same parity. They show that the complementary labeling of a  $\alpha$ -mean labeling is also an  $\alpha$ -mean labeling. They use graphs with  $\alpha$ mean labelings to construct new mean graphs. One construction consists of connecting a pair of corresponding vertices of two copies of an  $\alpha$ -mean graph by an edge. The other construction identifies a pair of suitable vertices from two  $\alpha$ -mean graphs. Barrientos and Krop also proved that every quadrilateral snake admits an  $\alpha$ -mean labeling. They conjecture that all trees of size n and maximum degree at most  $\lceil (n + 1)/2 \rceil$  are mean graphs and state some open problems. In [365] Barrientos proves that all trees with up to four end-vertices except  $K_{1,4}$  are mean graphs. Bailey and Barrientos [327] prove the following are mean graphs:  $C_n \cup C_m$ ,  $C_n \cup P_m$ ,  $K_2 + nK_1$ ,  $2K_2 + nK_1$ ,  $C_n \times K_2$ .

In [327] Bailey and Barrientos study several operations with mean graphs. They prove that the coronas  $G \odot K_1$  and  $G \odot K_2$  are mean graphs when G is an  $\alpha$ -mean graph. Also, if G and H are mean graphs with n vertices and n-1 edges and H is an  $\alpha$ -mean graph, then  $G \times H$  is a mean graph. They prove that given two mean graphs G and H, there exists a mean graph obtained by identifying an edge from G with an edge from H and uses this result to prove that the graphs  $R_n$   $(n \ge 2)$  of order 2n and size 4n-3 with vertex set  $V(R_n) = \{v_1, v_2, \ldots, v_{2n}\}$  and edge set  $E(R_n) = \{v_i v_{i+1} \mid 1 \le i \le n-1 \text{ and } n+1 \le i \le 2n-1\} \cup \{v_i v_{n+i} \mid 1 \le i \le n\} \cup \{v_i v_{n+i-1} \mid 2 \le i \le n\}$  (rigid ladders) are mean graphs.

Barrientos, Abdel-Aal, Minion, and Williams [366] use  $A_n$  to denote the set of all  $\alpha$ mean labeled graphs of size n such that the difference of the cardinalities of the bipartite sets of the verticies of the graphs is at most one. They prove that the class  $A_n$  is equivalent to the class of  $\alpha$ -labeled graphs of size n with bipartite sets that differ by at most one. They also prove that when  $G \in A_n$ , the coronas  $G \odot mK_1$ ,  $G \odot P_2$ , and  $G \odot P_3$  admit mean labelings.

In [2625] Vaidya and Bijukumar define two methods of creating new graphs from cycles as follows. For two copies of a cycle  $C_n$  the mutual duplication of a pair of vertices  $v_k$ and  $v'_k$  respectively from each copy of  $C_n$  is the new graph G such that  $N(v_k) = N(v'_k)$ . For two copies of a cycle  $C_n$  and an edge  $e_k = v_k v_{k+1}$  from one copy of  $C_n$  with incident edges  $e_{k-1} = v_{k-1}v_k$  and  $e_{k+1} = v_{k+1}v_{k+2}$  and an edge  $e'_m = u_m u_{m+1}$  in the second copy of  $C_n$  with incident edges  $e'_{m-1} = u_{m-1}u_m$  and  $e'_{m+1} = u_{m+1}u_{m+2}$ , the mutual duplication of a pair of edges  $e_k$  and  $e'_m$  respectively from two copies of  $C_n$  is the new graph G such that  $N(v_k) - v_{k+1} = N(u_m) - u_{m+1} = \{v_{k-1}, u_{m-1}\}$  and  $N(v_{k+1}) - v_k = N(u_{m+1}) - u_m =$  $\{v_{k+2}, u_{m+2}\}$ . They proved that the graph obtained by mutual duplication of a pair of edges from each copy of a cycle are mean graphs. Moreover, they proved that the shadow graphs of the stars  $K_{1,n}$  and bistars  $B_{n,n}$  are mean graphs.

Vasuki and Nagarajan [2710] proved the following graphs are admit mean labelings: the splitting graphs of paths and even cycles;  $C_m \odot P_n$ ;  $C_m \odot 2P_n$ ;  $C_n \cup C_n$ ; disjoint unions of any number of copies of the hypercube  $Q_3$ ; and the graphs obtained from by starting with m copies of  $C_n$  and identifying one vertex of one copy of  $C_n$  with the corresponding vertex in the next copy of  $C_n$ .) Jeyanthi and Ramya [1207] define the jewel graph  $J_n$  as the graph with vertex set  $\{u, x, v, y, u_i : 1 \leq i \leq n\}$  and edge set  $\{ux, vx, uy, vy, xy, uu_i, vu_i : 1 \leq i \leq n\}$ . They proved that the jewel graphs, jelly fish graphs (see §7.26 for the definition), and the graph obtained by joining any number of isolated vertices to the two endpoints of  $P_3$  are mean graphs. Ramya and Jeyanthi [2088] proved several families of graphs constructed from  $T_p$ -tree are mean graphs. Ahmad, Imran, and Semaničová-Feňovčiková [91] studied the relation between mean labelings and (a, d)-edge-antimagic vertex labelings. They show that two classes of caterpillars admit mean labelings. Revathi [2114] proved that the shadow graphs of bistars, combs, and the splitting graph of combs have mean labelings.

Recall from Section 2.7 that given connected graphs  $G_1, G_2, \ldots, G_n$ , Kaneria, Makadia, and Jariya [1296] define a cycle of graphs  $C(G_1, G_2, \ldots, G_n)$  as the graph obtained by adding an edge joining  $G_i$  to  $G_{i+1}$  for  $i = 1, \ldots, n-1$  and an edge joining  $G_n$  to  $G_1$ . (The resulting graph can vary depending on which vertices of the  $G_i$  are chosen.) When the n graphs are isomorphic to G the notation  $C(n \cdot G)$  is used. Also recall Kanneria and Makadia [1289] define a step grid graph  $St_n$  as the graph obtained by starting with paths  $P_n, P_{n-1}, \ldots, P_2$  ( $n \geq 3$ ) arranged vertically parallel with the vertices in the paths forming horizontal rows and edges joining the vertices of the rows. In [1320], [1306], and [1309], Kaneria, Viradia, and Makadia proved the following graphs are mean graphs: the path union of any number of copies of a mean graph;  $C(2t \cdot P_n)$ ;  $C(2t \cdot C_n)$ ;  $C(2t \cdot P_n \times P_m)$ ;  $C(2r \cdot B_{n,n}^2)$  ( $B_{n,n}^2$  is the square of the bistar  $B_{n,n}$ );  $C(2r \cdot M(C_n))$  ( $M(C_n)$  is the middle graph of  $C_n$ );  $C(2r \cdot (P_{2n} + 2K_1))$ ; step grid graphs; the path union of finitely copies of the step grid graphs; cycles of step grid graphs  $C(2r \cdot St_n)$ ; and  $C(2t \cdot K_{2,m})$ .

For a fixed vertex v of  $C_m$  Avadayappan and Vasuki [211] use  $(P_m; C_n)$  to denote the graph obtained from m copies of  $C_n$  and the path  $P_m: u_1u_2\cdots u_m$  by joining  $u_i$  with v of the *i*th copy of  $C_n$  with an edge for  $1 \leq i \leq m$ . They define  $(P_m; Q_3)$ ,  $(P_{2n}; S_m)$ ,  $(P_n; S_1)$  and  $(P_n; S_2)$ , where v is a fixed vertex of the cube  $Q_3$  and v is the center of the star  $S_k$ , in an analogous way. For  $C_n: v_1v_2\ldots v_nv_1$  they use  $[P_m; C_n]$  to denote the graph obtained from m copies of  $C_n$  with vertices  $v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, \ldots, v_{2n}, \ldots, v_{m_1}, \ldots, v_{m_n}$  by joining  $v_{ij}$  and  $v_{(i+1)j}$  with an edge, for some j and  $1 \leq i \leq m - 1$ . They define  $[P_m; Q_3]$  and  $[P_m; C_m^{(2)}]$ , where  $C_m^{(2)}$  is the friendship graph, similarly. In [211] they prove these families are mean graphs.

Ramya, Ponraj, and Jeyanthi [2091] called a mean graph super mean if vertex labels and the edge labels are  $\{1, 2, \ldots, p+q\}$ . They prove following graphs are super mean: paths, combs, odd cycles,  $P_n^2$ ,  $L_n \odot K_1$ ,  $C_m \cup P_n$   $(n \ge 2)$ , the bistars  $B_{n,n}$  and  $B_{n+1,n}$ . They also prove that unions of super mean graphs are super mean and  $K_n$  and  $K_{1,n}$  are not super mean when n > 3. In [1212] Jeyanthi, Ramya, and Thangavelu prove the following are super mean:  $nK_{1,4}$ ; the graphs obtained by identifying an endpoint of  $P_m$   $(m \ge 2)$ with each vertex of  $C_n$ ; the graphs obtained by identifying an endpoint of two copies of  $P_m$   $(m \ge 2)$  with each vertex of  $C_n$ ; the graphs obtained by identifying an endpoint of three copies of  $P_m$   $(m \ge 2)$ ; and the graphs obtained by identifying an endpoint of four copies of  $P_m$   $(m \ge 2)$ . In [1208] Jeyanthi and Ramya prove the following graphs have super mean labelings: the graph obtained by identifying the endpoints of two or more copies of  $P_5$ ; the graph obtained from  $C_n$   $(n \ge 4)$  by joining two vertices of  $C_n$  distance 2 apart with a path of length 2 or 3; Jeyanthi and Rama [1210] use S(G) to denote the graph obtained from a graph G by subdividing each edge of G by inserting a vertex. They prove the following graphs have super mean labelings:  $S(P_n \odot K_1), S(B_{n,n}), C_n \odot \overline{K_2}$ ; the graphs obtained by joining the central vertices of two copies of  $K_{1,m}$  by a path  $P_n$  (denoted by  $\langle B_{m,m}: P_n \rangle$ ); generalized antiprisms (see §6.2 for the definition), and the graphs obtained from the paths  $v_1, v_2, v_3, \ldots, v_n$  by joining each  $v_i$  and  $v_{i+1}$  to two new vertices  $u_i$  and  $w_i$ (double triangular snakes).

Lourdusamy and Seenivasan [1659] introduced the notion of super vertex mean labeling as follows. For a (p,q)-graph and an injective function f from the edges to the set  $\{1, 2, 3, \ldots, p+q\}$  that induces for each vertex v the label defined by  $f^*(v) = (\text{Round } \sum_{e \in E_v} f(e))/d(v)$ , where  $E_v$  denotes the set of edges in G that are incident to the vertex v, d(v) is the degree of v, and Round(x) is the integer nearest to x, such that the set of all edge labels and the induced vertex labels is  $\{1, 2, 3, \ldots, p+q\}$  is called a super vertex mean labeling of G and G is called a super vertex mean graph. In [1645] they investigated the all graphs of order up to 5 and regular graphs of order up to 7 for the property of being super vertex mean and proved that all linear triangular snakes are super vertex mean. Lourdusamy, George, and Seenivasan [1647] proved that all cycles except  $C_4$  are super vertex mean and Lourdusamy and George [1646] proved that linear  $C_n$  snakes with at least 2 blocks are super vertex mean graphs for the following cases:  $n = 4, 5, 6, \text{ and } 7; n \geq 8 \text{ even}; n \geq 9 \text{ and } n \equiv 1 \text{ mod } 4; \text{ and } n \geq 11 \text{ and } n \equiv 3 \text{ mod } 4.$  Inayah, Sudarsana, Musdalifah, and Mangesa [1070] have showed that the total graphs of paths and cycles are super mean graphs.

A graph G with q edges is called a k-mean graph if there is an injective function f from the vertices of G to  $\{0, 1, 2, ..., k + q - 1\}$  such that when each edge uv is labeled with (f(u) + f(v))/2 if f(u) + f(v) is even, and (f(u) + f(v) + 1)/2 if f(u) + f(v) is odd, the resulting edge labels are  $\{k, k + 1, k + 2, ..., k + q - 1\}$ . A graph G with q edges is said to have a restricted k-mean labeling if there is an injective function f from the vertices of G to  $\{k - 1, k, k + 1, ..., k + q - 1\}$  such that when each edge uv is labeled with  $\{k, k + 1, k + 2, ..., k + q - 1\}$ , the resulting edge labels  $\{k, k + 1, k + 2, ..., k + q - 1\}$  are distinct where k is a positive integer. A graph that admits a restricted k-mean labeling is called a restricted k-mean graph. Gayathri and Gopi proved some properties of k-mean labelings in [857]. In [858] they proved that if  $G_1$  and  $G_2$  are restricted k-mean graphs for all k, then  $G_1 \cup G_2$  is restricted k-mean for all k, and if  $G_1$  is a restricted k-mean graph for all  $k \ge k_1$  and  $G_2$  is a restricted k-mean graphs for all k, then  $G_1 \cup G_2$  is restricted k-mean for all  $k \ge k_1$ .

A mean graph is called k-super mean if vertex labels and the edge labels are  $\{k, k + 1, k+2, \ldots, p+q+k-1\}$ . Jeyanthi, Ramya, Thangavelu [1213] give super mean labelings for  $C_m \cup C_n$  and k-super mean labelings for a variety of graphs.

Vasuki and Nagarajan [2709] define  $H_n$ , called the *H*-graph of a path  $P_n$ , as the graph obtained from two copies of  $P_n$  with vertices  $v_1, v_2, \ldots, v_n$  and  $u_1, u_2, \ldots, u_n$  by joining the vertices  $v_{(n+1)/2}$  and  $u_{(n+1)/2}$  if *n* is odd, and the vertices  $v_{\frac{n}{2}+1}$  and  $u_{\frac{n}{2}}$  if *n* is even, and a cyclic snake  $mC_n$  as the the graph obtained from *m* copies of  $C_n$  by identifying the vertex  $v_{(k+2)_j}$  in the *j*th copy of the vertex  $v_{1_{j+1}}$  in the (j+1)th copy if n = 2k + 1 and identifying the vertex  $v_{(k+1)j}$  in the *j*th copy with the vertex  $v_{1j+1}$  in the (j + 1)th copy if n = 2k. They establish the super meanness of even cycles, *H*-graphs, the coronas of *H*-graphs, 2-coronas of *H*-graphs, coronas of cycles,  $mC_n$ -snakes  $(n \neq 4)$ , dragons  $P_n(C_m)$ for  $m \neq 4$ , and  $C_m \times P_n$  for m = 3 and 5. Vasuki, Sugirtha, and Venkateswari [2712] proved that the subdivision of the following graphs are super mean graphs:  $H_n, H_n \odot K_1$ ,  $H_n$  with two pendent edges attached to each vertex,  $C_n \odot K_1$   $(n \geq 3)$ , slanting ladders, triangular snakes with a pendent edge at each vertex, and  $C_m @C_n$  (the graph obtained by attaching paths  $P_n$  to  $C_m$  by identifying the endpoints of the paths with each successive pairs of vertices of  $C_m$ ).

Let G(V, E) be a simple graph of order p and size q. Then G is said to be a relaxed mean graph if it is possible to label the vertices  $x \in V$  with distinct elements f(x) from  $\{0, 1, 2, \ldots, q-1, q+1\}$  in such a way that when each edge uv is labeled with (f(u)+f(v))/2if f(u) + f(v) is even and (f(u) + f(v) + 1)/2 if f(u) + f(v) is odd, then the resulting edge labels  $\{1, 2, 3, \ldots, q\}$  are distinct. Such an f is called a relaxed mean labeling of G. Balaji, Ramesh, and Sudhaker [328] prove that the disjoint union of any path with n - 1 edges joining the pendent vertices of distinct paths is a relaxed mean graph and  $K_{1,m}$  is not a relaxed mean graph for  $m \geq 5$ . They also prove that the graph consisting of two stars  $K_{1,m}$  and  $K_{n,1}$  with an edge in common is a relaxed mean graph if and only if  $|m-n| \leq 5$ .

In [190] Arockiaraj, Rajesh Kannan, and Durai Baskar introduced the *F*-centroidal mean labeling of graphs by defining a function f to be an *F*-centroidal mean labeling of a graph G(V, E) with q edges if  $f: V(G) \to \{1, 2, 3, \ldots, q+1\}$  is injective and the induced function  $f^*: E(G) \to \{1, 2, 3, \ldots, q\}$  defined as  $f^*(uv) = \left\lfloor \frac{2 \left[f(u)^2 + f(u)f(v) + f(v)^2\right]}{3 \left[f(u) + f(v)\right]} \right\rfloor$  for all  $uv \in E(G)$  is bijective. A graph that admits an *F*-centroidal mean labeling is called an *F*-centroidal mean graph. In [182] they discussed the *F*-centroidal mean-ness of the tree  $P_n(X_1, X_2, \ldots, X_n)$  obtained from a path on n vertices by attaching  $X_i$  pendent vertices at each  $i^{th}$  vertex of the path for  $1 \leq i \leq n$ , the twig graph  $TW(P_n)$ , the graph  $P_n \circ S_m$  for  $m \leq 4$ ,  $P_m \times P_n$  for  $m \leq 3$ , ladders,  $P_n \circ K_2$ ,  $P_a^b$  for  $a \geq 2$  and  $b \leq 3$ , the middle graphs and splitting graphs of paths, the total graphs of paths,  $P_n^2$ , and  $P(1, 2, \ldots, n-1)$  the graph obtained by replacing each  $i^{th}$  edge of  $P_n$  by identifying its end vertices with the vertices of the two element component of  $K_{2,i}$ .

Arockiaraj, Rajesh Kannan, and Durai Baskar introduced super F-centroidal mean graphs [189] as follows. Let G be a graph and  $f: V(G) \to \{1, 2, 3, \ldots, p+q\}$  be an injection. With  $f^*$  defined as for the F-centroidal case f is called a super F-centroidal mean labeling if  $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$ . A graph that admits a super F-centroidal mean labeling is called a super F-centroidal mean graph. They proved that the following graphs are super F-centroidal mean graphs: paths, cycles, the union of any number of paths, the mirror graph of  $P_n$ ,  $P_n \circ S_m$ ,  $TW(P_n)$ ,  $P_n \cup C_m$ ,  $P_n^2$ , and dragons  $P_n(C_m)$  the graph obtained from  $C_m$  by identifying an end vertex of  $P_n$  at a vertex of  $C_m$ .

In [331] and [332] Balaji, Ramesh, and Subramanian use the term "Skolem mean" labeling for super mean labeling. They prove:  $P_n$  is Skolem mean;  $K_{1,m}$  is not Skolem mean if  $m \ge 4$ ;  $K_{1,m} \cup K_{1,n}$  is Skolem mean if and only if  $|m-n| \le 4$ ;  $K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is Skolem mean if |m-n| = 4+l for  $l = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots$ , and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+l for  $l = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots, n \geq l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is Skolem mean if |m-n| = 4+2l for  $l = 2, \ldots, m = 2, 3, 4 \ldots, n = 2l+m+4$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+l for  $l = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots, n \geq l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+l for  $l = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots, n \geq l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,m}$  is not Skolem mean if |m-n| > 4+2l for  $l = 2, \ldots, m = 2, 3, 4 \ldots, n \geq 2l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+2l for  $l = 2, \ldots, m = 2, 3, 4 \ldots, n \geq 2l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+2l for  $l = 2, \ldots, m = 2, 3, 4 \ldots, n \geq 2l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+2l for  $l = 2, \ldots, m = 2, 3, 4 \ldots, n \geq 2l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+2l for  $l = 2, \ldots, m = 2, 3, 4 \ldots, n \geq 2l+m+5$  and  $l \leq m < n$ ;  $K_{1,l} \cup K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is not Skolem mean if |m-n| > 4+2l for m < n < n. Balaji [330] proved that  $K_{1,l} \cup K_{1,m} \cup K_{1,n}$  is Skolem mean if |m-n| < 4+l for integers  $1, m \geq 1$  and  $l \leq m < n$ . In [2305] Shainy and Balaji determined necessary and sufficient conditions for the disjoint union of three stars to be Skolem mean.

In [1234] Jeyanthi, Selvi, and Ramya prove that  $C_m \cup C_n$ ,  $(P_n + K_1) \cup (n-2)K_2$  (n > 2),  $(P_n + K_2) \cup (2n - 3)K_2$   $(n \ge 2)$  and  $W_n \cup (n - 1)K_2$   $(n \ge 3)$  are Skolem difference mean graphs. In [1235] they show that the union of any finite number of paths, the union of any finite number of stars,  $G \cup nK_2$  where G is Skolem difference mean and all the vertex labels are odd,  $C_m \cup P_m$   $(m \ge 2)$ ,  $K_{m,n} \cup (m-1)(n-1)K_2$ , and  $K_{1,1,n} \cup (n-1)K_2$ . are skolem difference mean graphs.

In [1214] Jeyanthi, Ramya, and Thangavelu proved the following graphs have super mean labelings: the one point union of any two cycles, graphs obtained by joining any two cycles by an edge (dumbbell graphs),  $C_{2n+1} \odot C_{2m+1}$ , graphs obtained by identifying a copy of an odd cycle  $C_m$  with each vertex of  $C_n$ , the quadrilateral snake  $Q_n$ , where n is odd, and the graphs obtained from an odd cycle  $u_1, u_2, \ldots, u_n$  by joining the vertices  $u_i$ and  $u_{i+1}$  by the path  $P_m$  (m is odd) for  $1 \le i \le n-1$  and joining vertices  $u_n$  and  $u_1$  by the path  $P_m$ . Jeyanthi, Ramya, Thangavelu, and Aditanar [1212] give super mean labelings of  $C_m \cup C_n$  and  $T_p$ -trees. Vasuki and Arockiaraj [2707] proved that  $nC_4$ , n > 1, triangular grid graphs, the edge  $mC_n$ -snakes, and the braid graphs are super mean graphs. They further proved that the graphs obtained by identifying an edge of two cycles  $C_m$  and  $C_n$ is a super mean graph.

In [1206] Jeyanthi and Ramya define  $S_{m,n}$  as the graph obtained by identifying one endpoint of each of n copies of  $P_m$  and  $\langle S_{m,n} : P_m \rangle$  as a graph obtained by identifying one end point of a path  $P_m$  with the vertex of degree n of a copy of  $S_{m,n}$  and the other endpoint of the same path to the vertex of degree n of another copy of  $S_{m,n}$ . They prove the following graphs have super mean labelings: caterpillars,  $\langle S_{m,n} : P_{m+1} \rangle$ , and the graphs obtained from  $P_{2m}$  and 2m copies of  $K_{1,n}$  by identifying a leaf of *i*th copy of  $K_{1,n}$ with *i*th vertex of  $P_{2m}$ . They further establish that if T is a  $T_p$ -tree, then  $T \odot K_1$ ,  $T \odot \overline{K_2}$ , and, when T has an even number of vertices,  $T \odot \overline{K_n}$   $(n \geq 3)$  are super mean graphs.

Gopi [914] calls a graph G with p vertices and q edges a F-root square mean graph if there is a injective function f from the vertices of V(G) to  $\{1, 2, \ldots, q+1\}$  such that for each edge uv the induced function  $f^*(uv) = \lfloor \sqrt{(f(u)^2 + f(v)^2)/2} \rfloor$  is bijective. He proved that triangular snakes  $T_n$   $(n \ge 2)$ ,  $A(T_n)$   $(n \ge 3)$ ,  $D(T_n)$   $(n \ge 2)$ , quadrilateral snakes,  $A(Q_n), D(Q_n)$   $(n \ge 3)$  are F-root square mean graphs.

Kannan, Vikrama Prasad, and Gopi [1332] call a graph G with p vertices and q

edges a super root mean graph if there is an injective function f from the vertices of G to  $\{1, 2, \ldots, p + q\}$  such that for each edge uv the induced function  $f^*(uv) = \lfloor \sqrt{(f(u)^2 + f(v)^2)/2} \rfloor$  or  $f^*(uv) = \lceil \sqrt{(f(u)^2 + f(v)^2)/2} \rceil$  yields the set of vertex labels and edge labels  $\{1, 2, \ldots, p + q\}$ . They proved the following are super root square mean graphs:  $P_m \cup P_m$   $(m, n \ge 3)$ ;  $P_m \cup (P_n \cdot K_1)$   $(m, n \ge 3)$ ;  $(P_m \cdot K_1) \cup (P_n \cdot K_1)$   $(m, n \ge 3)$ ; the union of a path and a triangular snake; and the union of  $P_n \cdot K_1$  and a triangular snake. Gopi and Kalaiyarasi [916] prove that the following graphs have a super root square mean labeling:  $P_n^2$   $(n \ge 4)$ , slanting ladders  $SL_n(n \ge 3)$ , triangular snakes with a pendent edge attached to each vertex, and quadrilateral snakes with a pendent edge attached to each vertex.

Let G be a graph and let  $f: V(G) \to \{1, 2, ..., n\}$  be a function such that the label of the edge uv is (f(u) + f(v))/2 or (f(u) + f(v) + 1)/2 according as f(u) + f(v) is even or odd and  $f(V(G)) \cup \{f^*(e) : e \in E(G)\} \subseteq \{1, 2, ..., n\}$ . If n is the smallest positive integer satisfying these conditions together with the condition that all the vertex and edge labels are distinct and there is no common vertex and edge labels, then n is called the super mean number of a graph G and it is denoted by  $S_m(G)$ . Nagarajan, Vasuki, and Arockiaraj [1827] proved that for any graph of order p,  $S_m(G) \leq 2^p - 2$  and provided an upper bound of the super mean number of the graphs:  $K_{1,n}$   $n \geq 7$ ;  $tK_{1,n}$ ,  $n \geq 5$ , t > 1; the bistar B(p, n), p > n; the graphs obtained by identifying a vertex of  $C_m$  and the center of  $K_{1,n}$ ,  $n \geq 5$ ; and the graphs obtained by identifying a vertex of  $C_m$  and the vertex of degree 1 of  $K_{1,n}$ . They also gave the super mean number for the graphs  $C_n, tK_{1,4}$ , and B(p, n) for p = n and n + 1.

Manickam and Marudai [1695] defined a graph G with q edges to be an odd mean graph if there is an injective function f from the vertices of G to  $\{1, 3, 5, \ldots, 2q - 1\}$  such that when each edge uv is labeled with (f(u) + f(v))/2 if f(u) + f(v) is even, and (f(u) + f(v) + 1)/2 if f(u) + f(v) is odd, then the resulting edge labels are distinct. Such a function is called a odd mean labeling. For integers a and b at least 2, Vasuki and Nagarajan [2711] use  $P_a^b$  to denote the graph obtained by starting with vertices  $y_1, y_2, \ldots, y_a$  and connecting  $y_i$  to  $y_{i+1}$  with b internally disjoint paths of length i + 1 for  $i = 1, 2, \ldots, a - 1$  and  $j = 1, 2, \ldots, b$ . For integers  $a \ge 1$  and  $b \ge 2$  they use  $P_{\langle 2a \rangle}^b$  to denote the graph obtained by starting  $y_i$  to  $y_{i+1}$  with b internally disjoint paths of length i + 1 for  $i = 1, 2, \ldots, a - 1$  and  $j = 1, 2, \ldots, b$ . For integers  $a \ge 1$  and  $b \ge 2$  they use  $P_{\langle 2a \rangle}^b$  to denote the graph obtained by starting  $y_i$  to  $y_{i+1}$  with b internally disjoint paths of length  $i = 1, 2, \ldots, b$ . They proved that the graphs  $P_{2r,m}, P_{2r+1,2m+1}$ , and  $P_{\langle 2r \rangle}^m$  are odd mean graphs for all values of r and m.

Jeyanthi and Gomathi [1141] proved the edge linked cyclic snake  $EL(kC_n)$   $(n \ge 6)$  is an odd mean graph. In [1141] they constructed new families of odd mean graphs from linking existing odd mean graphs.

For a  $T_p$ -tree T with m vertices  $T@P_n$  is the graph obtained from T and m copies of  $P_n$  by identifying one pendent vertex of ith copy of  $P_n$  with ith vertex of T. For a  $T_p$ -tree T with m vertices  $T@2P_n$  is the graph obtained from T by identifying the pendent vertices of two vertex disjoint paths of equal lengths n-1 at each vertex of T. Ramya, Selvi and Jeyanthi [2093] prove that  $P_m \odot \overline{K_n}$  ( $m \ge 2, n \ge 1$ ) is an odd mean graph,  $T_p$ trees are odd mean graphs, and, for any  $T_p$  tree T, the graphs  $T@P_n$ ,  $T@2P_n$ ,  $\langle T\tilde{o}K_{1,n} \rangle$  are odd mean graphs.

For a  $T_p$ -tree T with m vertices let  $T \hat{o} C_n$  denote the graph obtained from T and m copies of  $C_n$  by identifying a vertex of  $i^{th}$  copy of  $C_n$  with  $i^{th}$  vertex of T and  $T \tilde{o} C_n$  denote the graph obtained from T and m copies of  $C_n$  by joining a vertex of  $i^{th}$  copy of  $C_n$  with  $i^{th}$  vertex of T by an edge. In [1237] Selvi, Ramya, and Jeyanthi prove that for a  $T_p$  tree T the graphs  $T \hat{o} C_n$   $(n > 3, n \neq 6)$  and  $T \tilde{o} C_n$ ,  $(n > 3, n \neq 6)$  are odd mean graphs.

Ramya, Selvi, and Jeyanthi [2092] prove that for a  $T_p$ -tree T the following graphs are odd mean graphs:  $T@P_n$ ,  $T@2P_n$ ,  $P_m \odot \overline{K_n}$ , and the graph obtained from T and m copies of  $K_{1,n}$  by joining the central vertex of *i*th copy of  $K_{1,n}$  with *i*th vertex of T by an edge.

For a graph G and some fixed vertex v of G, Pooranam, Vaski, and Suganthi [2008] proved the following graphs have odd mean labelings: graphs obtained from a path  $P_m$ :  $u_1u_2\cdots u_m$  and G by joining  $u_i$  to v to the *i*th copy of G. Their results include the cases where  $G = C_{4n}$ , a star, or the cube  $Q_3$ . For a graph G and some fixed vertex v of Gthey also proved the existence of odd mean labelings for graphs obtained from a path  $P_m: u_1u_2\cdots u_m$  and G by identitying  $u_i$  with v in the *i*th copy of G, where G is  $Q_3$  or  $C_{4n}^{(2)}$  and v is the vertex of  $C_{4n}^{(2)}$  of degree 4.

A graph G is said to be *vertex odd mean* graphvertex odd mean if there exist an injective function f: V(G) to  $\{1, 3, 5, \ldots, 2|E(G)|-1\}$  such that the induced mapping  $f^*$ : E(G) to the set of positive integers defined by  $f^*(uv) = (f(u) + f(v))/2$  is injective. Such a function is called a *vertex odd mean* labeling. A graph G is called a *vertex even mean* graph if there exist an injective function f: V(G) to  $\{2, 4, 6, \dots, 2|E(G)|\}$  such that the induced mapping  $f^*: E(G)$  to the set of positive integers defined by  $f^*(uv) = (f(u) + f(v))/2$ is injective. Such a function is called a *vertex even mean* labeling. A bijective mapping f: V(G) to  $\{0, 1, 2, \ldots, |V(G)| - 1\}$  is said to be a square sum labeling if the induced function  $f^*$  from E(G) to the positive integers defined by  $f^*(xy) = (f(x)^2 + (f(y))^2)$  is injective. A graph that has a square sum labeling is called a square sum graph. Maheswari and Srividya [1685] proved that every cycle  $C_n$   $(n \ge 6)$  with parallel  $P_3$  chords admit a vertex odd mean labeling, a vertex even mean labeling, and a square sum labeling. In [1684] they proved that the following graphs are square sum graphs: cycles with parallel chords, graphs obtained by attaching an arbitrary number of pendant edges at a vertex of degree 2 of a cycle with parallel chords, duplication of a vertex of degree 2 of a cycle with parallel chords, crowns with parallel chords, chains of even cycles with parallel chords, and graphs obtained from copies of  $C_n$  by joining a vertex from each copy of  $C_n$  to a common vertex.

Gayathri and Amuthavalli [840] (see also [150]) say a (p,q)-graph G has a (k,d)-odd mean labeling if there exists an injection f from the vertices of G to  $\{0, 1, 2, \ldots, 2k - 1 + 2(q-1)d\}$  such that the induced map  $f^*$  defined on the edges of G by  $f^*(uv) = [(f(u)+f(v))/2]$  is a bijection from edges of G to  $\{2k-1, 2k-1+2d, 2k-1+4d, \ldots, 2k-1+2(q-1)d\}$ . When d = 1 a (k,d)-odd mean labeling is called k-odd mean. For  $n \geq 2$ they prove the following graphs are k-odd mean for all k:  $P_n$ ; combs  $P_n \odot K_1$ ; crowns  $C_n \odot K_1$   $(n \geq 4)$ ; bistars  $B_{n,n}$ ;  $P_m \odot \overline{K_n}$   $(m \geq 2)$ ;  $C_m \odot \overline{K_n}$ ;  $K_{2,n}$ ;  $C_n$  except for n = 3 or 6; the one-point union of  $C_n$   $(n \geq 4)$  and an endpoint of any path; grids  $P_m \times P_n$   $(m \geq 2)$ ;  $(P_n \times P_2) \odot K_1$ ; arbitrary unions of paths; arbitrary unions of stars; arbitrary unions of cycles; the graphs obtained by joining two copies of  $C_n$   $(n \ge 4)$  by any path; and the graph obtained from  $P_m \times P_n$  by replacing each edge by a path of length 2. They prove the following graphs are not k-odd mean for any k:  $K_n$ ;  $K_n$  with an edge deleted;  $K_{3,n}$   $(n \ge 3)$ ; wheels; fans; friendship graphs; triangular snakes; Möbius ladders; books  $K_{1,m} \times P_2$   $(m \ge 4)$ ; and webs. For  $n \ge 3$  they prove  $K_{1,n}$  is k-odd mean if and only if  $k \ge n - 1$ . Gayathri and Amuthavalli [841] prove that the graph obtained by joining the centers of stars  $K_{1,m}$  and  $K_{1,n}$  are k-odd mean for m = n, n+1, n+2 and not k-odd mean for m > n + 2. For  $n \ge 2$  the following graphs have a (k, d)-mean labeling [862]:  $C_m \cup P_n$   $(m \ge 4)$  for all k; arbitrary unions of cycles for all k;  $P_{2m}$ ;  $P_{2m+1}$  for  $k \ge d$ ;  $(P_{2m+1} \text{ is not } (k, d)$ -mean when k < d); combs  $P_n \odot K_1$  for all k;  $K_{1,n}$  for  $k \ge d$ ;  $K_{2,n}$  for  $k \ge d$ ; bistars for all k;  $nC_4$  for all k; and quadrilateral snakes for  $k \ge d$ .

In [2249] Seoud and Salim [2250] proved that a graph has a k-odd mean labeling if and only if it has a mean labeling. In [2249] Seoud and Salim give upper bounds of the number of edges of graphs with a (k, d)-odd mean labeling

Pricilla [2038] defines an *even mean labeling* of a graph G as an injective function f from the verticies of G to  $\{2, 4, \ldots, 2|E(G)|\}$  such that the edge labels given by (f(u) + f(v))/2are distinct. Vaidya and Vyas [2694] proved that  $D_2(P_n)$ ,  $M(P_n)$ ,  $T(P_n)$ ,  $S'(P_n)$ ,  $P_n^2$ ,  $P_n^3$ , switching of pendent vertex in  $P_n$ ,  $S'(B_{n,n})$ , double fans, and duplicating each vertex by an edge in paths are even mean graphs.

Gayathri and Gopi [849] defined a graph G with q edges to be an k-even mean graph if there is an injective function f from the vertices of G to  $\{0, 1, 2, \dots, 2k + 2(q-1)\}$ such that when each edge uv is labeled with (f(u) + f(v))/2 if f(u) + f(v) is even, and (f(u) + f(v) + 1)/2 if f(u) + f(v) is odd, then the resulting edge labels are  $\{2k, 2k + 1\}$  $2, 2k+4, \ldots, 2k+2(q-1)$ . Such a function is called a k-even mean labeling. In [849] they proved that the graphs obtained by joining two copies of  $C_n$  with a path  $P_m$  are k-even mean for all k and all  $m, n \geq 3$  when  $n \equiv 0, 1 \pmod{4}$  and for all  $k \geq 1, m \geq 7$ , and  $n \geq 3$ . In [851] Gayathri and Gopi proved that various graphs obtained by joining two copies of stars  $K_{1,m}$  and  $K_{1,n}$  with a path by identifying the one endpoint of the path with the center of one star and the other endpoint of the path with the center of the other star are k-even mean. In [850] they proved that various graphs obtained by appending a path to a vertex of a cycle are k-even mean. In [852] they proved that  $C_n \cup P_m$ ,  $n \ge 4$ ,  $m \ge 2$ , is k-even mean for all k. Gayathri and Gopi [855] proved the following are k-even mean graphs: shadow graphs of stars with at least 3 vertices; edge duplication graphs of cycles with at least 4 vertices; and vertex duplication graphs of paths and cycles with at least 4 vertices.

Gayathri and Gopi [853] say graph G with q edges has a (k, d)-even mean labeling if there exists an injection f from the vertices of G to  $\{0, 1, 2, \ldots, 2k + 2(q-1)d\}$  such that the induced map f<sup>\*</sup> defined on the edges of G by  $f^*(uv) = (f(u) + f(v))/2$  if f(u) + f(v)is even and  $f^*(uv) = (f(u) + f(v) + 1)2$  if f(u) + f(v) is odd is a bijection from edges of G to  $\{2k, 2k + 2d, 2k + 4d, \ldots, 2k + 2(q-1)d\}$ . A graph that has a (k, d)-even mean labeling is called a (k, d)-even mean graph. They proved that  $P_m \oplus nK_1 (m \ge 3, n \ge 2)$ has a (k, d)-even mean labeling in the following cases: all (k, d) when m is even; all (k, d)when m is odd and n is odd; and m is odd, n is even and  $k \ge d$ . Kalaimathy [1267] investigated conditions under which a mean labeling for a graph G will yield a (k, d)-even mean labeling for G and vice versa. He also gave conditions under which two graphs that have (1, 1)-mean labelings can be joined by an single edge to obtain a new graph that has a (1, 1)-even mean labeling. Gopi's Ph. D. thesis [911] has a large number of results about mean, k-mean, k-odd mean, k-even mean, (k, d)-odd mean , and (k, d)-mean labelings.

A (p,q)-graph is said to have an even vertex odd mean labeling if there exists an injective function f from V(G) to  $\{0, 2, 4, \dots, 2q - 2, 2q\}$  such that the induced map  $f^*: E(G)$  to  $\{1, 3, 5, \dots, 2q - 1\}$  defined by  $f^*(uv) = (f(u) + f(v))/2$  is a bijection. A graph that admits an even vertex odd mean labeling is called an *even vertex odd* mean graph. Kannan, Vikrama Prasad, Gopi [1333] proved the following graphs have an even vertex odd mean labeling: slanting ladders  $SL_n (n \geq 3)$ ; double triangular snakes; alternative double triangular snakes; graphs obtained by starting with a tree G with at least 3 vertices and a mean labeling and a copy G' of G by joining each vertex of G to its corresponding vertex in G' with an edge; graphs obtained by starting with a path  $v_1v_2\cdots v_n$   $(n \ge 4)$  and joining  $v_1$  and  $v_3$  to an isolated vertex; graphs obtained by starting with a path  $v_1v_2\cdots v_n$   $(n \ge 4)$  and appending two edges to each of  $v_2, v_3, \ldots, v_{n-1}$ ; and graphs obtained from a quadrilateral snake and appending an edge at each vertex. The H-graph of a path  $P_n$  is the graph obtained from two copies of  $P_n$  with vertices  $v_1, v_2, \ldots, v_n$  and  $u_1, u_2, \ldots, u_n$  by joining the vertices  $v_{n/2+1}$  and  $u_{n/2+1}$  by an edge if n is odd and the vertices  $v_{(n+1)/2}$  and  $u_{(n/2)}$  by an edge if n is even. Kannan, Vikrama Prasad, and Gopi [1334] prove that the H-graph of  $P_n$   $(n \ge 3)$  and the graph  $H \odot K_1$ have even vertex odd mean labelings where H is the H graph of  $P_n$   $(n \ge 3)$ . In [1335] and [1336] Kannan, Vikrama Prasad, and Gopi proved the following are even vertex odd mean graphs: graphs obtained by joining the centers of two stars  $K_{1,m}$  and  $K_{1,n}$  by a path  $P_t$   $(m, n, t \ge 2)$ , graphs obtained by duplicating an edge of  $C_n$   $(n \ge 4)$ , graphs obtained by joining each endpoint of  $P_3$  to n isolated vertices, shadow graphs of stars, shadow graphs of bistars B(n, n), mirror graphs of paths, and the graphs obtained taking two copies of  $P_n \times P_2$  and joining each vertex of one with the matching vertex in the other with an edge. Prasad, Kannan, and Gopi [2036] proved that  $C_{4m} \odot K_{1,4n}$ ,  $P_m \odot P_n$ , and  $K_2 + K_n$  have even vertex odd mean labelings. In [2030] Prajapati and Raval proved that quadrilateral snakes, pentagonal snakes, and alternating quadrilateral snakes are vertex even and odd mean graphs. They also proved that even vertex odd mean graphs are even mean graphs. Jeyanthi, Ramya, and Selvi [1211] prove that  $T_P$ -trees (transformed trees),  $T@P_n$ ,  $T@2P_n$ , and  $\langle T \hat{o} K_{1,n} \rangle$  (where T is a  $T_P$  tree) are even vertex odd mean graphs.

For a graph G(V, E) a bijection f from  $V(G) \cup E(G)$  onto  $\{1, 2, \ldots, |V(G)| + |V(E)|\}$ is said to be a *total mean labeling* if the values of  $f^*(uv) = \lceil (f(u) + f(v) + f(uv))/3 \rceil$ taken over all edges are distinct. A graph G is said to be a *total mean labeling* graph if it admits a total mean labeling. Karuppasamy and Kaleeswari [1343] proved that  $P_n, P_n^+, K_{1,n}, K_{2,n}, C_n, B_{m,n}$ , triangular snakes, and alternate triangular snakes are total mean labeling graphs.

Murugan and Subramanian [1799] say a (p,q)-graph G has a Skolem difference mean labeling if there exists an injection f from the vertices of G to  $\{1, 2, ..., p+q\}$  such that the induced map  $f^*$  defined on the edges of G by  $f^*(uv) = (|f(u) - f(v)|)/2$  if |f(u) - f(v)|is even and  $f^*(uv) = (|f(u) - f(v)| + 1)/2$  if |f(u) + f(v)| is odd is a bijection from edges of G to  $\{1, 2, \ldots, q\}$ . A graph that has a Skolem difference mean labeling is called a *Skolem difference mean* graph. They show that the graphs obtained by starting with two copies of  $P_n$  with vertices  $v_1, v_2, \ldots, v_n$  and  $u_1, u_2, \ldots, u_n$  and joining the vertices  $v_{(n+1)/2}$ and  $u_{(n+1)/2}$  if n is odd and the vertices  $v_{n/2+1}$  and  $u_{n/2}$  if n is even are Skolem difference mean. Parmar and Vaghela [1898] proved the following graphs have Skolem difference mean labelings: brooms  $B_{n,d}$   $(n \ge 4, d \ge 2)$  (the graph with n vertices obtained from  $P_d$ by appending n-d edges at an endpoint), combs  $P_n \odot K_1$   $(n \ge 2)$ ,  $K_{1,m} \cup K_{1,n}$   $(m, n \ge 2)$ , and  $K_{1,3} * K_{1,n}$   $(n \ge 2)$  obtained from  $K_{1,3}$  by attaching root of a star  $K_{1,n}$  at each pendent vertex of  $K_{1,3}$ . Jeyanthi [1132] proved  $\cup P_{n_i}$   $(n_i \ge 2)$ ;  $\cup K_{1,n_i}$   $(n_i \ge 2)$ ;  $C_n \cup P_m$   $(n \ge$  $3, m \ge 2)$ ;  $K_{m,n} \cup (m-1)(n-1)K_2$ ; and  $K_{1,1,n} \cup (n-1)K_2$  are Skolem difference mean graphs. She also proved that if G is a Skolem difference mean graph, then  $G \cup nK_2$  is a Skolem difference mean graph.

Let  $L_0, L_1, \ldots$  denote the sequence of Lucas numbers. In [1936] Ponmoni, Navaneetha Krishnan, and Nagarajan introduce the following graph labeling method. A graph G with p vertices and q edges is said to have a *Skolem difference Lucas mean* labeling if there is an injective function f from the vertices to  $\{1, 2, \ldots, L_{p+q}\}$  such that when the edge uv is labeled with |f(u) - f(v)|/2 if |f(u) - f(v)| is even, and (|f(u) - f(v)| + 1)/2 if |f(u) - f(v)| is odd, then the resulting edge labels are distinct and belong to  $\{L_1, L_2, \ldots, L_q\}$ . A graph that admits a Skolem difference Lucas mean labeling is called a *Skolem difference Lucas mean* graph. They proved the graphs obtained from  $K_{1,m}$  by identifying the center of  $K_{1,n}$  with the endpoint of each non-center vextex of  $K_{1,m}$ , bistars,  $K_{1,m} \odot 2P_n$  and  $\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(m)} \rangle$  are Skolem difference Lucas mean graphs.

Selvi, Ramya, and Jeyanthi [2209] prove that  $C_n@P_n$   $(n \ge 3, m \ge 1)$ ,  $K_n(n \le 3)$ , the shrub  $St(n_1, n_2, \dots, n_m)$ , and the banana tree  $Bt(n, n, \dots, n)$  are Skolem difference mean graphs. They show that if G is a (p,q) graph with q > p then G is not a Skolem difference mean graph and prove that  $K_n$   $(n \ge 4)$  is not a Skolem difference mean graph. A skolem difference mean labeling for which all the labels are odd is called an *extra Skolem difference mean* labeling. They also prove that the graph  $T \langle K_{1,n_1} : K_{1,n_2} : \dots : K_{1,n_m} \rangle$ , obtained from the stars  $K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_m}$  by joining the central vertex of  $K_{1,n_m}$ , and  $K_{1,n_{j+1}}$  to a new vertex  $w_j$  for  $1 \le j \le m-1$  and the graph  $T \langle K_{1,n_1} \circ K_{1,n_2} \circ \dots \circ K_{1,n_m} \rangle$ , obtained from  $K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_m}$  by joining a leaf of  $K_{1,n_{j+1}}$  to a new vertex  $w_j$  for  $1 \le j \le m-1$  by an edge are extra Skolem difference mean graphs. Jeyanthi, Selvi, and Ramya [1235] proved that the union of any number of paths, any number of stars,  $G \cup nK_2$  where G is an extra Skolem difference mean tree,  $C_n \cup P_m$   $(n \ge 3, m \ge 2)$ ,  $K_{m,n} \cup (m-1)(n-1)K_2$ , and  $K_{1,1,n} \cup (n-1)K_2$  have Skolem difference mean labelings.

Let G(V, E) be a graph with p vertices and q edges. Ramya, Kalaiyarasi, and Jeyanthi [2090] say G is a Skolem odd difference mean if there exists an injective function f:  $V(G) \rightarrow \{0, 1, 2, 3, \ldots, p+3q-3\}$  such that the induced map  $f^* : E(G) \rightarrow \{1, 3, 5, \ldots, 2q-1\}$  denoted by  $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$  is a bijection. A graph that admits a Skolem odd difference mean labeling is called a *odd difference mean* graph. They prove that  $P_n$ ,  $C_n$   $(n \geq 4)$ ,  $K_{1,n}$ ,  $P_n \odot K_{1,n}$ , coconut trees T(n,m) obtained by identifying the central

vertex of the star  $K_{1,m}$  with a pendent vertex of  $P_n$ ,  $B_{m,n}$ , caterpillars  $S(n_1, n_2, \ldots, n_m)$ ,  $P_m@P_n$  and  $P_m@2P_n$  are Skolem odd difference mean graphs. ( $P_m@P_n$  is obtained from  $P_m$  and m copies of  $P_n$  by identifying one pendent vertex of the *i*-th copy of  $P_n$  with the *i*-th vertex of  $P_m$ ;  $P_m@2P_n$  is defined analogously.) They establish that  $K_n$ , n > 3 and  $K_{2,n}$   $(n \geq 3)$  are not Skolem odd difference mean graphs. They also prove that  $K_{2,n}$  is a Skolem odd difference mean graph if  $n \leq 2$ . In [1159] Jeyanthi, Kalaiyarasi, Ramya, and Saratha Devi prove that bistars B(m, n),  $mP_n$ ,  $mP_n \cup tP_s$ ,  $mK_{1,n} \cup tK_{1,s}$  and the graph  $\langle P_m \tilde{o} S_n \rangle$  obtained from  $P_m$  and m copies of  $K_{1,n}$  by joining the central vertex of  $i^{th}$  copy of  $K_{1,n}$  with  $i^{th}$  vertex of  $P_m$  by an edge admit Skolem odd difference mean labelings. They also prove that if G(p,q) is a Skolem odd differences mean graph then  $p \ge q$  and that wheels, umbrellas, books, and ladders are not Skolem odd difference mean graphs. They call a Skolem odd difference mean labeling a Skolem even vertex odd difference mean labeling if all the vertex labels are even. They prove that  $P_n, K_{1,n}, P_n \odot K_1$ , the coconut tree T(n,m) obtained by identifying the central vertex of  $K_{1,m}$  with a pendent vertex of a path  $P_n$ , B(m, n), caterpillars  $S(n_1, n_2, \ldots, n_m)$ ,  $P_m@P_n$  are  $P_m@2P_n$  are even vertex odd difference mean and  $C_n$  is not a Skolem even vertex odd difference mean graph. In [1269] Kalaiyarasi, Ramya, and Jeyanthi prove the following graphs have Skolem odd difference mean labelings: graphs obtained from a  $T_p$  tree with m vertices and m copies of  $K_{1,n}$  by identifying the central vertex of *i*th copy of  $K_{1,n}$ , with *i*th vertex of T; graphs obtained by connecting an isolated vertex to central vertex of each of a number of stars; the banana trees obtained by connecting an isolated vertex to one leaf of each of any number of  $K_{1,n}$ ; graphs obtained from  $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}$  by joining the central vertices of  $K_{1,n_j}$  and  $K_{1,n_{j+1}}$  to a new vertex  $w_j$  for  $1 \leq j \leq m-1$ ; graphs obtained from  $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}$ by joining a leaf of  $K_{1,n_i}$  and a leaf of  $K_{1,n_{i+1}}$  to a new vertex  $w_j$  for  $1 \le j \le m-1$ .

Lau, Jeyanthi, Ramya, and Kalaiyarasi [1467] say a (p,q)-graph G(V, E) is a Skolem even difference mean if there exists an injective function  $f: V(G) \to \{0, 1, 2, 3, \ldots, p + 3q - 1\}$  such that the induced map  $f^*: E(G) \to \{2, 4, \ldots, 2q\}$  defined by  $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$  is a bijection. A graph that admits a Skolem even difference mean labeling is called a *even difference mean*. They prove: the disjoint union of paths of length at least 2 and  $K_{2,n} \cup (n-1)K_2$   $(n \geq 2)$  are Skolem even vertex odd difference mean graphs; if G is a Skolem even vertex odd difference mean (q + 1, q)-graph, then  $G \cup nK_2, G \cup P_n$ , and  $G \cup K_{1,n}$  are Skolem odd difference mean graphs;  $C_m \cup P_n$   $(n \geq 2)$  is a Skolem even vertex even difference mean graph;  $P_m@P_n, mP_n, K_{m,n} \cup (m-1)(n-1)K_2$   $(m, n \geq 2)$ ,  $K_{1,n} \cup nK_2$ , and  $K_{1,1,n} \cup nK_2$  are Skolem even difference mean graphs; and if G is a Skolem even difference mean (q+1,q)-graph, then  $G \cup nK_2$  is a Skolem even difference mean graph. They conclude with the following open problem: Establish that  $G \cup nK_2$ where G is a (complete) multipartite graph is a Skolem even difference mean graph.

Kalaiyarasi, Ramya, and Jeyanthi [1268] say a graph G(V, E) with p vertices and q edges has a *centered triangular mean* labeling if it is possible to label the vertices with distinct elements f(x) from S, where S is a set of non-negative integers in such a way that for each edge e = uv,  $f^*(e) = \lceil (f(u) + f(v))/2 \rceil$  and the resulting edge labels are the first q centered triangular numbers. A graph that admits a centered triangular mean

labeling is called a *centered triangular mean* graph. They prove that  $P_n$ ,  $K_{1,n}$ , bistars  $B_{m,n}$ , coconut trees, caterpillars  $S(n_1, n_2, n_3, \ldots, n_m)$ ,  $St(n_1, n_2, n_3, \ldots, n_m)$ , banana trees  $Bt(n, n, \ldots, n)$  and  $P_m@P_n$  are centered triangular mean graphs.

Selvi, Ramya, and Jeyanthi [2208] define a triangular difference mean labeling of a graph G(p,q) as an injection  $f: V \longrightarrow Z^+$ , such that when the edge labels are defined as  $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$  the values of the edges are the first q triangular numbers. A graph that admits a triangular difference mean labeling is called a triangular difference mean graph. They prove that the following are triangular difference mean graphs:  $P_n, K_{1,n}, P_n \odot K_1$ , bistars  $B_{m,n}$ , graphs obtained by joining the roots of different stars to the new vertex, trees T(n,m) obtained by identifying a central vertex of a star with a pendent vertex of a path, the caterpillar  $S(n_1, n_2, \ldots, n_m)$  and the graph  $C_n@P_m$ .

A graph G(V, E) with p vertices and q edges is said to have centered triangular difference mean labeling if there is an injective mapping f from V to  $Z^+$  such that the edge labels induced by  $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$  are the first q centered triangular numbers. A graph that admits a centered triangular difference mean labeling is called a centered triangular difference mean graph. Ramya, Selvi, and Jeyanthi [1238] prove that  $P_n$ ,  $K_{1,n}$ ,  $C_n \odot K_1$ , bistars  $B_{m,n}$ ,  $C_n$  (n > 4), coconut trees, caterpillars  $S(n_1, n_2, n_3, \ldots, n_m)$ ,  $C_n @P_m$  (n > 4) and  $S_{m,n}$  are centered triangular difference mean graphs.

Gayathri and Tamilselvi [862] say a (p,q)-graph G has a (k,d)-super mean labeling if there exists an injection f from the vertices of G to  $\{k, k+d, \ldots, k+(p+q)d\}$  such that the induced map  $f^*$  defined on the edges of G by  $f^*(uv) = \lfloor (f(u) + f(v))/2 \rfloor$  has the property that the vertex labels and the edge labels together are the integers from k to k + (p+q)d. When d = 1 a (k, d)-super mean labeling is called k-super mean. For  $n \ge 2$  they prove the following graphs are k-super mean for all k: odd cycles;  $P_n$ ;  $C_m \cup P_n$ ; the one-point union of a cycle and the endpoint of  $P_n$ ; the union of any two cycles excluding  $C_4$ ; and triangular snakes. For  $n \ge 2$  they prove the following graphs are (k, d)-super mean for all k and d:  $P_n$ ; odd cycles; combs  $P_n \odot K_1$ ; and bistars. In [1214] Jeyanthi, Ramya, and Thangavelu proved the following graphs have k-super mean labelings:  $C_{2n}$ ,  $C_{2n+1} \times P_m$ , grids  $P_m \times P_n$ with one arbitrary crossing edge in every square, and antiprisms on 2n vertices (n > 4). (Recall an antiprism on 2n vertices has vertex set  $\{x_{1,1},\ldots,x_{1,n},x_{2,1},\ldots,x_{2,n}\}$  and edge set  $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$  where subscripts are taken modulo n). Jeyanthi, Ramya, Thangavelu [1213] give k-super mean labelings for a variety of graphs. Jeyanthi, Ramya, Thangavelu, and Aditanar [1212] show how to construct k-super mean graphs from existing ones. For  $n \geq 3$  Gayathri and Tamilselvi [862] proved the following graphs are k-super edge mean for all k: paths; cycles; combs  $P_n \odot K_1$ ; triangular snakes; crowns  $C_n \odot K_1$ ; the one-point union of  $C_3$  and an endpoint of  $P_n$ ; and  $P_n \odot K_2$ .

In [2179] Sandhya, Somasundaram, and Ponraj call a graph with q edges a harmonic mean graph if there is an injective function f from the vertices of the graph to the integers from 1 to q + 1 such that when each edge uv is labeled with  $\lceil 2f(u)f(v)/(f(u) + f(v)) \rceil$ or  $\lfloor 2f(u)f(v)/(f(u) + f(v)) \rfloor$  the edge labels are distinct. They prove the following graphs have such a labeling: paths, ladders, triangular snakes, quadrilateral snakes,  $C_m \cup$  $P_n$   $(n > 1); C_m \cup C_n; nK_3; mK_3 \cup P_n$   $(n > 1); mC_4; mC_4 \cup P_n; mK_3 \cup nC_4;$  and  $C_n \odot K_1$  (crowns). They also prove that wheels, prisms, and  $K_n$  (n > 4) with an edge deleted are not harmonic mean graphs. In [2177] Sandhya, Somasundaram, and Ponraj investigated the harmonic mean labeling for a polygonal chain, square of the path and dragon and enumerate all harmonic mean graph of order at most 5. In [1109] Jayasekaran and David Raj prove that some disconnected graphs are harmonic mean graphs. In [2060] Raj, Jayasekaran, and Sandhya investigate some new families of harmonic mean graphs. Seoud and Salim [2254] provided upper bounds of the number of edges of graphs of given orders with harmonic mean labelings and showed that all graphs of order at most 9 have harmonic mean labelings using the floor function portion of the definition. Meena and Sivasakth [1736] prove that subdivision graphs of  $P_n \odot K_1, P_n \odot \overline{K_2}, H$ -graphs,  $C_n \odot K_1, C_n \odot \overline{K_2}$ , quadrilateral snakes, and triangular snakes are harmonic mean.

Sandhya, Somasundaram, and Ponraj [2178] proved that the following graphs have harmonic mean labelings: graphs obtained by duplicating an arbitrary vertex or an arbitrary edge of a cycle; graphs obtained by joining two copies of a fixed cycle by an edge; the one-point union of two copies of a fixed cycle; and the graphs obtained by starting with a path and replacing every other edge by a triangle or replacing every other edge by a quadrilateral.

Vaidya and Barasara [2612] proved that the following graphs have harmonic mean labelings: graphs obtained by the duplication of an arbitrary vertex or arbitrary edge of a path or a cycle; the graphs obtained by the duplication of an arbitrary vertex of a path or cycle by a new edge; and the graphs obtained by the duplication of an arbitrary edge of a path or cycle by a new vertex.

In [1833] Narasimhan and Sampathkumar called a graph with p vertices a contra harmonic mean graph if there is an injective function f from the vertices of the graph to the integers from 1 to p such that when each edge uv is labeled with  $f(uv) = \lceil (f(u))^2 + (f(v))^2/(f(u) + f(v)) \rceil$  or  $f(uv) = \lfloor (f(u))^2 + (f(v))^2/(f(u) + f(v)) \rfloor$  the edge labels are distinct. They prove the following graphs have such a labeling: paths, cycles,  $C_m \cup P_n, C_m \cup C_n, nK_3, nK_3 \cup P_m$ , and  $nK_3 \cup C_m$ . Gopi [915] called a graph with q edges a k-contra harmonic mean graph if there is an bijective function f from the edges of the graph to the integers from k - 1 to k + q + 1 such that each edge uv is labeled with  $f(uv) = \lceil (f(u))^2 + (f(v))^2/(f(u) + f(v)) \rceil$  or  $f(uv) = \lfloor (f(u))^2 + (f(v))^2/(f(u) + f(v)) \rfloor$ . He proves that triangular snakes, double triangular snakes, quadrilateral snakes, and double quadrilateral snakes have k-contra harmonic mean labelings.

Gopi and Suba [919] say a graph G with p verticies and q edges is a super Lehmer-3 mean graph if there is an injective function f from the vertices of G to  $\{1, 2, \ldots, q+1\}$  such that for each edge uv the induced function  $f^*(uv) = \lfloor (f(u)^3 + f(v)^3)/(f(u)^2 + f(v^2)) \rfloor$  or  $f^*(uv) = \lceil (f(u)^3 + f(v)^3)/(f(u)^2 + f(v^2)) \rceil$  yields the set of vertex labels and edge labels being  $\{1, 2, \ldots, p\}$ . They prove that  $P_m \odot K_{1,n}$  and the graph obtained by identifying each endpoint of a path with an endpoint of the star  $K_{1,n}$  have a super Lehmer-3 labeling. In [918] Gopi and Nirmala provide Lehmer-3 mean labelings for  $P_m \odot C_n$   $(m, n \ge 3)$  and  $P_m \odot K_1 \odot C_n$   $(m, n \ge 3)$ .

An *F*-geometric mean labeling of a graph G with q edges, is an injective function from the vertex set of G to  $\{1, 2, \ldots, q+1\}$  such that the edge labels obtained from the

floor function of geometric mean of the vertex labels of the end vertices of each edge, are all distinct and the set of edge labels is  $\{1, 2, \ldots, q\}$ . Durai Baskar, Arockiaraj, and Rajendran [700] proved that the following graphs are *F*-geometric mean: graphs obtained by identifying a vertex of consecutive cycles (not necessarily of the same length) in a particular way; graphs obtained by identifying an edge of consecutive cycles (not necessarily of the same length) in a particular way; graphs obtained by joining consecutive cycles (not necessarily of the same length) by paths (not necessarily of the same length) in a particular way;  $C_n \odot K_1$ ;  $P_n \odot K_1$ ;  $L_n \odot K_1$ ;  $G \odot K_1$  where *G* is the graph obtained by joining two copies of  $P_n$  by an edge in a particular way; graphs obtained by appending two edges at each vertex of graphs obtained by joining two copies of  $P_n$  by an edge in a particular way; graphs obtained from  $C_n$  by appending two edges at each vertex of  $C_n$ ; graphs obtained from ladders by appending two edges at each vertex of the ladders; graphs obtained from  $P_n$  by appending an end point of the star  $S_2$  to each vertex of  $P_n$ ; and graphs obtained from  $P_n$  by appending an end point of the star  $S_3$  to each vertex of  $P_n$ .

A *C*-geometric mean labeling of a graph *G* with *q* edges, is an injective function from the vertex set of *G* to  $\{1, 2, 3, \ldots, q + 1\}$  such that the edge labels obtained from the ceiling function of the geometric mean of the vertex labels of the end vertices of each edge are all distinct and the set of edge labels is  $\{2, 3, 4, \ldots, q + 1\}$ . A graph is said to be a *C*-geometric mean graph if it admits a *C*-geometric mean labeling. In [702] Durai Baskar and Arockiaraj study the *C*-geometric meanness of some cycle related graphs such as cycle, union of a path and a cycle, unions of two cycles, the graphs  $C_3 \times P_n$ , corona of cycle, the graphs  $P_{a,b}$ ,  $P_b^a$  and some chain graphs.

A geometric mean labeling f of G(V, E) is called a super geometric mean labeling if  $f(V) \cup f(E) = \{1, 2, ..., |V| + |E|\}$ . Sandhya, Merly, and Shiny [2174] [2175] prove that the subdivision graphs of the following graphs have super geometric mean labelings: alternate quadrilateral snakes, double quadrilateral snakes, alternate double quadrilateral snakes, triple quadrilateral snakes, and subdivisions of alternate triple quadrilateral snakes. In [2176] they prove that the following graphs have super geometric mean labelings: triangular ladders, triangular snakes, alternate triangular snakes, quadrilateral snakes, and alternate quadrilateral snakes. Hemalatha and Selvi [1008] prove that following graphs have super geometric mean labelings: snakes, and graphs obtained by connecting any number of copies of  $C_n$  where each joined to the next with an edge.

Durai Basker and Arockiaraj [699] study the F-geometric meanness of cycles, stars, complete graphs, combs, ladders, triangular ladders, middle graphs of paths, graphs obtained from duplicating arbitrary vertex by a vertex as well as arbitrary edge by an edge in cycles, and subdivisions of combs and stars.

Arockiaraj and Meena Kumari introduced the F-Face magic mean labeling of graphs in [700]. This motivated Meena Kumari and Arockiaraj [1734] to introduce the (1,0,0)-F-face magic mean labeling of graphs as follows. A bijection  $\phi$  from V(G) to  $\{1, 2, \ldots, |V(G)|\}$  is called a (1,0,0)-F-face magic mean labeling of G if the induced face labeling  $\phi^*(f_i) = \lfloor (\text{sum of the labels of the vertices in the boundary of } f_i)/\deg(f_i) \rfloor$  is a constant for each

face  $f_i$ , including the exterior face of G, where deg  $f_i$  is the number of edges that bound the face. A graph that admits an (1,0,0)-F-Face magic mean labeling is called (1,0,0)-Fface magic mean. In [1734] Arockiaraj and Meena Kumar showed that  $P_n + K_1$   $(n \ge 2)$ , cycles with certain cords,  $C_m \times P_n$  where m and n are even, and graphs obtained by duplicating every edge of a cycle by a vertex admit (1,0,0)-F-Face magic mean labelings.

In [2541] Sundaram, Ponraj, and Somasundaram introduced a new labeling parameter called the *mean number* of a graph. Let f be a function from the vertices of a graph to the set  $\{0, 1, 2, ..., n\}$  such that the label of any edge uv is (f(u) + f(v))/2 if f(u) + f(v)is even and (f(u) + f(v) + 1)/2 if f(u) + f(v) is odd. The smallest integer n for which the edge labels are distinct is called the *mean number* of a graph G and is denoted by m(G). They proved that for a graph G with p vertices  $m(tK_{1,n}) \leq t(n+1) + n - 4$ ;  $m(G) \leq 2^{p-1} - 1$ ;  $m(K_{1,n}) = 2n - 3$  if n > 3; m(B(p, n)) = 2p - 1 if p > n + 2 where B(p, n) is a bistar; m(kT) = kp - 1 for a mean tree T,  $m(W_n) \leq 3n - 1$ , and  $m(C_3^{(t)}) \leq 4t - 1$ .

Let f be a function from V(G) to  $\{0, 1, 2\}$ . For each edge uv of G, assign the label  $\lceil (f(u) + f(v))/2 \rceil$ . Ponraj, Sivakumar, and Sundaram [2005] say that f is a mean cordial labeling of G if  $|v_{f(i)} - v_{f(j)}| \leq 1$  for i and j in  $\{0, 1, 2\}$  where  $v_{f(x)}$  and  $e_{f(x)}$  denote the number of vertices and edges labeled with x, respectively. A graph with a mean cordial labeling is called a mean cordial graph. Observe that if the range set of f is restricted to  $\{0, 1\}$ , a mean cordial labeling coincides with that of a product cordial labeling. Ponraj, Sivakumar, and Sundaram [2005] prove the following: every graph is a subgraph of a connected mean cordial graph;  $K_{1,n}$  is mean cordial if and only  $n \leq 2$ ;  $C_n$  is mean cordial if and only  $n \equiv 1, 2 \pmod{3}$ ;  $K_n$  is mean cordial if and only  $n \leq 2$ ;  $W_n$  is not mean cordial for all  $n \geq 3$ ; the subdivision graph of  $K_{1,n}$  is mean cordial; the comb  $P_n \odot K_1$  is mean cordial;  $P_n \odot 2K_1$  is mean cordial; and  $K_{2,n}$  is a mean cordial if and only  $n \leq 2$ . Seoud and Salim [2254] provided upper bounds of the number of edges of graphs of given orders with mean cordial labelings and proved that  $P_{2t} \times P_2$  is mean cordial if and only if  $t \equiv 2 \mod 3$  and  $C_n \odot K_1$  is mean cordial if and only if  $n \equiv 1$  or 2 mod 3.

In [1997] Ponraj and Sivakumar proved the following graphs are mean cordial: mGwhere  $m \equiv 0 \pmod{3}$ ;  $C_m \cup P_n$ ;  $P_m \cup P_n$ ;  $K_{1,n} \cup P_m$ ;  $S(P_n \odot K_1)$ ;  $S(P_n \odot 2K_1)$ ;  $P_n^2$  if and only if  $n \equiv 1 \pmod{3}$  and  $n \ge 7$ ; and the triangular snake  $T_n (n > 1)$  if and only if  $n \equiv 0 \pmod{3}$ . They also proved that if G is mean cordial then mG,  $m \equiv 1 \pmod{3}$  is mean cordial. Deshmukh and Shaikh [666] prove the graph  $\langle K_{1,n} : 2 \rangle$  and the path union of n copies of  $K_{1,m}$  are mean cordial graphs.

In [1968] Ponraj and Sathish Narayanan proved double triangular snakes  $D(T_n)$  are mean cordial if and only if n > 3 and obtained partial results on mean cordial labelings of alternate triangular snakes, double alternate triangular snakes.

In [1985] Ponraj, Sathish Narayanan, and Ramasamy introduced the notion of total mean cordial labeling. A total mean cordial labeling of a graph G(V, E) is a function  $f: V(G) \to \{0, 1, 2\}$  such that when each edge xy is assigned the label  $\lceil (f(x) + f(y))/2 \rceil$ we have  $|ev_f(i) - ev_f(j)| \leq 1$ ,  $i, j \in \{0, 1, 2\}$ , where  $ev_f(x)$  denotes the total number of vertices and edges labeled with x. A graph with a total mean cordial labeling is called *total* mean cordial. In [1985], [1986], and [1987], Ponraj, Sathish Narayanan, and Ramasamy determined the total mean cordiality of the following graphs:  $P_n$ ;  $C_n$ ;  $K_{1,n}$ ;  $W_n$ ;  $K_2$  +  $mK_1$ ; combs  $P_n \odot K_1$ ; double combs  $P_n \odot 2K_1$ ; crowns; flowers; lotuses inside a circle; bistars; quadrilateral snakes;  $K_{2,n}$ ; olive trees;  $S(P_n \odot K_1)$ ;  $S(K_{1,n})$  (S(G) denotes the subdivision of G); triangular snakes;  $P_n^2$ ; fans  $F_n$ ; umbrellas; butterflies; and dumbbells. In [1967], [1969], and [1970], Ponraj and Sathish Narayanan determined the total mean cordiality of  $K_n^c + 2K_2$ ; prisms; gears; helms;  $P_1 \cup P_2 \cup \cdots \cup P_n$ ;  $L_n \odot K_1$ ;  $S(W_n)$ ;  $S(P_n \odot 2K_1)$ ; and graphs obtained by subdividing each step of a ladder exactly once.

Let G be a (p,q)-graph. Ponraj and Sathish Narayanan [1972] and [1973] proved the following. If G satisfies any one of the following three conditions then  $G \odot 2K$  is total mean cordial: (1) G is a tree, (2) G is a unicycle, (3) q = p + 1. If G satisfies any one of the following three conditions then the shadow graph of G is total mean cordial: (1) G is a tree, (2) G is a unicycle, (3) q = p + 1. They also proved that the following are total mean cordial graphs:  $C_n \odot K_2$ ,  $C_n^{(2)}$ , dragons, splitting graphs of stars, splitting graphs of combs, books, ladders,  $P_n \odot K_2$  if and only if  $n \neq 1$ , and  $G \cup P_n$   $(n \neq 3)$ .

Ponraj, Sathish Narayanan, and Kala introduced the concept of radio mean labeling in [1982]. A radio mean labeling of a connected graph G is a one-to-one map f from V(G) to the set of natural numbers such that for each pair of distinct vertices u and v of G,  $d(u, v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 1 + \text{diam } (G)$ . The radio mean number of f, rmn(f), is the maximum number assigned to any vertex of G. The radio mean number of G, rmn(G), is the minimum value of rmn(f) taken over all radio mean labelings f of G. They proved  $rmn(G) \ge |V(G)|$ ; if G is a (p, q)-graph with diameter  $d \ge 2$ , then  $rmn(G) \le p + d - 2$ ; and if G is a (p, q)-connected graph with diameter 2 or 3, then rmn(G) = p. They also determine the radio mean number of  $K_n$ ,  $K_{m,n}$ , sunflowers, helms, gears, lotuses inside a circle, and graphs obtained by identifying any two vertices of two wheels of the same size,

In [1983] and [1984] Ponraj, Sathish Narayanan, and Kala determine the radio mean numbers of  $S(K_{m,n})$  (m > 1, n > 1);  $K_{m,n} \odot P_t$ ;  $C_6^{(t)}$ ;  $W_n \odot P_m$ ; graphs obtained by joining the rim vertices of the two wheels with an edge; and graphs obtained from a wheel by subdividing each spoke by a vertex. In [1988] Ponraj, Sathish Narayanan, and Kala give the radio mean number of graphs with diameter three, lotuses inside a circle, helms, and sunflower graphs.

In [1989] and [1974] Ponraj and Sathish Narayanan give the radio mean number of the following graphs: subdivisions of stars, subdivisions of wheels, subdivisions of  $K_2 + mK_1$ , subdivisions of bistars, jelly fish, subdivisions of jelly fish, books with pentagonal pages, graphs obtained by taking m disjoint copies of  $K_{1,n}$  and joining a new vertex to the centers of the m copies of  $K_{1,n}$ .

A radio mean D-distance labeling of a connected graph G is an injective map f from V(G) to the natural numbers such that for two distinct vertices u and v of G,  $d^{D}(u, v) + [(f(u) + f(v))/2] \ge 1 + diam^{D}(G)$ , where  $d^{D}(u, v)$  denotes the distance D between u and v and  $diam^{D}(G)$  denotes the D-diameter of G. The radio mean D-distance number of f,  $rmn^{D}(f)$ , is the maximum label assigned to any vertex of G. The radio mean D-distance number distance number of G,  $rmn^{D}(G)$ , is the minimum value of  $rmn^{D}(f)$  taken over all radio mean D-distance labeling f of G. Nicholas and Bosco [1843] determined the radio mean D-distance number of cycles, wheels, gears, helms, fans, and friendship graphs.

In [1971] Ponraj and Sathish Narayanan proved that the following graphs are not mean cordial:  $K_2 + \overline{K}_m$ ;  $\overline{K}_n + 2K_2$ ;  $P_n \times K_2$ ; flower graphs; sunflower graphs;  $C_n \odot K_2$ . Also they proved the following: the Mongolian tent  $MT_{m,n}$  is mean cordial if and only if  $m \equiv 0 \pmod{3}$  or  $n \equiv 0 \pmod{3} (MT_{m,n}$  is the graph obtained from  $P_m \times P_n$ , n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of  $P_m \times P_n$  to the new vertex); the book  $B_m$  is mean cordial if and only if m = 1; books with n pentagonal pages are mean cordial if and only if  $n \equiv 1 \pmod{3}$ ;  $P_n \odot K_2$  is mean cordial if and only if  $n \equiv 0 \pmod{3}$ ; quadrilateral snakes are mean cordial; alternate quadrilateral snakes  $A(Q_n)$  are mean cordial if and only if the square starts from second vertex of the path  $P_n$ , ends with  $(n-1)^{th}$  vertex and  $n \equiv 0, 2 \pmod{3}$ , or the square starts from first vertex, ends with  $n^{th}$  vertex and  $n \equiv 0, 1 \pmod{3}$ .

Kaneria, Khoda, and Karavadiya [1285] prove: the path union of n copies of a graph G is a mean cordial when  $n \equiv 0 \pmod{3}$ ; if G is balanced mean cordial, then  $P_n \times G$  and  $C_n \times G$  are balanced mean cordial; and if  $f : V(G) \longrightarrow \{0, 1, 2\}$  is a balanced mean cordial labeling for G, then  $G^*$  is also a balanced mean cordial graph.

In [1166] Jeyanthi and Maheswari define a one modulo three mean labeling of a graph G with q edges as an injective function  $\phi$  from the vertices of G to  $\{a \mid 0 \leq a \leq 3q-2 \text{ where } a \equiv 0 \pmod{3} \text{ or } a \equiv 1 \pmod{3} \}$  and  $\phi$  induces a bijection  $\phi^*$  from the edges of G to  $\{a \mid 1 \leq a \leq 3q-2 \text{ where } a \equiv 1 \pmod{3} \}$  given by  $\phi^*(uv) = \lceil (\phi(u) + \phi(v))/2 \rceil$ . They proved that  $P_{2n}$ , combs, bistars  $B_{n,n}$ ,  $T_p$ -trees with an even number of vertices,  $C_{4n+1}$ , ladders,  $K_{1,2n} \times P_2$  are one modulo three mean graphs. They also proved that bistars  $B_{m,n}$   $(m \neq n), K_{1,n}$  (n > 3), and  $K_n$ , (n > 3) are not one modulo three mean graphs. In [1174] Jeyanthi, Maheswari, and Pandiaraj [1174] proved that  $DA(Q_n), DA(Q_2) \odot nK_1, DA(Q_m) \odot nK_1, DA(T_2) \odot nK_1, DA(T_m) \odot nK_1, \overline{S}(DA(T_n)), \overline{S}(DA(Q_n))$ , and  $mP_n$  are one modulo three mean graphs.

Jeyanthi, Maheswari, and Pandiaraj [1173] prove that following graphs have one modulo three mean labelings: books  $K_{1,2n} \times P_2$ ; splitting graphs  $S'(P_{2n})$ ; vertex duplication graphs D(G, v'); edge duplication graphs D(G, e'); nth alternate quadrilateral snake graphs  $NA(Q_m)$ ; graphs obtained by joining the endpoints of paths  $P_{4m}$  to n isolated verticies; and extended jewel graphs  $EJ_n$  with vertex set  $\{u, v, x, y, w, z, u_i : 1 \leq i \leq n\}$ and edge set  $\{uv, ux, xy, yz, vw, wz, vu_i, zu_i : 1 \leq i \leq n\}$ .

For graphs  $G_1$  and  $G_2$ ,  $G_1 \circ G_2$  is the graph obtained from  $G_1$  and  $|V(G_1)|$  copies of  $G_2$  by joining a vertex of *i*th copy of  $G_2$  with the *i*th vertex of  $G_1$  by an edge. Jeyanthi, Maheswari, and Pandiaraj [1176] prove that the graphs  $T \odot \overline{K_n}$ ,  $T \circ K_{1,n}$ ,  $T \circ P_n$ , and  $T \circ 2P_n$  are one modulo three mean graphs.

A graph G is said to be one modulo three root square mean graph if there is an injective function  $\phi$  from the vertex set of G to the set  $\{0, 1, 3, \ldots, 3q-2, 3q\}$  where q is the number of edges of G and  $\phi$  induces a bijection  $\phi^*$  from the edge set of G to  $\{1, 4, \ldots, 3q-2\}$  given by  $\phi^*(uv) = \left[\sqrt{\frac{[\phi(u)]^2 + [\phi(v)]^2}{2}}\right]$  or  $\left\lfloor\sqrt{\frac{[\phi(u)]^2 + [\phi(v)]^2}{2}}\right\rfloor$  and the function  $\phi$  is called a one modulo three root square mean labeling of G. In [1111] Jayasekaran and Jaslin Melbha investigated some path related graphs that have one modulo three root square mean

labelings.

Somasundaram, Vidhyarani, and Ponraj [2450] introduced the concept of a geometric mean labeling of a graph G with p vertices and q edges as an injective function  $f: V(G) \rightarrow$  $\{1, 2, \ldots, q+1\}$  such that the induced edge labeling  $f^*: E(G) \rightarrow \{1, 2, \ldots, q\}$  defined as  $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$  or  $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$  is bijective. Among their results are: paths, cycles, combs, ladders are geometric mean graphs and  $K_n$  (n > 4) and  $K_{1,n}$  (n > 5)are not geometric mean graphs. Somasundaram, Vidhyarani, and Sandhya [2451] proved  $C_m \cup P_n, \ C_m \cup C_n, \ nK_3, \ nK_3 \cup P_n, \ nK_3 \cup C_m, \ P_n^2$ , and crowns are geometric mean graphs. Vaidya and Barasara [2615] investigated geometric mean labelings in context of duplication of graph elements in cycle  $C_n$  and path  $P_n$ . Durai Baskar, Arockiyaraj, and Rajendran investigate the geometric meanness of some graphs obtained from paths.

In Jeyanthi, Maheswari, and Pandiaraj [1175] define a graph G to be a one modulo three geometric mean graph if there is an injective function  $\phi$  from the vertex set of G to the set  $\{a|1 \leq a \leq 3q-2 \text{ and either } a \equiv 0 \pmod{3} \text{ or } a \equiv 1 \pmod{3}\}$  where q is the number of edges of G and  $\phi$  induces a bijection  $\phi^*$  from the edge set of G to  $\{a|1 \leq a \leq 3q-2 \text{ and } a \equiv 1 \pmod{3}\}$  given by  $\phi^*(uv) = \left\lceil \sqrt{\phi(u)\phi(v)} \right\rceil$  or  $\left\lfloor \sqrt{\phi(u)\phi(v)} \right\rfloor$ the function  $\phi$  is called one modulo three geometric mean labeling of G. They proved paths, cycles with length at least 5, ladders,  $P_n \odot K_1, P_n \odot P_2, P_n \odot \overline{P_2}$ , subdivision graphs  $S(P_n \odot K_1)$ , and subdivision graphs  $S(P_n \odot K_2)$  are one modulo three geometric graphs. They also prove that  $K_{1,n}$   $(n \geq 3)$  and graphs in which every edge lies on a triangle are not one modulo three geometric mean graph.

Jeyanthi, Selvi, and Ramya [1236] define a restricted triangular difference mean labeling of a graph G with p vertices and q edges as an injection  $f: V \to \{1, 2, 3, ..., pq\}$  such that for each edge uv, the edge labels defined by  $f^*(uv) = \lceil |f(u) - f(v)|/2 \rceil$  are the first q triangular numbers. A graph that admits a restricted triangular difference mean labeling is called a restricted triangular difference mean graph. Jeyanthi, Selvi, and Ramya [1236] investigate the restricted triangular difference mean behaviors of the paths, combs,  $K_n$ , bistars  $B_{m,n}$ , caterpillars  $S(n_1, n_2, \ldots, n_m)$ ,  $K_{m,n}$ , wheels, and graphs obtained by joining the centers of different stars to the new vertex. They also give a necessary condition for a graph to be a restricted triangular difference mean graph.

A (p,q) graph G(V, E) is said to be an *analytic mean* graph if it is possible to injectively label the vertices with  $\{0, 1, 2, \ldots, p-1\}$  in such a way that when each edge uv is labeled with  $|(f(v)^2 - (f(u))^2|/2$  when  $|(f(v)^2 - (f(u))^2|$  is even and  $|(f(v)^2 - (f(u))^2 + 1|/2$  when  $|(f(v)^2 - (f(u))^2| + 1$  is odd and the edge labels are distinct. In this case, f is called an *analytic mean* labeling of G. Raj and Vivek [2063] proved that  $P_m \cup C_n \cup K_{1,s}$   $(m, s \ge$ 2,  $n \ge 3)$ ,  $(P_m + K_1) \cup K_{1,n}$   $(m, n \ge 2)$ , graphs obtained by identifying the apex vertices of  $K_{1,m}$  and  $K_{1,n}$  and one vertex of two copies of  $C_s$  where  $m, n \ge 2, c \ge 3$  are analytic mean graphs.

Let G = (V, E) be a graph with p vertices and q edges. A graph G is analytic odd mean if there exist an injective function  $f : V \to \{0, 1, 3, 5 \dots, 2q - 1\}$  with an induce edge labeling  $f^* : E \to Z$  such that for each edge uv with  $f(u) < f(v), f^*(uv) = \left\lceil \frac{f(v)^2 - (f(u) + 1)^2}{2} \right\rceil$  if  $f(u) \neq 0$ , and  $f^*(uv) = \left\lceil \frac{f(v)^2}{2} \right\rceil$  if f(u) = 0 is injective. In this

case we say that f is an *analytic odd mean labeling* of G. Jeyanthi, Gomathi, and Lau [1156] proved that fans, double fans, double wheels, closed helms, total graphs of cycles, total graphs of paths, armed crowns  $C_n \Theta P_m$ , generalized Petersen graphs GP(n,2) are analytic odd mean graphs. In [1145] they prove that  $P_n$ ,  $C_n$ ,  $P_n \odot K_1$ , bistars, fans,  $C_n \odot K_1, L_n \odot K_1, C_m \cup S_m$ , two copies of  $C_n$  sharing a common edge, and  $C_m \cup C_n$  are analytic odd mean graphs. In [1144] they prove that wheels, flower graphs, some splitting graphs, and multiples of graphs are analytic odd mean graphs. In [1146] they prove that quadrilateral snakes, double quadrilateral snakes, coconut trees, fire cracker graphs, some star graphs, splitting graphs, complete bipartite graphs, unicyclic graphs, and the graphs obtained from a path of vertices  $v_1, v_2, v_3, \ldots, v_n$  by joining i pendent vertices at each of ith vertex  $1 \leq i \leq n$  (denoted  $P_n(1, 2, \ldots, n)$ ) are analytic odd mean graphs. Jeyanthi, Gomathi, and Lau [1157] proved the square graphs of  $P_n, C_n, B_{n,n}$  H-graphs, and  $H \odot mK_1$  admit analytic odd mean labelings. Jeyanthi, Gomathi, and Lau [1158] proved that quadrilateral snakes, double quadrilateral snakes, coconut trees, fire cracker graphs, and  $P_n(1,2,\ldots,n)$  are analytic odd mean graphs. In [1142] Jeyanthi and Gomathi proved that the subdivision and super subdivision of the following graphs are analytic odd mean: cycles, stars, combs, and graphs obtained from  $P_n \odot K_1$  by subdividing the each edge of  $P_n$ . Jeyanthi, Gomathi, and Lau [1147] proved that  $C_n \odot K_1$ , prisms, helms, banana trees, perfect binary trees, unicyclic graphs, certain caterpillars, and spiders are analytic odd mean graphs. Jeyanthi and Gomathi [1143] proved that the graphs  $TL_n$ , (the triangular ladder obtained from  $L_n$ ),  $TL_n \odot K_1$ ,  $T_n \odot K_1$ , and  $Q_n \odot K_1$  admit an analytic odd mean labelings.

Let G be a (p,q) graph and f a injective function from V(G) to  $\{k, k+1, \ldots, p+q+k-1\}$  For each edge uv, let  $f^* = \lceil (2f(u)f(v)/(f(u)+f(v)) \rceil$  or  $\lfloor (2f(u)f(v)/(f(u)+f(v)) \rfloor$ . We say f is a k-super harmonic mean if  $f(V) \cup \{f^*(uv) \mid uv \in E(G)\} = \{k, k+1, \ldots, p+q+k-1\}$ . A graph that admits a k-super harmonic mean labeling is called a k-super harmonic mean graph. In the case that k = 1 the labeling is called a super harmonic mean labeling. For all n > 1 Tamilselvi and Revathi [2570] prove that the following graphs have k-super harmonic mean labelings:  $P_n, nP_m$   $(m > 1), P_n \odot K_1, P_n \odot \overline{K_2}, P_n \odot \overline{K_3}, P_n^2$   $(n \ge 4)$ , the subdivision graph of  $P_n \odot K_1$ , and the middle graph of  $P_n$ .

A graph G = (V, E) with p vertices and q edges is said to be a (k, d)-Heronian mean graph if it is possible to label the vertices  $x \in V$  with distinct labels f(x) from  $k, k + d, k + 2d, \ldots, k + qd$  in such a way that when each edge uv is labeled with  $f^*(uv) = \lfloor (f(u) + f(v) + \sqrt{f(u)f(v)})/3 \rfloor$  or  $\lceil (f(u) + f(v) + \sqrt{f(u)f(v)})/3 \rceil$ , then the resulting edge labels are distinct. In this case f is called a (k, d)-Heronian mean labeling of G. In the case k = 1 and d = 1, the labeling is called Heronian mean labeling . Akilandeswari and Tamilselvi [109] proved that paths, ladders, and  $P_n \odot mK_1$  for  $n \ge 2$ ,  $1 \le m \le 4$ , are k-Heronian mean graphs. In [109] Akilandeswari and Tamilselvi proved that the following graphs have (k, d)-Heronian mean labelings: paths,  $(P_n \times P_2) \odot K_1$ ,  $T_n \odot K_1$  ( $T_n$  is the triangular snake obtained from  $P_n$ ),  $Q_n \odot K_1$ ,  $TL_n \odot K_1$  ( $TL_n$  is the triangular ladder obtained from  $L_n$ ), Peterson graphs, and the graphs obtained from two copies of  $P_n$  with vertices  $v_1, v_2, \ldots, v_n$  and  $u_1, u_2, \ldots, u_n$  by joining the vertices  $u_{(n+1)/2}$  and  $v_{(n+1)/2}$  if n is odd and  $u_{n/2+1}$  and  $v_{n/2+1}$  if n is even. Sampath, Narasimhan, and Nagaraja [2170] proved cycles,  $K_{1,n}$  if and only if  $n \leq 4$ ,  $C_m \cup P_n$ ,  $C_m \cup C_n$ ,  $nK_3$ ,  $nK_3 \cup P_m$ ,  $nK_3 \cup C_m$ ,  $mC_4$ , crowns  $C_n \odot K_1$ , dragons  $C_n @P_m$ , and  $P_n^2$  admit (1, 1)-Heronian mean labelings. Anitha, Selvam, and Thirusangu [174] provide Heronian mean labelings for the extended duplicate graph of the kite graph.

Arockiaraj and Meena [188] say a planar graph has an *F*-face magic mean labeling it there exists an assignment of labels to the edges that induces an assignment of labels to the faces of the graph such that the mean weight of each face is constant. They proved that the following graphs have *F*-face magic mean labelings:  $P_{2n} + K_1$ , the one-point union of *m* copies of  $C_n$ ,  $mC_n$ -snakes, and graphs obtained identifying the endpoints of any number of copies  $P_n$ . Amara Jothi, Baskar Babujee, and David [144] investigated face magic labeling of planar graphs of types (1, 0, 1), (1, 1, 0), (0, 1, 1) and (1, 1, 1) on duplication graphs.

#### 7.13 Pair Sum and Pair Mean Graphs

For a (p,q) graph G Ponraj and Parthipan [1955] define an injective map f from V(G) to  $\{\pm 1, \pm 2, \ldots, \pm p\}$  to be a pair sum labeling if the induced edge function  $f_{em}$  from E(G) to the nonzero integers defined by  $f_e(uv) = f(u) + f(v)$  is one-one and  $f_e(E(G))$  is either of the form  $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\}$  or  $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\} \cup \{k_{\frac{q+1}{2}}\}$ , according as q is even or odd. A graph with a pair sum labeling is called pair sum graph. In [1955] and [1956] they proved the following are pair sum graphs:  $P_n$ ,  $C_n$ ,  $K_n$  if and only if  $n \leq 4$ ,  $K_{1,n}$ ,  $K_{2,n}$ , bistars  $B_{m,n}$ , combs  $P_n \odot K_1$ ,  $P_n \odot 2K_1$ , and all trees of order up to 9. Also they proved that  $K_{m,n}$  is not pair sum graph if  $m, n \geq 8$  and enumerated all pair sum graphs of order at most 5.

In [1958], [1959], [1960], and [1961] Ponraj, Parthipan, and Kala proved the following are pair sum graphs:  $K_{1,n} \cup K_{1,m}$ ,  $C_n \cup C_n$ ,  $mK_n$  if  $n \leq 4$ ,  $(P_n \times K_1) \odot K_1$ ,  $C_n \odot K_2$ , dragons  $D_{m,n}$  for n even,  $\overline{K_n} + 2K_2$  for n even,  $P_n \times P_n$  for n even,  $C_n \times P_2$  for n even,  $(P_n \times P_2) \odot K_1$ ,  $C_n \odot K_2$  and the subdivision graphs of  $P_n \times P_2$ ,  $C_n \odot K_1$ ,  $P_n \odot K_1$ , triangular snakes, and quadrilateral snakes.

A (p,q)-graph G is said to be a super pair sum if there exists a bijection f from  $V(G) \cup E(G)$  to  $\{0, \pm 1, \pm 2, \ldots, \pm (\frac{p+q-1}{2})\}$  when p+q is odd and from  $V(G) \cup E(G)$  to  $\{\pm 1, \pm 2, \ldots, \pm (\frac{p+q}{2})\}$  when p+q is even such that f(uv) = f(u) + f(v). A graph that admits a super pair sum labeling is called a super pair sum graph. Vasuki, Velmurugan, and Sugirtha [2713] prove that the graphs  $H_n \odot mK_1$ ,  $(H_n$  is obtained from two copies of  $P_n$   $(n \geq 3)$  with vertices  $v_1, v_2, \ldots, v_n$  and  $u_1, u_2, \ldots, u_n$  by joining  $v_{(n+1)/2}$  and  $u_{(n+1)/2}$  if n is odd and  $v_{n/2}$  and  $u_{(n+2)/2}$  if n is even);  $(P_{2n}; S_m)$ ,  $S'(P_{2n})$ ,  $< B_{m,n} : P_k >$  for  $m \geq 1$ ,  $n \geq 1$ ,  $k \equiv 2 \pmod{4}$ ,  $< B(m) : P_k >$  for  $m \geq 1$   $k \equiv 0, 2 \pmod{4}$  and  $2B_{m,n}$   $(m \geq 1, n \geq 1)$  are super pair sum graphs.

Jeyanthi and Sarada Devi [1215] define an injective map f from E(G) to  $\{\pm 1, \pm 2, \ldots, \pm q\}$  as an *edge pair sum labeling* of a graph G(p,q) if the induced function of  $f^*$  from V(G) to  $Z - \{0\}$  defined by  $f^*(v) = \sum f(e)$  taken over all edges e incident to v is one-one and  $f^*(V(G))$  is either of the form  $\{\pm k_1, \pm k_2, \ldots, \pm k_{p/2}\}$  or  $\{\pm k_1, \pm k_2, \ldots, \pm k_{(p-1)/2}\} \cup \{k_{p/2}\}$  according as p is even or odd. A graph with an edge

pair sum labeling is called an *edge pair sum graph*. They proved that  $P_n, C_n$ , triangular snakes,  $P_m \cup K_{1,n}$ , and  $C_n \odot \overline{K_m}$  are edge pair sum graphs.

Jeyanthi, Sarada Devi, and Lau [1225] proved that the following graphs have edge pair sum labelings: triangular snakes  $T_n$ ,  $C_n \cup C_n$ ,  $K_{1,n} \cup K_{1,m}$ , and bistars  $B_{m,n}$ . They also proved that every graph is a subgraph of a connected edge pair sum graph. Jeyanthi and Sarada Devi [1216] showed that  $P_{2n} \times P_2$  and the graphs  $P_n(+)N_m$  obtained from a path  $P_n$  by joining its endpoints to m isolated vertices are edge pair sum graphs. Jeyanthi and Sarada Devi [1218] proved that the following graphs have edge pair sum labeling: shadow graphs  $S_2(P_n)$ ,  $S_2(K_{1,n})$ , total graphs  $T(C_{2n})$  and  $T(P_n)$ , the one-point union of any number of copies of  $C_n$ , the one-point union of  $C_m$  and  $C_n$ ,  $P_{2n-1}^2$ , and full binary trees in which all leaves are at the same level and every parent has two children. Jeyanthi and Sarada Devi [1217] proved the spiders  $SP(1^m, 2^t)$ ,  $SP(1^m, 2^t, 3)$ ,  $SP(1^m, 2^t, 4)$ , and for t even  $SP(1^m, 3^t, 3)$  are edge pair sum graphs. In [1216] Jeyanthi and Sarada Devi prove some cycle related graphs are edge pair sum graphs, total graphs, and  $P_n^2$  admit edge pair sum graph. In [1224] Jeyanthi and Sarada provide edge pair sum labelings for jewel graphs, gears, triangular ladders, balanced lobsters, and double wheels  $2C_n + K_1$ .

The tree WT(n) is obtained from  $K_{1,n+2}$  with central vertex  $c_1$  and end vertices  $x_i : 1 \le i \le n+2$  and another  $K_{1,n+2}$  with central vertex  $c_2$  and end vertices  $y_j : 1 \le j \le n+2$  by identifying vertex  $x_{n+2}$  and  $y_{n+2}$  and denoting the identified vertices by w. A w-tree WT(n:k) is obtained from k copies of WT(n) by joining a new vertex a to vertex w of each copy of WT(n). Jeyanthi, Sarada Devi, and Lau [1226] proved that the graphs WT(n:k) trees have edge pair sum labelings (see also [1227]).

In [1220], [1226], [1219], [1223] Jeyanthi and Sarada Devi prove the following graphs are edge pair sum graphs: shell graphs; some butterfly graphs; jelly fish; Y-trees; theta graphs; wheels with subdivided spokes,  $P_m + 2K_1$ ;  $C_4 \times P_m$ ;  $P_n \odot \overline{K}_m$ ;  $(P_2 \times P_m) \odot \overline{K}_n$ ;  $P_m \times C_3$ ; books; graphs obtained from the path  $P_n$  having an even fixed even number quadrilaterals on each edge of the path;  $K_2+mK_1$ ; graphs obtained by identifying one end point from each of m copies of  $P_n$ ; closed helms; graphs that are two copies of generalized Petersen graphs joined by a path  $P_n$ ,  $n \geq 5$ ; and graphs that two copies of fan  $P_n \odot K_1$ joined by a path  $P_n$ ,  $n \geq 5$ .

In [1221] Jeyanthi and Sarada Devi prove the following graphs admit edge pair sum labelings:  $K_{2,n}$ , double triangular snakes, wheels, flowers,  $\langle C_m, K_{1,n} \rangle$   $(m \ge 4, n \text{ odd})$ obtained from  $C_m$  and  $K_{1,n}$  by identifying any vertex of  $C_m$  with the central vertex of  $K_{1,n}$ , and  $\langle C_m * K_1 \rangle$   $(m \ge 4)$  the graphs obtained from  $C_m$  and  $K_{1,n}$  by identifying any vertex of  $C_m$  with an endpoint vertex of  $K_{1,n}$ . In [1222] they prove that the subdivision of graph of bistars  $B_{m,n}$ ,  $P_n \odot K_1$ , triangular snakes when the path has an odd number of verticies, double triangular snakes, double quadrilateral snakes, double alternative triangular snakes, and double alternative quadrilateral snakes are edge pair sum graph.

For a (p,q) graph G Ponraj and Parthipan [1957] define an injective map f from V(G) to  $\{\pm 1, \pm 2, \ldots, \pm p\}$  to be a *pair mean* labeling if the induced edge function  $f_{em}$  from E(G) to the nonzero integers defined by  $f_{em}(uv) = (f(u) + f(v))/2$  if f(u) + f(v) is even and  $f_{em}(uv) = (f(u) + f(v) + 1)/2$  if f(u) + f(v) is odd is one-one and

 $f_{em}(E(G)) = \{\pm k_1, \pm k_2, \dots, \pm k_{q/2}\}$  or

 $f_{em}(E(G)) = \{\pm k_1, \pm k_2, \ldots, \pm k_{(q-1)/2}\} \cup \{k_{(q+1)/2}\}, \text{ according as } q \text{ is even or odd. A graph with a pair mean labeling is called a$ *pair mean* $graph. They proved the following graphs have pair mean labelings: <math>P_n$ ,  $C_n$  if and only if  $n \leq 3$ ,  $K_n$  if and only if  $n \leq 2$ ,  $K_{2,n}$ , bistars  $B_{m,n}, P_n \odot K_1, P_n \odot 2K_1$ , and the subdivision graph of  $K_{1,n}$ . Also they found the relation between pair sum labelings and pair mean labelings.

The graph  $G@P_n$  is obtained by identifying an end vertex of a path  $P_n$  with any vertex of G. A graph G(V, E) with q edges is called a (k + 1)-equitable mean graph if there is a function f from V to  $\{0, 1, 2, \dots, k\}$   $(1 \le k \le q)$  such that the induced edge that labeling  $f^*$  from E to  $\{0, 1, 2, ..., k\}$  given by  $f^*(uv) - \lceil (f(u) + f(v))/2 \rceil$  has the properties  $|v_f(i) - v_f(j)| \le 1$  and  $|e_{f^*}(i) - e_{f^*}(j)| \le 1$  for i, j = 0, 1, 2, ..., k where  $v_f(x)$ and  $e_{f^*}(x)$  are the number of vertices and edges of G respectively with the label x. In [1131] Jeyanthi proved the following: a connected graph with q edges is a (q+1)-equitable mean graph if and only if it is a mean graph; a graph is 2-equitable mean graph if and only if it is a product cordial graph; for every graph G, the graph 3mG is a 3-equitable mean graph; for every 3-equitable mean graph G, the graph (3m+1)G is a 3-equitable mean graph;  $C_n$  is a 3-equitable mean graph if and only if  $n \not\equiv 0 \pmod{3}$ ;  $P_n$  is a 3-equitable mean graph for all  $n \geq 2$ ; if G is a 3-equitable mean graph then  $G@P_n$  is a 3-equitable mean graph for  $n \equiv 1 \pmod{3}$ ; the bistar B(m, n) with  $m \geq n$  is a 3-equitable mean graph if and only if  $n \ge |q/3|$ ;  $K_{1,n}$  is a 3-equitable mean graph if and only if  $n \le 2$ ; and for any graph H and 3m copies  $H_1, H_2, \ldots, H_{3m}$  of H, the graph obtained by identifying a vertex of  $H_i$  with a vertex of  $H_{i+1}$  for  $1 \le i \le 3m-1$  is a 3-equitable mean graph.

In [1459] Lakshmi and Nagarajan introduced the notion of geometric mean cordial labeling of graphs as follows. Let G = (V, E) be a graph and f be a mapping from V(G)to  $\{0, 1, 2\}$ . The graph G is called *geometric mean cordial* if each edge uv can be assigned the label  $\lceil \sqrt{f(u)f(v)} \rceil$  in such a way that and  $|v_f(i) - v_f(j)| \le 1$  and  $|e_f(i) - ef_(j)| \le 1$ , where  $v_f(x)$  and  $e_f(x)$  denote the number of vertices and edges labeled with x and  $x \in$  $\{0, 1, 2\}$  They proved that  $P_n$ ,  $C_n$   $(n \equiv 1, 2 \pmod{3})$  and  $K_{1,n}$  are geometric mean cordial graphs and  $K_n(n > 2)$ ,  $K_{2,n}$  (n > 2),  $K_{n,n}$  (n > 2) and wheels are not geometric mean cordial graphs. In [1305] Kaneria, Meera, and Maulik call these graphs geometric mean  $\beta$ -equitable. They proved:  $K_{mn}$   $(m, n \ge 4)$  is not a geometric mean 3-equitable graph, caterpillars  $S(x_1, x_2, \ldots, x_t)$  and  $C_n \odot K_1$   $(t \ge 2)$  are geometric mean 3-equitable graphs, and  $C_n \odot K_1$  is a geometric mean 3-equitable graph if and only if  $n \equiv 1, 2 \pmod{3}$ .

# 7.14 Irregular Total Labelings

In 1988 Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba [565] defined an *irregular* labeling of a graph G with no isolated vertices as an assignment of positive integer weights to the edges of G in such a way that the sums of the weights of the edges at each vertex are distinct. The minimum of the largest weight of an edge over all irregular labelings is called the *irregularity strength* s(G) of G. If no such weight exists,  $s(G) = \infty$ . Chartrand et al. gave a lower bound for  $s(mK_n)$ . Faudree, Jacobson, and Lehel [752] gave an upper bound for  $s(mK_n)$  when  $n \geq 5$  and proved that for graphs G with  $\delta(G) \geq n-2 \geq 1$ ,  $s(G) \leq 3$ . They also proved that if G has order n and  $\delta(G) = n - t$  and  $1 \le t \le \sqrt{n/18}$ ,  $s(G) \le 3$ . Aigner and Triesch proved  $s(G) \le n+1$  for any graph G with  $n \ge 4$  vertices for which s(G) is finite. In [2045] Przybylo proved that  $s(G) < 112n/\delta + 28$ , where  $\delta$  is the minimum degree of G and G has n vertices. The best bound of this form is currently due to Kalkowski, Karońki, and Pfender, who showed in [1270] that  $s(G) \le 6[n/\delta] < 6n/\delta+6$ . In [750] Faudree and Lehel conjectured that for each  $d \ge 2$ , there exists an absolute constant c such that  $s(G) \le n/d+c$  for each d-regular graph of order n. In Przybylo [2044] showed that for d-regular graphs s(G) < 16n/d + 6. In 1991 Cammack, Schelp, and Schrag [542] proved that the irregularity strength of a full d-ary tree (d = 2, 3) is its number of pendent vertices and conjectures that the irregularity strength of a tree with no vertices of degree two is its number of pendent vertices. This conjecture was proved by Amar and Togni [143] in 1998. Muthu Guru Packiam, Manimaran, and Thuraiswamy [1810] prove the following:  $s(C_n \odot mK_1) = mn, s(P_n \odot K_2) = n + 1, s(C_n \odot K_2) = n + 1, s(P_n \odot K_3) = n + 1$ , and  $sC_n \odot K_3) = n + 1$ . In [1255] Jinnah and Kumar determined the irregularity strength of triangular snakes.

Motivated by the notion of the irregularity strength of a graph and various kinds of other total labelings, Bača, Jendrol, Miller, and Ryan [277] introduced the total edge *irregularity strength* of a graph as follows. For a graph G(V, E) a labeling  $\partial : V \cup$  $E \to \{1, 2, \dots, k\}$  is called an *edge irregular total k-labeling* if for every pair of distinct edges uv and xy,  $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$ . Similarly,  $\partial$  is called an vertex irregular total k-labeling if for every pair of distinct vertices u and v,  $\partial(u) + \partial(u)$  $\sum \partial(e)$  over all edges e incident to  $u \neq \partial(v) + \sum \partial(e)$  over all edges e incident to v. The minimum k for which G has an edge (vertex) irregular total k-labeling is called the total edge (vertex) irregularity strength of G. The total edge (vertex) irregular strength of G is denoted by tes(G) (tvs(G)). They prove: for G(V, E), E not empty,  $\lceil (|E|+2)/3 \rceil \leq 1$  $tes(G) \le |E|; tes(G) \ge [(\Delta(G) + 1)/2]$  and  $tes(G) \le |E| - \Delta(G), if \Delta(G) \le (|E| - 1)/2;$  $\operatorname{tes}(P_n) = \operatorname{tes}(C_n) = [(n+2)/3]; \operatorname{tes}(W_n) = [(2n+2)/3]; \operatorname{tes}(C_3^n)$  (friendship graph)  $= [(3n+2)/3]; \operatorname{tvs}(C_n) = [(n+2)/3]; \text{ for } n \ge 2, \operatorname{tvs}(K_n) = 2; \operatorname{tvs}(K_{1,n}) = [(n+1)/2]; \text{ and}$  $\operatorname{tvs}(C_n \times P_2) = [(2n+3)/4]$ . Ahmad, Nurdin, and Baskoro [97] determined the exact value of the total edge (vertex) irregularity strength of generalized Halin graphs. Al-Mushayt, Ahmad, and Siddiqui [135] determined the exact values of the total edge irregular strength of hexagonal grid graphs. The (m, n)-lollipop graph denoted by  $L_{m,n}$  is a graph obtained by joining a complete graph  $K_m$  to a path graph  $P_n$  with a bridge. Ni'mah and Indriati [1846] determined tvs( $L_{m,n}$ ) for  $m \geq 3$  and  $n \geq 1$ . Aftiana and Indriati [55] proved that for  $n \geq 3$  the total edge irregularity strength of the graph obtained by joining two copies of  $K_n$  (barbell graph) with an edge is  $\lceil (n^2 - n + 3)/3 \rceil$ . In [1874] Nurdin and Hye consider the splitting graph of stars as a land transportation system and give the exact value of their total vertex irregularity strength. For  $m, n \geq 3$  Indriati, Widodo, and Sugeng [1074] determined the exact value of the total vertex irregularity strength for generalized helm graphs  $H_n^m$  (obtained from  $W_n$  by attaching  $P_m$  vertices at each vertex of the *n*-cycle) and for prisms with outer pendent edges.

In [90] Ahmad, Ibrahim, and Siddiqui determined the total irregularity strength of generalized Petersen graphs. In [137] Al-Mushayt, Ahmad, and Siddiqui [137] determined

the total edge (vertex) irregularity strength for convex polytope graphs having the same diameter. In [2379] Siddiqui determined the irregularity strength of six classes of convex polytope graphs with pendent edges. Ramdani, Salman, Assiyatum, and Semaničová-Feňovčíková [2086] establish upper bounds for the total vertex (edge) irregularity strength and total irregularity strength for disjoint union of arbitrary graphs. Naeem and Siddiqui [1819] determined the total irregularity strength of disjoint union of isomorphic copies of the generalized Petersen graph. In [162] Anholcer et al. presented some upper bound on group irregularity strength for all graphs. Moreover they gave the exact values and bounds on  $s_g(G)$  for disconnected graphs without a star as a component. In [2847] Yang, Siddiqui, Ibrahim, Ahmad, and Ahmad determined the exact value of the total irregularity strength of three planar graphs.

Jendrol, Miškul, and Soták [1112] (see also [1113]) proved:  $tes(K_5) = 5$ ; for  $n \ge 6$ ,  $\operatorname{tes}(K_n) = \left[ (n^2 - n + 4)/6 \right];$  and that  $\operatorname{tes}(K_{m,n}) = \left[ (mn + 2)/3 \right].$  They conjecture that for any graph G other than  $K_5$ , tes $(G) = \max\{\lfloor (\Delta(G) + 1)/2 \rfloor, \lfloor (|E| + 2)/3 \rfloor\}$ . Ivančo and Jendrol [1083] proved that this conjecture is true for all trees. Jendrol, Miškuf, and Soták [1112] prove the conjecture for complete graphs and complete bipartite graphs. The conjecture has been proven for the categorical product of two paths [77], the categorical product of a cycle and a path [2377], the categorical product of two cycles [84], the Cartesian product of a cycle and a path [317], the subdivision of a star [2378], and the toroidal polyhexes [282]. In [99] Ahmad, Siddiqui, and Afzal proved the conjecture is true for graphs obtained by starting with m vertex disjoint copies of  $P_n$   $(m, n \ge 2)$  arranged in m horizontal rows with the jth vertex of row i+1 directly below the jth vertex row i for  $1 = 1, 2, \ldots, m-1$  and joining the *j*th vertex of row *i* to the *j*+1th vertex of row *i*+1 for  $1 = 1, 2, \ldots, m-1$  and  $j = 1, 2, \ldots, n-1$  (the *zigzag* graph). Siddiqui, Ahmad, Nadeem, and Bashir [2381] proved the conjecture for the disjoint union of p isomorphic sun graphs (i. e.,  $C_n \odot K_1$ ) and the disjoint union of p sun graphs in which the orders of the n-cycles are consecutive integers. They pose as an open problem the determination of the total edge irregularity strength of disjoint union of any number of sun graphs. Brandt, Misškuf, and Rautenbach [498] proved the conjecture for large graphs whose maximum degree is not too large relative to its order and size. In particular, using the probabilistic method they prove that if G(V, E) is a multigraph without loops and with nonzero maximum degree less than  $|E|/10^3\sqrt{8|V|}$ , then tes $(G) = (\lceil |E| + 2)/3 \rceil$ . As corollaries they have: if G(V, E) satisfies  $|E| \ge 3 \cdot 10^3 |V|^{3/2}$ , then  $\operatorname{tes}(G) = \left[(|E|+2)/3\right]$ ; if G(V, E) has minimum degree  $\delta > 0$  and maximum degree  $\Delta$  such that  $\Delta < \delta \sqrt{|V|}/10^3 \cdot 4\sqrt{2}$  then tes(G) = [(|E|+2)/3]; and for every positive integer  $\Delta$  there is some  $n(\Delta)$  such that every graph G(V, E) without isolated vertices with  $|V| \ge n(\Delta)$  and maximum degree at most  $\Delta$  satisfies tes $(G) = \left[ (|E| + 2)/3 \right]$ . Notice that this last result includes d-regular graphs of large order. They also prove that if G(V, E) has maximum degree  $\Delta \geq 2|E|/3$ , then G has an edge irregular total k-labeling with  $k = \lfloor (\Delta + 1)/2 \rfloor$ . Pfender [1921] proved the conjecture for graphs with at least  $7 \times 10^{10}$  edges and proved for graphs G(V, E) with  $\Delta(G) \leq E(G)/4350$  we have tes $(G) = (\lceil |E| + 2)/3 \rceil$ . Murhu Guru Packiam, Manimaran, and Thuraiswamy [1797] investigate how the addition of a new edge affects the total edge irregularity strength of a graph. Laurence and Kathiresan [1472] determined the total

edge irregular strength of path union of cycles.

In [1239] Jeyanthi and Sudha investigated the total edge irregularity strength of the disjoint union of wheels. They proved the following:  $\operatorname{tes}(2W_n) = \lceil (4n+2)/3 \rceil, n \geq 3$ ; for  $n \geq 3$  and  $p \geq 3$  the total edge irregularity strength of the disjoint union of p isomorphic wheels is  $\lceil (2(pn+1)/3 \rceil;$  for  $n_1 \geq 3$  and  $n_2 = n_1 + 1$ ,  $\operatorname{tes}(W_{n_1} \cup W_{n_2}) = \lceil (2(n_1+n_2+1)/3) \rceil;$  for  $n_1, n_2, n_3$  where  $n_1 \geq 3$  and  $n_{i+1} = n_1 + i$  for i = 1, 2,  $\operatorname{tes}(W_{n_1} \cup W_{n_2} \cup W_{n_3}) = \lceil (2(n_1+n_2+n_3+1)/3) \rceil;$  the total edge irregularity strength of the disjoint union of  $p \geq 4$  wheels  $W_{n_1} \cup W_{n_2} \cup \cdots \cup W_{n_p}$  with  $n_{i+1} = n_1 + i$  and  $N = \sum_{j=1}^p n_j + 1$  is  $\lceil 2N/3 \rceil;$  and the total edge irregularity strength of  $p \geq 3$  disjoint union of wheels  $W_{n_1} \cup W_{n_2} \cup \cdots \cup W_{n_p}$  and  $N = \sum_{j=1}^p n_j + 1$  is  $\lceil (2N/3) \rceil$  if  $\max\{n_i \mid 1 \leq i \leq p\} \leq \frac{1}{2} \lceil (2N/3) \rceil$ . The complete star of a graph G is the graph obtained from p+1 copies of the graph G

The complete star of a graph G is the graph obtained from p+1 copies of the graph G by joining each vertex of  $G^{(0)}$  with all corresponding vertices of all the copies  $G^{(1)}, \ldots, G^{(p)}$ . Susanti, Khotimah, Hidayati, and Wahyujati [2547] determined the total edge irregularity strength of snowflake graphs, water bears graphs, the complete star of  $C_n$ , and two other families of ladder related graphs.

In [1240], [1242], [1243], and [1241] Jeyanthi and Sudha determine the total edge irregularity strength of fans, helms, closed helms, webs, flowers, gears, sun flowers, tadpoles, armed crowns, split graphs of cycles, split graph of paths, disjoint unions of isomorphic double wheels, and disjoint unions of consecutive non-isomorphic double wheels. Bokhary, Ali, and Maqbool [488] determined the exact values for the total vertex and edge irregularity strength of three wheel related families of graphs. Ibrahim, Asif, Ahmad, and Siddiqui [1039] investigated the total irregularity strength of fans, helms, closed helms, webs, ower graphs, gears, and sunflowers.

A generalized helm  $H_n^m$  is a graph obtained by inserting m vertices in every pendent edge of a helm  $H_n$ . Indriati, Widodo, and Sugeng [1072] proved that for  $n \ge 3$ ,  $\operatorname{tes}(H_n^1) = \lceil (4n+2)/3 \rceil$ ,  $\operatorname{tes}(H_n^2) = \lceil (5n+2)/3 \rceil$ , and  $\operatorname{tes}(H_n^m) = \lceil ((m+3)n+2))/3 \rceil$  for  $m \equiv 0 \mod 3$ . They conjecture that  $\operatorname{tes}(H_n^m) = \lceil ((m+3)n+2))/3 \rceil$ , for all  $n \ge 3$  and  $m \ge 10$ .

A cactus graph G is a connected graph where no edge lies in more than one cycle. A cactus graph consisting of some blocks where each block is  $C_n$  with same n is called an n-uniform cactus graph. If each cycle of the cactus graph has no more than two cut-vertices and each cut-vertex is shared by exactly two cycles, then G is called n-uniform cactus chain graph. Rosyida and Indriati [2145] determined the tes of n-uniform cactus chain graphs of length r for some  $n \equiv 0 \mod 3$ . They also investigated the tes of tadpole chain graphs. Rosyida and Indriati [2144] determined the total edge irregularity strength of the triangular cactus chain with length r and r + 1 pendant vertices  $(TC_r^{r+1})$  is  $\lceil (4r+3)/3 \rceil$ . A para-squares cactus is a graph each of whose blocks is a 4-cycle and two or more squares share a cut vertex. The para-squares cactus chain graph is a cactus graph such that each of the two squares has one common cut vertex. Rosyida and Indriati determined that total edge irregularity strength of para square cactus chain graph with length r and r pendant vertices  $(Q_r^r)$  is  $\lceil (5r+2)/3 \rceil$ .

Nurdin, Baskoro, Salman, and Gaos [1875] determine the total vertex irregularity strength of trees with no vertices of degree 2 or 3; improve some of the bounds given

in [277]; and show that  $\operatorname{tvs}(P_n) = \lceil (n+1)/3 \rceil$ . In [1878] Nurdin, Salman, Gaos, and Baskoro prove that for  $t \geq 2$ ,  $\operatorname{tvs}(tP_1) = t$ ;  $\operatorname{tvs}(tP_2) = t + 1$ ;  $\operatorname{tvs}(tP_3) = t + 1$ ; and for  $n \geq 4$ ,  $\operatorname{tvs}(tP_n) = \lceil (nt+1)/3 \rceil$ . Ahmad, Bača, and Bashir [78] proved that for  $n \geq 3$ and  $t \geq 1$ ,  $\operatorname{tvs}((n,t)\text{-kite}) = \lceil (n+t)/3 \rceil$ , where the (n,t)-kite is a cycle of length n with a *t*-edge path (the tail) attached to one vertex. In [939] Guo, Chen, Wang, and Yao give the total vertex irregularity strength of certain complete *m*-partite graphs.

Anholcer, Kalkowski, and Przybylo [171] prove that for every graph with  $\delta(G) > 0$ , tvs $(G) \leq \lceil 3n/\delta \rceil + 1$ . Majerski and Przybylo [1691] prove that the total vertex irregularity strength of graphs with n vertices and minimum degree  $\delta \geq n^{0.5} \ln n$  is bounded from above by  $(2 + o(1))n/\delta + 4$ . Their proof employs a random ordering of the vertices generated by order statistics. Anholcer, Karonński, and Pfender [170] prove that for every forest F with no vertices of degree 2 and no isolated vertices  $\operatorname{tvs}(F) = \lceil (n_1 + 1)/2 \rceil$ , where  $n_1$  is the number of vertices in F of degree 1. They also prove that for every forest with no isolated vertices and at most one vertex of degree 2,  $\operatorname{tvs}(F) = \lceil (n_1 + 1)/2 \rceil$ . Anholcer and Palmer [172] determined the total vertex irregularity strength  $C_n^k$ , which is a generalization of the circulant graphs  $C_n(1, 2, \ldots, k)$ . They prove that for  $k \geq 2$ and  $n \geq 2k + 1$ ,  $\operatorname{tvs}(C_n^k = \lceil (n+2k)/(2k+1) \rceil$ . Przybylo [2045] obtained a variety of upper bounds for the total irregularity strength of graphs as a function of the order and minimum degree of the graph.

In [2593] Tong, Lin, Yang, and Wang give the exact values of the total edge irregularity strength and total vertex irregularity strength of the toroidal grid  $C_m \times C_n$ . In [2382] Siddiqui, Miller, and Ryan determine the exact values of the total edge irregularity strength of octagonal grid graph. In [85] Ahmad, Bača, and Siddiqui gave the exact value of the total edge and total vertex irregularity strength for disjoint union of prisms and for disjoint union of cycles. In [83] Ahmad, Bača, and Numan showed that  $\operatorname{tes}(\bigcup_{j=1}^m F_{n_j}) = 1 + \sum_{j=1}^m n_j$  and  $\operatorname{tvs}(\bigcup_{j=1}^m F_{n_j}) = \lceil (2+2\sum_{j=1}^m n_j)/3 \rceil$ , where  $\bigcup_{j=1}^m F_{n_j}$ denotes the disjoint union of friendship graphs. Chunling, Xiaohui, Yuansheng, and Liping, [608] showed  $\operatorname{tvs}(K_p) = 2$  ( $p \ge 2$ ) and for the generalized Petersen graph P(n,k)they proved  $\operatorname{tvs}(P(n,k)) = \lceil n/2 \rceil + 1$  if  $k \le n/2$  and  $\operatorname{tvs}(P(n,n/2)) = n/2 + 1$ . They also obtained the exact values for the total vertex strengths for ladders, Möbius ladders, and Knödel graphs. For graphs with no isolated vertices, Przybylo [2044] gave bounds for  $\operatorname{tvs}(G)$  in terms of the order and minimum and maximum degrees of G. For d-regular (d > 0) graphs, Przybylo [2045] gave bounds for  $\operatorname{tvs}(G)$  in terms d and the order of G.

Ahmad, Ahtsham, Imran, and Gaig [68] determined the exact values of the total vertex irregularity strength for five families of cubic plane graphs. In [75] Ahmad and Bača determine that the total edge-irregular strength of the categorical product of  $C_n$  and  $P_m$  where  $m \ge 2$ ,  $n \ge 4$  and n and m are even is  $\lceil (2n(m-1)+2)/3 \rceil$ . They leave the case where at least one of n and m is odd as an open problem. In [84] and [85] Ahmad, Bača, and Siddiqui determine the exact values of the total edge irregularity strength of the categorical product of two cycles, the total edge (vertex) irregularity strength for the disjoint union of prisms, and the total edge (vertex) irregularity strength for the disjoint union of flowers, helms, generalized friendship graphs, and web graphs.

Indriati, Widodo, Wijayanti, Sugeng, and Bača [1071] determine the exact value of the total edge irregularity strength of the generalized web graph W(n, m) and two families of related graphs. Ahmad, Bača, and Numan [83] determined the exact values of the total vertex irregularity strength and the total edge irregularity strength of a disjoint union of friendship graphs. Bokhary, Ahmad, and Imran [487] determined the exact value of the total vertex irregularity strength of cartesian and categorical product of two paths. Koam and Ahmad [1384] determined the total vertex irregularity strength for all theta graphs and certain values of the total vertex irregularity strength of the conjecture for the lower bound of total vertex irregularity strength of the centralized uniform theta graphs. In [489] Bokhary and Faheem proved the conjecture of Bokhary, Ahmam, and Imran [487] that the  $tvs(P_m \Box P_n) = \lfloor \frac{mn+2}{5} \rfloor$  for  $m, n \geq 2$  for  $5 \leq m \leq 10$  and  $n \geq 1$ . the state graph for Tower of Hanoi problems with three towers.

In [1877] Nurdin, Salman, and Baskoro determine the total edge-irregular strength of the following graphs: for any integers  $m \ge 2$ ,  $n \ge 2$ ,  $tes(P_m \odot P_n) = \lceil (2mn+1)/3 \rceil$ ; for any integers  $m \ge 2$ ,  $n \ge 3$ ,  $tes(P_m \odot C_n) = \lceil ((2n+1)m+1)/3 \rceil$ ; for any integers  $m \ge 2$ ,  $n \ge 2$ ,  $tes(P_m \odot K_{1,n}) = \lceil (2m(n+1)+1)/3 \rceil$ ; for any integers  $m \ge 2$  and  $n \ge 3$ ,  $tes(P_m \odot G_n) = \lceil (m(5n+2)+1)/3 \rceil$  where  $G_n$  is the gear graph obtained from the wheel  $W_n$  by subdividing every edge on the *n*-cycle of the wheel; for any integers  $m \ge 2$ ,  $n \ge 2$ ,  $tes(P_m \odot F_n) = \lceil m(5n+2)+1 \rceil$ , where  $F_n$  is the friendship graph obtained from  $W_{2m}$  by subdividing every other rim edge; for any integers  $m \ge 2$  and  $n \ge 3$ ; and  $tes(P_m \odot W_n) = \lceil ((3n+2)m+1)/3 \rceil$ .

In [2069], [2070], and [2068] Rajasingh, Rajan, Teresa Arockiamary, and Quadras provide the total edge irregularity strengths of honeycomb mesh networks, hexagonal networks, butterfly networks, benes networks, and series compositions of uniform theta graphs.

In [1876] Nurdin, Baskoro, Salman, and Gaos proved: the total vertex-irregular strength of the complete k-ary tree  $(k \ge 2)$  with depth  $d \ge 1$  is  $\lceil (k^d + 1)/2 \rceil$  and the total vertex-irregular strength of the subdivision of  $K_{1,n}$  for  $n \ge 3$  is  $\lceil (n+1)/3 \rceil$ . They also determined that if G is isomorphic to the caterpillar obtained by starting with  $P_m$  and m copies of  $P_n$  denoted by  $P_{n,1}, P_{n,2}, \ldots, P_{n,m}$ , where  $m \ge 2$ ,  $n \ge 2$ , then joining the *i*-th vertex of  $P_m$  to an end vertex of the path  $P_{n,i}$ ,  $\operatorname{tvs}(G) = \lceil (mn+3)/3 \rceil$ . They conjectured that the total vertex irregularity strength of any tree T is determined only by the number of vertices of degrees 1, 2 and 3 in T. This conjecture was confirmed by Susilawati, Baskoro, and Simanjuntak [2550] by considering all trees with maximum degree five. They also characterized all such trees having the total vertex irregularity strength either  $t_1, t_2$  or  $t_3$ , where  $t_i = \lceil (1 + \sum_{j=1}^i n_j)/(i+1)\rceil$  and  $n_i$  is the number of vertices of degree i.

Ahmad and Bača [76] proved  $\operatorname{tvs}(J_{n,2}) = \lceil (n+1)/2 \rceil$   $(n \ge 4)$  and conjectured that for  $n \ge 3$  and  $m \ge 3$ ,  $\operatorname{tvs}(J_{n,m}) = \max\{\lceil (n(m-1)+2)/3 \rceil, \lceil (nm+2)/4 \rceil\}$ . They also proved that for the circulant graph (see §5.1 for the definition)  $C_n(1,2), n \ge 5$ ,  $\operatorname{tvs}(C_n(1,2)) = \lceil (n+4)/5 \rceil$ . They conjecture that for the circulant graph  $C_n(a_1, a_2, \ldots, a_m)$  with degree r

at least 5 and  $n \ge 5$ ,  $1 \le a_i \le \lfloor n/2 \rfloor$ ,  $\operatorname{tvs}(C_n(a_1, a_2, \ldots, a_m) = \lceil (n+r)/(1+r) \rceil$ . Ahmad, Arshadb, and Ižaríková [73] determine  $\operatorname{tes}(G)$  where G is the generalized helm and  $\operatorname{tvs}(G)$  where G is the generalized sun graph.

Slamin, Dafik, and Winnona [2420] consider the total vertex irregularity strengths of the disjoint union of isomorphic sun graphs, the disjoint union of consecutive nonisomorphic sun graphs,  $\operatorname{tvs}(\cup_{i=1}^{t}S_{i+2})$ , and disjoint union of any two nonisomorphic sun graphs. (Recall  $S_n = C_n \odot K_1$ .) Rajasingh and Annamma [2067] determine the total vertex irregularity strength of 1-fault tolerant Hamiltonian graphs CH(n), H(n), and W(m). Indriati, Widodo, Wijayanti, Sugeng, Bača, and Semaničová-Feňovčíková [1073] determine the exact value of the total vertex irregularity strength for generalized helm graphs and for prisms with outer pendent edges. In [207] Asim and Hasni provided an upper bound for  $\operatorname{es}(K_n)$  that is far better than the previously known upper bound.

In [64] Ahmad shows that the total vertex irregularity strength of the antiprism graph  $A_n$   $(n \ge 3)$  is  $\lceil (2n+4)/5 \rceil$  (see §5.7 for the definition) and gives the vertex irregularity strength of three other families convex polytope graphs. Al-Mushayt, Arshad, and Siddiqui [136] determined an exact value of the total vertex irregularity strength of some convex polytope graphs. Ahmad, Baskoro, and Imran [88] determined the exact value of the total vertex irregularity strength of some the total vertex irregularity strength of disjoint union of helm graphs.

For  $n \geq 3, m \geq 2$  Jeyanthi and Sudha [1244] determine the total vertex irregularity strength of  $P_n \odot K_1, P_n \odot K_2, C_n \odot K_2, L_n \odot K_1, P_2 \odot C_n, P_n \odot \overline{K_m}, (C_n \times P_2) \odot K_1$ , and  $C_n \odot \overline{K_m}$ . In [1245] they determine the total vertex irregularity strength for the graph obtained from a cycle by identifying the endpoint of a path and the vertex of a cycle,  $C_n \odot P_m$ , the split graph of a cycle, and split graph of a path. In [1245] they determine the total vertex irregularity strength for quadrilateral snakes, sunflowers, double wheels, triangular books, quadrilateral books, and graphs obtained from the wheel  $W_n$ and attaching n pendent edges to the center. In [1247] Jeyanthi and Sudha determined the total irregularity strength of the n-crossed prism, m copies of crossed prism, necklace and m copies of necklace graph and that these graphs admit totally irregular total k-labeling.

In [63] Agustin, Dafik, Marsidi, and Albirri introduced a natural extension of the notation of the total *H*-irregularity strength of graphs by considering the evaluation of the weight that is not only for each edge but also the weight on each subgraph *H* of *G*. They say a total  $\alpha$ -labeling is a *total H*-irregular  $\alpha$ -labeling of the graph *G* if for a subgroup *H* of *G*, the total *H*-weights  $W(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$  are distinct. The minimum  $\alpha$  for which the graph *G* has a total *H*-irregular  $\alpha$ -labeling is called the *total H*-irregularity strength of *G*, denoted by tHs(G). They study the *tHs* of shackles and amalgamations of any graphs and their bounds.

The notion of an irregular labeling of an Abelian group  $\Gamma$  was introduced Anholcer, Cichacz and Milanič in [164]. They defined a  $\Gamma$ -irregular labeling of a graph G with no isolated vertices as an assignment of elements of an Abelian group  $\Gamma$  to the edges of Gin such a way that the sums of the weights of the edges at each vertex are distinct. The group irregularity strength of G, denoted  $s_g(G)$ , is the smallest integer s such that for every Abelian group  $\Gamma$  of order s there exists  $\Gamma$ -irregular labeling of G. They proved that if Gis connected, then  $s_g(G) = n + 2$  when  $\cong K_{1,3^{2q+1}-2}$  for some integer  $q \ge 1$ ;  $s_g(G) = n + 1$  when  $n \equiv 2 \pmod{4}$  and  $G \not\cong K_{1,3^{2q+1}-2}$  for any integer  $q \geq 1$ ; and  $s_g(G) = n$  otherwise. Moreover, Anholcer and Cichacz [163] showed that if G is a graph of order n with no component of order less than 3 and with all the bipartite components having both color classes of even order. Then  $s_g(G) = n$  if  $n \equiv 1 \pmod{2}$ ;  $s_g(G) = n + 1$  if  $n \equiv 2 \pmod{4}$ ; and  $s_g(G) \leq n + 1$  if  $n \equiv 0 \pmod{4}$ .

Marzuki, Salman, and Miller [1723] introduced a new irregular total k-labeling of a graph G called total irregular total k-labeling, denoted by ts(G), which is required to be at the same time both vertex and edge irregular. They gave an upper bound and a lower bound of ts(G); determined the total irregularity strength of cycles and paths; and proved  $ts(G) \ge \max\{tes(G), tvs(G)\}$ . For  $n \ge 3$ , Ramdani and Salman [2083] proved  $ts(S_n \times P_2) = n + 1$ ;  $ts((P_n + P_1) \times P_2) = \lceil (5n + 1)/3 \rceil$ ,  $ts(P_n \times P_2) = n$ ; and  $ts(C_n \times P_2) = n$ . In [2084] Ramdani, Salman, and Assiyatun prove that for a regular graph G  $ts(mG) \le m(ts(G)) - \lfloor (m-1)/2 \lfloor$ ,  $ts(mC_n) = \lceil (mn+2)/3 \rfloor$  for  $n \equiv 3 \mod 3$ , and  $ts(m(C_n \times P_2) = mn + 1$ . In [2085] Ramdani, Salman, Assiyatun, Semaničová-Feňovčíková, and Bača estimate the upper bound of the total irregularity strength of graphs. In [1179] Jeyanthi and Sudha determined the total irregularity strength of double fans  $DF_n$   $(n \ge 3)$ , double triangular snakes  $DT_p$   $(p \ge 3)$ , joint-wheel graphs  $WH_n$   $(n \ge 3)$ , and  $P_m + \overline{K_m}$   $(m \ge 3)$ . In addition, they show that these graphs admit totally irregular total k-labeling and they determined the exact ts value for each.

In [1809] Muthgu Guru Packiam defines a face irregular total k-labeling f from  $V \cup E \cup F$  to  $\{1, 2, \ldots, k\}$  of a 2-connected plane graph G(V, E, F) as a labeling of vertices and edges such that different faces have different weights. The minimum k for which a plane graph G has a face irregular total k-labeling is called total face irregularity strength of G and is denoted by tfs(G). He provides a bound on this parameter and the exact values for shell graphs and a family of planar graphs consisting of an even number of 5-sided faces and one external infinite face. In [280] Bača, Lascsková, Naseem, and Semaničová-Feňovčíková estimate the lower and upper bounds of the entire face irregularity strength for the disjoint union of multiple copies of a plane graph and prove the sharpness of the lower bound in two cases.

Recall that an *edge-covering* of G is a family of subgraphs  $H_1, H_2, \ldots, H_t$  such that each edge of E(G) belongs to at least one of the subgraphs  $H_i, i = 1, 2, \ldots, t$ . In this case we say that G admits an  $(H_1, H_2, \ldots, H_t)$ -(edge) covering. If every subgraph  $H_i$ is isomorphic to a given graph H, we say that G admits an H-covering. Motivated by the irregularity strength and the edge irregularity strength of a graph G, Ashraf, Bača, Kimáková, and Semaničová-Feňovčíková [203] introduced two new parameters, edge (vertex) H-irregularity strengths, as the natural extensions of the parameters s(G) and es(G) as follows. Let G be a graph admitting an H-covering. For the subgraph H of G under the edge k-labeling  $\beta$  from E(G) to  $\{1, 2, \ldots, k\}$ , the associated H-weight is defined as wt $_{\beta}(H) = \sum \beta(e)$  over all edges e. An edge k-labeling  $\beta$  is called an Hirregular edge k-labeling of the graph G if for every two different subgraphs H' and H'' isomorphic to H we have  $wt_{\beta}(H') \neq wt_{\beta}(H'')$ . The edge H-irregularity strength of a graph G, denoted by ehs(G, H), is the smallest integer k such that G has an H-irregular edge k-labeling. Ashraf et al. define the vertex H-irregularity strength of a graph G, vhs(G, H), analogously. They estimate the bounds of the parameters ehs(G, H) and vhs(G, H) and determine the exact values of the edge (vertex) H-irregularity strength for paths, ladders, and fans in order to prove the sharpness of lower bounds of these parameters. Nisviasari, Dafik, and Agustin [1848] provided the total H-irregularity strength of triangular ladders when H is a windmill or triangular ladder.

In [204] Ashraf, Bača, Lascsáková, and Semaničová-Feňovčíková estimated the bounds for the total *H*-irregularity strength of a graph and determind the exact values of the total *H*-irregularity strength for paths ladders and fans. Ashrafa, Bača, Semaničová-Feňovčíková, and Shabbirc [205] investigated the total (respectively, edge and vertex) *G*-irregularity strengths of the graphs that contains exactly n subgraphs isomorphic to G.

In [71] Ahmad, Al-Mushayt, and Bača define a vertex k-labeling  $\phi$  of a graph G from V(G) to  $\{1, 2, \ldots, k\}$  to be edge irregular k-labeling if for every two distinct edges e and f, there is  $w_{\phi}(e) \neq w_{\phi}(f)$ , where the weight of an edge e = xy is  $w_{\phi}(xy) = \phi(x) + \phi(y)$ . The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by es(G). They estimated the bounds of the edge irregularity and determined its exact values for paths, cycles, stars, double stars and  $P_m \times P_n$ . Tarawneh, Hasni, and Ahmad [2572] determined the exact value of the edge irregularity strength of the corona product of graphs with paths. Tarawneh, Hasni, and Ahmad [2573] determine the exact value of edge irregularity strength of corona graphs  $C_n \odot mK_1$  ( $m \ge 2$ ). Ahmad [65] determined the exact value of es( $C_n \odot K_1$ ). In [81] Ahmad, Bača, and Nadeen determine the exact value of the edge irregularity strength for several classes of Toeplitz graphs. Tarawneh, Hasni, Siddiqui, and Asim [2575] determined the exact value of edge irregularity strength for several classes of Toeplitz graphs.

A chain graph  $C[B_1, B_2, \ldots, B_n]$  is a graph with blocks  $B_1, B_2, \ldots, B_n$  such that  $B_i$ and  $B_{i+1}$  have a common vertex in such a way that the block-cut vertex graph is a path. Ahmad, Gupta, and Simanjuntak [89] prove the following:  $es(C[C_4^{(n)}]) = 2n + 1$ ; if  $H_m$  is an  $mK_3$ -path, then  $es(H_m)$  has lower bound  $\lceil \frac{3m+3}{2} \rceil$  and upper bound 2m + 1;  $es(mK_4$ -path) = 3m + 2; and  $es(K_{1,n} + \overline{K_1}) = n + 2$  for  $n \ge 3$ . They obtained bounds for  $es(P_m + \overline{K_n})$  and determined that the edge irregularity strength of a graph obtained by joining the vertex of degree m in  $K_{1,m}$  to each vertex in  $K_{1,n}$ , and the vertex of degree nin  $K_{1,n}$  to each vertex in  $K_{1,m}$  is m+n+2. They posed the open problems of determining  $es(mK_3$ -path),  $es(mK_n$ -path) ( $m \ge 2$ ) and  $n \ge 5$ , and  $es(P_m + \overline{K_n})$  for  $n \ge 1$  and  $m \ge 7$ . Tarawneh, Hasni, and Asim [2574] determined the exact value of edge irregularity strength for disjoint union of a star graph and the subdivision of a star graph.

The strong product of graphs  $G_1$  and  $G_2$  has as vertices the pairs (x, y) where  $x \in V(G_1)$ and  $y \in V(G_2)$ . The vertices  $(x_1, y_1)$  and  $(x_2, y_2)$  are adjacent if either  $x_1x_2$  is an edge of  $G_1$  and  $y_1 = y_2$  or if  $x_1 = x_2$  and  $y_1y_2$  is an edge of  $G_2$ . For  $m, n \ge 2$  Ahmad, Bača, Bashir, Siddiqui [79] proved that the total edge irregular strength of the strong product of  $P_m$  and  $P_n$  is  $\lceil 4(mn+1)/3 \rceil - (m+n)$ . Al-Mushayt [134] determined the edge irregularity strength of cartesian product of a star and  $P_2$  and a cycle and  $P_2$ , and the strong product of path  $P_n$  with  $P_2$ . Conjectures for the exact value of  $K_{1,n} \times P_m$  and  $C_n \times P_m$  are stated. Bača and Siddiqui [318] determine the exact value of the total edge irregularity strength of the strong product of any two cycles.

An edge  $e \in G$  is called a *total positive edge* or *total negative edge* or *total stable edge* of G if tvs(G + e) > tvs(G) or tvs(G + e) < tvs(G) or tvs(G + e) = tvs(G), respectively. If all edges of  $\overline{G}$  are total stable (total negative) edges of G, then G is called a *total stable* (*total negative*) graph. Otherwise G is called a *total mixed* graph.

Muthu Guru Packiam and Kathiresan [1882] showed that  $K_{1,n}$   $n \ge 4$ , and the disjoint union of  $t \ge 2$  copies of  $K_3$  are total negative graphs and that the disjoint union of  $t \ge 2$  copies of  $P_3$  is a total mixed graph.

For a simple graph G with no isolated edges and at most one isolated vertex Anholcer [160] calls a labeling  $w : E(G) \to \{1, 2, ..., m\}$  product-irregular, if all product degrees  $pd_G(v) = \prod_{e \ni v} w(e)$  are distinct. Analogous to the notion of irregularity strength the goal is to find a product-irregular labeling that minimizes the maximum label. This minimum value is called the product irregularity strength of G and is denoted by ps(G). He provides bounds for the product irregularity strength of paths, cycles, cartesian products of paths, and cartesian products of cycles. In [161] Anholcer gives the exact values of ps(G) for  $K_{m,n}$  where  $2 \le m \le n \le (m+2)(m+1)/2$ , some families of forests including complete d-ary trees, and other graphs with d(G) = 1. Skowronek-Kaziów [2417] proves that for the complete graphs  $ps(K_n) = 3$ . Darda and Hujdurović [646] proved that  $ps(X) \le |V(X)| - 1$ for any graph X with more than 3 vertices and gave a connection between the product irregularity strength and the multidimensional multiplication table problem.

In [4] Abdo and Dimitrov introduced the total irregularity of a graph. For a graph G, they define  $\operatorname{irr}_t(G) = (1/2) \sum_{u,v \in V} |d_G(u) - d_G(v)|$ , where  $d_G(w)$  denotes the vertex degree of the vertex w. For G with n vertices they proved  $\operatorname{irr}_t(G) \leq (1/12)(2n^3 - 3n^2 - 2n + 3)$ . For a tree G with n vertices they prove  $\operatorname{irr}_t(G) \leq (n-1)(n-2)$  and equality holds if and only if  $G \approx S_n$ . You, Yang, and You [2870] determined the graph with the maximal total irregularity among all unicyclic graphs.

Inspired by the concept of distant chromatic numbers Przybylo [2046] calls a labeling f from the edges of a graph G to  $\{1, 2, 3, \ldots, k\}$  r-distant irregular, if for every vertex v, the weights of the set of all vertices that are at distance less than or equal to r from v are pairwise distinct, where the weight of the vertex is the sum of the labels of the edges that are incident with that vertex. The minimum k for which there exists an r-distant irregular labeling of G is called r-distant irregularity strength of G and is denoted by  $s_r(G)$ . Muthu Guru Packiam, Manimaran, and Thuraiswamy [1811] proved the following:  $s_1(P_n) = 2$  for  $n = 3, 4, 5; s_1(P_n) = 3$  if  $n > 5; s_1(C_n) = 3; s_1(K_{m,n}) = s(K_{m,n}); s_1(F_n) = s(F_n) = [(n+1)/3]$  for  $n > 2; s_1(K_{m,n}) = 3$  when  $1 < n/2 \le m < n; s_1(P_n \times K_2) = 3; s_1(C_n \times K_2) = 3; s_1(C_m \odot \overline{K_n})$  for all m and n. They also prove that if G is d-regular with n vertices, then  $s_1(G) = s(G) \le [n/2] + 1$  for  $d \ge n/2$ .

#### 7.15 Geometric Labelings

If a and r are positive integers at least 2, we say a (p,q)-graph G is (a,r)-geometric if its vertices can be assigned distinct positive integers such that the value of the edges obtained as the product of the endpoints of each edge is  $\{a, ar, ar^2, \ldots, ar^{q-1}\}$ . Hegde [982] has shown the following: no connected bipartite graph, except the star, is (a, a)-geometric where a is a prime number or square of a prime number; any connected (a, a)-geometric graph where a is a prime number or square of a prime number, is either a star or has a triangle;  $K_{a,b}$ ,  $2 \le a \le b$  is (k, k)-geometric if and only if k is neither a prime number nor the square of a prime number; a caterpillar is (k, k)-geometric if and only if k is neither a prime number nor the square of a prime number;  $K_{a,b,1}$  is (k,k)-geometric for all integers  $k \geq 2$ ;  $C_{4t}$  is (a, a)-geometric if and only if a is neither a prime number nor the square of a prime number; for any positive integers t and  $r \ge 2$ ,  $C_{4t+1}$  is  $(r^{2t}, r)$ -geometric; for any positive integer t,  $C_{4t+2}$  is not geometric for any values of a and r; and for any positive integers t and  $r \ge 2$ ,  $C_{4t+3}$  is  $(r^{2t+1}, r)$ -geometric. Hegde [984] has also shown that every  $T_p$ -tree and the subdivision graph of every  $T_p$ -tree are (a, r)-geometric for some values of a and r (see Section 3.2 for the definition of a  $T_p$ -tree). He conjectures that all trees are (a, r)-geometric for some values of a and r.

Hegde and Shankaran [993] prove: a graph with an  $\alpha$ -labeling (see §3.1 for the definition) where m is the fixed integer that is between the endpoints of each edge has an  $(a^{m+1}, a)$ -geometric for any a > 1; for any integers m and n both greater than 1 and m odd,  $mP_n$  is  $(a^r, a)$ -geometric where r = (mn + 3)/2 if n is odd and  $(a^r, a)$ -geometric where r = (m(n+1)+3)/2 if n is even; for positive integers  $k > 1, d \ge 1$ , and odd n, the generalized closed helm (see §5.3 for the definition) CH(t, n) is  $(k^r, k^d)$ -geometric where r = (n-1)d/2; for positive integers  $k > 1, d \ge 1$ , and odd n, the generalized web graph (see §5.3 for the definition) W(t, n) is  $(k^r, a)$ -geometric where  $a = k^d$  and r = (n-1)d/2; for positive integers  $k > 1, d \ge 1$ , the generalized n-crown  $(P_m \times K_3) \odot K_{1,n}$  is (a, a)-geometric where  $a = k^d$ ; and  $n = 2r + 1, C_n \odot P_3$  is  $(k^r, k)$ -geometric.

If a and r are positive integers and r is at least 2 Arumugan, Germina, and Anadavally [194] say a (p,q)-graph G is additively (a,r)-geometric if its vertices can be assigned distinct integers such that the value of the edges obtained as the sum of the endpoints of each edge is  $\{a, ar, ar^2, \ldots, ar^{q-1}\}$ . In the case that the vertex labels are nonnegative integers the labeling is called additively (a, r)\*-geometric. They prove: for all a and r every tree is additively (a, r)\*-geometric; a connected additively (a, r)-geometric graph is either a tree or unicyclic graph with the cycle having odd size; if G is a connected unicyclic graph and not a cycle, then G is additively (a, r)-geometric if and only if either a is even or a is odd and r is even; connected unicyclic graphs are not additively (a, r)\*-geometric; if a disconnected graph is additively (a, r)-geometric, then each component is a tree or a unicyclic graph with an odd cycle; and for all even a at least 4, every disconnected graph for which every component is a tree or unicyclic with an odd cycle has an additively (a, r)-geometric labeling.

Vijayakumar [2731] calls a graph G (not necessarily finite) *arithmetic* if its vertices can be assigned distinct natural numbers such that the value of the edges obtained as
the sum of the endpoints of each edge is an arithmetic progression. He proves [2730] and [2731] that a graph is arithmetic if and only if it is (a, r)-geometric for some a and r.

### 7.16 Strongly Multiplicative Graphs

Beineke and Hegde [432] call a graph with p vertices strongly multiplicative if the vertices of G can be labeled with distinct integers  $1, 2, \ldots, p$  such that the labels induced on the edges by the product of the end vertices are distinct. They prove the following graphs are strongly multiplicative: trees; cycles; wheels;  $K_n$  if and only if  $n \leq 5$ ;  $K_{r,r}$  if and only if  $r \leq 4$ ; and  $P_m \times P_n$ . They then consider the maximum number of edges a strongly multiplicative graph on n vertices can have. Denoting this number by  $\lambda(n)$ , they show:  $\lambda(4r) \leq 6r^2$ ;  $\lambda(4r+1) \leq 6r^2 + 4r$ ;  $\lambda(4r+2) \leq 6r^2 + 6r + 1$ ; and  $\lambda(4r+3) \leq 6r^2 + 10r + 3$ . Adiga, Ramaswamy, and Somashekara [52] give the bound  $\lambda(n) \leq n(n+1)/2 + n - 2 - \lfloor (n+2)/4 \rfloor - \sum_{i=2}^{n} i/p(i)$  where p(i) is the smallest prime dividing i. For large values of n this is a better upper bound for  $\lambda(n)$  than the one given by Beineke and Hegde. It remains an open problem to find a nontrivial lower bound for  $\lambda(n)$ .

Seoud and Zid [2265] prove the following graphs are strongly multiplicative: wheels;  $rK_n$  for all r and n at most 5;  $rK_n$  for  $r \ge 2$  and n = 6 or 7;  $rK_n$  for  $r \ge 3$  and n = 8 or 9;  $K_{4,r}$  for all r; and the corona of  $P_n$  and  $K_m$  for all n and  $2 \le m \le 8$ . In [2243] Seoud and Mahran [2243] give some necessary conditions for a graph to be strongly multiplicative.

In Kanani and Chhaya [1273] and [1274] prove the following graphs are strongly multiplicative: the total graph, splitting graph, and shadow graph of paths; triangular snakes; splitting graphs of stars and bistars, the degree splitting graph of the bistars  $B_{n,n}$ , and restricted square graph  $B_{m,n}^2$ . In [1277] and [1278] Kanani and Chhaya prove the following graphs are strongly multiplicative: helms, flowers, fans, friendship graphs, bistars, gears, double triangular snakes, double fans, double wheels, snakes, double alternate quadrilateral snakes, double quadrilateral snakes, braid graphs, and triangular ladders.

Germina and Ajitha [868] (see also [32]) prove that  $K_2 + \overline{K_t}$ , quadrilateral snakes, Petersen graphs, ladders, and unicyclic graphs are strongly multiplicative. Acharya, Germina, and Ajitha [32] have shown that  $C_k^{(n)}$  (see §2.2 for the definition) is strongly multiplicative and that every graph can be embedded as an induced subgraph of a strongly multiplicative graph. Germina and Ajitha [868] define a graph with q edges and a strongly multiplicative labeling to be *hyper strongly multiplicative* if the induced edge labels are  $\{2, 3, \ldots, q + 1\}$ . They show that every hyper strongly multiplicative graph has exactly one nontrivial component that is either a star or has a triangle and every graph can be embedded as an induced subgraph of a hyper strongly multiplicative graph.

Vaidya, Dani, Vihol, and Kanani [2634] prove that the arbitrary supersubdivisions of tree,  $K_{mn}$ ,  $P_n \times P_m$ ,  $C_n \odot P_m$ , and  $C_n^m$  are strongly multiplicative. Vaidya and Kanani [2640] prove that the following graphs are strongly multiplicative: a cycle with one chord; a cycle with twin chords (that is, two chords that share an endpoint and with opposite endpoints that join two consecutive vertices of the cycle; the cycle  $C_n$  with three chords that form a triangle and whose edges are the edges of two 3-cycles and a n - 3-cycle. duplication of an vertex in cycle (see §2.7 for the definition); and the graphs obtained from  $C_n$  by identifying of two vertices  $v_i$  and  $v_j$  where  $d(v_i, v_j) \ge 3$ . In [2643] the same authors prove that the graph obtained by an arbitrary supersubdivision of path, a star, a cycle, and a tadpole (that is, a cycle with a path appended to a vertex of the cycle.

Krawec [1408] calls a graph G on n edges modular multiplicative if the vertices of G can be labeled with distinct integers  $0, 1, \ldots, n-1$  (with one exception if G is a tree) such that the labels induced on the edges by the product of the end vertices modulo nare distinct. He proves that every graph can be embedded as an induced subgraph of a modular multiplicative graph on prime number of edges. He also shows that if G is a modular multiplicative graph on prime number of edges p then for every integer  $k \geq 2$ there exist modular multiplicative graphs on  $p^k$  and kp edges that contain G as a subgraph. In the same paper, Krawec also calls a graph G on n edges k-modular multiplicative if the vertices of G can be labeled with distinct integers  $0, 1, \ldots, n+k-1$  such that the labels induced on the edges by the product of the end vertices modulo n + k are distinct. He proves that every graph is k-modular multiplicative for some k and also shows that if p = 2n + 1 is prime then the path on n edges is (n + 1)-modular multiplicative. He also shows that if p = 2n + 1 is prime then the cycle on n edges is (n + 1)-modular multiplicative if there does not exist  $\alpha \in \{2, 3, ..., n\}$  such that  $\alpha^2 + \alpha - 1 \equiv 0 \mod p$ . He concludes with four open problems. In [1409] Krawec shows that every graph is a subgraph of a modular multiplicative graph. He also defines k-modular multiplicative graphs and proves that certain families of paths and cycles admit such a labeling.

### 7.17 k-sequential Labelings

In 1981 Bange, Barkauskas, and Slater [342] defined a k-sequential labeling f of a graph G(V, E) as one for which f is a bijection from  $V \cup E$  to  $\{k, k+1, \ldots, |V \cup E| + k - 1\}$  such that for each edge xy in E, f(xy) = |f(x) - f(y)|. This generalized the notion of simply sequential where k = 1 introduced by Slater. Bange, Barkauskas, and Slater showed that cycles are 1-sequential and if G is 1-sequential, then  $G + K_1$  is graceful. Hegde and Shetty [992] have shown that every  $T_p$ -tree (see §4.4 for the definition) is 1-sequential. In [2423], Slater proved:  $K_n$  is 1-sequential if and only if  $n \leq 3$ ; for  $n \geq 2$ ,  $K_n$  is not k-sequential for any  $k \ge 2$ ; and  $K_{1,n}$  is k-sequential if and only if k divides n. Acharya and Hegde [37] proved: if G is k-sequential, then k is at most the independence number of G;  $P_{2n}$  is nsequential for all n and  $P_{2n+1}$  is both n-sequential and (n+1)-sequential for all n;  $K_{m,n}$  is k-sequential for k = 1, m, and n;  $K_{m,n,1}$  is 1-sequential; and the join of any caterpillar and  $K_t$  is 1-sequential. Acharya [23] showed that if G(E, V) is an odd graph with  $|E| + |V| \equiv 1$ or 2 (mod 4) when k is odd or  $|E| + |V| \equiv 2$  or 3 (mod 4) when k is even, then G is not k-sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig, and Turgeon [448] we have:  $mK_4$  is not k-sequential for any k when m is odd and  $mK_2$ is not k-sequential for any odd k when  $m \equiv 2$  or 3 (mod 4) or for any even k when  $m \equiv 1$ or 2 (mod 4). He further noted that  $K_{m,n}$  is not k-sequential when k is even and m and n are odd, whereas  $K_{m,k}$  is k-sequential for all k. Acharya points out that the following result of Slater's [2424] for k = 1 linking k-graceful graphs and k-sequential graphs holds in general: A graph is k-sequential if and only if G + v has a k-graceful labeling f with

f(v) = 0. Slater [2423] also proved that a k-sequential graph with p vertices and q > 0 edges must satisfy  $k \le p-1$ . Hegde [979] proved that every graph can be embedded as an induced subgraph of a simply sequential graph. In [23] Acharya conjectured that if G is a connected k-sequential graph of order p with  $k > \lfloor p/2 \rfloor$ , then k = p-1 and  $G = K_{1,p-1}$  and that, except for  $K_{1,p-1}$ , every tree in which all vertices are odd is k-sequential for all odd positive integers  $k \le p/2$ . In [979] Hegde gave counterexamples for both of these conjectures.

In [990] Hegde and Miller prove the following: for n > 1,  $K_n$  is k-sequentially additive if and only if (n, k) = (2, 1), (3, 1) or (3, 2);  $K_{1,n}$  is k-sequentially additive if and only if k divides n; caterpillars with bipartition sets of sizes m and n are k-sequentially additive for k = m and k = n; and if an odd-degree (p, q)-graph is k-sequentially additive, then  $(p+q)(2k+p+q-1) \equiv 0 \pmod{4}$ . As corollaries of the last result they observe that when m and n are odd and k is even  $K_{m,n}$  is not k-sequentially additive and if an odd-degree tree is k-sequentially additive then k is odd.

In [2240] Seoud and Jaber proved the following graphs are 1-sequentially additive: graphs obtained by joining the centers of two identical stars with an edge;  $S_n \cup S_m$  if and only if nm is even;  $C_n \odot \overline{K_m}$ ;  $P_n \odot \overline{K_m}$ ;  $kP_3$ ; graphs obtained by joining the centers of kcopies of  $P_3$  to each vertex in  $\overline{K_m}$ ; and trees obtained from K by replacing each edge by a path of length 2 when  $n \equiv 0, 1 \pmod{4}$ . They also determined all 1-sequentially additive graphs of order 6.

### 7.18 IC-colorings

For a subgraph H of a graph G with vertex set V and a coloring f from V to the natural numbers define  $f_s(H) = \Sigma f(v)$  over all  $v \in H$ . The coloring f is called an IC-coloring if for any integer k between 1 and  $f_s(G)$  there is a connected subgraph H of G such that  $f_s(H) = k$ . The IC-*index* of a graph G, M(G), is max $\{f_s | f_s \text{ is an IC-coloring of } G\}$ . Salehi, Lee, and Khatirinejad [2163] obtained the following:  $M(K_n) = 2^n - 1$ ; for  $n \geq 2^n - 1$ 2,  $M(K_{1,n}) = 2^n + 2$ ; if  $\Delta$  is the maximum degree of a connected graph G, then  $M(G) \geq 2^n$  $2^{\Delta} + 2$ ; if  $ST(n; 3^n)$  is the graph obtained by identifying the end points of n paths of length 3, then  $ST(n; 3^n)$  is at least  $3^n + 3$  (they conjecture that equality holds for  $n \ge 4$ ); for  $n \ge 2$ ,  $M(K_{2,n}) = 3 \cdot 2^n + 1$ ;  $M(P_n) \ge (2 + \lfloor n/2 \rfloor)(n - \lfloor n/2 \rfloor) + \lfloor n/2 \rfloor - 1$ ; for  $m, n \geq 2$ , the IC-index of the double star DS(m, n) is at least  $(2^{m-1}+1)(2^{n-1}+1)$  (they conjecture that equality holds); for n > 3,  $n(n+1)/2 < M(C_n) < n(n-1) + 1$ ; and for  $n \geq 3, 2^n + 2 \leq M(W_n) \leq 2^n + n(n-1) + 1$ . They pose the following open problems: find the IC-index of the graph obtained by identifying the endpoints of n paths of length b; find the IC-index of the graph obtained by identifying the endpoints of n paths; and find the IC-index of  $K_{m,n}$ . Shiue and Fu [2371] completed the partial results by Penrice [1914] Salehi, Lee, and Khatirinejad [2163] by proving  $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$ for any  $2 \leq m \leq n$ .

#### 7.19 Minimal k-rankings

A k-ranking of a graph is a labeling of the vertices with the integers 1 to k inclusively such that any path between vertices of the same label contains a vertex of greater label. The rank number of a graph G,  $\chi_r(G)$ , is the smallest possible number of labels in a ranking. A k-ranking is *minimal* if no label can be replaced by a smaller label and still be a k-ranking. The concept of the rank number arose in the study of the design of very large scale integration (VLSI) layouts and parallel processing (see [649], [1573] and [2216]). Ghoshal, Laskar, and Pillone [892] were the first to investigate minimal k-rankings from a mathematical perspective. Laskar and Pillone [1462] proved that the decision problem corresponding to minimal k-rankings is NP-complete. It is HP-hard even for bipartite graphs [662]. Bodlaender, Deogun, Jansen, Kloks, Kratsch, Müller, Tuza [479] proved that the rank number of  $P_n$  is  $\chi_r(P_n) = |\log_2(n)| + 1$  and satisfies the recursion  $\chi_r(P_n) = 1 + \chi_r(P_{\lceil (n-1)/2 \rceil})$  for n > 1. The following results are given in [662]:  $\chi_r(S_n) = 2; \ \chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2; \ \chi_r(W_n) = \lfloor \log_2(n-3) \rfloor + 3(n > 3);$  $\chi_r(K_n) = n$ ; the complete t-partite graph with n vertices has ranking number n+1 - the cardinality of the largest partite set; and a split graph with n vertices has ranking number n+1 - the cardinality of the largest independent set (a *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set.) Wang proved that for any graphs G and H  $\chi_r(G+H) = \min\{|V(G)| + \chi_r(H), |V(H) + \chi_r(G)\}.$ 

In 2009 Novotny, Ortiz, and Narayan [1871] determined the rank number of  $P_n^2$  from the recursion  $\chi_r(P_n^2) = 2 + \chi_r(P(\lceil (n-2)/2 \rceil))$  for n > 2. They posed the problem of determining  $\chi_r(P_m \times P_n)$  and  $\chi(P_n^k)$ . In 2009 [142] and [141] Alpert determined the rank numbers of  $P_n^k$ ,  $C_n^k$ ,  $P_2 \times C_n$ ,  $K_m \times P_n$ ,  $P_3 \times P_n$ , Möbius ladders and found bounds for rank numbers of general grid graphs  $P_m \times P_n$ . About the same time as Alpert and independently, Chang, Kuo, and Lin [553] determined the rank numbers of  $P_n^k$ ,  $C_n^k$ ,  $P_2 \times P_n$ ,  $P_2 \times C_n$ . Chang et al. also determined the rank numbers of caterpillars and proved that for any graphs G and  $H - \chi_r(G[H]) = \chi_r(H) + |V(H)|(\chi_r(G) - 1)$ .

In 2010 Jacob, Narayan, Sergel, Richter, and Tran [1090] investigated k-rankings of paths and cycles with pendent paths of length 1 or 2. Among their results are: for any caterpillar  $G \ \chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n) + 1$  and both cases occur; if  $2^m \leq n \leq 2^{m+1}$  then for any graph G obtained by appending edges to an n-cycle we have  $m+2 \leq \chi_r(G) \leq m+3$ and both cases occur; if G is a lobster with spine  $P_n$  then  $\chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n) + 2$ and all three cases occur; if G a graph obtained from the cycle  $C_n$  by appending paths of length 1 or 2 to any number of the vertices of the cycle then  $\chi_r(P_n) \leq \chi(G) \leq \chi(P_n) + 2$ and all three cases occur; and if G the graph obtained from the comb obtained from  $P_n$  by appending one path of length m to each vertex of  $P_n$  then  $\chi_r(G) = \chi_r(P_n) + \chi_r(P_{m+1}) - 1$ .

Sergel, Richter, Tran, Curran, Jacob, and Narayan [2266] investigated the rank number of a cycle  $C_n$  with pendent edges, which they denote by  $CC_n$ , and call a *caterpillar cycle*. They proved that  $\chi(CC_n) = \chi_r(C_n)$  or  $\chi(CC_n) = \chi_r(C_n)$ +1 and showed that both cases occur. A *comb tree*, denoted by C(n, m), is a tree that has a path  $P_n$  such that every vertex of  $P_n$  is adjacent to an end vertex of a path  $P_m$ . In the comb tree C(n, m)  $(n \ge 3)$  there are 2 pendent paths  $P_{m+2}$  and n-2 paths  $P_{m+1}$ . They proved  $\chi_r(C(n, m)) = \chi_r(P_{m+1}) - 1$ . They define a *circular lobster* as a graph where each vertex is either on a cycle  $C_n$  or at most distance two from a vertex on  $C_n$ . They proved that if G is a lobster with longest path  $P_n$ , then  $\chi_r(P_n) \leq \chi_r(G) \leq \chi_r(P_n) + 2$  and determined the conditions under which each true case occurs. If G is circular lobster with cycle  $C_n$ , they showed that  $\chi_r(C_n) \leq \chi_r(G) \leq \chi_r(C_n) + 2$  and determined the conditions under which each true case occurs. An *icicle graph*  $I_n$   $(n \geq 3)$  has three pendent paths  $P_2$  and is comprised of a path  $P_n$  with vertices  $v_1, v_2, \ldots, v_n$  where a path  $P_{i-1}$  is appended to vertex  $v_i$ . They determine the rank number for icicle graphs.

Richter, Leven, Tran, Ek, Jacob, and Narayan [2116] define a reduction of a graph  $G_S^*$  such that  $V(G_S^*) = V(G) \setminus S$  and, for vertices u and v, uv is an edge of  $G_S^*$  if and only if there exists a u - v path in G with all internal vertices belonging to S. A vertex separating set of a connected graph G is a set of vertices whose removal disconnects G. They define a bent ladder  $BL_n(a, b)$  as the union of ladders  $L_a$  and  $L_b$  (where  $L_n = P_n \times P_2$ ) that are joined at a right angle with a single  $L_2$  so that n = a+b+2. A staircase ladder  $SL_n$  is a graph with n-1 subgraphs  $G_1, G_2, \ldots, G_{n-1}$  each of which is isomorphic to  $C_4$ . (They are ladders with a maximum number of bends.) Richter et al. [2116] prove:  $\chi_r(BL_n(a, b)) = \chi_r(L_n) - 1$  if  $n = 2^k - 1$  and  $a \equiv 2$  or 3 (mod 4) and is equal to  $\chi_r(L_n)$  otherwise; and for any ladder  $L_n$  with multiple bends, the rank number is either  $\chi_r(L_n)$  or  $\chi_r(L_n) + 1$ ).

The arank number of a graph G is the maximum value of k such that G has a minimal k-ranking. Eyabi, Jacob, Laskar, Narayan, and Pillone [745] determine the arank number of  $K_n \times K_n$ , and investigated the arank number of  $K_m \times K_n$ .

### 7.20 Set Graceful and Set Sequential Graphs

The notions of set graceful and set sequential graphs were introduced by Acharaya in 1983 [24]. A graph is called *set graceful* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge, the assignment of sets to the vertices is injective, and the assignment to the edges is bijective. A graph is called *set sequential* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph such that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge and the the assignment of sets to the vertices and the edges is bijective. The following has been shown:  $P_n$  (n > 3) is not set graceful [983];  $C_n$  is not set sequential [38];  $C_n$  is set graceful if and only if  $n = 2^m - 1$ [985] and [24];  $K_n$  is set graceful if and only if n = 2, 3 or 6 [1781];  $K_n$   $(n \ge 2)$  is set sequential if and only if n = 2 or 5 [985];  $K_{a,b}$  is set sequential if and only if (a+1)(b+1)is a positive power of 2 [985]; a necessary condition for  $K_{a,b,c}$  to be set sequential is that a, b, and c cannot have the same parity [983];  $K_{1,b,c}$  is not set sequential when b and c even [985];  $K_{2,b,c}$  is not set sequential when b and c are odd [983]; no theta graph is set graceful [983]; the complete nontrivial n-ary tree is set sequential if and only if n+1 is a power of 2 and the number of levels is 1 [983]; a tree is set sequential if and only if it is set graceful

[983]; the nontrivial plane triangular grid graph  $G_n$  is set graceful if and only if n = 2[985]; every graph can be embedded as an induced subgraph of a connected set sequential graph [983]; every graph can be embedded as an induced subgraph of a connected set graceful graph [983], every planar graph can be embedded as an induced subgraph of a set sequential planar graph [985]; every tree can be embedded as an induced subgraph of a set sequential tree [985]; and every tree can be embedded as an induced subgraph of a set graceful tree [985]. Hegde conjectures [985] that no path is set sequential. Hegde's conjecture [986] that every complete bipartite graph that has a set graceful labeling is a star was proved by Vijayakumar [2732]. Shahida and Sunitha [2304] prove that the concept of set-gracefulness is equivalent to topologically set-gracefulness in trees and almost all finite trees are not set-graceful. Using this they characterize topologically set-graceful stars and topologically set-graceful paths. In [29] Acharya and Germina survey results on set-valuations of graphs and give open problems and conjectures.

Germina, Kumar, and Princy [867] prove: if a (p, q)-graph is set-sequential with respect to a set with n elements, then the maximum degree of any vertex is  $2^{n-1} - 1$ ; if G is setsequential with respect to a set with n elements other than  $K_5$ , then for every edge uvwith d(u) = d(v) one has  $d(u) + d(v) < 2^{n-1} - 1$ ;  $K_{1,p}$  is set-sequential if and only if phas the form  $2^{n-1} - 1$  for some  $n \ge 2$ ; binary trees are not set-sequential; hypercubes  $Q_n$ are not set-sequential for n > 1; wheels are not set-sequential; and uniform binary trees with an extra edge appended at the root are set-graceful and set graceful.

Vijayakumar [2732] and Gyri, Balister, and Schelp [267] proved that if a complete bipartite graph G has a set-graceful labeling, then it is a star. Abhishek [6] described a method for constructing a set-graceful bipartite graph of arbitrarily large order and size beginning with a set-graceful bipartite graph. Acharya, Germina, Princy, and Rao [34] proved that  $K_{1,m,n}$  is set-graceful if and only if  $m = 2^s - 1$  and  $n = 2^t - 1$  and almost all graphs are not set-graceful. In [7] Abhishek surveys results on set-valued graphs. Many open problems and conjectures are included.

Acharya [24] has shown: a connected set graceful graph with q edges and q+1 vertices is a tree of order  $p = 2^m$  and for every positive integer m such a tree exists; if G is a connected set sequential graph, then  $G + K_1$  is set graceful; and if a graph with p vertices and q edges is set sequential, then  $p + q = 2^m - 1$ . Acharya, Germina, Princy, and Rao [34] proved: if G is set graceful, then  $G \cup \overline{K_t}$  is set sequential for some t; if G is a set graceful graph with n edges and n + 1 vertices, then  $G + \overline{K_t}$  is set graceful if and only if m has the form  $2^t - 1$ ;  $P_n + \overline{K_m}$  is set graceful if n = 1 or 2 and m has the form  $2^t - 1$ ;  $K_{1,m,n}$  is set graceful if and only if m has the form  $2^t - 1$  and n has the form  $2^s - 1$ ;  $P_4 + \overline{K_m}$  is not set graceful when m has the form  $2^t - 1$  ( $t \ge 1$ );  $K_{3,5}$  is not set graceful; if G is set graceful, then graph obtained from G by adding for each vertex v in G a new vertex v' that is adjacent to every vertex adjacent to v is not set graceful; and  $K_{3,5}$  is not set graceful.

Acharya, Germina, Abhishek, and Slater [31] prove  $C_m$  is set-graceful if and only if  $m = (4^n - 1)/3$ ;  $mK_2$  is set-sequential if and only if  $m = (4^n - 1)/3$ ; and, for  $r + s = 2^{n-1}$  the bistar B(r, s) is set-sequential if and only if r and s are odd. They also prove that connected planar graphs with an even number of faces, regular polyhedrons, and cacti

containing an odd number of cycles are not set-sequential.

Abhishek [6] proved that if G is a set-sequential bipartite graph and H is 2k-setsequential, then  $4^k G \cup H$  is set-sequential. As a corollary, he gets  $mP_3$  is set-sequential if and only if  $m = (16^n - 1)/5$ . Abhishek and Agustine [9] characterized the set-sequential caterpillars of diameter four and give a necessary condition for a graph to be set-sequential. Abhishek [8] characterized the set-sequential caterpillars of diameter five.

In [1738] Mehra and Puneet introduce a topological integer additive set-labeling of signed graphs as follows. Let S = (V, E, s) be a signed graph with corresponding graph G = (V, E) and the signature function s. Here, G is an integer additive set-labeled graph having an injective function  $f : V(G) \to P(X) - \{\emptyset\}$  that produces another injective function  $g_f : E(G) \to P(X) - \{\emptyset, \{0\}\}$  defined by  $g_f(uv) = f(u) + f(v)$  for every edge uv, where X is the subset of non-negative integers, P(X) is its power set, and the signature function defined as  $s : E(G) \to \{+, -\}$  is such that  $s(uv) = -1^{|f(u)+f(v)|}$  for all edges uv. If  $f(V(G)) \cup \{\emptyset\}$  forms a topology on X then the signed graph S is called a *topological integer additive set-labeled signed graph* (T-IASL). They proved the following graphs have T-IASL labelings: paths, stars, double stars, tadpoles, and graphs obtained by identifying an end of a path with the center of a star.

### 7.21 Vertex Equitable Graphs

Given a graph G with q edges and a labeling f from the vertices of G to the set  $\{0, 1, 2, \dots, \lceil q/2 \rceil\}$  define a labeling  $f^*$  on the edges by  $f^*(uv) = f(u) + f(v)$ . If for all i and j and each vertex the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the edge labels induced by  $f^*$  are  $1, 2, \ldots, q$ , Lourdusamy and Seenivasan [1656] call a f a vertex equitable labeling of G. They proved the following graphs are vertex equitable: paths, bistars, combs, n-cycles for  $n \equiv 0$  or 3 (mod 4),  $K_{2,n}$ ,  $C_3^{t}$  for  $t \geq 2$ , quadrilateral snakes,  $K_2 + mK_1$ ,  $K_{1,n} \cup K_{1,n+k}$  if and only if  $1 \le k \le 3$ , ladders, arbitrary super divisions of paths, and *n*-cycles with  $n \equiv 0$ or 3 (mod 4). They further proved that  $K_{1,n}$  for  $n \geq 4$ , Eulerian graphs with n edges where  $n \equiv 1$  or 2 (mod 4), wheels,  $K_n$  for n > 3, triangular cacti with  $q \equiv 0$  or 6 or 9 (mod 12), and graphs with p vertices and q edges, where q is even and  $p < \lceil q/2 \rceil + 2$ are not vertex equitable. Lourdusamy and Patrick [1651] prove that triangular ladders  $TL_n, L_n \odot mK_1, Q_n \odot K_1, TL_n \odot K_1$ , and alternate triangular snakes  $A(T_n)$  are vertex equitable graphs. In [47] Acharya, Jain, and Kansal introduced vertex equitable labelings of signed graphs and studied vertex equitable behavior of signed paths, signed stars, and signed complete bipartite graphs  $K_{2,n}$ .

Jeyanthi and Maheswari [1171] proved that the following graphs have vertex equitable labelings: the square of the bistar  $B_{n,n}$ ; the splitting graph of the bistar  $B_{n,n}$ ;  $C_4$ -snakes; connected graphs for in which each block is a cycle of order divisible by 4 (they need not be the same order) and whose block-cut point graph is a path;  $C_m \odot P_n$ ; tadpoles; the one-point union of two cycles; and the graph obtained by starting friendship graphs,  $C_{n_1}^{(2)}, C_{n_2}^{(2)}, \ldots, C_{n_k}^{(2)}$  where each  $n_i \equiv 0 \pmod{4}$  and joining the center of  $C_{n_i}^{(2)}$  to the center of  $C_{i+1}^{(2)}$  with an edge for  $i = 1, 2, \ldots, k - 1$ . In [1161] Jeyanthi and Maheswari prove that  $T_p$  trees, bistars B(n, n + 1),  $C_n \odot K_m$ ,  $P_n^2$ , tadpoles, certain classes of caterpillars, and  $T \odot \overline{K_n}$  where T is a  $T_p$  tree with an even number of vertices are vertex equitable. Jeyanthi and Maheswari [1164] gave vertex equitable labelings for graphs constructed from  $T_p$  trees by appending paths or cycles. Jeyanthi and Maheswari [1160] proved that graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle, total graphs of a paths, splitting graphs of paths, and the graphs obtained identifying an edge of one cycle with an edge of another cycle are vertex equitable (see §2.7 for the definitions of duplicating vertices and edges, a total graph, and a splitting graph.)

For a graph H with vertices  $v_1, v_2, \ldots, v_n$  and n copies of a graph G,  $H \circ G$  is a graph obtained by identifying a vertex  $u_i$  of the *i*th copy of G with a vertex  $v_i$  of H for  $1 \leq i \leq n$ . The graph  $H \circ G$  is a graph obtained by joining a vertex  $u_i$  of the *i*th copy of G with a vertex  $v_i$  of H by an edge for  $1 \leq i \leq n$ . Jeyanthi, Maheswari, and Laksmi prove [1187] that the graphs  $L_m \circ nC_4$ ,  $L_m \circ nC_4$ ,  $C_m \circ nC_4$ , and  $P_m \circ nC_4$  are vertex equitable graphs. The graph  $S^*(G)$  is obtained from a graph G by replacing every edge e of G with  $K_{2,m}$  ( $m \geq 2$ ) with the endpoints of e merged with the two vertices of the 2-vertices part of  $K_{2,m}$  after removing the edge e from G. Jeyanthi, Maheswari, and Vijaya Laksmi [1183] prove the graphs  $S^*(P_n \cdot K_1)$ ,  $S^*(B(n, n))$ ,  $S^*(P_n \times P_2)$ , and  $S^*(Q_n)$  of the quadrilateral snake are vertex equitable.

In [1168] Jeyanthi and Maheswari proved the double alternate triangular snake  $DA(T_n)$ obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to two new vertices  $v_i$  and  $w_i$  is vertex equitable; the double alternate quadrilateral snake  $DA(Q_n)$ obtained from a path  $u_1, u_2, \ldots, u_n$  by joining  $u_i$  and  $u_{i+1}$  (alternatively) to two new vertices  $v_i, x_i$  and  $w_i, y_i$  respectively and then joining  $v_i, w_i$  and  $x_i, y_i$  is vertex equitable; and NQ(m) the  $n^{th}$  quadrilateral snake obtained from the path  $u_1, u_2, \ldots, u_m$  by joining  $u_i, u_{i+1}$  with 2n new vertices  $v_j^i$  and  $w_j^i, 1 \le i \le m - 1, 1 \le j \le n$  is vertex equitable. Jeyanthi and Maheswari [1181] prove  $DA(T_n) \odot K_1$ ,  $DA(T_n) \odot 2K_1$ ,  $DA(T_n)$ ,  $DA(Q_n) \odot$  $K_1$ ,  $DA(Q_n) \odot 2K_1$ , and  $DA(Q_n)$  are vertex equitable.

In [1167] and [1169] Jeyanthi and Maheswari show a number of families of graphs have vertex equitable labelings. Their results include: armed crowns  $C_m \odot P_n$ , shadow graphs  $D_2(K_{1,n})$ ; the graph  $C_m * C_n$  obtained by identifying a single vertex of a cycle graph  $C_m$ with a single vertex of a cycle graph  $C_n$  if and only if  $m + n \equiv 0,3 \pmod{4}$ ; for  $n \equiv 0 \pmod{4}$  the graph obtained from m copies of  $C_n * C_n$  and  $P_m$  by joining each vertex of  $P_m$  with the cut vertex in one copy of  $C_n * C_n$ ; and graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle; the total graph of  $P_n$ ; the splitting graph of  $P_n$ ; and the fusion of two edges of  $C_n$ .

Jeyanthi, Maheswari, and Vijayalaksmi [1182] proved the following graphs are vertex equitable: jewel graphs  $J_n$  with vertex set  $\{u, v, x, y, u_i : 1 \leq i \leq n\}$  and edge set  $\{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \leq i \leq n\}$ ; jelly fish graphs  $(JF)_n$  with vertex set  $\{u, v, u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n-2\}$  and edge set  $\{uu_i : 1 \leq i \leq n\} \cup \{vv_j : 1 \leq i \leq n-2\} \cup \{u_{n-1}u_n, vu_n, vu_{n-1}\}$ ; lobsters constructed from the path  $a_1, a_2, \ldots, a_n$  with vertices  $a_{i1}$  and  $a_{i2}$  adjacent to  $a_i$  for  $1 \leq i \leq n$  and pendent vertices  $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^k$ joining  $a_{ij}$  for  $1 \leq i \leq n$  and j = 1, 2;  $L_n \odot \overline{K_m}$ ; and the graph obtained from ladder a  $L_n$  and 2n copies of  $K_{1,m}$  by identifying a non-central vertex of *i*th copy of  $K_{1,m}$  with *i*th vertex of  $L_n$ .

Jeyanthi, Mahewari, and Vijaya Laksmi [1178] prove the following graphs are vertex equitable: graphs obtained by joining a vertex of a cycle to a degree 2 vertex of a comb  $(P_n \odot K_1)$  with an edge; path unions of quadrilateral snakes; cycle unions of n copies of  $mC_4$ -snakes where  $n \equiv 0, 3 \mod 4$ ; the graphs obtained from a path  $u_1, u_2, \ldots, u_m$  by joining the end points of each edge  $u_i u_{i+1}$  to 2n isolated vertices  $v_j^i, w_j^i$  for  $1 \le m-1, 1 \le j \le n$ , where n is even (the nth quadrilateral snake).

Jeyanthi, Maheswari, and Vijaya Laksmi [1178] prove that subdivisions of double triangular snakes S(D(Tn)), subdivisions of double quadrilateral snakes  $S(D(Q_n))$ , subdivisions of double alternate triangular snakes  $S(DA(T_n))$ , subdivisions of double alternate quadrilateral snakes  $S(DA(Q_n))$ ,  $DA(Q_m) \odot nK_1$ , and  $DA(T_m) \odot nK_1$  admit vertex equitable labelings.

The super subdivision graph  $S^*(G)$  of a graph G is the graph obtained from G by replacing every edge uv of G by  $K_{2,m}$  (m may vary for each edge) and identifying u and v with the two vertices in  $K_{2,m}$  that form the partite set with exactly two members. Jeyanthi, Maheswari, and Vijayalaksmi [1183] prove that super subdivision graphs of  $P_n \odot K_1$ , bistars B(n, n),  $P_n \times P_2$ , and quadrilateral snakes are vertex equitable.

For a graph H with vertices  $v_1, v_2, \ldots, v_n$  and n copies of a graph G,  $H \circ G$  is a graph obtained by identifying a vertex  $u_i$  of the *i*th copy of G with a vertex  $v_i$  of H for  $1 \leq i \leq n$ . The graph  $H \circ G$  is a graph obtained by joining a vertex  $u_i$  of the *i*th copy of G with a vertex  $v_i$  of H by an edge for  $1 \leq i \leq n$ . Jeyanthi, Maheswari, and Laksmi [1187] prove that the graphs  $L_m \circ nC_4$ ,  $L_m \circ nC_4$ ,  $C_m \circ nC_4$  and  $P_m \circ nC_4$  are vertex equitable graphs.

For a graph G with p vertices and q edges and  $A = \{1, 3, \dots, q\}$  if q is odd or  $A = \{1, 3, \dots, q\}$  $\{1, 3, \ldots, q+1\}$  if q is even Jeyanthi, Maheswari and Vijaya Laksmi [1177] say a vertex labeling f from V(G) to A is an odd vertex equitable even labeling if the induced edge labeling  $f^*$  defined by  $f^*(uv) = f(u) + f(v)$  for all edges uv has the property that for all u and v in A the number of vertices labeled with u and the number of vertices labeled with v differ by at most 1 and the induced edge labels are  $2, 4, \ldots, 2q$ . A graph that admits odd vertex equitable even labeling is called an *odd vertex equitable even* graph. They show that the following graphs have odd vertex equitable even lableings: paths, graphs obtained by identifying an endpoint of  $P_m$  with each vertex of  $P_n$ ,  $K_{1,n}$  if and only if n = 1 or 2,  $K_{1,n} \cup K_{1,n-2}$   $(n \ge 3)$ ,  $K_{2,n}$ ,  $T_p$ -trees,  $C_n$  when  $n \equiv 0$  or 1 mod 4, quadrilateral snakes, ladders  $L_n$ ,  $L_n \odot K_1$ , and arbitrary super subdivision of paths. They prove that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph. As a corollary of this they get that the following are not odd vertex equitable even graphs:  $K_n$   $(n \geq 3)$ , wheels, triangular snakes, double triangular snakes, triangular ladders, flower graphs, fans  $P_n \odot K_1$   $(n \ge 2)$ , double fans  $P_n \odot K_2$ ,  $(n \ge 2)$ , friendship graphs  $C_n^3$ , windmills  $K_m^n$  (m > 3),  $K_2 + mK_1$ ,  $B_{n,n}^2$ , total graphs  $T(P_n)$ , and composition graphs  $P_n[P_2]$ . They also show that if G is a (p,q) graph with  $p \leq p$  $\lceil q/2 \rceil + 1$ , then G is not an odd vertex equitable even graph. Jeyanthi, Maheswari, and Vijava Laksmi [1188] gave odd vertex equitable even labelings for ladder related families of graphs. Jeyanthi and Maheswari [1172] proved that the subdivision of double triangular

snakes and the subdivision of double quadrilateral snakes are odd vertex equitable even graphs. Lourdusamy and Patrick [1650] proved that  $P_n \odot mK_1$ , the quadrilateral snake attached to each vertex of path  $P_n$ , the super splitting graph  $S^*(P_n \odot K_1)$ , the super splitting graphs of ladders and the bistars  $B_{n,n}$ ,  $B_{n,n}^2$ , and the splitting  $S'(B_{n,n})$  admit even vertex equitable even labelings. Lourdusamy, Wency, and Patrick [1662] prove that  $S(D(Q_n)), S(D(T_n)), DA(Q_m) \odot nK_1, DA(T_m) \odot nK_1, S(DA(Q_n))$  and  $S(DA(T_n))$  are an even vertex equitable even graphs.

For graphs  $G_1$  and  $G_2$  that graph  $G_1OG_2$  is obtained from  $G_1$  and  $|VG_1\rangle|$  copies of  $G_2$ by identifying one vertex of *i*th copy of  $G_2$  with *i*th vertex of  $G_1$ . Jeyanthi, Maheswari, and Vijayalakshmi [1190] proved the following graphs have odd vertex equitable even labelings: subdivision graphs of ladders,  $L_m \hat{O}P_n$ ,  $L_n \odot \overline{K_m}$  (m > 1),  $C_n$  if and only if  $n \equiv 0$  or 1 (mod 4),  $K_{1,n+k} \cup K_{1,n}$  i if and only if k = 1, 2, and  $\langle L_n \hat{O}K_{1,m} \rangle$ .

Motivated by the concept of vertex equitable labeling first defined by Lourdusamy and Seenivasan in [1656], Lourdusamy, Mary, and Patrick [1648] introduced the concept of even vertex equitable even labeling as follows. Let G be a graph with p vertices and q edges and  $A = \{0, 2, 4, \dots, q+1\}$  if q is odd or  $A = \{0, 2, 4, \dots, q\}$  if q is even. A graph G is said to be an even vertex equitable even labeling if there exists a vertex labeling f from V(G)to A that induces an edge labeling f defined by  $f^*(uv) = f(u) + f(v)$  for all edges uv such that for all a and b in A,  $|v_f(a) - v_f(b)| \leq 1$  and the induced edge labels are  $2, 4, \ldots, 2q$ , where  $v_f(a)$  is the number of vertices v with f(v) = a for  $a \in A$ . A graph that admits even vertex equitable even labeling is called an *even vertex equitable even* graph. They proved that paths, combs, complete bipartite graphs, cycles,  $K_2 + mK_1$ , bistars, ladders,  $(P_n \times P_2) \odot K_1$ , and the subdivision graphs of ladders and bistars  $B_{n,n}$  admit an even vertex equitable even labeling. In [1654] Lourdusamy and Patrick proved that  $C_m \odot P_n$ ,  $C_{4n}$  and  $C_{4n+3}$  with a quadrilateral snake attached to each vertex of the cycle, the graphs obtained by indentifying an edge of  $C_m$  and  $C_n$ , and the graphs obtained by duplicating an arbitrary vertex and edge of a cycle admit an even vertex equitable even labeling. Lourdusamy, Shobana Mary, and Patrick [1655] proved  $P_n^2$ ,  $S(P_n \odot K_1)$ ,  $S'(P_n)$ ,  $T(P_n)$ , graphs obtained by duplication of an egde of a path, quadrilateral snakes,  $D(Q_n)$ ,  $A(T_n)$ , and  $DA(T_n)$  have even vertex equitable even labelings. Lourdusamy and Patrick [1650] proved that  $P_n \odot mK_1$ , the quadrilateral snake attached to each vertex of path  $P_n$ , the super splitting graph  $S^*(P_n \odot K_1)$ , the super splitting graphs of ladders and the bistars  $B_{n,n}, B_{n,n}^2$ , and the splitting  $S'(B_{n,n})$  admit even vertex equitable even labelings.

### 7.22 Sequentially Additive Graphs

Bange, Barkauskas, and Slater [343] defined a k-sequentially additive labeling f of a graph G(V, E) to be a bijection from  $V \cup E$  to  $\{k, \ldots, k + |V \cup E| - 1\}$  such that for each edge xy, f(xy) = f(x) + f(y). They proved:  $K_n$  is 1-sequentially additive if and only if  $n \leq 3$ ;  $C_{3n+1}$  is not k-sequentially additive for  $k \equiv 0$  or 2 (mod 3);  $C_{3n+2}$  is not k-sequentially additive for  $k \equiv 1$  or 2 (mod 3);  $C_n$  is 1-sequentially additive if and only if  $n \equiv 0$  or 1 (mod 3); and  $P_n$  is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive. Hegde [981] proved that  $K_{1,n}$  is k-sequentially additive if and

only if k divides n.

Hajnal and Nagy [947] investigated 1-sequentially additive labelings of 2-regular graphs. They prove:  $kC_3$  is 1-sequentially additive for all k;  $kC_4$  is 1-sequentially additive if and only if  $k \equiv 0$  or 1 (mod 3);  $C_{6n} \cup C_{6n}$  and  $C_{6n} \cup C_{6n} \cup C_3$  are 1-sequentially additive for all n;  $C_{12n}$  and  $C_{12n} \cup C_3$  are 1-sequentially additive for all n. They conjecture that every 2-regular simple graph on n vertices is 1-sequentially additive where  $n \equiv 0$  or 1 (mod 3).

Acharya and Hegde [39] have generalized k-sequentially additive labelings by allowing the image of the bijection to be  $\{k, k+d, \ldots, (k+|V\cup E|-1)d\}$ . They call such a labeling additively (k, d)-sequential.

## 7.23 Difference Graphs

Analogous to a sum graph, Harary [958] calls a graph a difference graph if there is an bijection f from V to a set of positive integers S such that  $xy \in E$  if and only if  $|f(x) - f(y)| \in S$ . Bloom, Hell, and Taylor [474] have shown that the following graphs are difference graphs: trees,  $C_n$ ,  $K_n$ ,  $K_{n,n}$ ,  $K_{n,n-1}$ , pyramids, and n-prisms. Gervacio [872] proved that wheels  $W_n$  are difference graphs if and only if n = 3, 4, or 6. Sonntag [2457] proved that cacti (that is, graphs in which every edge is contained in at most one cycle) with girth at least 6 are difference graphs and he conjectures that all cacti are difference graphs. Suggeng and Ryan [2499] provided difference labelings for cycles; fans; cycles with chords; graphs obtained by the one-point union of  $K_n$  and  $P_m$ ; and graphs made from any number of copies of a given graph G that has a difference labeling by identifying one vertex the first with a vertex of the second, a different vertex of the second with the third and so on.

Hegde and Vasudeva [1005] call a simple digraph a mod difference digraph if there is a positive integer m and a labeling L from the vertices to  $\{1, 2, \ldots, m\}$  such that for any vertices u and v, (u, v) is an edge if and only if there is a vertex w such that  $L(v) - L(u) \equiv L(w) \pmod{m}$ . They prove that the complete symmetric digraph and unidirectional cycles and paths are mod difference digraphs.

In [2239] Seoud and Helmi provided a survey of all graphs of order at most 5 and showed the following graphs are difference graphs:  $K_n$ ,  $(n \ge 4)$  with two deleted edges having no vertex in common;  $K_n$ ,  $(n \ge 6)$  with three deleted edges having no vertex in common; gear graphs  $G_n$  for  $n \ge 3$ ;  $P_m \times P_n$   $(m, n \ge 2)$ ; triangular snakes;  $C_4$ -snakes; dragons (that is, graphs formed by identifying the end vertex of a path and any vertex in a cycle); graphs consisting of two cycles of the same order joined by an edge; and graphs obtained by identifying the center of a star with a vertex of a cycle.

### 7.24 Square Sum Labelings and Square Difference Labelings

Ajitha, Arumugam, and Germina [133] call a labeling f from a graph G(p,q) to  $\{1, 2, \ldots, q\}$  a square sum labeling if the induced edge labeling  $f^*(uv) = (f(u))^2 + (f(v))^2$  is injective. They say a square sum labeling is a strongly square sum labeling if the q edge

labels are the first q consecutive integers of the form  $a^2 + b^2$  where a and b are less than p and distinct. They prove the following graphs have square sum labelings: trees; cycles;  $K_2 + mK_1$ ;  $K_n$  if and only if  $n \leq 5$ ;  $C_n^{(t)}$  (the one-point union of t copies of  $C_n$ ); grids  $P_m \times P_n$ ; and  $K_{m,n}$  if  $m \leq 4$ . They also prove that every strongly square sum graph except  $K_1, K_2$ , and  $K_3$  contains a triangle.

In [881] Ghodasara and Patel gave a counterexample to the conjecture by Germina and Sebastian[871] that if  $G_1$  and  $G_2$  are square sum graphs then  $G_1 \cup G_2$  is a square sum graph. They proved that the duplication graphs of any vertex of the following graphs are square sum graphs:  $K_n$  if and only if  $n \leq 7$ , the Petersen graph P(5,2),  $K_{1,n}$ , and  $C_n$ They also proved that cycle  $C_n$  with  $[\frac{n}{2}]$  concurrent chords is a square sum graph.

In [878] Ghodasara and Patel proved that the following constructions based on the bistar  $B_{n,n}$  are square sum graphs: the restricted square, the splitting graph, the shadow graph, the degree splitting graph, the arbitrary super subdivision graph, and the duplication of any vertex of  $B_{n,n}$ . They defined restricted total graph of  $B_{n,n}$  as a graph with vertex set =  $V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v_i, u'_i, v'_i/1 \leq i \leq n\}$ , where u and v are apex vertices,  $u_i$  and  $v_i$  are pendent vertices,  $w, u'_i$  and  $v'_i$  are vertices corresponding to the edges of  $B_{n,n}$  and edge set =  $E(B_{n,n}) \cup \{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \leq i \leq n\}$ . They also defined restricted middle graph of  $B_{n,n}$  as a graph with vertex set =  $V(B_{n,n}) \cup E(B_{n,n}) = \{u, v, w, u_i, v'_i/1 \leq i \leq n\}$ , where u and v are apex vertices,  $u_i$  and  $v_i$  are pendent vertices,  $w, u'_i$  are vertices corresponding to the edges of  $B_{n,n}$  and edge set =  $\{uv, w, u_i, v_i, u'_i, v'_i/1 \leq i \leq n\}$ , where u and v are apex vertices,  $u_i$  and  $v_i$  are pendent vertices,  $w, u'_i$  are vertices corresponding to the edges of  $B_{n,n}$  and edge set =  $\{uw, vw, wu'_i, wv'_i, uu'_i, vv'_i, u_iu'_i, v_iv'_i, /1 \leq i \leq n\}$ . They proved that restricted total graph and restricted middle graph of  $B_{n,n}$  are square sum graphs.

Germina and Sebastian [870] proved that the following graphs are square sum graphs: trees; unicyclic graphs;  $mC_n$ ; cycles with a chord; the graphs obtained by joining two copies of cycle  $C_n$  by a path  $P_k$ ; and graphs that are a path union of k copies of  $C_n$  and the path is  $P_2$ . In [2227] Seoud and Al-Harere give several necessary conditions for a graph to be a square sum graph and show that  $2C_n$ ,  $P_{2n}$ , and  $C_{2n}$  are square sum graphs. Huilgol and Sriram [1034] prove that if  $G_1$  and  $G_2$  are square sum, then  $G_1 \cup G_2 \cup G_3$  is also square sum, where  $G_3$  is a set of isolated vertices.

In [2444] Somashekara and Veena used the term "square sum labeling" to mean "strongly square sum labeling." They proved that the following graphs have strongly square sum labelings: paths,  $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$ , complete *n*-ary trees, and lobsters obtained by joining centers of any number of copies of a star to a new vertex. They observed that that if every edge of a graph is an edge of a triangle then the graph does not have strongly square sum labeling. As a consequence, the following graphs do not have a strongly square sum labelings:  $K_n, n \ge 3$ ; wheels; fans  $P_n + K_1$   $(n \ge 2)$ ; double fans  $P_n + K_2$   $(n \ge 2)$ ; friendship graphs  $C_3^{(n)}$ ; windmills  $K_m^{(n)}$  (m > 3); triangular ladders; triangular snakes; double triangular snakes; and flowers. They also proved that helms are not strongly square sum graphs and the graphs obtained by joining the centers of two wheels to a new vertex are not strongly square sum graphs.

In [2453] Sonchhatra and Ghodasara call a (p, q)-graph G = (V, E) sum perfect square if there exists a bijection f from V to  $\{0, 1, 2, ..., p-1\}$  such that the function  $f^*$  from Edefined by  $f^*(uv) = (f(u)) + (f(v))^2$  for all edges uv is an injection. Such an f is called a sum perfect square labeling of G. In a series of four papers the following graphs are proved to be sum perfect square graphs: cycles, cycles with one chord, cycles with twin chords, trees [2453]; several snake related graphs [2454];  $K_{1,n} + K_1$ ,  $K_2 + mK_1$ ,  $C_n \odot K_1$ , graphs obtained from  $K_{1,n}$  with endpoint verticies  $v_1, v_2, \ldots, v_n$  by joining  $v_i$  and  $v_{i+1}$  with an edge for  $i = 1, 2, \ldots, v_{\lfloor n/2 \rfloor}$  ("half wheel"), the middle graphs of paths, the total graphs of paths [2455];  $P^2$  (n > 1),  $mK_{1,n}$ ,  $mC_n$ ,, and the splitting graph and the shadow graph of a star [2452]. In [2455] they prove that the union of two stars and that for any sum perfect square graph  $G, G \cup P_n$  is sum perfect square. They conjecture that the union of any two sum perfect square graphs is sum perfect square.

Ajitha, Princy, Lokesha, and Ranjini [105] defined a graph G(p,q) to be a square difference graph if there exist a bijection f from V(G) to  $\{0, 1, 2, \ldots, p-1\}$  such that the induced function  $f^*$  from E(G) to the natural numbers given by  $f^*(uv) = |(f(u))^2 - (f(v))^2|$  for every edge uv of G is a bijection. Such a the function is called a square difference labeling of the graph G. They proved that following graphs have square difference labelings: paths, stars, cycles,  $K_n$  if and only if  $n \leq 5$ ,  $K_{m,n}$  if  $m \leq 4$ , friendship graphs  $C_3^{(n)}$ , triangular snakes, and  $K_2 + mK_1$ . They also prove that every graph can be embedded as a subgraph of a connected square difference graph and conjecture that trees, complete bipartite graphs and  $C_k^{(n)}$  are square difference graphs.

Tharmaraj and Sarasija [2585] proved that following graphs have square difference labelings: fans  $F_n$   $(n \ge 2)$ ;  $P_n + \overline{K_2}$ ; the middle graphs of paths and cycles; the total graph of a path; the graphs obtained from m copies of an odd cycle and the path  $P_m$  with consecutive vertices  $v_1, v_2, \ldots, v_m$  by joining the vertex  $v_i$  to a vertex of the  $i^{th}$  copy of the odd cycle; and the graphs obtained from m copies of the star  $S_n$  and the path  $P_m$  by joining the vertex  $v_i$  of  $P_m$  to the center of the  $i^{th}$  copy of  $S_n$ . Sebastian and Germina [2196] proved that certain planar graphs and higher order level joined planar grid admit square sum labeling. They also study square sum properties of several classes of graphs with many odd cycles.

Vaghela and Parmar [2717] say a graph G admits a difference perfect square cordial labeling if there is a bijection  $f: V(G) \to \{1, 2, \ldots, |V(G|\}\)$  such that for each edge uv the induced map  $f^*: E(G) \to \{0, 1\}$  defined by  $f^*(uv) = 1$  if  $u^2 - 2uv + v^2 = 1$ , and 0 otherwise, has the property that the number of edges labeled with 0 and the number of edges labeled with 1 difference by at most 1. A graph that admits a difference perfect square cordial labeling is said to be a difference perfect square cordial graph. They obtained difference perfect square cordial labelings for paths, cycles, wheels, fans, combs, crowns,  $(C_m \odot K_1) \cup (P_n \odot K_1)$ ,  $D_2(P_n)$ ,  $P_n^2$ ,  $K_2 \odot C_n$ , graphs consisting of two copies of  $C_n$  that share a common edge, the vertex switching of  $C_n$ , and the graph obtained by starting with  $P_n$   $(n \ge 6)$  and two new vertices u and v of  $P_n$  and joining v to first two vertices and last two vertices of  $P_n$  and joining u to the remaining vertices of  $P_n$  (called the shipping graph). In [2718] Vaghela and Parmar provided difference perfect square cordial labelings of the H-graphs of paths, some corona graphs, total graphs of paths, and graphs obtained from  $P_n \times P_2$  where  $P_n$  has conscutive vertices  $v_1, v_2, \ldots, v_n$  by joining  $v_i$  in left  $P_n$  to  $v_{i+1}$  in the right copy of  $P_n$  with an edge for  $i = 1, 2, \ldots, n - 1$ .

Sharon Philomena and Thirusangu [1929] proved the cycle cactus graph  $C_n^{(3)}$ , the tree

of diameter 4 obtained from the bistar  $B_{n,n}$  by subdividing the middle edge with a new vertex, and the graph obtained by joining one vertex of a cycle and one vertex of degree 2 of a comb by an edge have square and cube difference labelings (that is, the absolute cube difference of end-vertices of the edges are distinct). Sherman [2327] proved the path union of  $nC_3$  and the disjoint union of m stars  $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}$  are square difference graphs

Subashini, Bhuvaneswari, and Manimekalai [2475] proved the following graphs have square difference labelings: theta graphs, the duplication of any vertex of degree 3 in the cycle of a theta graph, the one point union of any number of theta graphs, the path union of any number of copies of a theta graph, the fusion of any two vertices in the cycle of a theta graph, and the switching of a central vertex of a theta graph.

### 7.25 Permutation and Combination Graphs

Hegde and Shetty [999] define a graph G with p vertices to be a *permutation graph* if there exists a injection f from the vertices of G to  $\{1, 2, 3, \ldots, p\}$  such that the induced edge function  $g_f$  defined by  $g_f(uv) = f(u)!/|f(u) - f(v)|!$  is injective. They say a graph G with p vertices is a *combination graph* if there exists a injection f from the vertices of G to  $\{1, 2, 3, \ldots, p\}$  such that the induced edge function  $g_f$  defined as  $g_f(uv) =$ 

f(u)!/|f(u) - f(v)|!f(v)! is injective. They prove:  $K_n$  is a permutation graph if and only if  $n \leq 5$ ;  $K_n$  is a combination graph if and only if  $n \leq 5$ ;  $C_n$  is a combination graph for n > 3;  $K_{n,n}$  is a combination graph if and only if  $n \leq 2$ ;  $W_n$  is a not a combination graph for  $n \leq 6$ ; and a necessary condition for a (p,q)-graph to be a combination graph is that  $4q \leq p^2$  if p is even and  $4q \leq p^2 - 1$  if p is odd. They strongly believe that  $W_n$  is a combination graph for  $n \geq 7$  and all trees are combinations graphs. Baskar Babujee and Vishnupriya [414] prove the following graphs are permutation graphs:  $P_n$ ;  $C_n$ ; stars; graphs obtained adding a pendent edge to each edge of a star; graphs obtained by joining the centers of two identical stars with an edge or a path of length 2); and complete binary trees with at least three vertices. Seoud and Salim [2251] determine all permutation graphs of order at most 9 and prove that every bipartite graph of order at most 50 is a permutation graph. Seoud and Mahran [2242] give an upper bound on the number of edges of a permutation graph. They show that these conditions are not sufficient for a graph to be a permutation graph.

Ghodasara and Patel [880] proved that the following graphs are permutation graphs: the Petersen graph P(5, 2), trees,  $K_{3,n}$   $(n \ge 1)$  for n+3 prime,  $W_n$   $(n \ge 3)$  for n+1 prime, shell graph  $S_n$   $(n \ge 3)$  for prime n, dumbbell graph  $D_{n,k,2}$   $(n,k \ge 3)$ ,  $C_n \odot K_1$   $(n \ge 3)$ , and the one point union  $C_n^{(k)}$   $(k \ge 2, n \ge 3)$  of k copies of cycle  $C_n$ . A t-ply  $P_t(u, v)$ is a graph with t paths, each of length at least two and such that no two paths have a vertex in common except for the end vertices u and v. Ghodasara and Patel defined  $t^*$ -ply  $P_{t^*}(u, v)$  as a special case of t-ply  $P_t(u, v)$  graph with every t path have same length and proved that  $t^*$ -ply  $P_{t^*}(u, v)$  is a permutation graph.

The graph obtained from two copies of an (m, n) kite graph by connecting the degree

1 vertex of one copy to the vertex of degree 3 and the second coppy is called the 1-*join* (m,n) kite. The graph ontained by repeating this construction with t copies of an (m,n) kite is called a is called the t-*join* (m,n) kite. Sriramr and Govindarajan [2465] proved t-join (m,n) kites are permutation graphs.

Ghodasara and Patel [879] proved that the following graphs are combination graphs:  $C_n \times P_2$  for  $n \ge 6$ , umbrella graph U(m, n) for m, n > 2, armed crown  $C_n \oplus P_m$  for  $n \ge 4$  and  $m \ge 1$ , the graphs obtained by joining  $C_{2m}$  ( $m \ge 2$ ) to each pendent vertex of  $K_{1,n}$  ( $n \ge 2$ ), the duplication of any rim vertex of  $W_n$  for  $n \ge 7$ ,  $C_n$  with  $\left[\frac{n-4}{2}\right]$  concurrent chords for  $n \ge 6$ , and the duplication of vertex in  $C_n$  for  $n \ge 5$ .

Hegde and Shetty [999] say a graph G with p vertices and q edges is a strong kcombination graph if there exists a bijection f from the vertices of G to  $\{1, 2, 3, \ldots, p\}$ such that the induced edge function  $g_f$  from the edges to  $\{k, k + 1, \ldots, k + q - 1\}$  defined by  $g_f(uv) = f(u)!/|f(u) - f(v)|!f(v)!$  is a bijection. They say a graph G with p vertices and q edges is a strong k-permutation graph if there exists a bijection f from the vertices of G to  $\{1, 2, 3, \ldots, p\}$  such that the induced edge function  $g_f$  from the edges to  $\{k, k + 1, \ldots, k + q - 1\}$  defined by  $g_f(uv) = f(u)!/|f(u) - f(v)|!$  is a bijection. Seoud and Anwar [2229] provided necessary conditions for combination graphs, permutation graphs, strong k-combination graphs, and strong k-permutation graphs.

Seoud and Al-Harere [2228] showed that the following families are combination graphs: graphs that are two copies of  $C_n$  sharing a common edge; graphs consisting of two cycles of the same order joined by a path; graphs that are the union of three cycles of the same order; wheels  $W_n$   $(n \ge 7)$ ; coronas  $T_n \odot K_1$ , where  $T_n$  is the triangular snake; and the graphs obtained from the gear  $G_m$  by attaching n pendent vertices to each vertex which is not joined to the center of the gear. They proved that a graph G(n,q) having at least 6 vertices such that 3 vertices are of degree 1, n - 1, n - 2 is not a combination graph, and a graph G(n,q) having at least 6 vertices such that there exist 2 vertices of degree n - 3, two vertices of degree 1 and one vertex of degree n - 1 is not a combination graph.

Seoud and Al-Harere [2226] proved that the following families are combination graphs: unions of four cycles of the same order; double triangular snakes; fans  $F_n$  if and only if  $n \ge 6$ ; caterpillars; complete binary trees; ternary trees with at least 4 vertices; and graphs obtained by identifying the pendent vertices of stars  $S_m$  with the paths  $P_{n_i}$ , for  $1 \le n_i \le m$ . They include a survey of trees of order at most 10 that are combination graphs and proved the following graphs are not combination graphs: bipartite graphs with two partite sets with  $n \ge 6$  elements such that n/2 elements of each set have degree n; the splitting graph of  $K_{n,n}$   $(n \ge 3)$ ; and certain chains of two and three complete graphs. Seoud and Anwar [2229] proved the following graphs are combination graphs: dragon graphs (the graphs obtained from by joining the endpoint of a path to a vertex of a cycle); triangular snakes  $T_n$   $(n \ge 3)$ ; wheels; and the graphs obtained by adding kpendent edges to every vertex of  $C_n$  for certain values of k.

In [2225] and [2226] Seoud and Al-Harere proved the following graphs are non-combination graphs:  $G_1 + G_2$  if  $|V(G_1)|, |V(G_2)| \ge 2$  and at least one of  $|V(G_1)|$  and  $|V(G_2)|$  is greater than 2; the double fan  $\overline{K_2} + P_n$ ;  $K_{l,m,n}$ ;  $K_{k,l,m,n}$ ;  $P_2[G]$ ;  $P_3[G]$ ;  $C_3[G]$ ;  $C_4[G]$ ;  $K_m[G]$ ;  $W_m[G]$ ; the splitting graph of  $K_n$   $(n \ge 3)$ ;  $K_n$   $(n \ge 4)$  with an edge deleted;  $K_n$   $(n \ge 5)$  with three edges deleted; and  $K_{n,n}$   $(n \ge 3)$  with an edge deleted. They also proved that a graph G(n,q)  $(n \ge 3)$  is not a combination graph if it has more than one vertex of degree n-1.

In [2587] and [2586] Tharmaraj and Sarasija defined a graph G(V, E) with p vertices to be a beta combination graph if there exist a bijection f from V(G) to  $\{1, 2, \ldots, p\}$ such that the induced function  $B_f$  from E(G) to the natural numbers given by  $B_f(uv) =$ (f(u) + f(v))!/f(u)!f(v)! for every edge uv of G is injective. Such a function is called a beta combination labeling. They prove the following graphs have beta combination labelings:  $K_n$  if and only if  $n \leq 8$ ; ladders  $L_n$   $(n \geq 2)$ ; fans  $F_n$   $(n \geq 2)$ ; wheels; paths; cycles; friendship graphs;  $K_{n,n}$   $(n \geq 2)$ ; trees; bistars;  $K_{1,n}$  (n > 1); triangular snakes; quadrilateral snakes; double triangular snakes; alternate triangular snakes (graphs obtained from a path  $v_1, v_2, \ldots, v_n$ , where for each odd  $i \leq n - 1$ ,  $v_i$  and  $v_{i+1}$  are joined to a new vertex  $u_{i,i+1}$ ; alternate quadrilateral snakes (graphs obtained from a path  $v_1, v_2, \ldots, v_n$ , where for each odd  $i \leq n - 1$ ,  $v_i$  and  $v_{i+1}$  are joined to two new vertices  $u_{i,i+1,1}$  and  $u_{i,i+1,2}$ ; helms; gears; combs  $P_n \odot K_1$ ; and coronas  $C_n \odot K_1$ .

## 7.26 Strongly \*-graphs

A variation of strong multiplicity of graphs is a strongly \*-graph. A graph of order n is said to be a *strongly* \*-graph if its vertices can be assigned the values  $1, 2, \ldots, n$  in such a way that, when an edge whose vertices are labeled i and j is labeled with the value i + j + ij, all edges have different labels. Adiga and Somashekara [53] have shown that all trees, cycles, and grids are strongly \*-graphs. They further consider the problem of determining the maximum number of edges in any strongly \*-graph of given order and relate it to the corresponding problem for strongly multiplicative graphs. In [2244] and [2245] Seoud and Mahan give some technical necessary conditions for a graph to be strongly \*-graph,

Baskar Babujee and Vishnupriya [414] have proved the following are strongly \*-graphs:  $C_n \times P_2, (P_2 \cup \overline{K}_m) + \overline{K}_2$ , windmills  $K_3^{(n)}$ , and *jelly fish* graphs J(m, n) obtained from a 4-cycle  $v_1, v_2, v_3, v_4$  by joining  $v_1$  and  $v_3$  with an edge and appending m pendent edges to  $v_2$  and n pendent edges to  $v_4$ .

Baskar Babujee and Beaula [398] prove that cycles and complete bipartite graphs are vertex strongly \*-graphs. Baskar Babujee, Kannan, and Vishnupriya [408] prove that wheels, paths, fans, crowns,  $(P_2 \cup mK_1) + \overline{K_2}$ , and umbrellas (graphs obtained by appending a path to the central vertex of a fan) are vertex strongly \*-graphs.

In [2246] Seoud, Roshdy, and AboShady gave an upper bound for the number of edges of any graph in terms of the number of vertices to be a strongly \*-graph and some new families to be strongly\*- graphs. They also provided an algorithm for checking if a graph is a strongly \*-graph or not.

### 7.27 Triangular Sum Graphs

Hegde and Shankaran [994] call a labeling of graph with q edges a triangular sum labeling if the vertices can be assigned distinct non-negative integers in such a way that, when an edge whose vertices are labeled i and j is labeled with the value i + j, the edges labels are  $\{k(k + 1)/2 | k = 1, 2, ..., q\}$ . They prove the following graphs have triangular sum labelings: paths, stars, complete *n*-ary trees, and trees obtained from a star by replacing each edge of the star by a path. They also prove that  $K_n$  has a triangular sum labeling if and only if n is 1 or 2 and the friendship graphs  $C_3^{(t)}$  do not have a triangular sum labeling. They conjecture that  $K_n$   $(n \ge 5)$  are forbidden subgraphs of graph with triangular sum labelings. They conjectured that every tree admits a triangular sum labeling. They show that some families of graphs can be embedded as induced subgraphs of triangular sum graphs. They conclude saying "as every graph cannot be embedded as an induced subgraph of a triangular sum graph, it is interesting to embed families of graphs as an induced subgraph of a triangular sum graph". In response, Seoud and Salim [2248] showed the following graphs can be embedded as an induced subgraph of a triangular sum graphs.

Vaidya, Prajapati, and Vihol [2661] showed that cycles, cycles with exactly one chord, and cycles with exactly two chords that form a triangle with an edge of the cycle can be embedded as an induced subgraph of a graph with a triangular sum labeling. They proved that several classes of graphs do not have triangular sum labelings. Among them are: helms, graphs obtained by joining the centers of two wheels to a new vertex, and graphs in which every edge is an edge of a triangle. As a corollary of the latter result they have that  $P_m + \overline{K_n}$ ,  $W_m + \overline{K_n}$ , wheels, friendship graphs, flowers, triangular ladders, triangular snakes, double triangular snakes, and flowers. do not have triangular sum labelings.

Seoud and Salim [2248] proved the following are triangular sum graphs:  $P_m \cup P_n$ ,  $m \ge 4$ ; the union of any number of copies of  $P_n$ ,  $n \ge 5$ ;  $P_n \odot \overline{K_m}$ ; symmetrical trees; the graph obtained from a path by attaching an arbitrary number of edges to each vertex of the path; the graph obtained by identifying the centers of any number of stars; and all trees of order at most 9.

For a positive integer *i* the *i*th pentagonal number is i(3i - 1)/2. Somashekara and Veena [2445] define a pentagonal sum labeling of a graph G(V, E) as one for which there is a one-to-one function *f* from V(G) to the set of nonnegative integers that induces a bijection  $f^+$  from E(G) to the set of the first |E| pentagonal numbers. A graph that admits such a labeling is called a pentagonal sum graph. Somashekara and Veena [2445] proved that the following graphs have pentagonal sum labelings: paths,  $K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$ , complete *n*-ary trees, and lobsters obtained by joining centers of any number of copies of a star to a new vertex. They conjecture that every tree has a pentagonal sum labeling and as an open problem they ask for a proof or disprove that cycles have pentagonal labelings. They observed that if every edge of a graph is an edge of a triangle then the graph does not have pentagonal sum labeling. As was the case for triangular sum labelings the following graphs do not have a pentagonal sum labeling:  $P_m + \overline{K_n}$ , and  $W_m + \overline{K_n}$  wheels, friendship graphs, flowers, triangular ladders, triangular snakes, double triangular snakes, and flowers. Somashekara and Veena [2445] also proved that helms and the graphs obtained by joining the centers of two wheels to a new vertex are not pentagonal sum graphs.

### 7.28 Divisor Graphs

Santhosh and Singh [2187] call a graph G(V, E) a *divisor graph* if V is a set of integers and  $uv \in E$  if and only if u divides v or vice versa. They prove the following are divisor graphs: trees;  $mK_n$ ; induced subgraphs of divisor graphs; cocktail party graphs  $H_{m,n}$  (see Section 7.1) for the definition); the one-point union of complete graphs of different orders; complete bipartite graphs;  $W_n$  for n even and n > 2; and  $P_n + \overline{K_t}$ . They also prove that  $C_n$   $(n \ge 4)$  is a divisor graph if and only if n is even and if G is a divisor graph then for all n so is  $G + K_n$ .

Chartrand, Muntean, Saenpholphat, and Zhang [568] proved complete graphs, bipartite graphs, complete multipartite graphs, and joins of divisor graphs are divisor graphs. They also proved if G is a divisor graph, then  $G \times K_2$  is a divisor graph if and only if Gis a bipartite graph; a triangle-free graph is a divisor graph if and only if it is bipartite; no divisor graph contains an induced odd cycle of length 5 or more; and that a graph Gis divisor graph if and only if there is an orientation D of G such that if (x, y) and (y, z)are edges of D then so is (x, z).

In [116] and [118] Al-Addasi, AbuGhneim, and Al-Ezeh determined precisely the values of n for which  $P_n^k$  ( $k \ge 2$ ) are divisor graphs and proved that for any integer  $k \ge 2$ ,  $C_n^k$ is a divisor graph if and only if  $n \le 2k + 2$ . In [119] they gave a characterization of the graphs G and H for which  $G \times H$  is a divisor graph and a characterization of which block graphs are divisor graphs. (Recall a graph is a *block graph* if every one of its blocks is complete.) They showed that divisor graphs form a proper subclass of perfect graphs and showed that cycle permutation graphs of order at least 8 are divisor graphs if and only if they are perfect. (Recall a graph is *perfect* if every subgraph has chromatic number equal to the order of its maximal clique.) In [117] Al-Addasi, AbuGhneim, and Al-Ezeh proved that the contraction of a divisor graph along a bridge is a divisor graph; if e is an edge of a divisor graph that lies on an induced even cycle of length at least 6, then the contraction along e is not a divisor graph; and they introduced a special type of vertex splitting that yields a divisor graph when applied to a cut vertex of a given divisor graph.

AbuHijleh, AbuGhneim, and Al-Ezeh [20] prove that for any tree T,  $T^2$  is a divisor graph if and only if T is a caterpillar and the diameter of T is less than six. For any caterpillar T and a positive integer k with diam(T) < 2k, they show that  $T^k$  is a divisor graph. Moreover, for a caterpillar T and  $k \ge 3$  with diam(T) = 2k or diam(T) = 2k + 1, they show that  $T^k$  is a divisor graph if and only if the centers of T have degree two. In [21] AbuHijleh, AbuGhneim, and Al-Ezeh prove that the k-th power  $Q_n^k$  of  $Q_n$  is a divisor graph if and only if n = 2, 3 or  $n \ge 4$  and  $k \ge n-1$  hold. In the case of the n-dimensional folded-hypercube  $FQ_n$  (that is, the graph obtained from  $Q_n$  by adding to it a perfect matching that connects opposite pairs of the vertices of  $Q_n$ ) they show that  $FQ_n$  is a divisor graph for odd n, but not for even  $n \ge 4$ . They also prove  $(FQ_n)^k$  is not a divisor graph if and only if  $2 \le k \le \lfloor n/2 \rfloor$ , where  $n \ge 5$ .

Ganesan and Uthayakumar [821] proved that  $G \odot H$  is a divisor graph if and only if G is a bipartite graph and H is a divisor graph. Frayer [771] proved  $K_n \times G$  is a divisor graph for each n if and only if G contains no edges and  $\overline{K_n} \times \overline{K_2}$   $(n \ge 3)$  is a divisor graph. Vinh [2748] proved that for any n > 1 and  $0 \le m \le n(n-1)/2$  there exists a divisor graph of order n and size m. She also gave a simple proof of the characterization of divisor graphs due to Chartrand, Muntean, Saenpholphat, and Zhang [568]. Gera, Saenpholphat, and Zhang [865] established forbidden subgraph characterizations for all divisor graphs that contain at most three triangles. Tsao [2602] investigated the vertex-chromatic number, the clique number, the clique cover number, and the independence number of divisor graphs have some necessary and sufficient conditions for a graph to be divisor graph.

# References

- M. E. Abdel-Aal, Odd harmonious labelings of cyclic snakes, Internat. J. Appl. Graph Theory Wireless Adhoc Networks and Sensor Networks, 5 (3) (2013) 1-13.
- [2] M. E. Abdel-Aal, New classes of odd graceful graphs, Internat. J. Appl. Graph Theory in Wireless ad hoc Networks and Sensor Networks, 5(2) June 2013.
- [3] M. E. Abdel-Aal, New families of odd harmonious graphs, Internat. J. Soft Comput., Math. Control, 3(1) (2014) 1-13.
- [4] H. Abdo and D. Dimitrov, The total irregularity of a graph, arXiv:1207.5267v1 July 2012.
- [5] R. J. R. Abel and M. Buratti, Some progress on (v, 4, 1) difference families and optical orthogonal codes, J. Combin. Theory Ser. A, **106** (2004) 59-75.
- [6] K. Abhishek, A note on set-indexed graphs, J. Disc. Math. Sci. Crypt. 18 (2015) 31-40.
- [7] K. Abhishek, Set-valued graphs: a survey, J. Disc. Math. Sci. Crypt. 18 (2015) 55-80.
- [8] K. Abhishek, Set-valued graphs II, J. Fuzzy Set Valued Analysis, 2013 (2013) 1-16.
- K. Abhishek and G. K. Agustine, Set-valued graphs, J. Fuzzy Set Valued Analysis, 2012 (2012) 17 pages.
- [10] V. J. Abhyankar, Direct Methods of Gracefully Labeling Graphs, Ph. D. Thesis, University of Mumbai, 2002.
- [11] V. J. Abhyankar and V. N. Bhat-Nayak, Easiest graceful labeling of olive trees, Bull., Bombay Math. Coll., 14 (2000) 16-25.
- [12] V. J. Abhyankar and V. N. Bhat-Nayak, personal communication.
- [13] J. Abrham, Perfect systems of difference sets-A survey, Ars Combin., 17A (1984)
  5-36.
- [14] J. Abrham, Graceful 2-regular graphs and Skolem sequences, *Discrete Math.*, 93 (1991) 115-121.
- [15] J. Abrham, M. Carter, and K. Eshghi, personal communication.
- [16] J. Abrham and A. Kotzig, Extensions of graceful valuations of 2-regular graphs consisting of 4-gons, Ars Combin., 32 (1991) 257-262.

- [17] J. Abrham and A. Kotzig, Two sequences of 2-regular graceful graphs consisting of 4-gons. Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity (Prachatice, 1990), 14, Ann. Discrete Math., 51, North-Holland, Amsterdam, 1992.
- [18] J. Abrham and A. Kotzig, All 2-regular graphs consisting of 4-cycles are graceful, Discrete Math., 135 (1994) 1-14.
- [19] J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.*, **150** (1996) 3-15.
- [20] E. A. AbuHijleh, O. A. AbuGhneim, and H. Al-Ezeh, Characterizing when powers of a caterpillar are divisor graphs, Ars Combin., 113A (2014) 85-89.
- [21] E. A. AbuHijleh, O. A. AbuGhneim, and H. Al-Ezeh, Characterizing which powers of hypercubes and folded hypercubes are divisor graphs, *Discuss. Math. Graph Theory*, 35(2) (2015) 301-311.
- [22] B. D. Acharya, Construction of certain infinite families of graceful graphs from a given graceful graph, *Def. Sci. J.*, **32** (1982) 231-236.
- [23] B. D. Acharya, On *d*-sequential graphs, J. Math. Phy. Sci., 17 (1983) 21-35.
- [24] B. D. Acharya, Set valuations of a graph and their applications, MRI Lecture Notes in Applied Mathematics, No. 2, Mehta Research Institute, Allahabad, 1983.
- [25] B. D. Acharya, Are all polyominoes arbitrarily graceful?, Proc. First Southeast Asian Graph Theory Colloquium, Ed. K. M. Koh and H. P. Yap, Springer-Verlag, N. Y. 1984, 205-211.
- [26] B. D Acharya and S. Arumugarn, Labeling of discrete structures in Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 1-14.
- [27] B. D. Acharya, S. Arumugarn, and A. Rosa, Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008.
- [28] B. D. Acharya and K. A. Germina, Strongly k-indexable unicyclic graphs, Graph Theory Notes N. Y., 55 (2008) 45-49.
- [29] B. D. Acharya and K. A. Germina, Set-valuations of graphs and their applications: A survey, Annals Pure Appl. Math., 4(1) (2013) 8-42. https://pdfs. semanticscholar.org/e438/81bc8a4f04508693916f527945ac0b7b1b6a.pdf
- [30] B. D. Acharya and K. A. Germina, Maximal strongly k-indexable graphs, J. Combin. Math. Combin. Comput., 76 (2011) 75-99.

- [31] B. D. Acharya, K. A. Germina, K. Abhishek, and P. J. Slater, Some new results on set-graceful and set-sequential graphs, *Combin. System Sci.*, 37(2-4) (2012) 229-239.
- [32] B. D. Acharya, K. A. Germina, and V. Ajitha, Multiplicatively indexable graphs, in Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 29-40.
- [33] B. D. Acharya, K. A. Germina, and T. M. K. Anandavally, Some new perspectives on arithmetic graphs, *in* Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 41-46.
- [34] B. D. Acharya, K. A. Germina, K. L. Princy, and S. B. Rao, On set-valuations of graphs, *in* Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 47-56.
- [35] B. D. Acharya, K. A. Germina, K. L. Princy, and S. B. Rao, Graph labellings, embedding and NP-completeness theorems, J. Combin. Math. Combin. Comput., 67 (2008) 163-180.
- [36] B. D. Acharya and M. K. Gill, On the index of gracefulness of a graph and the gracefulness of two-dimensional square lattice graphs, *Indian J. Math.*, 23 (1981) 81-94.
- [37] B. D. Acharya and S. M. Hegde, Further results on k-sequential graphs, Nat. Acad. Sci. Lett., 8 (1985) 119-122.
- [38] B. D. Acharya and S. M. Hegde, Set sequential graphs, Nat. Acad. Sci. Lett., 8 (1985) 387-390.
- [39] B. D. Acharya and S. M. Hegde, Arithmetic graphs, J. Graph Theory, 14 (1990) 275-299.
- [40] B. D. Acharya and S. M. Hegde, Strongly indexable graphs, Discrete Math., 93 (1991) 123-129.
- [41] B. D. Acharya and S. M. Hegde, On certain vertex valuations of a graph I, Indian J. Pure Appl. Math., 22 (1991) 553-560.
- [42] B. D. Acharya, S. B. Rao, and S. Arumugan, Embeddings and NP-complete problems for graceful graphs, *in* Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 57-62.
- [43] B. D. Acharya, S. B. Rao, T. Singh, and V. Parameswaran, Some sigma labelled graphs, Presented at 19th Annual Conference of Ramanujan Mathematical Society, held at Dr. B. R. Ambedkar University, Agra, July 21-24, 2004.

- [44] B. D. Acharya, S. B. Rao, T. Singh, and V. Parameswaran, Neighborhood magic graphs, 2004, unpublished.
- [45] M. Acharya and V. Bhat-Nayak, Minimal 4-equitability of  $C_{2n} \odot K_1$ , Ars Combin., **65** (2002) 209-236.
- [46] M. Acharya and V. Bhat-Nayak, k-equitability of  $C_{2n} \odot K_1$ , k = 2, 2n and associated graphs. Util. Math., 68 (2005) 109-130.
- [47] M. Acharya, R. Jain, and S. Kansal, Vertex equitable labeling of signed graphs, *Elect. Notes Discr. Math.*, 63 (2017) 461-468.
- [48] M. Acharya and T. Singh, Graceful signed graphs, Czechoslovak Math. J., 54(129) (2004) 291-302.
- [49] D. G. Adalja and G. V. Ghodasara, Sum divisor cordial labeling in the context of duplication of graph elements, *Mathematics Today*, **35** (2019) 61-83. http: //mathematicstoday.org/currentisue/V35\_Dec\_2019\_8.pdf
- [50] M. Adamaszek, Efficient enumeration of graceful permutations, J. Combin. Math. Combin. Comput., 87 (2013) 191-197. http://front.math.ucdavis.edu/math. CO/0608513.
- [51] T. C. Adefokun and D. O. Ajayi, On radio number of stacked-book graphs, arXiv:1901.00355 [math.CO] 2Jan2019 1-9.
- [52] C. Adiga, H. Ramashekar, and D. Somashekara, A note on strongly multiplicative graphs, *Discuss. Math.*, 24 (2004) 81-83.
- [53] C. Adiga and D. Somashekara, Strongly  $\star$ -graphs, Math. Forum, **13** (1999/00) 31-36.
- [54] C. Adiga, C. K. Subbaraya, A. S. Shrikanth and M. A. Sriraj, On vertex balance index set of some graphs, *Bull. Iran. Math. Soc.*, 39 (4) (2013) 627-634.
- [55] M. Aftiana and D. Indriati, On edge irregular total k-labeling and total edge irregularity etrength of barbell graphs, J. Physics Conference Ser., (2019) 1306:012031 1-7. DOI:10.1088/1742-6596/1306/1/012031
- [56] I. H. Agustin, F. Susanto, Dafik, R. M. Prihandini, R. Alfarisi, and I. Sudarsana, On H-supermagic labelings of m-shadow of paths and cycles, Int. J. Math. Math. Sci., (2019) Art. ID 8780329, 7 pp. https://www.hindawi.com/journals/ijmms/ 2019/8780329/
- [57] H. U. Afzal On super edge-magicness of two special families of graphs, Util. Math. 97 (2015) 97-108.

- [58] H. U. Afzal, M. Ather, A. Q. Baig, and A. Maheshwari, Further results on super edge-magic graphs, Util. Math., 110 (2019) 283-291.
- [59] H. U. Afzal and M. Aslam, On the deficiency of disjoint union of further planar classes of graphs, Util. Math., 105 (2017) 207-218.
- [60] A. Agarwal and M. Lopez, Representation number for complete graphs minus stars, J. Combin. Math. Combin. Comput., 87 (2013) 33-41.
- [61] A. Aguado and S. El-Zanati, On σ-labeling the union of three cycles, J. Combin. Math. Combin. Comput., 64 (2008) 33-48.
- [62] A. Aguado, S. El-Zanati, H. Hake, J. Stob, and H. Yayla, On ρ-labeling the union of three cycles, Australas. J. Combin., 37 (2007) 155-170.
- [63] I. H. Agustin, Dafik, Marsidi, E. R. Albirri, On the total *H*-irregularity strength of graphs: A new notion, *J. Physics: Conf. Series*, 855 (2017) 012004 doi:10.1088/1742-6596/855/1/012004
- [64] A. Ahmad, On vertex irregular total labeling of convex polytope graphs, Util. Math., 89 (2012) 69-78.
- [65] A. Ahmad, Computing the edge irregularity strength of certain unicyclic graphs, unpublished.
- [66] A. Ahmad, Computing 3-total edge product cordial labeling of generalized Petersen graphs P(n, m), Ars Combin., **137** (2018) 263-271.
- [67] S. Ahmad, On the evaluation of a subdivision of the ladder graph, Punjab Univ. J. Math. (Lahore), 47(1) (2015) 15-19. http://pu.edu.pk/images/journal/maths/ PDF/1\_4-02-09.pdf
- [68] A. Ahmad, S. Ahtsham, H. B. M. Imran, and A. Q. Baig, Vertex irregular labelings of cubic graphs, Util. Math., 91 (2013) 287-299.
- [69] Y. Ahmad, U. Ali, M. Bilal, S. Zafar, Z. Zahid, Some new standard graphs labeled by 3-total edge product cordial labeling, *Appl. Math. Nonlinear Sci.*,2(1) (2017) 61-72. https://content.sciendo.com/view/journals/amns/2/ 1/article-p61.xml
- [70] A. Ahmad, K. Ali, M. Bača, P. Kovář, and A. Semaničová-Feňovčíková, On vertexantimagicness of regular graphs, *Acta Math. Sinica*, English series, 28(9) (2012) 1865-1874.
- [71] A. Ahmad, O. Al-Mushayt, M. Bača, On edge irregular strength of graphs, Appl. Math. Comput., 243 (2014) 607-610.

- [72] A. Ahmad, K. Ali, and E. Baskoro, On super edge-magic total labelings of a forest of banana trees, Util. Math., 83 (2010) 323-332
- [73] A. Ahmad, M. Arshadb, and G. Ižaríková, Irregular labelings of helm and sun graphs, AKCE Internat. J. Graphs Combin., 12 (2015) 161-168. https://www. tandfonline.com/doi/full/10.1016/j.akcej.2015.11.010
- [74] A. Ahmad, K. M. Awan, I. Javaid, and Slamin, Total vertex irregularity strength of wheel related graphs, Australas. J. Combin., 51 (2011) 147-156.
- [75] A. Ahmad and M. Bača, On edge irregular total labeling of certain family of graphs, AKCE J. Graphs Combin., 6 (2009) 21-29. https://www.researchgate.net/publication/261951606\_Edge\_irregular\_ total\_labeling\_of\_certain\_family\_of\_graphs#fullTextFileContent
- [76] A. Ahmad and M. Bača, On vertex irregular total labeling, Ars Combin., 112 (2013) 129-139.
- [77] A. Ahmad and M. Bača, Total edge irregularity strength of the categorical product of two paths, Ars Combin., 114 (2014) 203-212.
- [78] A. Ahmad, M. Bača, and Y. Bashir, Total vertex irregularity strength of certain classes of unicyclic graphs, Bull. Math. Soc. Sci. Math. Roumanie, 57(2) (2014) 147-152. https://www.researchgate.net/publication/262933043\_Total\_ vertex\_irregularity\_strength\_of\_certain\_classes\_of\_unicyclic\_graphs
- [79] A. Ahmad, M. Bača, Y. Bashir, and M. K. Siddiqui, Total edge irregularity strength of strong product of two paths, Ars Combin., 106 (2012) 449-459.
- [80] A. Ahmad, M. Bača, M. Lascsáková, and A. Semaničová-Feňovčíková, Super magic and antimagic labelings of disjoint union of plane graphs, *Science Internat.*, 24(1) (2012) 21-25.
- [81] A. Ahmad, M. Bača, and M. F. Nadeen, On edge irregularity strength of Toeplitz graphs, U.P.B. Sci. Bull., Series A, 78(4) (2016) 155-162.
- [82] A. Ahmada, M. Bača, M. Naseem, and A. Semaničová-Feňovčíková, On 3-total edge product cordial labeling of honeycomb, AKCE Internat. J. Graphs Combin., 14 (2017) 149-157.
- [83] A. Ahmad, M. Bača and M. Numan, On irregularity strength of disjoint union of friendship graphs, *Elect. J. Graph Th. App.*, 1(2) (2013) 100-108.
- [84] A. Ahmad, M. Bača, and M. K. Siddiqui, On edge irregular total labeling of categorical product of two cycles, *Th. Comp. Systems*, 54 (2014) 1-12.
- [85] A. Ahmad, M. Bača, and M. K. Siddiqui, Irregular total labeling of disjoint union of prisms and cycles, Australas. J. Combin., 59(1) (2014) 98-106.

- [86] A. Ahmad, A. Q. Baig, and M. Imran, On super edge-magicness of graphs, Util. Math., 89 (2012) 373-380.
- [87] M. A. Ahmed, J. Baskar Babujee, M. Bača, and A. Semaničová-Feňovčíková, Util. Math., 111 (2019) 161-173.
- [88] A. Ahmad, E. T. Baskoro, and M. Imran, Total vertex irregularity strength of disjoint union of Helm graphs, *Discuss. Math. Graph Theory*, **32**(3) (2012) 427-434.
- [89] A. Ahmad, A. Gupta, and R. Simanjuntak, Computing the edge irregularity strengths of chain graphs and the join of two graphs, *Electron. J. Graph Theory Appl.*, (EJGTA) 6(1) (2018) 201-207.
- [90] A. Ahmad, M. Ibrahim, and M. K. Siddiqui, On the total irregularity strength of generalized Petersen graph, *Math. Rep. (Bucur.)*, 18(68) no. 2 (2016) 197-204.
- [91] A. Ahmad, M. Imran, and A. Semaničová-Feňovčíková, Relation between mean labeling and (a; d)-edge antimagic vertex labeling, Sci. Internat., 24(1)(2012) 7-9.
- [92] A. Ahmad, I. Javaid, and M. F. Nadeem, Further results on super edge magic deficiency of unicyclic graphs, Ars Combin., 99 (2011) 129-138.
- [93] A. Ahmad, I. Javaid, M. F. Nadeem, and R. Hasni, On super edge magic deficiency of some families related to ladder graphs, *Australas. J. Combin.*, **51** (2011) 201-208.
- [94] A. Ahmad, S.C. López, F.A. Muntaner-Batle, and M. Rius-Font, Enumerating super edge-magic labeling for the union of non-isomorphic graphs, *Bull. Aust. Math. Soc.*, 84 (2011)(2) 310-321.
- [95] A. Ahmad and R. Marinescu-Ghemeci, Radio labeling of some ladder-related graphs, *Math. Rep. (Bucur.)* 19(69) (2017), no. 1, 107-119.
- [96] A. Ahmad, M. Nadeem, and A. Gupta, On super edge-magic deficiency of certain Toeplitz graphs, *Hacet. J. Math. Stat.*, 47(3) (2018) 513-519.
- [97] A. Ahmad, Nurdin, and E. T. Baskoro, On total irregularity strength of generalized Halin graph, Ars Comb., 122 (2015) 319-332.
- [98] A. Ahmad, A. Semaničová-Feňovčíková, and M. K. Siddiqui, Construction of αtree from smaller graceful trees, Util. Math., 99 (2016) 175-186.
- [99] A. Ahmad, M. K. Siddiqui, and D. Afzal, On the total edge irregularity strength of zigzag graphs, Australas. J. Combin., 54 (2012) 141-149.
- [100] A. Ahmad, M. K. Siddiqui, M. F. Nadeem, and M. Imran, On super edge magic deficiency of kite graphs, Ars Combin., 107 (2012) 201-208.

- [101] A. Ahmad and I. Tomescu, On vertex-magic total labelings of some families of rotationally-symmetric graphs, Util. Math., 86 (2011) 347-357.
- [102] M. A. Ahmed and J. Baskar Babujee, Totally antimagic total labeling of complete bipartite graphs, Rom. J. Math. Comput. Sci., 7(1) (2017) 21-28.
- [103] T. Ahmed and H. Snevily, The α-labeling number of comets is 2, Bull. Inst. Combin. Appl., 72 (2014) 25-40.
- [104] M. Aigner and E. Triesch, Irregular assignments of trees and forests, SIAM J. Discrete Math., 3(4) (1990) 439-449.
- [105] V. Ajitha. K. L. Princy, V. Lokesha, and P. S. Ranjini, On square difference graphs, Math. Combin. Internat. Book Ser., 1 (2012) 31-40.
- [106] S. Akbari, N. Ghareghani, G. Khosrovshahi, and S. Zare, A note on zero-sum 5-flows in regular graphs, *Electron. J. Combin.*, **19**(2) (2012), Paper 7, 5 pp.
- [107] S. Akbari, M. Kano, S. Zare, 0-Sum and 1-sum flows in regular graphs, *Electronic J. Combin.*, 23(2) (2016) #P2.37 10pp. https://www.combinatorics.org/ojs/ index.php/eljc/article/view/v23i2p37/pdf
- [108] S. Akbari, F. Rahmati, and S. Zare, Zero-sum magic labelings and null sets of regular graphs. *Electronic J. Combin.*, 21(2) (2014) Paper 2.17, 9 pp.
- [109] K. Akilandeswari and M. Tamilselvi, On k-Heronian mean labeling, Internat. J. Innovative Res. Sci., Eng. Tech. 6(9) (2017) 18019-18023.
- [110] K. Akilandeswari and M. Tamilselvi, k-Heronian mean labeling of some graphs, Internat. J. Pure Appl. Math., 117(12) (2017) 173-180.
- [111] D. G. Akka and N. S. Warad, Super magic strength of a graph, Indian J. Pure Appl. Math., 41(4) (2010) 557-568.
- [112] R. Akhtar, A. B. Evans, and D. Pritikin, Representation numbers of stars, *Integers*, 10 (2010), A54, 733-745.
- [113] R. Akhtar, A. B. Evans, and D. Pritikin, Representation numbers of complete multipartite graphs, *Discrete Math.*, **312**(6) (2012) 1158-1165.
- [114] R. Akhtar, The representation number of some sparse graphs, *Discrete Math.*, 312(22) (2012) 3417-3423.
- [115] A. D. Akwu and D. O. A. Ajayi, On totally antimagic total lableings of complete bipartite graphs, *Roman. J. Math. Comput. Sci.*, 7(1) 21-28. arXiv:1601.02112v1 [math. CO] 9 Jan 2016.
- [116] S. Al-Addasi, O. A. AbuGhneim and H. Al-Ezeh, Divisor orientations of powers of paths and powers of cycles. Ars Combin. 94 (2010) 371-380.

- [117] S. Al-Addasi, O. A. AbuGhneim and H. Al-Ezeh, Merger and vertex splitting in divisor graphs, *Internat. Math. Forum*, 5(38) (2010) 1861-1869.
- [118] S. Al-Addasi, O. A. AbuGhneim and H. Al-Ezeh, Characterizing powers of cycles that are divisor graphs, Ars Combin., 97A (2010) 447-451.
- [119] S. Al-Addasi, O. A. AbuGhneim and H. Al-Ezeh, Further new properties of divisor graphs, J. Combin. Math. Combin. Comput., 81 (2012) 261-272.
- [120] R. E. L. Aldred and B. D. McKay, Graceful and harmonious labellings of trees. http://users.cecs.anu.edu.au/~bdm/papers/graceful.pdf
- [121] R. E. L. Aldred, J. Siráň and M. Siráň, A note on the number of graceful labelings of paths, *Discrete Math.*, **261** (2003) 27-30.
- [122] M. Alfalayleh, L. Brankovic, H. Giggins, M. Z. Islam, Towards the Graceful Tree Conjecture: A survey, In: *Proceed. AWOCA 2004*, 7-9 July, Ballina, Austrtalia (2004).
- [123] G. Ali, M. Bača, F. Bashir, and A. Semaničová, On super vertex-antimagic labeling of disjoint union of paths, AKCE J. Graphs Combin., 6 (2009) 11-20.
- [124] G. Ali, M. Bača, F. Bashir, and A. Semaničová, On face antimagic labeling of disjoint union of prisms, Util. Math., 85 (2011) 97-112.
- [125] G. Ali, M. Bača, Y. Lin, and A. Semaničová-Feňovčiková, Super-vertex-antimagic total labelings of disconnected graphs, *Discrete Math.*, **309** (2009) 6048-6054.
- [126] K. Ali, M. Hussain, A. Ahmad and M. Miller, Magic Labelings of type (a, b, c) of families of wheels, *Math. Computer Sci.*, 7 (2013) 315-319.
- [127] K. Ali, M. Hussain, and Razzaq, Super edge-magic total labelings of a tree, Util. Math., 91 (2013) 355-364.
- [128] K. Ali, M. Hussain, H. Shaker, and M. Javaid, Super edge-magic total labeling of subdivided stars, Ars Combin., 120 (2015) 161-167.
- [129] K. Ali, S. T. R. Rizvi, and A. Semaničová-Feňovčiková, C<sub>4</sub>-supermagic labelings of disjoint union of prisms, *Math. Rep. (Bucur.)*, **18**(68) (2016) no. 3, 315-320.
- [130] M. Ali, M.T Rahim, G. Ali, and M. Farooq, An upper bound for the radio number of generalized gear graph, Ars Combin., 107 (2012) 161-168.
- [131] S. Alikhani, W. Kocay, G.-C. Lau, and S.-M. Lee, On the k-edge magic graphs, Elect. Notes Disc. Math., 45 (2014) 35-41.

- [132] M. M. Ali Al-Shamiri, A. Elrokh, Y. ElMashtawye, and S. Euat Tallah, The cordial labeling for the Cartesian product between paths and cycles, *Internat. J. Res.,*- Granthaalayah, 8(3) (2020) 331-341. https://doi.org/10.5281/zenodo. 3742934.
- [133] V. Ajtha, S. Arumugam, and K. A. Germina, On square sum graphs, AKCE J. Graphs Combin., 6 (2009) 1-10.
- [134] O. Al-Mushayt, On the edge irregularity strength of products of certain families with  $P_2$ , Ars Combin., **135** (2017) 323-334.
- [135] O. Al-Mushayt, A. Ahmad, and M. K. Siddiqui, On the total edge irregularity strength of hexagonal grid graphs, *Australas. J. Combin.*, 53 (2012) 263-271.
- [136] O. Al-Mushayt, A. Arshad, and M. K. Siddiqui, Total vertex irregularity strength of convex polytope graphs, *Acta Math. Univ. Comenian. (N.S.)*, **82**(1) (2013) 29-37.
- [137] O. Al-Mushayt, A. Ahmad, and M. K. Siddiqui, Irregular total labelings of convex polytope graphs having the same diameter, *Util. Math.*, **101** (2016) 295-307.
- [138] N. Alon, Combinatorial nullstellensatz, Combinatorics, Probabability and Computing, 8 (1999) 7-29.
- [139] N. Alon, G. Kaplan, A. Lev, Y. Roditty, and R. Yuster, Dense graphs are antimagic, J. Graph Theory, 47 (2004) 297-309.
- [140] N. Alon and E. Scheinerman, Generalized sum graphs, Graphs and Combin., 8 (1992) 23-29.
- [141] H. Alpert, Rank numbers of grid graphs, *Discrete Math.*, **310** (2010) 3324-3333.
- [142] H. Alpert, Rank numbers of path powers and grid graphs, personal communication.
- [143] D. Amar and O. Togni, Irregularity strength of trees, Discrete Math., 190(1-3) (1998) 15-38.
- [144] A. Amara Jothi, J. Baskar Babujee, and N. G. David, On face magic labeling of duplication graphs. http://oaji.net/articles/2017/296-1504963989.pdf
- [145] A. Amara Jothi, N. G. David, J. Baskar Babujee, Super edge bimagic labeling graphs, Int. J. Math. Sci. Comput., 5(1) (2015) 39-43.
- [146] B. Ambili and G. S. Singh, Graceful labelings Graceful labeling of arbitrary strong supersubdivisions of graphs, *Graph Theory Notes N. Y.*, **61** (2011) 31-36. http: //gtn.kazlow.info/GTN61.pdf

- [147] M. Amini and K. Eshghi, Constraint programming models and population-based simulated annealing algorithm for finding graceful and  $\alpha$ -labeling of quadratic graphs, *Iran J. Comput. Sci.*, **1**(3) (2018) 155-164.
- [148] S. Amutha and K. M. Kathiresan, Pendant edge extensions of two families of graphs, *Proceed. of the National Seminar on Algebra and Discrete Math.*, Univ. Kerala, Thiruvananthapuram, India, 146-150.
- [149] S. Amutha and M. Uma Devi, Super graceful labeling for some families of fan graphs, J. Comput. Math. Sci., 10(8) (2019) 1551-1562. http://www.compmath-journal.org/dnload/S-Amutha-and-M-Uma-Devi-/ CMJV10I08P1551.pdf
- [150] K. Amuthavalli, Graph Labeling and its Applications-Some Generalization of Odd Mean Labelings, Ph. D. Thesis, (2010) Mother Teresa Women's University, Kodaikanal Tamilnadu, India.
- [151] K. Amuthavalli and P. Sugapriya, Reverse super edge-trimagic labeling labeling for star related graphs, *Malaya J. Mat.*, 7(3) (2019) 519-525. https://www. malayajournal.org/articles/MJM07030025.pdf
- [152] S. Anusa, S. S. Sandhya, and S. Somasundaram, New classes of mean graphs, J. Disc. Math. Sci. Crypt., 22(3) (2019) 351-360. DOI:10.1080/09720529.2018.
   1534369
- [153] M. Andar, S. Boxwala, and N. Limaye, Cordial labelings of some wheel related graphs, J. Combin. Math. Combin. Comput., 41 (2002) 203-208.
- [154] M. Andar, S. Boxwala, and N. Limaye, A note on cordial labelings of multiple shells, *Trends Math.*, (2002) 77-80.
- [155] M. Andar, S. Boxwala, and N. Limaye, On the cordiality of the t-uniform homeomorphs–I, Ars Combin., 66 (2003) 313-318.
- [156] M. Andar, S. Boxwala, and N. Limaye, On the cordiality of the t-uniform homeomorphs–II (Complete graphs), Ars Combin., 67 (2003) 213-220.
- [157] M. Andar, S. Boxwala, and N. Limaye, New families of cordial graphs, J. Combin. Math. Combin. Comput., 53 (2005) 117-154.
- [158] M. Andar, S. Boxwala, and N. Limaye, On the cordiality of the *t*-ply  $P_t(u, v)$ , Ars Combin., **77** (2005) 245-259.
- [159] M. Andar, S. Boxwala, and N. Limaye, On the cordiality of corona graphs, Ars Combin., 78 (2006) 179-199.
- [160] M. Anholcer, Product irregularity strength of graphs, Discrete Math., 309 (2009) 6434-6439.

- [161] M. Anholcer, Product irregularity strength of certain graphs, Ars Math. Contemp., 7(1) (2014) 23-29.
- [162] M. Anholcer, S. Cichacz, R. Jura, A. Marczyk, Note on group irregularity strength of disconnected graphs, *Open Math.*, 16 (2018) 154-160, doi:10.1515/math-2018-0017.
- [163] M. Anholcer and S. Cichacz, Group irregular labelings of disconnected graphs, Contributions Discrete Math., 12(2) (2017) 158-166.
- [164] M. Anholcer, S. Cichacz, and M. Milanič, Group irregularity strength of connected graphs, J. Combin. Optim., (2013) **30**(1) (2015) 1-17. doi:10.1007/s10878-013-9628-6
- [165] M. Anholcer, S. Cichacz, I. Peterin, and A. Tepeh, Distance magic labeling and two products of graphs, *Graphs and Combin.*, **31**(5) (2015) 1117-1124. doi:10.1007/s00373-014-1455-8
- [166] M. Anholcer, S. Cichacz, I. Peterin, and A. Tepeh, Distance magic labeling and two products of graphs, *Graphs Combin.*, **31**(5) (2015) 1125-1136.
- [167] M. Anholcer, S. Cichacz, I. Peterin, and A. Tepeh, Group distance magic labeling of direct product of graphs, Ars Math. Contemp. 9 (2015) 93-108.
- [168] M. Anholcer, S. Cichacz, Note on distance magic products  $G \circ C_4$ , Graphs and Combin., **31**(5) (2015) 1117-1124. doi:10.1007/s00373-014-1453-x.
- [169] M. Anholcer, S. Cichacz, I. Peterin, and A. Tepeh, Group distance magic labeling of direct product of graphs, Ars Math. Contemp., 9(1) (2015) 93-107.
- [170] M. Anholcer, M. Karoński, and F. Pfender, Total vertex irregularity strength of forests, *Discrete Math.*, **312**(2) (2012) 229-237. arXiv:1103.2087v1 [math.CO] 10 Mar. 2011.
- [171] M. Anholcer, M. Karoński, and J. Przbylo, A new bound for the total vertex irregularity strength of graphs, *Discrete Math.*, **309** (2009) 6316-6317.
- [172] M. Anholcer and C. Palmer, Irregular labelings of circulant graphs, *Discrete Math.*, 312(23) (2012) 3461-3466.
- [173] D. J. Anick, Counting graceful labelings of trees: a theoretical and empirical study, Discrete Appl. Math., 198 (2016) 65-81.
- [174] K. Anitha, B. Selvam, and K. Thirusangu, k-Graceful, odd-even graceful, Heronian mean and analytic mean labeling for the extended duplicate graph of kite graph, *Internat. J. Engin. Tech.* 7(4.10) (2018) 355-359. https://www.sciencepubco. com/index.php/ijet/article/view/20934/9835

- [175] K. Annathurai, R. Ponraj, and R. Kala, Some 3-remainder cordial graphs, *Global J. Engin. Sci. Res.*, 5(6) (2018) 169-175.
- [176] K. Annathurai, R. Ponraj, and R. Kala, Further results on 3-remainder cordial labeling of graphs, *Internat. J. Management*, IT Engin., 8(8) (2018) 120-129.
- [177] S. Anusa, S. S. Sandhya, and S. Somasundaram, New classes of mean graphs, J. Discrete Math. Sci. Cryptogr., 22(3) (2019) 351-360.
- [178] R. Aravamudhan and M. Murugan, Numbering of the vertices of  $K_{a,1,b}$ , unpublished.
- [179] I. C. Arkut, R. C. Arkut, and A. Basak, Topology constrained label switching for multicast routing, *Proceed. Eighth IEEE Internat. Sympos. Comput. Comm.* (ISCC03), 1530-1346/03 1-8. http://www.academia.edu/4370947/
- [180] I. Arkut, R. Arkut, N. Ghani, Graceful label numbering in optical MPLS networks, In: *Proc. SPIE*, **4233** (2000) 1-8 OptiComm 2000: Optical Networking and Communications. Imrich Chlamtac: Ed (2000).
- [181] A. Armstrong and D. McQuillan, Vertex-magic total labelings of even complete graphs, *Discrete Math.*, **311** (2011) 676-683.
- [182] S. Arockiaraj, A. Durai Baskar, and A. Rajesh Kannan, F-root square mean labeling of line graph of some graphs, Utilitas Math., 112 (2019) 11-32.
- [183] S. Arockiaraj and P. Mahalakshmi, On odd sum graphs, Internat. J. Math. Combin., 4 (2013) 59-78.
- [184] S. Arockiaraj, P. Mahalakshmi, and P. Namasivayam, Odd sum labeling of some subdivision graphs, *Kragujevac J. Math.*, 38(1) (2014) 203-222. http://elib.mi. sanu.ac.rs/files/journals/kjm/42/kjom3801-16.pdf
- [185] S. Arockiaraj, P. Mahalakshmi, and P. Namasivayam, Odd sum labelings of some splitting graphs, Util. Math., 105 (2017) 53-73.
- [186] S. Arockiaraj, P. Mahalakshmi, and P. Namasivayam, Odd sum labeling of graphs obtained by duplicating any edge of some graphs. *Electron. J. Graph Theory Appl.* (*EJGTA*), 3(2) (2015) 197-215.
- [187] S. Arockiaraj, P. Mahalakshmi, and P. Namasivayam, Odd sum labeling of graphs obtained from some graph operations, J. Graph Labeling, 2(2) (2015) 217-233.
- [188] S. Arockiaraj and A. Meena, On F-face magic mean labeling of some planar graphs, Internat. J. Pure Appl. Math., 117(6) (2017) 1-8. https://acadpubl.eu/jsi/ 2017-117-5-6/articles/6/1.pdf

- [189] S. Arockiaraj, A. Rajesh Kannan, and A. Durai Baskar, Super F-centroidal mean graphs, Internat. J. Math. Combin., 3 (2019) 113-127.
- [190] S. Arockiaraj, A. Rajesh Kannan, and A. Durai Baskar, F-centroidal mean labeling of graphs obtained from paths, *Internat. J. Math. Combin.*, 4 (2019) 122-135.
- [191] S. Arumugam and J. Bagga, Graceful labeling algorithms and complexity–a survey, Indones. Math. Soc., Special Edition (2011) 1-9.
- [192] S. Arumugam, D. Fronček, and N. Kamatchi, Distance magic graphsa survey, J. Indonesian Math. Soc., Special Edition, 1126 (2011).
- [193] S. Arumugam and K. Germina, On indexable graphs, Discrete Math., 161 (1996) 285-289.
- [194] S. Arumugam, K. Germina, and T. Anandavally, On additively geometric graphs, J. Combin. Math. Combin. Comput., 70 (2009) 207-216.
- [195] S. Arumugam and N. Kamatchi, On (a, d)-distance antimagic graphs, Australas. J. Combin., 54 (2012) 279-287.
- [196] S. Arumugam, N. Kamatchi, and P. Kovář, Distance magic graphs, Util. Math., 99 (2016) 131-142.
- [197] S. Arumugam, M. Miller, O. Phanalasy, and J. Ryan, Antimagic labeling of generalized pyramid graphs, *English Series Acta Mathematica Sinica*, **30** (2) (2014) 283-290.
- [198] S. Arumugam and M. Nalliah, Super (a, d)-edge antimagic total labelings of friendship and generalized friendship graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), 103-110,
- [199] S. Arumugam and M. Nalliah, Super (a, d)-edge antimagic total labelings of friendship graphs, Australas. J. Combin., 53 (2012) 237-243.
- [200] S. Arumugam and M. Nalliah, Super (a, 3)-edge-antimagic total labelings for union of two stars, Util. Math., 101 (2016) 337-349.
- [201] S. Arumugam, K. Premalatha, M. Bača, and A. Semaničová-Feňovčíková, Local antimagic vertex coloring of a graph, *Graphs Combin.*, 33(2) (2017) 275-285.
- [202] Y. F. Ashari and A. N. M. Salman,  $(H_1, H_2)$ -supermagic labelings for some shackles of connected graphs  $H_1$  and  $H_2$ , (2019) J. Phys.: Conf. Ser., 1127 012059.
- [203] F. Ashraf, M. Bača, Z. Kimáková, and A. Semaničová-Feňovčíková, On vertex and edge *H*-irregularity strengths of graphs, *Discrete Math. Algorithms Appl.*, 8(4) (2016) 1650070, 13 pp.

- [204] F. Ashraf, M. Bača, M. Lascsáková, and A. Semaničová-Feňovčíková, On Hirregularity strengths of graphs, Disc. Math. Graph Th., 37 (2017) 1067-1078.
- [205] F. Ashraf, M. Bača, A. Semaničová-Feňovčíková, A. Shabbirc, On H-irregularity strengths of G-amalgamationm of graphs, *Elect. J. Graph Th. Appl.*, 5 (2) (2017) 325-334.
- [206] M. Asif, G. Ali, M. Numan, and A. Semaničová-Feňovčíková, Cycle-supermagic labeling for some families of graphs, *Util. Math.*, **103** (2017) 51-59.
- [207] M. A. Asim and R. Hasni, Iteractive algorithm for computing irregularity strength of complete graph, Ars Combin., 138 (2018) 17-24.
- [208] J. Asplund and N. B. Fox, Minimum coprime labelings for operations on graphs, *Integers.* 19 (2019) Paper No. A24, 26 pp.
- [209] S. Avadayappan, P. Jeyanthi, and R. Vasuki, Super magic strength of a graph, Indian J. Pure Appl. Math., 32 (2001) 1621-1630.
- [210] S. Avadayappan, R. Vasuki, and P. Jeyanthi, Magic strength of a graph, Indian J. Pure Appl. Math., 31 (2000) 873-883.
- [211] S. Avadayappan and R. Vasuki, New families of mean graphs, Internat. J. Math. Combin., 2 (2010) 68-80.
- [212] J. Ayel and O. Favaron, Helms are graceful, in Progress in Graph Theory (Waterloo, Ont., 1982), Academic Press, Toronto, Ont. (1984) 89-92.
- [213] A. Azaizeh, R. Hasni, G. C. Lau, and A. Ahmad, 3-total edge product cordial labeling of complete, bipartite and generalized friendship graphs, *Util. Math.*, 106 (2018) 259-270.
- [214] M. Baća, P. Jeyanthi, P. Selvagopal, N. T. Muthu Raja, and A. Semaničová-Feňovčíková, Ladders and fan graphs are cycle-antimagic, *Hacettepe J. Math. & Stat.*, 49(3) (2020) 1093-1106.
- [215] E. M. Badr, On graceful labeling of the generalization of cyclic snakes, J. Discrete Math. Sci. Cryptogr., 18(6) (2015) 773-783.
- [216] E. M. Badr, M. I. Moussa, and K. Kathiresan, Crown graphs and subdivision of ladders are odd graceful, *Internat. J. Computer Math.*, 88(17) (2011) 3570-3576.
- [217] A. Q. Baig and H. U. Afzal, Some super edge-magic (p,q)-graphs with magic constant 3p, Util. Math., **98** (2015) 53-63.
- [218] A. Q. Baig, H. U. Afzal, M. Imran, and I. Javaid, Super edge-magic labeling of volvox and pancyclic graphs, Util. Math., 93 (2014) 49-56.

- [219] A. Q. Baig, E. T. Baskoro, and A. Semaničová-Feňovčíková, On the super edgemagic deficiency of a star forest, Ars Combin., 115 (2014) 3-12.
- [220] D. Bantva, A lower bound for the radio number of graphs, Algorithms and Discrete Applied Mathematics. CALDAM 2019. Lecture Notes in Computer Science, vol 11394, 161-173. Springer, Cham
- [221] D. Bantva, Further results on the radio number of trees, *Elect. Notes Discr. Math.*, 63 (2017) 85-91.
- [222] D. Bantva, Radio number for middle graph of paths, *Elect. Notes Discr. Math.*, 63 (2017) 93-100.
- [223] M. V. Bapat, Some vertex prime graphs and a new type of graph labeling, Internat. J. Math. Trends and Tech., 47(1) (2017) 49-55.
- [224] M. V. Bapat, Some complete-graph related families of product cordial (pc) graphs, Aryabhatta J. Math. Informatics, 09(02) (2017) 133-140.
- [225] M. V. Bapat, Product cordial labeling of some fusion graphs, Internat. J. Math. Trends and Tech., 50(2) (2017) 125-129.
- [226] M. V. Bapat, Extended vertex edge additive cordial (EVEAC) labeling-a new labeling method for graphs, *Internat. J. Stat. Appl. Math.*, 2(6) (2017) 92-94.
- [227] M. V. Bapat, Extended edge vertex cordial labelings of graph, Internat. J. Math. Archive, 8(9) (2017) 17-19.
- [228] A. Basak, MPLS multicasting using caterpillars and a graceful labelling scheme, IEEE Conference Publications, Information Visualisation, 2004. IV 2004. Proceedings. Eighth International Conference. 382-387. doi:10.1109/IV.2004.1320172
- [229] R. L. Bras, C. P. Gomes, and B. Selman, Double-wheel graphs are graceful, IJ-CAI'13 Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence, AAAI Press (2013) 587-593.
- [230] M. Bača, On magic and consecutive labelings for the special classes of plane graphs, Util. Math., 32 (1987) 59-65.
- [231] M. Bača, Labelings of two classes of convex polytopes, Util. Math., 34 (1988) 24-31.
- [232] M. Bača, Labelings of *m*-antiprisms, Ars Combin., 28 (1989) 242-245.
- [233] M. Bača, On magic labelings of Möbius ladders, J. Franklin Inst., 326 (1989) 885-888.
- [234] M. Bača, On magic labelings of type (1, 1, 1) for three classes of plane graphs, Math. Slovaca, 39 (1989) 233-239.

- [235] M. Bača, On certain properties of magic graphs, Util. Math., 37 (1990) 259-264.
- [236] M. Bača, On magic labelings of *m*-prisms, *Math. Slovaca*, **40** (1990) 11-14.
- [237] M. Bača, On consecutive labelings of plane graphs, J. Franklin Inst., 328 (1991) 249-253.
- [238] M. Bača, On magic labelings of honeycomb, *Discrete Math.*, **105** (1992) 305-311.
- [239] M. Bača, On magic labelings of grid graphs, Ars Combin., **33** (1992) 295-299.
- [240] M. Bača, On magic labelings of type (1, 1, 1) for the special class of plane graphs, J. Franklin Inst., 329 (1992) 549-553.
- [241] M. Bača, On magic labellings of convex polytopes, Ann. Disc. Math., 51 (1992) 13-16.
- [242] M. Bača, Labelings of two classes of plane graphs, Acta Math. Appl. Sinica, 9 (1993) 82-87.
- [243] M. Bača, Face antimagic labelings of convex polytopes, Util. Math., 55 (1999) 221-226.
- [244] M. Bača, Antimagic labelings of antiprisms, J. Combin. Math. Combin. Computing, 35 (2000) 217-224.
- [245] M. Bača, Special face numbering of plane quartic graphs, Ars Combin., 57 (2000) 285-292.
- [246] M. Bača, Consecutive-magic labeling of generalized Petersen graphs, Util. Math., 58 (2000) 237-241.
- [247] M. Bača, On face antimagic labelings of plane graphs, J. Combin. Math. Combin. Comput., 53 (2005) 33-38.
- [248] M. Bača and C. Barrientos, Graceful and edge-antimagic labelings, Ars Combin., 96 (2010) 505-513.
- [249] M. Bača and C. Barrientos, On super edge-antimagic total labelings of  $mK_n$ , Discrete Math., **308** (2008) 5032-5037.
- [250] M. Bača and F. Bashir, On super *d*-antimagic labelings of disjoint union of prisms, AKCE J. Graphs Combin., 6 (2009) 31-39.
- [251] M. Bača, Y. Bashir, M. F. Nadeem, and A. Shabbir, On super edge-antimagic total labeling of Toeplitz graphs. Mathematics in the 21st century, 110, Springer Proc. Math. Stat., 98, Springer, Basel, 2015.
- [252] M. Bača, F. Bashir, and A. Semaničová, Face antimagic labeling of antiprisms, Util. Math., 84 (2011) 209-224.
- [253] M. Bača1 E. T. Baskoro, L. Brankovic, S. Jendrol, Y. Lin, O. Phanalasy, J. Ryan, A. Semaničová-Feňovčíková, Slamin, K. A. Sugeng, A survey of face-antimagic evaluations of graphs, *Australa. J. Combin.*, **69**(3) (2017) 382-393.
- [254] M. Bača, B. Baskoro, and Y. Cholily, Face antimagic labelings for a special class of plane graphs  $C_a^b$ , J. Combin. Math. Combin. Comput., **55** (2005) 5-15.
- [255] M. Bača, B. Baskoro, and Y. Cholily, On *d*-antimagic labelings for a special class of plane graphs, *J. Combin. Math. Combin. Comput.*, **61** (2007) 21-32.
- [256] M. Bača, B. Baskoro, Y. Cholily, S. Jendrol, Y. Lin, M. Miller, J. Ryan, R. Simanjuntak, Slamin, and K. Sugeng, Conjectures and open problems in face antimagic evaluations of graphs, J. Indonesian Math. Soc. (MIHMI), **11** (2005) 175-192.
- [257] M. Bača, B. Baskoro, S. Jendrol and M. Miller, Antimagic labelings of hexagonal plane maps, Util. Math., 66 (2004) 231-238.
- [258] M. Bača, B. Baskoro, and M. Miller, On *d*-antimagic labelings of honeycomb, *Proceedings 12th Australian Workshop on Combinatorial Algorithms*, Bandung, Indonezia (2001) 45-50.
- [259] M. Bača, E. T. Baskoro, and M. Miller, Antimagic valuations for the special class of plane graphs, *Lecture Note Computer Science-LNCS*, **3330** (2005) 58-64.
- [260] M. Bača, E. T. Baskoro, M. Miller, J. Ryan, R. Simanjuntak, and K. A. Sugeng, Survey of edge antimagic labelings of graphs, J. Indonesian Math. Soc., (MIHMI) 12 (2006) 113-130.
- [261] M. Bača, E. T. Baskoro, R. Simanjuntak, and K. A. Sugeng, Super edge-antimagic labelings of the generalized Petersen graph P(n, (n-1)/2), Util. Math., **70** (2006) 119-127.
- [262] M. Bača, F. Bertault, J. MacDougall, M. Miller, R. Simanjuntak, and Slamin, Vertex-antimagic total labelings of graphs, *Discuss. Math. Graph Theory*, 23 (2003) 67-83.
- [263] M. Bača, F. Bertault, J. MacDougall, M. Miller, R. Simanjuntak, and Slamin, Vertex-antimagic total labelings of (a, d)-antimagic and (a, d)-face antimagic graphs, preprint.
- [264] M. Bača and L. Brankovic, Edge antimagicness for a class of disconnected graphs, Ars Combin., 97A (2010) 145-152.

- [265] M. Bača, L. Brankovic, M. Lascsáková, O. Phanalasy, and A. Semaničová-Feňovčíková, On *d*-antimagic labelings of plane graphs, *Electronic J. Graph Theory Appl.*, 1(1) (2013) 28-39.
- [266] M. Bača, L. Brankovic, and A. Semaničová-Feňovčikovă, Labelings of plane graphs with determined face eights, Acta Mechanica Slovaca, 13(2) (2009) 64-71.
- [267] M. Bača, L. Brankovic, and A. Semaničová-Feňovčiková, Labelings of plane graphs containing Hamilton path, Acta Math. Sinica, 27(4) (2011) 701-714.
- [268] M. Bača, Dafik, M. Miller, and J. Ryan, Edge-antimagic total labeling of disjoint unions of caterpillars, J. Combin. Math. Combin. Computing, 65 (2008) 61-70.
- [269] M. Bača, Dafik, M. Miller, and J. Ryan, Antimagic labeling of disjoint unions of s-crowns, Util. Math., 79 (2009) 193-205.
- [270] M. Bača and I. Hollánder, Prime-magic labelings of  $K_{n,n}$ , J. Franklin Inst., **327** (1990) 923-926.
- [271] M. Bača and I. Hollánder, Labelings of a certain class of convex polytopes, J. Franklin Inst., 329 (1992) 539-547.
- [272] M. Bača and I. Holländer, On (a, d)-antimagic prisms, Ars Combin., 48 (1998) 297-306.
- [273] M. Bača and I. Holländer, On (a, b)-consecutive Petersen graphs, Acta Math. Appl. Sinica (English Ser.), 14 (1998) 265-270.
- [274] M. Bača, I. Holländer, and K. W. Lih, Two classes of super-magic quartic graphs, J. Combin. Math. Combin. Comput., 23 (1997) 113-120.
- [275] M. Bača, M. Irfan, A. Javad, and A. Semaničová-Feňočová, On total edge product cordial labeling of fullerenes, *Electron. J. Graph Theory Appl.*, (EJGTA) 6(2) (2018) 238-249. http://www.ejgta.org/index.php/ejgta/article/view/395/ pdf\_79
- [276] M. Bača, J. Jendrol, M. Miller, and J. Ryan, Antimagic labelings of generalized Petersen graphs that are plane, Ars Combin., 73 (2004) 115-128.
- [277] M. Bača, J. Jendrol, M. Miller, and J. Ryan, On irregular total labellings, *Discrete Math.*, 307 (2007) 1378-1388.
- [278] M. Bača, Z. Kimáková, A. Semaničová-Feňovčiková, and M. A. Umar, Treeantimagicness of disconnected graphs, *Math. Probl. Eng.*, 2015, Art. ID 504251, 4 pp.
- [279] M. Bača, P. Kovář, A. Semaničová, and M. K. Shafiq, On super (a, 1)- edgeantimagic total labelings of regular graphs, *Discrete Math.*, **310** (2010) 1408-1412.

- [280] M. Bača, M. Lascsková, M. Naseem, A. Semaničová-Feňovčíková, On entire face irregularity strength of disjoint union of plane graphs, *Appl. Math. Comput.*, **307** (2017) 232-238.
- [281] M. Bača, M. Lascsáková, and A. Semaničová, On connection between α-labelings and edge-antimagic labeling of disconnected graphs, Ars Combin., 106 (2012) 321-336.
- [282] M. Bača, M. Lascsáková and M. K. Siddiqui, Total edge irregularity strength of toroidal fullerene, *Math. Comput. Sci.*, 7, (2013) 487-492.
- [283] M. Bača, Y. Lin, and M. Miller, Valuations of plane quartic graphs, J. Combin. Math. Combin. Computing, 41 (2002) 209-221.
- [284] M. Bača, Y. Lin, and M. Miller, Antimagic labelings of grids, Util. Math., 72 (2007) 65-75.
- [285] M. Bača, Y. Lin, M. Miller, and J. Ryan, Antimagic labelings of Möbius grids, Ars Combin., 78 (2006) 3-13.
- [286] M. Bača, Y. Lin, M. Miller, and R. Simanjuntak, New constructions of magic and antimagic graph labelings, *Util. Math.*, 60 (2001) 229-239.
- [287] M. Bača, Y. Lin, M. Miller, and M. Z. Youssef, Edge-antimagic graphs, Discrete Math., 307 (2007) 1232-1244.
- [288] M. Bača, Y. Lin, and F. A. Muntaner-Batle, Super edge-antimagic of path-like trees, Util. Math., 73 (2007) 117-128.
- [289] M. Bača, Y. Lin, and F. A. Muntaner-Batle, A note on even disjoint union of paths, AKCE Int. J. Graphs Comb., 6(1) (2009) 41-46.
- [290] M. Bača, Y. Lin, F. Muntaner-Batle, and M. Rius-Font, Strong labeling of linear forests, Acta Math. Sinica, English Series, 25 (2009) 1951-1964.
- [291] M. Bača, Y. Lin, and F. Muntaner-Batle, Edge-antimagic labeling of forests, Util. Math., 81 (2010) 31-40.
- [292] M. Bača, J. MacDougall, M. Miller, Slamin, and W. Wallis, Survey of certain valuations of graphs, *Discussiones Mathematicae Graph Theory*, **20** (2000) 219-229.
- [293] M. Bača, Y.Lin, and A. Semaničová-Feňovčíková, Note on super antimagicness of disconnected graphs, AKCE J. Graphs Combin., 6 (2009) 47-55.
- [294] M. Bača and M. Miller, Antimagic face labeling of convex polytopes based on biprisms, J. Combin. Math. Combin. Comput., 36 (2001) 229-236.

- [295] M. Bača and M. Miller, Valuations of a certain class of convex polytopes, J. Combin. Math. Combin. Comput., 43 (2002) 207-218.
- [296] M. Bača and M. Miller, On *d*-antimagic labelings of type (1,1,1) for prisms, J. Combin. Math. Combin. Comput., 44 (2003) 199-207.
- [297] M. Bača and M. Miller, Super Edge-Antimagic Graphs: A Wealth of Problems and Some Solutions, BrownWalker Press, 2007, Boca Raton, FL, USA
- [298] M. Bača, M. Miller, O. Phanalasy, J. Ryan, A. Semaničová-Feňovčíková, and A. Abildgaard Sillasen, Totally antimagic total graphs, *Australas. J. Combin.*, 61 (2015) 42-56.
- [299] M. Bača, M. Miller, O. Phanalasy, and A. Semaničová-Feňovčíková, Super dantimagic labelings of disconnected plane graphs, Acta Math. Sin. (Engl. Ser.), 26 (2010) 2283-2294.
- [300] M. Bača, M. Miller, O. Phanalasy, and A. Semaničová-Feňovčíková, Constructions of antimagic labelings for some families of regular graphs, J. Algor. Computation, 44, (2013) 1-7.
- [301] M. Bača, M. Miller, O. Phanalasy and A. Semaničová-Feňočová, Antimagic labeling of join graphs, Math. Comput. Sci., 9(2) (2015) 139-143.
- [302] M. Bača, M. Miller, and J. Ryan, On *d*-antimagic labelings of prisms and antiprisms, *Proceedings 12th Australian Workshop on Combinatorial Algorithms*, Bandung, Indonezia (2001) 51-58.
- [303] M. Bača, M. Miller, J. Ryan, and A. Semaničová-Feňovčíková, On *H*-antimagicness of disconnected graphs, *Bull. Aust. Math. Soc.*, 94(92) (2016) 201-207.
- [304] M. Bača, M. Miller, J. Ryan, and A. Semaničová-Feňovčíková, Edge-magic total labelings. In: Magic and Antimagic Graphs. Developments in Mathematics, 60 (2-19). Springer, Cham.
- [305] M. Bača, M. Miller, J.Ryan, and A. Semaničová-Feňovčíková, Magic and Antimagic Graphs Attributes, Observations, and Challenges in Graph Labelings, 60 Springer Nature Switzerland AG 2019.
- [306] M. Bača, M. Miller, and Slamin, Vertex-magic total labeling of generalized Petersen graphs, Int. J. Comput. Math., 79 (2002) 1259-1263.
- [307] M. Bača, F. Muntaner-Batle, A. Semaničová-Feňovčíková, and M. K. Shafiq, On super (a, 2)-antimagic total labeling of disconnected graphs, Ars Combin., 113, (2014) 129-137.
- [308] M. Bača, M. Numan, and A. Semaničová-Feňovčíková, Super *d*-antimagic labelings of generalized prism, *Util. Math.*, **99** (2016) 101-119.

- [309] M. Bača, M. Numan, and A. Semaničová-Feňovčíková, Super d-antimagic labelings of disjoint union of generalized prisms, Util. Math., 103 (2017) 299-310.
- [310] M. Bača, M. Numan, and A. Shabbir, Labelings of type (1,1,1) for toroidal fullerenes, *Turkish J. Math.*, 37 (2013) 899-907.
- [311] M. Bača, M. Numan, and M. K. Siddiqui, Super face antimagic labelings of union of antiprisms, *Math. Comput. Sci.*, 7(2) (2013) 245-253.
- [312] M. Bača and M. Murugan, Super edge-antimagic labelings of cycle with a chord, Australas. J. Combin., 35 (2006) 253-261.
- [313] M. Bača, A. Semaničová-Feňovčíková, and M. K. Shafiq, A method to generate large classes of edge-antimagic trees, *Util. Math.*, **86** (2011) 33-43.
- [314] M. Bača, A. Semaničová-Feňovčikovă, M. A. Umar, and D. Welyyanti, On Hantimagicness of Cartesian product of graphs, *Turkish J. Math.*, 42(1) (2018) 339-348.
- [315] M. Bača, A. Semaničová-Feňovčíková, T. M. Wang, and G. H. Zhang, On (a, 1)vertex-antimagic edge labeling of regular graphs, J. Appl. Math., (2015), Art. ID 320616, 7 pp.
- [316] M. Bača, and A. Shabbir, Total labelings of toroidal polyhexes, Science Internat., 24(3,) (2012) 239-241.
- [317] M. Bača, and M. K. Siddiqui, Total edge irregularity strength of generalized prism, Applied Math. Comput., 235 (2014) 168-173.
- [318] M. Bača and M. K. Siddiqui, On total edge irregularity strength of strong product of two cycles, Util. Math., 104 (2017) 255-275.
- [319] M. Bača and M. Z. Youssef, Further results on antimagic graph labeling, Australas. J. Combin., 38 (2007) 163-172.
- [320] M. Bača and M. Z. Youssef, On harmonious labelings of corona graphs, J. Appl. Math., 2014 Art. ID 627248, 4 pp.
- [321] J. Bagga, L. P. Fotso, P. B. Max, and S. Arumugam, New classes of graceful unicyclic graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), *Electron. Notes Discrete Math.*, 48 (2015) 27-32.
- [322] J. Bagga and A. Heinz, Graceful labelings of unicyclic graphs. Advances in discrete mathematics and applications: Mysore, 2008, 6979, *Ramanujan Math. Soc. Lect. Notes Ser.*, 13, Ramanujan Math. Soc., Mysore, 2010.
- [323] P. Bahl, S. Lake, and A. Wertheim, Gracefulness of families of spiders, *Involve*, 3 (2010) 241-247.

- [324] A. Q. Baig, A. Ahmad, E. Baskoro, and R. Simanjuntak, On the super edge-magic deficiency of forests, Util. Math., 86 (2011) 147-159.
- [325] A. Q. Baig, E. Baskoro, and A. Semaničová-Feňovčiková, On the super edge-magic deficiency of a star forest, Ars Combin., 115 (2014) 3-12. https://www.researchgate.net/publication/287059730\_On\_the\_super\_ edge-magic\_deficiency\_of\_a\_star\_forest
- [326] A. Q. Baig, M. Imran, I. Javaid, and A. Semaničová-Feňovčiková, Super edgemagic deficiency of graphs, Util. Math., 87 (2012) 355-364.
- [327] S. Bailey and C. Barrientos, Operations with mean graphs, Congr. Numer., 217 (2013) 5-19.
- [328] V. Balaji, D. S. T. Ramesh, and S. Sudhakar, Further results on relaxed mean labeling, Int. J. Adv. Appl. Math. Mech., 3(3) (2016) 92-99.
- [329] C. Balbuena, E. Barker, K. C. Das, Y. Lin, M. Miller, J. Ryan, Slamin, K. Sugeng, and M. Tkac, On the degrees of a strongly vertex-magic graph, *Discrete Math.*, **306** (2006) 539-551.
- [330] V. Balaji, Solution of a conjecture on Skolem mean graph of stars  $K_{1,l} \cup K_{1,m} \cup K_{1,n}$ , Internat. J. Math. Combin., 4 (2011) 115-117.
- [331] V. Balaji, D. S. T. Ramesh and A. Subramanian, Skolem mean labeling, Bull. Pure and Applied Sci., 26E (2) (2007) 245-248.
- [332] V. Balaji, D. S. T. Ramesh and A. Subramanian, Some results on Skolem mean graphs, Bull. Pure and Applied Sci., 27E (1)(2008) 67-74.
- [333] R. Balakrishnan, Graph labelings, unpublished.
- [334] C. Balbuena, E. Barker, Y. Lin, M. Miller, and K. Sugeng, Consecutive magic graphs, *Discrete Math.*, **306** (2006) 1817-1829.
- [335] C. Balbuena, P. García-Vázquez, X. Marcote, and J. C. Valenzuela, trees having an even or quasi even degree sequence are graceful, *Applied Math. Letters*, **20** (2007) 370-375.
- [336] R. Balakrishnan and R. Kumar, Decomposition of complete graphs into isomorphic bipartite subgraphs, *Graphs and Combin.*, **10** (1994) 19-25.
- [337] R. Balakrishnan and R. Kumar, Existence and nonexistence of certain labellings for the graph  $K_n^c \bigvee 2K_2$ , Util. Math., 46 (1994) 97-102.
- [338] R. Balakrishnan and R. Sampathkumar, Decompositions of regular graphs into  $K_n^c \bigvee 2K_2$ , Discrete Math., **156** (1996) 19-28.

- [339] R. Balakrishnan, A. Selvam, and V. Yegnanarayanan, On felicitous labelings of graphs, Proceed. National Workshop on Graph Theory and its Appl. Manonmaniam Sundaranar Univ., Tiruneli (1996) 47-61.
- [340] R. Balakrishnan, A. Selvam, and V. Yegnanarayanan, Some results on elegant graphs, *Indian J. Pure Appl. Math.*, 28 (1997) 905-916.
- [341] C. Balbuena, E. Barker, Y. Lin, M. Miller, and K. Sugeng, Consecutive magic graphs, *Discrete Math.*, **306** (2006) 1817-1829.
- [342] D. Bange, A. Barkauskas, and P. Slater, Simply sequential and graceful graphs, Proc. of the 10th S.E. Conf. on Combinat., Graph Theory, and Computing, (Util. Math. Winnipeg, 1979) 155-162.
- [343] D. Bange, A. Barkauskas, and P. Slater, Sequentially additive graphs, *Discrete Math.*, 44 (1983) 235-241.
- [344] J. Bao, L. Zhao, Y. Yang, W. Feng, and Jirimutu, The generalized Peterson graph P(n,7) is  $\frac{3n+6}{2}$ , 3)-antimagic, *Util. Math.*, **100** (2016) 33-41.
- [345] M. V. Bapat and N. B. Limaye, Some families of 3-equitable graphs, J. Combin. Math. Combin. Comput., 48 (2004) 179-196.
- [346] M. V. Bapat and N. B. Limaye, A note on 3-equitable labelings of multiple shells, J. Combin. Math. Combin., Comput. 51 (2004) 191-202.
- [347] M. V. Bapat and N. B. Limaye, Edge-three cordial graphs arising from complete graphs, J. Combin. Math. Combin. Comput., 56 (2006) 147-169.
- [348] M. V. Bapat and N. B. Limaye,  $E_3$ -cordiality of some helm-related graphs, Ars Combin., **119** (2015) 429-443.
- [349] C. Barrientos, New families of equitable graphs, Util. Math., 60 (2001) 123-137.
- [350] C. Barrientos, Graceful labelings of cyclic snakes, Ars Combin., 60 (2001) 85-96.
- [351] C. Barrientos, Equitable labelings of corona graphs, J. Combin. Math. Combin. Comput., 41 (2002) 139-149.
- [352] C. Barrientos, Graceful labelings of chain and corona graphs, *Bull. Inst. Combin.* Appl., **34** (2002) 17-26.
- [353] C. Barrientos, Difference Vertex Labelings, Ph.D. Thesis, Universitat Politécnica de Catalunya, Barcelona, 2004.
- [354] C. Barrientos, The gracefulness of unions of cycles and complete bipartite graphs, J. Combin. Math. Combin. Comput. 52 (2005) 69-78.

- [355] C. Barrientos, Graceful graphs with pendant edges, Australas. J. Combin. **33** (2005) 99-107.
- [356] C. Barrientos, Graceful arbitrary supersubdivisions of graphs, Indian J. Pure Appl. Math., 38 (2007) 445-450.
- [357] C. Barrientos, Odd-graceful labelings, preprint.
- [358] C. Barrientos, Unicylic graceful graphs, preprint.
- [359] C. Barrientos, On graceful chain graphs, Util. Math., 78 (2009) 55-64.
- [360] C. Barrientos, Odd-graceful labelings of trees of diameter 5, AKCE Int. J. Graphs Comb., 6 (2009) 307-313.
- [361] C. Barrientos, Alpha graphs with different pendent paths, *Elect. J. Graph Th. Appl.*, to appear.
- [362] C. Barrientos, On additive vertex labelings, Indonesian J. Combin., 4(1) (2020) 34-52.
- [363] C. Barrientos and S. Minion, Some families of  $\alpha$ -labeled subgraphs of the integral grid, preprint.
- [364] C. Barrientos and S. Minion, Harmonious graphs from  $\alpha$ -trees, preprint.
- [365] C. Barrientos, Mean trees, Bull. Inst. Combin. Appl., **75** (2015) 8-18.
- [366] C. Barrientos, M. E. Abdel-Aal, S. Minion, and D. Williams, The mean labeling of some crowns, J. Algorithms and Comput., 45 (2014) 43-54.
- [367] C. Barrientos and S. Barrientos, On graceful supersubdivisions of graphs, Bull. Inst. Combin. Appl., 70 (2014) 77-85.
- [368] C. Barrientos, I. Dejter, and H. Hevia, Equitable labelings of forests, Combin. and Graph Theory, 1 (1995) 1-26.
- [369] C. Barrientos and H. Hevia, On 2-equitable labelings of graphs, Notas de la Sociedad de Matemática de Chile, XV (1996) 97-110.
- [370] C. Barrientos and E. Krop, Mean graphs, AKCE Int. J. Graphs Comb., 11(1) (2014) 13-26.
- [371] C. Barrientos and E. Krop, Improved bounds for relaxed graceful trees, *Graphs Combin.*, **33** (2017) 287-305.
- [372] C. Barrientos and S. Minion, Alpha labelings of snake polyominoes and hexagonal chains, Bull. Inst. Combin. Appl., 74 (2015) 73-83.

- [373] C. Barrientos and S. Minion, Three graceful operations, J. Algorithms and Comput., 45 (2014) 13-24.
- [374] C. Barrientos and S. Minion, Enumerating families of labeled graphs, J. Integer Sequences, 18 (2015) Article 15.1.7, 14 pages.
- [375] C. Barrientos and S. Minion, New attack on Kotzig's Conjecture, Electron. J. Graph Theory Appl., 4(2) (2016) 119-131.
- [376] C. Barrientos and S. Minion, Broader families of cordial graphs, to appear in J. Graph Labelings, preprint.
- [377] C. Barrientos and S. Minion, On the graceful Cartesian product of alpha-trees, Theory Appl. Graphs, 4(1) (2017) Art. 3, 12 pp. https://digitalcommons. georgiasouthern.edu/cgi/viewcontent.cgi?article=1030&context=tag
- [378] C. Barrientos and S. Minion, on the number of  $\alpha$ -labeled graphs, *Discuss. Math. Graph Th.*, **38**(1) (2018) 177-188.
- [379] C. Barrientos and S. Minion, Constructing graceful graphs with caterpillars, J. Algor. Computation, 48 (2016) 117-125.
- [380] C. Barrientos and S. Minion, New  $\alpha$ -trees and graceful unions of  $\alpha$ -graphs and linear forests, J. Combin. Math. Combin. Comput., **108** (2019) 205-220.
- [381] C. Barrientos and S. Minion, Snakes: from graceful to harmonious, Bull. Institute Combin. Appl., CA 79 (2017) 95-107.
- [382] C. Barrientos and S. Minion, Snakes and caterpillars in graceful graphs, J. Algorithms and Comput., 50 (2018) 37-47.
- [383] C. Barrientos and S. Minion, Folding trees gracefully, AKCE Int. J. Graphs Comb., to appear.
- [384] C. Barrientos, Special graceful labelings of irregular fences and lobsters, Universal J. Math. Appl., 2(1) (2019) 1-10.
- [385] C. Barrientos and S. Minion, New advances in Kotzigs conjecture, Fundamental J. Math. Appl., 2(2) (2019) 186-194.
- [386] C. Barrientos and S. Minion, The graceful coalescence of alpha cycles, Comm. Adv. Math. Sc., 2(2) (2019) 114-120.
- [387] C. Barrientos and S. Minion, Series-parallel operations with alpha-graphs, *Theory* Appl. Graphs, **6**(1), Article 4 (2019).
- [388] C. Barrientos and S. Minion, Counting and labeling grid related graphs, *Electron. J. Graph Th. Appl.*, 7(2) (2019) 349-363.

- [389] C. Barrientos and S. Minion, Some families of  $\alpha$ -labeled subgraphs of the integral grid, preprint.
- [390] C. Barrientos and S. Minion, Harmonious graphs from  $\alpha$ -trees, preprint.
- [391] M. Barrus, Antimagic labeling and canonical decomposition of graphs, Inform. Process. Lett., 110 (2010) 261-263.
- [392] J. Baskar Babujee, Bimagic labelings in path graphs, Math. Education, 38 (2004) 12-16.
- [393] J. Baskar Babujee, On edge bimagic labeling, J. Combin. Inf. Syst. Sci., 28 (2004) 239-244.
- [394] J. Baskar Babujee, Euler's phi function and graph labeling, Int. J. Contemp. Math. Sci., 5 (2010) 977-984.
- [395] J. Baskar Babujee, Prime labelings on graphs, Proc. Jangjeon Math. Soc., 10 (2007) 121-129.
- [396] J. Baskar Babujee and S. Babitha, On 1-vertex bimagic vertex labeling, Tamkang J. Math., 45 (2014)(3) 259-272.
- [397] J. Baskar Babujee, S. Babitha, and V. Vishnupriya, New constructions of super edge bimagic labeling, Proceed. Inter. Conf. Math. and Comput. Sci. ICMCS 2010, 5-6 Feb. 2010 79-82.
- [398] J. Baskar Babujee and C. Beaula, On vertex strongly\*-graph, *Proceed. Internat.* Conf. Math. and Comput. Sci., 25-26 July 2008, Loyola College, Chennai.
- [399] J. Baskar Babujee and R. Jagadesh, Super vertex total edge bimagic labeling for graphs with cycles, *Pacific-Asian J. Math.*, **2** (2008) 113-122.
- [400] J. Baskar Babujee and R. Jagadesh, Super edge bimagic labeling for disconnected graphs, Inter. Journal Appl. Math. Eng. Sci., 2 (2008) 171-175.
- [401] J. Baskar Babujee and R. Jagadesh, Super edge bimagic labeling for trees, Inter. J. Analyzing Methods Components Combin. Biol. Math., 1 (2008) 107-116.
- [402] J. Baskar Babujee and R. Jagadesh, Super edge bimagic labeling for some classes of connected graphs derived from fundamental graphs, *Inter. J. Combin. Graph Theory Appl.*, 1 (2008) 85-92.
- [403] J. Baskar Babujee and R. Jagadesh, Vertex consecutive edge bimagic labeling for star like graphs Global J. Appl. Math. and Math. Sci., 1 (2008) 197-202.
- [404] J. Baskar Babujee and R. Jagadesh, Prime labeling for some class of acyclic graphs, Internat. J. Analyzing of Components and Combin. Biology in Math, 1 (2009) 27-35.

- [405] J. Baskar Babujee and R. Jagadesh, On *a*-vertex consecutive edge bimagic total labeling for graphs with cycles, *Indian J. Math. and Math. Sci.*, **5** (2009) 149-158.
- [406] J. Baskar Babujee and R. Jagadesh, On *a*-vertex consecutive edge bimagic total labeling for certain classes of graphs, *Internat. J. Math. Sci.*, **9** (2010) 49-61.
- [407] J. Baskar Babujee, R. Jagadesh, and V. Vishnupriya, On *a*-vertex consecutive edge bimagic total labeling for some class of graphs derives from fundamental graphs, *Pacific-Asian J. Math.*, **3** (2009) 283-291.
- [408] J. Baskar Babujee, K. Kannan, and V. Vishnupriya, Vertex strongly\*-graphs, Internat. J. Analyzing Components and Combin. Biology in Math., 2 19-25.
- [409] J. Baskar Babujee and N. Rao, Edge-magic trees, Indian J. Pure Appl. Math., 33 (2002) 1837-1840.
- [410] J. Baskar Babujee and L. Shobana, Prime cordial labelings, Int. Review on Pure and Appl. Math., 5 (2009) 277-282.
- [411] J. Baskar Babujee and L. Shobana, Prime and prime cordial labeling for some special graphs, Int. J. Contemp. Math. Sciences, 5 (2010) 2347-2356.
- [412] J. Baskar Babujee and L. Shobana, On Z<sub>3</sub>-magic labeling and Cayley digraphs, Int. J. Contemp. Math. Sciences, 5 (2010) 2357-2368.
- [413] J. Baskar Babujee and V. Vishnupriya, Prime labelings on trees, Internat. Review Pure Appl. Math., 2 (2006) 159-162.
- [414] J. Baskar Babujee and V. Vishnupriya, Permutation labelings for some trees, Internat. J. Math. Comput. Sci., 3 (2008) 31-38.
- [415] J. Baskar Babujee, V. Vishnupriya, and R. Jagadesh, On a-vertex consecutive edge bimagic labeling for trees, *Internat. J. Comput. Math. and Numerical Simulation*, 2 67-78.
- [416] E. Baskoro and Y. Cholily, Expanding super edge-magic graphs, J. Math. Fund. Sci., [S.l.], 36(2) (2013) 117-125.
- [417] E. Baskoro and A. A. G. Ngurah, On super edge-magic total labelings, Bull. Inst. Combin. Appl., 37 (2003) 82-87.
- [418] E. T. Baskoro, I. W. Sudarsana, and Y. M. Cholily, How to construct new super edge-magic graphs from some old ones, J. Indones. Math. Soc., 11(2) (2005) 155-162.
- [419] J. Bass, personal communication.

- [420] D. Bantva, S. Vaidya, and S. Zhou, Radio number of trees, The Eighth International Workshop on Graph Labelings (IWOGL 2014), *Electron. Notes Discrete Math.*, 48 (2015) 135-141.
- [421] D. Bantva, S. Vaidya, and S. Zhou, Discrete Applied Math., 217 (2017) 110-122.
- [422] R. Beals, J. Gallian, P. Headley, and D. Jungreis, Harmonious groups, J. Combin. Th., Series A, 56 (1991) 223-238.
- [423] A. F. Beardon, The maximum degree in a vertex-magic graph, Austral. J. Combin., 30 (2004) 113-116.
- [424] A. F. Beardon, Magic labellings of infinite graphs, Austral. J. Combin., 30 (2004) 117-132.
- [425] A. F. Beardon, The average degree in vertex-magic graph, Austral. J. Combin., 35 (2006) 51-56.
- [426] B. Beavers, Golomb rulers and graceful graphs, http://webcourse.cs. technion.ac.il/236801/Spring2009/ho/WCFiles/Golomb\_Rulers\_and\_ Graceful\_Graphs.pdf
- [427] M. Beck and M. Farahmand, Partially magic labelings and the antimagic graph conjecture, Sm. Lothar. Combin., 78B (2017) Art. 86, 11 pp.
- [428] M. Beck and M. Jackanich, Bipartite graphs are weakly antimagic, arXiv:1306.1763v2 [math.CO] 18 Jun 2013.
- [429] S. Beena, On  $\Sigma$  and  $\Sigma'$  labelled graphs, Discrete Math., **309** (2009) 1783-1787.
- [430] R. Raziya Begam, M. Palanivelrajan, K. Gunasekaran, and A. R. Shahul Hameed, Graceful labeling of some theta related graphs, *Internat. J. Fuzzy Mathematical Archives*, 2 (2013) 78-84.
- [431] A. K. Behera, D. Mishra, and P. C. Nayak, Some new classes of even-even and oddeven graceful graphs, *Internat. J. Appl. Engin. Res.*, **10**(19) (2015) 40171-40176.
- [432] L. W. Beineke and S. M. Hegde, Strongly multiplicative graphs, Discuss. Math. Graph Theory, 21 (2001) 63-75.
- [433] J. R. Bell, Products of graceful bipartite graphs, Aequat. Math., 93 (2019) 1245-1260. DOI:10.1007/s00010-019-00649-6
- [434] J. R. Bell and D. F. Cummins, Graceful pairings, Australas. J. Combin., 75 (2019) 343356.
- [435] A. Benini and A. Pasotti, Decompositions of complete multipartite graphs, via generalized graceful labelings, *Australasian J. Combin.*, **59**(1) (2014) 120-143.

- [436] A. Benini and A. Pasotti, α-labelings of a class of generalized Petersen graphs, Discuss. Math. Graph Th., 35 (2015) 43-53.
- [437] K. Benson, M. Porter, and M. Tomova, The radio numbers of all graphs of order n and diameter n 2, *Matematiche (Catania)*, **68**(2) (2013) 167-190.
- [438] M. Benson and S. M. Lee, On cordialness of regular windmill graphs, Congr. Numer., 68 (1989) 45-58.
- [439] K. Bérczi, A. Bernäth, and M. Vizer, Regular graphs are antimagic, *Electron. J. Combin.*, 22(3) (2015) Paper 3.34, 1-6. arXiv:1504.08146
- [440] C. Berge, Regularisable graphs II, Discrete Math., 23 (1978) 91-95.
- [441] D. Bergstrand, F. Harary, K. Hodges, G. Jennings, L. Kuklinski, and J. Wiener, The sum numbering of a complete graph, *Bull. Malaysian Math. Soc.*, **12** (1989) 25-28.
- [442] D. Bergstrand, K. Hodges, G. Jennings, L. Kuklinski, J. Wiener, and F. Harary, Product graphs are sum graphs, *Math. Magazine*, 65 (1992) 262-264.
- [443] O. Berkman, M. Parnas, and Y. Roditty, All cycles are edge-magic, Ars Combin., 59 (2001) 145-151.
- [444] A. H. Berliner, N. Dean, J. Hook, A. Marr, A. Mbirika, and C. D. McBee, Coprime and prime labelings of graphs, J. Integer Seq., 19(5) (2016) article 16.5.8, 14 pp.
- [445] J. C. Bermond, Graceful graphs, radio antennae and French windmills, *Graph Theory and Combinatorics*, Pitman, London (1979) 18-37.
- [446] J. C. Bermond, A. E. Brouwer, and A. Germa, Systemes de triplets et differences associées, *Problems Combinatories et Théorie des Graphs*, Colloq. Intern. du Centre National de la Rech. Scient., 260, Editions du Centre Nationale de la Recherche Scientifique, Paris (1978) 35-38.
- [447] J. C. Bermond and G. Farhi, Sur un probleme combinatoire d'antennes en radioastronomie II, Annals of Discrete Math., 12 (1982) 49-53.
- [448] J. C. Bermond, A. Kotzig, and J. Turgeon, On a combinatorial problem of antennas in radioastronomy, in Combinatorics, A. Hajnal and V. T. Sós, eds., *Colloq. Math. Soc. János Bolyai*, 18, 2 vols. North-Holland, Amsterdam (1978) 135-149.
- [449] J. C. Bermond and D. Sotteau, Graph decompositions and G-design, Proc. 5th British Combin. Conf., 1975, Congr. Numer., XV (1976) 53-72.

- [450] F. Bertault, M. Miller, H. Pé-Rosés, R. Feria-Puron, and E. Vaezpour, A heuristic for magic and antimagic graph labellings, *Proceed. VII Spanish Congress on Metaheuristics, and Evolutive and Bioinspired Algorithms,* (MAEB 2010) V. Campos, A. Duarte, M. Gallego, F. Cortazar, R. Marti (eds). Ibergarceta Publicaciones, S.L., Madrid. 677-684. arXiv:1305.1880
- [451] D. Beutner and H. Harborth, Graceful labelings of nearly complete graphs, *Result. Math.*, **41** (2002) 34-39.
- [452] L. Bezegová and J. Ivančo, An extension of regular supermagic graphs, Discrete Math., 310(24) (2010) 3571-3578.
- [453] L. Bezegová and J. Ivančo, On conservative and supermagic graphs, Discrete Math., 311(21) (2011) 2428-2436.
- [454] L. Bezegová and J. Ivančo, A characterization of complete tripartite degree-magic graphs, Discuss. Math. Graph Theory 32(2) (2012) 243-253.
- [455] L. Bezegová and J. Ivančo, Number of edges in degree-magic graphs, Discrete Math., 313 (2013) 1349-1357.
- [456] P. Bhals, S. Lake, and A. Werthem, Gracefulness of families of spiders, *Involve*, 3(3) (2010) 241-247.
- [457] V. Bhat-Nayak and U. Deshmukh, New families of graceful banana trees, Proc. Indian Acad. Sci. Math. Sci., 106 (1996) 201-216.
- [458] V. Bhat-Nayak and U. Deshmukh, Gracefulness of  $C_{4t} \cup K_{1,4t-1}$  and  $C_{4t+3} \cup K_{1,4t+2}$ , J. Ramanujan Math. Soc., **11** (1996) 187-190.
- [459] V. Bhat-Nayak and U. Deshmukh, Skolem-graceful labelings of unions of paths, personal communication.
- [460] V. Bhat-Nayak and U. Deshmukh, Gracefulness of  $C_3 \cup P_n$ , unpublished.
- [461] V. Bhat-Nayak and U. Deshmukh, Gracefulness of  $C_{2x+1} \cup P_{x-2\theta}$ , Proc. International Conf. on Graph Theory and Number Theory, Trichy 1996.
- [462] V. N. Bhat-Nayak and S. K. Gokhale, Validity of Hebbare's conjecture, Util. Math., 29 (1986) 49-59.
- [463] V. N. Bhat-Nayak and A. Selvam, Gracefulness of *n*-cone  $C_m \vee K_n^c$ , Ars Combin., **66** (2003) 283-298.
- [464] V. N. Bhat-Nayak and S. Telang, Cahit-k-equitability of  $C_n \circ K_1$ , k = n to 2n 1,  $n \ge 3$ , Congr. Numer., 155 (2002) 131-213.
- [465] V. N. Bhat-Nayak and S. Telang, Cahit-equitability of coronas, Ars Combin., 71 (2004) 3-32.

- [466] A. A. Bhatti, M. Javaid, and M. Hussain, On super (a, d)-edge-antimagic total labeling of subdivided caterpillar, Util. Math., 98 (2015) 227-241.
- [467] A. A. Bhatti, M. Tahir and M. Javaid, Super (a, d)-EAT labeling of some wheel like graphs, J. Graph Labeling, 1(2) (2015) 137-149.
- [468] A. A. Bhatti, Q. Zahra, and M. Javaid, Further results on super (a, d)-EAT labeling of subdivided stars, Util. Math., 98 (2015) 113-126.
- [469] R. Binthiya and P. B. Sarasija, Some new even harmonious graphs, Internat. Math. Soft Comput., 4(2) (2014) 105-111.
- [470] A. Blinco, S. El-Zanati, and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, *Discrete Math.*, 284 (2004) 71-81.
- [471] G. S. Bloom, A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful, Ann. N.Y. Acad. Sci., 326 (1979) 32-51.
- [472] G. S. Bloom and S. W. Golomb, Applications of numbered undirected graphs, Proc. IEEE, 65 (1977) 562-570.
- [473] G. S. Bloom and S. W. Golomb, Numbered complete graphs, unusual rulers, and assorted applications, in *Theory and Applications of Graphs, Lecture Notes in Math.*, 642, Springer-Verlag, New York (1978) 53-65.
- [474] G. S. Bloom, P. Hell, and H. Taylor, Collecting autographs: n-node graphs that have n-integer signatures, Annals N.Y. Acad. Sci., 319 (1979) 93-102.
- [475] G. S. Bloom and D. F. Hsu, On graceful digraphs and a problem in network addressing, *Congr. Numer.*, **35** (1982) 91-103.
- [476] G. S. Bloom and D. F. Hsu, On graceful directed graphs that are computational models of some algebraic systems, *Graph Theory with Applications to Algorithms* and Computers, Ed. Y. Alavi, Wiley, New York (1985).
- [477] G. S. Bloom and D. F. Hsu, On graceful directed graphs, SIAM J. Alg. Discrete Meth., 6 (1985) 519-536.
- [478] G. S. Bloom, A. Marr, W. D. Wallis, Magic digraphs, J. Combin. Math. Combin. Comput., 65 (2008) 205-212.
- [479] H. Bodlaender, J. Deogun, K. Jansen, T. Kloks, D. Kratsch, H. Müller, and Z. Tuza, Rankings of graphs, SIAM J. Discrete Math., 11 (1998) 168-181.
- [480] R. Bodendiek, H. Schumacher, and H. Wegner, Über graziőse Graphen, Math.-Phys. Semesterberichte, 24 (1977) 103-106.
- [481] R. Bodendiek and G. Walther, Arithmetisch antimagische Graphen, Graphentheorie III, K. Wagner and R. Bodendiek (eds.), Mannhein, 1993.

- [482] R. Bodendiek and G. Walther, On number theoretical methods in graph labelings, *Res. Exp. Math.*, **21** (1995) 3-25.
- [483] R. Bodendiek and G. Walther, (A, D)-antimagic parachutes, Ars Combin., 42 (1996) 129-149.
- [484] R. Bodendiek and G. Walther, (a, d)-antimagic parachutes II, Ars Combin., 46 (1997) 33-63.
- [485] R. Bodendiek and G. Walther, On arithmetic antimagic edge labelings of graphs, Mitt. Math. Ges. Hamburg, 17 (1998) 85-99.
- [486] R. Bodendiek, H. Schumacher, and H. Wegner, Über eine spezielle Klasse groziőser Eulerscher Graphen, Mitt. Math. Gesellsch. Hamburg, 10 (1975) 241-248.
- [487] S. A. H. Bokhary, A. Ahmad, and M. Imran, On vertex irregular total labelings of cartesian products of two paths, *Util. Math.*, **90** (2013) 239-249.
- [488] S. A. H. Bokhary, U. Ali, and S. Maqbool, Irregular total labeling of wheel related graphs, Util. Math., 107 (2018) 231-242.
- [489] S. A. Bokhary and H. Faheem, Vertex irregular total labeling of grid graph, Palest. J. Math., 8(1) (2019) 52-62.
- [490] J. Boland, R. Laskar, C. Turner, and G. Domke, On mod sum graphs, Congr. Numer., 70 (1990) 131-135.
- [491] J. Bondy and U. Murty, *Graph Theory with Applications*, North-Holland, New York (1976).
- [492] R. Boonklurb, A. Narissayaporn, and S. Singhun, Super edge-magic labeling of mnode k-uniform hyperpaths and m-node k-uniform hypercycles, AKCE Internat. J. Graphs and Combin., 13 (2016) 218-229.
- [493] I. Borosh, D. Hensley, and A. Hobbs, Vertex prime graphs and the Jacobsthal function, *Congr. Numer.*, **127** (1997) 193-222.
- [494] D. Bouchard, P. Clark, S. M. Lee, S.-P. Lo, and H.-H. Su, On balance index sets of generalized book and ear expansion graphs, 24th MCCCC. J. Combin. Math. Combin. Comput., 82 (2012) 3-15.
- [495] D. Bouchard, P. Clark, and H.-H. Su, On edge-balance index sets of L-product of cycles, J. Indones. Math. Soc., Special Edition (2011) 27-38.
- [496] D. Bouchard, P. Clark, and H.-H. Su, On edge-balance index sets of L-product of cycles with stars, Part II, 24th MCCCC. J. Combin. Math. Combin. Comput., 82 (2012) 199-209.

- [497] S. A. Boxwala and P. Vashishta, Some new families of graceful graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014) 127-133, *Electron. Notes Discrete Math.*, 48, Elsevier Sci. B. V., Amsterdam, 2015.
- [498] S. Brandt, J. Miškuf, and D. Rautenbach, On a conjecture about edge irregular total labelings, J. Graph Theory, 57 (2008) 333-343.
- [499] L. Brankovic, C. Murch, J. Pond, and A. Rosa, Alpha-size of trees with maximum degree three and perfect matching. In: *Proceedings of AWOCA2005*, 18-21 September, Ballarat, Australia, (2005) 47-56.
- [500] L. Brankovic, A. Rosa, and J. Širáň, Labellings of trees with maximum degree three-an improved bound., J. Combin. Math. Combin. Comput., 55 (2005) 159-169.
- [501] L. Brankovic and I. M. Wanless, Graceful labelling: State of the art, applications and future directions, *Math Comput. Sci.*, **5** (2011) 11-20.
- [502] B. Brešar and S. Klavžar, Θ-graceful labelings of partial cubes, Discrete Math. 306 (2006) 1264-1271.
- [503] G. Brinkmann, S. Crevals, H. Mélot, L.J. Rylands, and E. Steffan, α-labelings and the structure of trees with nonzero α-deficit. Discrete Math. Theor. Comput. Sci., 14(1) (2012) 159-174.
- [504] H. Broersma and C. Hoede, Another equivalent of the graceful tree conjecture, Ars Combin., 51 (1999) 183-192.
- [505] C. Bu, Gracefulness of graph  $K_n + \overline{K_m}$ , J. Harbin Shipbuilding Eng. Inst., **15** (1994) 91-93.
- [506] C. Bu, On edge-gracefulness of graphs, *Heilongjiang Daxue Ziran Kexue Xuebao*, 12 (1995) 18-20.
- [507] C. Bu, Sequential labeling of the graph  $C_n \odot \overline{K}_m$ , unpublished.
- [508] C. Bu and C. Cao, The gracefulness for a class of disconnected graphs, J. Natural Sci. Heilongjiang Univ., 12 (1995) 6-8.
- [509] C. Bu and L. Chen, Some conclusions about graceful graphs, J. Harbin Shipbuilding Eng. Inst., 14 (1993) 103-108.
- [510] C. Bu and W. Feng, Some composite theory about graceful graphs, J. Harbin Eng. Univ., 16 (1995) 94-97.
- [511] C. Bu, Z. Gao, and D. Zhang, On k-gracefulness of  $r p_n \times p_2$ , J. Harbin Shipbuilding Eng. Inst., **15** (1994) 85-89.

- [512] C. Bu and B. He, The k-gracefulness of some graphs, J. Harbin Eng. Univ., 14 (1993) 92-95.
- [513] C. Bu and J. Shi, A class of (k, d)-arithmetic graphs, J. Harbin Eng. Univ., 16 (1995) 79-81.
- [514] C. Bu and J. Shi, Some conclusions about indexable graphs, J. Harbin Eng. Univ., 16 (1995) 92-94.
- [515] C. Bu and J. Zhang, The properties of (k, d)-graceful graphs, unpublished.
- [516] C. Bu, D. Zhang, and B. He, k-gracefulness of  $C_n^m$ , J. Harbin Shipbuilding Eng. Inst., **15** (1994) 95-99.
- [517] R. Bunge, S. El-Zanati, W. O'Hanlon, and C. Vanden Eynden, On  $\gamma$ -labeling almost-bipartite graphs, Ars Combin., **107** (2012) 65-80.
- [518] R. Bunge, A. Chantasartrassmee, S. El-Zanati, and C. Vanden Eynden, On cyclic decompositions of complete graphs into tripartite graphs, J. Graph Theory, 72 (2013) 90-111.
- [519] M. Buratti, G. Rinaldi, and T. Traetta, Some results on 1-rotational Hamiltonian cycle systems, J. Combin. Designs, 22(6) (2014) 231-251.
- [520] M. Burzio and G. Ferrarese, The subdivision graph of a graceful tree is a graceful tree, *Discrete Math.*, 181 (1998) 275-281.
- [521] S. I. Butt, M. Numan, I. A. Shah, and S. Ali, Face labelings of type (1,1,1) for generalized prism, Ars Combin., 137 (2018) 41-52.
- [522] C. P. Bonnington and J. Širáň, Bipartite labelings of trees with maximum degree three, J. Graph Theory, 31 (1999) 79-91.
- [523] M. I. Bosmia and K. K. Kanani, Divisor cordial labeling in the context of corona product, 9th National Level Science Symposium. February 14, 2016, Organized by Christ College, Rajkot, Sponsored by GUJCOST, Gandhinagar, Mathematics & Statistics, 3 (2016) 178-182.
- [524] M. I. Bosmia and K. K. Kanani, Divisor cordial labeling in the context of graph operations on bistar, *Global J. Pure and Appl. Math.*, **12**(3) (2016) 2605-2618.
- [525] S. Cabaniss, R. Low, and J. Mitchem, On edge-graceful regular graphs and trees, Ars Combin., 34 (1992) 129-142.
- [526] L. Caccetta and R. Jia, Positive binary labelings of graphs, Austral. J. Combin., 14 (1996) 133-148.
- [527] I. Cahit, Elegant valuation of the paths, Ars Combin., 16 (1983) 223-227.

- [528] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, Ars Combin., 23 (1987) 201-207.
- [529] I. Cahit, On cordial and 3-equitable labellings of graphs, Util. Math., **37** (1990) 189-198.
- [530] I. Cahit, Status of graceful tree conjecture in 1989, in *Topics in Combinatorics and Graph Theory*, R. Bodendiek and R. Henn (eds), *Physica-Verlag*, Heidelberg 1990.
- [531] I. Cahit, Recent results and open problems on cordial graphs, Contemporary Methods in Graph Theory, R. Bodendiek (ed.), Wissenschaftsverlag, Mannheim, 1990, 209-230.
- [532] I. Cahit, Equitable tree labellings, Ars Combin., 40 (1995) 279-286.
- [533] I. Cahit, On harmonious tree labellings, Ars Combin., 41 (1995) 311-317.
- [534] I. Cahit, *H*-cordial graphs, *Bull. Inst. Combin. Appl.*, **18** (1996) 87-101.
- [535] I. Cahit, Some totally modular cordial graphs, Discuss. Math. Graph Theory, 22 (2002) 247-258.
- [536] I. Cahit, Graceful labelings of rooted complete trees, personal communication.
- [537] I. Cahit, A note on graceful directed trees, personal communication.
- [538] I. Cahit and R. Yilmaz,  $E_3$ -cordial graphs, Ars Combin., 54 (2000) 119-127.
- [539] H. Cai, L. X. Wei, X. R. Lu, Gracefulness of unconnected graphs  $(P_1 \vee P_n) \cup G_r, (P_1 \vee P_n) \cup (P_3 \vee \overline{K}_r)$  and  $W_n \cup St(m)$ , J. Jilin Univ. Sci., **45** (2007) 539-543.
- [540] B. Calhoun, K. Ferland, L. Lister, and J. Polhill, Totally magic labelings of graphs, Australas. J. Combin., 32 (2005) 47-59.
- [541] N. Cairnie and K. Edwards, The computational complexity of cordial and equitable labelling, *Discrete Math.*, **216** (2000) 29-34.
- [542] L. A. Cammack, Schelp and G. Schrag, Irregularity strength of full d-ary trees Congr. Numer., 81 (1991) 113-119.
- [543] D. Canales, M. Tomova, and C. Wyels, A gap in the achievable radio number line, AKCE Int. J. Graphs Combin., 10(4) (2013) 349-357.
- [544] K. Carlson, Generalized books and  $C_m$ -snakes are prime graphs, Ars Combin. 80 (2006) 215-221.
- [545] M. R. Cerioli, C. G. Fernandes, O. Lee, C. Lintzmayer, G. O. Mota, and C. N. da Silva, On Edge-magic labelings of forests, *Elect. Notes Th. Comput. Sci.*, 346 (2019) 299307. https://www.researchgate.net/publication/336175787\_On\_Edge-magic\_Labelings\_of\_Forests/references#fullTextFileContent

- [546] Y. Caro, Y. Roditty, and J. Schőnheim, Starters for symmetric (n, G, 1)-designs.  $\rho$ -labelings revisited, unpublished.
- [547] R. Cattell, Vertex magic total labelings of complete multipartite graphs, J. Combin. Math. Combin. Computing, 55 (2005) 187-197.
- [548] R. Cattell, Graceful labellings of paths, *Discrete Math.*, **307** (2007) 3161-3176.
- [549] N. Cavenagh, D. Combe, and A. M. Nelson, Edge-magic group labellings of countable graphs, *Electronic J. Combin.*, **13** (2006) #R92 (19 pages).
- [550] M. R. Cerioli, C. G. Fernandes, O. Lee, C. N. Lintzmayer, On edge-magic labelings of forests, *Elect. Notes Theor. Comput. Sci.*, **346**(30) (2019) 299-307. https: //www.sciencedirect.com/science/article/pii/S157106611930077
- [551] W. H. Chan, R. M. Low, and W. C. Shiu, Group-antimagic labelings of graphs, Congr. Numberrantim, 217 (2013) 21-31.
- [552] B. Chandra and R. Kala, Group S<sub>3</sub> cordial prime labeling of graphs, Malaya J. Matematik, S(1) (2019) 403-407. https://www.researchgate.net/publication/ 334700018\_Group\_S\_3\_cordial\_prime\_labeling\_of\_graphs
- [553] C.-W. Chang, D. Kuo, and H.-C. Lin, Ranking numbers of graphs, *Inform. Process. Letters* **110** (2010) 711-716.
- [554] F.-H. Chang, H.-B. Chen, W.-T. Li, and Z. Pan, Shifted-antimagic labelings for graphs, arXiv:1806.06019.
- [555] F. Chang, Y.-C. Liang, Z. Pan, and X. Zhu, Antimagic labeling of regular graphs, J. Graph Theory, 82 (2016) 339-349. arXiv:1505.07688
- [556] F.-H. Chang, P. Chin, W.-T. Li, and Z. Pan, The strongly antimagic labelings of double spiders, arXiv:1712.09477
- [557] G. J. Chang, Strong sum graphs, Bull. Inst. Combin. Appl., 7 (1993) 47-52.
- [558] G. J. Chang, D. F. Hsu, and D. G. Rogers, Additive variations on a graceful theme: some results on harmonious and other related graphs, *Congr. Numer.*, **32** (1981) 181-197.
- [559] G. Chartrand, D. Erwin, D. VanderJagt, and P. Zhang, γ-labelings of graphs, Bull. Inst. Combin Appl., 44 (2005) 51-68.
- [560] G. Chartrand, D. Erwin, D. VanderJagt, and P. Zhang, On γ-labeling of trees, Discuss. Math. Graph Theory, 25 (2005) 363-383.
- [561] G. Chartrand, D. Erwin, and P. Zhang, Radio antipodal colorings of graphs, Math. Bohem., 127 (2002) 57-69.

- [562] G. Chartrand, D. Erwin, and P. Zhang, A graph labeling problem suggested by FM channel restrictions, Bull. Inst. Combin. Appl., 43 (2005) 43-57.
- [563] G. Chartrand, D. Erwin, P. Zhang, and F. Harary, Radio labelings of graphs, Bull. Inst. Combin. Appl., 33 (2001) 77-85.
- [564] G. Chartrand, H. Hevia, and O.R. Oellermann, The chromatic number of a factorization of a graph, Bull. Inst. Combin. Appl., 20 (1997) 33-56.
- [565] G. Chartrand, M. Jacobson, J. Lehel, O. Oellermann, S. Ruiz, and F. Saba, Irregular networks, *Congr. Numer.*, 64 (1988) 187-192.
- [566] G. Chartrand, S. M. Lee, and P. Zhang, Uniformly cordial graphs, *Discrete Math.*, 306 (2006) 726-737.
- [567] G. Chartrand and L. Lesniak, *Graphs & Digraphs* 4th ed. CRC Press (2005).
- [568] G. Chartrand, R. Muntean, V. Saenpholphat, and P. Zhang, Which graphs are divisor graphs?, *Congr. Numer.*, **151** (2001) 189-200.
- [569] G. Chartrand, L. Nebesky, and P. Zhang, Radio k-colorings of paths, Discuss. Math. Graph Theory, 24 (2004) 5-21.
- [570] P. D. Chawathe and V. Krishna, Odd graceful labelings of countably infinite locally finite bipartite graphs, *Conference on Graph Theory and its Applications*, March 2001, School of Mathematics, Anna University, Chennai.
- [571] P. D. Chawathe and V. Krishna, Antimagic labelings of complete *m*-ary trees, Number theory and discrete mathematics (Chandigarh, 2000), 77-80, Trends Math., Birkhäuser, Basel, 2002.
- [572] T. T. Chelvam, N. M. Rilwan, and G. Kalaimurugan, Antimagic and magic labelings in Cayley digraphs, Australas. J. Combin., 55 (2013) 65-71.
- [573] D. L. Chen and C. J. Jiang, The K-gracefulness of the rhomb-ladder graph  $\nabla_n^m$ , Shandong Kuangye Xueyuan Xuebao, **11** (1992) 196-199, 208.
- [574] K.-J. Chen, S.-M. Lee, and Y.-C. Wang, On the edge-graceful indices of the L-product of (p, p + 1) graphs and  $K_2$ , unpublished.
- [575] W. C. Chen, H. I. Lü, and Y. N. Yeh, Operations of interlaced trees and graceful trees, Southeast Asian Bull. Math., 21 (1997) 337-348.
- [576] L.-C. Chen, On harmonious labelings of the amalgamation of wheels, Master's Thesis, Chung Yuan Christian University, Taiwan.
- [577] Z. Chen, Harary's conjectures on integral sum graphs Discrete Math., 160 (1996) 241-244.

- [578] Z. Chen, Integral sum graphs from identification, *Discrete Math.*, **181** (1998) 77-90.
- [579] Z. Chen, On super edge-magic graphs. J. Combin. Math. Combin. Comput., 38 (2001) 55-64.
- [580] Z. Chen, On integral sum graphs, *Discrete Math.*, **306** (2006) 19-25.
- [581] Z.-Z. Chen, A generalization of the Bodendiek conjecture about graceful graphs, *Topics in Combinatorics and Graph Theory*, R. Bodendiek and R. Henn, eds., Physica-Verlag, Heidelberg, 1990, 737-746.
- [582] Z. Chen, On integral sum graphs with a saturated vertex, Czechoslovak Math. J., 60(135) (2010) 669-674.
- [583] Y. Cheng, Lattice grids and prisms are antimagic, Theoret. Comput. Sci., 374 (2007) 66-73. arXiv:0603106 [math.CO].
- [584] Y. Cheng, A new class of antimagic Cartesian product graphs, *Discrete Math.*, 308 (2008) 6441-6448.
- [585] H. Cheng, B. Yao, X. Chen, and Z. Zhang, On graceful generalized spiders and caterpillars, Ars Combin. 87 (2008) 181-191.
- [586] C. Chithra, G. Marimuthu, and G. Kumar, Some cycle-supermagic labeling of graphs, Internat. J. Comput. Eng. Res., (IJCER) 8(9) (2018) 79-83.
- [587] A. M. Chitre and N. B. Limaye, On 5-equitability of one point union of shells, AKCE J. Graphs Combin., 6 (2009) 57-68.
- [588] A. M. Chitre and N. B. Limaye, On edge-3-equitability of  $\overline{K_n}$ -union of gears, J. Combin. Math. Combin. Comput., 83 (2012) 129-150.
- [589] A. M. Chitre and N. B. Limaye, On edge-3-equitability of  $\overline{K_n}$ -union of helms, J. Combin. Math. Combin. Comput., to appear.
- [590] J. O. Choi, J. P. Georges, and D. Mauro, David, Relating edge-coverings to the classification of Z<sup>k</sup><sub>2</sub>-magic graphs, *Discrete Math.*, **312**(19) (2012) 2938-2945.
- [591] D. Chopra, R. Dios, and and S. M. Lee, On the edge-magicness of Zykov sum of graphs, personal communication.
- [592] D. Chopra and S. M. Lee, On super edge-magic graphs which are weak magic, J. Combin. Math. Combin. Comput., 62 (2007) 177-187.
- [593] D. Chopra and S. M. Lee, On Q(a)P(b)-super edge-graceful graphs, J. Combin. Math. Combin. Comput., 58 (2006) 135-152.

- [594] D. Chopra and S. M. Lee, On the integer magic spectra of the generalized theta graphs, unpublished.
- [595] D. Chopra, S. M. Lee, and H. H. Su, On edge-balance index sets of fans and wheels, unpublished.
- [596] C.-C. Chou and S. M. Lee, On  $Z_3$ -magic graphs, unpublished.
- [597] C.-C. Chou, M. Galiardi, M. Kong, S. M. Lee, D. Perry, and H.-H. Su, On edgebalance index sets of *L*-product of cycles with stars, Part I. J. Combin. Math. Combin. Comput., 78 (2011) 195-211.
- [598] D. Chopra, S. M. Lee, and H. H. Su, On edge-balance index sets of fans and broken fans, Proceedings of the Fortieth Southeastern International Conference on Combinatorics, Graph Theory and Computing, *Congr. Numer.*, **196** (2009) 183-201.
- [599] S. A. Choudum and S. P. M. Kishore, All 5-stars are Skolem graceful, Indian J. Pure and Appl. Math., 27 (1996) 1101-1105.
- [600] S. A. Choudum and S. P. M. Kishore, Graceful labellings of the union of paths and cycles, *Discrete Math.*, 206 (1999) 105-117.
- [601] S. A. Choudum and S. P. M. Kishore, On Skolem gracefulness of k-stars, Ramanujan Mathematical Society Inter. Conf. Discrete Math. and Number Theory, 1996.
- [602] S. A. Choudum and S. P. M. Kishore, Graceful labelling of the union of cycles and stars, unpublished.
- [603] F. R. K. Chung and F. K. Hwang, Rotatable graceful graphs, Ars Combin., 11 (1981) 239-250.
- [604] P.-T. Chung and S. M. Lee, On the super edge-graceful spiders of even order, J. Combin. Math. Combin. Comput., 64 (2008) 3-17.
- [605] P.-T. Chung and S. M. Lee, A note on the edge-balance index sets for graphs, Cong. Numer., 203 (2010) 105-129.
- [606] P.-T. Chung, S. M. Lee, W.-Y. Gao, More trees of even orders which are super edge-graceful, *Congr. Numer.*, **206** (2010) 41-58.
- [607] P.-T. Chung, S. M. Lee, W.-Y. Gao, and K. Schaffer, On the super edge-graceful tress of even orders, Proceedings of the Thirty-Seventh Southeastern International Conference on Combinatorics, Graph Theory and Computing, *Congr. Numer.*, 181 (2006) 5-17.

- [608] T. Chunling, L. Xiaohui, Y. Yuansheng, and W. Liping, Irregular total labellings of some families of graphs, *Indian J. Pure Appl. Math.*, 40(3)(2006) 155-181.
- [609] S. Cichacz, Group distance magic labeling of some cycle-related graphs, Australas. J. Combin., 57 (2013) 235-243.
- [610] S. Cichacz, Note on group distance magic complete bipartite graphs, Central European J. Math., 12 (3) (2014) 529-533.
- [611] S. Cichacz, Note on group distance magic graphs  $G[C_4]$ , Graphs Combin., **30** (3) (2014) 565-571. doi:10.1007/s00373-013-1294-z
- [612] S. Cichacz, Group distance magic graphs  $G \times C_n$ , Discrete Appl. Math. 177 (20) (2014) 80-87. doi:10.1016/j.dam.2014.05.044
- [613] S. Cichacz, Distance magic (r, t)-hypercycles, Util. Math., 101 (2016) 283-294.
- [614] S. Cichacz, A Γ-magic rectangle set and group distance magic labeling, Combinatorial algorithms, 122-127, Lecture Notes in Comput. Sci., 8986, Springer, Cham, 2015. doi:10.1007/978-3-319-19315-1\_11
- [615] S. Cichacz, On zero sum-partition of Abelian groups into three sets and group distance magic labeling, Ars Math. Contemporanea, 13(2) (2017) 417-425.
- [616] S. Cichacz, Zero sum partition into sets of the same order and its applications, *Electronic J. Combin.*, 25(1) (2018) #P1.20.
- [617] S. Cichacz, B. Freyberg, D. Fronček, Orientable  $\zeta_n$ -distance magic graphs, *Discuss.* Math. - Graph Theory, (2017), to appear. doi:10.7151/dmgt.2094.
- [618] S. Cichacz and D. Fronček, Distance magic circulant graphs, Discrete Math., 339(1) (2016) 84-94. http://www.ii.uj.edu.pl/documents/12980385/ 26042491/MD\_71.pdf
- [619] S. Cichacz and D. Fronček, E. Krop, and C. Raridan, Distance magic Cartesian products of graphs, *Discuss Math. Graph Theory*, **36** (2016) 299-308. doi:10.7151/dmgt.1852
- [620] S. D. Fronček, I. Singgih, label-Cichacz, Vertex magic total ings of 2-regular graphs, Discrete math.,  $\mathbf{340}$ (1)(2017)3117-3124. doi:10.1016/j.disc.2016.06.022
- [621] S. Cichacz, D. Fronček, K. Sugeng, and S. Zhou, Group distance magic and antimagic graphs, *Acta Math. Sinica*, English Series **32**(10) (2016) 1159-1176. doi:10.1007/s10114-016-4646-9
- [622] S. Cichacz, D. Fronček, A. Khodkar, and W. Xu, Super edge-graceful paths and cycles, Bull. ICA. 57 (2009) 79-90.

- [623] S. Cichacz, A. Görlich, Constant sum partition of set of integers and distance magic graphs, *Discuss. Math. Graph Theory*, 38(1) (2018) 97-106.
- [624] S. Cichacz, A. Görich, and Z. Tuza, Cordial labeling of hypertrees, *Disc. Math.*, 313 (22) (2013) 2518-2524. doi:10.1016/j.disc.2013.07.025
- [625] S. Cichacz and M. Nikodem, Union of distance magic graphs, Discuss. Math. Graph The., 37(1) 2017 239-249.
- [626] N. E. Clarke and A. Sanaei, Skolem labelling of generalised Dutch windmills, Australasian J. Combin., 66(3) (2016) 407-419.
- [627] B. Clemens, R. Coulibaly, J. Garvens, J. Gonnering, J. Lucas, and S. Winters, An introduction to the gracefulness of one-point and two-point union of graphs, *Congr. Numer.*, **171** (2004) 167-171.
- [628] A. Clifton and A. Khodkar Super edge-graceful labelings of kites, J. Combin. Math. Combin. Comput., 102 (2017) 277-295.
- [629] Z. Coles, A. Huszar, J. Miller, and Z. Szaniszlo, 4-equitable tree labelings, Congressus Numerantium 228 (2017) 51-63.
- [630] A. Collins, C. Magnant, H. Wang, Tight super-edge-graceful labelings of trees and their applications, AKCE Internat. J. Graphs Combin., 12 (2015) 113-118.
- [631] D. Combe, Magic labelings of graphs over finite abelian groups, Austral. J. Comb. 29 (2004) 259-271.
- [632] D. Craft and E. H. Tesar, On a question by Erdős about edge-magic graphs, Discrete Math., 207 (1999) 271-276.
- [633] D. W. Cranston, Regular bipartite graphs are antimagic, J. Graph Theory, 60 (2009) 173-182.
- [634] D. W. Cranston, Y. Liang and X. Zhu, Odd degree regular bipartite graphs are anti-magic, J. Graph Theory, 80(1) (2015) 28-33.
- [635] Dafik, M. Miller, and J. Ryan, Super edge-magic total labelings of  $mK_{n,n,n}$ , Ars Combin., **101** 97-107.
- [636] Dafik, M. Miller, J. Ryan, and M. Bača, On antimagic labelings of disjoint union of complete s-partite graphs, J. Combin. Math. Combin. Comput., 65 (2008) 41-49.
- [637] Dafik, M. Miller, J. Ryan, and M. Bača, On super (a, d)-edge-antimagic total labelings of disconnected graphs, *Discrete Math.*, **309** (2009) 4909-4915.
- [638] Dafik, M. Miller, J. Ryan, and M. Bača, Super edge-antimagic labeling of  $mK_{n,n,n}$ , Ars Combin., **101** (2011) 97-107.

- [639] Dafik, M. Miller, J. Ryan, and M. Bača, Antimagic labelings of the union of two stars, Australas. J. Combin., 42 (2008) 35-44.
- [640] Dafik, Slamin, D. Tana, A. Semaničová-Feňovčíková, and M. Bača, Constructions of *H*-antimagic graphs using smaller edge-antimagic graphs, *ArsComb.*, **133** (2017) 233-245.
- [641] Dafik, S. Slamin, R. W. Romdhani, and I. Y. Arianti, On super antimagicness of generalized flower and disk brake graphs, *Util. Math.*, **100** (2016) 127-135.
- [642] H. Dao, H. Hua, M. Ngo, and C. Raridan, On the edge-balanced index sets of complete even bipartite graphs, J. Graph Labeling, 2(1) (2016) 79-87.
- [643] S. N. Daoud, Edge odd graceful labeling of cylinder and torus grid graphs, *IEEE*, 7 (2019) 10568-10592. doi:10.1109/ACCESS.2018.2889293.
- [644] S. N. Daoud, Edge even graceful labeling of polar grid graphs, Symmetry, 11(38):22 (2019) 1-28. https://www.mdpi.com/2073-8994/11/1/38
- [645] S. N. Daoud, Vertex odd graceful labeling, Ars Combin., 142 (2019) 65-87.
- [646] R. Darda and A. Hujdurović, On bounds for the product irregularity strength of graphs, *Graphs and Combin.*, 31(5) (2015) 1347-1357.
- [647] 5. S. Das, S. Ghosh, S. Nandi, S. Sen, A lower bound technique for radio k-coloring, Discrete Math., 340 (2017) 855-861.
- [648] G. K. Dayanand and S. Ahmed, The super magic properties of connected and disconnected graphs, J. Inf. Optim. Sci., 36(3) (2015) 231-246.
- [649] P. de la Torre, R. Greenlaw, and T. Przytycka, Optimal tree ranking is in NC, Parallel Process. Lett., 2 (1992) 31-41.
- [650] Daili, Z-H Wang, and Z. Xie, Gracefulness of  $P_{2r,2m}$ , Ars Combin., **104** (2012) 185-191.
- [651] J. W. Daykin, C. S. Iliopoulos, M. Miller, and O. Phanalasy, Antimagicness of generalized corona and snowflake graphs, *Math. Comput. Sci.*, 9(1) (2015) 105111.
- [652] N. Dean, Proof of the prime ladder conjecture, Integers, A40 (2017) 9 pages.
- [653] P. Deb and N. B. Limaye, On elegant labelings of triangular snakes, J. Combin. Inform. System Sci., 25 (2000) 163-172.
- [654] P. Deb and N. B. Limaye, Some families of elegant and harmonius graphs, Ars Combin., 61 (2001) 271-286.
- [655] P. Deb and N. B. Limaye, On harmonius labelings of some cycle related graphs, Ars Combin., 65 (2002) 177-197.

- [656] A. Delman, S. Koilraj, and P. L. Raj, A neighbourhood prime labeling of some special graphs, Inter. J. Pure Appl. Math., 120(4) (2018) 601-607. https://ijpam. eu/contents/2018-120-4/10/10.pdf
- [657] C. Delorme, Two sets of graceful graphs, J. Graph Theory, 4 (1980) 247-250.
- [658] C. Delorme, M. Maheo, H. Thuillier, K. M. Koh, and H. K. Teo, Cycles with a chord are graceful, J. Graph Theory, 4 (1980) 409-415.
- [659] K. Deng and Y. Li, Caterpillars with maximum degree 3 are antimagic, Discrete Math., 342(6) (2019) 1799-1801.
- [660] Y. X. Deng, Harmoniousness of the graphs  $C_n + \overline{K}_t$ , J. Math. Res. Exposition, 15 (1995), suppl., 79-81.
- [661] G. Denham, M. G. Leu, and A. Liu, All 4-stars are Skolem-graceful, Ars Combin., 36 (1993) 183-191.
- [662] D. Dereniowski, Rank Coloring of Graphs, in: M. Kubale (Ed.), Graph Colorings, Contemporary Mathematics, 352, Amer. Math. Soc., (2004) 79-93.
- [663] T. Deretsky, S. M. Lee, and J. Mitchem, On vertex prime labelings of graphs, in Graph Theory, Combinatorics and Applications Vol. 1, J. Alavi, G. Chartrand, O. Oellerman, and A. Schwenk, eds., Proceedings 6th International Conference Theory and Applications of Graphs (Wiley, New York, 1991) 359-369.
- [664] U. Derings and H. Hünten, Magic graphs A new characterization, Report No. 83265 - OR, Universität Bonn April 1983, ISSN 0724-3138.
- [665] U. Deshmukh, Skolem graceful graphs A survey, *in* Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 165-170.
- [666] U. Deshmukh and V. Y. Shaikh, Mean cordial labelling of some star-related graphs, Internat. J. Math. Combin.. 3 (2016) 146-157.
- [667] J. Devaraj, On edge-cordial graphs, Graph Theory Notes of New York, XLVII (2004) 14-18.
- [668] J. Devaraj, On consecutive labelings of ladder graphs, Bull. Pure Appl. Sci. Sect. E Math. Stat., 26 (2007) 1-10.
- [669] S. K. Devi and B. Gayathri, k-even edge-graceful labeling of the graph  $P_n@K_{1,m}$ , Bull. Pure Appl. Sci., **29** E(2)387-407.
- [670] A. T. Diab, Study of some problems of cordial graphs, Ars Combin., 92 (2009) 255-261.
- [671] A. T. Diab, On cordial labelings of the second power of paths with other graphs, Ars Combin., 97A (2010) 327-343.

- [672] A. T. Diab, On cordial labelings of wheels with other graphs, Ars Combin., 100 (2011) 265-279.
- [673] A. T. Diab, Generalization of some results on cordial graphs, Ars Combin., 99 (2011) 161-173.
- [674] A. T. Diab, On cordial labelings of the second power of cycles with other graphs, Util. Math., 97 (2015) 65-84.
- [675] A. T. Diab and E. Elsakhawi, Some results on cordial graphs, Proc. Math. Phys. Soc. Egypt, No. 77 (2002) 67-87.
- [676] A. T. Diab and S. A. E. Mohammed, On cordial labelings of fans with other graphs, Ars Combin., 106 (2012) 263-275.
- [677] A. T. Diab and S. Nada, Some results on specific graphs, Ars Combin. 132 (2017) 93-103.
- [678] N. Diefenderfer, M. Hastings, L. N. Heath, H. Prawzinsky, B. Preston, E. White, and A. Whittemore, Prime vertex labelings of families of unicyclic graphs, *Rose-Hulman Undergrad. Math J.*, 16(1) (2015) 253-269.
- [679] N. Diefenderfer, D. C. Ernst, M. G. Hastings, L. N. Heath, H. Prawzinsky, B. Preston, E. White, and A. Whittemore, Prime vertex labelings of several families of graphs, *Involve*, 9(4) (2016) 667-688.
- [680] Dafik, A. I. Kristiana, S. Setiawani, and K. M. F. Azizah, Generalized shackle of fans is a super (a, d)-edge-antimagic total graph, J. Graph Label., 2 (1) (2016) 59-68.
- [681] J. Dinitz and P. Rodney, Disjoint difference families with block size 3, Util. Math., 52 (1997) 153-160.
- [682] D.M.T.B. Dissanayake, R.A.S.T. Abeysekara, K.D.E. Dhananjaya, A.A.I. Perera, and P.G.R.S. Ranasinghe, Prime labeling of complete tripartite graphs of the form K<sub>1,m,n</sub>, Elixir Appl. Math., **130** (2019) 53092-53094. https://www.researchgate.net/publication/333357422\_Prime\_labeling\_ of\_complete\_tripartite\_graphs\_of\_the\_form\_K\_1mn
- [683] K. Driscoll, E. Krop, and M. Nguyen, All trees are six-cordial, arXiv:1604.02105.
- [684] J. Doma, Unicyclic Graceful Graphs, M. S. Thesis, McMaster Univ., 1991.
- [685] D. Donovan, S. El-Zanati, C. Vanden Eynden, and S. Sutinuntopas, Labelings of unions of up to four uniform cycles, Australas. J. Combin., 29 (2004) 323-336.
- [686] M. Doob, On the construction of magic graphs, Congr. Numer., 10 (1974) 361-374.

- [687] M. Doob, Generalizations of magic graphs, J. Combin. Theory, Ser. B, 17 (1974) 205-217.
- [688] M. Doob, Characterizations of regular magic graphs, J. Combin. Theory, Ser. B, 25 (1978) 94-104.
- [689] S. Drajnová, J. Ivančo, A. Semaničová, Numbers of edges in supermagic graphs, J. Graph Theory, 52 (2006) 15-26.
- [690] K. Driscol, All trees are seven-cordial, arXiv:1909.12351.
- [691] W. Dou, On the mod sum number of  $H_{m,n}$ , J. Comb. Optim., 26(3) (2013) 465-471.
- [692] W. Dou and J. Gao, The (mod, integral) sum numbers of fans and  $K_{n,n} E(nK_2)$ , Discrete Math., **306** (2006) 2655-2669.
- [693] W. Dou and J. Gao, Some results on mod (integral) sum graphs, Ars Combin., 82 (2007) 3-31.
- [694] G. M. Du, Cordiality of complete k-partite graphs and some special graphs, Neimenggu Shida Xuebao Ziran Kexue Hanwen Ban, (1997) 9-12.
- [695] G. M. Du, On the cordiality of the union of wheels, J. Inn. Mong. Norm. Univ. Nat. Sci., 37 (2008) 180-181, 184.
- [696] Z.-t. Du and H.-q. Sun, The gracefulness of the digraph  $n \cdot \vec{C}_{2p}$ , J. Beijing Univ. Posts and Telecommunications, **17** (1994) 85-88.
- [697] G. Duan and Y. Qi, k-gracefulness of two classes of graphs, (Chinese) J. Zhengzhou Univ. Nat. Sci. Ed., 34 (2002) 36-38, 49.
- [698] M. Dufour, Sur la Decomposition d'un Graphe Complet en Arbres Isomorphes, Ph.D. Thesis, Universite de Montreal, 1995.
- [699] A. Durai Baskar and S. Arockiaraj, F-geometric mean graphs, Appl. Appl. Math., 10(2) (2015) 937-952.
- [700] A. Durai Baskar, S. Arockiaraj, and B. Rajendran, F-geometric mean labeling of some chain graphs and thorn graphs, *Kragujevac J. Math.* 37(1) (2013) 163-186.
- [701] A. Durai Baskar, S. Arockiyaraj, and B. Rajendran, Geometric meanness of graphs obtained from paths, *Util. Math.*, **101** (2016) 45-68.
- [702] A. Durai Baskar and S. Arockiaraj, C-Geometric mean labeling of some cycle related graphs, Internat. J. Math. Combin. 2 (2019) 69-85. https: //search.proquest.com/openview/16137f5f82c5f19a9ad9700490d080e5/1? cbl=51283&pq-origsite=gscholar

- [703] D. Dyer, I. Payne, N. Shalaby, and B. Wicks, On the graceful conjecture for triangular cacti, Australas. J. Combin., 53 (2012) 151-170.
- [704] P. Dyrlaga, K. Szopa, Orientable  $Z_n$ -distance magic regular graphs, arXiv:1712.02676 [math.CO] (2017).
- [705] T. Eccles, Graphs of large linear size are antimagic, J. Graph Theory, 81(3) (2016) 236-261.
- [706] M. Edwards and L. Howard, A survey of graceful trees, Atlantic Elect.J. Math., 1 (2006) 5-30.
- [707] S. Ehard, S. Glock, and F. Joos, A rainbow blow-up lemma for almost optimally bounded edge-colourings, arXiv:1907.09950 [math.CO], 28 pages.
- [708] P. Eldergill, Decomposition of the Complete Graph with an Even Number of Vertices, M. Sc. Thesis, McMaster University, 1997.
- [709] M. N. Ellingham, Sum graphs from trees, Ars Combin., **35** (1993) 335-349.
- [710] A. Elumalai, On Graceful, Cordial and Elegant Labelings of Cycles Related and Other Graphs, Ph. D. dissertation, Anna University, 2004, Chennai, India.
- [711] A. Elumalai and G. Sethuraman, Cordialness of cycles with parallel  $P_k$ -chords and multiple subdivision graphs, Ars Combin., 85 (2007) 85-98.
- [712] A. Elumalai and G. Sethuraman, Gracefulness of union of cycle with parallel chords and complete bipartite graphs or paths, J. Discrete Math. Sci. Cryptogr., 12 (2009) 245-252.
- [713] A. Elumalai and G. Sethuraman, Elegant labeled graphs, J. Inform. Math. Sci., 2(1) (2010) 45-49.
- [714] A. Elumali and G. Sethuraman, Gracefulness of a cycle with parallel chords and parallel  $P_k$ -chords of different lenghts, Ars Combin., **104** (2012) 143-148.
- [715] A. Elsonbaty and S. N. Daoud, Edge even graceful labeling of some path and cycle related graphs, Ars Combin,. 30 (2017) 79-96.
- [716] A. Elsonbaty and K. Mohamed, On some variants of gracefulness of cycle graphs, Ars Combin., 135 (2017) 39-50.
- [717] A. Elsonbaty and S. Nagy Daoud, Edge even graceful labeling of cylinder grid graph, Symmetry (2019), 11, 584. 30 pages.
- [718] S. El-Zanati, H.-L. Fu, and C.-L. Shiue, A note on the α-labeling number of bipartite graphs, *Discrete Math.*, **214** (2000) 241-243.

- [719] S. El-Zanati and D. I. Gannon, On ρ-labeling 2-regular graphs consisting of 5cycles, Int. J. Math. Comput. Sci., 6(1) (2011) 13-20.
- [720] S. El-Zanati, M. Kenig, and C. Vanden Eynden, Near α-labelings of bipartite graphs, Australas. J. Combin., 21 (2000) 275-285.
- [721] S. El-Zanati and C. Vanden Eynden, Decompositions of  $K_{m,n}$  into cubes, J. Combin. Designs, 4 (1996) 51-57.
- [722] S. El-Zanati and C. Vanden Eynden, On graphs with strong  $\alpha$ -valuations, Ars Combin., 56 (2000) 175-188.
- [723] S. El-Zanati and C. Vanden Eynden, On  $\alpha$ -valuations of disconnected graphs, Ars Combin., **61** (2001) 129-136.
- [724] S. El-Zanati and C. Vanden Eynden, On Rosa-type labelings and cyclic graph decompositions, *Math. Slovaca*, **59** (2009) 1-18.
- [725] S. El-Zanati, C. Vanden Eynden, and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.*, 24 (2001) 209-219.
- [726] H. Enomoto, A. S. Llado, T. Nakamigawa, and G. Ringel, Super edge-magic graphs, SUT J. Math., 34 (1998) 105-109.
- [727] H. Enomoto, K. Masuda, and T. Nakamigawa, Induced graph theorem on magic valuations, Ars Combin., 56 (2000) 25-32.
- [728] A. A. Ephremnath and A. Elumalai, Every cycle with chord Hamiltonian path is harmonious and elegant, *Int. J. Adv. Engg. Tech.*, **VII** (II) (2016) 1-4.
- [729] P. Erdős and A. B. Evans, Representations of graphs and orthogonal Latin squares, J. Graph Theory, 13 (1989) 593-595.
- [730] K. Eshghi, The Existence and Construction of α-valuations of 2-Regular Graphs with 3 Components, Ph. D. Thesis, Industrial Engineering Dept., University of Toronto, 1997.
- [731] K. Eshghi, α-valuations of special classes of quadratic graphs, Bull. Iranian Math. Soc., 28 (2002) 29-42.
- [732] K. Eshghi, Holey  $\alpha$ -labelings of graphs, Util. Math., 64 (2003) 139-147.
- [733] K. Eshghi, Extension of  $\alpha$ -labelings of quadratic graphs, *IJMMS* **11** (2004) 571-578.
- [734] K. Eshghi, An integer programming model and a Tabu search algorithm to generate  $\alpha$ -labeling of special classes of quadratic qraphs, *Iranian J. Operations Res.*, **7**(1) (2016) 1-17.

- [735] K. Eshghi, An introduction to the graceful graphs, Sharif University of Technology 2002) 65 pp. http://sharif.edu/~eshghi/Graceful%20Graphs-Final% 20Edition-89-12-15.pdf
- [736] K. Eshghi and P. Azimi, Applications of mathematical programming in graceful labeling of graphs, J. Appl. Math., 1(1) (2004) 1-8.
- [737] K. Eshghi and P. Azimi, An algorithm for finding a feasible solution of graph labeling problems, Util. Math., 72 (2007) 163-174.
- [738] K. Eshghi and M. Carter, Construction of α-valuations of special classes of 2regular graphs, Topics in Applied and Theoretical Mathematics and Computer Science, Math. Comput. Sci. Eng., WSEAS, Athens (2001) 139-154.
- [739] K. Eshghi and M. Salarrezaie, Existence and construction of  $\alpha$ -labeling for quadratic graph Q(7, 4k) and its extensions, *Int. J. Graph. Theory Appl.*, (IJGTA), to appear.
- [740] K. Eshghi and M. Salarrezaie, An integer programming model and a Tabu search algorithm to generate  $\alpha$ -labeling of special classes of quadratic graph, *Iran. J. Oper. Res.*, to appear.
- [741] A. B. Evans, Representations of disjoint unions of complete graphs, unpublished.
- [742] A. B. Evans, G. H. Fricke, C. C. Maneri, T. A. McKee, and M. Perkel, Representations of graphs modulo n, J. Graph Theory 18 (1994) 801-815.
- [743] A. B. Evans, G. Isaak, and D. A. Narayan, Representations of graphs modulo n, Discrete Math., 223 (2000) 109-123.
- [744] G. Exoo, A. Ling, J. McSorley, N. Phillips, and W. Wallis, Totally magic graphs, Discrete Math., 254 (2002) 103-113.
- [745] G. Eyabi, J. Jacob, R. Laskar, D. Narayan, and D. Pillone, Minimal rankings of the Cartesian product  $K_n \Box K_m$ , Discuss. Math. Graph Th., **32**(4) (2012) 725-735.
- [746] L. Fan and Z. Liang, On Lee's conjecture and some results, Discuss. Math. Graph Theory, 29 (2009) 481-498.
- [747] W. Fang, A computational approach to the graceful tree conjecture, arXiv:1003.3045v1 [cs.DM].
- [748] W. Fang, New computational result on harmonious trees, arXiv:1106.3490v1 [cs.DM] 17 June 2011.
- [749] H. Farida and D. Indriati, On the total edge irregularity strength of three towers hanoi graph, J. Physics Confer. Ser., (2019) 1211:012010 https://iopscience. iop.org/article/10.1088/1742-6596/1211/1/012010/pdf

- [750] R. J. Faudree and J. Lehel, Bound on the irregularity strength of regular graphs. *Combinatorics* (Eger, 1987) 247-256, Colloq. Math. Soc. Jănos Bolyai, 52, North-Holland, Amsterdam, 1988.
- [751] R. J. Faudree, M. S. Jacobson, J. Lehel and R. H. Schelp, Irregular networks, regular graphs and integer matrices with distinct row and column sums, *Discrete Math.*, **76** (1989) 223-240.
- [752] R. J. Faudree, M. S. Jacobson, L. Kinch, and J. Lehel, Irregularity strength of dense graphs. *Discrete Math.*, 91(1) (1991) 45-59.
- [753] W. Feng, C. Xu, and Jirimutu, A survey of the gracefulness of digraphs, Int. J. Pure Appl. Math., 69 (2011) 245-253.
- [754] W. Feng, X. Hong, Y. Yang, and Jirimutu, On (a, d)-antimagic labelings of generalized Petersen graphs P(n, 5), Util. Math. **95** (2014) 349-356.
- [755] C. Fernandez, A. Flores, M. Tomova, and C. Wyels, The radio number of gear graphs, arXiv:0809.2623
- [756] H. Fernau, J. F. Ryan, and K. A. Sugeng, A sum labelling for the generalised friendship graph, *Discrete Math.*, **308** (2008) 734-740.
- [757] R. Figueroa-Centeno and R. Ichishima, The n-dimensional cube is felicitous, Bull. Instit. Combin. Appl., 41 (2004) 47-50.
- [758] R. M. Figueroa-Centeno and R. Ichishima, On the sequential number and super edge-magic deficiency of graphs, Ars Combin., 129 (2016) 157-163.
- [759] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, Bertrand's postulate and magic product labelings, *Bull. Instit. Combin. Appl.*, **30** (2000) 53-65.
- [760] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, The place of super edgemagic labelings among other classes of labelings, *Discrete Math.*, 231 (2001) 153-168.
- [761] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, On super edge-magic graphs, Ars Combin., 64 (2002) 81-95.
- [762] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, Magical coronations of graphs, Australas. J. Combin., 26 (2002) 199-208.
- [763] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, Labeling the vertex amalgamation of graphs, *Discuss. Math. Graph Theory*, **23** (2003) 129-139.
- [764] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, On edge-magic labelings of certain disjoint unions of graphs, Australas. J. Combin. 32 (2005) 225-242.

- [765] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, J. Combin. Math. Combin. Comput., 55 (2005) 17-31.
- [766] R. Figueroa-Centeno, R. Ichishima, and F. Muntaner-Batle, On the super edgemagic deficiency of graphs, Ars Combin., 78 (2006) 33-45.
- [767] R. Figueroa-Centeno, R. Ichishima, F. Muntaner-Batle and A. Oshima, A magical approach to some labeling conjectures, *Discussiones Math. Graph Theory*, **31** (2011) 79-113.
- [768] R. Figueroa-Centeno, R. Ichishima, F. Muntaner-Batle, and A. Oshima, Gracefully cultivating trees on a cycle, *Electron. Notes Discrete Math.*, **48** (2015) 143-150.
- [769] R. Figueroa-Centeno, R. Ichishima, F. Muntaner-Batle, and M. Rius-Font, Labeling generating matrices, Labeling generating matrices, J. Combin. Math. Combin. Comput., 67 (2008) 189-216.
- [770] F. Firmansah and T. Tasari, Odd harmonious labeling on edge amalgamation from double quadrilateral graphs, *Desimal: J. Mat.*, 3(1) (2020) 65-72. DOI:10.24042/ djm.v3i1.5712
- [771] C. Frayer, Properties of divisor graphs, Technical Report, Depart. Math., Grand Valley State University, Allandale, Michigan, USA.
- [772] S. Freeda and R. S. Chellathurai, H- and H<sub>2</sub>-cordial labeling of some graphs, Open J. Discrete Math., 2 (2012) 149-155. doi:10.4236/ojdm.2012.24030
- [773] B. Freyberg and M. Keranen, Orientable  $Z_n$ -distance magic labeling of the Cartesian product of many cycles, *Electron. J. Graph Theory Appl.*, 5(2) (2017) 304-311.
- [774] B. Freyberg and M. Keranen, Orientable  $Z_n$ -distance magic labeling of the Cartesian product of two cycles, Australas. J. Combin., **69** (2017) 222-235.
- [775] B. Freyberg and M. Keranen, Orientable  $Z_n$ -distance magic magic graphs via products, Australasian J. Combin., **70**(3) (2018) 319-328.
- [776] D. Fronček, Bi-cyclic decompositions of complete graphs into spanning trees, *Discrete Math.*, **307** (2007) 1317-1322.
- [777] D. Fronček, Recent advances in Rosa-type labelings of graphs, AKCE J. Graphs Combin., 6 (2009) 69-78.
- [778] D. Fronček,  $\alpha_2$ -labeling of graphs, *Opuscula Math.*, **29** (2009) 393-397.
- [779] D. Fronček, Group distance magic labeling of Cartesian product of cycles, Australas. J. Combin., 55 (2013) 167-174.

- [780] D. Fronček, Handicap distance antimagic graphs and incomplete tournaments, AKCE Int. J. Graphs Comb., 16(2) (2013) 119-127.
- [781] D. Fronček, Alpha labelings of full hexagonal caterpillars, AKCE Internat. J. Graphs Combin., 13 (2016) 85-89.
- [782] D. Fronček, Handicap incomplete tournaments and ordered distance antimagic graphs, https://www.researchgate.net/publication/267092003\_Handicap\_ incomplete\_tournaments\_and\_ordered\_distance\_antimagic\_graphs
- [783] D. Fronček, Regular incomplete tournaments with handicap two, *Congr. Numer.*, **227** (2016) 277-285.
- [784] D. Fronček, Regular handicap graphs of odd order, J. Combin. Math. Combin. Comput., 102 (2017) 253-266.
- [785] D. Fronček, Full spectrum of regular incomplete 2-handicap tournaments of order  $n \equiv 0 \pmod{16}$ , J. Combin. Math. Combin. Comput., **106** (2018) 175-184.
- [786] D. Fronček, O. Kingston, and K. Vezina, Alpha labelings of straight simple polynomial caterpillars, *Congr. Numer.*, **222** (2014) 57-64.
- [787] D. Fronček, P. Kovář, and T. Kovářová, Vertex magic total labeling of products of cycles, Australas. J. Combin., 33 (2005) 169-181.
- [788] D. Fronček, P. Kovář, and T. Kovářová, Constructing distance magic graphs from regular graphs, J. Combin. Math. Combin. Comput., 78 (2011) 349-354.
- [789] D. Fronček and M. Kubesa, Factorizations of complete graphs into spanning trees, Congr. Numer., 154 (2002) 125-134.
- [790] D. Fronček and J. Qiu, Supermagic graphs with many odd degrees, *Combin. Algor.*, 229236, Lecture Notes in Comput. Sci., 11638, Springer, Cham, 2019.
- [791] D. Froncek and A. Shepanik, Regular handicap graphs of order  $n \equiv 0 \pmod{8}$ , Electronic J. Graph Th. Appl. 6 (2) (2018) 208-218.
- [792] D. Fronček and L. Tollefson, Decompositions of complete graphs into kayak paddles, J. Indones. Math. Soc., Special Edition (2011) 30-44.
- [793] D. Fronček and L. Tollefson, Decompositions of complete graphs into kayak paddles with two odd cycles, Proceedings of the Forty-Second Southeastern International Conference on Combinatorics, Graph Theory and Computing. *Congr. Numer.*, 207 (2011) 161-169.
- [794] D. Fronček and S. Winters, Another family of gracious bipartite graphs, Congr. Numer., 170 (2004) 65-70.

- [795] R. Frucht, Graceful numbering of wheels and related graphs, Ann. N.Y. Acad. Sci., 319 (1979) 219-229.
- [796] R. Frucht, On mutually graceful and pseudograceful labelings of trees, Scientia Series A, 4 (1990/1991) 31-43.
- [797] R. Frucht, Nearly graceful labelings of graphs, *Scientia*, 5 (1992-1993) 47-59.
- [798] R. Frucht and J. A. Gallian, Labeling prisms, Ars Combin., 26 (1988) 69-82.
- [799] R. Frucht and L. C. Salinas, Graceful numbering of snakes with constraints on the first label, Ars Combin., 20 (1985), B, 143-157.
- [800] C.-M. Fu, N.-H. Jhuang and Y.-L. Lin, Integer-magic spectra of sun graphs, Ars Combin., 103 (2012) 55-64.
- [801] H. L. Fu and K. C. Huang, On prime labelling, *Discrete Math.*, **127** (1994) 181-186.
- [802] H. L. Fu and S. L. Wu, New results on graceful graphs, J. Combin. Info. Sys. Sci., 15 (1990) 170-177.
- [803] Y. Fukuchi, Graph labelings in elementary abelian groups, Discrete Math., 189 (1998) 117-122.
- [804] Y. Fukuchi, A recursive theorem for super edge-magic labelings of trees, SUT J. Math., 36 (2000) 279-285.
- [805] Y. Fukuchi, Edge-magic labelings of generalized Petersen graphs P(n, 2), Ars Combin., **59** (2001) 253-257.
- [806] Y. Fukuchi, Edge-magic labelings of wheel graphs, Tokyo J. Math., 24 (2001) 153-167.
- [807] Y. Fukuchi and A. Oshima, Super-edge-magic labelings of a family of trees with diameter 4, Adv. Appl. Discrete Math., 1 (2008) 149-157.
- [808] Y. Fukuchi, and A. Oshima, Super-edge-magic labeling of trees with large diameter, Far East J. Math. Sci., 28 (2008) 497-571.
- [809] S. J. Gajjar and A. K. Des, Cordial labeling of generalized Jahangir graph, Internat. J. Math. Appl., 4(1D) (2016) 21-33. https://www.researchgate. net/publication/329626405\_Cordial\_Labeling\_of\_Generalized\_Jahangir\_ Graph
- [810] J. A. Gallian, Labeling prisms and prism related graphs, Congr. Numer., 59 (1989) 89-100.
- [811] J. A. Gallian, A survey: recent results, conjectures and open problems on labeling graphs, J. Graph Theory, 13 (1989) 491-504.
- [812] J. A. Gallian, Open problems in grid labeling, Amer. Math. Monthly, 97 (1990) 133-135.
- [813] J. A. Gallian, A guide to the graph labeling zoo, *Discrete Appl. Math.*, **49** (1994) 213-229.
- [814] J. Gallian, unpublished.
- [815] J. A. Gallian and D. S. Jungreis, Labeling books, *Scientia*, **1** (1988) 53-57.
- [816] J. A. Gallian, J. Prout, and S. Winters, Graceful and harmonious labelings of prisms and related graphs, Ars Combin., 34 (1992) 213-222.
- [817] J. A. Gallian and L. A. Schoenhard, Even harmonious graphs, AKCE J. Graphs Combin., 11(1) (2014) 27-49.
- [818] J. A. Gallian and D. Stewart, Properly even harmonious labelings of disconnected graphs, AKCE J. Graphs Combin., 12(2-3) (2015) 193-203.
- [819] J. A. Gallian and D. Stewart, Even harmonious labelings of disjoint graphs with a small component, AKCE J. Graphs Combin., 12(2-3) (2015) 204-215.
- [820] J. A. Gallian and D. Stewart, Even harmonious labelings of disjoint unions with even sequential graphs, J. Graph Labeling, 1(1) (2015) 1-10.
- [821] S. Ganesan and D. Uthayakumar, Corona of bipartite graphs with divisor graphs produce new divisor graphs, *Bull. Kerala Math. Assoc.*, **9**(1) (2012) 219-226.
- [822] T. Gangopadhyay and S. P. Rao Hebbare, Bigraceful graphs-I, Util. Math., 17 (1980) 271-275.
- [823] D. I. Gannon and S. I. El-Zanati, All 2-regular graphs with uniform odd components admit ρ-labelings, Australas. J. Combin., 53 (2012) 207-219.
- [824] A-B. Gao, X-D. Zhang, and L-J. Xu, On (super) vertex-graceful labeling of graphs, Ars Combin., 126 (2016) 121-131.
- [825] M. Gao, The edge different sets of the direct product of two paths, MS thesis, Fuzhou University, 2010.
- [826] Y. Z. Gao, On gracefulness of  $C_m \cup P_2$ , J. Hebei Teachers College, No. 3 (1993) 13-21.
- [827] Y. Z. Gao and Z. H. Liang, private communication.
- [828] Z.-B. Gao, The labelings of a variation of banana trees, Ars Combin., 94 (2010) 175-181.

- [829] Z.-B. Gao, G. C. Lau, and S. M. Lee, On friendly index and product-cordial index sets of Möbius-liked graph, J. Discrete Math. Sci. Crypt., 21(3) (2018) 647-659 DOI: 10.1080/09720529.2016.1220089
- [830] Z.-B. Gao, G.-Y. Sun, Y.-N. Zhang, Y. Meng, and G. C. Lau, Product cordial and total product cordial labelings of P<sup>n</sup><sub>n+1</sub>, J. Discrete Math., (2015) Art. ID 512696, 6 pp.
- [831] Z.-B. Gao, Odd graceful labelings of some union graphs, J. Nat. Sci. Heilongjiang Univ., 24 (2007) 35-39.
- [832] Z.-B. Gao, The super vertex-graceful labeling of graphs, Ars Combin., 124 (2016) 389-400.
- [833] Z.-B. Gao and C. Fan, Some discussion on super edge-magic labelings of  $St(a_1, \ldots, a_n)$ , Ars Combin., **108** (2013) 187-192.
- [834] Z.-B. Gao, G.-C. Lau, and S.-M. Lee, On friendly index and product-cordial index sets of Möbius-liked graph, J. Discrete Math. Sci. Cryptogr., 21(3) (2018) 647-659.
- [835] Z.-B. H. Gao, Ruo-Yuan, S.-M. Lee, H.-N. Ren, and G.-Y. Sun, Discussions on full friendly index sets of a class of cubic graphs, Ars Combin., 146 (2019) 375-389.
- [836] Z-B. Gao, G-Y. Sun, S-M. Lee, On full friendly index sets of 1-level and 2-levels N-grids, Discrete Appl. Math., 211 (2016) 68-78.
- [837] Z.-B. Gao and X. Zhang, A note on super edge-graceful labelings of caterpilars, J. Shanghai Jiaotong Univ., 42 (2008) 493-495.
- [838] Z.- B. Gao, X.-D. Zhang, and L.-J. Xu, Odd graceful labelings of graphs, *Discrete Math. Algor. Appl.*, 1(3) (2009) 377-388.
- [839] M. Gardner, Mathematical games: the graceful graphs of Solomon Golomb, or how to number a graph parsimoniously, *Scientific Amer.*, **226**3 (1972) 108-112; **226**4 (1972) 104; **226**(6) (1972) 118.
- [840] B. Gayathri and K. Amuthavalli, personal communication.
- [841] B. Gayathri and K. Amuthavalli, k-odd mean labeling of the graph  $\langle K_{1,n} : K_{1,m} \rangle$ , Acta Ciencia Indica Math., **XXIV M** (2008) 827-834.
- [842] B. Gayathri and K. Amuthavalli, k-odd mean labeling of prism, Trans. Comb., 4(1) (2015) 49-56.
- [843] B. Gayathri and S. K. Devi, k-edge-graceful labeling and k-global edge-graceful labeling of some graphs, Internat. J. Engin. Sci., Adv. Comput. and Bio-Tech., 2(1) (2010) 25-37.

- [844] B. Gayathri and S. K. Devi, k-even edge-graceful labeling of the bistar graph  $B_{n,n}$ , Proceed. Internat. Conf. Math. and Comput. Sci. ICMCS 2011, 525-532.
- [845] B. Gayathri and S. K. Devi, k-even edge-graceful labeling of some graphs, Internat. J. Math. and Comput. Sci., 6(1) (2011) 27-34.
- [846] B. Gayathri and S. K. Devi, k-even edge-graceful labeling of some cycle related graphs, Internat. J. Engin. Sci. Invention, 2(10) (2013) 1-17.
- [847] B. Gayathri and M. Duraisamy, personal communication.
- [848] B. Gayathri, M. Duraisamy, and M. Tamilselvi, Even edge graceful labeling of some cycle related graphs, Int. J. Math. Comput. Sci., 2 (2007) 179-187.
- [849] B. Gayathri and R. Gopi, k-even mean labeling of  $D_{m,n}@C_n$ , Internat. J. Enger. Sci., Adv. Comput. Bio-Tech., 1(3) (2010) 137-145.
- [850] B. Gayathri and R. Gopi, k-even mean labeling of  $D_{m,n}$ , Acta Ciencia Indica, XXXVII M(2) (2011) 291-300.
- [851] B. Gayathri and R. Gopi, k-even mean labeling of  $T_{n,m,t}$ , Internat. J. Enger. Sci., Adv. Comput. Bio-Tech., 1(3) (2011) 137-145.
- [852] B. Gayathri and R. Gopi, k-even mean labeling of  $C_n \cup P_m$ , Elixir Appl. Math., **36** (2011) 3102-3105.
- [853] B. Gayathri and R. Gopi, (k, d)-even mean labeling of  $P_m \oplus nK_1$ , Internat. J. Math. Soft Comput., 1(1) (2011) 17-23. http://citeseerx.ist.psu.edu/ viewdoc/download?doi=10.1.1.922.5279&rep=rep1&type=pdf
- [854] B. Gayathri and R. Gopi, Necessary condition for mean labeling, Internat. J. Engin. Sci., Adv. Comput. Bio-Tech., 4(3) July-Sept. (2013) 43-52.
- [855] B. Gayathri and R. Gopi, k-even mean labeling of some graph operations, Internat. J. Engin. Sci., Adv. Comput. and Bio-Tech., 4(3) Oct-Dec. (2013) 59-70.
- [856] B. Gayathri and R. Gopi, Cycle related mean graphs, *Elixir Internat. J. Applied Sci.*, No. 71, (2014) 25116-25124.
- [857] B. Gayathri and R. Gopi, Some properties of k-mean labeling, Internat. J. Multidisciplinary Res. Publication (IJMRAP), 1(3) (2018) 1-2.
- [858] B. Gayathri and R. Gopi, Some general results on restricted k-mean graph, J. Discrete Math. Sci. Cryptography, (2019) 1-11. DOI:10.1080/09720529.2019. 1630936
- [859] B. Gayathri and V. Hemalatha, Even sequential harmonious graphs, personal communication.

- [860] B. Gayathri and D. Muthurramakrishnan, Some results on k-even sequential harmonious labeling of graphs, *Elixir Internat. J. Applied Sci.*, **47** (2012) 9054-9057.
- [861] B. Gayathri and M. Subbiah, Strong edge graceful labelings of some graphs, *Bull. Prue Appl. Sci.*, **27E** (2008) 1-10.
- [862] B. Gayathri and M. Tamilselvi, personal communication.
- [863] G. Ge, Y. Miao and X. Sun, Perfect difference families, perfect difference matrices and related combinatorial structures, J. Combin. Des., 18 (2010) 415-449.
- [864] J. P. Georges, D. W. Mauro, and Y. Wang, On the structures of  $V_4$ -magic and  $Z_4$ -magic graphs, J. Combin. Math. Combin. Comput., **75** (2010) 137-152.
- [865] R. Gera, V. Saenpholphat, and P. Zhang, Divisor graphs with triangles, Congr. Numer., 165 (2003) 51-64.
- [866] K. A. Germina, More on classes of strongly indexable graphs, European J. Pure and Applied Math., 3-2 (2010) 269-281.
- [867] K. A. Germina, A. Kumar, and K. L. Princy, Further results on set-valued graphs, J. Discrete Math. Sci. Cryptogr., 11 (2008) 559-566.
- [868] K. A. Germina and V. Ajitha, Strongly multiplicative graphs, unpublished.
- [869] K. A. Germina and T.M.K. Anandavally, On arithmetic embeddings of graphs, unpublished.
- [870] K. A. Germina and R. Sebastian, On square sum graphs, *Proyecciones* **32**(2) (2013) 107-117.
- [871] K. A. Germina and R. Sebastian, Futher results on square sum graph, Internat. Math. Forum, 8(1) (2013) 47-57.
- [872] S. V. Gervacio, Which wheels are proper autographs?, Sea Bull. Math., 7 (1983) 41-50.
- [873] S. V. Gervacio, personal communication.
- [874] H. V. Getzimah and K. Palani, Antimagic total labeling on  $Z_{p+q}$ , J. Comp. Math. Sci., **9**(11) (2018) 1653-1660.
- [875] M. Ghebleh and R. Khoeilar, A note on: "H-cordial graphs," Bull. Inst. Combin. Appl., 31 (2001) 60-68.
- [876] G. V. Ghodasara and I. I. Jadav, New grid related cordial graphs, Internat. J. Appl. Math., 28(2) (2013) 1244-1248.

- [877] G. V. Ghodasara and J. P. Jena, Prime cordial labeling of the graphs related to cycle with one chord, twin chords and triangle, *Internat. J. Pure and Appl. Math.*, 89(1) (2013) 79-87. http://dx.doi.org/10.12732/ijpam.v89i1.9
- [878] G. V. Ghodasara and M. Patel, Some bistar related square sum graphs, Internat. J. Math. Trend and Tech., 8(1) (2013) 47-57.
- [879] G. V. Ghodasara and M. Patel, Some new combination graphs, Internat. J. Math. Appl., 5 (2A) (2017) 153-161.
- [880] G. V. Ghodasara and M. Patel, More on permutation labeling of graphs, Internat. J. Appl. Graph Th., 1(2) (2017) 30-42.
- [881] G. V. Ghodasara and M. Patel, Innovative results on square sum labeling of graphs, J. Graph Labeling, 4(1) (2018) 15-24.
- [882] G. V. Ghodasara, A. H. Rokad and I. I. Jadav, Cordial labeling of grid related graphs, Internat. J. Comb. Graph Th. and App., 6(2)(2013) 55-62.
- [883] G. V. Ghodasara and A. H. Rokad, Cordial labeling of  $K_{n,n}$  related graphs, *Internat. J. Sci. Res.*, **2**(5) (2013) 74-77.
- [884] G. V. Ghodasara and A. H. Rokad, Cordial labeling in context of vertex switching of special graphs, *Internat. J. Math. Sci.*, 33(2) (2013) 1389-1396.
- [885] G. V. Ghodasara and A. H. Rokad, Cordial Labeling in context of barycentric subdivision of special graphs, *Internat. J. Math. Res.*, **5**(5) (2013) 475-483.
- [886] G. V. Ghodasara and S. G. Sonchhatra, Cordial labeling of fan related graphs, Internat. J. Sci. Eng. Res., 4(8) (2013) 470-476.
- [887] G. V. Ghodasara and S. M. Vaghasiya, Product cordial labeling of graphs related to cycle with one chord, twin chords and triangle, *Internat. J. Math. Comput. Modelling*, 18(2) (2013) 1109-1113.
- [888] G. V. Ghodasara and S. M. Vaghasiya, Product cordial labeling of graphs related to helm, closed helm and gear graph, *Internat. J. Pure and Appl. Math.*, 91(4) (2014) 495-504. http://dx.doi.org/10.12732/ijpam.v91i4.6
- [889] E. Ghorbani and S. Kamali, Prime labeling of ladders, arXiv:1610.08849v1 [math.CO].
- [890] S. Ghosh, On certain classes of graceful lobsters, Ars Combin., 136 (2018) 67-96.
- [891] S. Ghosh, S. Mohanty, and A. Pal, Cordial labeling of Cartesian product between two balanced bipartite graphs, J. Informatics Math. Sci., 11(2) (2019) 179-193.
- [892] J. Ghoshal, R. Laskar, and D. Pillone, Minimal rankings, *Networks*, **28** (1996) 45-53.

- [893] J. Ghoshal, R. Laskar, D. Pillone, and G. Fricke, Further results on mod sum graphs, *Congr. Numer.*, **101** (1994) 201-207.
- [894] R. B. Gnanajothi, Topics in Graph Theory, Ph. D. Thesis, Madurai Kamaraj University, 1991.
- [895] E. K. Gnang, On graceful labelings of trees, arXiv:1811.07614
- [896] R. Godbold and P. J. Slater, All cycles are edge-magic, Bull. Inst. Combin. Appl., 22 (1998) 93-97.
- [897] L. Goddyn, R. B. Richter, and J. Siráň, Triangular embeddings of complete graphs from graceful labellings of paths, *J. Combin. Theory Ser. B*, **97** (2007) 964-970.
- [898] A. Godinho and T. Singh, Group distance magic labeling of  $C_n^r$ , Algorithms and Discrete Applied Math., 187-192, Lecture Notes in Comput. Sci., 10156, Springer, Cham, 2017.
- [899] A. Godinho and T. Singh, Some distance magic graphs, *Elect. Notes Discr. Math.*,
  63 (2017) 311-315.
- [900] A. Godinho, T. Singh, and S. Arumugam, On nearly distance magic graphs, *Theoretical Comput. Sci. Disc. Math.*, 76-82, Lecture Notes in Comput. Sci., 10398, Springer, Cham, 2017.
- [901] C. G. Goh and C. K. Lim, Graceful numberings of cycles with consecutive chords, 1992, unpublished.
- [902] S. W. Golomb, How to number a graph, in *Graph Theory and Computing*, R. C. Read, ed., Academic Press, New York (1972) 23-37.
- [903] S. W. Golomb, Polyominoes: Puzzles, Patterns, Problems, and Packings, second ed. Princeton University Press, Princeton, MA, 1996.
- [904] V. Lakshmi (alias) Gomathi, A. Nagarajan, A. NellaiMurugan, On felicitous labelings of  $P_{r,2m+1}$ ,  $P_r^{2m+1}$  and  $C_n \times P_m$ , Internat. J. Math. Soft Computing, **3**(1) (2012) 97-105.
- [905] V. L. A. Gomathi and A. Nagarajan, Felicitous labelings of some graphs, OUT-REACH IX (2016) 132-139.
- [906] J. Gómez, Solution of the conjecture: If  $n \equiv 0 \pmod{4}$ , n > 4, then  $K_n$  has a super vertex-magic total labeling, *Discrete Math.*, **307** (2007) 2525-2534.
- [907] J. Gómez, Two new methods to obtain super vertex-magic total labelings of graphs, Discrete Math., **308** (2008) 3361-3372.
- [908] J. Gómez and P. Kovář On super vertex-magic total labeling of the disjoint union of k copies of  $K_n$ , Ars Combin., **113** (2014) 175-192.

- [909] J. Goodell, A. Beveridge, M. Gallagher, D. Goodwin, J. Gyori, and A. Joseph, Sum graphs, unpublished.
- [910] V. Lakshmi (alias) Gomathi, Felicitous labelings of some graphs, Academic J. Applied Math. Sci., 2(8) (2016) 93-97.
- [911] R. Gopi, A Study on Different Kinds of Mean Graphs, Ph. D. Thesis, Bharathiasan University, Trichy, 2013.
- [912] R. Gopi, Odd sum labeling of alternative quadrilateral snake, Internat. J. Engin. Sci., Advan. Comput. and Bio-Tech., 7(3) (2016) 73-77.
- [913] R. Gopi, Odd sum labeling of tree related graphs, Internat. J. Math. and its Applications., 4(4) (2016) 11-16.
- [914] R. Gopi, F-root square mean labeling of some graphs, Internat. J. Math. Combin., 1 (2019) 146-149. https://www.researchgate.net/publication/332173176\_ F-Root\_Square\_Mean\_Labeling\_of\_Some-\_Graphs
- [915] R. Gopi, k-Contra harmonic mean labeling of snake related graphs, Math. J. Interdiscip. Sci., 7(2) (2019) 77-80. https://mjis.chitkara.edu.in/index.php/ mjis/article/view/175/114
- [916] R. Gopi, Super root square mean labeling of some graphs, Internat. J. Engin. Sci., Advanced Comput. and Bio-Tech., 8(2) (2017) 75-81. DOI:10.26674/ijesacbt/ 2017/49175
- [917] R. Gopi and A. Irudaya Mary, Odd sum labelings of some more graphs, Internat. J. Engin. Sci., Adv. Comput. and Bio-Tech., 7(4) (2016) 95-103.
- [918] R. Gopi and C. Nirmala, Lehmer-3 mean labeling of cycle related graphs, Int. J. Res. Instinct., 5 (2018) 30-35.
- [919] R. Gopi and C. Suba, Super Lehmer-3 mean labeling of tree related graphs, Internat. J. Math. Appl., 5 3-A (2017) 25-27.
- [920] R. J. Gould and V. Rödl, Bounds on the number of isolated vertices in sum graphs, Graph Theory, Combin. and Appl., 1 (1991) 553-562.
- [921] T. Grace, Graceful, Harmonious, and Sequential Graphs, Ph. D. Thesis, University Illinois at Chicago Circle, 1982.
- [922] T. Grace, On sequential labelings of graphs, J. Graph Theory, 7 (1983) 195-201.
- [923] T. Grace,  $K_4$  snakes are sequential, Technical Report, Department of Computer Science, Illinois Institute of Technology (1986).
- [924] A. J. Graham, D. A. Pike, and N. Shalaby, Skolem labelled trees and  $P_s \Box P_t$ Cartesian products, Australas. J. Combin., **38** (2007) 101-115.

- [925] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Discrete Methods, 1 (1980) 382-404.
- [926] M. Grannell, T. Griggs, and F. Holroyd, Modular gracious labelings of trees, *Discrete Math.*, 231 (2001) 199-219.
- [927] I. Gray, New Construction Methods for Vertex-magic Total Labelings of Graphs, Ph.D. thesis, University of Newcastle, 2006.
- [928] I. Gray, Vertex-magic total labellings of regular graphs, SIAM J. Discrete Math. 21 (2007) 170-177.
- [929] I. Gray and J. MacDougall, Sparse semi-magic squares and vertex-magic labelings, Ars Combin., 80 (2006) 225-242.
- [930] I. Gray and J. MacDougall, Sparse anti-magic squares and vertex-magic labelings of bipartite graphs, *Discrete Math.*, **306** (2006) 2878-2892.
- [931] I. Gray and J. A. MacDougall, Vertex-magic labelings of regular graphs. II, Discrete Math., 309 (2009) 5986-5999.
- [932] I. Gray and J. MacDougall, Vertex-magic labeling of non-regular graphs, Australas. J. Combin., 46 (2010) 173-183.
- [933] I. Gray and J. A. MacDougall, Vertex-magic labelings: Mutations, Austral. J. Combin., 45 (2009) 189-206.
- [934] I. Gray and J. MacDougall, Vertex-magic labeling of regular graphs: disjoint unions and assemblages. *Discrete Appl. Math.*, 160 (2012) 1114-1125.
- [935] I. Gray, J. MacDougall, R. Simpson, and W. Wallis, Vertex-magic total labeling of complete bipartite graphs, Ars Combin., 69 (2003) 117-127.
- [936] I. Gray, J. MacDougall, J. McSorley, and W. Wallis, Vertex-magic labeling of trees and forests, *Discrete Math.*, 261 (2003) 285-298.
- [937] I. Gray, J. MacDougall, and W. Wallis, On vertex-magic labeling of complete graphs, *Bull. Inst. Combin. Appl.*, **38** (2003) 42-44.
- [938] R. P. Grimaldi, Graphs from rings, Congr. Numer., 71 (1990) 95-104.
- [939] J. Guo, X. Chen, Z. Wang, and B. Yao, Total vertex irregularity strength of certain equitable complete *m*-partite graphs, Ars Comb., **123** (2015) 407-418.
- [940] W. F. Guo, Gracefulness of the graph B(m, n), J. Inner Mongolia Normal Univ., (1994) 24-29.
- [941] W. F. Guo, Gracefulness of the graph B(m, n, p), J. Math. (PRC), **15** (1995) 345-351.

- [942] S. Guruswamy and M. Varadhan, Generating graceful unicyclic graph from a given graceful tree, *Ars Combin.*, to appear.
- [943] A. Gutiérrez and A. Lladó, Magic coverings, J. Combin. Math. Combin. Comput., 55 (2005) 43-56.
- [944] P. Gvozjak, On the Oberwolfach Problem for Cycles with Multiple Lengths, Ph.D. thesis, Burnaby: Simon Fraser University, 2004.
- [945] A. Gyárfás and J. Lehel, A method to generate graceful trees, in Colloque C.N.R.S. Problèmes Combinatories et Théorie des Graphes, Orsay, 1976 (1978) 207-209.
- [946] E. Gyri, P. N. Balister, and R. H. Schelp, Coloring vertices and edges of a graph by nonempty subsets of a set, *Eur. J. Comb.*, 4 (2011) 533-537.
- [947] P. Hajnal and G. Nagy, Simply sequentially additive labelings of 2-regular graphs, Discrete Math., 310 (2010) 922-928.
- [948] S. Hall, K. Hillesheim, E. Kocina, and M. Schmit, personal communication.
- [949] N. Han and Z. Liang, On the graceful permutation graphs conjecture, J. Discrete Math. Sci. Cryptogr., 11 (2008) 501-526.
- [950] A. K. Handa, A. Godinho, and T. Singh, Distance antimagic labeling of the ladder graph, *Elect. Notes Discr. Math.*, 63 (2017) 317-322.
- [951] A. K. Handa, A. Godinho, and T. Singh, Some distance antimagic labeled graphs. In: S. Govindarajan and A. Maheshwari A. (eds), Algorithms and Discrete Applied Mathematics. Lecture Notes in Computer Science, 9602 190-200. Springer, Cham 2016.
- [952] T. Hao, On sum graphs, J. Combin. Math. Combin. Computing, 6 (1989) 207-212.
- [953] K. M. M. Haque, X. Lin, Y. Yang, and J. Zhang, Prime cordial labeling of flower snark and related graphs, Ars Combin., 105 (2012) 45-52.
- [954] K. M. M. Haque, X. Lin, Y. Yang, and P. Zhao, On the prime labeling of generalized Petersen graph P(n, 1), Util. Math., 83 (2010) 95-106.
- [955] K. M. M. Haque, X. Lin, Y. Yang, and P. Zhao, On the prime cordial labeling of generalized Petersen graph, Util. Math., 82 (2010) 71-79.
- [956] K. M. M. Haque, X. Lin, Y. Yang, and P. Zhao, On the prime labeling of generalized Petersen graph P(n,3), Int. J. Contemp. Math. Sci., 6 (2011) 1783-1800.
- [957] K. M. M. Haque, X. Lin, Y. Yang, and P. Zhao, Prime labeling on Knödel graphs  $W_{3,n}$ , Ars Combin., **109** (2013) 113-128.
- [958] F. Harary, Sum graphs and difference graphs, Congr. Numer., 72 (1990) 101-108.

- [959] F. Harary, Sum graphs over all the integers, *Discrete Math.*, **124** (1994) 99-105.
- [960] F. Harary, I. Hentzel, and D. Jacobs, Digitizing sum graphs over the reals, Caribb. J. Math. Comput. Sci., 1 (1991) 1-4.
- [961] F. Harary and D. Hsu, Node-graceful graphs, Comput. Math. Appl., 15 (1988) 291-298.
- [962] M. Harminc, On a characterization of graphs by average labelings, *Discuss. Math. Graph Theory*, **17** (1997) 133-136.
- [963] M. Harminc and R. Soták, Graphs with average labellings, Graph theory (Prague, 1998) Discrete Math., 233 (2001) 127-132.
- [964] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, San Diego, 1990.
- [965] N. Hartsfield and W. F. Smyth, The sum number of complete bipartite graphs, in *Graphs and Matrices* (ed. R. Rees), Marcel Dekker (1992) 205-211.
- [966] J. Haslegrave, Proof of a local antimagic conjecture, Discrete Math. Theor. Comput. Sci., 20(1) (2018), Paper No. 18, 14 pp.
- [967] P. Haxell, O. Pikhurko, and A. Taraz, Primality of trees J. Combinatorics, 2 (2011) 481-500.
- [968] M. He, The gracefulness of the graph  $2C_n$ , Neimenggu Daxue Xuebao Ziran Kexue, 26 (1995) 247-251.
- [969] W. He, L. Wang, H. Mi, Y. Shen, and X. Yu, Integral sum graphs from a class of trees, Ars Combin., 70 (2004) 197-205.
- [970] W. He, X. Yu, H. Mi, Y. Sheng, and L. Wang, The (integral) sum number of the graph  $K_n \setminus E(K_r)$  for  $K_r \subset K_n$ , Discrete Math., **243** (2002) 241-252.
- [971] Y. He, L. Shen, Y. Wang, Y. Chang, Q. Kang, and X. Yu, The integral sum number of complete bipartite graphs K<sub>r,s</sub>, Discrete Math., 239 (2001) 137-146.
- [972] S. P. Rao Hebbare, Graceful cycles, Util. Math., 10 (1976) 307-317.
- [973] D. Hefetz, Anti-magic graphs via the combinatorial nullstellensatz, J. Graph Theory, 50 (2005) 263-272.
- [974] D. Hefetz, T. Mütze, and J. Schwartz, On antimagic directed graphs, J. Graph Theory, 64 (2010) 219-232.
- [975] D. Hefetz, A. Saluz, and H. Tran, An application of the combinatorial nullstellensatz to a graph labeling problem, *J. Graph Theory*, **65** (2010) 70-82.

- [976] A. I. Hefnawy, E. A. Elsid, and S. Euat Tallah, Cordial labeling for the sum and union of the second power of paths and stars, *Util. Math.*, **108** (2018) 283-291.
- [977] S. M. Hegde, Additively graceful graphs, Nat. Acad. Sci. Lett., **12** (1989) 387-390.
- [978] S. M. Hegde, On indexable graphs, J. Combin. Inf. Sci. Sys., 17 (1992) 316-331.
- [979] S. M. Hegde, On K-sequential graphs, Nat. Acad. Sci. Lett., 16 (1993) 299-301.
- [980] S. M. Hegde, On harmonious labelings of graphs, KREC Research Bull., 5 (1996) 15-18.
- [981] S. M. Hegde, On (k, d)-graceful graphs, J. Combin., Inform. Sys. Sci., 25 (2000) 255-265.
- [982] S. M. Hegde, Geometric graphs, KREC Research Bulletin, 9 (2000) 27-34.
- [983] S. M. Hegde, Further results on set sequential and set graceful graphs, *Electronic Notes Discrete Math.*, 15 (2003) 100-104.
- [984] S. M. Hegde, On multiplicative labelings of graphs, in Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 83-96.
- [985] S. M. Hegde, On set labelings of graphs, in Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 97-108.
- [986] S. M. Hegde, Set colorings of graphs, Europ. J. Combin., **30** (2009) 986-995.
- [987] S. M. Hegde, On sequentially additive graphs, unpublished.
- [988] S. M. Hegde and S. M. Kumudakshi, Construction of graceful digraphs using algebraic structures, J. Discrete Math. Sci. Cryptogr. 99 (2016) 175-186.
- [989] S. M. Hegde and S. M. Kumudakshi, Further results on graceful directed graphs, *Electronic Notes Discrete Math.*, 53 (2016) 149-154.
- [990] S. M. Hegde and M. Miller, Further results on sequentially additive graphs, Discuss. Math. Graph Theory, 27 (2007) 251-268.
- [991] S. M. Hegde and T. S. Murthy, A partial solution to cordial tree conjecture, J. Discrete Math. Sci. Cryptogr., 17(3) (2014) 257-263.
- [992] S. M. Hegde and S. Shetty, Sequential and magic labeling of a class of trees, Nat. Acad. Sci. Letters, 24 (2001) 137-141.
- [993] S. M. Hegde and P. Shankaran, Geometric labeled graphs, AKCE J. Graphs Combin., 5 (2008) 83-97.

- [994] S. M. Hegde and P. Shankaran, On triangular sum labelings of graphs, in Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 109-115.
- [995] S. M. Hegde and S. Shetty, On graceful trees, Appl. Math. E-Notes, 2 (2002) 192-197.
- [996] S. M. Hegde and S. Shetty, On arithmetic graphs, Indian J. Pure Appl. Math., 33 (2002) 1275-1283.
- [997] S. M. Hegde and S. Shetty, On magic strengths of graphs, KREC Research Bull., 11 (2002) 15-21.
- [998] S. M. Hegde and S. Shetty, On magic graphs, Australas. J. Combin., 27 (2003) 277-284.
- [999] S. M. Hegde and S. Shetty, Combinatorial labelings of graphs, Applied Math. E-Notes, 6 (2006) 251-258.
- [1000] S. M. Hegde and S. Shetty, Strongly indexable graphs and applications, *Discrete Math.*, **309** (2209) 6160-6168.
- [1001] S. M. Hegde and S. Shetty, Strongly k-indexable and super edge magic labelings are equivalent, unpublished
- [1002] S. M. Hegde, S. Shetty, and P. Shankran, Further results on super edge-magic deficiency of graphs, Ars Combin., 99 (2011) 487-502.
- [1003] S. M. Hegde and Shivarajkumar, On graceful unicyclc wheels, Ars Combin., 117 (2014) 47-64.
- [1004] S. M. Hegde and Shivarajkumar, On k-graceful digraphs, Util. Math. 95 (2014) 161-173.
- [1005] S. M. Hegde and Vasudeva, On mod difference labelings of digraphs, AKCE J. Graphs Combin., 6 (2009) 79-84.
- [1006] S. M. Hegde and Vasudeva, An algorithm for sum labeling of graphs, unpublished.
- [1007] K. Heinrich and P. Hell, On the problems of bandsize, Graphs and Combin., 3 (1987) 279-284.
- [1008] V. Hemalatha and V. M. Selvi, Super geometric mean labeling of some cycle related graphs, Internat. J. Sci. Engin. Res., 6(11) (2015) 522-529.
- [1009] H. Hendy, K. A. Sugeng, A. N. M. Salman, and N. Ayunda, Another H-super magic decompositions of the lexicographic product of graphs, *Indonesian J. Combin.*, 2(2) (2018) 72-81. http://dx.doi.org/10.19184/ijc.2018.2.2.2

- [1010] A. Hertz and C. Picouleau, On graceful dierence labelings of disjoint unions of circuits, arXiv:1908.113100v1 [math.CO].
- [1011] K. Ezhilarasi Hilda and J. Jeba Jesintha, Shell flower graphs: felicitous, Internat. J. Pure Appl. Math., 120(8) (2018) 179-188. https://acadpubl.eu/hub/ 2018-120-8/2/19.pdf
- [1012] Y. S. Ho and S. M. Lee, Some initial results on the supermagicness of regular complete k-partite graphs, J. Combin. Math. Combin. Computing, **39** (2001) 3-17.
- [1013] Y.S. Ho, S. M. Lee, and H. K. Ng, On friendly index sets of root-unions of stars by cycles, J. Combin. Math. Combin. Comput., 62 (2007) 97-120.
- [1014] Y. S. Ho, S. M. Lee, and H. K. Ng, On the friendly index sets of regular windmills, unpublished.
- [1015] Y. S. Ho, S. M. Lee, and E. Seah, on the edge-graceful (n, kn)-multigraphs conjecture, J. Combin. Math. Combin. Computing, (1991) 141-147.
- [1016] Y. S. Ho, S. M. Lee, and S. C. Shee, Cordial labellings of unicyclic graphs and generalized Petersen graphs, *Congr. Numer.*, 68 (1989) 109-122.
- [1017] Y. S. Ho, S. M. Lee, and S. C. Shee, Cordial labellings of the cartesian product and composition of graphs, Ars Combin., 29 (1990) 169-180.
- [1018] C. Hoede and H. Kuiper, All wheels are graceful, Util. Math., 14 (1987) 311.
- [1019] J. Holden, D. McQuillan, and J. M. McQuillan, A conjecture on strong magic labelings of 2-regular graphs, *Discrete Math.*, **309** (2009) 4130-4136.
- [1020] M. Horton, Graceful Trees: Statistics and Algorithms, 2003, Univ. Tasmania, http: //eprints.utas.edu.au/19/1/GracefulTreesStatisticsAndAlgorithms.pdf.
- [1021] M. Hovey, A-cordial graphs, Discrete Math., 93 (1991) 183-194.
- [1022] P. Hrnčiar and A. Haviar, All trees of diameter five are graceful, Discrete Math., 233 (2001) 133-150.
- [1023] P. Hrnčiar and G. Monoszova, A new family of graceful trees, preprint.
- [1024] C-C. Hsiao, On Graph Labeling Problems of Antimagic Type, M. S. Thesis, TungHai University, 2006.
- [1025] D. F. Hsu, Harmonious labelings of windmill graphs and related graphs, J. Graph Theory, 6 (1982) 85-87.
- [1026] D. F. Hsu and A. D. Keedwell, Generalized complete mappings, neofields, sequenceable groups and block designs, I, *Pacific J. Math.*, **111** (1984) 317-332.

- [1027] D. F. Hsu and A. D. Keedwell, Generalized complete mappings, neofields, sequenceable groups and block designs, II, *Pacific J. Math.*, **117** (1985) 291-312.
- [1028] H. L. Hu, New graceful labelings of the *n*-cycle  $C_n$  and its *r*-crowns, *Pure Appl. Math.*, (Xi'an) **26** (2010) 454-457.
- [1029] H. Hua and C. Raridan, On the edge-balanced index sets of odd/even complete bipartite graphs, *Congr. Numer.*, **219** (2014) 227-232.
- [1030] Q. Huang, Harmonious labeling of crowns  $C_n \odot K_1$ , unpublished
- [1031] C. Huang, A. Kotzig, and A. Rosa, Further results on tree labellings, Util. Math., 21c (1982) 31-48.
- [1032] J. Huang and S. Skiena, Gracefully labeling prisms, Ars Combin., 38 (1994) 225-242.
- [1033] P.-Y. Huang, T.-L. Wong, and X. Zhu, Weighted-1-antimagic graphs of prime power order, *Discrete Math.*, **312**(14) (2012) 2162-2169.
- [1034] M. I. Huilgol and V. Sriram, Square sum labeling of disjoint union of graphs, J. Graph Labeling, 2(2) (2016) 103-106.
- [1035] M. Hussain, K. Ali, M.T. Rahim, and E. T. Baskoro, On (a, d)-vertex-antimagic labelings of Harary graphs, Util. Math., 83 (2010) 73-80.
- [1036] M. Hussain, E.T. Baskoro, and K. Ali, On super antimagic total labeling of Harary graph, Ars Combin., 104 (2012) 225-233.
- [1037] M. Hussain, E. Baskoro, and Slamin, On super edge-magic total labeling of banana trees, Util. Math., 79 (2009) 243-251.
- [1038] M. Hussainn and A. Tabraiz, Super *d*-anti-magic labeling of subdivided  $kC_5$ , Turkish J. Math., **39**(5) (2015) 773-783.
- [1039] M. Ibrahim, M. Asif, A. Ahmad, and M. K. Siddiqui, Computing the total irregularity strength of wheel related graphs, Util. Math., 108 (2018) 321-338. https://www.researchgate.net/publication/286456461\_ Total\_Edge\_Irregularity\_Strength\_of\_Wheel\_Related\_Graphs
- [1040] R. Ichishima, S. C. López, and F. A. Muntaner-Batle, On the beta-number of forests with isomorphic components, *Discuss. Math. Graph Theory*, 38 (2018) 683-701.
- [1041] R. Ichishima, S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, The power of digraph products applied to labelings, *Discrete Math.*, **312** (2012) 221-228.
- [1042] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, Enlarging the classes of super edge-magic 2-regular graphs, AKCE Int. J. Graphs Comb., **10**(2) (2013) 129-146.

- [1043] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, The measurements of closeness to graceful graphs, Australas. J. Combin., 62 (2015) 197-210.
- [1044] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, The consecutively super edge-magic deficiency of graphs and related concepts, preprint.
- [1045] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, On the beta-number of joins of graphs, Australas. J. Combin., 69 (3) (2017) 402-409.
- [1046] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, The harmonious number of graphs and related graph related parameters, *Bull. Inst. Combin. Appl.*, 82 (2018) 80-95.
- [1047] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, New parameters for studying graceful properties of graphs, *Elect. Notes Discrete Math.*, **60** (2017) 3-10.
- [1048] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, On the beta-number of the joins of graphs, Australas. J. Combin., 69 (2017) 402-409.
- [1049] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, Bounds for the strength of graphs, Australasian J. Combin., 72(3) (2018) 492-508.
- [1050] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, On the strength of some trees, AKCE Internat. J. Graphs Combin., to appear. https://doi.org/10.1016/j. akcej.2019.06.002
- [1051] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, On the strong beta-number of galaxies with three and four components, AKCE Internat. J. Graphs Combin., to appear. https://doi.org/10.1016/j.akcej.2019.03.004
- [1052] R. Ichishima and F. A. Muntaner-Batle, On the strong sequential number of graphs, preprint.
- [1053] R. Ichishima and F. A. Muntaner-Batle, On the strong beta-number of galaxies with five components, preprint.
- [1054] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, On the beta-number of the coronas of graphs, preprint.
- [1055] R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, On the beta-number and gamma-number of galaxies, *Discrete Math. Lett.*, 1 (2019) 49-53.
- [1056] R. Ichishima, F. A. Muntaner-Batle, and M. Rius-Font, Embedding trees into super edge-magic trees, Bull. Inst. Combin. Appl., 52 (2008) 51-60.
- [1057] R. Ichishima, F. A. Muntaner-Batle, and M. Rius-Font, Bounds on the size of super edge-magic graphs depending on the girth, *Ars Combin.*, **119** (2016) 129-133.

- [1058] R. Ichishima and A. Oshima, On the beta-number of linear forests with an even number of components, AKCE Int. J. Graphs Comb., 15(3) (2018) 238-241.
- [1059] R. Ichishima and A. Oshima, On partitional labelings of graphs, Math. Comput. Sci., 3 (2010) 39-45.
- [1060] R. Ichishima and A. Oshima, On partitional and other related graphs, Math. Comput. Sci., 5 (2011) 41-50.
- [1061] R. Ichishima and A. Oshima, On the super edge-magic deficiency of 2-regular graphs with two components, Ars Combin., **129** (2016) 437-447.
- [1062] R. Ichishima and A. Oshima, On the super edges-magic deficiency and  $\alpha$ -valuations of graphs, J. Indones. Math. Soc., Special Edition (2011) 59-69.
- [1063] R. Ichishima and A. Oshima, Bounds for the gamma-number of graphs, *Util. Math.*, to appear.
- [1064] R. Ichishima, A. Oshima, and Y. Takamashi, The strength sum of graphs and related invariants, (2020), prepreint.
- [1065] M.Imran, H. U. Afzal, and A. Q. Baig, On super edge-magic deficiency of volvox and dumbbell graphs, AKCE Internat. J. Graphs Combin., 13 (2016) 112-119.
- [1066] M. Imran, M. K. Siddiqui, and M. Numan, Super d-antimagic labeling of uniform subdivision of wheel, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, 77 (2015)(2) 227-240.
- [1067] N. Inayah, A. Lladó, J. Moragas, Magic and antimagic H-decompositions, Discrete Math., 312(7) (2012) 1367-1371.
- [1068] N. Inayah, A. N. M. Salman, and R. Simanjuntak, On (a, d)-H-antimagic coverings of graphs, J. Combin. Math. Combin. Comput., 71 (2009) 273-281
- [1069] N. Inayah, R. Simanjuntak and A. N. M. Salman, Super (a, d) H-antimagic total labelings for shackles of a connected graph H, Australasian J. Combin., 57 (2013) 127-138.
- [1070] N. Inayah, I. W. Sudarsana, S. Musdalifah, and N. D. Mangesa, On super mean labeling for total graph of path and cycle, *Internat. J. Math. Math. Sci.*, **7-8** (2018) 1-5.
- [1071] D. Indriati, Widodo, I. E. Wijayanti, K. A. Sugeng, and M. Bača, On total edge irregularity strength of generalized web graphs and related graphs, *Math. Comput. Sci.*, 9(2) (2015) 161-167.
- [1072] D. Indriati, I. E. W. Widodo, and K. A. Sugeng, On the total edge irregularity strength of generalized helm, AKCE Int. J. Graphs Comb., 10(2) (2013) 147-155.

- [1073] D. Indriati, E. W. Widodo, I. E. Wijayanti, K. A. Sugeng, M. Bača, and A. Semaničová-Feňovčíková, The total vertex irregularity strength of generalized helm graphs and prisms with outer pendant edges, *Austral. J. Combin.*, 65(1) (2016) 14-26.
- [1074] D. Indriati, W. I. E. Widodo, and K. A. Sugeng, On the total edge irregularity strength of generalized helm, AKCE Int. J. Graphs Comb., 10(2) (2013) 147-155. https://ajc.maths.uq.edu.au/pdf/65/ajc\_v65\_p014.pdf
- [1075] P. Inpoonjai, Degree-magic labelings on the join and composition of complete tripartite graphs, Comm. Math. Appl., textbf10(3) (2019) 391-402. https://www. rgnpublications.com/journals/index.php/cma/article/view/1157/0
- [1076] M. Irfan and A. Semaničová-Feňovčová, On total labelings of graphs with prescribed weights, AKCE Internat. J. Graphs Combin., 13 (2016) 191-199.
- [1077] G. Isaak, R. Jamison, and D. Narayan, Greedy rankings and arank numbers, *In-form. Process. Lett.*, **109** (2009) 825-827.
- [1078] J. Ivančo, On supermagic regular graphs, Math. Bohemica, 125 (2000) 99-114.
- [1079] J. Ivančo, Magic and supermagic dense bipartite graphs, Discuss. Math. Graph Theory, 27 (2007) 583-591.
- [1080] J. Ivančo, A construction of supermagic graphs, AKCE J. Graphs Combin., 6 (2009) 91-102.
- [1081] J. Ivančo, Supermagic generalized double graphs, Discuss. Math. Graph Th., 36 (2016) 211-225.
- [1082] J. Ivančo, On k-total edge product cordial graphs, Australasian J. Comb., 67(3) (2017) 476-485. https://ajc.maths.uq.edu.au/pdf/67/ajc\_v67\_p476.pdf
- [1083] J. Ivančo, and S. Jendrol, Total edge irregularity strenght of trees, Discuss. Math. Graph Theory, 26 (2006) 449-456.
- [1084] J. Ivančo, P. Kovář, and A. Semaničová-Feňovčová, On the existence of regular supermagic graphs, J. Combin. Math. Combin. Comput., 71 (2009) 49-64.
- [1085] J. Ivančo, Z. Lastivkova, and A. Semaničova, On magic and supermagic line graphs, Math. Bohemica, 129 (2004) 33-42.
- [1086] J. Ivančo and I. Lučkaničová, On edge-magic disconnected graphs, SUT J. Math., 38 (2002) 175-184.
- [1087] Ivančo and Polláková, Supermagic graphs having a saturated vertex, *Discuss. Math. Graph Th.*, **34**(1) (2014) 75-84.

- [1088] J. Ivančo and A. Semaničova, Some constructions of supermagic non-regular graphs, Australas. J. Combin., 38 (2007) 127-139.
- [1089] J. Ivančo and A. Semaničova, Some constructions of supermagic graphs using antimagic graphs, SUT J. Math., 42 (2007) 177-186.
- [1090] J. Jacob, D. Narayan, E. Sergel, P. Richter, and A. Tran, personal communication.
- [1091] R. Jagadesh and J. Baskar Babujee, Super edge bimagic labeling, Proceed. 3rd Inter. Conf. Math. Comp. Sci., 5-6th Jan. 2009, Loyola College, Chennai, India, (2009) 79-82.
- [1092] R. Jagadesh and J. Baskar Babujee, Edge vertex prime labeling for some class of graphs, National Conference on Recent Trends in Mathematics and its Applications, 24-25 February 2017, SRM University, Vadapalani, Chennai, India.
- [1093] P. Jampachon, K. Nakprasit, and T. Poomsa-ard, Graceful labeling of some classes of spider graphs with three legs greater than one, *Thai J. Math.*, **12**(3) (2014) 621-630.
- [1094] M. M. Jariya, Index of labeling, Internat. J. Math. Trends Tech. 657 (2019) 87-96. https://www.ijmttjournal.org/Volume-65/Issue-8/IJMTT-V6518P509.pdf
- [1095] I. Javaid, A. Ismail, M. Salman, A. Ahmad, and Slamin, Labeling of chordal rings, Util. Math., 90 (2013) 61-75.
- [1096] I. Javaid, F. Khalid, A. Ahmad, and M. Imran, On a weaker version of sum labeling of graphs, *Math. Rep. (Bucur.)* 16(66) (2014), no. 3, 413-420.
- [1097] M. Javaid, On super edge-antimagic total labeling of subdivided stars, Discuss. Math. Graph Theory, 34(4) (2014) 691-705.
- [1098] M. Javaid, Super (a, d)-EAT labeling of subdivided stars, AKCE Int. J. Graphs Comb., **12**(1) (2015) 14-18.
- [1099] M. Javaid and A. A. Bhatti, On anti-magic vertex labeling of disjoint union of star hyper-graphs, Canadian J. Sci. and Eng. Math., 3(5) (2012) 202-209.
- [1100] M. Javaid, A. A. Bhatti, and M. Hussain, On (a, d)-edge-antimagic total labelings of extended w-trees, Util. Math., 87 (2012) 293-303.
- [1101] M. Javaid and A. A. Bhatti, On super (a, d)-edge-antimagic total labelings of generalized extended w-trees, AKCE Int. J. Graphs Comb., 11 (2014) 115-126.
- [1102] M. Javaid, A. A. Bhatti, M. Hussain, and K. Ali, Super edge-magic total labeling on a forest of extended w-trees, Util. Math., 91 (2013) 155-162.
- [1103] M. Javaid, M. Hussain, K. Ali, and K. H. Dar, Super edge-magic total labeling on w-trees, Util. Math., 86 (2011) 183-191.

- [1104] M. Javaid, M. Hussain, K. Ali, and M. Shaker, On super edge-magic total labeling on subdivision of trees, *Util. Math.* 89 (2012) 169-177.
- [1105] A. Javed and M. K. Jamil, 3-total edge product cordial labeling of rhombic grid, AKCE Int. J. Graphs Comb., 16(2) (2019) 213221.
- [1106] S. Javed, A. Riasat, and S. Kanwal, On super edge-magicness and deficiencies of forests, Util. Math., 98 (2015) 149-169.
- [1107] J. Jayapriya, 0-edge magic labeling of slitting graph, Global J. Pure and Appl. Math., 12(1) (2016) 421-426.
- [1108] J. Jayapriya and K. Thirusangu, 0-edge magic labeling for some class of graphs, Indian J. Comput. Sci. Engin.(IJCSE), 3(3) (2012) 425-427. http://citeseerx. ist.psu.edu/viewdoc/download?doi=10.1.1.300.9750&rep=rep1&type=pdf
- [1109] C. Jayasekaran and C. David Raj, Harmonic mean labeling of disconnected graphs, J. Discrete Math. Sci. Cryptogr., 19(1) (2016) 1-12.
- [1110] C. Jayasekaran and J. L. Flower, Edge trimagic total lableings of some shadow and splitting graphs, Internat. J. Pure Appl. Math., 120(3) (2018) 303-313. http: //www.ijpam.eu
- [1111] C. Jayasekaran and M. Jaslin Melbha, Some more results on one modulo three root square mean graphs, Int. Electron. J. Pure Appl. Math., 12(1) (2018) 141-149.
- [1112] S. Jendrol, J. Misšuf, and R. Soták, Total edge irregularity strength of complete graphs and complete bipartite graphs, *Elec. Notes Discrete Math.*, 28 (2007) 281-285.
- [1113] S. Jendrol, J. Misšuf, and R. Soták, Total edge irregularity strength of complete graphs and complete bipartite graphs, *Discrete Math.*, **310** (2010) 400-407.
- [1114] R. H. Jeurissen, Magic graphs, a characterization, Europ. J. Combin., 9 (1988) 363-368.
- [1115] J. Jeba Jesintha, New Classes of Graceful Trees, Ph. D. Thesis, Anna University, Chennai, India, 2005.
- [1116] J. Jeba Jesintha and K. Ezhilarasi Hilda, Butterfly graphs with shell orders m and 2m + 1 are graceful, *Bonfring Internat. J. Research Communication Engin.*, **2**(2) (2012) 1-5.
- [1117] J. Jeba Jesintha and K. Ezhilarasi Hilda, Shell-butterfly graphs are harmonious, Adv. Appl. Disc. Math., 14(1) (2014) 39-51.
- [1118] J. Jeba Jesintha and K. Ezhilarasi Hilda, Variation of graceful labeling on disjoint union of two subdivided shell graphs, Annals Pure Applied Math., 8(2) (2014) 19-25.

- [1119] J. Jeba Jesintha and K. Ezhilarasi Hilda, Subdivided uniform shell bow graphs are one modulo three graceful, *Math. Sci. Internat. Res. J.*, 3(2) (2014) 645-646.
- [1120] J. Jeba Jesintha and K. Ezhilarasi Hilda, Subdivided uniform shell bow graphs and subdivided shell graphs are odd graceful, *Eduventure Res. J. Queen Marys College*, 7(2) (2014) 103-105.
- [1121] J. Jeba Jesintha and K. Ezhilarasi Hilda, All uniform bow graphs are graceful, Math. Comput. Sci., 9 (2015)(2) 185-191.
- [1122] J. Jaba Jesintha and K. Ezhilarasi Hilda, Double shells with two pendant edges at the apex are k-graceful, J. Combin. Math. Combin. Comput., 92 (2015) 47-57.
- [1123] J. Jeba Jesintha and K. Ezhilarasi Hilda, Shell butterfly graphs are graceful, Internat. J. Pure Appl. Math., 101(6) (2015) 949-956.
- [1124] J. Jeba Jesintha and K. Ezhilarasi Hilda,  $\rho^*$  labeling of paths and shell butterfly graphs, *Internat. J. Pure and Appl. Math.*, **101**(5) (2015) 645-653.
- [1125] J. Jeba Jesintha and K. Ezhilarasi Hilda, Shell butterfly graphs are edge odd graceful, Internat. J. Pure and Appl. Math., 109(7) (2016) 159-166.
- [1126] J. Jeba Jesintha and K. Ezhilarasi Hilda, The graph SSG(2) is odd graceful and odd harmonious, Internat. J. Comp. Aided Engin. and Tech., (2016) to appear.
- [1127] J. Jeba Jesintha and G. Sethuraman, A new class of graceful rooted trees, A new class of graceful rooted trees, J. Discrete Math. Sci. Cryptogr., 11 (2008) 421-435.
- [1128] J. Jeba Jesintha and G. Sethuraman, Generation of graceful trees, Internat. J. Algor. Comput. Math., 2 (2009) 33-38.
- [1129] J. Jeba Jesintha and G. Sethuraman, All arbitrarily fixed generalized banana trees are graceful, Math. Comput. Sci., 5(1) (2011) 51-62.
- [1130] J. Jeba Jesintha and K. Subashini, Gracefulness in path union of vertex switching of even cycles in increasing order, South East Asian J. Math. Math. Sci., 14(3) (2018) 15-20.
- [1131] P. Jeyanthi, k-equitable mean labeling, J. Algor. Computation, 44(1) (2013) 21-30.
- [1132] P. Jeyanthi, Skolem difference mean labeling of union of graphs, Util. Math., 111 (2019) 139-150.
- [1133] P. Jeyanthi, N. Angel Benseera and M. Immaculate Mary, On totally magic cordial labeling, SUT J. Math., 49(1) (2013) 13-18.
- [1134] P. Jeyanthi and N. Angel Benseera, Totally vertex-magic cordial labeling, SUT J. Math., 49(1) (2013) 1-12.

- [1135] P. Jeyanthi and N. Angel Benseera, Totally magic cordial labeling of one-point union of n copies of a graph, *Opuscula Math.*, **34**(1) (2014) 115-133.
- [1136] P. Jeyanthi and N. Angel Benseera, Totally magic cordial labeling of some graphs, J. Algorithms Comput., 46 (2015) 1-8.
- [1137] P. Jeyanthi, N. Angel Benseera, and I. Cahit, Totally magic cordial labeling of  $mP_n$  and  $mK_n$ , Proyectiones J. Math., **35**(4) (2016) 371-381.
- [1138] P. Jeyanthi, N. Angel Benseera and G.-C, Lau, On k-totally magic cordial labeling of graphs, Disc. Math., Algor. Appl., 7(3)(2015) 1550024, 7 pp. DOI:10.1142/ S179383091550024X
- [1139] P. Jeyanthi and N. Angel Benseera, Totally magic cordial deficiency of some graphs, Utilitas Math., 105 (2017) 87-94.
- [1140] P. Jeyanthi, N. Angel Benseera, and G.-C. Lau, Hypo-k-totally magic cordial labeling of graphs, *Proyecciones J. Math.*, 34(4) (2015) 347-355.
- [1141] P. Jeyanthi and R. Gomathi, Odd mean labeling of chain of graphs, *Electron. Notes Disc. Math.*, 48 (2015) 157-160.
- [1142] P. Jeyanthi and R. Gomathi, Analytic odd mean labeling of super subdivision and H-super subdivision of graphs, J. Emerging Tech. Innov. Res. (JETIR), 6(6) (2019) 541-551. http://www.jetir.org/papers/JETIR1908A25.pdf
- [1143] P. Jeyanthi and R. Gomathi, Analytic odd mean labeling of corona graphs, Malaya J. Mat., 8(3)(2020) 898-902.
- [1144] P. Jeyanthi, R. Gomathi, and G.-C. Lau, Analytic odd mean labelings of some graphs, *Palestine J. Math.*, 8(2) (2019) 392-399.
- [1145] P. Jeyanthi, R. Gomathi, and G.-C. Lau, preprint.
- [1146] P. Jeyanthi, R. Gomathi, and G.-C. Lau, Some results on analytic odd mean labelings of graphs, Bull. Internat. Math. Virtual Inst., 9 (2019) 487-499.
- [1147] P. Jeyanthi, R. Gomathi, and G.-C. Lau, Further results on analytic odd mean labeling of graphs, *Internat. J. Math. Combin.*, 4 (2019) 136-148. https://www.researchgate.net/publication/338448124\_Further\_Results\_ on\_Analytic\_Odd\_Mean\_Labeling\_of\_Graphs
- [1148] P. Jeyanthi and K. Jeya Daisy,  $Z_k$ -magic labeling of subdivision graphs, *Discrete* Math., Algor. Appl., 8(3) (2016) 1650046 (19 pages).
- [1149] P. Jeyanthi and K. Jeya Daisy, Certain classes of magic graphs, J. Graph Labeling, 4(1) (2018) 38-47.

- [1150] P. Jeyanthi and K. Jeya Daisy,  $Z_k$ -magic labeling of some families of graphs, J. Algor. Computation, 50(2) (2018) 1-12.
- [1151] P. Jeyanthi and K. Jeya Daisy,  $Z_k$ -Magic labeling of star of graphs, *Proyecciones J. Math.*, **39**(1) (2020) 31-51.
- [1152] P. Jeyanthi and K. Jeya Daisy,  $Z_k$ -Magic labeling of open star of graphs, Bull. Int. Math. Virtual Inst., 7(2) (2017) 243-255.
- [1153] P. Jeyanthi, K. Jeya Daisy, and A. Semaničová-Feňovčíková,  $Z_k$ -Magic labeling of path union of graphs, *CUBO J. Math.*, **21**(2) (2019) 15-35.
- [1154] P. Jeyanthi and K. Jeya Daisy, Some results on  $Z_k$ -magic labeling, Palestine J. Math., 8(2) (2019) 400-412.
- [1155] P. Jeyanthi and K. Jeya Daisy,  $Z_k$ -magic labeling of cycle of graphs, Internat. J. Math. Combin., 1 (2019) 88-102.
- [1156] P. Jeyanthi, R. Gomathi and G.-C. Lau, Analytic odd mean labeling of some standard graphs, *Proceed. Internat. Confer. Algebra Discrete Math.*, ICADM (2018) 62-68.
- [1157] P. Jeyanthi, R. Gomathi, and G.-C. Lau, Analytic odd mean labeling of square and H graphs, Internat. J. Math. Combin., Special Issue 1 (2018) 61-67.
- [1158] P. Jeyanthi, R. Gomathi, and G.-C. Lau, Some results on analytic odd mean labeling of graph, Bull. Inter. Math. VIRTUAL INST., 9 (2019) 487-499. DOI: 10.7251/BIMVI1903487J
- [1159] P. Jeyanthi, R. Kalaiyarasi, D. Ramya, and T. Saratha Devi, Some results on Skolem odd difference mean labeling, *Proyecciones J. Math.*, **35**(4) (2016) 407-417.
- [1160] P. Jeyanthi and A. Maheswari, Vertex equitable labeling of cycle and path related graphs, Util. Math., 98 (2015) 215-226.
- [1161] P. Jeyanthi and A. Maheswari, Some results on vertex equitable labeling, Open J. Discrete Math., 2 (2012) 51-57. http://dx.doi.org/10.4236/ojdm.2012.22009 NEW Nov. 2016
- [1162] P. Jeyanthi and A. Maheswari, 3-product cordial labeling, SUT J. Math., 48 (2) (2012) 231-240.
- [1163] P. Jeyanthi and A. Maheswari, 3-product cordial labeling of some graphs, Internat. J. Math. Combin., 1 (2012) 96-105.
- [1164] P. Jeyanthi and A. Maheswari, Vertex equitable labeling of transformed trees, J. Algor. Computation, 44 (1) (2013) 9-20. https://jac.ut.ac.ir/article\_7912\_ b7a385d110f5ec4c2d805c82bcf3079e.pdf

- [1165] P. Jeyanthi and A. Maheswari, Some results on 3-product cordial labeling, Util. Math., 99 (2016) 215-229.
- [1166] P. Jeyanthi and A. Maheswari, One modulo three mean labeling of graphs, Amer. J. Appl. Math. Stat., 2(5) (2014) 302-306.
- [1167] P. Jeyanthi and A. Maheswari, Vertex equitable labeling of graphs, Graph Theory Notes New York, LXVII (2014) 34-42.
- [1168] P. Jeyanthi and A. Maheswari, Further results on vertex equitable graphs, *Electron. Notes Disc. Math.*, 48 (2015) 161-164.
- [1169] P. Jeyanthi and A. Maheswari, Vertex equitable labeling of cycle and path related graphs, Util. Math., 98 (2015) 215-226.
- [1170] P. Jeyanthi and A. Maheswari, 3-product cordial labeling of star graphs, Southeast Asian Bull. Math., 39(3) (2015) 429-437.
- [1171] P. Jeyanthi and A. Maheswari, Vertex equitable labeling of cyclic snakes and bistar graphs, J. Scientific Res., 6(1) (2014) 79-85.
- [1172] P. Jeyanthi and A. Maheswari, Odd vertex equitable even labeling of cycle related graphs, CUBO A Math. J., 20 (02) (2018) 13-21. http://dx.doi.org/10.4067/ S0719-06462018000200013
- [1173] P. Jeyanthi, A. Maheswari, and P. Pandiaraj, One modulo three mean labeling of cycle related graphs, *Internat. J. Pure and Appl. Math.*, **103**(4)(2015) 625-633.
- [1174] P. Jeyanthi, A. Maheswari, and P. Pandiaraj, On one modulo three mean labeling of graphs, J. Discrete Math. Science & Cryptography, 19:2 (2016) 375-384.
- [1175] P. Jeyanthi, A. Maheswari, and P. Pandiaraj, One modulo three geometric mean graphs, J. Algor. Comput., 50(1) (2018) 101-108.
- [1176] P. Jeyanthi, A. Maheswari, and P. Pandiaraj, One modulo three mean labeling of transformed trees, *Proyectiones J. Math.*, 35(3) (2016) 277-289.
- [1177] P. Jeyanthi, A. Maheswari and M. VijayaLaksmi, Odd vertex equitable even labeling of graphs, *Proyecciones J. Math.*, 36(1) (2017) 1-11. https://scielo. conicyt.cl/pdf/proy/v36n1/art01.pdf
- [1178] P. Jeyanthi, A. Maheswari, and M. VijayaLaksmi, New results on vertex equitable labeling, J. Algebra Combin. Discrete Structures Appl., 3(2)(2016) 97-104. https: //dergipark.org.tr/en/download/article-file/267792
- [1179] P. Jeyanthi and A. Sudha, On the total irregularity strength of some graphs, Bull. Inter. Math. Virtual Inst., 9(2) (2019) 393-401. http://www.imvibl.org/ buletin/bulletin\_imvi\_9\_2\_2019/bulletin\_imvi\_9\_2\_2019\_393\_401.pdf

- [1180] P. Jeyanthi, A. Maheswari and M. VijayaLaksmi, Vertex equitable labeling of union of cyclic snake related graphs, *Proyecciones J. Mathematics*, **35**(1) (2016) 177-186. https://scielo.conicyt.cl/pdf/proy/v35n2/art03.pdf
- [1181] P. Jeyanthi, A. Maheswari, and M. VijayaLaksmi, Vertex equitable labeling of double alternate snake graph, J. Algor. Comput., 46 (2015) 27-34. https://jac. ut.ac.ir/article\_7923\_76d1a6298f68c081c967627653edc287.pdf
- [1182] P. Jeyanthi, A. Maheswari and M. VijayaLaksmi, Vertex equitable labeling of cycle and star related graphs, J. Scientific Res., 7(3)(2015) 33-42. https://www. banglajol.info/index.php/JSR/article/view/22810
- [1183] P. Jeyanthi, A. Maheswari, and M. VijayaLaksmi, Vertex equitable labelings of super subdivision graphs, *Scientific Internat.*, **27**(5) (2015) 3881-3883. https://www.researchgate.net/profile/P\_Jeyanthi2/publication/ 284182322\_VERTEX\_EQUITABLE\_LABELING\_OF\_SUPER\_SUBDIVISION\_GRAPHS/ links/564ee37a08aefe619b103a26.pdf
- [1184] P. Jeyanthi, A. Maheswari, and M. VijayaLakshmi, 3-Product cordial labeling of some snake graphs, *Proyectiones J. Math.* 38(1) (2019) 13-30.
- [1185] P. Jeyanthi, A. Maheswari, and M. VijayaLaksmi, Further results on 3-product cordial labeling, *Proyecciones J. Math.*, 38(2) (2019) 191-202.
- [1186] P. Jeyanthi, A. Maheswari, and M. VijayaLakshmi, Vertex switching in 3-product cordial graphs, J. Algor. Computation, 50(1) (2018) 185-193.
- [1187] P. Jeyanthi, A. Maheswari, and M. VijayaLaksmi, Vertex equitable labeling of cyclic snake related graphs, *Proyectiones J. Math.*, 37(4) (2018) 613-625.
- [1188] P. Jeyanthi, A. Maheswari, and M. VijayaLaksmi, Odd vertex equitable even labeling of ladder graphs, *Jordon J. Math. Stat.*, **12**(1) (2019) 75-87..
- [1189] P. Jeyanthi, A. Maheswari, and M. Vijayalaksmi, 3-product cordial labeling of some snake graphs, *Proyectiones J. Math.*, 38 DOI:10.4067/ S0716-09172019000100013
- [1190] P. Jeyanthi, A. Maheswari, and M. Vijayalakshmi, Odd vertex equitable even labeling of ladder graphs, *Jordan J. Math. Stat.*, **12**(1) (2019) 75-87. https://www.researchgate.net/publication/332062041\_0dd\_vertex\_ equitable\_even\_labeling\_of\_ladder\_graphs
- [1191] P. Jeyanthi and N. T. Muthuraja, Some cycle-super magic graphs, Internat. J. Math. Soft Comput., 4(2) (2014) 137-144.
- [1192] P. Jeyanthi, N. T. Muthuraja, A. Semaničová-Feňovčíková and S. J. Dharshikha, More classes of cycle super antimagic graphs, *Australasian J. Combin.*, 67(1) (2017) 46-64.

- [1193] P. Jeyanthi and S. Philo, Odd harmonious labeling of some new families of graphs, *Electron. Notes Disc. Math.*, 48 (2015) 165-168.
- [1194] P. Jeyanthi and S. Philo, Odd harmonious labeling of some cycle related graphs, *Proyecciones J. Math.*, 35(1) 2016) 85-98.
- [1195] P. Jeyanthi and S. Philo, Odd harmonious labeling of plus graphs, Bull. Int. Math. Virtual Inst., 7 (2017)(3) 515-526.
- [1196] P. Jeyanthi and S. Philo, Odd harmonious labeling of some new graphs, Southeast Asian Bull. Math., 43 (2019) 509-523. https://www.researchgate. net/publication/329626405\_Cordial\_Labeling\_of\_Generalized\_Jahangir\_ Graph
- [1197] P. Jeyanthi and S. Philo, Odd harmonious labeling of some super subdivision graphs, *Proyecciones J. Math.*, 38(1) (2019) 1-11.
- [1198] P. Jeyanthi and S. Philo, Odd harmonious labeling of complete bipartite graphs, preprint.
- [1199] P. Jeyanthi and S. Philo, Odd harmonious labeling of line and disjoint union of graphs, preprint.
- [1200] P. Jeyanthi and S. Philo, Odd harmonious labeling of subdivided shell graphs, Internat. J. Comput. Sci. Eng., 7(5) (2019) 77-80.
- [1201] P. Jeyanthi and S. Philo, Odd harmonious labeling of step ladder graphs, *Util. Math.*, to appear.
- [1202] P. Jeyanthi and S. Philo, Odd harmonious labelings of certain graphs, JASC: J. Appl. Sci. Comput., VI(IV) (2019) 1224-1232. https://www.researchgate. net/profile/P\_Jeyanthi2/publication/332320709\_ODD\_HARMONIOUS\_ LABELING\_OF\_CERTAIN\_GRAPHS/links/5cadc56492851ccd4ac087b5/ ODD-HARMONIOUS-LABELING-OF-CERTAIN-GRAPHS.pdf
- [1203] P. Jeyanthi and S. Philo, Some results on odd harmonious labeling of graphs, Bull. Internat. Math. Virtual Inst., 9 (2019) 567-576. DOI:10.7251/BIMVI1903567J
- [1204] P. Jeyanthi and S. Philo, Odd harmonious labeling of some classes of graph, CUBO, A Math. J., 22(3) (2020) 299-314.
- [1205] P. Jeyanthi, S. Philo and M. Z. Youssef, Odd harmonious labeling of grid graphs, Proyec. J. Math., 38(3) (2019) 411-428. https://scielo.conicyt.cl/pdf/proy/ v38n3/0716-0917-proy-38-03-00411.pdf
- [1206] P. Jeyanthi and D. Ramya, Some results on super mean graphs, Util. Math., 92 (2013) 149-160.

- [1207] P. Jeyanthi and D. Ramya, On construction of mean graphs, J. Scientific Research 5(2) (2013) 265-273.
- [1208] P. Jeyanthi and D. Ramya, Super mean graphs, Util. Math., 96 (2015) 101-109.
- [1209] P. Jeyanthi, D. Ramya, and A. Maheswari, On mean labeling of some trees, Util. Math., 93 (2014) 161-191.
- [1210] P. Jeyanthi and D. Ramya Super mean labeling of some classes of graphs, International J. Math. Combin., 1 (2012) 83-91.
- [1211] P. Jeyanthi, D. Ramya, and M. Selvi, Even vertex odd mean labeling of transformed trees, TWMS J. App. Eng. Math., 10(2) 2020 338-345.
- [1212] P. Jeyanthi, D. Ramya, and P. Thangavelu, On super mean graphs, AKCE J. Graphs Combin., 6(1) (2009) 103-112.
- [1213] P. Jeyanthi, D. Ramya, and P. Thangavelu, Some constructions of k-super mean graphs, Inter. J. Pure Applied Math., 56 (2009) 77-86.
- [1214] P. Jeyanthi, D. Ramya, and P. Thangavelu, On super mean labeling of some graphs, SUT J. Math., 46 (2010) 53-66.
- [1215] P. Jeyanthi and T. Sarada Devi, Edge pair sum labeling, J. Scientific Research, 5
  (3) (2013) 457-467.
- [1216] P. Jeyanthi and T. Sarada Devi, On edge pair sum labeling of graphs, Internat. J. Math. Trends Tech., 7(2) (2014) 106-113.
- [1217] P. Jeyanthi and T. Sarada Devi, Edge pair sum labeling of spider graph, J. Algor. Comput., 45 (2014) 25-34.
- [1218] P. Jeyanthi and T. Sarada Devi, Some edge pair sum graphs, J. Discr. Math. Science & Crypt., 18(5) (2015) 481-493.
- [1219] P. Jeyanthi and T. Sarada Devi, Edge pair sum labeling of some Cartesian product graphs, Discrete Math., Algor. Appl., 8 (23) (2016) 16500245 (10 pages).
- [1220] P. Jeyanthi and T. Sarada Devi, Edge pair sum labeling of butterfly graph with shell order, *Malaya J. Matematik*, 4(2) (2016) 205-210.
- [1221] P. Jeyanthi and T. Sarada Devi, Edge pair sum labeling of cycle related graphs, J. Algor. Comput., 48 (2016) 57-68.
- [1222] P. Jeyanthi and T. Sarada Devi, Edge pair sum labeling of some subdivision of graphs, Bull. Internat. Math. Virtual Inst., 8 (2018) 67-80.
- [1223] P. Jeyanthi and T. Saraha Devi, Some new families of edge pair sum graphs, *Palettine J. Math.*, 8(2) (2019) 61-69.

- [1224] P. Jeyanthi and T. Sarada Devi, Edge pair sum labeling of some classes of graphs, J. Discr. Math. Sci. Crypt., to appear.
- [1225] P. Jeyanthi, T. Sarada Devi, and G.-C Lau, Some results on edge pair sum labeling, *Electronic Notes Disc. Math.*, 48 (2015) 169-173.
- [1226] P. Jeyanthi, T. Sarada Devi, and G.-C. Lau, New results on edge pair sum graphs, Internat. J. Math. Appl., 4(1-B) (2016) 57-65.
- [1227] P. Jeyanthi, T. Saratha Devi, and G.-C. Lau, Edge pair sum labeling of WT(n:k) tree, *Global J. Pure and Appl. Math.*, **11**(3) (2015) 1523-1539.
- [1228] P. Jeyanthi and P. Selvagopal, Magic covering of chain of an arbitrary 2-connected simple graph, SUT J. Math., 44 (2008) 23-31.
- [1229] P. Jeyanthi and P. Selvagopal, More classes of H-supermagic graphs, Internat. J. Algor., Comput. and Math., 3 (2010) 93-108.
- [1230] P. Jeyanthi and P. Selvagopal, H-supermagic strength of some graphs, Tokyo J. Math., 33 (2010) 499-507.
- [1231] P. Jeyanthi and P. Selvagopal, Some  $C_4$ -supermagic graphs, Ars Combin., 111 (2013) 129-136.
- [1232] P. Jeyanthi and P. Selvagopal, Supermagic coverings of some simple graphs, Internat. J. Math. Combin., 1 (2011) 33-48.
- [1233] P. Jeyanthi, P. Selvagopal, and S. Soma Sundaram, Some  $C_3$ -supermagic graphs, Util. Math., 89 (2012) 357-366.
- [1234] P. Jeyanthi, M. Selvi, and D. Ramya, Skolem difference mean labeling of disconnected graphs, *Proyecciones J. Math.*, 36(2) (2017) 347-361.
- [1235] P. Jeyanthi, M. Selvi, and D. Ramya, Skolem difference mean labeling of union of graphs, Util. Math., 111 (2019) 139-150.
- [1236] P. Jeyanthi, M. Selvi, and D. Ramya, Restricted triangular difference mean graphs, *Proyecciones J. Math.*, **39**(2) (2020) 275-286.
- [1237] P. Jeyanthi, M. Selvi, and D. Ramya, Odd mean labeling of  $T \hat{o} C_n$  and  $T \tilde{o} C_n$ , Util. Math., **107** (2018) 115-130.
- [1238] P. Jeyanthi, M. Selvi, and D. Ramya, Centered triangular difference mean graphs, Util. Math., 112 (2019) 125-138.
- [1239] P. Jeyanthi and A. Sudha, Total edge irregularity strength of disjoint union of wheel graphs, *Elect. Notes Disc. Math.*, 48 (2015) 175-182.

- [1240] P. Jeyanthi and A. Sudha, Total edge irregularity strength of wheel related graphs, J. Graph Labeling, 2(1) (2015) 45-57.
- [1241] P. Jeyanthi and A. Sudha, On the total irregularity strength of wheel related graphs, Utilitas Math., 110 (2019) 131-134.
- [1242] P. Jeyanthi and A. Sudha, Some results on edge irregular total labeling, J. Internat. Math. Virtual Institute, 9 (2019) 73-91.
- [1243] P. Jeyanthi and A. Sudha, Total edge irregularity strength of disjoint union of double wheel graphs, *Proyectiones J. Math.* 35(3) (2016) 251-262.
- [1244] P. Jeyanthi and A. Sudha, Total vertex irregularity strength of corona product of some graphs, J. Algorithms Comput., 48(1) (2016) 127-140.
- [1245] P. Jeyanthi and A. Sudha, Total edge irregularity strength of some families of graphs, Util. Math., 109 (2018) 139-153.
- [1246] P. Jeyanthi and A. Sudha, Total vertex irregularity strength of some graphs, Palestine J. Math., 7(2) (2018) 139-153.
- [1247] P. Jeyanthi and A. Sudha, Total irregularity strength of disjoint union of crossed prism and necklace graphs, *Util. Math.*, to appear.
- [1248] S. Jezný and M. Trenklér, Characterization of magic graphs, Czechoslovak Math. J., 33 (1983) 435-438.
- [1249] T.-S. A. Jiang, The radio number of grid graphs, arXiv:1401.6583v1 [math.CO] 25 Jan 2014.
- [1250] D. J. Jin, S. Z. Liu, S. H. Lee, H. L. Liu, X. G. Lu, and D. Zhang, The joint sum of graceful trees, *Comput. Math. Appl.*, 26 (1993) 83-87.
- [1251] D. J. Jin, F. H. Meng, and J. G. Wang, The gracefulness of trees with diameter 4, Acta Sci. Natur. Univ. Jilin., (1993) 17-22.
- [1252] M. I. Jinnah and S. Beena, On  $E_4$ -cordial graphs, Ars Combin., **119** (2015) 413-422.
- [1253] M. I. Jinnah and A. V. Nair, On-cordial labelling of graphs, Adv. Appl. Disc. Math., 14(2) 75-94.
- [1254] M. I. Jinnah and G. S. Singh, A note on arthmetic numberings of graphs, Proc. Symposium on Graphs and Combinatorics, Kochi, Kerala, India (1991) 83-87.
- [1255] M. I. Jinnah and K. R. Santhosh Kumar, Irregularity strength of triangular snake and double triangular snake, *Adv. Appl. Discrete Math.* **9** (2012)(2) 83-92.

- [1256] Jirimutu, On k-gracefulness of r-crown  $I_r(K_{1,n})$   $(n \ge 2, r \ge 2)$  for complete bipartite graph, J. Inner Mongolia Univ. Nationalities, **2** (2003) 108-110.
- [1257] Jirimuta, Y-L Bao, and F-L Kong, On k-gracefulness of r-crown for complete bipartite graphs, International J. Pure and Applied Math., **15** (2004) 81-86.
- [1258] Jirimuta, J. Wang, and X. Xirong, on the gracefulness of the digraphs  $n \cdot \overrightarrow{C_m}$ , Internat.J. Pure and Appl. Math., **23** (2005) 393-400.
- [1259] Jirimuta and J. Wang, On (a, d)-antimagic labelings of generalized Petersen graphs P(n, 2), Ars Combin., **90** (2009) 161-174.
- [1260] R. P. Jones, Gracelessness, Proc. 10th S-E Conf. Combinatorics, Graph Theory, and Computing, Congr. Numer., XXIII-XXIV, Util. Math., (1979) 547-552.
- [1261] J. Joshi and D. Parmar,  $H_k$ -Cordial labeling of some snake graph, J. Xidian Univer. 14(3) (2020) 844-858.
- [1262] J. R. Joshi and D. Parmar, H<sub>k</sub> Cordial labeling of m-polygonal snake graphs, Alochang Chakra J., IX(IV) (2020) 1924-1938. https: //www.researchgate.net/publication/340923339\_HK-\_Cordial\_labeling\_ of\_m\_-11990-1-12234-129054Polygonal\_Snake\_Graphs/references# fullTextFileContent
- [1263] Jirimutu, X. Xu, W. Feng, and Y. Bao, Proof of a conjecture on the gracefulness of a digraph, Util. Math., 81 (2010) 255-264.
- [1264] G. Jothi, Highly vertex prime graphs, personal communication.
- [1265] J. S-T. Juan and D-F. Liu, Antipodal labelings of cycles, Ars Combin., 103 (2012) 81-96.
- [1266] D. Jungreis and M. Reid, Labeling grids, Ars Combin., 34 (1992) 167-182.
- [1267] S. Kalaimathy, Some results on (k, d)-even mean labeling, Inter. J. Math. Soft Comput.,  $\mathbf{2}(2)$  (2012) 103-112.
- [1268] P. Jeyanthi, R. Kalaiyarasi and D. Ramya, Centered triangular mean graphs, Inter. J. Math. Comb., 1 (2019) 126-133. https://www.researchgate.net/ publication/332230070\_Centered\_Triangular\_Mean\_Graphs
- [1269] R. Kalaiyarasi, D. Ramya, and P. Jeyanthi, Skolem odd difference mean labeling of tress, *Global J. Pure and Appl. Math.*, **11**(2) (2015) 887-898.
- [1270] M. Kalkowski, M. Karoński, and F. Pfender, A new upper bound for the irregularity strength of graphs, SIAM J. Discrete Math., 25 (2011) 1319-1321.

- [1271] N. Kamatchi, G. R. Vijayakumar, A. Ramalakshmi, S. Nilavarasi, and S. Arumugam, Distance antimagic labelings of graphs, *Theoretical Comput. Sci. Discr. Math.*, 113-118, Lecture Notes in Comput. Sci., 10398, Springer, Cham, 2017.
- [1272] K. K. Kanani and M. I. Bosmia, On cube divisor cordial graphs, Internat. J. Math. Comput. Appl. Res. (IJMCAR), 5(4) (2015) 117-128.
- [1273] K. K. Kanani and T. M. Chhaya, Strongly multiplicative labeling of some path related graphs, Internat. J. Math. Comput. Appl. Res. (IJMCAR), 5(5) (2015) 1-6.
- [1274] K. K. Kanani and T. M. Chhaya, Some new families of strongly multiplicative graphs, 9th National Level Science Symposium. February 14, 2016, Organized by Christ College, Rajkot, Sponsored by GUJCOST, Gandhinagar, Mathematics & Statistics, 3 (2016) 197-200
- [1275] K. K. Kanani and M. V. Modha, 7-cordial labeling of standard graphs, Internat. J. Appl. Math. Res., 3(4) (2014) 547-560.
- [1276] K. K. Kanani and N. B. Rathod, Some new 4-cordial graphs, J. Math. Comput. Sci., 4(5) (2014) 834-848.
- [1277] K. K. Kanani and T. M. Chhaya, Strongly multiplicative labeling of some standard graphs, Internat. J. Math. Soft Comput, 7(1) (2017) 13-21.
- [1278] K. K. Kanani and T. M. Chhaya, K. K. Kanani and T. M. Chhaya, Strongly multiplicative labeling of some snake related graphs, *Internat. J. Math. Trends* and Tech. (IJMTT), 45(1) (2017) 53-56.
- [1279] V. J. Kaneria, A. M. Gohil, and H. M. Makadia, Graceful related labeling and its applications, Int. J. Math. Res., 7(1) (2015) 47-54.
- [1280] V. J. Kaneria, M. M. Jariya, and H. Karavadiya, Cordially of the complete graphs, J. Graph Labeling, 2(2) (2016) 89-101.
- [1281] V. J. Kaneria and M. M. Jariya, Semi smooth graceful graph and construction of new graceful trees, *Elixir Appl. Math.*, 76 (2014) 28,536-28,538.
- [1282] V. J. Kaneria and M. M. Jariya, Smooth graceful graphs and its applications to construct graceful graphs, Int. J. Sci. and Res., (IJSR) 3(8) (2014) 909-912.
- [1283] V. J. Kaneria, M. M. Jariya, and H. M. Makadia, Graceful labeling of arrow graphs and double arrow graphs, *Malaya J. Matematik*, 3(4)(2015) 382-386. https: //www.malayajournal.org/articles/paper2\_2015.pdf
- [1284] V. J. Kaneria, M. M. Jariya, and M. Meghpara, Graceful labeling for some star related graphs, Int. Math. Forum, 9(26) (2014) 1289-1293. http://citeseerx. ist.psu.edu/viewdoc/download?doi=10.1.1.685.6438&rep=rep1&type=pdf

- [1285] V. J. Kaneria, M J Khoda, and H M Karavadiya, Balanced mean cordial labeling and graph operations, Int. J. Math. Appl., 4(3A) (2016) 181-184. http://www. ijmaa.in/v4n3-a/181-184.pdf
- [1286] V. J. Kaneria, M. Meghpara and M. Khoda, Semi smooth graceful labeling and its application to produce  $\alpha$ -labeling, J. Graph Labeling, 2(2) (2016) 153-160.
- [1287] V. J. Kaneria and H. M. Makadia, Some graceful graphs, J. of Math. Res., 4 (1) (2012) 54-57.
- [1288] V. J. Kaneria and H. M. Makadia, Graceful labeling for tensor product of two path of odd lengths and some grid related graphs, Int. J. Innovation Sci. Math., 2(5) (2014) 470-472.
- [1289] V. J. Kaneria and H. M. Makadia, Graceful labeling for step grid graph, J. Advances Math., 9(5) (2014) 2647-2654.
- [1290] V. J. Kaneria and H. M. Makadia, Some results on graceful labeling for step grid related graphs, AKCE Int. J. Math. Trends Tech. (IJMTT) 65 (11) (2-19) 29-38. http://www.ijmttjournal.org/Volume-65/Issue-11/IJMTT-V65I11P503.pdf
- [1291] V. J. Kaneria and H. M. Makadia, Graceful labeling for double step grid graph, Int. J. Math. Appl., 3(1) (2015) 33-38.
- [1292] V. J. Kaneria and H. M. Makadia, Graceful labeling for plus graph, Int. J. Current Rec. Sci. Tech., 1(3) (2015) 15-20.
- [1293] V. J. Kaneria and H. M. Makadia, Some results on graceful labeling for families of plus graph, Int. J. Current Rec. Sci. Tech., 1(4) (2015) 17-23.
- [1294] V. J. Kaneria and H. M. Makadia, Graceful labeling for swastik graph, Int. J. Math. Appl., 3(3-D) (2015) 25-29 http://ijmaa.in/v3n3-d/25-29.pdf
- [1295] V. J. Kaneria and H. M. Makadia, Some results on graceful labeling for families of swastik graphs, Advances and Appl. Discrete Math., 16(2) (2015) 161-172.
- [1296] V. J. Kaneria, H. M. Makadia, and M. M. Jariya, Graceful labeling for cycle of graphs, Int. J. Math. Res., 6 (2) (2014) 173-178.
- [1297] V. J. Kaneria, H. M. Makadia, M. M. Jariya, and M. Meghpara, Graceful labeling for complete bipartite graphs, *Appl. Math. Sci.*, 8 (103) (2014) 5099-5104.
- [1298] V. J. Kaneria, H. M. Makadia, and M. Meghpara, Some graceful graphs, Int. J. Math. Soft Comp., 4(2) (2014) 165-172.
- [1299] V. J. Kaneria, H. M. Makadia, and M. Meghpara, Gracefulness of cycle of cycles and cycle of complete bipartite graphs, *Int. J. Math. Trend Tech.*, **12**(1) (2014) 19-26.

- [1300] V. J. Kaneria, H. M. Makadia, and M. Meghpara, Cordiality of a star of the complete graph and a cycle graph  $C(n \cdot K_n)$ , J. Math. Res., **6**(4) (2014) 18-28.
- [1301] V. J. Kaneria, H. M. Makadia, and M. Meghpara, Graceful labeling for grid related graphs, Int. J. Math. Soft Comput., 5(1) (2015), 111-117. https://citeseerx. ist.psu.edu/viewdoc/download?doi=10.1.1.884.8585&rep=rep1&type=pdf
- [1302] V. J. Kaneria, H. M. Makadia and R. V. Viradia, Graceful labeling for disconnected grid related graphs, Bull. Math. Sci. Appl., 4(1) (2015) 6-11.
- [1303] V. J. Kaneria, H. M. Makadia and R. V. Viradia, Some results on graceful labeling for double step grid related graphs, Int. J. Math. Sci. Appl., 9(1) (2015) 117-127.
- [1304] V. J. Kaneria, H. M. Makadia and R. V. Viradia, Various graph operation on semi smooth graceful graphs, Int. J. Math. and Soft Computing, 6(1) (2016 57-79. https://www.researchgate.net/publication/321164406\_Various\_Graph\_ Operations\_on\_Semi\_Smooth\_Graceful\_Graphs
- [1305] V. J. Kaneria, M. Meera, and K. Maulik, Geometric mean 3-equitable labeling of some graphs, *Internat. J. Sci. Res. Reviews*, IJSRR (2018) 7(1) Suppl. 245-250.
- [1306] V. J. Kaneria and M. Meghpara, Mean labeling for some cycle of graphs, Int. J. Math. Sci. Eng. Appl., 9(2) (2015) 267-274.
- [1307] V. J. Kaneria and M. Meghpara, Semi smooth graceful labeling on some graphs, Int. J. Math. Appl., 3 (3C) (2015) 1-5.
- [1308] V. J. Kaneria and M. Meghpara, Graceful labeling for one point union for path of graphs, Int. J. Math. Appl., 3(1) (2015) 49-55.
- [1309] V. J. Kaneria, M. Meghpara, and H. M. Makadia, Mean labeling for step grid graph, Advan. Appl. Math. Sci., 14(1) (2015) 111-114. https://www.researchgate.net/publication/318636734\_MEAN\_LABELING\_ FOR\_STEP\_GRID\_GRAPH
- [1310] V. J. Kaneria, M. Meghpara, and H. M. Makadia, Graceful labeling for one point union of path and barycentric subdivision of a grid graph, *Int. J. Math. Comp. Res.*, 2(9) (2014) 624-629.
- [1311] V. J. Kaneria, M. Meghpara, and H. M. Makadia, Graceful labeling for open star of graphs, Int. J. Math. Stat. Invention, (IJMSI), 2(9) (2014) 19-23.
- [1312] V. J. Kaneria, M. Meghpara, and H. M. Makadia, Cordial labeling for cycle of complete bipartite graphs and cycle of wheels, *Int. J. Pure Appl. Math.*, (IJPAM), 101(1) (2015) 1-8. https://www.researchgate.net/publication/275240630\_ Cordial\_labeling\_for\_cycle\_of\_complete\_bipartite\_graphs\_and\_cycle\_ of\_wheels

- [1313] V. J. Kaneria, K. M. Patadiya, and J. R. Teraiya, Balanced cordial labeling and its application to produce new cordial families, *Int. J. Math. Appl.*, 4 (1-C) (2016) 65-68.
- [1314] V. J. Kaneria, O. Teraiya and M. Meghpara, Double path union of  $\alpha$ -graceful graph and its  $\alpha$ -labeling, J. of Graph Labeling, 2(2) (2016) 107-114.
- [1315] V. J. Kaneria and J. R. Teraiya, Complete star of a graph and its balanced cordial labeling, Internat. J. Math. Soft Comput., 7(1) (2017) 89-94.
- [1316] V. J. Kaneria, J. R. Teraiya, and K. M. Patadiya, Some result on balanced cordial graphs, Int. J. Math. Appl., 4 (2-A), (2016) 85-87.
- [1317] V. J. Kaneria and S. K. Vaidya, Index of cordiality for complete graphs and cycle, Inter. J. Applied Math. and Computation, 2(4) (2010) 38-46.
- [1318] V. J. Kaneria, S. K. Vaidya, G. V. Ghodasara, and S. Srivastav, Some classes of disconnected graceful graphs, Proc. First Internat. Conf. Emerging Technologies and Appl. Engin. Tech. Sci., (2008) 1050-1056.
- [1319] V. J. Kaneria, R. V. Viradia, M. M. Jariya, and H. M. Makadia, Various labeling for the graph  $C(t \cdot P_n)$ , Int. J. Math. Comp. Res., 2(11) (2014) 745-751.
- [1320] V. J. Kaneria, R. V. Viradia and H. M. Makadia, Mean labeling for path union and cycle of graphs, *AKCE Int. J. Graphs Comb.*, to appear.
- [1321] A. Kanetkar, Prime labeling of grids, AKCE J. Graphs Combin., 6 (2009) 135-142.
- [1322] A. Kanetkar, S. S. Sane, Graceful labeling of a family of quasistars with paths in arithmetic progression, Ars Combin., 83 (2007) 307-320.
- [1323] S. M. Kang, S. Nazeer, W. Nazeer, and B.-Y. Lee, Radio number of caterpillar graphs, *Wulfenia*, 22(5) (2015) 48-58.
- [1324] Q. D. Kang, The k-gracefulness of the product graphs  $P_m \times C_{4n}$ , J. Math. Res. Exposition, **9** (1989) 623-627.
- [1325] Q. Kang, S. M. Lee, and L. Wang, On the edge-graceful spectra of the wheel graphs, unpublished.
- [1326] Q. D. Kang, Z.-H. Liang, Y.-Z. Gao, and G.-H. Yang, On the labeling of some graphs, J. Combin. Math. Combin. Comput., 22 (1996) 193-210.
- [1327] Q. D. Kang and X. Zhao, Strongly harmonious labelings of windmill graphs, J. Hebei Normal College, 2 (1992) 1-7.
- [1328] S. Kanwal, A. Azam, and Z. Iftikhar, SEMT labelings and deficiencies of forests with two components (II), *Punjab U. of Math.*, **51**(4)(2019) 1-12.

- [1329] S. Kanwal, Z. Iftikhar, and A. Azam, SEMT labelings and deficiencies of forests with two components, *Punjab Univ. J. Math. (Lahore)* 51(5) (2019) 137-149.
- [1330] S. Kanwal, M. Imtiaz, Z. Iftikhar, R. Ashraf, M. Arshad, R. Irfan, and I. Sumbal, Embedding of supplementary results in strong EMT valuations and strength, *Open Math.*, 17(1) (2019) 527-543.
- [1331] S. Kanwal, S. Javed, and A. Riasat, On the super edge-magicness and the deficiency of some families of acyclic graphs, *Util. Math.*, **100** (2016) 323-356.
- [1332] M. Kannan, R. Vikrama Prasad, and R. Gopi, Super root square mean labeling of disconnected graphs, *Internat. J. Math. Appl.*, 4(1C) (2016) 93-98. https://www.researchgate.net/publication/303814124\_Super\_Root\_ Square\_Mean\_Labeling\_Of\_Disconnected\_Graphs
- [1333] M. Kannan, R. Vikrama Prasad, and R. Gopi, Even vertex odd mean labeling of some graphs, *Global J. Pure and Applied Math.*, 13(3) (2017) 1019-1034.
- [1334] M Kannan, R. Vikrama Prasad, and R. Gopi, Even vertex odd mean labelings of H-graph, Internat. J. Math. Archive, 8(8) (2017) 162-167.
- [1335] M. Kannan, R. Vikrama Prasad, and R. Gopi, A notion of even vertex odd mean labeling graphs, Internat. J. Math. Trends Tech., (IJMTT) 50(3) (2017) 153
- [1336] M. Kannan, R. Vikrama Prasad, and R. Gopi, Some graph operations of even vertex odd mean labeling graphs, *Internat. J. Appl. Eng. Res.*, **12**(18) (2017) 7749-7753.
- [1337] S. Kanwal and I. Kanwal, SEMT valuations of disjoint union of combs, stars and banana trees, *Punjab Univ. J. Math. (Lahore)*, **50**(3) (2018) 131-144.
- [1338] S. Kanwal, A. Riasat, M. Imtiaz, Z. Iftikhar, S. Javed, and S. Ashraf, Bounds of strong EMT strength for certain subdivision of star and bistar, *Open Math.*, 16 (2018) 1313-1325.
- [1339] G. Kaplan, A. Lev, and Y. Roditty, Bertrand's postulate, the prime number theorem and product anti-magic graphs, *Discrete Math.*, **308** (2008) 787-794.
- [1340] G. Kaplan, A. Lev, and Y. Roditty, On zero-sum partitions and anti-magic trees, Discrete Math., 309 (2009) 2010-2014.
- [1341] N. Karst, J. Langowitz, J. Oehrlein, and D. S. Troxell, Radio k-chromatic number of cycles for large k, Discrete Math. Algorithms Appl., 9(3) (2017), 1750031, 20pp.
- [1342] C. Karthikeyan1, S. Arthi, M. Abinaya, R. Swathi, A. Madhumathi, Super Fibonacci graceful labelling of some cycle related graphs, *JSRD -Internat. J. Sci. Res. Develop.*, 5(12) (2018) 2321-0613.

- [1343] K. Karuppasamy and S. Kaleeswari, Total mean labeling graphs, Internat. J. Recent Tech. Engin. (IJRTE), 8(4S4) (2019) 90-92. https://www.ijrte.org/ wp-content/uploads/papers/v8i4s4/D10361284S419.pdf
- [1344] K. Kathiresan, Subdivisions of ladders are graceful, Indian J. Pure Appl. Math., 23 (1992) 21-23.
- [1345] K. Kathiresan, Two classes of graceful graphs, Ars Combin., 55 (2000) 129-132.
- [1346] K. Kathiresan, Graceful labeling of ladders with pendant edges, unpublished.
- [1347] K. Kathiresan, Odd graceful graphs, unpublished.
- [1348] K. Kathiresan and S. Amutha, Arbitrary supersubdivisions of stars are graceful, Indian J. Pure Appl. Math., 35 (2004) 81-84.
- [1349] K. M. Kathiresan and S. Amutha, Fibonacci graceful graphs, Ars Combin., 97 (2010) 41-50.
- [1350] K. Kathiresan and R. Ganesan, A labeling problem on the plane graphs  $P_{a,b}$ , Ars Combin., **73** (2004) 143-151.
- [1351] K. Kathiresan and R. Ganesan, *d*-antimagic labelings of plane graphs  $P_a^b$ , *J. Combin. Math. Combin. Comput.*, **52** (2005) 89-96.
- [1352] K. Kathiresan and S. Gokulakrishnan, On magic labelings of type (1,1,1) for the special classes of plane graphs, Util. Math., 63 (2003) 25-32.
- [1353] K. M. Kathiresan and S. D. Laurence, On super (a, d)-H-antimagic total covering of star related graphs, *Discuss. Math. Graph Theory*, 35(4) (2015) 755-764.
- [1354] K. Kathiresan, S. Muthuvel, and V. Nagasubbu, Consecutive labelings for two classes of plane graphs, Util. Math., 55 (1999) 237-241.
- [1355] Km. Kathiresan and S. Sabarimalai Madha, Star super edge-magic deficiency of graphs, *Contrib. Discrete Math.*, **12**(1) (2017) 143-156.
- [1356] K. Kathiresan and R. Sumathi, Solution to an open problem in gracefulness of arbitrary supersubdivisions of graphs, Util. Math., 84 (2011) 333-338.
- [1357] K. Kayathri and R. Amutha, Edge-graceful labelings of connected graphs, *Electronic Notes Discrete Math.*, 53 (2016) 287-296.
- [1358] M. Kchikech, R. Khennoufa, and O. Togni, Linear and cyclic radio k-labelings of trees, Discuss. Math. Graph Theory, 27 (2007) 105-123.
- [1359] M. Kchikech, R. Khennoufa, and O. Togni, Radio k-labelings for cartesian products of graphs, Discuss. Math. Graph Theory, 28 (2008) 165-178.

- [1360] M. Khalid, S. T. R. Rizvi, and K. Ali, Note on cycle-(super)magic labelings of disconnected graphs, Util. Math., 104 (2017) 315-320. arXiv:1506.06087
- [1361] E. Khairunnisa and K. A. Sugeng, Graceful labelling of corona product of aster flower graph, Adv. Intel. Systems Res., (AISR), 157 (2018) 68-71. Mathematics, Informatics, Science, and Education International Conference (MISEIC 2018). https://download.atlantis-press.com/article/25905010.pdf
- [1362] N. Khan, Cordial labelling of cycles, Annals Pure Appl. Math.,  $\mathbf{1}(2)$  (2012) 117-130.
- [1363] S. Khatun and Sk. Md. Abu Nayeem, Graceful labeling of some zero divisor graphs, Elect. Notes Discr. Math., 63 (2017) 189-196.
- [1364] R. Khennoufa and O. Togni, The radio antipodal and radio number of the hypercube, Ars Combin., 102 (2011) 447-461.
- [1365] J. Keene and A. Simoson, Balanced strands for asymmetric, edge-graceful spiders, Ars Combin., 42 (1996) 49-64.
- [1366] A. Kézdy,  $\rho$ -valuations for some stunted trees, *Discrete Math.*, **306** (2006) 2786-2789.
- [1367] R. Khennoufa and O. Togni, A note on radio antipodal colourings of paths, Math. Bohem., 130 (2005) 277-282.
- [1368] A. Khodkar, S. Nolen, and J. Perconti, Super edge-graceful labelings of complete bipartite graphs, Australas. J. Combin., 46 (2010) 241-261.
- [1369] A. Khodkar, Super edge-graceful labelings of complete tripartite graphs, J. Combin. Math. Combin. Comput., 76 (2011) 137-158.
- [1370] A. Khodkar, The union of vertex disjoint 3-cycles is super edge-graceful, Bull. Inst. Combin. Appl., 69 (2013) 43-46.
- [1371] A. Khodkar, R. Rasi, and S. M. Sheikholeslami, Super edge-gracefulness of complete graphs, Util. Math., 87 (2012) 207-233.
- [1372] A. Khodkar and K. Vinhage, Super edge-graceful labelings of total stars and total cycles, J. Combin. Math. Combin. Comput., 76 (2011) 233-247.
- [1373] B. M. Kim, W. Hwang, and B. C. Song, Radio number for the product of a path and a complete graph, J. Comb. Optim., 30(1) (2015) 139-149.
- [1374] R. Y. Kim, S-M. Lee, and H. K. Ng, On balancedness of some graph constructions, J. Combin. Math. Combin. Comp., 66 (2008) 3-16.
- [1375] R. Y. Kim, S-M. Lee, and H. K. Ng, On balancedness of some families of graphs, preprint.
- [1376] S.-R. Kim and J. Y. Park, On super edge-magic graphs, Ars Combin., 81 (2006) 113-127.
- [1377] Y.-H. Kim, On constructions of new super edge-magic graphs from some old ones by attaching some pendants, *Commun. Korean Math. Soc.*, **32**(1) (2017 225-231.
- [1378] J. S. Kimberley and J. A. MacDougall, Mutation of vertex-magic regular graphs, 24th MCCCC. J. Combin. Math. Combin. Comput., 82 (2012) 157-177.
- [1379] W.W. Kirchherr, On the cordiality of some specific graphs, Ars Combin., 31 (1991) 127-138.
- [1380] W.W. Kirchherr, NEPS operations on cordial graphs, Discrete Math., 115 (1993) 201-209.
- [1381] S. P. Kishore, Graceful Labellings of Certain Disconnected Graphs, Ph. D. Thesis, Indian Institute of Technology, Madras, 1996.
- [1382] S. Klee, H. Lehmann, and A. Park, Prime labeling of families of trees with Gaussian integers, AKCE Internat. J. Graphs Combin., 13 (2016) 165-176.
- [1383] D. Knuth, personal communication, 2020.
- [1384] A. N. Koam and A. Ahmad, Computation of total vertex irregularity strength of theta graphs, *IEESAccess*, 7 (2019) 113826-113831. https://ieeexplore.ieee. org/stamp/stamp.jsp?arnumber=8786105
- [1385] K. M. Koh, M. Miller, W. F. Smyth, and Y. Wang, On optimum summable graphs, AKCE Int. J. Graphs Comb., 3(1) (2006) 45-57.
- [1386] K. M. Koh, L. Y. Phoon, and K. W. Soh, The gracefulness of the join of graphs, *Electron. Notes Discrete Math.*, 48 (2015) 57-64.
- [1387] K. M. Koh, L. Y. Phoon, and K. W. Soh, The gracefulness of the join of graphs (II), AKCE Internat. J. Graphs Combin., 12 (2015) 180-185.
- [1388] K. M. Koh and N. Punnim, On graceful graphs: cycles with 3-consecutive chords, Bull. Malaysian Math. Soc., 5 (1982) 49-63.
- [1389] K. M. Koh, D. G. Rogers, P. Y. Lee, and C. W. Toh, On graceful graphs V: unions of graphs with one vertex in common, *Nanta Math.*, **12** (1979) 133-136.
- [1390] K. M. Koh, D. G. Rogers, and C. K. Lim, On graceful graphs: sum of graphs, Research Report 78, College of Graduate Studies, Nanyang University (1979).
- [1391] K. M. Koh, D. G. Rogers, and T. Tan, On graceful trees, Nanta Math., 10 (1977) 27-31.

- [1392] K. M. Koh, P. Y. Lee, and T. Tan, Fibonacci trees, Southeast Asian Bull. Math., 2(1) (1978) 45-47.
- [1393] K. M. Koh, D. G. Rogers, and T. Tan, A graceful arboretum: a survey of graceful trees, in *Proceed. Franco-Southeast Asian Confer.*, Singapore, May 1979, 2 278-287.
- [1394] K. M. Koh, D. G. Rogers, and T. Tan, Products of graceful trees, Discrete Math., 31 (1980) 279-292.
- [1395] K. M. Koh, D. G. Rogers, and T. Tan Another class of graceful trees, J. Austral. Math. Soc. Ser. A, 31 (1981) 226-235.
- [1396] K. M. Koh, D. G. Rogers, H. K. Teo, and K. Y. Yap, Graceful graphs: some further results and problems, *Congr. Numer.*, **29** (1980) 559-571.
- [1397] K. M. Koh, T. Tan, and D. R. Rogers, Interlaced trees: a class of graceful trees, Combinatorial Mathematics, VI (Proc. Sixth Austral. Conf., Univ. New England, Armidale, 1978), 65-78, Lecture Notes in Math., 748 Springer, Berlin, 1979.
- [1398] K. M. Koh, T. Tan, and D. G. Rogers, Two theorems on graceful trees, *Discrete Math.*, 25 (1979) 141-148.
- [1399] K. M. Koh and K. Y. Yap, Graceful numberings of cycles with a P<sub>3</sub>-chord, Bull. Inst. Math. Acad. Sinica, 12 (1985) 41-48.
- [1400] S. R. Kola and P. Panigrahi, Radio numbers of some caterpillars, The Eighth International Workshop on Graph Labelings (IWOGL 2014) 289-296, *Electron. Notes Discrete Math.*, 48, Elsevier Sci. B. V., Amsterdam, 2015.
- [1401] T. Kojima On  $C_4$ -supermagic labelings of the Cartesian product of paths and graphs, *Discrete Math.*, **313** (2013) 164-173.
- [1402] S. R. Kola and P. Panigrahi, Nearly antipodal chromatic number  $ac'(P_n)$  of the path  $P_n$ , Math. Bohem., **134** (2009) 77-86.
- [1403] M. Konečný, S. Kučera, J. Novotná, J. Pekárek, Š. Šimsa, and M. Töpfer, Minimal sum labeling of graphs, Combinatorial algorithms, 252-263, *Lecture Notes in Comput. Sci.*, 10765, Springer, Cham, 2018.
- [1404] M. Kong, A. C. Lee, and S. M. Lee, On the balance index sets of the homeomorph of regular graphs, *Congr. Numer.*, **204** (2010) 193-203.
- [1405] M. C. Kong, Y.-C. Wang, and S.-M. Lee. On edge-balanced index sets of some complete k-partite graphs. Proceed. Fortieth Southeastern Internat. Confer. Combin., Graph Th. and Comput., 196 (2009) 71-94.
- [1406] M. Koppendrayer, personal communication.

- [1407] V. Kostyuk, D. Narayan, and V. Williams, Minimal rankings and the arank number of a path, *Discrete Math.*, **306** (2006) 1991-1996.
- [1408] W. O. Krawec, Modular multiplicative graphs, Graph Theory Notes N. Y., 64 (2013) 45-48.
- [1409] W. O. Krawec, Modular multiplicative graphs, Ars Combin., **124** (2016) 33-40.
- [1410] H. K. Krishnappa, K. Kopthapalli, and V. C. Venkaiah, Vertex magic total of complete graphs, AKCE J. Graphs Combin., 6 (2009) 143-154.
- [1411] M. C. Kong, S. M. Lee, E. Seah, and A. Tang, A complete characterization of balanced graphs, J. Combin. Math. and Combin. Comput., 66 (2008) 225-236.
- [1412] M. C. Kong, S. M. Lee, and H. S. H. Sun, On magic strength of graph, Ars Combin., 45 (1997) 193-200.
- [1413] A. Kotzig, Decompositions of a complete graph into 4k-gons (in Russian), Matematický Casopis, 15 (1965) 229-233.
- [1414] A. Kotzig, On certain vertex valuations of finite graphs, Util. Math., 4 (1973) 67-73.
- [1415] A. Kotzig,  $\beta$ -valuations of quadratic graphs with isomorphic components, *Util.* Math., 7 (1975) 263-279.
- [1416] A. Kotzig, Decomposition of complete graphs into isomorphic cubes, J. Combin. Theory, Series B, 31 (1981) 292-296.
- [1417] A. Kotzig, Recent results and open problems in graceful graphs, Congr. Numer., 44 (1984) 197-219.
- [1418] A. Kotzig and A. Rosa, Magic valuations of finite graphs, Canad. Math. Bull., 13 (1970) 451-461.
- [1419] A. Kotzig and A. Rosa, Magic valuations of complete graphs, Centre de Recherches Mathematiques, Universite de Montreal, (1972) CRM-175.
- [1420] A. Kotzig and J. Turgeon, β-valuations of regular graphs with complete components, Colloq. Math. Soc. János Bolyai 18, Combinatorics, Keszthély, Hungary, 1976.
- [1421] P. Kovář, Vertex magic total labeling of products of regular VMT graphs and regular supermagic graphs, J. Combin. Math. Combin. Comput., 54 (2005) 21-31.
- [1422] P. Kovář, Magic labelings of regular graphs, AKCE Inter. J. Graphs and Combin., 4 (2007) 261-275.

- [1423] P. Kovář, Unified approach to magic labeling of copies of regular graphs, Proceed. Thirty-Fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing. Congr. Numer., 168 (2004) 197-205.
- [1424] P. Kovář, D. Fronček, and T. Kovářová, A note on 4-regular distance magic graphs Australas. J. Combin., 54 (2012) 127-132.
- [1425] P. Kovář, M. Kravčenko, M. Krbeček, and A. Silber, Supermagic graphs with many different degrees, *Discuss. Math. Graph Theory*, to appear.
- [1426] P. Kovár and A. Silber, Distance magic graphs of high regularity, AKCE Int. J. Graphs Comb., 9(2) (2012) 213-219.
- [1427] J. Kratochvil, M. Miller, and N. H. Minh, Sum graph labels-An upper bound and related problems. In: *Proceed. Twelfth Australasian Workshop Combin. Algor.*, Bandun, Indonesia (2001).
- [1428] J. Kratochvil, M. Miller, and H. Nguyen, Sum graph labels–An upper bound and related problems, *Proc. of AWOCA 2001*, Institut Teknologi Bandung, Indonesia, (2001) 126-131.
- [1429] A. K. Krishnaa, A study of the major graph labelings of trees, *Informatica (Vilnius)* 15 (2004) 515-524.
- [1430] A. K. Krishnaa, On antimagic labelings of some cycle related graphs, J. Discrete Math. Sci. Cryptogr., 15 (2012) (4-5) 225-235.
- [1431] A. K. Krishnaa, Formulas and algorithms of antimagic labelings of some helm related graphs, J. Discrete Math. Sci. Cryptogr., 19(2) (2016) 425-434.
- [1432] A. K. Krishnaa and M. S. Dulawat, Labelling paths, J. Rajasthan Acad. Phy. Sci., 5 (2006) 99-104.
- [1433] A. K. Krishnaa, M. S. Dulawat, and G. S. Rathore, Computational complexity in decision problems, presented in Conf. of Raj. Parishad, Dec. 14-15, 2001, Udaipur, India.
- [1434] V. H. Krisnawati, A. A. G. Ngurah, N. Hidayat, and A. R. Alghofari, On the super edge-magic deficiency of forests, December 2019AIP Conference Proceedings 2192(1):040007 https://aip.scitation.org/doi/10.1063/1.5139133
- [1435] E. Krop, Lobsters with an almost perfect matching are graceful, Bull. Inst. Combin. Appl., 74 (2015) 21-24.
- [1436] E. Krop, S. M. Lee, and C. Raridan, On the number of unlabeled vertices in edge-friendly labelings of graphs, *Discrete Math.*, **312** (2012) 574-577.

- [1437] E. Krop, S. M. Lee, and C. Raridan, On the edge-balanced index sets of product graphs, J. Indones. Math. Soc., (2011) Special edition, 71-78.
- [1438] E. Krop, S. Minion, P. Patel, and C. Raridan, A solution to the edge-balanced index set problem for complete odd bipartite graphs, *Bull. Inst. Combin. Appl.*, 70 (2014) 119-125.
- [1439] E. Krop, F. Mutiso, and C. Raridan, On super edge-graceful trees of diameter four, Congr. Numer., 210 (2011) 151-167
- [1440] E. Krop and K. Sikes, On the edge-balanced index sets of complete bipartite graphs, Congr. Numer., 207 (2011) 23-32.
- [1441] J. W. Krussel, Equitable labelling of complete bipartite graphs, Ars Combin., to appear.
- [1442] Q. Kuang, S. M. Lee, J. Mitchem, and A-G. Wang, On edge-graceful unicyclic graphs, Congr. Numer., 61 (1988) 65-74.
- [1443] S. S. Kumar and G. Marimuthu, *H-E*-super magic decomposition of complete bipartite graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), *Electron. Notes Discrete Math.*, 48 (2015) 297-300.
- [1444] N. Kumari and S. Mehra, On vertex product cordial labeling, Internat. J. Pure Appl. Math., 114(3) (2017) 553-570. https://ijpam.eu/contents/2017-114-3/ 11/index.html
- [1445] D. Kuo, G. Chang, and Y.-H. Kwong, Cordial labeling of  $mK_n$ , Discrete Math., 169 (1997) 121-131.
- [1446] S. Kuppusamy and S. Guruswamy, Graceful and cordial labeling of subdivision of graphs, *Electronic Notes Disc. Math.*, 53 (2016) 123-131.
- [1447] H. Kwong, On balance index sets of rooted trees, Ars Combin., **91** (2009) 373-382.
- [1448] H. Kwong, and S. M. Lee, On the integer-magic spectra of the corona of two graphs, 36th Southeastern International Conference on Combinatorics, Graph Theory, and Computing, *Congr. Numer.* **174** (2005) 207-222.
- [1449] H. Kwong, and S. M. Lee, On friendly index sets of generalized books, J. Combin. Math. Combin. Comput., 66 (2008) 43-58.
- [1450] H. Kwong, and S. M. Lee, On edge-balance index sets of generalized theta graphs, Cong. Numer., 198 (2009) 157-168.
- [1451] H. Kwong, and S. M. Lee, On edge-balance index sets of flower graphs, Bull. Instit. Combin. Appl., 62 (2011) 107-116.

- [1452] H. Kwong, and S. M. Lee, S.-P. Lo, and Y.C. Wang, On uniformly balanced graphs, Australasian J. Combin., 53 (2012) 83-95.
- [1453] H. Kwong, S-M. Lee, and K. K. Ng, On friendly index set of 2-regular graphs, Discrete Math. 308 (2008) 5522-5532.
- [1454] H. Kwong, S. M. Lee, and H. K. Ng, On product-cordial index sets of cylinders, Congr. Numer., 206 (2010) 139-150.
- [1455] H. Kwong, S. M. Lee, H. K. Ng, On product-cordial index sets and friendly index sets of 2-regular graphs and generalized wheels, *Acta Math. Sinica*, (Engl. Ser.) 28 (4) (2012) 661-674.
- [1456] H. Kwong, S.-M. Lee, and H. K. Ng, On product-cordial index sets of Möbius ladders and their bridge join with  $K_4$ , J. Graph Label.,  $\mathbf{2}(1)$  (2016) 25-43.
- [1457] H. Kwong, S-M. Lee, and D. G. Sarvate, On balance index sets of one-point unions of graphs, J. Combin. Math. Combin. Comput., 66 (2008) 113-127.
- [1458] Y.-L. Lai, Review of "Some results on total product cordial labeling of graphs", J. Indian Acad. Math., 28 (2006) 309-320 by M. Sundaram, R. Ponraj, and S. Somasundaram, MR2313075 (2008a:05241).
- [1459] K. Chitra Lakshmi and K. Nagarajan, Geometric mean cordial labeling of graphs, Intnat. J. Math.Soft Comput., 7(1) (2017)75-87.
- [1460] D. R. Lakshmi and S. Vangipuram, A note on the graceful numbering of a class of trees, *Def. Sci. J.*, **35** (1985) 65-70.
- [1461] D.R. Lashmi and S. Vangipuram, An  $\alpha$ -valuation of quadratic graph Q(4, 4k), *Proc. Nat. Acad. Sci. India* Sec. A, **57** (1987) 576-580.
- [1462] R. Laskar and D. Pillone, Theoretical and complexity results for minimal rankings, Recent advances in interdisciplinary mathematics (Portland, ME, 1997). J. Combin. Inform. System Sci., 25 (2000) 17-33.
- [1463] G.-C. Lau, Every graph is local antimagic total, arXiv:1906.10332.
- [1464] G.-C. Lau, S. Alikhani, S. M. Lee, and W. Kocay, On k-edge-magic labelings of maximal outerplanar graphs, AKCE Int. J. Graphs Comb., 12(1) (2015) 40-46.
- [1465] G.-C. Lau, H. H. Chu, N. Suhadak, F. Y. Foo, and H. K. Ng, On SD-prime cordial graphs, *Internat. J. Pure Appl. Math.*, (2016) 1017-1028. https://ijpam. eu/contents/2016-106-4/4/4.pdf
- [1466] G.-C. Lau, Z.-B. Gao, S. M. Lee, and G.-Y. Sun, On friendly index sets of the edge-gluing of complete graph and cycles, AKCE Internat. J. Graphs Combin., 13 (2016) 107-111.

- [1467] G. C. Lau, P. Jeyanthi D. Ramya and R. Kalaiyarasi, On Skolem odd and even difference mean graphs, J. King Saud Univer. Science, 30 (2018) 286-291.
- [1468] G.-C. Lau, S. M. Lee, and W. C. Shiu, On edge-prime cubic graphs with small components, Open J. Discret. Appl. Math., 2(2) (2019) 48-58. https://www.researchgate.net/publication/335502722\_On\_edge-prime\_ cubic\_graphs\_with\_small\_components
- [1469] G.-C. Lau, W. C. Shiu, and H.-K. Ng, Further results on cube divisor cordial labeling, AKCE Internat. J. Graphs and Combin. 13 (2016) 200-209.
- [1470] G.-C. Lau, W. C. Shiu, H.-K. Ng, and P. Jeyanthi, Further results on SD-prime labeling, *JCMMCC* 98 (2016) 151-170.
- [1471] S. D. Laurence and K. M. Kathiresan, On super (a, d)- $P_n$ -antimagic total labeling of stars, AKCE Int. J. Graphs Comb., **12**(1) (2015) 54-58.
- [1472] S. D. Laurence and K. M. Kathiresan, The total edge irregular strength of path union of cycles, Util. Math., 105 (2017) 125-131.
- [1473] Q. Laurejas and A. Pedrano, On 0-edge magic labeling of some graphs, Internat. J. Math. Appl., 5 Issue 2C (2017) 329-337.
- [1474] H.-F. Law, Full friendly index sets of spiders, Ars Combin., 119 (2015) 23-31.
- [1475] A. N-T. Lee and S. M. Lee, On (k, 1)-strongly indexable graphs associated with sequences of positive integers, *Congr. Numer.*, **199** (2009) 205-215.
- [1476] A. N-T. Lee and S. M. Lee, On a construction of (k, d)-strongly indexable graphs, unpublished.
- [1477] A. N-T. Lee, S. M. Lee, and H. K. Ng, On balance index sets of graphs, J. Combin. Math. Combin. Comput., 66 (2008) 135-150.
- [1478] A. C. Lee, S. M. Lee, and H.-H. Su, On the balance index sets of generalized friendship graphs, envelope graphs of cycles and cubic trees, Proceedings of the Fortieth Southeastern International Conference on Combinatorics, Graph Theory and Computing, *Congr. Numer.*, **196** (2009) 3-22.
- [1479] C. Lee, Minimum coprime graph labelings, arXiv:1907.12670, 17 pages
- [1480] H. Y. Lee, H. M. Lee, and G. J. Chang, Cordial labelings of graphs, *Chinese J. Math.*, **20** (1992) 263-273.
- [1481] L. M. Lee, S. M. Lee, and G. Murthy, On edge-graceful labelings of complete graphs-solution of Lo's conjecture, *Congr. Numer.*, 62 (1988) 225-233.
- [1482] M.-J. Lee, On super (a, 1)-edge-antimagic total labeling of grids and crowns, Ars Combin., 104 (2012) 97-105.

- [1483] M.-J. Lee, W.-H. Tsai, and C. Lin, Super (a, 1)-cyclic-antimagic labelings of the grid, Ars Combin., 112 (2013) 3-12.
- [1484] M.-J. Lee, C. Lin, and W.-H. Tsai, On antimagic labeling for power of cycles, Ars Combin., 98 (2011) 161-165.
- [1485] M.-J. Lee, W.-H. Tsai, and C. Lin, On super (a, 1)-edge-antimagic total labelings of subdivision of stars, Util. Math., 88 (2012) 355-365.
- [1486] P-S Lee, On  $\alpha$ -labelings of prism graphs and gear graphs, unpublished.
- [1487] S. M. Lee, k-graceful labelling of Mongolian tents and related graphs, Congr. Numer., 50 (1985) 85-96.
- [1488] S. M. Lee, A conjecture on edge-graceful trees, *Scientia*, **3** (1989) 45-47.
- [1489] S. M. Lee, New directions in the theory of edge-graceful graphs, Proc. 6th Caribbean Conf. Combin. & Computing (1991) 216-231.
- [1490] S. M. Lee, On constructions which yield fully magic graphs, Ars Combin. 107 (2012) 385-409.
- [1491] S. M. Lee, E. Chen, E. Yera, and L. Wang, On super edge-graceful (p, p+1)-graphs, Congr. Numer., **171** (2004) 51-65.
- [1492] S. M. Lee, K-J. Chen, and Y-C. Wang, On the edge-graceful spectra of cycles with one chord and dumbbell graphs, *Congr. Numer.*, **170** (2004) 171-183.
- [1493] S. M. Lee, K. Ho, and H.-H. Su, On balance index sets of k-level wheel graphs, Congr. Numer., 229 (2017) 313-324.
- [1494] S. M. Lee, Y. S. Ho, S. K. Tan, and H. H. Su, Edge-magic indices of stars, Proceedings of the Forty-Third Southeastern International Conference on Combinatorics, Graph Theory and Computing. *Congr. Numer.*, **213** (2012) 15-26.
- [1495] S. M. Lee, M. Kitagaki, J. Young, and W. Kocay, On edge-graceful and edge-magic maximal outerplanar graphs, J. Combin. Math. Combin. Comput., 59 (2006) 119-129.
- [1496] S. M. Lee and M. C. Kong, On super edge-magic n-stars, J. Combin. Math. Combin. Comput. 42 (2002) 87-96.
- [1497] S. M. Lee, K. Y. Lai, Y. S. Wang, and M. K. Kiang, On the graceful permutation graphs conjecture, *Congr. Numer.*, 103 (1994) 193-201.
- [1498] S. M. Lee and A. N-T Lee, On super edge-magic unicyclic graphs, unpublished.

- [1499] S. M. Lee and A. N-T Lee, On super edge-magic graphs with many odd cycles, Proceedings of the Thirty-Fourth Southeastern International Conference on Combinatorics, Graph Theory and Computing. *Congr. Numer.*, **163** (2003) 65-80.
- [1500] S. M. Lee, A. Lee, H. Sun, and I. Wen, On the integer-magic spectra of graphs, J. Combin. Math. Combin. Comput., 42 (2002) 77-86.
- [1501] S. M. Lee and E. Leung, and H. K. Ng, On super vertex-graceful trees, Proceedings of the Thirty-Fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing, *Congr. Numer.*, **167** (2004) 3-26.
- [1502] S. M. Lee, E. Leung, and H. K. Ng, On super vertex-graceful unicylic graphs, *Czechoslovak Math. J.*, 59(134) (2009) 1-22.
- [1503] S. M. Lee, C. Levesque, S-P. B. Lo, and K. Schaffer, On the edge-graceful spectra of the cylinder graphs (I), J. Combin. Math. Combin. Comput., 66 (2008) 195-214.
- [1504] S. M. Lee and A. Liu, A construction of k-graceful graphs from complete bipartite graphs, SEA Bull. Math., 12 (1988) 23-30.
- [1505] S. M. Lee and A. Liu, A construction of cordial graphs from smaller cordial graphs, Ars Combin., 32 (1991) 209-214.
- [1506] S. M. Lee, A. Liu, and S. K. Tan, On balanced graphs, Proceedings of the Twentyfirst Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, MB, 1991), Congr. Numer., 87 (1992) 59-64.
- [1507] S. M. Lee and S-P. Lo, On (1,2)-strongly indexable spiders, J. Combin. Math. Combin. Comput., 72 (2010) 101-113.
- [1508] Sin-Min Lee, R. M. Low, H. K. Ng, and Y-C. Wang, On friendly index sets of kgalaxies, *Elect. J. Graph Theory and Appl.*, 7(1) (2019) 110. DOI:10.5614/ejgta. 2019.7.1.1
- [1509] S. M. Lee, P. N. Ma, L. Valdés, and S.-M Tong, On the edge-graceful grids, Congr. Numer., 154 (2002) 61-77.
- [1510] S. M. Lee, T. Min-Fang, and S. P. B. Lo, On the edge-balance index set of some trees, unpublished.
- [1511] S. M. Lee and K. C. Ng, Every Young tableau graph is d-graceful, Combinatorial Math. Annal., New York Acad. Sci., 555 (1989) 296-302.
- [1512] S. M. Lee, M. Kong, and Y. C. Wang, On edge-balance index sets of some complete k-partite graphs, Congr. Numer., 196 (2009) 71-94.
- [1513] S. M. Lee and H. K. Ng, On friendly index sets of total graphs of trees, Util. Math., 73 (2007) 81-95.

- [1514] S. M. Lee and H. K. Ng, On friendly index sets of bipartite graphs, Ars Combin., 86 (2008) 257-271.
- [1515] S. M. Lee and H. K. Ng, A class of k-graceful bipartite planar graphs, unpublished.
- [1516] S. M. Lee and H. K. Ng, On friendly index sets of cycles with parallel chords, Ars Combin., 97A (2010) 253-267.
- [1517] S. M. Lee and H. K. Ng, On friendly index sets of graphs, unpublished.
- [1518] S.M. Lee and H. K. Ng, On friendly index sets of prisms and Möbius ladders, J. Combin. Math. Combin. Comput., 90 (2014) 59-74.
- [1519] S. M. Lee, H. K. Ng, Y.-S. Ho, and F. Saba, On edge-graceful edge-splitting extensions of paths and spiders, unpublished.
- [1520] S. M. Lee, H. K. Ng, and G.-C. Lau, On friendly index sets of spiders, *Malays. J. Math. Sci.*, 8(1) (2014) 47-68.
- [1521] S. M. Lee, H. K. Ng, and H. Sun, On super vertex-graceful caterpillars, Proceed. Forty-First Southeastern Inter. Conf. Combin., Graph Th. and Compu., Congr. Numer., 204 (2010) 33-44.
- [1522] S. M. Lee, H. K. Ng, and S. M. Tong, On the balance index of the chain-sum graphs of cycles, Util. Math., 77 (2008) 113-123.
- [1523] S. M. Lee, H. K. Ng, S. M. Tong, On friendly index sets of broken wheels with three spokes, J. Combin. Math. Combin. Comput., 74 (2010) 13-31.
- [1524] S. M. Lee, H. K. Ng, and Y. Wen, On the edge-magic indices of (v, v + 1)-graphs, Util. Math., **72** (2007) 97-110.
- [1525] S. M. Lee, Y. C. Pan, and M. C. Tsai, On vertex-graceful (p, p+1)-graphs, Congr. Numer., **172** (2005) 65-78.
- [1526] S. M. Lee, W. M. Pigg, and T. J. Cox, On edge-magic cubic graphs conjecture, Congr. Numer., 105 (1994) 214-222.
- [1527] S. M. Lee, L. Quach, and S. Wang, On Skolem-gracefulness of graphs which are disjoint union of paths and stars, *Congr. Numer.*, **61** (1988) 59-64.
- [1528] S. M. Lee, F. Saba, E. Salehi, and H. Sun, On the  $V_4$ -magic graphs, Congr. Numer., **156** (2002) 59-67.
- [1529] S. M. Lee, F. Saba, and G. C. Sun, Magic strength of the kth power of paths, Congr. Numer., 92 (1993) 177-184.
- [1530] S. M. Lee and E. Salehi, Integer-magic spectra of amalgamations of stars and cycles, Ars Combin., 67 (2003) 199-212.

- [1531] S. M. Lee, E. Salehi and H. Sun, Integer-magic spectra of trees with diameter at most four, J. Combin. Math. Combin. Comput., 50 (2004) 3-15.
- [1532] S. M. Lee, E. Schmeichel, and S.C. Shee, On felicitous graphs, Discrete Math., 93 (1991) 201-209.
- [1533] S. M. Lee and E. Seah, On edge-gracefulness of kth power cycles, Congr. Numer., 71 (1990) 237-242.
- [1534] S. M. Lee and E. Seah, Edge-gracefulness labelings of regular complete K-partite graphs, Congr. Numer., 75 (1990) 41-50.
- [1535] S. M. Lee and E. Seah, On edge-gracefulness of composition of step graphs and null graphs, Graph Theory, Combinatorics, Algorithms, and Applications (San Francisco, 1989), SIAM (1991) 325-330.
- [1536] S. M. Lee and E. Seah, On edge-graceful triangular snakes and sunflower graphs, unpublished.
- [1537] S. M. Lee, E. Seah, and S.-P. Lo, On edge-graceful 2-regular graphs, J. Combin. Math. Combin. Comput., 12 (1992) 109-117.
- [1538] S. M. Lee, E. Seah, and S. K. Tan, On edge-magic graphs, Congr. Numer., 132 (1992) 179-191.
- [1539] S. M. Lee, E. Seah, and P.-C. Wang, On edge-gracefulness of kth power graphs, Bull. Inst. Math. Acad. Sinica, 18 (1990) 1-11.
- [1540] S. M. Lee, and Q. X. Shan, All trees with at most 17 vertices are super edge-magic, 16th MCCCC Conference, Carbondale, University Southern Illinois, Nov. 2002.
- [1541] S. M. Lee and S.C. Shee, On Skolem-graceful graphs, Discrete Math., 93 (1991) 195-200.
- [1542] S. M. Lee, S. L. Song, and L. Valdeés, On Q(a)P(b)-super edge-graceful wheels, unpublished.
- [1543] S. M. Lee and H.-H. Su, On the balance index sets of permutation graphs, unpublished.
- [1544] S. M. Lee, H.-H. Su, and H. Todt, On the edge-balance index sets of broken wheels, Congr. Numer., 221 (2014) 31-42.
- [1545] S. M. Lee, H.-H. Su, and Y.-C. Wang, On the integer-magic spectra of honeycomb graphs, Cong. Numer., 193 (2008) 49-65.
- [1546] S. M. Lee, H.-H. Su, and Y.-C. Wang, On balance index sets of disjoint graphs, unpublished.

- [1547] S. M. Lee, H.-H. Su, and Y.-C. Wang, On k-edge-magic Halin graphs, Proceed. Forty-First Southeastern Inter. Conf. Combin., Graph Th. and Comput., Congr. Numer., 204 (2010) 129-145.
- [1548] S. M. Lee, H.-H. Su, and Y.-C. Wang, On the edge-balance index sets of (p, p+1)-graphs, Int. J. Contemp. Math. Sci., 7 (2012)(29-32) 1429-1447.
- [1549] S. M. Lee, H.-H. Su, and Y.-C. Wang, On k-edge-magic cubic graphs, 24th MC-CCC. J. Combin. Math. Combin. Comput., 82 (2012) 87-103.
- [1550] S. M. Lee, H.-H. Su, and W. Wei, On a family of the super edge-graceful trees, J. Combin. Math. Combin. Comput., 106 (2018) 23-36.
- [1551] S. M. Lee, H. Sun, W. Wei, Y. Wen and P. Yiu, The super edge-gracefulness of two infinite families of trees, *Congr. Numer.*, **190** (2008) 109-128.
- [1552] S. M. Lee, H. Sun, and I Wen, On group-magic graphs, J. Combin. Math. Combin. Computing, 38 (2001) 197-207.
- [1553] S. M. Lee and S. K. Tan, A class of arbitrarily graceful planar bipartite graphs, J. Combin. Math. Combin. Comput., 9 (1991) 119-127.
- [1554] S. M. Lee, M.-F. Tao, and B. Lo, On the edge-balance index sets of some trees, unpublished.
- [1555] S. M. Lee, S. M. Tong, and E. Seah, On the edge-magic and edge-graceful total graphs conjecture, *Congr. Numer.*, 141 (1999) 37-48.
- [1556] S. M. Lee, L. Valdés, and Y. S. Ho, On group-magic trees, double trees and abbreviated double trees, J. Combin. Math. Combin. Computing, 46 (2003) 85-95.
- [1557] S. M. Lee and G. Wang, All pyramids, lotuses and diamonds are k-graceful, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.), 32 (1988) 145-150.
- [1558] S. M. Lee and L. Wang, On k-edge-graceful trees, unpublished.
- [1559] S. M. Lee and P. Wang, On the k-gracefulness of the sequential join of null graphs, Congr. Numer., 71 (1990) 243-254.
- [1560] S. M. Lee, J. Y-C Wang, On super edge-magicness of chain graphs whose blocks are complete, unpublished.
- [1561] S. M. Lee, L. Wang, H. Ng, Y-C Wang, On the edge-graceful spectra of the corona of (p, p + 1)-graphs with  $K_1$ , unpublished.
- [1562] S. M. Lee, L. Wang, K. Nowak, and W. Wei, On the edge-graceful trees conjecture, J. Combin. Math. Combin. Comput., 54 (2005) 83-98.

- [1563] S. M. Lee, L. Wang, and Y. Wen, On the edge-magic cubic graphs and multigraphs, Congr. Numer., 165 (2003) 145-160.
- [1564] S. M. Lee, S. Wang, and I. Wui, On Skolem-gracefulness of 4-stars, Congr. Numer., 62 (1988) 235-239.
- [1565] S. M. Lee and W. Wei, On the super vertex-gracefulness of cartesian products of graphs, Congr. Numer., 180 (2006) 175-192.
- [1566] S. M. Lee, L. Wang, and E. R. Yera, On super edge-graceful Eulerian graphs, Congr. Numer., 174 (2005) 83-96.
- [1567] S. M. Lee, I. Wen, and H. Hugo, On group-magic graphs, J. Combin. Math. Combin. Comput. 38 (2001) 197-207.
- [1568] S. M. Lee and H. Wong, On the integer spectra of the power of paths, J. Combin. Math. Combin. Comput., 42 (2002) 187-194.
- [1569] S. M. Lee and R. Wong, On P(a)Q(1)-super vertex-graceful unicyclic graphs, Congr. Numer., **173** (2005) 79-96.
- [1570] S. M. Lee, Y. S. Wong, and M. K. Kiang, On graceful permutations graphs conjecture, *Congr. Numer.*, **103** (1994) 193-201.
- [1571] S. M. Lee and I. Wui, On Skolem-gracefulness of 2-stars and 3-stars, Bull. Malaysian Math. Soc., 10 (1987) 15-20.
- [1572] S. M. Lee, I. Wui and J. Yeh, On the amalgamation of prime graphs, Bull. Malaysian Math. Soc. (Second Series), 11 (1988) 59-67.
- [1573] C. E. Leiserson, Area efficient graph layouts for VLSI, in: Proc. 21st Ann. IEEE Symposium, FOCS, (1980) 270-281.
- [1574] P. C. Li, Antimagic labelings of cycle powers, Ars Combin., 124 (2016) 341-351.
- [1575] T. Li, Z-X. Song, G. Wang, D. Yang, and C-Q. Zhang, Antimagic orientations of even regular graphs, J. Graph Theory, 90 (2019) 46-53.
- [1576] W. Z. Li, G. H. Li, and Q. T. Yan, Study on some labelings on complete bipartite graphs, Adv. Comput. Sci., Envir., Ecoinforma., and Ed., Comm. Comput. Inf. Sci., 214 (2011) 297-301.
- [1577] X. Li, V. Mak, and S. Zhou, Optimal radio labellings of complete *m*-ary trees. Discrete Appl. Math., **158**(5) (2010) 507-515.
- [1578] H. X. Liang, and C. F. Liu, On k-gracefulness of graphs, Dongbei Shida Xuebao, 33 (1991) 41-44.

- [1579] Y.-C. Liang, T.-L. Wong, and X. Zhu, Anti-magic labeling of trees, Discrete Math., 331 (2014) 9-14.
- [1580] Y.-C. Liang and X. Zhu, Anti-magic labelling of Cartesian product of graphs, *Theoret. Comput. Sci.* 477 (2013) 1-5.
- [1581] Y.-C. Liang and X. Zhu, Antimagic labeling of cubic graphs, J. Graph Th., 75 (2014) 31-36.
- [1582] Z. Liang, The harmoniousness of book graph  $St(4k + 1) \times P_2$ , Southeast Asian Bull. Math., **21** (1997) 181-184.
- [1583] Z. Liang, On the gracefulness of the graph  $C_m \cup P_n$ , Ars Combin., **62** (2002) 273-280.
- [1584] Z. Liang, On the graceful conjecture of permutation graphs of paths, Ars Combin., 91 (2009) 65-82.
- [1585] Z. Liang, Cycle-supermagic decompositions of complete multipartite graphs, Discrete Math., 312(22) (2012) 3342-3348.
- [1586] Z. Liang, On the strongly c-harmoniousness of cycle with some chords, Ars Combin., 109 (2013) 143-160.
- [1587] Z. -H. Liang, G-supermagic coverings of graphs, Acta Math. Appl. Sin., 37(5) (2014) 857-864.
- [1588] Z. Liang and Z.-L. Bai, On the odd harmonious graphs with applications, J. Appl. Math. Comput., (2009) 29 105-116.
- [1589] Z. Liang and Y. Miao, On Lee's conjecture, Util. Math., 87 (2012) 305-329.
- [1590] Z. Liang, D. Q. Sun, and R. J. Xu, k-graceful labelings of the wheel graph  $W_{2k}$ , J. Hebei Normal College, 1 (1993) 33-44.
- [1591] Z. Liang, H. Zhang, N. Xu, S. Ye, Y. Fan, and H. Ge, Gracefulness of five permutation graphs of paths. *Util. Math.*, 72 (2007) 241-249.
- [1592] Z. Liang and H. Zuo, On the gracefulness of the graph  $P_{2m,2n}$ , Appl. Anal. Discrete Math., 4 (2010) 175-180.
- [1593] S.-C. Liaw, D. Kuo, and G. Chang, Integral sum numbers of graphs, Ars Combin., 54 (2000) 259-268.
- [1594] K.-W Lih, On magic and consecutive labelings of plane graphs, Util. Math., 24 (1983) 165-197.
- [1595] N. B. Limaye, k-equitable graphs, k = 2, 3, in Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 117-133.

- [1596] C.-M. Lin and T-M. Wang, On zero magic sums of integer magic graphs, Ars Combin., 118 (2015) 119-134.
- [1597] Y. Lin, A. Ahmad, M. Miller, K. Sugeng, and M. Bača, Further results on dantimagic labelings of antiprisms, *Proceed. 15-th Australasian Workshop Combin. Alg.*, (2004) 103-108, Ballina, Australia.
- [1598] Y. Lin and M. Miller, Vertex-magic total labelings of complete graphs, Bull. Inst. Combin. Appl., 33 (2001) 68-76.
- [1599] Y. Lin, M. Miller, R. Simanjuntak, and Slamin, Magic and antimagic labelings of wheels, unpublished.
- [1600] Y. Lin, Slamin, M. Bača, and M. Miller, On *d*-amtimagic labelings of prisms, Ars Combin., 72 (2004) 65-76.
- [1601] Y. Lin and K. Sugeng, Face antimagic labelings of plane graphs  $p_a^b$ , Ars Combin., 80 (2006) 259-273.
- [1602] Z. Lingqi, Z. Linna, R. Yuan, The generalized Petersen graph P(n,3) is (5n + 5/2, 2)-antimagic, Util. Math., **96** (2015) 17-25.
- [1603] A. Litersky, Graceful kayak paddles, M.S. Thesis, University of Minnesota Duluth, 2011.
- [1604] B. Liu, Some new results on graceful graphs, Math. Appl., 3 (1990) 108-110.
- [1605] B. Liu, Sums of squares and labels of graphs, Math. Practice Theory, (1994) 25-29.
- [1606] B. Liu and X. Zhang, On a conjecture of harmonious graphs, Systems Science and Math. Sciences, 4 (1989) 325-328.
- [1607] B. Liu and X. Zhang, On harmonious labelings of graphs, Ars Combin., 36 (1993) 315-326.
- [1608] D. Liu, Radio number for trees, *Discrete Math.*, **308** (2008) 1153-1164.
- [1609] D. Liu, Multi-level distance labelings for trees, unpublished.
- [1610] D. Liu and M. Xie, Radio number for square of cycles, Congr. Numer., 169 (2004) 105-125.
- [1611] D. Liu and M. Xie, Radio number for square of paths, Ars Combin., 90 (2009) 307-319.
- [1612] D. Liu and X. Zhu, Multi-level distance labelings for paths and cycles, SIAM J. Discrete Math., 19 (2005) 610-621.

- [1613] J. B. Liu, T. Zou, Y. Lu, Gracefulness of Cartesian product graphs, *Pure Appl. Math. (Xi'an)*, **28** (2012)(3) 329-332, 362.
- [1614] J. B. Liu, L. Wang, and Y. Lu, Odd graceful labeling and algorithm of bicyclic graphs with a common edge, J. Hefei Univ. Technol. Nat. Sci., 35(6) (2012) 857-859.
- [1615] R. Y. Liu, On Bodendiek's conjecture for graceful graphs, Chinese Quart. J. Math., 4 (1989) 67-73.
- [1616] Y. Liu, The gracefulness of the star graph with top sides, J. Sichuan Normal Univ., 18 (1995) 52-60.
- [1617] Y. Liu, Proof of a conjecture concerning gracefulness of graphs, Huaihua Shizhuan Xuebao, 15 (1996) 13-25.
- [1618] Y. Liu, Crowns graphs  $Q_{2n}$  are harmonious graphs, Hunan Annals Math., **16** (1996) 125-128.
- [1619] Y. Liu, All crowns and helms are harmonious, unpublished.
- [1620] Z. S. Liu, A class of graceful graphs and two classes of graceful graphs derived from it, Neimenggu Daxue Xuebao, 16 (1985) 183-186.
- [1621] Q. Liu and Z. Liu, A necessary and sufficient condition for a graph to be  $E_2$ -cordial, Ars Combin., **123** (2015) 159-167.
- [1622] Q. Liu, Z. Liu, and H. Wu, On the  $E_k$ -cordiality of some graphs, Ars Combin., 120 (2015) 193-198.
- [1623] Z. Liu and B. Zhu, A necessary and sufficient condition for a 3-regular graph to be cordial, Ars Combin., 84 (2007) 225-230.
- [1624] A. Lladó and J. Moragas, Cycle-magic graphs, Discrete Math., 307 (2007) 2925-2933.
- [1625] M.-L. Lo and L. V. Alegria, Radio number for fourth power paths, *Involve* 9(2) (2016) 317-332.
- [1626] S. Lo, On edge-graceful labelings of graphs, Congr. Numer., 50 (1985) 231-241.
- [1627] S. Locke and A. Niedziałomski,  $K_n \Box P$  is radio graceful, *Matematiche (Catania)* 73 (2018) (1) 127-137.
- [1628] S. C. López and F. A. Muntaner-Batle, A new application of the  $\otimes_h$ -product to  $\alpha$ -labelings, *Discrete Math.*, **338** (2015) 839-843.
- [1629] S. C. López and F. A. Muntaner-Batle, Langford sequences and a product of digraphs, *European J. Combin.*, 53 (2016) 86-95.

- [1630] S. C. López, F. A. Muntaner-Batle, and M. Prabu, A new labeling construction from the  $\otimes_h$ -product. *Discrete Math.*, **340**(8) (2017) 1903-1909.
- [1631] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, Super edge-magic models, Math. in Comput. Sci., 5 (2011) 63-68.
- [1632] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, Enumerating super edgemagic labelings for some types of pathlike trees, *Utilitas Math.* to appear. https: //core.ac.uk/download/pdf/41759896.pdf
- [1633] S. C. López, F. A. Muntaner-Batle, M. Rius-Font, Perfect super edge-magic graphs, Bull. Math. Soc. Sci. Math. Roumanie, 55 (103)(2) (2012) 199-208.
- [1634] S. C. López, F. A. Muntaner-Batle, M. Rius-Font, Perfect edge-magic labelings, Bull. Math. Soc. Sci. Math. Roumanie, 57 (105)(1) (2014) 81-91.
- [1635] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, Bi-magic and other generalizations of super edge-magic labeling, Bull. Aust. Math. Soc., 84 (2011) 137-152.
- [1636] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font Bi-magic and other generalizations of super edge-magic labelings, [corrigendum to S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, Bi-magic and other generalizations of super edge-magic labeling, *Bull. Aust. Math. Soc.*, 84 (2011) 137-152.]
- [1637] S. C. López, F. A. Muntaner-Batle and M. Rius-Font, New problems related to the valences of (super) edge-magic labelings, AKCE Int. J. Graphs Combin., 10(2) (2013) 169-181.
- [1638] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, On super edge-magic decomposable graphs, *Indian J. Pure Appl. Math.* 43(5) (2012) 455-473.
- [1639] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, Open problems involving super edge-magic labelings and related topics, *Bull. Inst. Combin. Appl.*, 65 (2012) 43-56.
- [1640] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, A problem on edge-magic labelings of cycles, *Canadian Math. Bull.*, 57 (2014) 375-380. doi:10.4153/CMB-2013-036-1
- [1641] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, Labeling constructions using digraph products, *Discrete Appl. Math.*, 161 (2013) 3005-3016.
- [1642] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, The jumping knight and other (super) edge-magic constructions, *Mediterr. J. Math.*, **11**(2) (2014) 217-235.
- [1643] S. C. López, F. A. Muntaner-Batle, and M. Rius-Font, Magic coverings and the Kronecker product, Util. Math., 95 (2014) 73-84.

- [1644] A. Lourdusamy and S. George, Super vertex mean labeling of cycles through different ways, *Proyecciones*, 37(2) (2018) 181-198.
- [1645] A. Lourdusamy and S. George, Super vertex mean graphs of order  $\leq 7$ , J. Appl. Math. Inform., **35**(5-6) (2017) 565-586.
- [1646] A. Lourdusamy and S. George, Linear cyclic snakes as super vertex mean graphs, Internat. J. Math. Sci. Comput., 1 (2018) 109-126.
- [1647] A. Lourdusamy, S. George and M. Seenivasan, An extension of mean labeling, preprint.
- [1648] A. Lourdusamy, J. S. Mary and F. Patrick, Even vertex equitable even labeling, Sciencia Acta Xaveriana, 7 (2016) 37-46. https://www.researchgate. net/profile/Lourdusamy\_Arockiam2/publication/303564979\_Even\_ Vertex\_Equitable\_Even\_Labeling/links/574876d908ae2e0dd30162ed/ Even-Vertex-Equitable-Even-Labeling.pdf
- [1649] A. Lourdusamy and F. Patrick, Sum divisor cordial graphs, Proyectiones, 35(1) (2016) 119-136.
- [1650] A. Lourdusamy and F. Patrick, Even vertex equitable even labeling of path and bistar related graphs, *Informa. Sci. Comput.*, **2016** (2016) 1-9. https://www.researchgate.net/publication/313160627\_Even\_Vertex\_ Equitable\_Even\_Labeling\_of\_Path\_and\_Bistar\_Related\_Graphs
- [1651] A. Lourdusamy and F. Patrick, Vertex equitable labeling for ladder and snake related graphs, J. Prime Research Math., 13 (2017) 1-7.
- [1652] A. Lourdusamy and F. Patrick, New construction on SD-prime cordial labeling, *Theoretical Computer Science and Discrete Mathematics*, 134-143, Lecture Notes in Comput. Sci., 10398, Springer, Cham, 2017.
- [1653] A. Lourdusamy and F. Patrick, Some results on SD-prime cordial labeling, Proyecciones, 36(4) (2017) 601-614.
- [1654] A. Lourdusamy and F. Patrick, Even vertex equitable even labeling for cycle related graphs, Kragujevac J. Math. 43(3) (2019) 427-441. https://imi.pmf.kg. ac.rs/kjm/pdf/accepted-finished/db52db9c7ff1618dd35125495d743075\_ 2536\_07152018\_043122/kjm\_43\_3-7.pdf
- [1655] A. Lourdusamy, J. Shobana Mary, and F. Patrick, Even vertex equitable even labeling for path related graphs, *Sciencia Acta Xaveriana*, 7(1) (2018) 47-56.
- [1656] A. Lourdusamy and M. Seenivasan, Vertex equitable labeling of graphs, J. Discrete Math. Sci, Crypt., 11 (2008) 727-735.

- [1657] A. Lourdusamy and M. Seenivasan, Mean labeling of cyclic snakes, AKCE Int. J. Graphs Combin., 8 (2011) 105-113.
- [1658] A. Lourdusamy and M. Seenivasan, Mean labeling of edge linked cyclic snakes, AKCE Int. J. Graphs Combin., 10(4) (2013) 391-403.
- [1659] A. Lourdusamy and M. Seenivasan, Vertex-mean graphs, Internat. J. Math. Combin., 3 (2011) 114-120.
- [1660] A. Lourdusamy, M. Seenivasan, S. George and R. Revathy, Super vertex-mean graphs, *Sciencia Acta Xaver.*, 5(2) (2014) 39-46.
- [1661] A. Lourdusamy, S. J. Wency, and F. Patrick, On SD-prime cordial labeling, Internat. J. Pure Appl. Math., (2017) 221-228.
- [1662] S. J. Lourdusamy, S. J. Wency, and F. Patrick, Even vertex equitable even labeling for snake related graphs, *Proyecciones*, 38(1) (2019) 177-189. https://scielo.conicyt.cl/scielo.php?script=sci\_arttext&pid= S0716-09172019000100177
- [1663] R. M. Low and S. M. Lee, On group-magic Eulerian graphs, J. Combin. Math. Compin. Comput., 50 (2004) 141-148.
- [1664] R. M. Low and S. M. Lee, On integer-magic Eulerian graphs, personal communication.
- [1665] R. M. Low and W. C. Shiu, On the integer-magic spectra of graphs, Congr. Numer., 191 (2008) 193-203.
- [1666] R. M. Low and L. Sue, Some new results on the integer-magic spectra of tessellation graphs, Australas. J. Combin., 38 (2007) 255-266.
- [1667] A. Lozano, M. Mora, C. Seara, and J. Tey, Caterpillars are antimagic, arXiv:1812.06715v1 [math.CO] pp 9.
- [1668] A. Lozano, M. Mora, and C. Seara, Antimagic labeling of caterpillas, Appl. Math. Comput., 347 (2017) 734-740.
- [1669] A. Lozano, M. Mora, C. Seara, and J. Tey, Trees whose even-degree vertices induce a path are antimagic, arXiv: 1905.06595 [math.CO] May 16 2019 7pp
- [1670] H.-C. Lu, On the constructions of sequential graphs, Taiwanese J. Math., 10 (2006) 1095-1107.
- [1671] H.-C. Lu, On large harmonious graph, Ars Combin., 91 (2009) 447-458.
- [1672] X. Lu and X. F. Li,  $P_1 \bigvee T_m$  graphs and a certification of its gracefulness, Gongcheng Shuxue Xuebao, **13** (1996) 109-113.

- [1673] X. Lu, W. Pan, and X. Li, k-gracefulness and arithmetic of graph  $St(m) \cup K_{p,q}$ , J. Jilin Univ., **42** (2004) 333-336.
- [1674] K. J. Ma and C. J. Feng, About the Bodendiek's conjecture of graceful graph, J. Math. Research and Exposition, 4 (1984) 15-18.
- [1675] K. J. Ma and C. J. Feng, On the gracefulness of gear graphs, Math. Practice Theory, (1984) 72-73.
- [1676] X. D. Ma, Some classes of graceful graphs, J. Xinjiang Univ. Nat. Sci., 3 (1986) 106-107.
- [1677] X. Ma, A graceful numbering of a class of graphs, J. Math. Res. and Exposition, (1988) 215-216.
- [1678] X. Ma, Y. Liu, and W. Liu, Graceful graphs: cycles with (t-1) chords, *Math.* Appl., **9** (1990), suppl., 6-8.
- [1679] J. A. MacDougall, M. Miller, Slamin, and W. D. Wallis, Vertex-magic total labelings of graphs, Util. Math., 61 (2002) 3-21.
- [1680] J. A. MacDougall, M. Miller, and K. Sugeng, Super vertex-magic total labelings of graphs, *Proceedings Australasian Workshop Combin. Algorithm 2004*, Balina, NSW (2004) 222-229.
- [1681] J. A. MacDougall, M. Miller, and W. D. Wallis, Vertex-magic total labelings of wheels and related graphs, Util. Math., 62 (2002) 175-183.
- [1682] J. A. MacDougall and W. D. Wallis, Strong edge-magic labeling of a cycle with a chord, Australas. J. Combin., 28 (2003) 245-255.
- [1683] J. A. MacDougall and W. D. Wallis, Strong edge-magic graphs of maximum size, Discrete Math., 308 (2008) 2756-2763.
- [1684] A. U. Maheswari and V. Srividya, New families of square sum graphs, J. Comput. Math. Sci., 10(4) (2019) 907-914.
- [1685] A. U. Maheswari and V. Srividya, Some labelings on cycles with parallel P<sub>3</sub> chords, JASC: J. Appl. Sci. Comput., VI (I) (2019) 469-475. https://pdfs.semanticscholar.org/f1f2/bc5bc33c50a47e0c3bea04d6957c6536cf96.pdf? \_ga=2.208011094.537058225.1571009368-2077171013.1571009368
- [1686] H. Mahmoudzadeh and K. Eshghi, Metaheuristic approach to the graceful labeling problem, Int. J. Appl. Metaheuristic Comput., IJAMC 1 (2010) 42-57.
- [1687] M. Maheo, Strongly graceful graphs, Discrete Math., 29 (1980) 39-46.
- [1688] M. Maheo and H. Thuillier, On *d*-graceful graphs, Ars Combin., **13** (1982) 181-192.

- [1689] A. Mahmoody, A note on graceful graphs with large chromatic numbers, Ars Combin., 90 (2009) 423-424.
- [1690] H. Mahmoudzadeh and K. Eshghi, A metaheuristic approach to the graceful labeling problem, Modeling, Analysis, and Applications in Metaheuristic Computing: Advancements and Trends. IGI Global, 2012, (2014) 217-232. doi:10.4018/978-1-4666-0270-0.ch013
- [1691] P. Majerski and J. Przybylo, Total vertex irregularity strength of dense graphs, J. Graph Th., 76(1) (2014) 34-41.
- [1692] P. Majerski and J. Przybyo, On the irregularity strength of dense graphs, SIAM J. Discrete Math., 28(1) (2014) 1, 197-205.
- [1693] H. M. Makadia, H. M. Karavadiya, and V. J. Kaneria, Some results on α-graceful graphs, *Internat. J. of Math. Combin.*, to appear.
- [1694] H. M. Makadia, H. M. Karavadiya, and V. J. Kaneria, Graceful centers of graceful graphs and universal graceful graphs, *Proyecciones J. Math.*, 38 (2019)(2) 305-314.
- [1695] K. Manickam and M. Marudai, Odd mean labelings of graphs, Bull. Pure Appl. Sci., 25E 1 (2006) 149-153.
- [1696] K. Manickam and M. Marudai, Edge magic labeling of graphs Util. Math., 79 (2009) 181-187.
- [1697] K. Manickam, M. Marudai, and R. Kala, Some results on felicitous labeling of graphs, J. Combin. Math. Combin. Comput., 81 (2012) 273-279.
- [1698] J. M. Manulang and K. A. Sugeng, Graceful labeling on torch graph, Indon. J. Combin., 2(1) (2018) 14-19.
- [1699] H. T. Marbun and A. N. M. Salman, Wheel-supermagic labelings for a wheel kmultilevel corona with a cycle, AKCE Int. J. Graphs Comb., 10(2)(2013) 183-191.
- [1700] G. T. Marimuthu and M. Balakrishnan, E-super vertex magic labelings of graphs, Discrete Appl. Math., 160(12) (2012) 1766-1774.
- [1701] G. Marimuthu and M. Balakrishnan, Super edge magic graceful graphs, Information Sciences, 287(10) (2014) 140-151.
- [1702] G. Marimuthu, S. Kavitha, and M. Balakrishnan, Super edge magic graceful labeling of generalized Petersen graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), *Electron. Notes Discrete Math.*, 48 (2015) 235-241.
- [1703] G. Marimuthu and P. Krishnaveni, Super edge-antimagic gracefulness of disconnected graphs, *Theoretical Comput. Sci. Disc. Math.*, 152-155, Lecture Notes in Comput. Sci., 10398, Springer, Cham, 2017.

- [1704] G. T. Marimuthu and G. Kumar, Solution to some open problems on E-super vertex magic labeling of disconnected graphs, Appl. Math. Comput., 268 (2015) 657-663.
- [1705] G. Marimuthu and G. Kumar, V-super and E-super vertex-magic total labelings of graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), *Electron. Notes Discrete Math.*, 48 (2015) 223-230.
- [1706] G. Marimuthu and S. S. Kumar, *H-V*-super magic decomposition of complete bipartite graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), *Electron. Notes Discrete Math.*, 48 (2015) 231-233.
- [1707] G. T. Marimuthu, M. S. Raja Durga, and G. Durga Devi, Solution to some open problems on *E*-super vertex magic total labeling of graphs, *Appl. Appl. Math.*, 10(2) (2015) 1104-1112.
- [1708] G. T. Marimuthu, M. S. Raja Durga, and G. Durga Devi, V-Super vertex inantimagic total labelings of digraphs, J. Graph Labeling, 1(1) (2015) 21-30.
- [1709] R. Marinescu-Ghemeci, Radio number for some thron graphs, Discuss. Math. Graph Theory, 30 (2010) 210-222.
- [1710] A. M. Marr, Labelings of Directed Graphs, Ph.D. thesis, Southern Illinois University at Carbondale, 2007, 63 pages.
- [1711] A. M. Marr, Graceful labelings of directed graphs, J. Combin. Math. Combin. Comput., 66 (2008) 97-103.
- [1712] A. M. Marr, N. C. K. Phillips, and W. D. Wallis, Bimagic labelings, AKCE J. Graphs Combin., 6 (2009) 155-160.
- [1713] A. M. Marr, S. Ochel, and B. Perez, In-magic total labelings of digraphs, J. Graph Labeling, 1(2) (2015) 81-93.
- [1714] P. Martinez, J. Ortiz, M. Tomova, and C. Wyels, Radio numbers for generalized prism graphs, *Discuss. Math. Graph Theory* **31** (2011) 45-62. arXiv:1007.5346v1 [math.CO].
- [1715] T. K. Maryati, E. T. Baskoro, and A. N. M. Salman, P<sub>h</sub>-(super) magic labelings of some trees, J. Combin. Math. Combin. Comput., 65 (2008) 197-204.
- [1716] T. K. Maryati, E. T. Baskoro, A. N. M. Salman, and Irawati, On the path-(super)magicness of a cycle with some pendants, Util. Math., 96 (2015) 319-330.
- [1717] T. K. Maryati, A. N. M. Salman, and E. T. Baskoro, Supermagic coverings of the disjoint union of graphs and amalgamations, *Discrete Math.*, **313**(4) (2013) 397-405.

- [1718] T. K. Maryati, A. N. M. Salman, and E. T. Baskoro, and Irawati, The supermagicness of a disjoint union of isomorphic connected graphs, *Proceed. 4th IMT-GT Internat. Conf. Math. Stat. Appl.*, 3 (2008) 1-5.
- [1719] T. Maryati, A. N. M. Salman, E. T. Baskoro, J. Ryan, M. Miller, On H-supermagic labelings for certain shackles and amalgamations of a connected graph, Util. Math., 83 (2010) 333-342.
- [1720] M. Matamala and J. Zamora, Graphs admitting antimagic labeling for arbitrary sets of positive integers, LAGOS'17IX Latin and American Algorithms, Graphs and Optimization, *Electron. Notes Discrete Math.*, **62** (2017) 159-164.
- [1721] M. Matamala and J. Zamora, Weighted antimagic labeling, Discrete Appl. Math., 245 (2018) 194-201.
- [1722] M. Mavronicolas and L. Michael, A substitution theorem for graceful trees and its applications *Discrete Math.*, **309** (2009) 3757-3766.
- [1723] C. C. Marzuki, A. N. M. Salman, and M. Miller, On the total irregularity strength of cycles and paths, *Far East J. Math. Sci.*, (to appear).
- [1724] O. Mbianda, Properly Even Harmonious Graphs, M. S. Thesis, University of Minnesota Duluth, 2016.
- [1725] D. McQuillan, A technique for constructing magic labelings of 2-regular graphs, J. Combin. Math. Combin. Comput., 75 (2010) 129-135.
- [1726] D. McQuillan, Edge-magic and vertex-magic total labelings of certain cycles, Ars Combin., 90 (2009) 257-266.
- [1727] D. McQuillan and J. McQuillan, Magic labelings of triangles, Discrete Math., 309 (2009) 2755-2762.
- [1728] D. McQuillan and J. McQuillan, Strong vertex-magic and edge-magic labelings of 2-regular graphs of odd order using Kotzig completion, *Discrete Math.*, **341**(1) (2018) 194-202.
- [1729] D. McQuillan and K. Smith, Vertex-magic total labeling of odd complete graphs, Discrete Math., 305 (2005) 240-249.
- [1730] D. McQuillan and K. Smith, Vertex-magic total labeling of multiple complete graphs, Congr. Numer., 180 (2006) 201-205.
- [1731] J. McSorley and W. Wallis, On the spectra of totally magic labelings, Bull. Inst. Combin. Appl., 37 (2003) 58-62.
- [1732] J. McSorley and J. Trono, On k-minimum and m-minimum edge-magic injections of graphs, Discrete Math., 310 (2010) 56-69.

- [1733] K. McTavish, personal communication.
- [1734] A. Meena Kumari and S. Arockiaraj, On (1,1,0)-F-face magic mean labeling of some graphs, Utilitas Math., 103 (2017) 139-159.
- [1735] S. Meena and J. Naveen, Some results on prime labeling of graphs, Internat. J. Math. Comput. Res., 2(9) (2014) 606-618.
- [1736] S. Meena and M. Sivasakthi, Harmonic mean labeling Of subdivision graphs, IJRAR- Internat. J. Res. Analytical Reviews, 6(1) (2019) 196-204. http://ijrar. com/upload\_issue/ijrar\_issue\_20542941.pdf
- [1737] S. Meena and K. Vaithilingam, Prime labeling of friendship graphs, Internat. J. Engin. Res. Tech., (IJERT) 1(10) (2012).
- [1738] S. Mehra and Puneet, 2017, Topological integer additive set-labeling of signed graphs, Int. J. Recent Sci. Res.8(12) (2017) 22423-22427. http://dx.doi.org/ 10.24327/ijrsr.2017.0812.126
- [1739] A. Meissner and K. Zwierzyński, Vertex-magic total labeling of a graph by distributed constraint solving in the Mozart System, *in* Parallel Processing and Applied Mathematics, Lecture Notes in Computer Science, **3911** Springer Berlin/Heidelberg, 2006.
- [1740] L. S. Melnikov and A. V. Pyatkin, Regular integral sum graphs, Discrete Math., 252 (2002) 237-245.
- [1741] E. Mendelsohn and N. Shalaby, Skolem labelled graphs, Discrete Math., 97 (1991) 301-317.
- [1742] E. Mendelsohn and N. Shalaby, On Skolem labelling of windmills, Ars Combin., 53 (1999) 161-172.
- [1743] Y. Miao and Z. Liang, On the strongly c-harmoniousness cycle with  $P_2$  or  $P_3$  chord, Ars Combin., **102** (2011) 101-128.
- [1744] G. Michael and M. Z. Youssef, On prime self-complementary graphs, J. Disc. Math. Sci. and Cryptography, 17(3) (2014) 239-256.
- [1745] M. Miller, Open problems in graph theory: labelings and extremal graphs, personal communication.
- [1746] M. Miller and M. Bača: Antimagic valuations of generalized Petersen graphs, Australas. J. Combin., 22 (2000) 135-139.
- [1747] M. Miller, M. Bača, and Y. Lin: On two conjectures concerning (a, d)-antimagic labelings of antiprisms, J. Combin. Math. Combin. Comput., 37 (2001) 251-254.

- [1748] M. Miller, M. Bača, and J. A. MacDougall: Vertex-magic total labeling of generalized Petersen graphs and convex polytopes, J. Combin. Math. Combin. Comput., 59 (2006) 89-99.
- [1749] M. Miller, D. Patel, J. Ryan, K. Sugeng, Slamin, and M. Tuga, Exclusive sum labeling of graphs, J. Combin. Math. Combin. Comput., 55 (2005) 149-158.
- [1750] M. Miller, O. Phanalasy, J. Ryan, All graphs have antimagic total labelings, *Electronic Notes Disc. Math.*, 38(1) (2011) 645-650.
- [1751] M. Miller, O. Phanalasy, J. Ryan, and L. Rylands, Antimagicness of some families of generalized graphs, Australas. J. Combin., 53 (2012) 179-190.
- [1752] M. Miller, O Phanalasy, J. Ryan, and L. Rylands, A note on antimagic labelings of trees, Bull. Inst. Combin. Appl., 72 (2014) 94-100.
- [1753] M. Miller, O. Phanalasy, J. Ryan, and L. Rylands, Sparse graphs with vertex antimagic edge labelings, *AKCE Int. J. Graphs Comb.*, **10**(2 (2013) 193-198.
- [1754] M. Miller, J. Ryan, K. Sugeng, Slamin, and M. Tuga, Exclusive sum labeling of graphs, J. Combin. Math. Combin. Comput., 55 (2005) 137-148.
- [1755] M. Miller, C. Rodger, and R. Simanjuntak, Distance magic labelings of graphs, Australas. J. Combin., 28 (2003) 305-315.
- [1756] M. Miller, J. Ryan, and Slamin, Integral sum numbers of cocktail party graphs and symmetric complete bipartite graphs, Bull. Inst. Combin. Appl., 25 (1999) 23-28.
- [1757] M. Miller, J. Ryan, Slamin, and W. Smyth, Labelling wheels for the minimum sum number, J. Combin. Math. Combin. Comput., 28 (1998) 289-297.
- [1758] M. Miller, J. Ryan, Slamin, K. Sugeug, and M. Tuga, Open problems in exclusive sum graph labeling, preprint. http://theory.snu.ac.kr/awoca2003/ppt/ OpenProblemsinExclusiveSumLabeling.ppt
- [1759] M. Miller, J. Ryan, and W. Smyth, The sum number of the cocktail party graph, Bull. Inst. Combin. Appl., 22 (1998) 79-90.
- [1760] D. Mishra and A. C. Panda, Some new family of graceful lobsters, Adv. Appl. Discrete Math., 14 (2014)(1) 1-24.
- [1761] D. Mishra and A. C. Panda, A Class of diameter six graceful trees, J. Advances Math., 9(5) (2014) 2677- 2686.
- [1762] D. Mishra and A. C. Panda, A class of diameter six trees with graceful labelings, Inter. J. Math. Trends Tech., 9(1) (2014) 1-11.

- [1763] D. Mishra and A. C. Panda, A family of graceful diameter six trees generated by component moving techniques, *British J. Math. Comput. Sci.*, 21(1) Article no.BJMCS.3107, (2017) 1-15.
- [1764] D. Mishra and P. Panigrahi, Graceful lobsters obtained by component moving of diameter four trees, Comput. Math. Appl., 50 (2005) 367-380.
- [1765] D. Mishra and P. Panigrahi, Some graceful lobsters with all three types of branches incident on the vertices of the central path, *Comput. Math. Appl.* 56 (2008) 1382-1394.
- [1766] D. Mishra and P. Panigrahi, Some new classes of graceful lobsters obtained from diameter four trees, Math. Bohem., 135 (2010) 257-278.
- [1767] J. Mitchem and A. Simoson, On edge-graceful and super-edge-graceful graphs, Ars Combin., 37 (1994) 97-111.
- [1768] S. Mitra and S. Bhoumik, Graceful labeling of triangular extension of complete bipartite graph, *Electron. J. Graph Theory Appl. (EJGTA)*, 7(1) (2019) 11-30.
- [1769] M. V. Modha and K. K. Kanani, Some new families of 5-cordial graphs, Internat. J. Math. and Soft Computing, 5(1) (2015) 129-141.
- [1770] M. V. Modha and K. K. Kanani, k-cordial labeling of fan and double fan, Internat. J. Applied Math. Res., 4(2) (2015) 362-369.
- [1771] M. V. Modha and K. K. Kanani, Some wheel related 7-cordial graphs, Internat. J. Emerging Tech. Appl. Eng., Tech., Sci., 9 (2016) 71-78.
- [1772] M. V. Modha and K. K. Kanani, k-cordiality of wheel, path related and cycle related graphs, Internat. J. Math. Sci. Comput., 5(2) (2015) 79-82.
- [1773] M. V. Modha and K. K. Kanani, k-cordiality wheel, path related and cycle related graphs, Internat. J. Math. Sci. Comput., 5(2) (2015) 70-82.
- [1774] M. V. Modha and K. K. Kanani, On k-cordial labeling of some graphs, British J. Math. Comput Sci., 13(3) (2016) 1-7.
- [1775] M. V. Modha and K. K. Kanani, k-cordial labeling of some prisms, Internat. J. Math. Trends and Tech. (IJMTT), 35(3) (2016) 174-176.
- [1776] M. V. Modha and K. K. Kanani, 5-cordiality of some path, cycle and wheel related graphs, 9th National Level Science Symposium. February 14, 2016, Organized by Christ College, Rajkot, Sponsored by GUJCOST, Gandhinagar, Mathematics & Statistics, 3 (2016) 183-187.
- [1777] V. Mohan and A. Sugumaran, Some new divisor cordial graphs, J. Emerging Tech. and Innovative Res., 5(8)(2018) 229-238. https://ijmttjournal.org/ Volume-12/number-2/IJMTT-V12P513.pdf

- [1778] V. Mohan and A. Sugumaran, Difference cordial labeling of some graphs related to square graph, J. Emerging Tech. and Innovative Res., 5(11)(2018) 589-594.
- [1779] V. Mohan and A. Sugumaran, Difference cordial labeling for plus and hanging pyramid graphs, Internat. J. Res. Analytical Reviews, 6(1)(2019) 1106-1114.
- [1780] M. Mollard and C. Payan, Elegant labelings and edge-colorings: A proof of two conjectures of Hartman, and Chang, Hsu, Rogers, Ars Combin., 36 (1993) 97-106.
- [1781] M. Mollard, C. Payan, and S. Shixin, Graceful problems, Seventh Hungarian Colloquium on Finite and Infinite Combinatorics, Budapest July 1987.
- [1782] K. M. Mominul Haque, X. Lin, Y. Yang, and J. Zhang, Prime cordial labeling of flower snark and related graphs, Ars Combin., 105 (2012) 45-52.
- [1783] R. Montgomery, A. Pokrovskiy, and B. Sudakov, A proof of Ringel's Conjecture, arXiv:2001.02665, 2020
- [1784] D. Morgan, All lobsters with perfect matchings are graceful, *Electron. Notes Discrete Math.*, **11** (2002) 6 pp.
- [1785] D. Morgan, Gracefully labeled trees from Skolem sequences, Congr. Numer., 142 (2000) 41-48.
- [1786] D. Morgan and R. Rees, Using Skolem and Hooked-Skolem sequences to generate graceful trees, J. Combin. Math. Combin. Comput., 44 (2003) 47-63.
- [1787] M. Morris-Rivera, M. Tomova, C. Wyels, A. Yeager, The radio number of  $C_n \Box C_n$ , Ars Combin., **120** (2015) 7-21.
- [1788] M. I. Moussa and E. M. Badr, Odd graceful labelings of crown graphs, 1st Internat. Conf. Comp. Sci. Algor. Appl., 2009 1-5 Cairo, Egypt.
- [1789] M. I. Moussa, Some simple algorithms for some odd graceful labeling graphs, Proceed. 9th WSEAS Internat. Conf. Applied Informatics and Communications (AIC '09) August, 2009, Moscow, Russia.
- [1790] M. I. Moussa, An algorithm for odd graceful labeling of the union of paths and cycles, Internat. J. Appl. Graph Theory in Wireless ad hoc Networks (Graph Hoc), 2(1) (2010) 112-119.
- [1791] M. I. Moussa, An algorithm for odd gracefulness of the tensor product of two line graphs, Inter. J. Applications of Graph Theory in Wireless ad hoc Networks and Sensor Networks (GRAPH-HOC), 3 (2011).
- [1792] M. I. Moussa and E. M. Badr, Ladder and subdivision of ladder graphs with pendant edges are odd graceful, *Internat. J. Appl. Graph Theory Wireless Ad hoc Networks and Sensor Networks*, (GRAPH-HOC) textbf8(1) (2016) 1-8. https: //arxiv.org/ftp/arxiv/papers/1604/1604.02347.pdf

- [1793] D. Moulton, Graceful labelings of triangular snakes, Ars Combin., 28 (1989) 3-13.
- [1794] J. Mülbacher, Magische Quadrate und ihre Verallgemeinerung: ein graphentheoretisches Problem in: Graphs, Data Structures, Algorithms, Hensen Verlag, München, 1979.
- [1795] A. Munia, J. Maowa, S. Tania, M. Kaykobad, A new class of graceful tree, Internat. J. Eng. Sci. and Res., 5 (2014) 1112-1115.
- [1796] F. A. Muntaner-Batle, Special super edge-magic labelings of bipartite graphs, J. Combin. Math. Combin. Comput., 39 (2001) 107-120.
- [1797] K. Murhu Guru Packiam, T. Manimaran, and A. Thuraiswamy, On total edge irregularity strength of graph, Ars Combin., 129 (2016) 173-183.
- [1798] M. Murugan, Almost-magic, relaxed-magic and magic strength of a graph, Util. Math., 65 (2004) 53-64.
- [1799] K. Murugan and A. Subramanian, Skolem difference mean labeling of Hgraphs, Internat. J. Math. Soft Comput., 1(1) (2011) 115-129. http://oaji.net/ articles/2015/296-1433770540.pdf
- [1800] M. Murugan and G. Arumugan, Bi-graceful graphs, Number theory and discrete mathematics (Chandigarh, 2000) *Trends Math.*, Birkhäuser 243-249.
- [1801] M. Murugan and G. Arumugan, Are banana trees graceful?, Math. Ed. (Siwan), 35 (2001) 18-20.
- [1802] M. Murugan and G. Arumugan, On graceful numberings of  $nC_5$  with a common edge, unpublished.
- [1803] M. Murugan and G. Arumugan, An algorithm to find graceful numberings of a Spl. class of banana trees, unpublished.
- [1804] S. Murugan and S. Chandra Kumar, H-V-super magic labeling of H-factorable graphs, Malaya J. Mat., 7(2) (2019) 138-141.
- [1805] N. Murugesan and R. Senthil Amutha, Bimagic total labeling for bistar  $B_{n,n}$ , Internat. J. Sci. Innov. Math. Res. (IJSIMR), **2**(9) (2014) 764-769.
- [1806] S. Murugesan, D. Jayaraman, and J. Shiama, Square divisor cordial graphs, *Internat. J. Comput. Appl.*, 64 (22) (2013) 1-4.
- [1807] A. Muthaiyan and P. Pugalenthi, Vertex odd divisor cordial labeling of graphs, IJISET - Internat. J. Innovative Sci., Engin. Tech., 2(10) (2015) 395-400. http: //ijiset.com/vol2/v2s10/IJISET\_V2\_I10\_53.pdf

- [1808] S. Mutharasu and D. Kumar, V<sub>k</sub>-Super vertex magic labeling of graphs, Malaya J. Mat., 6(4) (2018) 795-799. https://www.malayajournal.org/articles/ MJM06040014.pdf
- [1809] K. Muthgu Guru Packiam, Total face irregularity strength of plane graphs, J. Graph Label., 2(1) (2016) 69-77.
- [1810] K. Muthu Guru Packiam, T. Manimaran, and A. Thuraiswamy, Irregularity strength of corona of two graphs, *Theoretial Comput. Sci. Disc. Math.*, 175-181, Lecture Notes in Comput. Sci., 10398, Springer, Cham, 2017.
- [1811] K. Muthu Guru Packiam, T. Manimaran, and A. Thuraiswamy, 1-distant irregularity strength of graphs, *Theoretical Comput. Sci. Disc. Math.*, 182-190, Lecture Notes in Comput. Sci., 10398, Springer, Cham, 2017.
- [1812] D. Muthuramakrishnan, k-even Sequential Harmonious Labeling of Graphs, PhD thesis, Bharathidasan University, 2013.
- [1813] D. Muthuramakrishnan and S. Sutha, Pell graceful labeling of graphs, Malaya J. Matematik, 7(3) (2019) 508-512. http://www.malayajournal.org/articles/ MJM07030023.pdf
- [1814] H. Nagamochi, M. Miller, and Slamin, Bounds on the number of isolates in sum graph labeling, *Discrete Math.*, 240 (2001) 175-185.
- [1815] S. Nada, A. T. Diab, A. Elrokh, and D. E. Sabra, The corona between cycles and paths, J. Egyptian Math. Soc., 25(2) (2017) 111-118.
- [1816] S. Nada, A. T. Diab, A. Elrokh, and D. E. Sabra, The corona between paths and cycles, Ars Combin., 139 (2018) 269-281.
- [1817] S. Nada, A. Elrokh, and E. Elshafey, The cordiality of the join and union of the second power of fans, Appl. Comput. Math., 7(6) (2018) 219-224. doi:10.11648/ j.acm.20180706.11
- [1818] U. Nadzima and T. S. Martini, Super (a, d) H-antimagic total labeling on wheel edge corona product with a path and a cycle, ICMETA 2018 IOP Conf. Series: J. Physics: Conf. Series 1306 (2019) 012007 2414-2423. doi:10.1088/1742-6596/ 1306/1/012007
- [1819] M. Naeem and M. K. Siddiqui, Total irregularity strength of disjoint union of isomorphic copies of generalized Petersen graph, *Discrete Math. Algor. Appl.*, 9(6) (2017) 1750071, 9 pp.
- [1820] CT. Nagaraj, C. Y. Ponnappan, and G. Prabakaran, Even vertex magic total labeling, Internat. J. Pure Appl. Math., 115 (9)(2017) 363-375.

- [1821] CT. Nagaraj, C. Y. Ponnappan, and G. Prabakaran, Internat. J. Math Trends Tech., Odd vertex magic total labeling of trees, 52(6) (2017) 374-379.
- [1822] CT. Nagaraj, C. Y. Ponnappan, and G. Prabakan, Even vertex magic total labeling of isomorphic and non isomorphic suns, *Internat. J. Math Trends Tech.*, **52** (7) (2017) 458-467.
- [1823] CT. Nagaraj, C. Y. Ponnappan, and G. Prabakaan, Even vertex magic total labeling of some 2-regular graphs, *Internat. J. Math Trends Tech.*, 54(1) (2018) 52-59.
- [1824] CT. Nagaraj, C. Y. Ponnappan, and G. Prabakaan, Odd vertex magic total labeling of some graphs, *Internat. J. Pure Appl. Math.*, **115** (10)(2018) 97-109. DOI:10. 12732/ijpam.v118i10.53.
- [1825] CT. Nagaraj, C. Y. Ponnappan, and G. Prabakan, Odd vertex magic total labeling of isomorphic and non isomorphic suns, *Internat. J. Math Trends Tech.*, **52**(1) (2018) 34-41.
- [1826] CT. Nagaraj, C. Y. Ponnappan, and G. Prabakaran, Multiple magic graphs, Int. J. Math. Appl., 6(1) (2018) 109-115. (Special Issue)
- [1827] A. Nagarajan, R. Vasuki, and S. Arockiaraj, Super mean number of a graph, *Kragujevac J. Math.* 36(1) (2012) 93-107.
- [1828] M. Nalliah, Antimagic labeling of digraphs, J. Indones. Math. Soc., 22(1) (2016) 61-69.
- [1829] M. Nalliah, A survey on recent open problems on super (a, d)-edge antimagic total labelings J. Graph Labeling, 1(2) (2015) 129-136.
- [1830] M. Nalliah and S. Arumugam, Super (a, 3)-edge antimagic total labeling for union of two stars, *Theoretical Computer Science and Discrete Mathematics*, 203-211, Lecture Notes in Comput. Sci., 10398, Springer, Cham, 2017.
- [1831] D. Narayan, Problem 380. Representations of graphs modulo n, Discrete Math., **257** (2002) 614.
- [1832] D. Narayan, An upper bound for the representation number of graphs of fixed order, Integers, 3 (2003) A12 4 pages.
- [1833] G. Narasimhan and R. Sampathkumar, (2013). Contra harmonic mean labeling of graphs, Mapana-J. Sciences, 12(8) 2329. https://doi.org/10.12723/mjs.26.7
- [1834] A. Naseem, K. Shabbir, and H. Shaker, The radio number of edge-joint graphs, Ars Combin., 139 (2018) 337-315.

- [1835] S. Nazeer, M. S. Khan, I. Kousar, and W. Nazeer, Radio labeling and radio number for generalized caterpillar graphs, J. Appl. Math. Informatics, 34 (5-6) (2016) 451-465.
- [1836] S. Nazeer, I. Kousar, and W. Nazeer, Radio and radio antipodal labelings for circulant graphs  $G(4k+2; \{1,2\})$ , J. Appl. Math. Inform., **33**(1-2) (2015) 173-183.
- [1837] W. Nazeer, S. Nazeer, I. Kousar, and M. Munir, The radio and radio antipodal numbers of non-bipartite cubic graphs of order 2<sup>k</sup>, Ars Combin., 145 (2019) 301-312.
- [1838] P. Nedumaran and K. Thirusangu, On 0-edge magic labeling in certain graphs, Internat. J. Appl. Research, 1(12) (2015) 874-877.
- [1839] N. Neela and C. Selvaraj, Total magic cordial labeling of complete multipartite graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), 251254, *Electron. Notes Discrete Math.*, 48, Elsevier Sci. B. V., Amsterdam, 2015.
- [1840] N. Neela and C. Selvaraj, Conjecture on odd graceful graphs, J. Combin. Math. Combin. Comput., 97 (2016) 65-82.
- [1841] J. Nešetřil and A. Pultr, A Dushnik-Miller type dimension of graphs and its complexity, in: M. Karpinski, Ed., Fundamentals of Computation Theory, Lecture Notes in Computer Science, 56, Springer, Berlin, 1977, 482-493.
- [1842] C.-m Ni Z.-s Liu, and F.-l Lu, On the E<sub>3</sub>-cordiality of some graphs, J. Combin. Math. Combin. Comput., 97 (2016) 3-10.
- [1843] T. Nicholas and K. John Bosco, Radio mean D-distance labeling of cycle-relted graphs, Internat. J. Math. Trends Tech., (IJMTT) 40(3) (2016) 247-251.
- [1844] A. J. Niedziałomski, Consecutive radio labelings and the Cartesian product of graphs, Ph. D. Thesis, University of Iowa, August 2013.
- [1845] A. J. Niedzialomski, Radio graceful Hamming graphs, Discuss. Math. Graph Theory, 36(4) (2016) 1007-1020.
- [1846] S. A. N. Ni'mah and D. Indriati, On vertex irregular total k-labeling and total vertex irregularity strength of lollipop graphs, J. Physics Confer. Series 1306:012025 July 2019
- [1847] R. Nirmalasari Wijaya, J. Ryan, and T. Kalinowski, Cube-magic labelings of grids, arXiv:1702.02639.
- [1848] R. Nisviasari, Dafik, and I. H. Agustin, The total H-irregularity strength of triangular ladder graphs, J. Phys.: Conf. Ser., 1465 012026 (2020) https: //iopscience.iop.org/article/10.1088/1742-6596/1465/1/012026

- [1849] H. K. Ng,  $\alpha$ -valuations and k-gracefulness, Notices AMS, 7 (1986) 247.
- [1850] H. K. Ng, Gracefulness of a class of lobsters, *Notices AMS*, 7 (1986) 825-05-294.
- [1851] A. A. G. Ngurah, On (a, b)-edge-antimagic total labeling of odd cycle. J. Indones. Math. Soc., 9 (2003) 9-12.
- [1852] A. A. G. Ngurah and Adiwijaya, New results on the (super) edge-magic deficiency of chain graphs, Int. J. Math. Math. Sci., (2017), Art. ID 5156974, 6 pp.
- [1853] A. A. G. Ngurah, On the (super) edge-magic deficiency of chain graphs, J. Combin. Math. Combin. Comput., 103 (2017) 225-236.
- [1854] A. A. G. Ngurah and E. Baskoro, On magic and antimagic total labelings of generalized Petersen graph, Util. Math., 63 (2003) 97-107.
- [1855] A. A. G. Ngurah, E. Baskoro, and R. Simamjuntak, On antimagic total labelings of generalized Petersen graph, J. Combin. Math. Combin. Comput., 55 (2005) 57-70.
- [1856] A. A. G. Ngurah, E. Baskoro, and R. Simamjuntak, On (a, d)-edge-antimagic total labelings of  $mC_n$ , Bull. Inst. Combin. Appl., 48 (2006) 35-44.
- [1857] A. A. G. Ngurah, E. Baskoro, and R. Simamjuntak, On the super edge-magic deficiencies of graphs, Australas. J. Combin., 40 (2008) 3-14.
- [1858] A. A. G. Ngurah, E. Baskoro, and R. Simamjuntak, On new families of (super) edge-magic graphs, Util. Math., 74 (2007) 111-120.
- [1859] A. A. G. Ngurah, E. T. Baskoro, and I. Tomescu, Magic graphs with pendant eges, Ars Combin., 99 (2011) 149-160.
- [1860] A. A. G. Ngurah, A. N. M. Salman, and I. W. Sudarsana, On supermagic coverings of fans and ladders, SUT J. Math., 46 (2010) 67-78.
- [1861] A. A. G. Ngurah, A. N. M. Salman, and L. Susilowati, *H*-supermagic labelings of graphs, *Discrete Math.*, **310** (2010) 1293-1300.
- [1862] A. A. G. Ngurah and R. Simanjuntak, Super edge-magic deficiency of join-product graphs, Util. Math., 105 (2017) 279-289.
- [1863] A. A. G. Ngurah and R. Simanjuntak, On the super edge-magic deficiency of join product and chain graphs, *Electron. J. Graph Theory Appl. (EJGTA)*, 7(1) (2019) 157-167.
- [1864] A. A. G. Ngurah, R. Simamjuntak, and E. Baskoro, On (super) edge-magic total labeling of subdivision of  $K_{1,3}$ , SUT J. Math., 43 (2007) 127-136.

- [1865] T. Nicholas and S. Somasundaram, More results on integral sum graphs, Graph Theory and its Applications, Editors: R. Balakrishnan et al., Narosa Publishing House, New Delhi, India (2004) 75-83.
- [1866] T. Nicholas and S. Somasundaram, More results on sum graphs, Proceedings of the Conference on Graph Theory and its Applications held at Anna University, Chennai, (2001) 73-83.
- [1867] T. Nicholas, S. Somasundaram, and V. Vilfred, On (a, d)-antimagic special trees, unicyclic graphs and complete bipartite graphs, Ars Combin., **70** (2004) 207-220.
- [1868] T. Nicholas and V. Vilfred, Sum graph and edge reduced sum number, unpublished.
- [1869] R. W. Nirmalasari Wijaya, J. Ryan, and T. Kalinowski, H-supermagic labelings for firecrackers, banana trees and flowers, Australasian J. Combin., 69(3) (2017) 442-451. arXiv:1607.07911
- [1870] R. Nistyawati and T. S. Martini, On super (a, d) H-antimagic total labeling of edge corona product and corona product of gear graph with path and cycle, IOP Conf. Series: J. Physics: Conf., Series 1306 (2019) 012010 1-7. https:// iopscience.iop.org/article/10.1088/1742-6596/1306/1/012010/pdf
- [1871] S. Novotny, J. Ortiz, and D. Narayan, Minimal k-rankings and the rank number of  $P_n^2$ , Inform. Process. Lett., **109** (2009) 193-198.
- [1872] T. Noviati, T. S. Martini, and D. Indriati, On H-supermagic labeling of friendship edge corona product with path and sun edge corona product with path, J. Physics Conference Series, 1306:012008 (2019) 1-6. DOI:10.1088/1742-6596/ 1306/1/012008
- [1873] R. Nowakowski and C. Whitehead, Ordered graceful labellings of the 2-star, Graph theory (Prague, 1998), Discrete Math., 233 (2001) 183-191.
- [1874] K. Nurdin and K. Hye, Irregular labeling on transportation network of splitting graphs of stars, Proc. Jangjeon Math. Soc., 22(1) (2019) 103-108. http: //jangjeonopen.or.kr/public/upload/1549515921-pjms22-1-11.pdf
- [1875] S. Nurdin, E. T. Baskoro, A. N. M. Salman, and N. N. Gaos, On the total vertex irregularity strength of trees. *Discrete Math.*, **310** (2010) 3043-3048.
- [1876] S. Nurdin, E. T. Baskoro, A. N. M. Salman, and N. N. Gaos, On total vertexirregular labellings for several types of trees, Util. Math., 83 (2010) 277-290.
- [1877] S. Nurdin, A. N. M. Salman, and E. T. Baskoro, The total edge-irregular strengths of the corona product of paths with some graphs, J. Combin. Math. Combin. Comput., 65 (2008) 163-175.

- [1878] S. Nurdin, A. N. M. Salman, N. N. Gaos, and E. T. Baskoro, On the total vertexirregular strength of a disjoint union of t copies of a path, J. Combin. Math. Combin. Comput., 71 (2009) 227-233.
- [1879] D. E. Nurvazly and K. A. Sugeng, Graceful labelling of edge amalgamation of cycle graph, J. Physics Confer. Ser., 1108 012047 (2018) http://iopscience. iop.org/article/10.1088/1742-6596/1108/1/012047/pdf
- [1880] M. A. Ollis, New complete Latin squares of odd order, European J. Combin., 41 (2014) 35-46.
- [1881] A. Ovais, M. Umar, M. Bača, and A. Semaničová-Feňovčíková, Fans are cycleantimagic, Australas. J. Combin., 68 (2017) 94-105.
- [1882] L. Packiam and K. Kathiresan, On total vertex irregularity strength of graphs, Discuss. Math. Graph Theory, 32(1) (2012) 39-45.
- [1883] W. Pan and X. Lu, The gracefulness of two kinds of unconnected graphs  $(P_2 \vee \overline{K_n}) \cup St(m)$  and  $(P_2 \vee \overline{K_n}) \cup T_n$ , J. Jilin Univ., **41** (2003) 152-154.
- [1884] A. C. Panda and D. Mishra, Some new classes of graceful diameter six trees, Turkic World Math. Soc. J. Appl. Engin. Math., 5(2) (2015) 269-275.
- [1885] A. C. Panda, D. Mishra, R. B. Dash, A class of diameter six trees exhibiting graceful labeling. J. Discrete Math. Sci. Cryptogr., 19(5-6) (2016) 947-963.
- [1886] J. Pandimadevi and S. P. Subbiah, Vertex antimagic total labeling of digraphs, *Kyungpook Math. J.*, 55(2) (2015) 267-277.
- [1887] P. Pandya and N. P. Shrimali, Vertex-edge neighborhood prime labeling of some graphs, Internat. J. Scien. Res. Review, 7(10) 2018 735-743. http://www. dynamicpublisher.org/gallery/82-ijsrr-d1002.pdf
- [1888] P. Panigrahi, A survey on radio k-colorings of graphs, AKCE J. Graphs Combin., 6 (2009) 161-169.
- [1889] P. Panigrahi and D. Mishra, Graceful lobsters obtained from diameter four trees using partitioning technique, Ars Combin., 87 (2008) 291-320.
- [1890] P. Panigrahi and J. Saha, On harmoniousness of hypercubes, AKCE J. Graphs Combin., 5 (2008) 189-198.
- [1891] A. Panpa and T. Poomsa-ard, On graceful spider graphs with at most four legs of lengths greater than one, J. Appl. Math., 3 (2016) 1-5.
- [1892] A. Parestu, D. R. Silaban, and K. A. Sugeng, Pelabelan simpul-ajaib total dari gabungan graf matahari, prosising Seminar Nasional Matematika, Universitas Parahyangan, Bandung, 3 (2008) 407-414.

- [1893] A. Parestu, D. R. Silaban, and K. A. Sugeng, Vertex antimagic total labelings of the union of suns, J. Combin. Math. Combin. Comput., 71 (2009) 179-188.
- [1894] J. Y. Park, J. H. Choi, and J. H. Bae, On super edge-magic labeling of some graphs, Bull. Korean Math. Soc., 45 (2008) 11-21.
- [1895] D. Parmar and J. Joshi,  $H_k$  -Cordial labeling of triangular snake graph, J. Appl. Sci. Computat., **6**(3) (2019) 2118-2123.
- [1896] Y. Parmar, Edge vertex prime labeling for wheel, fan and friendship graph, Internat. J. Math. Stat. Invention (IJMSI), 5(8) (2017) 23-29.
- [1897] Y. Parmar, Edge vertex prime labeling for  $K_{2,n}$  and  $K_{3,n}$  graphs, Math. J. Interdisciplinary Sci., **6**(2) (2018) 167-180.
- [1898] D. Parmar and U. Vaghela, Skolem difference mean labeling of some path related graphs, *Pramana Res. J.*, 9(5) (2019) 651-657.
- [1899] A. Pasotti, Constructions for cyclic Moebius ladder systems, Discrete Math., 310 (2010) 3080-3087.
- [1900] A. Pasotti, On d-divisible  $\alpha$ -labelings of  $C_{4k} \times P_m$ , Util. Math., **90** (2013) 135-148.
- [1901] A. Pasotti, On *d*-graceful labelings, Ars Combin., **111** (2013) 207-223.
- [1902] A. M. Pastel, and H. Raynaud, Numerotation gracieuse des oliviers, in *Colloq. Grenoble*, Publications Université de Grenoble, (1978) 218-223.
- [1903] S. K. Patel, Neighborhood-prime labeling of some generalized Petersen graphs, Internat. J. Math. Soft Comput., 7(2) (2017) 111-118.
- [1904] M. Patel and G. V. Ghodasara, Neighborhood-prime labeling of some cycle related graphs, *Math. Today*, **34** (2018) 159-173.
- [1905] S. K. Patel, U. M. Prajapati, and A. N. Kansagara, Product cordial labeling of extensions of barbell graph, *Math. Combin. Book Ser.*, 4 (2018) 146-159.
- [1906] S. K. Patel and N. P. Shrimali, Neighborhood-prime labeling, Internat. J. Math. and Soft Comput., 5(2) (2015) 135-143.
- [1907] S. K. Patel and N. P. Shrimali, Neighborhood-prime labeling of some union graphs, Internat. J. Math. Soft Comput., 6(1) (2016) 39-47.
- [1908] S. K. Patel and N. P. Shrimali, Neighborhood-prime labeling of some product graphs, Algebra Discrete Math., 25(1) (2018) 118-129.
- [1909] S. K. Patel and J. Vasava, On prime labeling of some union graphs, Kragujevac J. Math., 42(3) (2018) 441-452.

- [1910] L. Pattabiraman, T. Loganathan, and C. V. Rao, Graceful labeling of hexagonal snakes, Graceful Labeling of Hexagonal Snakes AIP Confer. Proceed. 2112, 020056 (2019) 1-3. https://doi.org/10.1063/1.5112241
- [1911] O. Pechenik and J. Wise, Generalized graph cordialty, Discuss. Math. Graph Th., 32(3) (2012) 557-567.
- [1912] A. C. Pedrano and R. Rulete, On the total product cordial labeling on the cartesian product of  $P_m \times C_n$ ,  $C_m \times C_n$  and the generalized Petersen graph P(m, n), Malaya J. Mat., 5(2) (2017) 194-201. https://www.malayajournal.org/articles/Paper\_7\_2017.pdf
- [1913] A. Pedrano and R. Rulete, On total product cordial labeling of some crown graphs, Internat. J. Math. Appl., 5(2B) (2017) 273-284.
- [1914] S. G. Penrice, Some new graph labeling problems: a preliminary report, DIMACS Tech. Rep. 95-29 (1995).
- [1915] J. Pereira, T. Singh, and S. Arumugam, m-gracefulness of graphs, Algor. Discrete Appl. Math., Lecture Notes in Comput. Sci., 9602, Springer, [Cham], (2016) 289-298.
- [1916] J. Pereira, T. Singh, and S. Arumugam, On (k, d)-Skolem graceful graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), 81-88, *Elec*tron. Notes Discrete Math., 48, Elsevier Sci. B. V., Amsterdam, 2015.
- [1917] J. Pereira, T. Singh, and S. Arumugam, On (k, d) hooked Skolem graceful graphs, arXiv:1705.06736v1
- [1918] M. A. Perumal, S. Navaneethakrishnan, A. Nagarajan, S. Arockiaraj, Super graceful labeling for some simple graphs, *Internat. J. Mathematics and Soft Computing*, 2 (2012) 35-49.
- [1919] M. A. Perumal, S. Navaneethakrishnan, S. Arockiaraj, A. Nagarajan, Super graceful labeling for some special graphs, *Int. J. Res. Rev. Appl. Sci.* 9(3) (2011) 382-404.
- [1920] K. Petrie and B. Smith, Symmetry breaking in graceful graphs, in Proc. CP'03, LNCS, 930-934 Springer (2003)
- [1921] F. Pfender, Total edge irregularity strength of large graphs, Discrete Math., 312(2) 229-237. arXiv:1006.4501
- [1922] O. Phanalasy, Antimagic labeling of generalized sausage graphs, J. Indones. Math. Soc., 20(2) (2014) 95-110.
- [1923] O. Phanalasy, M. Miller, C. S. Iliopoulos, S. P. Pissis, and E. Vaezpour, Construction of antimagic labeling for the Cartesian product of regular graphs, *Math. in Comput. Sci.*, 5(1) (2011) 81-87.
- [1924] O. Phanalasy, M. Miller, J. Ryan and S. Arumugam, Antimagic labeling of generalized pyramid graphs, Acta. Math. Sin.-English Ser., 30 (2014) 283.
- [1925] O. Phanalasy, M. Miller, L. J. Rylands, and P. Lieby, On a relationship between completely separating systems and antimagic labeling of regular graphs, In C. S. Iliopoulos and W. F. Smyth, editors, *Proceedings of IWOCA10, Lecture Notes in Computer Science*, 6460 (2011) 238-241.
- [1926] N. C. K. Phillips, R. S. Rees, and W. D. Wallis, Edge-magic total labelings of wheels, Bull. Inst. Combin. Appl., 31 (2001) 21-30.
- [1927] N. C. K. Phillips, R. S. Rees, and W. D. Wallis, personal communication.
- [1928] V. Sharon Philomena and K. Thirusangu, Some new classes of 3-total product cordial graphs, *Internat. J. Comput. Algor.*, 3 (2014) 647-650.
- [1929] V. Sharon Philomena and K. Thirusangu, Square and cube difference labeling of cycle cactus, special tree and a new key graph, Annals Pure Appl. Math., 8(2) (2014) 115-122.
- [1930] D. A. Pike, A. Sanaei and N. Shalaby, Pseudo-Skolem sequences and Graph Skolem Labelling, Math. Scandinavica, 120 (2017) 17-38.
- [1931] O. Pikhurko, Every tree with at most 34 vertices is prime, Util. Math., 62 (2002) 185-190.
- [1932] O. Pikhurko, Dense edge-magic graphs and thin additive bases, *Discrete Math.*, 306 (2006) 2097-2107.
- [1933] O. Pikhurko, Trees are almost prime, *Discrete Math.*, **307** (2007) 1455-1462.
- [1934] O. Pikhurko, Characterization of product anti-magic graphs of large order, Graphs and Combin., 23 (2007) 681-689.
- [1935] S. Poljak and M. Sûra, An algorithm for graceful labeling of a class of symmetrical trees, Ars Combin., 14 (1982) 57-66.
- [1936] A. Ponmoni, S. Navaneetha Krishnan and A. Nagarajan, Skolem difference Lucas mean labeling for star related graphs, *Internat. J. Math. Trends Tech.*, 62(2) (2018) 104-109. http://www.ijmttjournal.org/archive/ijmtt-v62p515
- [1937] C. Y. Ponnappan, CT. Nagaraj, and G. Prabakaran, Odd vertex magic labeling, Internat. J. Comput. Organization Trends, 9(1) (2014) 86-89. https://pdfs. semanticscholar.org/88c4/f87f384255dddab2fca95e4fc0644424f8cc.pdf
- [1938] R. Ponraj,  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial labeling of graphs, Varahmihir J. Math. Sci., 8 (2008) 137-145.

- [1939] R. Ponraj, Further results on  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial labeling of graphs, J. Indian Acad. Math., **31** (2009) 157-163.
- [1940] R. Ponraj and M. Maria Adaickalam, Quotient cordial labeling of star related graphs, J. Indian Acad. Math., 37(2)(2015) 313-324.
- [1941] R. Ponraj and M. Maria Adaickalam, 3-difference cordiality of some graphs, Palestine. J. Mathematics, 6 (2017) 202-210.
- [1942] R. Ponraj and M. Maria Adaickalam, 3-difference cordiality of some union of graphs, *Palestine J. Math.*, 2 (2017) 141-148.
- [1943] R. Ponraj and M. Maria Adaickalam, 3-difference cordial labeling of some cycle related graphs, J. Algor. Comput., 47 (2016) 1-10.
- [1944] R. Ponraj, M. Maria Adaickalam, and R. Kala, 3-difference cordiality of some special graphs, *Jordan J. Math. Stat.*, **11**(2) (2018) 145-167.
- [1945] R. Ponraj, M. Maria Adaickalam and R. Kala, Quotient cordial labeling of graphs, Inter. J. Math. Combin., 1 (2016) 101-108.
- [1946] R. Ponraj, M. Maria Adaickalam and R. Kala, k-difference cordial labeling of graphs, Internat. J. Math. Comb., 2 (2016) 121-131.
- [1947] R. Ponraj, M. Adaickalam, and R. Kala, 3-difference cordiality of corona of double alternate snake graph, Bull. Int. Math. Virtual Inst., 8(2) (2018) 245-258.
- [1948] R. Ponraj, K. Annathurai, and R. Kala, Remainder cordial labeling of graphs, J. Algor. Comput., 49 (2017) 17-30.
- [1949] R. Ponraj, K. Annathurai, and R. Kala, k-remainder cordial graphs, J. Algor. Comput., 49(2) (2017) 41-52.
- [1950] R. Ponraj, K. Annathurai, and R. Kala, 4-remainder cordial labeling of some special graphs, *Internat. J. Pure Appl. Math.*, **118**(6) (2018) 399-405.
- [1951] R. Ponraj, K. Annathurai, and R. Kala, 4-remainder cordial labeling of some graphs, Internat. J. Math. Combin., 1 (2018) 138-145.
- [1952] R. Ponraj, K. Annathurai, and R. Kala, 4-reminder cordial labeling graphs obtained from ladder, *Internat.l J. Mathematical Combin.*, to appear.
- [1953] R. Ponraj, K. Annathurai, and R. Kala, Remainder cordiality of some graphs, *Palestine J. Math.*, 8(1) (2019) 367-372.
- [1954] R. Ponraj, K. Annathurai, and R. Kala, 4-remainder cordial labeling of some new families graphs, *Palestine J. Math.*, 8(2) (2019) 78-86.

- [1955] R. Ponraj and J. V. X. Parthipan, Pair sum labeling of graphs, J. Indian Acad. Math., 32(2) (2010) 587-595.
- [1956] R. Ponraj and J. V. X. Parthipan, Further results on pair sum labeling of trees, Appl. Math., 2(10) (2011) 1270-1278.
- [1957] R. Ponraj and J. V. X. Parthipan, Pair mean labeling of graphs, J. Indian Acad. Math., 33(2) (2011) 375-383.
- [1958] R. Ponraj, J. V. X. Parthipan, and R. Kala, Some results on pair sum labeling of graphs, Inter. J. Mathematical Combin., 4 (2010) 53-61.
- [1959] R. Ponraj, J. V. X. Parthipan, and R. Kala, A note on pair sum graphs, J. Scientific Research, 3(2) (2011) 321-329.
- [1960] R. Ponraj, J. V. X. Parthipan and R. Kala, Further results on pair sum graphs, *Appl. Math.*, 3(3) (2012) 267-275.
- [1961] R. Ponraj, J. V. X. Parthipan and R. Kala, Some new families of pair sum graphs, J. Math. Comput. Sci., 2(3) (2012) 747-758.
- [1962] R. Ponraj and S. Sathish Narayanan, Further results on difference cordial labeling of corona graphs, J. Indian Academy Math., 35 (2013) 217-235.
- [1963] R. Ponraj and S. Sathish Narayanan, Difference cordiality of some graphs obtained from double alternate snake graphs, *Global J. Math. Sciences: Theory and Practi*cal, 5 (2013) 167-175.
- [1964] R. Ponraj and S. Sathish Narayanan, Difference cordiality of some snake graphs, J. Appl. Math. and Informatics, 32 (2014) 377-387.
- [1965] R. Ponraj and S. Sathish Narayanan, Difference cordiality of some derived graphs, Internat. J. Math. Combin., 4 (2013) 37-48.
- [1966] R. Ponraj and S. Sathish Narayanan, Difference cordial labeling of graphs obtained from triangular snakes, Applications Applied Math., 9(2) (2014) 811-825.
- [1967] R. Ponraj and S. Sathish Narayanan, Total mean cordiality of  $K_n^c + 2K_2$ , Palestine J. Math., **4**(2) (2015) 431-438.
- [1968] R. Ponraj and S. Sathish Narayanan, Mean cordiality of some snake graphs, *Pales-tine J. Math.*, 4(2) (2015) 439-445.
- [1969] R. Ponraj and S. Sathish Narayanan, Total mean cordial labeling of some cycle related graphs, J. Appl. Math. & Informatics, 33 (1-2) (2015) 101-110.
- [1970] R. Ponraj and S. Sathish Narayanan, Some results on total mean cordial labeling of graphs, *International J. Math. Combin.*, 2 (2015) 122-132.

- [1971] R. Ponraj and S. Sathish Narayanan, Existence and non existence of mean cordial labeling of certain graphs, J. Prime Research Math., 11 (2015) 123-136.
- [1972] R. Ponraj and S. Sathish Narayanan, Total mean cordiality of some derived graphs, Bull. inter. Math. Virtual Institute, 5 (2015) 181-190.
- [1973] R. Ponraj and S. Sathish Narayanan, Further results on total mean cordial labeling of graphs, J. Algor. Comput., 46 (2015) 73-83.
- [1974] R. Ponraj and S. Sathish Narayanan, Radio mean number of certain graphs, Inter. J. Math. Combin., 2 (2016) 51-64.
- [1975] R. Ponraj, S. Sathish Narayanan, and R. Kala, Difference cordial labeling of graphs, Global J. Math. Sciences: Theory and Practical, 3 (2013) 192-201.
- [1976] R. Ponraj, S. Sathish Narayanan, and R. Kala, Difference cordial labeling of graphs obtained from double snakes, *Internat. J. of Math. Research*, **5** (2013) 317-322.
- [1977] R. Ponraj, S. Sathish Narayanan, and R. Kala, Difference cordial labeling of corona graphs, J. Math. Comput. Sci., 3 (2013) 1237-1251.
- [1978] R. Ponraj, S. Sathish Narayanan, and R. Kala, Difference cordial labeling of subdivided graphs, *Scientia Magna*, 9(3) (2013) 57-66.
- [1979] R. Ponraj, S. Sathish Narayanan, and R. Kala, Difference cordial labeling of subdivision of snake graphs, Universal J. Appl. Math., 2(1) (2014) 40-45.
- [1980] R. Ponraj, S. Sathish Narayanan, and R. Kala, Difference cordiality of product related graphs, *Tbilisi Math. J.*, 8(2) (2015) 41-47.
- [1981] R. Ponraj, S. Sathish Narayanan, and R. Kala, A note on difference cordial graphs, *Palestine J. Math.*, 4(1) (2015) 189-197.
- [1982] R. Ponraj, S. Sathish Narayanan, and R. Kala, Radio mean labeling of a graph, AKCE Internat. J. Graphs Combin., 12 (2015) 224-228.
- [1983] R. Ponraj, S. Sathish Narayanan, and R. Kala, On radio mean number of some graphs, *Internat. J.Math. Combin.*, **3** (2014) 41-48.
- [1984] R. Ponraj, S. Sathish Narayanan, and R. Kala, Radio mean number of some wheel related graphs, *Jordan J. Math. and Stat. (JJMS)*, 7(4) (2014) 273-286.
- [1985] R. Ponraj, S. Sathish Narayanan and A. M. S. Ramasamy, Total mean cordial labeling of graphs, *International J.Math. Combin.*, 4 (2014) 56-68.
- [1986] R. Ponraj, S. Sathish Narayanan and A. M. S. Ramasamy, Total mean cordial labeling of some graphs, *Util. Math.*, to appear.

- [1987] R. Ponraj, S. Sathish Narayanan and A. M. S. Ramasamy, Total mean cordiality of umbrella, butterfly and dumbbell graphs, *Jordan J. Math. and Stat. (JJMS)*, 8(1)(2015) 59-77.
- [1988] R. Ponraj, S. Sathish Narayanan, and R. Kala, Radio mean labeling of a graph AKCE Internat. J. Graphs Combin., 12 (2015) 224-228.
- [1989] R. Ponraj, S. Sathish Narayanan and R. Kala, Radio mean number of some subdivision graphs, Jordan J. Math. and Stat. (JJMS), 9(1) (2016) 45-64.
- [1990] R. Ponraj, S. Sathish Narayanan and A. M. S. Ramasamy, Parity combination cordial labeling of graphs, Jordan J. Math. and Stat. (JJMS), 8(4)(2015) 293-308.
- [1991] R. Ponraj and R. Singh, 4-Prime cordial graphs obtained from 4-prime cordial graphs, Bull. Int. Math. Virtual Inst. 8(1) (2018) 1-9.
- [1992] R. Ponraj, R. Singh, R. Kala and S. Sathish Narayanan, k-Prime cordial graphs, J. Appl. Math. & Informatics, 34(3-4) (2016) 227-237.
- [1993] R. Ponraj, R. Singh, and S. Sathish Narayanan, A note on 3-prime cordial graphs, J. Algor. Comput., 48 (2016) 45-55.
- [1994] R. Ponraj, R. Singh, and R. Kala, 4-Prime cordial labeling of some special graphs, Bull. Int. Math. Virtual Inst. 8(1) (2018) 89-97.
- [1995] R. Ponraj, R. Singh, and R. Kala, 4-Prime cordiality of m copies of some graphs, Palestine J. Math., 8(1) (2018) 307-311.
- [1996] R. Ponraj, R. Singh, and S. Sathish Narayanan, On parity combination cordial graphs, *Palestine J. Math.*, 6(1) (2017) 211-218. https://pjm.ppu.edu/sites/ default/files/papers/PJM\_MAR\_2017\_25.pdf
- [1997] R. Ponraj and M. Sivakumar, On mean cordial graphs, Internat. J. Math. Combin., 3 (2013) 78-84.
- [1998] R. Ponraj, M. Sivakumar, and M. Sundaram, k-Product cordial labeling of graphs, Int. J. Contemp. Math. Sci., 7 (2012) 733-742.
- [1999] R. Ponraj, M. Sivakumar, and M. Sundaram, On 4-product cordial graphs, Inter. J. Math. Archive, 7 (2012) 2809-2814.
- [2000] R. Ponraj, M. Sivakumar, and M. Sundaram, k-Total product cordial labeling of graphs, Appl. Appl. Math.: An Inter. J., 7 (2012) 708-716.
- [2001] R. Ponraj, M. Sivakumar, and M. Sundaram, On 3-total product cordial graphs, Int. Math. Forum, 7(29-32) (2012) 1537-1546.
- [2002] R. Ponraj, M. Sivakumar, and M. Sundaram, New families of 3-product cordial graphs, Inter. J. Math. Archive, 5 (2012) 1985-1990.

- [2003] R. Ponraj, M. Sivakumar, and M. Sundaram, 3-Total product cordial labeling of some sub divided graphs, *Inter. J. Math. Research*, 5 (2012) 517-526.
- [2004] R. Ponraj and M. Sivakumar, A note on k-Total product cordial graphs, Global J. Math. and Math. Sciences, 1 (2012) 37-44.
- [2005] R. Ponraj, M. Sivakumar, and M. Sundaram, Mean cordial labeling of graphs, Open J. Discrete Math., 2 (2012) 145-148. doi:10.4236/ojdm.2012.24029
- [2006] R. Ponraj and S. Somasundaram, Further results on mean graphs, Proc. SACOEF-ERENCE, National Level Conference, Dr. Sivanthi Aditanar College of Engineering, (2005) 443-448.
- [2007] R. Ponraj and S. Somasundaram, Mean labeling of graphs obtained by identifying two graphs, J. Discrete Math. Sci. Cryptogr., 11(2) (2008) 239-252.
- [2008] G. Pooranam, R. Vasuki, and S. Suganthi, New families of odd mean graphs, Internat. J. Math. Combin., 1 (2019) 134-145. https://www.researchgate. net/publication/338178781\_International\_Journal\_of\_Mathematical\_ Combinatorics\_vol\_12019
- [2009] P. Pradhan and K. Kamesh, On k-graceful labeling of some graphs, Math. Inform., 34(1-2) (2016) 9-17.
- [2010] P. Pradhan and A. Kumar, Graceful hairy cycles with pendent edges and some properties of cycles and cycle related graphs, *Bull. Calcutta Math. Soc.*, **103**(3) (2011) 233-246.
- [2011] T. R. Pradipta and A. N. M. Salman, Some cycle-supermagic labelings of the calendula graphs, J. Physics Conference Series, 948(1):012071 (2018) doi:10.1088/1742-6596/948/1/012071
- [2012] U. M. Prajapati and S. J. Gajjar, Some results on prime labeling Open J. Discrete Math., 4 (2014) 60-66. http://dx.doi.org/10.4236/ojdm.2014.43009
- [2013] U. M. Prajapati and S. J. Gajjar, Prime labeling of generalized Petersen graph, Internat. J. Math. Soft Comput., 5(1) (2015) 65-71. http://www.ijmsc.com/index. php/ijmsc/article/download/329/ijmsc-5-1-8
- [2014] U. M. Prajapati and S. J. Gajjar, Prime cordial labeling of generalized prism graph  $Y_{m,n}$ , Ultra Scientist, **27**(3)A (2015) 189-204.
- [2015] U. M. Prajapati and R. M. Gajjar, Cordial labeling for complement of some graphs, Math. Today, 30 (2015) 99-118.
- [2016] U. M. Prajapati and R. M. Gajjar, Cordial labeling in the context of duplication of some graph elements, *Internat. J. Math. Soft Comput.*, 6(2) (2016) 65-73.

- [2017] U. M. Prajapati and R. M. Gajjar, Cordiality in the context of duplication in crown related graphs, J. Math. Comput. Sci., 6(6) (2016) 1058-1073. http://scik.org/ index.php/jmcs/article/view/2896
- [2018] U. M. Prajapati and R. M. Gajjar, Cordiality in the context of duplication in flower related graphs, Internat. J. Math. Soft Comput., 7(2) (2017) 9-101. www. ijmsc.com/index.php/ijmsc/article/view/ijmsc-7-2-8/pdf
- [2019] U. M. Prajapati and R. M. Gajjar, Cordiality in the context of duplication in web and armed helm, *Internat. J. Math. Combin.*, 3 (2017) 90-105. fs.gallup.unm. edu/IJMC-3-2017.pdf
- [2020] U. M. Prajapati and R. M. Gajjar, Some labeling techniques of holiday star graph, Internat. J. Math. Trends Tech., 49(6) (2017) 339-344. www.ijmttjournal.org/ 2017/Volume-49/number-6/IJMTT-V49P554.pdf
- [2021] U. M. Prajapati and R. M. Gajjar, Some labeling techniques of Kusudama flower graph, Math. Today, 34(A) (2017) 211-217. http://mathematicstoday.org/ currentisue/V34(A)\_2018\_22.pdf
- [2022] . U. M. Prajapati and S. J. Gaijjar, Some labeling techniques of braided star graph, Internat. J. Math. Appl., 5(4-C) (2017) 361-369. http://ijmaa.in/v5n4-c/ 361-369.pdf
- [2023] U. M. Prajapati and R. M. Gajjar, Cordiality in the context of duplication in helm and closed helm, Internat. J. Math. Appl., 6(1-A) (2018) 139-145. http: //ijmaa.in/v6n1-a/139-145.pdf
- [2024] U. M. Prajapati and R. M. Gajjar, Personal communcation.
- [2025] U. M. Prajapati and N. B. Patel, Edge product cordial labeling of some cycle related graphs, Open J. Discrete Math., 6(6) (2016) 268-278. http://dx.doi. org/10.4236/ojdm.2016.64023
- [2026] U. M. Prajapati and N. B. Patel, Edge product cordial labeling of some graphs, J. Appl. Math. Comput. Mech., 18(1) (2019) 69-76. http://amcm.pcz.pl/?id= view&volume=18&issue=1&article=6
- [2027] U. M. Prajapati and K. K. Raval, Product cordial labeling in context of some one point union graphs, *IJRAR- Internat. Res. Analytical Reviews*, 5I(4) (2018) 490-495.
- [2028] U. M. Prajapati and K. K. Raval, Product cordial graph in the context of some graph operations on gear graph, Open J. Discrete Math., 6 (2016) 259-267.
- [2029] U. M. Prajapati and K. K. Raval, Some graphs in the context of product cordial labeling, *Math. Today*, **33** (2017) 58-66. http://mathematicstoday.org/ archives/V33\_2017\_9.pdf

- [2030] U. M. Prajapati and K. K. Raval, Vertex even and odd mean labeling in the context of some cyclic snake graphs, *Internat. J. Emerging Tech. Innovat, Res.*, 5(7) (2018) 319-324. http://www.jetir.org/papers/JETIRC006365.pdf
- [2031] U. M. Prajapati and P. D. Shah, Some edge product cordial graphs in the context of duplication of some graph elements, Open J. Discrete Math., 6 (2016) 248-258. http://dx.doi.org/10.4236/ojdm.2016.64021
- [2032] U. M. Prajapati and K. P. Shah, On odd prime labeling, *IJRAR- Internat. J. Res. Analy. Rev.*, 5(4) (2018) 2348-1269.
- [2033] U. M. Prajapati and K. P. Shah, Odd prime labeling of graphs with duplication of graph elements, JASC: J. Appl. Sci. Comp., textV(XII) (2018) 1281-1281.
- [2034] U. M. Prajapati and B. M. Suthar, Prime labeling in the context of duplication of graph elements in  $K_{2,n}$ , Internat. J. Math. Soft Comput., 7(1) (2017) 117-125.
- [2035] U. M. Prajapati and A. V. Vantiya, SD-prime cordial labeling of some snake graphs, JASC: J. Appl. Sci. Comput., VI(IV) (2019) 1857-1868.
- [2036] R. V. Prasad, M. Kannan, and R. Gopi, Some results on even vertex odd mean labeling graphs, *Internat. J. Mechanical Enginer. Tech.*, (IJMET), 9(2) (2018) 615-621.
- [2037] D. Prathap and J. Baskar Babujee, Magic and bimagic labeling for star graphs, Internat. Review Pure Appl. Math., 5 (2009) 67-76.
- [2038] B. Nirmala Gnanam Pricilla, A Study On New Classes Of Graphs In Variations Of Graceful Graph, Ph.D. Thesis, Bharath University, Chennai, 2008. http:// shodhganga.inflibnet.ac.in/handle/1w0603/33
- [2039] Q. Pu, M. Numan, S. I. Butt, M. Asif, F. Rafique, and Z. Shao, Cyclic super magic labelings for toroidal and Klein-bottle fullerene, *IEEE Access*, 7 (2019) 145928-145933, doi:10.1109/ACCESS.2019.2943368
- [2040] J.-F. Puget, Breaking symmetries in all different problems, *in* Proceedings of Sym-Con04, the 4th International Workshop on Symmetry in Constraints, 2004, 6 pp.
- [2041] N. Punnim and N. Pabhapote, On graceful graphs: cycles with a  $P_k$ -chord,  $k \ge 4$ , Ars Combin., **23A** (1987) 225-228.
- [2042] P. R. L. Pushpam and A. Saibulla, On super (a, d)-edge antimagic total labeling of certain families of graphs, *Discuss. Math. Graph Theory* **32**(3) (2012) 535-543.
- [2043] A. Pyatkin, New formula for the sum number for the complete bipartite graphs, Discrete Math., 239 (2001) 155-160.

- [2044] J. Przybylo, Irregularity strength of regular graphs, *Electron. J. Combin.*, 15(1) (2008), P82, 10 pp.
- [2045] J. Przybylo, Linear bound on the irregularity strength and the total vertex irregularity strength of graphs, SIAM J. Discrete Math., 23(1) (2008/09) 511-516.
- [2046] J. Przybylo, Distant irregularity strength of graphs, Discrete Math., 313(24) (2013) 2875-2880.
- [2047] J. Quadras and S. Teresa Arockiamary, Total edge irregularity strength of hexagonal networks, J. Combin. Math. Combin. Comput., 92 (2015) 131-138.
- [2048] J. Qian, On some conjectures and problems in graceful labelings graphs, unpublished.
- [2049] P. Ragukumara and G. Sethuraman, Binomial trees are graceful, AKCE Internat. J. Graphs Combin., 1711 632-636. DOI:10.1016/j.akcej.2018.06.005
- [2050] A. Raheem, On super (a, d)-edge-antimagic total labeling of a subdivided stars, Ars Combin., **136** (2018) 169-179.
- [2051] A. Raheem, M. Javaid, and A. Q. Baig, On super edge-antimagicness of subdivided stars, *Discuss. Math. Graph Th.*, 4 (2015) 663-673.
- [2052] A. Raheem and A. Q. Baig, Super edge-antimagicness of subdivided star trees, Discuss. Math. Graph Th., 1 (2016) 27-33.
- [2053] M. T. Rahim, K. Ali, Kashif and I. Javaid, On antimagic total labeling of some families of graphs, Ars Combin., 95 (2010) 225-234.
- [2054] M. T. Rahim, M. Farooq, M. Ali, and S. Jan, Multi-level distance labelings for generalized gear graphs Internat. J. Math. and Soft Comput., 2(1) (2012) 57-63.
- [2055] M. T. Rahim and Slamin, Vertex-magic total labelings of the union of suns, Ars Combin., 103 (2012) 305-310.
- [2056] M. T. Rahim and Slamin, Most wheel related graphs are not vertex magic, Util. Math., 77 (2008) 193-199.
- [2057] M. T. Rahim and Slamin, Vertex-magic total labeling of the union of suns, Ars Combin., 103 (2012) 305-310.
- [2058] M. T. Rahim, I. Tomescu, and Slamin, On vertex-magic total labeling of some wheel related graphs, Util. Math., 73 (2007) 97-104.
- [2059] S. Rahmawati, K. A. Sugeng, D.R. Silaban, M. Miller, and M. Bača, Construction of new larger (a, d)-edge antimagic vertex graphs by using adjacency matrices, *Austral. J. Combin.*, 56, (2013) 257-272.

- [2060] C. D. Raj, C. Jayasekaran, and S. S. Sandhya, Few families of harmonic mean graphs, J. Combin. Math. Combin. Comput., 96 (2016) 235-243.
- [2061] P. L. R. Raj and S. Koilraj, Cordial labeling for the splitting graph of some standard graphs, Internat. J. Math. Soft Comput., 1(1)(2011) 105-114. http: //oaji.net/articles/2015/296-1433770428.pdf
- [2062] P. L. R. Raj and R. Valli, Some new families of divisor cordial graphs, Internat. J. Math. Trends Tech., 7 (2) (2014) 94-102. http://ijmttjournal.org/Volume-7/ number-2/IJMTT-V7P512.pdf
- [2063] P. L. R. Raj and K. Vivek, Some new families of analytic mean graphs, Internat. J. Anal. Exper. Modal Anal., VI(X) (2019) 358-361. https://www.researchgate. net/publication/336847676\_Some\_New\_Families\_of\_Analytic\_Mean\_Graphs
- [2064] I. Rajasingh and P. R. L. Pushpam, Strongly harmonious labeling of helms, personal communication.
- [2065] I. Rajasingh and P. R. L. Pushpam, On graceful and harmonious labelings of t copies of  $K_{m,n}$  and other special graphs, personal communication.
- [2066] I. Rajasingh, B. Rajan, and V. Annamma, On total vertex irregularity strength of triangle related graphs, Annals Pure Appl. Math., 1(2) (2012) 108-116. www.researchmathsci.org
- [2067] I. Rajasingh and V. Annamma, Total vertex irregularity strength of 1-fault tolerant hamiltonian graphs, *Math. Comput. Sci.*, **9**(2) (2015) 151-160.
- [2068] I. Rajasingh, B. Rajan, and S. Teresa Arockiamary, Irregular total labeling of butterfly and benes networks, In: Abd Manaf A., Sahibuddin S., Ahmad R., Mohd Daud S., El-Qawasmeh E. (eds), Informatics Eng. Infor. Sci., ICIEIS 2011, Comm. Comput. Information Sci., 253 (2011) Springer, Berlin, Heidelberg. https://doi. org/10.1007/978-3-642-25462-8\_25
- [2069] I. Rajasingh and S. Teresa Arockiamary, Total edge irregularity strength of honeycomb networks, *Internat. J. Comput. and Applied Math.*, 8(3) (2013) 213-220.
- [2070] I. Rajasingh and S. Teresa Arockiamary, Total edge irregularity strength of series parallel graphs, *Internat. J. Pure Appl. Math.*, **99**(1) (2015) 11-21.
- [2071] T. J. Rajesh Kumar T.J. and T. K. Mathew Varkey, A note on total neighborhood labelings, Internat. J. Pure and Appl. Math., 118(4) (2018) 1007-1013.
- [2072] T. J. Rajesh Kumar and T. K. Mathew Varkey, Gaussian neighborhood-prime labeling of some trees and other graphs, Annals Pure and Appl. Math., 16(1) (2018) 133-140. http://www.researchmathsci.org/APAMart/apam-v16n1-15.pdf

- [2073] T. J. Rajesh Kumar and T. K. Mathew Varkey, A note on total neighborhood labelings, Internat. J. Pure and Appl. Math., 118(4) (2018) 1007-1013. https: //ijpam.eu/contents/2018-118-4/14/14.pdf
- [2074] V. Ramachandran and C. Sekar, One modulo N gracefulness of acyclic graphs, Ultra Scientist, 25(3)A (2013) 417-424.
- [2075] V. Ramachandran and C. Sekar, One modulo N gracefulness of splitting graphs and subdivision of double triangle graphs, *Scientia* **125** (2014) 1-9.
- [2076] V. Ramachandran and C. Sekar, One modulo N gracefulness of arbitrary supersubdivisions of graphs, Internat. J. Math. Combin., 2 (2014) 36-46.
- [2077] V. Ramachandran and C. Sekar, Gracefulness and one modulo N gracefulness of  $L_n \otimes S_m$ , Scientia Magna, **10**(3) (2014) 66-76.
- [2078] V. Ramachandran and C. Sekar, One modulo N gracefulness of regular bamboo tree and coconut tree, Internat. J. Appl. Graph Theory Wireless ad hoc Networks Sensor Networks, (GRAPH-HOC) 6(2) (2014) 1-10. doi:10.5121/jgraphoc.2014.6201
- [2079] V. Ramachandran and C. Sekar, Graceful labelling of supersubdivision of ladder, Internat. J. Math. Appl., 2(2) (2014) 29-36.
- [2080] V. Ramachandran and C. Sekar, One modulo N gracefulness of crowns, armed crowns and chain of even cycles, Ars Combin., 138 (2018) 143-159.
- [2081] K. Ramanjaneyulu, V. C. Venkaiah, and K. Kothapalli, Cordial labelings of a class of planar graphs AKCE J. Graphs Combin., 6 (2009) 171-181.
- [2082] J. L. Ramírez-Alfonsín, Gracefulness of replicated paths and cycles, Ars Combin., 53 (1999) 257-269.
- [2083] R. Ramdani and A. N. M. Salman, On the total irregularity strength of some Cartesian product graphs, AKCE Int. J. Graphs Comb., 10(2) (2013) 199-209.
- [2084] R. Ramdani, A. N. M. Salman, and H. Assiyatun, On the total irregularity strength of regular graphs, J. Math. Fundam. Sci., 47(3) (2015) 281-295.
- [2085] R. Ramdani, A. N. M. Salman, H. Assiyatun, A. Semaničová-Feňovčíková, and Bača, Total irregularity strength of three families of graphs, *Math. Comput. Sci.*, 9(2) (2015) 229-237.
- [2086] R. Ramdani, A, N. M. Salman, A. Assiyatum, and A. Semaničová-Feňovčíková, On the total irrgularity strenght of disjoint union of graphs, *Math. Reports*, 18(68), 4 (2016) 469-482.

- [2087] V. Ramachandran and C. Sekar, One modulo N gracefulness of n-polygonal snakes,  $C_n^{(t)}$  and  $P_{a,b}$ , Internat. J. Engineering Res. & Tech., (IJERT) **2**(10) (2013) 3514-3529.
- [2088] D. Ramya and P. Jeyanthi, Mean labeling of some graphs, SUT J. Math., 47(2) (2011) 129-141.
- [2089] D. Ramya and P. Jeyanthi, New super mean graphs, J. Disc. Math. Sci. Cryptogr., 17(5-6) (2014) 395-418.
- [2090] D. Ramya, R. Kalaiyarasi and P. Jeyanthi, Skolem odd difference mean graphs, J. Algor. and Comput., 45 (2014) 1-20.
- [2091] D. Ramya, R. Ponraj, and P. Jeyanthi, Super mean labeling of graphs, Ars Combin., 112 (2013) 65-72.
- [2092] D. Ramya, M. Selvi, and P. Jeyanthi, Odd mean labeling of some trees, Util. Math., 106 (2018) 65-78.
- [2093] D. Ramya, M. Selvi and P. Jeyanthi, Odd mean labeling of some graphs, preprint.
- [2094] Ramya1 and B. Sooryanarayana, Radio multiplicative number of certain classes of transformation graphs, Int. J. Math. Appl., 7(4)(2019) 127-148. http://ijmaa. in/v7n4/127-148.pdf
- [2095] N. Sridharan and R. Umarani,  $E_k$ -cordial labellings of graphs, *Elixir Appl. Math.*, **38** (2011) 4564-4567.
- [2096] S. B. Rao, Sigma graphs: A survey, in Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 135-140.
- [2097] S. B. Rao and U. K. Sahoo, Embeddings in Eulerian graceful graphs, Australasian J. Comb., 62(1) (2015) 128-139.
- [2098] S. B. Rao, T. Singh, and V. Parameswaran, Some sigma labelled graphs: I, Graphs, Combinatorics, Algorithms and Applications, S. Armugam, B. D. Acharya and S. B. Rao, eds., (2004), 125-133, Narosa Publishing House, New Delhi.
- [2099] S. N. Rao, Prime labelling, In: R. C. Bose Centenary Symposium on Discrete Math. and Applications, Kolkata (2002).
- [2100] N. B. Rathod and K. K. Kanani, Some path related 4-cordial graphs, Internat. J. Math. and Soft Computing, 5(2) (2015) 21-27.
- [2101] N. B. Rathod and K. K. Kanani, V<sub>4</sub>-cordial labeling of quadrilateral snakes, Internat. J. Emerging Tech. Appl. Eng., Tech., Sci., 9 (2016) 45-51.

- [2102] N. B. Rathod and K. K. Kanani, On V<sub>4</sub>-cordial labeling of graphs, British J. Math. Comput Sci., 13(4) (2016) 1-15.
- [2103] N. B. Rathod and K. K. Kanani, 4-cordial labeling of star, book and fan related graphs, *Proceedings of 8th National Level Science Symposium*, Rajkot, India, 2 (2015) 38-42.
- [2104] N. B. Rathod and K. K. Kanani, 5-cordial labeling of some standard graphs, Proceedings of 8th National Level Science Symposium, Rajkot, India, 2 (2015) 43-48.
- [2105] N. B. Rathod and K. K. Kanani, 4-cordiality of some new path related graphs, Internat. J. Math. Trends Techno. (IJMTT) 34(1) (2016) 5-8.
- [2106] N. B. Rathod and K. K. Kanani, V<sub>4</sub>-cordial labeling of some alternate triangular snake related graphs, 9th National Level Science Symposium. February 14, 2016, Organized by Christ College, Rajkot, Sponsored by GUJCOST, Gandhinagar, Mathematics & Statistics, 3 (2016) 188-196.
- [2107] J. Ratilal and P. Parmar, H<sub>k</sub>-Cordial labeling of some snake graphs, J. Xidian Univ., 14(3) (2020) 844-858. https://www.researchgate.net/publication/ 340115606\_HK\_Cordial\_labeling\_of\_Some\_Snake\_Graphs/references
- [2108] M. A. Razzaq, S. T. Rizvi, and K. Ali, *H*-groupmagic total labelings of families of fan graphs, *Util. Math.*, **108** (2018) 213219.
- [2109] T. A. Redl, Graceful graphs and graceful labelings: two mathematical programming formulations and some other new results, *Congressus Numer.*, 164 (2003) 17-32.
- [2110] M. Reid, personal communication.
- [2111] M. Regees, Super edge trimagic total labeling of some star type graphs, J. Discrete Math. Sci. Cryptogr., 20(3) (2017) 747-754.
- [2112] M. Regees and C. Jayasekaran, Super edge trimagic total labeling of generalized prism and web graphs, J. Discrete Math. Sci. Cryptogr., 19(1) (2016) 81-92.
- [2113] J. Renuka, P. Balaganesan, P. Selvaraju, On harmonious labeling, Internat. J. Advance in Math. Math. Sci., 1(2) (2012) 65-70.
- [2114] N. Revathi, Mean labeling of some graphs, Internat. J. Sci. Res. (IJSR), 4(10) (2015) 1921-1922. https://www.ijsr.net/archive/v4i10/23101501.pdf
- [2115] A. Riasat, S. Kanwal, and S. Javed, On odd-graceful labeling of disjoint union of graphs, Util. Math., 101 (2016) 189-214.
- [2116] P. Richter, E. Leven, A. Tran, B. Ek, J. Jacob, and D. A. Narayan, Rank numbers for bent ladders, *Discuss. Math. Graph Th.* 34(2) (2014) 309-329.

- [2117] G. Ringel, Problem 25, in Theory of Graphs and its Applications, Proc. Symposium Smolenice 1963, Prague (1964) 162.
- [2118] G. Ringel and A. Llado, Another tree conjecture, Bull. Inst. Combin. Appl., 18 (1996) 83-85.
- [2119] G. Ringel, A. Llado, and O. Serra, Decomposition of complete bipartite graphs into trees, DMAT Research Report 11/96, Univ. Politecnica de Catalunya.
- [2120] A. Riskin, Cordial deficiency, Bull. Malays. Math. Sci. Soc., 30 (2007) 201-204. arXiv:math/0610760v1 [math.CO] 25 Oct 2006.
- [2121] A. Riskin, On the cordial deficiency of complete multipartite graphs, arXiv:0706.2431v1 [math. CO] 16 June 2007.
- [2122] A. Riskin,  $Z_2^2$ -cordiality of  $K_n$  and  $K_{m,n}$ , arXiv:0709.0290v1 [math.CO] 3 Sep. 2007.
- [2123] A. Riskin and G. Weidman, On edge graceful labelings of disjoint unions of 2r-regular edge graceful graphs, Bull. Inst. Combin. Appl., 52 (2008) 45-50. arXiv:math/0605234v1.
- [2124] S. T. R. Rizvi, K. Ali, and M. Hussain, Cycle-supermagic labelings of the disjoint union of graphs, Util. Math., 104 (2017) 215-226. arXiv:1506.06087.
- [2125] S. T. R. Rizvi, K. Ali, N. Iqbal, and A. Gulraze, Super edge-magicness of stars like graphs, Ars Combin., 136 (2018) 405-417.
- [2126] S. T. R. Rizvi, M. Khalid, K. Ali, M. Miller, and J. Ryan, On cycle-supermagicness of subdivided graphs, *Bull. Aust. Math. Soc.*, 92(1) (2015)11-18.
- [2127] L. Robertson and B. Small, On Newman's conjecture and prime trees, Integers: The Elect. J. Combin. Number Theory, 9 (2009) A10 112-128.
- [2128] E. Robeva, An Extensive Survey of Graceful Trees, Undergraduate Honors Thesis, Stanford University, 2011.
- [2129] Y. Roditty and T. Bachar, A note on edge-magic cycles, Bull. Inst. Combin. Appl., 29 (2000) 94-96.
- [2130] D. G. Rogers, A graceful algorithm, Southeast Asian Bull. Math., 2 (1978) 42-44.
- [2131] A. H. Rokad, 3-Equitable labeling in context of ring sum of graphs Research & Reviews: Discrete Math. Structures: Open Journal Systems, 2(3) (2015) 6272-6275.
- [2132] A. H. Rokad, Product cordial labeling of double wheel and double fan related graphs, Kragujevac J. Math., 43(1) (2019) 7-13. http://elib.mi.sanu.ac.rs/ files/journals/kjm/55/kjmn55p7-13.pdf

- [2133] A. H. Rokad and K. M. Patadiya, Cordial labeing of some graphs, Aryabhatta J. Math. Informatics, 09 (01) (2017) 589-597.
- [2134] Q. Rong and D. Xiong, The gracefulness of graph  $P_{2r,b}$ , J. Systems Sci. Math. Sci., **30**(5) (2010) 703-709.
- [2135] D. Ropp, Graceful labelings of cycles and prisms with pendant points, Congress. Numer., 75 (1990) 218-234.
- [2136] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.
- [2137] A. Rosa, Labelling snakes, Ars Combin., 3 (1977) 67-74.
- [2138] A. Rosa, Cyclic Steiner triple systems and labelings of triangular cacti, Scientia, 1 (1988) 87-95.
- [2139] A. Rosa and J. Siráň, Bipartite labelings of trees and the gracesize, J. Graph Theory, 19 (1995) 201-215.
- [2140] T. Rose and H.-H. Su, An algorithm to calculate the balance index set of a graph, Proceedings of the Forty-Third Southeastern International Conference on Combinatorics, Graph Theory and Computing. *Congr. Numer.*, **213** (2012) 107-121.
- [2141] M. Roswitha, E.T. Baskoro, T. K. Maryati, N. A. Kurdhi, and I. Susanti, Further results on cycle-supermagic labeling, AKCE Int. J. Graphs Comb., 10(2) (2013) 211-220.
- [2142] M. Roswitha and E.T. Baskoro, *H*-magic covering of some classes of graphs, *AIP Conf. Proc. on ICREM5*, ITB Bandung, **1450** (2012) 135-138.
- [2143] M. Roswitha, T. S. Martini, and S. A. Nugroho, (a, d)-H-Anti magic total labeling on double cones graph, ICMETA 2018 IOP Conf. Series: J. Physics: Conf. Series 1306 (2019) 012012 1-5.
- [2144] I. Rosyida and D. Indriati, On total edge irregularity strength of some cactus chain graphs with pendant vertices, *IOP Conf. Series: J. Physics: Conf. Series* 1211 (2019) 012016. doi:10.1088/1742-6596/1211/1/012016
- [2145] I. Rosyida and D. Indriati, Computing total edge irregularity strength of some n-uniform cactus chain graphs and related chain graphs, Indon. J. Combin., 4(1) (2020) 53-75. https://www.researchgate.net/publication/ 342523731\_Computing\_total\_edge\_irregularity\_strength\_of\_some\_ n-uniform\_cactus\_chain\_graphs\_and\_related\_chain\_graphs/comments# fullTextFileContent

- [2146] P. L. Rozario Raj and M. M. Sheriff, Some more results on neighbourhoodprime labeling of graphs, *Internat. J. Anal. Exper. Modal Anal.*, XI(X) Iss (2019) 38-43. https://www.researchgate.net/publication/336847767\_Some\_ More\_Results\_on\_Neighbourhood-Prime\_Labeling\_of\_Graphs
- [2147] J. Ryan, Exclusive sum labelings of graphs: A survey, AKCE J. Graphs Combin., 6 (2009) 113-126.
- [2148] J. Ryan, O. Phanalasy, M. Miller and L. Rylands, On antimagic labeling for generalized web and flower graphs, In C. S. Iliopoulos and W. F. Smyth, editors, *Proceed. IWOCA10, Lecture Notes in Computer Science*, 6460 (2011) 303-313.
- [2149] J. Ryan, O. Phanalasy, L. Rylands and M. Miller, On antimagic labeling for generalized web and flower graphs, Combinatorial algorithms, 303-313, Lecture Notes in Comput. Sci., 6460, Springer, Heidelberg, 2011.
- [2150] L. Rylands, O. Phanalasy, J. Ryan and M. Miller, Construction for antimagic generalized web graphs, Int. J. Graphs and Combinatorics, 8 (2011) 141-149.
- [2151] L. Saha and P. Panigrahi, Antipodal number of some powers of cycles, Discrete Math., 312(9) (2012) 1550-1557.
- [2152] L. Saha and P. Panigrahi, On the radio number of toroidal grids, Australasian J. Combin., 55 (2013) 273-288.
- [2153] L. Saha and P. Panigrahi, On the radio number of square of graphs, The Eighth International Workshop on Graph Labelings (IWOGL 2014), 205-212, *Electron. Notes Discrete Math.*, 48, Elsevier Sci. B. V., Amsterdam, 2015. 10.
- [2154] M. Salarrezaei, Mathematical programming models for graceful labeling problem of 2-regular graphs. Master of Science thesis, Sharif University of Technology (2012).
- [2155] E. Salehi, Zero-sum magic graphs and their null sets, Ars Combin., 82 (2007) 41-53.
- [2156] E. Salehi, PC-labelings of a graphs and its PC-sets, Bull. Inst. Combin. Appl., 58 (2010) 112-121.
- [2157] E. Salehi and D. Bayot, The friendly index set of  $P_m \times P_n$ , Util. Math., 81 (2010) 121-130.
- [2158] E. Salehi and P. Bennett, On integer-magic spectra of caterpillars, J. Combin. Math. Combin. Comput., 61 (2007) 65-71.
- [2159] E. Salehi, S. Churchman, T. Hill, and J. Jordan, Product cordial sets of trees, Congr. Numer., 220 (2014) 183-193.

- [2160] E. Salehi and S. De, On a conjecture concerning the friendly index sets of trees, Ars Combin., **90** (2009) 371-381.
- [2161] E. Salehi and S.-M. Lee, On friendly index sets of trees, Cong. Numer., 178 (2006) 173-183.
- [2162] E. Salehi and S. M. Lee, Integer-magic spectra of functional extensions of graphs, J. Combin. Math. Combin. Comput., 64 (2008) 127-139.
- [2163] E. Salehi, S. M. Lee, and M. Khatirinejad, IC-colorings and IC-indices of graphs, Discrete Math., 299 (2005) 297-310.
- [2164] E. Salehi and D. McGinn, On friendly index set of hypercube  $Q_n$ , J. Combin. Math. Combin. Comput., **106** (2018) 109-124.
- [2165] E. Salehi and Y. Mukhin, Product cordial sets of long grids, Ars Combin., 107 (2012) 339-351.
- [2166] E. Salehi, Y. Mukhin, and S. W. Saputro, Cordial sets of hypercubes, Bull. Instit. Combin. Appl., 75 (2015) 95-106.
- [2167] A. N. M. Salman and T. K. Maryati, On graph-(super)magic labeling of a pathamalgamation of isomorphic graphs, in: The Proceedings of the 6th IMT-GT International Conference on Mathematics, Statistics and its Applications, 2010, 228-233.
- [2168] A. N. M. Salman, A. A. G. Ngurah, and N. Izzati, On (super) edge-magic total labelings of a subdivision of a star  $S_n$ , Util. Math., **81** (2010) 275-284.
- [2169] H. Salmasian, A result on prime labelings of trees, Bull. Inst. Combin. Appl., 28 (2000) 36-38.
- [2170] R. K. Sampath, G. Narasimhan, and K. M. Nagaraja, Heron mean labelling of graphs, *Internat. J. Recent Sci. Res.*, 8(9) (2017) 19808-19811. http://dx.doi. org/10.24327/ijrsr.2017.0809.0763
- [2171] L. Šándorová and M.Trenklér, On a generalization of magic graphs, Colloquia Math. Societatis J.Bolyai, 52 Combinatorics, North-Holland, Amsterdam 1988, 447-452.
- [2172] Y. Sanaka, On  $\gamma$  labelings of complete bipartite graphs, Ars Combin., **111** (2013) 251-256.
- [2173] S. Saduakdee and K. Khemmani, The unique  $\gamma$ -min labelings of graphs, *Thai J.* Math., 2018, Special issue, 187-203.
- [2174] S. S. Sandhya, E. R. Merly, and B. Shiny, Subdivision of super geometric mean labeling for quadrilateral snake graphs, *Internat. J. Math. Trends Tech.*, 24(1) (2015) 1-16.

- [2175] S. S. Sandhya, E. R. Merly, and B. Shiny, Super geometric mean labeling on double quadrilateral snake graphs, *Asian Pacific J. Res.*, **1** (XXI) (2015) 128-135.
- [2176] S. S. Sandhya, E. R. Merly, and B. Shiny, New classes of super geometric mean graphs, J. Discrete Math. Sci. Cryptogr., 19(1) (2016) 117-140.
- [2177] S. S. Sandhya, S. Somasundaram, and R. Ponraj, Some more results on harmonic mean graphs, J. Math. Res., 4(1) (2012) 21-29.
- [2178] S. S. Sandhya, S. Somasundaram, and R. Ponraj, Harmonic mean labeling of some cycle related graphs. Int. J. Math. Anal., (Ruse) 6(37-40) (2012) 1997-2005. http://www.m-hikari.com/ijma/ijma-2012/ijma-37-40-2012/ sandhyaIJMA37-40-2012.pdf
- [2179] S. S. Sandhya, S. Somasundaram, R. Ponraj, Some results on harmonic mean graphs, Int. J. Contemp. Math. Sci., 7 (2012) 197-208.
- [2180] I. P. Sandy, A. Rizal, E. N. Manurung, and K. A. Sugeng, Alternative construction of graceful symmetric trees, J. Phys.: Conf. Ser., 1008 (2018) 012031
- [2181] K. Sankar and G. Sethuramam, Graceful and cordial labeling of subdivision of graphs, *Electronic Notes Discrete Math.*, 53 (2016) 123-131.
- [2182] G. Santhosh, Sequential coronations of graphs, Natl. Acad. Sci. Lett., 28 (2005) 269-270.
- [2183] G. Santhosh and G. Singh, A note on the super edge-magic deficiency of graphs, Far East J. Appl. Math. 24 (2006) 233-242.
- [2184] G. Santhosh and G. Singh, On super magic strength of graphs, Far East J. Appl. Math., 18 (2005) 199-207.
- [2185] G. Santhosh, On weak magic graphs, Ars Combin., **92** (2009) 245-254.
- [2186] G. Santhosh and G. Singh, A note on subdivision of integral sum graphs, unpublished.
- [2187] G. Santhosh and G. Singh, On divisor graphs, unpublished.
- [2188] G. A. Saputri, K. A. Sugeng, and D. Fronček, The odd harmonious labeling of dumbbell and generalized prism graphs, AKCE Int. J. Graphs Comb., 10(2) (2013) 221-228.
- [2189] P. B. Sarasija and R. Binthiya, Even harmonious graphs with applications, Internat. J. Comput. Sci. Infor. Security, 9 (2011) 161-163.

- Τ. [2190] G. Sathiamoorthy, С. Natarajan, S. K. Avyaswamy, and Ν. Janakiraman, Graceful labeling of graph splitting of a caterpillar, Internat. J. Mech. Engin. Tech., (IJMET) 8(11) (2017)663-667. https: //www.iaeme.com/MasterAdmin/uploadfolder/IJMET\_08\_11\_067/IJMET\_ 08\_11\_067.pdf
- [2191] G. Sathiamoorthy, Graceful labeling of nova cloud, Internat. J. Mech. Engin. Tech., (IJMET) 8(11) (2017) 1-9. http://www.iaeme.com/MasterAdmin/ UploadFolder/IJMET\_08\_11\_001/IJMET\_08\_11\_001.pdf
- [2192] K. Schaffer and S. M. Lee, Edge-graceful and edge-magic labelings of Cartesian products of graphs, *Congr. Numer.*, **141** (1999) 119-134.
- [2193] S. A. Schluchter, J. Z. Schroeder, K. Cokus, R. Ellingson, H. Harris, E. Rarity, and T. Wilson, Prime labelings of generalized Petersen graphs, *Involve*, 10(1) (2017) 109-124.
- [2194] S.A. Schluchter, and T. Wilson, Prime labelings of bipartite generalized Petersen graphs and other prime cubic bipartite graphs, *Congr. Numer.*, **226** (2016) 227241.
- [2195] J. Z. Schroeder, Every cubic bipartite graph has a prime labeling except  $K_{3,3}$ , Graphs Combin., **35**(1) (2019) 119-140.
- [2196] R. Sebastian and K. A. Germina, Square sum labeling of class of planar graphs, *Proyecciones*, 34(1) (2015) 55-68.
- [2197] J. Sedláček, Problem 27, in Theory of Graphs and its Applications, Proc. Symposium Smolenice, June, (1963) 163-167.
- [2198] J. Sedláček, On magic graphs, Math. Slov., 26 (1976) 329-335.
- [2199] M. Seenivasan and A. Lourdusamy, Absolutely harmonious labeling of graphs, Internat. J. Math. Combin., 2 (2011) 40-51.
- [2200] C. Sekar, Studies in Graph Theory, Ph. D. Thesis, Madurai Kamaraj University, 2002.
- [2201] C. Sekar and V. Ramachandren, Graceful labelling of arbitrary supersubdivision of disconnected graph, Ultra Scientist, 25(2)A (2013) 315-318.
- [2202] P. Selvaraju, New classes of graphs with  $\alpha$ -valuation, harmonious and cordial labelings, Ph. D. Thesis, Anna University, 2001. Madurai Kamaraj University, 2002.
- [2203] P. Selvagopal and P. Jeyanthi, On  $C_k$ -supermagic graphs, Inter. J. Math. Comput. Sci., **3** (2008) 25-30.
- [2204] P. Selvagopal, P. Jeyanthi, N. P. T. Muthuraja, and A. Semaničová-Feňovčiková Super (a, d)-star-antimagic graphs, *Hacet. J. Math. Stat.*, 48 (2019)(3) 700-710.

- [2205] P. Selvaraju and C. Moha, Prime valuation of union of cycles and cycle related graphs, Internat. J. Advan. Res. Comput. Sci. Software Engin., 7(6) (2017) 839-842.
- [2206] P. Selvaraju and G. Sethurman, Decomposition of complete graphs and complete bipartitie graphs into copies of  $P_n^3$  or  $S_2(P_n^3)$  and harmonious labeling of  $K_2 + P_n$ , J. Indones. Math. Soc., Special Edition (2011) 109-122.
- [2207] T. M. Selvarajan and R. Subramoniam, Prime graceful labeling, Internat. J. Engin. Tech., 7(4.36) (2018) 750-752.
- [2208] M. Selvi, D. Ramya and P. Jeyanthi, Triangular difference mean graphs, Internat. J. Math. Combin., 2 (202)) 109-117.
- [2209] M. Selvi, D. Ramya and P. Jeyanthi, Skolem difference mean graphs, Proyectiones J. Math., 34(3) (2015) 243-254.
- [2210] A. Semaničová, On magic and supermagic circulant graphs, Discrete Math., 306 (2006) 2263-2269.
- [2211] A. Semaničová-Feňovčíková, M. Bača, and M. Lascsáková, Two constructions of H-antimagic graphs AKCE Internat. J. Graphs Combin., 14(1) (2017) 42-47.
- [2212] A. Semaničová-Feňovčíková, M. Bača, M. Lascsáková, M. Miller, and J. Ryan, Wheels are cycle-antimagic, *Electronic Notes Disc. Math.*, 48 (2015) 11-18. The Eighth International Workshop on Graph Labelings (IWOGL 2014)
- [2213] M. F. Semenyuta, (a, d)-Distance antimagic labeling of some types of graphs, Cybernetics Systems Anal., 52 Issue 6 (2016) 950-955.
- [2214] M. F. Semenyuta, On the (a, d)-distance antimagic and 1-vertex bimagic vertex labelling of some types of graphs, *Kibernet. Sistem. Anal.*, **54** (2018)(2) 134-141; translation in *Cybernet. Systems Anal.*, **54** (2018) (2) 295-301.
- [2215] M. F. Semenyuta, S. M. Nedilko, and V. M. Nedilko, Bimagic vertex labelings, *Cybernet. Systems Anal.*, 54(5) (2018) 771-778.
- [2216] A. Sen, H. Deng, and S. Guha, On a graph partition problem with application to VLSI Layout, *Inform. Process. Lett.*, 43 (1992) 87-94.
- [2217] R. Senthil Amutha and N. Murugesan, Characterization of connected vertex magic total labeling graphs in topological ideals, J. Inform. Eng. Appl., 1(1) (2011) 22-25.
- [2218] A. Seoud, G. M. Abd El Hamid, and M. S. Abo Shady, Indexable and strongly indexable graphs, *Proceed. Pakistan Acad. Sci.*, 49(2) (2012) 139-144.
- [2219] M. A. Seoud and M. E. Abdel-Aal, On odd graceful graphs, Ars Comb., 108 (2013) 161-185.

- [2220] M. A. Seoud and A. E. I. Abdel Maqsoud, On 3-equitable and magic labelings, Proc. Math. Phys. Soc. Egypt, 7 (2000) 67-76.
- [2221] M. A. Seoud and A. E. I. Abdel Maqsoud, On cordial and balanced labelings of graphs, J. Egyptian Math. Soc., 7 (1999) 127-135.
- [2222] M. A. Seoud, A. E. I. Abdel Maqsoud and Y. I. Aldiban, New classes of graphs with and without 1-vertex magic vertex labeling, *Proc. Pakistan Acad. Sci.*, 46 (2009) 159-174.
- [2223] M. A.Seoud, A. E. I. Abdel Maqsoud and J. Sheehan, Harmonious graphs, Util. Math., 47 (1995) 225-233.
- [2224] M. A. Seoud, A. E. I. Abdel Maqsoud, and J. Sheehan, Gracefulness of the union of cycles and paths, Ars Combin., 54 (2000) 283-292.
- [2225] M. A. Seoud and M. N. Al-Harere, Some notes on non combination graphs, Applied Math. Sciences, 6(131) (2012) 6515-6520.
- [2226] M. A. Seoud and M. N. Al-Harere, Some notes on combination graphs, Ars Combin., to appear
- [2227] M. A. Seoud and M. N. Al-Harere, Further results on square sum graphs, Nat. Acad. Sci. Lett., 37(5) (2014) 473-475.
- [2228] M. A. Seoud and M. A. Anwar, On combination and permutation graphs, Util. Math., 98 243-255. https://www.researchgate.net/publication/295579013\_ On\_Combination\_and\_Permutation\_Graphs
- [2229] M. A. Seoud and M. A. Anwar, Some families of combination and permutation graphs, Ars Combin., 127 (2016) 435-443. https://www.researchgate.net/ publication/307928064\_Some\_Families\_of\_Combination\_and\_Permutation\_ Graphs
- [2230] M. A. Seoud, A. T. Diab, and E. A. Elsahawi, On strongly-C harmonious, relatively prime, odd graceful and cordial graphs, Proc. Math. Phys. Soc. Egypt, no. 73, (1998) 33-55.
- [2231] A. Seoud, A. El Sonbaty, and S. S. Abd El Rehim, Some methods of labeling graphs, J. Egyptian Math. Soc., 18(2) (2010) 199-221.
- [2232] M. A. Seoud and E. A. Elsahawi, On almost graceful, felicitous and elegant graphs, J. Egyptian Math. Soc., 7 (1999) 137-149.
- [2233] M. A. Seoud and E. A. El Sahawi, On variations of graceful labelings, Ars Combin., 87 (2008) 127-138.
- [2234] M. A. Seoud and E. A. El Sahawi, On strongly *c*-elegant graphs, preprint.

- [2235] M. A. Seoud, A. El Sonbaty, and A. E. A. Mahran, On divisor graphs, J. Egyptian Math. Soc., 18(2) (2010) 187-198.
- [2236] M. A. Seoud A. El Sonbaty, and A. E. A. Mahran, Primality of some graphs, Ars Combin., 112 (2013) 459-469.
- [2237] M. A. Seoud and E. F. Helmi, On product cordial graphs, Ars Combin., 101 (2011) 519-529.
- [2238] M. A. Seoud and E. F. Helmi, Some α-graphs and odd graceful graphs, Ars Comb., 101 (2011) 385-404.
- [2239] M. A. Seoud and E. F. Helmi, On difference graphs, J. Combin. Math. and Combin. Comput. 76 (2011) 189-199.
- [2240] M. A. Seoud and H. Jaber, On 1-sequentially additive graphs, Ars Combin., to appear.
- [2241] M. A. Seoud and H. Jaber, Prime cordial and 3-equitable prime cordial graphs, Util. Math., 111 (2019) 95125.
- [2242] M. A. Seoud and A. E. A. Mahran, On permutation graphs, J. Egyptian Math. Soc. 20 (2012) 57-63.
- [2243] M. A. Seoud and A. E. A. Mahran, On strongly multiplicative graphs, Ars Combin., 118 (2015) 155-165.
- [2244] M. A. Seoud and A. E. A. Mahran, Necessary conditions for strongly \*-graphs, AKCE Internal. J. Graphs Comb., 9(2) (2012) 115-122.
- [2245] M. A. Seoud and A. E. A. Mahran, Some notes on strongly \*-graphs, Proceed. Pakistan Acad. Sci.,: Pakistan Acad. Sci., A. Physical Comput. Sci., 54(2) (2017) 179-195. https://www.paspk.org/wp-content/uploads/2017/07/ A-06-PS-410-On-Strongly-Graphs.pdf
- [2246] M. A. Seoud, E. M. Roshdy, and M. S. AboShady, On strongly \*-graphs, Proc. Pak. Acad. Sci., A 54(2) (2017) 179-195.
- [2247] M. A. Seoud and M. A. Salim, Two upper bounds of prime cordial graphs, *JCMCC*, 75 (2010) 95-103.
- [2248] M. A. Seoud and M. A. Salim, Further results on triangular sum graphs, Internat. Math. Forum, 7(48) (2012) 2393-2405. http://www.m-hikari.com/imf/ imf-2012/45-48-2012/seoudIMF45-48-2012.pdf
- [2249] M. A. Seoud and M. A. Salim, Upper bounds of four types of graph labelings, 127 (2016) 271-278.

- [2250] M. A. Seoud and M. A. Salim, On odd mean graphs, Util. Math., 84 (2011) 119-130.
- [2251] M. A. Seoud and M. A. Salim, On permutation labeling, J. Egyptian Math. Soc., 19(3) (2011) 134-136.
- [2252] M. A. Seoud and M. A Salim, On mean graphs, Ars Combin., 115 (2014) 13-34.
- [2253] M. A. Seoud and M. A. Salim, Further results on edge-odd graceful graphs, Turkish J. Math., 40(3) (2016) 647-656.
- [2254] M. A. Seoud and M. A. Salim, Some results and remarks on harmonic mean and mean cordial labelings, Ars Combin., 133 (2017) 329-340.
- [2255] M. A. Seoud, A. El Sonbaty, and A. E. A. Mahran, On prime graphs, Ars Combin., 104 (2012) 241-260.
- [2256] M. A. Seoud and R. J. Wilson, Some disgraceful graphs, Int. J. Math. Ed. Sci. Tech., 24 (1993) 435-441.
- [2257] M. A. Seoud and M. Z. Youssef, On labelling complete tripartite graphs, Int. J. Math. Ed. Sci. Tech., 28 (1997) 367-371.
- [2258] M. A. Seoud and M. Z. Youssef, On prime labelings of graphs, Congr. Numer., 141 (1999) 203-215.
- [2259] M. A. Seoud and M. Z. Youssef, Families of harmonious and non-harmonious graphs, J. Egyptian Math. Soc., 7 (1999) 117-125.
- [2260] M. A. Seoud and M. Youssef, New families of graceful disconnected graphs, Ars Combin., 57 (2000) 233-245.
- [2261] M. A. Seoud and M. Z. Youssef, On harmonious graphs of order 6, Ars Combin., 65 (2002) 155-176.
- [2262] M. A. Seoud and M. Z. Youssef, The effect of some operations on labelling of graphs, Proc. Math. Phys. Soc. Egypt, 73 (2000) 35-49.
- [2263] M. A. Seoud and M. Z. Youssef, Harmonious labellings of helms and related graphs, https://www.researchgate.net/publication/312218305\_ Harmonious\_labelling\_of\_helms\_and\_related\_graphs.
- [2264] M. A. Seoud and M. Z. Youssef, On gracefulness of disconnected graphs, https://www.researchgate.net/publication/312218547\_On\_gracefulness\_ of\_disconnected\_graphs.
- [2265] M. A. Seoud and A. Zid, Strong multiplicativity of unions and corona of paths and complete graphs, Proc. Math. Phys. Soc. Egypt, 74 (1999) 59-71.

- [2266] E. Sergel, P. Richter, A. Tran, P. Curran, J. Jacob, and D. A. Narayan, Rank numbers for some trees and unicycle graphs, *Aequat. Math.*, 82(1-2) (2011) 65-79.
- [2267] G. Sethuraman and R. Dhavamani, Graceful numbering of an edge-gluing of shell graphs, *Discrete Math.*, 218 (2000) 283-287.
- [2268] G. Sethuraman and R. Dhavamani, Graceful numbering of union of shell graphs, unpublished.
- [2269] G. Sethuraman and A. Elumalai, On graceful graphs: Pendant edge extensions of a family of complete bipartite and complete tripartite graphs, *Indian J. Pure Appl. Math.*, **32** (2001) 1283-1296.
- [2270] G. Sethuraman and A. Elumalai, Gracefulness of a cycle with parallel  $P_k$ -chords, Australas. J. Combin., (2005) **32**) 205-211.
- [2271] G. Sethuraman and A. Elumalai, Packing of any set of graphs into a graceful/harmonious/elegant graph, Ars Combin., 76 (2005) 297-301.
- [2272] G. Sethuraman and A. Elumalai, Graceful, harmonious and elegant labellings on star extension graphs, unpublished.
- [2273] G. Sethuraman and A. Elumalai, Every graph is a vertex induced subgraph of a graceful graph and elegant graph, unpublished.
- [2274] G. Sethuraman and J. Jesintha, A new class of graceful lobsters, J. Combin. Math. Combin. Computing, 67 (2008) 99-109.
- [2275] G. Sethuraman and J. Jesintha, Gracefulness of a family of rooted trees, Far East J. Appl. Math., 30 (2008) 143-159.
- [2276] G. Sethuraman and J. Jesintha, A new family of graceful rooted trees, Proc. National Conf. Appl. Math., (2008) 74-80.
- [2277] G. Sethuraman and J. Jesintha, A new class of graceful rooted trees, J. Disc. Math. Sci. Crypt., 11 (2008) 421-435.
- [2278] G. Sethuraman and J. Jesintha, Generating new graceful trees, Proc. Inter. Conf. Math. Comput. Sci., July (2008) 67-73.
- [2279] G. Sethuraman and J. Jesintha, Generation of graceful trees, Proc. Inter. Conf. Math. Comput Sci., 1 (2009) 1-3.
- [2280] G. Sethuraman and J. Jesintha, All extended banana trees are graceful, Proc. Internat. Conf. Math. Comput. Sci., 1 (2009) 4-8.
- [2281] G. Sethuraman and J. Jesintha, All banana trees are graceful, Advances Appl. Disc. Math., 4 (2009) 53-64.

- [2282] G. Sethuraman and S. P. M. Kishore, On graceful graphs: Union of n copies of edge deleted subgraphs of  $K_4$ , Indian J. Pure Appl. Math., **30** (1999) 801-808.
- [2283] G. Sethuraman and V. Murugan, Generating graceful trees from caterpillars by recursive attachment, *Electronic Notes Discrete Math.*, **53** (2016) 133-147.
- [2284] G. Sethuraman and V. Murugan, Generating graceful unicyclic graph from a given graceful tree, Ars. Combin., (2019), to appear.
- [2285] G. Sethuraman and V. Murugan, Generating graceful unicyclic graphs from a given forest, AKCE Internat. J. Graphs Combin., to appear. https://doi.org/ 10.1016/j.akcej.2019.02.007
- [2286] G. Sethuraman and P. Ragukumar, Every tree is a subtree of graceful tree, graceful graph and alpha-labeled graph, Ars Combin., 132 (2017) 105-109.
- [2287] G. Sethuraman and P. Ragukumar, Towards optimal embedding of an arbitrary tree in a graceful tree, The Eighth International Workshop on Graph Labelings (IWOGL 2014), 73-80, *Electron. Notes Discrete Math.*, 48, Elsevier Sci. B. V., Amsterdam, 2015.
- [2288] G. Sethuraman, P. Ragukumar, and P. J. Slater, Embedding an arbitrary tree in a graceful tree, Bull. Malays. Math. Sci. Soc., 39(1) (2016) suppl., S341-S360.
- [2289] G. Sethuraman, P. Ragukumar, and P. J. Slater, Any tree with m edges can be embedded in a graceful tree with less than 4m edges and in a graceful planar graph, *Discrete Math.*, **340**(2) (2017) 96-106.
- [2290] G. Sethuraman and K. Sankar, On graceful and cordial labeling of shell graphs, Ars Combin., 108 (2013) 515-532.
- [2291] G. Sethuraman and K. Sankar, Subdivision of wheel is graceful and cordial, Internat. J. Math. Anal., textbf9(17) (2015) 817-822. http://www.m-hikari.com/ ijma/ijma-2015/ijma-17-20-2015/01IsankarIJMA17-20-2015-08.pdf
- [2292] G. Sethuraman and P. Selvaraju, Gracefulness of arbitrary supersubdivisions of graphs, Indian J. Pure Appl. Math., 32 (2001) 1059-1064.
- [2293] G. Sethuraman and P. Selvaraju, On graceful graphs: one vertex unions of nonisomorphic complete bipartite graphs, *Indian J. Pure Appl. Math.*, **32** (2001) 975-980.
- [2294] G. Sethuraman and P. Selvaraju, One edge union of shell graphs and one vertex union of complete bipartite graphs are cordial, *Discrete Math.*, **259** (2002) 343-350.
- [2295] G. Sethuraman and P. Selvaraju, Decompositions of complete graphs and complete bipartite graphs into isomorphic supersubdivision graphs, *Discrete Math.*, 260 (2003) 137-149.

- [2296] G. Sethuraman and P. Selvaraju, Super-subdivisions of connected graphs are graceful, unpublished.
- [2297] G. Sethuraman and P. Selvaraju, On graceful graphs I: Union of non-isomorphic complete bipartite graphs with one vertex in common, J. Combin. Inform. System Sci., 26 (2001) 23-32.
- [2298] G. Sethuraman and P. Selvaraju, New classes of graphs on graph labeling, preprint.
- [2299] G. Sethuraman and P. Selvaraju, On harmonious and felicitous graphs: Union of n-copies of edge deleted subgraphs of  $K_4$ , unpublished.
- [2300] G. Sethuraman, P. Selvaraju, and A. Elumalai, On harmonious, felicitous, elegant and cordial graphs: Union of n copies of edge deleted subgraphs of  $K_4$ , preprint.
- [2301] G. Sethuraman and S. Venkatesh, Decomposition of complete graphs and complete bipartite graphs into  $\alpha$ -labelled trees, Ars Combin., **93** (2009) 371-385.
- [2302] H. Shaabani and I. N. Kamalabadi, An efficient population-based simulated annealing algorithm for the multi-product multi-retailer perishable inventory routing problem, *Comput. Ind. Eng.*, **99** (2016)
- [2303] M. K. Shafiq, G. Ali, and R. Simanjuntak, Distance magic labeling of a union of graphs, AKCE J. Graph Combin., 6 (2009) 191-200.
- [2304] A. T. Shahida and M. S. Sunitha, Characterization for topologically set-graceful stars and paths, Adv. Appl. Discrete Math., 15(1) (2015) 18.
- [2305] V. S. Shainy and V. Balaji, On limits of Skolem mean labeling for star graphs  $K_{1,a_1} \cup K_{1,b_1} \cup K_{1,c_1}$ . JASC: J. Appl. Sci. Comput., V(XII) (2018) 1-6.
- [2306] A. R. Shahul Hameed, M. Palanivelrajan, K. Gunasekaran, and R. Raziya Begam, On graceful labeling of some bicyclic graphs, *Internat. J. Fuzzy Mathematical Archives*, 3 (2013) 1-8.
- [2307] A. R. Shahul Hameed, M. Palanivelrajan, K. Gunasekaran and R. Raziya Begam, Graceful labeling of extended komodo dragon graphs, *Bull. Math. Stat. Res.*, 2(4) (2014) 454-459.
- [2308] A. R. Shahul Hameed, M. Palanivelrajan, K. Gunasekaran and R. Raziya Begam, Graceful labeling of extended squid graphs, *Bull. Math. Stat. Res.*, 2(4) (2014) 465-473.
- [2309] A. R. Shahul Hameed, M. Palanivelrajan, K. Gunasekaran and R. Raziya Begam, Graceful labeling of komodo dragon graphs, J. Sci. Res. Physical & Math. Sci., 2(2) (2015) 17-27.

- [2310] A. R. Shahul Hameed, M. Palanivelrajan, K. Gunasekaran and R. Raziya Begam, Graceful labeling of squid graphs, J. Sci. Res. Physical & Math. Sci., 2(2) (2015) 28-38.
- [2311] H. Shaker, A. Rana, M. H. Zobair, and M. Hussain, Super edge-magic total labeling of subdivision of stars, Ars Combin., 116 (2014) 177-183.
- [2312] S. Shan and X. Yu, Antimagic orientation of biregular bipartite graphs, *Electron. J. Combin.*, 24 (2017) Paper 4.31.
- [2313] J.-L. Shang, Spiders are antimagic, Ars Combin., **118** (2015) 367-372.
- [2314] J.-L. Shang,  $P_2$ ,  $P_3$ ,  $P_4$ -free linear forests are antimagic, Util. Math., **101** (2016) 13-22.
- [2315] J.-L. Shang, Antimagic labeling of linear forests, Util. Math., 106 (2018) 23-37.
- [2316] J.-L Shang, C. Lin, and S-C Liaw, On the antimagic labeling of star forests, Util. Math., 97 (2015) 373-385.
- [2317] Z. Shao, F. Deng, Z. Li, and A. Vese, Graceful labelings of the generalized Petersen graphs, Commun. Combin. Opt., 2(2) (2017) 149-159.
- [2318] A. Sharary, Integal sum graphs from complete graphs, cycles and wheels, Arab Gulf Sci. Res., 14-1 (1996) 1-14.
- [2319] S. C. Shee, On harmonious and related graphs, Ars Combin., 23 (1987) A, 237-247.
- [2320] S. C. Shee, Some results on  $\lambda$ -valuation of graphs involving complete bipartite graphs, *Discrete Math.*, **28** (1991) 73-80.
- [2321] S. C. Shee and Y. S. Ho, The cordiality of one-point union of n-copies of a graph, Discrete Math., 117 (1993) 225-243.
- [2322] S. C. Shee and Y. S. Ho, The cordiality of the path-union of n copies of a graph, Discrete Math., 151 (1996) 221-229.
- [2323] S. Shee and S. Lee, On harmonious and felicitious labeling of graphs, Congr. Numer., 68 (1989) 155-170.
- [2324] Y. Shen, J. Dong, G. Zheng, and L. Guo, The radio number of standard caterpillars, Ars Combin., 125 (2016) 257-269.
- [2325] V. Shendra Shainy and V. Balaji, Even felicitous labeling, Malaya J. Mat., S(1) (2020) 45-47. https://doi.org/10.26637/MJM0S20/0009https://doi.org/ 10.26637/MJM0S20/0009
- [2326] D. A. Sheppard, The factorial representation of major balanced labelled graphs, Discrete Math., 15 (1976) 379-388.

- [2327] Z. Sherman, Square difference labeling of some union and disjoint union graphs, Algebra Discrete Math., **25**(2) (2018) 294-302.
- [2328] H. Shimazu, Graceful labelling of the union of cycles and paths, Internat. J. Physics Appl., 3(2) (2011) 205-208.
- [2329] W. C. Shiu, Super-edge-graceful labelings of some cubic graphs, Acta Math. Sin., 22 (2006) 1621-1628.
- [2330] W. C. Shiu, Edge-magic labeling matrices of the composition of paths and null graphs, *Congr. Numer.*, 187 (2007) 55-68.
- [2331] W. C. Shiu, The non-edge-magic simple connected cubic graph of order 10, JCMCC 83 (2012) 225-241.
- [2332] W. C. Shiu, Product-cordial index set for Cartesian product of a graph with a path, J. Combin. Number Th., 4(3) (2013) 177-194.
- [2333] W. C. Shiu, Extreme edge-friendly indices of complete bipartite graphs, Trans. Combin., 5(3) (2016) 11-21.
- [2334] W. C. Shiu, Full edge-friendly index sets of complete bipartite graphs, Trans. Combin., 6(2) (2017) 7-17.
- [2335] W. C. Shiu, Integer-antimagic spectra of disjoint unions of cycles, Theory Appl. Graphs, 5(2) (2018) Art. 3, 12 pp.
- [2336] W. C. Shiu and M.-H. Ho, Full friendly index sets of slender and flat cylinder graphs, *Transactions Combin.*, 2 (4) (2013) 63-80.
- [2337] W. C. Shiu and M.-H. Ho, Full Friendly index sets and full product-cordial index sets of some permutation Petersen graphs, J. Combin. and Number Theory, 5(3) 227-244.
- [2338] W. C. Shiu and H. Kwong, Full friendly index sets of  $P_2 \times P_n$ , Discrete Math., **308** (2008) 3688-3693.
- [2339] W. C. Shiu and H. Kwong, An algebraic approach for finding balance index sets, Australasian J. Combin., 45 (2009) 139-155.
- [2340] W. C. Shiu and H. Kwong, Product-cordial index and friendly index of regular graphs, *Transactions Combin.*, 1 (2012) 15-20.
- [2341] W. C. Shiu, P. C. B. Lam, Super-edge-graceful labelings of multi-level wheel graphs, fan graphs and actinia graphs, *Congr. Numer.*, **174** (2005) 49-63.
- [2342] W. C. Shiu, P. C. B. Lam, and H. L. Cheng, Supermagic labeling of an s-duplicate of  $K_{n,n}$ , Congr. Numer., **146** (2000) 119-124.

- [2343] W. C. Shiu, P. C. B. Lam, and H. L. Cheng, Edge-gracefulness of the composition of paths with the null graphs, *Discrete Math.*, 253 (2002) 63-76.
- [2344] W. C. Shiu, P. C. B. Lam, and S. M. Lee, Edge-magic index sets of (p, p)-graphs, Congr. Numer., 137 (1999) 97-107.
- [2345] W. C. Shiu, P. C. B. Lam, and S. M. Lee, On a construction of supermagic graphs, J. Combin. Math. Combin. Comput., 42 (2002) 147-160.
- [2346] W. C. Shiu, P. C. B. Lam, and S. M. Lee, Edge-magic indices of (n, n-1)-graphs, Electron. Notes Discrete Math., **11** (2002) 443-458.
- [2347] W. C. Shiu, P. C. B. Lam, and S. M. Lee, Edge-magic index sets of square of paths, Util. Math., 97 (2015) 271-286.
- [2348] W. C. Shiu, P. C. B. Lam, and P. K. Sun, Construction of group-magic graphs and some A-magic graphs with A of even order, Congr. Numer., 167 (2004) 97-107.
- [2349] W. C. Shiu and G.-C. Lau, Some results on k-edge-magic broken wheel graphs, J. Graph Labeling, 1(2) (2015) 65-79.
- [2350] W.-C. Shiu, G.-C. Lau, and S.-M. Lee, On (semi-)edge-primality of graphs, Iranian J. Math. Sci. and Informatics, 12(2) (2017) 1-14.
- [2351] W. C. Shiu and S. M. Lee, Some edge-magic cubic graphs, J. Combin. Math. Combin. Comput., 40 (2002) 115-127.
- [2352] W. C. Shiu and S. M. Lee, Full friendly index sets and full product-cordial index sets of twisted cylinders, J. Combin. Number Theory, 3 (3) (2012) 209-216.
- [2353] W. C. Shiu, S. M. Lee, and K. Schaffer, Some k-fold edge-graceful labelings of (p, p-1)-graphs, J. Combin. Math. Combin. Comput., **38** (2001) 81-95.
- [2354] W. C. Shiu and M. H. Ling, Full friendly index sets of Cartesian products of two cycles, Acta Mathematica Sinica, English Series, 26 (2010) 1233-1244.
- [2355] W. C. Shiu, M. H. Ling, and R. M. Low, The entire edge-graceful spectra of cycles with one chord, Proc. Thirty-Seventh Southeastern International Conf. on Combin., Graph Theory and Comput., Congr. Numer., 183 (2006) 213-219.
- [2356] W. C. Shiu, M. H. Ling, and R. M. Low, The edge-graceful spectra of connected bicyclic graphs without pendant, *JCMCC*, 66 (2008) 171-185.
- [2357] W. C. Shiu and R. M. Low, Group magicness of complete N-partite graphs, J. Combin. Math. Combin. Comput., 58 (2006) 129-134.
- [2358] W. C. Shiu and R. M. Low, Integer-magic spectra of sun graphs, J. Comb. Optim., 14 (2007) 309-321.

- [2359] W. C. Shiu and R. M. Low, Ring-magic labelings of graphs, Australasian J. Combin., 41 (2008) 147-158.
- [2360] W. C. Shiu and R. M. Low,  $Z_k$ -magic labeling of fans and wheels with magic-value zero, Australasian J. Combin., 45 (2009) 309-316.
- [2361] W. C. Shiu and R. Low, The integer-magic spectra of bicyclic graphs without pendant, Proceedings of the Forty-Third Southeastern International Conference on Combinatorics, Graph Theory and Computing. *Congr. Numer.*, **214** (2012) 65-73.
- [2362] W. C. Shiu and R. M. Low, Group-magic labelings of graphs with deleted edges, Australasian J. Combin., 57 (2013) 3-19.
- [2363] W. C. Shiu and R. M. Low, Integer-antimagic spectra of complete bipartite graphs and complete bipartite graphs with a deleted edge, *Bull. Inst. Combin. Appl.*, 76 (2016) 54-68.
- [2364] W. C. Shiu and R. M. Low, The integer-antimagic spectra of dumbbell graphs, Bull. ICA, 77 (2016) 89-110.
- [2365] W. C. Shiu and R. M. Low, The integer-magic spectra and null sets of the Cartesian product of trees, Austral. J. Combin., 70(1) (2018) 157-167.
- [2366] W. C. Shiu, P. K. Sun, and R. M. Low, Integer-antimagic spectra of tadpole and lollipop graphs, preprint.
- [2367] W. C. Shiu and F. S. Wong, Extreme friendly indices of  $C_m \times P_n$ , Cong. Numer., **197** (2009) 65-75.
- [2368] W. C. Shiu and F. S. Wong, Strong vertex-graceful labeling for some double cycles, Congr. Numer., 20 (2010) 17-24.
- [2369] W. C. Shiu and F. S. Wong, Full friendly index sets of cylinder graphs, Australasian J. Combin., 52 (2012) 141-162.
- [2370] C.-L. Shiue and H.-L. Fu,  $\alpha$ -labeling number of trees, *Discrete Math.*, **306** (2006) 3290-3296.
- [2371] C.-L. Shiue and H.-L. Fu, The IC-indices of complete bipartite graphs, *Electron. J. Combin.*, 15(1) (2008) Research paper 43, 13 pages.
- [2372] C.-L. Shiue and H.-C. Lu, Trees which admit no  $\alpha$ -labelings, Ars Combin., 103 (2012) 453-463.
- [2373] S. Shixin and S. Yu, On Hamming-graceful graphs, unpublished.
- [2374] N. P. Shrimali and P. B. Pandya, Total neighborhood prime labeling of some graphs, Internat. J. Sci. Res. Math. Stat. Sci., 5(6) (2018) 157-163.

- [2375] N. P. Shrimali and Y. M. Parmar, Edge vertex prime labeling for some trees, JASC: J. Appl. Sci. Comput., VI(I) (2019) 2136-1249. https://www.researchgate.net/ publication/330959323\_Edge\_vertex\_prime\_labeling\_for\_some\_trees
- [2376] N. P. Shrimali, A. K. Rathod, and P. L. Vihol, Neighborhood-prime labeling for some graphs, *Malaya J. Mat.*,7(1) (2019) 108-112. http://www.malayajournal. org/selected\_article.php?id=538
- [2377] M. K. Siddiqui, On the total edge irregularity strength of a categorical product of a cycle and a path, AKCE J. Graphs, Combin., 9(1) (2012) 43-52.
- [2378] M. K. Siddiqui, On edge irregularity strength of subdivision of star  $S_n$ , Internat. J. Math. Soft Comput., **2**(1) (2012) 75-82.
- [2379] M. K. Siddiqui, On irregularity strength of convex polytope graphs with certain pendent edges added, Ars Combin., 129 (2016) 199-210.
- [2380] M. K. Siddiqui, On  $Z_{3k}$ -magic labeling of certain families of graphs, unpublished.
- [2381] M. K. Siddiqui, A. Ahmad, M. F. Nadeem, and Y. Bashir, Total edge irregularity strength of the disjoint union of sun graphs, *Internat. J. Math. Soft Comput.*, 3(1) (2013) 21-27.
- [2382] M. K. Siddiqui, M. Miller, and J. Ryan, Total edge irregularity strength of octagonal grid graph, Util. Math., 103 (2017) 277-287.
- [2383] M. K. Siddiqui, M. Numan, and M. A. Umar, Face antimagic labeling of Jahangir graph, *Math. Comput. Sci.*, 7(2) (2013) 237-243.
- [2384] D. R. Silaban, A. Parestu, B. N. Herawati, K. A. Sugeng, and Slamin, Vertexmagic total labelings of unions of generalized Petersen graphs and union of special circulant graphs, J. Combin. Math. Combin. Comput., 71 (2009) 201-207.
- [2385] D. R. Silaban and K. A. Sugeng, Edge antimagic total labelings on paths and uncycles, J. Combin. Math. Combin. Comput., 65 (2007) 1-8. https://www.researchgate.net/publication/265587900\_Edge\_antimagic\_ total\_labeling\_on\_paths\_and\_unicycles#fullTextFileContent
- [2386] R. Simanjuntak, F. Bertault, and M. Miller, Two new (a, d)-antimagic graph labelings, Proc. Eleventh Australia Workshop Combin. Algor., Hunrer Valley, Australia (2000) 179-189.
- [2387] R. Simanjuntak and K. Wijaya, On distance antimagic graphs, arXiv:1312.7405.
- [2388] M. Simaringa and S. Muthukumaran, Edge vertex prime labeling of some graphs, Malaya J. Matematik, 7(2) (2019) 264-268 http://www.malayajournal.org/ articles/MJM07020018.pdf

- [2389] M. Simaringa and S. Muthukumaran, Edge vertex prime labeling of graphs, Malaya J. Mat., 7(3) (2019) 572-578. https://www.malayajournal.org/articles/ MJM07030034.pdf
- [2390] G. J. Simmons, Synch-sets: a variant of difference sets, Proc. 5th Southeastern Conference on Combinatorics, Graph Theory and Computing, Util. Math. Pub. Co., Winnipeg (1974) 625-645.
- [2391] D. Sinha and J. Kaur, Edge-friendly labelings of graph-II, Util. Math., 99 (2016) 251-271.
- [2392] D. Sinha and J. Kaur, Region indices of friendly labeling of a planar graph, Util. Math., 101 (2016) 149-178.
- [2393] I. Singgih, New Methods for Magic Total Labelings of Graphs, Masters Thesis, Department of Mathematics and Statistics, U. Minnesota Duluth, 2015.
- [2394] G. S. Singh, A note on graceful prisms, Nat. Acad. Sci. Lett., 15 (1992) 193-194.
- [2395] G. S. Singh, Subdivisions of ladders are arithmetic, Multidisc Plinary Research Review, 2 (1992) 23-25.
- [2396] G. S. Singh, A note on sequential crowns, Nat. Acad. Sci. Lett., 16 (1993) 243-245.
- [2397] G. S. Singh, A note on labeling of graphs, Graphs and Combin., 14 (1998) 201-207.
- [2398] G. S. Singh, Some generalities on arithmetic graphs, Graph Theory Notes of New York, XXXVIII (2000) 12-16.
- [2399] G. S. Singh, personal communication.
- [2400] G. S. Singh, Divisor graphs-I, unpublished.
- [2401] G. S. Singh and J. Devaraj, On triangular graceful graphs, unpublished.
- [2402] G. S. Singh and G. Santhosh, A note on integral sum crowns, Ars Combin., 66 (2003) 65-77.
- [2403] G. S. Singh and V. Vilfred, Some results on arithmetic trees, unpublished.
- [2404] N. M. Singhi, G. R. Vijayakumar and N. Usha Devi, Set-magic labelings of infinite graphs. Ars Combin., 63 (2002) 305-310.
- [2405] S. Singhun, Graphs with edge-odd graceful labelings, Int. Math. Forum, 8(9-12) (2013) 577-582. http://www.m-hikari.com/imf/imf-2013/9-12-2013/ singhunIMF9-12-2013.pdf
- [2406] S. Singhun, R. Boonklurb, and C. Charnsamorn, A supermagic labeling of finite copies of Cartesian product of cycles, *East-West J. Math.*, **17**(1) (2015) 61-69.

- [2407] D. Sinha and J. Kaur, Full friendly index set-I, Discrete Appl. Math., 161(9) (2013) 1262-1274.
- [2408] D. Sinha and J. Kaur, Full friendly index set-II, J. Combin. Math. Combin. Comput., 79 (2011) 65-75.
- [2409] D. Sinha and J. Kaur, Edge friendly labelings of graph-I, Ars Combin., 129 (2016) 139-156..
- [2410] D. Sinha and J. Kaur, Region indices of friendly labeling of a planar graph, Util. Math., 101 (2016) 149-178.
- [2411] D. Sinha and J. Kaur, Region indices of edge-friendly labeling of a plane graph, Util. Math., 99 (2016) 187-213.
- [2412] Siqinbate and W. Feng, Wei, Gracefulness of the digraph  $3\vec{C}_{2p}$ , Pure Appl. Math. (Xi'an) **29**(2) (2013) 111-117.
- [2413] Siqinqimuge, Jirimutu, and L. Zhao, On the gracefulness of the digraphs  $n-\dot{C}_{19}$  for even n, Util. Math., 85 (2011) 193-199.
- [2414] M. Sivakumar, On 4-total product cordiality of some corona graphs, Internat. J. Math. Combin., 3 (2016) 99-106.
- [2415] R. Sivaraman, Graceful graphs and its applications, Internat. J. Current Research, 8(11) (2016) 41062-41067.
- [2416] T. Skolem, On certain distribution of integers into pairs with given differences, Math. Scand., 5 (1957) 57-68.
- [2417] J. Skowronek-Kaziów, Multiplicative vertex-colouring weightings of graphs, Inform. Process. Lett., 112(5) (2012) 191-194.
- [2418] S. Slamet, K. Sugeng, and M. Miller, Sum graph based access structure in a secret sharing scheme, J. Prime Research Math., 2 (2006) 113-119.
- [2419] Slamin, M. Bača, Y. Lin, M. Miller, and R. Simanjuntak, Edge-magic total labelings of wheels, fans and friendship graphs, *Bull. ICA*, **35** (2002) 89-98.
- [2420] Slamin, Dafik, and W. Winnona, Total vertex irregularity strength of the disjoint union of sun graphs, Int. J. Comb., 2012 Art. ID 284383, 9 pp.
- [2421] Slamin and M. Miller, On two conjectures concerning vertex-magic total labelings of generalized Petersen graphs, Bull. Inst. Combin. Appl., 32 (2001) 9-16.
- [2422] Slamin, A. C. Prihandoko, T. B.Setiawan, V. Rosita, and B. Shaleh, Vertexmagic total labelings of disconnected graphs, J. Prime Research in Math., 2 (2006) 147-156. Researchgate.net/publication/234058013\_Vertex-magic\_ Total\_Labelings\_of\_Disconnected\_Graphs

- [2423] P. J. Slater, On k-sequential and other numbered graphs, Discrete Math., 34 (1981) 185-193.
- [2424] P. J. Slater, On k-graceful graphs, Proc. of the 13th S.E. Conf. on Combinatorics, Graph Theory and Computing, (1982) 53-57.
- [2425] P. J. Slater, On k-graceful, locally finite graphs, J. Combin. Theory, Series B, 35 (1983) 319-322.
- [2426] P. J. Slater, Problems in graph theory: graceful and sequential numbering of infinite graphs, Southeast Asian Bull. Math., 9 (1985) 15-22.
- [2427] P. J. Slater, On k-graceful, countably infinite graphs, *Discrete Math.*, **61** (1986) 293-303.
- [2428] R. Slíva, Antimagic labeling graphs with a regular dominating subgraph, Inform. Process. Lett., 112(21) (2012) 844-847.
- [2429] D. Small, Regular (even) spider graphs are edge-graceful, Congr. Numer., 74 (1990) 247-254.
- [2430] B. M. Smith, Set of symmetry breaking constraints, (2005) https://pdfs. semanticscholar.org/c2cf/ccdb6f07044b87bb4e5a1a152d24939c7540.pdf
- [2431] B. M. Smith and J.-F Puget, Constraint models for graceful graphs, Constraints, 15 (1) (2010) 64-92.
- [2432] W. Smyth, Sum graphs of small sum number, Colloquia Mathematica Societatis János Bolyai, 60 (1991) 669-678.
- [2433] W. Smyth, Sum graphs: New results, new problems, Bull. Inst. Combin. Appl., 2 (1991) 79-81.
- [2434] H. Snevily, Combinatorics of Finite Sets, Ph. D. Thesis, U. Illinois, 1991.
- [2435] H. Snevily, New families of graphs that have  $\alpha$ -labelings, *Discrete Math.*, **170** (1997) 185-194.
- [2436] B. Sokolowsky, Achieving all radio numbers, AKCE Int. J. Graphs Comb., 10(1) (2013) 77-95.
- [2437] A. Solairaju and M. A. Arockiasamy, Gracefulness of k-step staircase graphs, J. Analysis Computation, 6 (2010) 109-114.
- [2438] A. Solairaju and M. A. Arockiasamy, Graceful mirror-staircase graphs, Inter. J. Contemporary Math. Sci., 5 (2010) 2433-2441.
- [2439] A. Solairaju and M. A. Arockiasamy, A study of labeling techiques in relations to a new family of staircase graphs, unpublished.

- [2440] A. Solairaju and K. Chithra, New classes of graceful graphs by merging a finite number of  $C_4$ , Acta Cienc. Indica Math., **34** (2008) 959-965.
- [2441] A. Solairaju and K. Chithra, Edge-odd graceful labeling of some graphs, Proceedings of the ICMCS, 1 (2008) 101-107.
- [2442] A. Solairaju and K. Chithra, Edge odd graceful graphs, *Electronic Notes in Discrete Math.*, **33** (2009) 15-20.
- [2443] D. D. Somashekara and C. R. Veena, On strong vertex graceful graphs, Internat. Math. Forum, 5(56) (2010) 2751-2757.
- [2444] D. D. Somashekara and C. R. Veena, On square sum labelings of graphs, Proc. Jangjeon Math. Soc, 15(1) (2012) 69-78.
- [2445] D. D. Somashekara and C. R. Veena, On pentagonal sum labelings of graphs, Adv. Studies Contemp. Math., 22(3) (2012) 421-432.
- [2446] S. Somasundaram and R. Ponraj, Mean labelings of graphs, Natl. Acad, Sci. Let., 26 (2003) 210-213.
- [2447] S. Somasundaram and R. Ponraj, Non-existence of mean labeling for a wheel, Bull. Pure and Appl. Sciences (Mathematics & Statistics), **22E** (2003) 103-111.
- [2448] S. Somasundaram and R. Ponraj, Some results on mean graphs, Pure and Applied Math. Sci., 58 (2003) 29-35.
- [2449] S. Somasundaram and R. Ponraj, On mean graphs of order < 5, J. Decision Math. Sci., 9 (2004) 47-58.
- [2450] S. Somasundaram, P. Vidhyarani and R. Ponraj, Geometric mean labelings of graphs, Bull. Pure Appl. Sci., 30E(2) (2011) 153-160.
- [2451] S. Somasundaram, P. Vidhyarani and S. S. Sandhya, Some results on geometric mean graphs, *Internat. Math. Forum*, 7(28) (2012) (28) 1381-1391.
- [2452] S. Sonchhatra and G. V. Ghodasara, Sum perfect square graphs in context of some graph operations, *Internat. J. Math. Trends Tech. (IJMTT)*, **46**(2) (2017) 62-65.
- [2453] S. G. Sonchhatra and G. V. Ghodasara, Sum perfect square labeling of graphs, Internat. J. Scient. Innovative Math. Res., 4 (2016) 64-70.
- [2454] S. G. Sonchhatra and G. V. Ghodasara, Snakes related sum perfect square graphs, Internat. J. Mathematics Appl., 4 (2016) 157-168.
- [2455] S. G. Sonchhatra and G. V. Ghodasara, Some new sum perfect square graphs, Internat. J. Pure Appl. Math., textbf113(3) (2017) 489-499.

- [2456] M. Sonntag, Antimagic vertex-labelling of hypergraphs, Discrete Math., 247 (2002) 187-199.
- [2457] M. Sonntag, Difference labelling of cacti, Discuss. Math. Graph Theory, 23 (2003) 55-65.
- [2458] M. Sonntag and H.-M. Teichert, Sum numbers of hypertrees, Discrete Math., 214 (2000) 285-290.
- [2459] M. Sonntag and H.-M. Teichert, On the sum number and integral sum number of hypertrees and complete hypergraphs, *Discrete Math.*, 236 (2001) 339-349.
- [2460] B. Sooryanarayana and P. Raghunath, Radio labeling of cube of a cycle, Far East J. Appl. Math., 29 (2007) 113-147.
- [2461] B. Sooryanarayana, M. Vishu Kumar, K. Manjula, Radio number of cube of a path, Int. J. Math. Comb., 1 (2010) 5-29.
- [2462] D. Speyer and Z. Szaniszló, Every tree is 3-equitable, Discrete Math., 220 (2000) 283-289.
- [2463] R. Sridevi, S. Navaeethakrishnan, and K. Nagarajan, Super Fibonacci graceful labeling, Math. Combin. Book Ser., 3 (2010) 22-40.
- [2464] R. Sridevi, S. Navaeethakrishnan, A. Nagarajan, and K. Nagarajan, Odd-even graceful graphs, J. Appl. Math. Inform., 30(5-6) (2012) 913-923.
- [2465] S. Sriramr and R. Govindarajan, Permutation labeling of joins of kite graph, Internat. J. Comput. Engin. Tech., (IJCET) 10(3) (2019) 18, Article ID: IJCET\_10\_03\_001. http://www.iaeme.com/MasterAdmin/UploadFolder/IJCET\_ 10\_03\_001-2-3/IJCET\_10\_03\_001-2-3.pdf
- [2466] S. Stalin Kumar and G. T. Marimuthu, H V-super magic decomposition of complete bipartite graphs, Commun. Korean Math. Soc., 30(3) (2015) 313-325.
- [2467] R. P. Stanley, Linear homogeneous Diophantine equations and magic labelings of graphs, Duke Math. J., 40 (1973) 607-632.
- [2468] R. P. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Math. J.*, 43 (1976) 611-531.
- [2469] R. Stanton and C. Zarnke, Labeling of balanced trees, Proc. 4th Southeast Conf. Combin., Graph Theory, Comput., (1973) 479-495.
- [2470] V. L. Stella Arputha Mary, S. Navaneethakrishnan, and A. Nagarajan, Strong  $Z_{4p}$ -magic labeling, Ars Combin., **135** (2017) 197-211.
- [2471] B. M. Stewart, Magic graphs, Canadian J. Math., 18 (1966) 1031-1059.
- [2472] B. M. Stewart, Supermagic complete graphs, Canadian J. Math., 19 (1967) 427-438.
- [2473] A. Su, J. Buchanan, R. C. Bunge, E. Pelttari, G. Rasmuson, E. Sparks, and S. Tagaris, On decompositions of complete multipartite graphs into the union of two even cycles,
- [2474] H.-H. Su, and H. Todt, On the edge-balance index sets of broken wheels, Congr. Numer., 223 (2015) 205-218. J. Combin. Math. Combin. Comput., 96 (2016) 129-157.
- [2475] G. Subashini, Κ. Bhuvaneswari, and K. Manimekalai, Square dif-Internat. J. Engin. Res. ference labeling of theta graphs, Tech., (IJERT), 8(9)(2019)343-345. https://www.ijert.org/research/ square-difference-labeling-of-theta-graphs-IJERTV8IS090122.pdf
- [2476] M. Subbiah, A Study on Some Variations of Graph Labeling and its Applications in Various Fields, Ph. D. Thesis, Depart. Math., (2011), Bharathidasan University.
- [2477] S. P. Subbiah and J. Pandimadevi, H E-super magic decomposition of graphs, Electron. J. Graph Theory Appl. (EJGTA), 2(2) (2014) 115-128.
- [2478] I. W. Sudarsana, E. T. Baskova, D. Ismaimuza, and H. Assiyatun, Creating new super edge-magic total labelings from old ones, J. Combin. Math. Combin. Comput., 55 (2005) 83-90.
- [2479] I. W. Sudarsana, E. T. Baskoro, S. Uttunggadewa, and D. Ismaimuza, Expansion techniques on the super edge anti magictotal graphs, J. Combin. Math. Combin. Comput., 71 (2009) 189-199.
- [2480] I. W. Sudarsana, A. Hendra, Adiwijaya, D. Y. Setyawan, On super edge anti magic total labeling for t-joint copies of wheel, Far East J. Math. Sci., (FJMS) 69(2) (2012) 275-283.
- [2481] I. W. Sudarsana, D. Ismaimuza, E. Baskova, and H. Assiyatun, On super (a, d)antimagic total labeling of disconnected graphs, J. Combin. Math. Combin. Comput., 55 (2005) 149-158.
- [2482] S. Sudha, Some chain graphs are graceful, Math. Sci. Internat. Research J., 1(3) (2012) 855-859.
- [2483] S. Sudha, Gracefulness of joining complete bipartite graphs having a common vertex set, *Engin. Sci. Internat. Research J.*, 1(1) (2013) 84-85.
- [2484] M. Sudha and A. C. Babu, Even-even gracefulness of some families of graphs, IOSR J. Math., (IOSR-JM), 8 (6) (2013) 7-11.

- [2485] S. Sudha and V. Kanniga, Arbitrary supersubdivision of helms, centipedes and ladder graphs are graceful, Math, Sci. Internat. Research J., 1(3)(2012) 860-863.
- [2486] S. Sudha and V. Kanniga, Gracefulness of joining isolated vertices to a path, Global J. Math. Math. Sciences, 2(1) (2012) 91-94.
- [2487] S. Sudha and V. Kanniga, Gracefulness of some new class of graphs, Engin. Sci. Internat. Reser. J., 1(1) (2013) 81-83.
- [2488] S. Sudha and V. Kanniga, Graceful labeling on the combination of some graphs, Math. Sci. Internat. Research J., 2(2) (2013) 630-633.
- [2489] S. Sudha and V. Kanniga, Superposition of stars on cycles and n-centipedes are graceful, Math. Sci. Internat. Research J., 2(2) (2013) 634-636.
- [2490] K. A. Sugeng and N. H. Bong, Vertex (a, d)-antimagic total labeling on circulant graph  $C_n(1, 2, 3)$ , J. Indones. Math. Soc., Special Edition (2011) 79-89.
- [2491] K. A. Sugeng, D. Frončcek, M. Miller, J. Ryan, and J. Walker, On distance magic labelings of graphs, J. Combin. Math. Combin. Comput., 71 (2009) 39-48.
- [2492] K. A. Sugeng, B. N. Herawati, M. Miller, and M. Bača, On magicness and antimagicness of the union of 4-regular circulant graphs, *Austral. J. Combin.*, 50, (2011) 141-153.
- [2493] K. A. Sugeng and M. Miller, Relationship between adjacency matrices and super (a, d)-edge-antimagic-total labelings of graphs, J. Combin. Math. Combin. Comput., 55 (2005) 71-82.
- [2494] K. A. Sugeng and M. Miller, On consecutive edge magic total labelings of graphs, J. Discrete Algorithms, 6 (2008) 59-65.
- [2495] K. A. Sugeng, M. Miller, M. Bača, Super edge-antimagic total labelings, Util. Math., 71 (2006) 131-141.
- [2496] K. A. Sugeng, M. Miller, Y. Lin, and M. Bača, Super (a, d)-vertex-antimagic total labelings, J. Combin. Math. Combin. Comput., 55 (2005) 91-102.
- [2497] K. A. Sugeng, M. Miller, Y. Lin, and M. Bača, Face antimagic labelings of prisms, Util. Math., 71 (2006) 269-286.
- [2498] K. A. Sugeng, M. Miller, Slamin, and M. Bača, (a, d)-edge-antimagic total labelings of caterpillars, *Lecture Notes Comput. Sci.*, 3330 (2005) 169-180.
- [2499] K. A. Sugeng and J. Ryan, On several classes of monographs, Australas. J. Combin., 37 (2007) 277-284.
- [2500] K. A. Sugeng, J. Ryan, and H. Fernau, A sum labelling for the flower  $f_{p,q}$ , unpublished.

- [2501] K. A. Sugeng and D. R. Silaban, Super (a, d)-vertex antimagic total labeling on disjoint union of regular graphs, J. Combin. Math. Combin. Comput., 71 (2009) 217-225.
- [2502] K. A. Sugeng and W. Xie, Construction of super edge magic total graphs, in: Proceed. Sixteenth Australasian Workshop oCombin. Algor. 2005, Ballarat, Australia, (2005) 303-310/ https://core.ac.uk/download/pdf/213011717.pdf
- [2503] M. Sugiyama, Generalization of magic graphs and their properties, Inform. Process. Soc. Japan (IPSJ), 59(6) (2017) 1394-1404.
- [2504] A. Sugumaran and V. Mohan, Prime cordial labeling of some graphs related to H-graph, Annals Pure and Appl. Math., 16(1) (2018) 171-176.
- [2505] A. Sugumaran and V. Mohan, Further results on prime cordial labeling, Annals Pure and Appl. Math., 14(3) (2017) 489-496. http://www.researchmathsci. org/APAMart/apam-v14n3-16.pdf
- [2506] A. Sugumaran and V. Mohan, Difference cordial labeling of some special graphs and related to fan graphs, *Internat. J. Res. Engin. Appl. Management*, 4(12) (2019) 57-60. http://ijream.org/papers/IJREAMV04I1248014.pdf
- [2507] A. Sugumaran and K. Rajesh, Extended results on sum divisor cordial labeling, Proyecc. J. Math., 38(4) (2019) 653-663. https://www.revistaproyecciones. cl/index.php/proyecciones/article/view/3779/3239
- [2508] A. Sugumaran and K. Suresh, Some results on divisor cordial labeling, Internat. J. Applied Engin. Res, 10(72) (2015) 68-71.
- [2509] A. Sugumaran and K. Suresh, Some results on vertex odd divisor cordial labeling of graphs, Internat. J. Pure Appl. Math., 113(11) (2017) 199-207. https: //acadpubl.eu/jsi/2017-113-pp/articles/11/22.pdf
- [2510] A. Sugumaran and K. Suresh, A special class of prime cordial graphs, Internat. J. Pure Appl. Math., 114(6) (2017) 41-48. https://acadpubl.eu/jsi/ 2017-114-5/articles/2/5.pdf
- [2511] A. Sugumaran and K. Suresh, Further results on vertex odd divisor cordial labeling of some graphs, Annals Pure Appl. Math., 16(2) (2018) 365-372. http://www. researchmathsci.org/APAMart/apam-v16n2-13.pdf
- [2512] A. Sugumaran and K. Suresh, Some star related vertex odd divisor cordial labeling of graphs, Annals Pure Appl. Math., 16(2) (2018) 385-391. http://www. researchmathsci.org/APAMart/apam-v16n2-15.pdf

- [2513] A. Sugumaran and K. Suresh, Cycle related vertex odd divisor cordial labeling of graphs, J. Comput. Math. Sci., 9(3) (2018) 204-211. http://www.compmath-journal.org/dnload/A-Sugumaran-and-K-Suresh-/ CMJV09I03P0204.pdf
- [2514] A. Sugumaran and P. Vishnu Prakash, Some new results of prime cordial labeling, Internat. J. Pure Engin. Math., (IJPEM), 5(1) (2017) 9-14. http://www. ascent-journals.com/IJPEM/Vol5No1/2-vishnuprakash.pdf
- [2515] A. Sugumaran and P. Vishnu Prakash, Prime cordial labeling for theta graph, Annals Pure Appl. Math., 14(3) (2017) 379-386. https://pdfs.semanticscholar. org/38c3/54cd5af11aead326d7c7de6193719995481b.pdf
- [2516] A. Sugumaran and P. Vishnu Prakash, Some new results on prime cordial labeling for theta graph, J. Comput. Math. Sci., 8(11) (2017) 630-634.
- [2517] A. Sugumaran and P. Vishnu Prakash, Some new families of 3-equitable prime cordial graphs, Internat. J. Stat. Appl. Math., 3(1) (2018) 45-49. http://www. mathsjournal.com/pdf/2018/vol3issue1/PartA/2-6-12-620.pdf
- [2518] A. Sugumaran and P. Vishnu Prakash, E-cordial labeling for theta graph, Research Reviews: Discrete Math. Structures, 5(1)(2018) 26-32.
- [2519] A. Sugumaran and P. Vishnu Prakash, 3-equitable prime cordial labeling in theta graph, J. Emerging Tech. Innovative Res., 5(7) (2018) 332-338.
- [2520] A. Sugumaran and K. Rajesh, Some new results on sum divisor cordial graphs, Annals Pure Appl. Math., 14(1) (2017) 45-52. https://pdfs.semanticscholar. org/eee0/93c3f3bb98122a7551b08f1e1a26db538370.pdf
- [2521] A. Sugumaran and K. Rajesh, Sum divisor cordial labeling of theta graph, Annals Pure Appl. Math., 14(2) (2017) 313-320. https://pdfs.semanticscholar.org/ a540/180f1aeaf801a1e9ba53e817dfade5afcfc1.pdf
- [2522] A. Sugumaran and K. Rajesh, Sum divisor cordial labeling of Herschel graph, Annals Pure Appl. Math., 14(3) (2017) 465-472. https://pdfs.semanticscholar. org/6327/4d4d2ff4c6239308ac67cba007846e1688be.pdf
- [2523] A. Sugumaran and K. Rajesh, Further results on sum divisor cordial labeling, J. Comput. Math. Sci., 8(12) (2017) 802-807. http://www.compmath-journal.org/ dnload/A-Sugumaran-and-K-Rajesh/CMJV08I12P0802.pdf
- [2524] A. Sugumaran and K. Rajesh, Some graph operations on sum divisor cordial labeling related to H-graph, Internat. J. Stat. Appl. Math., 3(1) (2018) 389-395. http://www.mathsjournal.com/archives/2018/vol3/issue1/PartE/3-1-15

- [2525] A. Sugumaran, K. Rajesh, Sum divisor cordial labeling on some special graphs, Internat. J. Math. Appl., 6(2-B) (2018) 347-353. https://pdfs.semanticscholar. org/2105/7352842a3fb5d129c4e140c93775c65734a7.pdf
- [2526] A. Sugumaran and K. Rajesh, Sum divisor cordial labeling on some classes of graphs, J. Comput. Math. Sci., 10(2) (2019) 316-325. http://www.compmath-journal.org/dnload/A-Sugumaran-and-K-Rajesh/ CMJV10I02P0316.pdf
- [2527] A. Sugumaran and K. Rajesh, Perrin graceful graphs, Internat. J. Pure Appl. Math., 114(6) (2017) 131-137.
- [2528] A. Sugumaran and K. Rajesh, Lucas divisor cordial labeling, Internat. J. Pure Appl. Math., 113(6) (2017) 233-241. https://acadpubl.eu/jsi/2017-114-5/ articles/2/14.pdf
- [2529] R. G. Sun, Harmonious and sequential labelings of the book graphs  $B_m$ , Gaoxiao Yingyong Shuxue Xuebao Ser. A, **9** (1994) 335-337.
- [2530] D. Q. Sun and D.Y. Wang, personal communication.
- [2531] G. C. Sun, J. Guan, and S.-M. Lee, A labeling algorithm for magic graph, Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994). Congr. Numer. 102 (1994) 129-137.
- [2532] G. C. Sun and S. M. Lee, Construction of magic graphs, Congr. Numer., 103 (1994) 243-251.
- [2534] H. Sun, X. Zhang, and B. Yao, Construction of new graphical passwords with graceful-type labellings on trees, *Conference: 2018 2nd IEEE Adv. Infor. Management, Communicates, Elect. Automation Control Conf. (IMCEC)* https://www.researchgate.net/publication/328166886\_Construction\_of\_ New\_Graphical\_Passwords\_with\_Graceful-Type\_Labellings\_on\_Trees
- [2535] M. Sundaram, R. Ponraj and S. Somasundaram, Product cordial labeling of graphs, Bull. Pure and Appl. Sci. (Math. & Stat.), 23E (2004) 155-163.
- [2536] M. Sundaram, R. Ponraj, and S. Somasundram, Prime cordial labeling of graphs, J. Indian Acad. Math., 27 (2005) 373-390.
- [2537] M. Sundaram, R. Ponraj, and S. Somasundram, Some results on total product cordial labeling of graphs, J. Indian Acad. Math., 28 (2006) 309-320.

- [2538] M. Sundaram, R. Ponraj, and S. Somasundaram, On a prime labeling conjecture, Ars Combin., 80 (2006) 205-209.
- [2539] M. Sundaram, R. Ponraj, and S. Somasundram, Total product cordial labeling of graphs, Bull. Pure Appl. Sci. Sect. E Math. Stat., 25 (2006) 199-203.
- [2540] M. Sundaram, R. Ponraj and S. Somasundaram, On prime graphs of order  $\leq 6$ , Acta Ciencia Indica, **32** (2006) 859-871.
- [2541] M. Sundaram, R. Ponraj and S. Somasundaram, Mean number of a graph, Pure Appl. Math. Sci., 57 (2007) 93-101.
- [2542] M. Sundaram, R. Ponraj and S. Somasundaram, EP-cordial labeling of graphs, Varahmihir J. Math. Sci., 7 (2007) 183-194.
- [2543] M. Sundaram, R. Ponraj and S. Somasundaram, On graph labeling parameter, J. Discrete Math. Sci. Cryp., 11 (2008) 219-229.
- [2544] M. Sundaram, R. Ponraj, and S. Somasundaram, Some results on product cordial labeling, *Pure and Applied Math. Sciences*, to appear.
- [2545] N. Suparta and D. M. A. Ariawan, Some methods for constructing some classes of graceful uniform trees, *Indonesian J. Combin.*, 2(2) (2018) 123-135. http://dx. doi.org/10.19184/ijc.2018.2.2.7
- [2546] M. C. Superdock, The Graceful Tree Conjecture: A Class of Graceful Diameter-6 Trees, Senior Thesis, Department of Mathematics of Princeton University, 2013
- [2547] Y. Susanti, H. Khotimah, M. Hidayati, and M. F. Wahyujati, On total edge irregularity strength of some graphs and the construction of some labeling, *Far East J. Math. Sci. (FJMS)*, **104**(1) (2018) 49-64. http://www.pphmj.comhttp://dx.doi.org/10.17654/MS104010049
- [2548] Y. Susanti, I. Ernanto1, and B. Surodjo, On some new edge odd graceful graphs, Proceed. 8th SEAMS-UGM Internat. Conf. Math. Appl. 2019 AIP Conf. Proc. 2192, 040016-1040016-12. https://aip.scitation.org/doi/pdf/ 10.1063/1.5139142?class=pdf
- [2549] F. Susanto, On super (a, d)- $C_n$ -antimagic total labeling of disjoint union of cycles, AKCE Int. J. Graphs Comb., **15**(1) (2018) 22-26.
- [2550] S. Susilawati, E. T. Baskoro, and R. Simanjuntak, Total vertex irregularity strength of trees with maximum degree five, *Elect. J. Graph Th. Appl.* 6 (2) (2018) 250-257.
- [2551] L. Susilowati, T. Sania, and N. Estuningsih, Super (a, d)- $C_n$ -antimagic total labeling of ladder graph, Adv. Appl. Discrete Math., **10**(2) (2012) 77-93.

- [2552] M. Sutton, Sumable Graphs Labellings and Their Applications, Ph.D. Thesis, Dept. Computer Science, The University of Newcastle, 2001.
- [2553] M. Sutton, A. Draganova, and M. Miller, Mod sum numbers of wheels, Ars Combin., 63 (2002) 273-287.
- [2554] M. Sutton and M. Miller, Mod sum graph labelling of  $H_{n,n}$  and  $K_n$ , Australas. J. Combin., **20** (1999) 233-240.
- [2555] M. Sutton and M. Miller, On the sum number of wheels, Discrete Math., 232 (2001) 185-188.
- [2556] M. Sutton, M. Miller, J. Ryan, and Slamin, Connected graphs which are not mod sum graphs, *Discrete Math.*, **195** (1999) 287-293.
- [2557] V. Swaminathan and P. Jeyanthi, Super vertex-magic labeling, Indian J. Pure and Appl. Math., 34 (2003) 935-939.
- [2558] V. Swaminathan and P. Jeyanthi, On super vertex-magic labeling, J. Discrete Math. Sciences & Cryptography, 8 (2005) 217-224.
- [2559] V. Swaminathan and P. Jeyanthi, Super edge-magic strength of fire crackers, banana trees and unicyclic graphs, *Discrete Math.*, **306** (2006) 1624-1636.
- [2560] V. Swaminathan and P. Jeyanthi, Super edge-magic strength of generalized theta graph, Inter. J. Management and Systems, 22 (2006) 203-220.
- [2561] V. Swaminathan and P. Jeyanthi, Super edge-magic strength of some trees, Util. Math., 72 (2007) 199-210.
- [2562] V. Swaminathan and P. Jeyanthi, (a, d)-1-vertex-antimagic vertex labeling, Util. Math., 74 (2007) 179-186.
- [2563] V. Swaminathan and P. Jeyanthi, Super edge-magic labeling of some new classes of graphs, *Math. Education*, XLII(2) (2008) 91-94.
- [2564] V. Swaminathan and P. Jeyanthi, Strong super edge-magic graphs, Math. Education, XLII(3) (2008) 156-160.
- [2565] B. Swita, U. Rafflesia, N. Henni Ms, D. S. Adji, and M. Simanihuruk, On edge magic total labeling of (7,3)-cycle books, *Int. J. Math. Math. Sci.*, (2019) Art. ID 1801925, 8 pp. https://doi.org/10.1155/2019/1801925
- [2566] R. Sweetly and J. P. Joseph, The radio number of  $(W_n: 2)$  graphs, J. Discrete Math. Sci. Cryptogr., **12** (2009) 729-736.
- [2567] Z. Szaniszló, k-equitable labellings of cycles and some other graphs, Ars Combin., 37 (1994) 49-63.

- [2568] A. Tabraiz and M. Hussain, Magic and anti-magic total labeling on subdivision of grid graphs, J. Graph Label., 2(1) (2016) 9-24.
- [2569] A. Taimur, M. Numan, A. Mumtaz, and A. Semaničová-Feňovčíková,Super (a, d) H-antimagic labeling of subdivided graphs, Open Math., V. Vilfred, S. Somasundaram and T. Nicholas, Classes of prime graphs 16 (2018) 688-697.
- [2570] M. Tamilselvi and N. Revathi, ksuper harmonic mean labeling of some graphs, Aryabhatta J. Math. Informatics, **9**(1) (2017) 779-787.
- [2571] R. Tao, On k-cordiality of cycles, crowns and wheels, Systems Sci. Math. Sci., 11 (1998) 227-229.
- [2572] I. Tarawneh, R. Hasni, and A. Ahmad, On the edge irregularity strength of corona product of graphs with paths, *Appl. Math. E-Notes*, **16** (2016) 80-87.
- [2573] I. Tarawneh, R. Hasni, and A. Ahmad, On the edge irregularity strength of corona product of cycle with isolated vertices, AKCE Internat. J. Graphs Combin., 13 (2016) 213-217.
- [2574] I. Tarawneh, R. Hasni, and M. A. Asim, On the edge irregularity strength of disjoint union of star graph and subdivison of star graph, Ars Combin., 141 (2018) 93-100.
- [2575] I. Tarawneh, R. Hasni, M. K. Siddiqui, and M. A. Asim, On the edge irregularity strength of disjoint union of certain graphs, Ars Combin., 142 (2019) 239-249.
- [2576] M. Teffilia1 and J. Devaraj, On friendly index set of graphs, Internat. J. Sci. Res. Math. Stat. Sci., 5(4) (2018) 258-263. DOI:10.26438/ijsrmss/v5i4.258263
- [2577] H.-M. Teichert, The sum number of d-partite complete hypergraphs, Discuss. Math. Graph Theory, 19 (1999) 79-91.
- [2578] H.-M. Teichert, Classes of hypergraphs with sum number one, Discuss. Math. Graph Theory, 20 (2000) 93-103.
- [2579] H.-M. Teichert, Sum labellings of cycle hypergraphs, Discuss. Math. Graph Theory, 20 (2000) 255-265.
- [2580] S. Telang, private communication.
- [2581] A. Tenguria and R. Verma, 3-Total super sum cordial labeling for union of some graphs, *IJAIS*, Foundation of Computer Science, New York, USA, 8(4) (2015) 25-30.
- [2582] A. Tenguria and R. Verma. 3-Total super product cordial labeling for some graphs, Internat. J. Science Res., 4(2) (2015) 557-559.

- [2583] A. Tenguria and R. Verma, 3-total super sum cordial labeling by applying operations on some graphs, Appl. Appl. Math., 11 (2016)(1) 444-446.
- [2584] M. Tezer and I. Cahit, A note on (a, d)-vertex antimagic total labeling of paths and cycles, *Util. Math.*, **68** (2005) 217-221.
- [2585] T. Tharmaraj and P. B. Sarasija, Square difference labeling for certain graphs, Internat. J. Math. Archive, 4(8) (2013) 183-186.
- [2586] T. Tharmaraj and P. B. Sarasija, On beta combination labeling graphs, Internat. J. Computer Applications, 79(13) (2013) 26-29.
- [2587] T. Tharmaraj and P. B. Sarasija, Beta combination graphs, Internat. J. Computer Applications, 76(14) (2013) 1-5.
- [2588] K. Thejeshwi and S. Kirupa, Application of graceful graph in MPLS, IJSRD -Internat. J. Sci. Res. Development, 6(06) (2018) 2321-0613.
- [2589] K. Thirusangu and M. Madhu, Divisor cordial labeling in extended duplicate graph of star, bistar and double star, *JASC: J. Appl. Sci. Comput.*, **VI**(I) (2019) 19-24.
- [2590] K. Thirusangu, A. K. Nagar, and R. Rajeswari, Labelings in Cayley digraphs, European J. Combin., 32 (2011) 133-139.
- [2591] A. Tiwari and A. Tripathi, On the range of size of sum graphs & integral sum graphs of a given order, *Discrete Appl. Math.*, 161 (16-17) (2013) 2653-2661.
- [2592] H. Todt, On edge-balance index sets of distance two, Congr. Numer., 223 (2015) 105-117.
- [2593] C. Tong, X. Lin, Y. Yang, and L. Wang, Irregular total labellings of  $C_m \Box C_n$ , Util. Math., 81 (2010) 3-13.
- [2594] A. Tout, A. N. Dabboucy, and K. Howalla, Prime labeling of graphs, Nat. Acad. Sci. Letters, 11 (1982) 365-368.
- [2595] T. Traetta, A complete solution to the two-table Oberwolfach problems, J. Combin. Theory, Series A, 120(5) (2013) 984-997.
- [2596] M. Trenklér, Some results of magic graphs, graphs and other combinatorics topics, Teubner-Texte zur Mathematik - Band 59, Leipzig (1983) 328-332.
- [2597] M. Trenklér, Numbers of vertices and edges of magic graphs, Ars Combin., 55 (2000) 93-96.
- [2598] M. Trenklér, Super-magic complete n-partite hypergraphs, Graphs and Combin., 17 (2001) 171-175.
- [2599] M. Trenklér, Magic *p*-dimensional cubes, Acta Arithmetica, **96** (2001) 361-364.

- [2600] M. Trenklér and V. Vetchý, Magic powers of graphs, Math. Bohemica, 122 (1997) 121-124.
- [2601] M. Truszczyński, Graceful unicyclic graphs, Demonstatio Mathematica, 17 (1984) 377-387.
- [2602] Y. Tsao, A simple research of divisor graphs, The 29th Workshop Combin. Math. Computation Th., National Taipei College of Business, Institute of Information and Decision Sciences, Taipei, Taiwan, April 27-28, 2012, 186-190. http://par. cse.nsysu.edu.tw/~algo/paper/paper12/C2\_2.pdf
- [2603] M. Tuczyński, P. Wenus, and K. Węsek, On cordial labeling of hypertrees, *Discrete Math. Theor. Comput. Sci.*, **21**(4) (2019) Paper No. 1, 14 pp.
- [2604] M. Tuga, M. Miller, J. Ryan, and Z. Ryjáčk, Exclusive sum labelings of trees, J. Combin. Math. Combin. Comput., 55 (2005) 109-121.
- [2605] J. Uma and A. Mazudha Shanofer, A note on edgegraceful labeling for corona and flower graph, AIP Conf. Proceed. 2112, 020095, (2019) 1-5. https://aip. scitation.org/doi/pdf/10.1063/1.5112280
- [2606] N. Ujwala, Applications of graceful graph, Internat. J. Eng. Sci. Res. Tech., 4 (2015) 129-131.
- [2607] R. Ulfatimah, M. Roswitha, and T. A. Kusmayadi, *H*-supermagic labeling on shrubs graph and  $L_m \cdot P_n$ , *Internat. Confer. Math.: Edu., Th. Appl., IOP Conf.* Series: Journal of Physics: Conf. Series **855** (2017) 012055 1-8.
- [2608] S. K. Vaidya and C. M. Barasara, Product cordial labeling for some new graphs, J. Math. Research, 3(2) (2011) 206-211. http://www.ccsenet.org/journal/index. php/jmr/article/view/8703
- [2609] S. K. Vaidya and C. M. Barasara, Further results on product cordial labeling, International J. Math. Combin., 3 (2012) 64-71.
- [2610] S. K. Vaidya and C. M. Barasara, Some product cordial graphs, *Elixir Discrete Math.*, 41 (2011) 5948-5952.
- [2611] S. K. Vaidya and C. M. Barasara, Product cordial graphs in the context of some graph operations, *Internat. J. Math. Sci. Comput.*, 1(2) (2011) 1-6. https://www.researchgate.net/publication/265662133\_ Product\_Cordial\_Graphs\_in\_the\_Context\_of\_Some\_Graph\_Operations
- [2612] S. K. Vaidya and C. M. Barasara, Harmonic mean labeling in the context of duplication of graph elements, *Elixir Discrete Math.*, 48 (2012) 9482-9485.
- [2613] S. K. Vaidya and C. M. Barasara, Further results on product cordial graphs, Internat. J. Math. Soft Comput., 2(2) (2012) 67-74.

- [2614] S. K. Vaidya and C. M. Barasara, Edge product cordial labeling of graphs, J. Math. Comput. Science, 2(5) (2012) 1436-1450. http://scik.org/index.php/ jmcs/article/view/420/189
- [2615] S. K. Vaidya and C. M. Barasara, Geometric mean labeling in the context of duplication of graph elements, *Internat. J. Math. Sci. Engin. Appl.*, 6(6) (2012) 311-319. http://www.ascent-journals.com/IJMSEA/Vol6No6/29-S.K.Vaidya. pdf
- [2616] S. K. Vaidya and C. M. Barasara, Some new families of edge product cordial graphs, Advanced Modeling Optimization, 15(1) (2013) 103-111. http://camo. ici.ro/journal/vol15/v15a9.pdf
- [2617] S. K. Vaidya and C. M. Barasara, Some edge product cordial graphs, Internat. J. Math. Soft Comput., 3(3) (2013) 49-53.
- [2618] S. K. Vaidya and C. M. Barasara, Edge product cordial labeling in the context of some graph operations, *Internat. J. Math. Scientific Comput.*, 3(1) (2013) 4-7. https://veltech.edu.in/wp-content/uploads/2016/04/Paper-02-13.pdf
- [2619] S. K. Vaidya and C. M. Barasara, Total edge product cordial labeling of graphs, *Malaya J. Matematik*, 3(1) (2013) 55-63. http://www.malayajournal. org/articles/MJM039.pdf
- [2620] S. K. Vaidya and C. M. Barasara, On total edge product cordial labeling, Internat. J. Math. Scientific Comput., 3(2) (2013) 12-16.
- [2621] S. K. Vaidya and C. M. Barasara, On edge product cordial labeling of some product related graphs, Internat. J. Math. Appl., 2(2) (2014) 15-22. http://ijmaa.in/ v2n2.html
- [2622] S. K. Vaidya and C. M. Barasara, Product and edge product cordial labeling of degree splitting graph of some graphs, Adv. Appl. Discrete Math., 15(1) (2015) 61-74.
- [2623] S. K. Vaidya and C. M. Barasara, Product cordial labeling of line graph of some graphs, *Kragujevac J. Math.*, 40(2) (2016) 290-297.
- [2624] S. K. Vaidya and L. Bijukumar, Odd graceful labeling of some new graphs, Modern Appl. Sci., 4 (2010) 65-69.
- [2625] S. K. Vaidya and L. Bijukumar, New mean graphs, International J. Math. Combin., 3 (2011) 107-113.
- [2626] S. K. Vaidya and N. A. Dani, Some new star related graphs and their cordial as well as 3-equitable labeling, J. Science 1(1) (2010) 111-114.

- [2627] S. K. Vaidya and N. A. Dani, Cordial labeling and arbitrary super subdivision of some graphs, Inter. J. Information Sci. Comput. Math., 2(1) (2010) 51-60.
- [2628] S. K. Vaidya and N. A. Dani, Some new product cordial graphs, J. Applied Computer Science & Math., 8(4) (2010) 62-65. http://jacs.usv.ro/index.php
- [2629] S. K. Vaidya and N. A. Dani, Cordial and 3-equitable graphs induced by duplication of edge, *Math. Today*, 27 (2011) 71-82.
- [2630] S. K. Vaidya, N. Dani, K. Kanani, and P. Vihol, Cordial and 3-equitable labeling for some star related graphs, *Internat. Mathematical Forum*, 4 (2009) 1543-1553.
- [2631] S. K. Vaidya, N. Dani, K. Kanani, and P. Vihol, Cordial and 3-equitable labeling for some shell related graphs, J. Sci. Res., 1 (2009) 438-449.
- [2632] S. K. Vaidya, N. Dani, K. Kanani, and P. Vihol, Some wheel related 3-equitable graphs in the context of vertex duplication, Adv. and Appl. Disc. Math., 4 (2009) 71-85.
- [2633] S. K. Vaidya, N. A. Dani, K. K. Kanani, and P. L. Vihol, Cordial and 3-equitable labeling for some wheel related graphs, *IAENG International J. Applied Math.* 41(2) (2011) 99-105.
- [2634] S. K. Vaidya, N. A. Dani, P. L. Vihol, and K. K. Kanni, Strongly multiplicative labeling in the context of arbitrary supersubdivision, J. Math. Research, 2(2) (2010) 28-33.
- [2635] S. K. Vaidya, G. Ghodasara, S. Srivastav, and V. Kaneria, Cordial labeling for two cycle related graphs, *Math. Student*, **76** (2007) 237-246.
- [2636] S. K. Vaidya, G. V. Ghodasara, S. Srivastav, and V. J. Kaneria, Cordial and 3-equitable labeling of star of a cycle, *Math. Today*, **24** (2008) 54-64.
- [2637] S. K. Vaidya, G. Ghodasara, S. Srivastav, and V. Kaneria, Some new cordial graphs, Internat. J. Scientific Computing, 2 (2008) 81-92.
- [2638] S. K. Vaidya and K. Kanani, Some new results on cordial labeling in the context of arbitrary supersubdivision of graph, *Applied Math. Sci.*, 4(47) (2010) 2323-2329.
- [2639] S. K. Vaidya and K. K. Kanani, Some cycle related product cordial graphs, Internat. J. Algor., Comput. and Math., 3(1) (2010) 109-116.
- [2640] S. K. Vaidya and K. K. Kanani, Strongly multiplicative labeling for some cycle related graphs, *Modern Applied Science*, 4(7) (2010) 82-88.
- [2641] S. K. Vaidya and K. K. Kanani, Prime labeling for some cycle related graphs, J. Math. Research, 2(2) (2010) 98-103.

- [2642] S. K. Vaidya and K. K. Kanani, Some new product cordial graphs, Math. Today, 27 (2011) 64-70.
- [2643] S. K. Vaidya and K. K. Kanani, Some strongly multiplicative graphs in the context of arbitrary supersubdivision, *Internat. J. Applied Math. and Comput.*, 3(1) (2011) 60-64.
- [2644] S. K. Vaidya, V. J. Kaneria, S. Srivastav, and N. A. Dani, Gracefulness of union of two path graphs with grid graph and complete bipartite graph, *Proceedings of* the First International Conference on Emerging Technologies and Applications in Engineering Technology and Sciences (2008) 616-619.
- [2645] S. K. Vaidya, K. Kanani, S. Srivastav, and G. Ghodasara, Baracentric subdivision and cordial labeling of some cycle related graphs, *Proceedings of the First International Conference on Emerging Technologies and Applications in Engineering*, *Technology and Sciences*, (2008) 1081-1083.
- [2646] S. K. Vaidya, K. Kanani, P. Vihol, and N. Dani, Some cordial graphs in the context of barycentric subdivision, Int. J. Comtemp. Math. Sciences, 4 (2009) 1479-1492.
- [2647] S. K. Vaidya and N. J. Kothari, Some new families of line graceful graphs, Internat. J. Mathematics Sci. Comput., 1(2) (2011) 26-28.
- [2648] S. K. Vaidya and N. J. Kothari, Line gracefulness of some path related graphs, Internat. J. Math. Sci. Comput., 4(1) (2014) 15-18.
- [2649] S. K. Vaidya and B. Lekha, Odd graceful labeling of some new graphs, Modern Appl. Sci., 4 (10) (2010) 65-70.
- [2650] S. K. Vaidya and B. Lekha, Some new odd graceful graphs, Advances Appl. Disc. Math., 6(2). (2010) 101-108.
- [2651] S. K. Vaidya and B. Lekha, New families of odd graceful graphs, Internat. J. Open Problems Comp. Sci. Math., 3(5) (2010) 166-171. www.ijopcm.org/Vol/ 10/IJOPCM(vol.3.5.18.D.10).pdf
- [2652] S. K. Vaidya and B. Lekha, Some new graceful graphs, Internat. J. Math. Soft Comp., 1(1) (2011) 37-45. http://oaji.net/articles/2015/296-1433770065. pdf
- [2653] S. K. Vaidya and B. Lekha, Some new results on E-cordial labeling, Internat. J. Information Sci. Comput. Math., 3(1) (2011) 21-29.
- [2654] S. K. Vaidya and B. Lekha, Some new families of *E*-cordial graphs, *J. Math. Res.*,  $\mathbf{3}(4)$  (2011) 105-111.
- [2655] S. K. Vaidya and B. Lekha, Some new perspectives on odd sequential graphs, Internat. J. Math. Soft Comput., 4(1) (2014) 93-101.

- [2656] S. K. Vaidya and U. M. Prajapati, Some results on prime and k-prime labeling, J. Math. Research, 3(1) (2011) 66-75. http://www.ccsenet.org/journal/index. php/jmr/
- [2657] S. K. Vaidya and U. M. Prajapati, Some switching invariant prime graphs, Open J. Discrete Math., 2 (2012) 17-20. http://www.scirp.org/journal/ PaperInformation.aspx?paperID=17155
- [2658] S. K. Vaidya and U. M. Prajapati, Some new results on prime graphs, Open J. Discrete Math., 2 (2012) 99-104. doi:10.4236/ojdm.2012.23019
- [2659] S. K. Vaidya and U. M. Prajapati, Fibonacci and super Fibonacci graceful labelings of some cycle related graphs, *Inter. J. Math. Combin.*, 4 (2011) 59-69. http: //fs.gallup.unm.edu/IJMC-4-2011.pdf
- [2660] S. K. Vaidya and U. M. Prajapati, Prime labeling in the context of duplication of graph elements, *Internat. J. Math. Soft Comput.*, 3(1) (2013) 13-20. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1. 1.893.8500&rep=rep1&type=pdf
- [2661] S. K. Vaidya, U. M. Prajapati, and P. Vihol, Some important results on triangular sum graphs, Appl. Math. Sciences, 3 (2009) 1763-1772.
- [2662] S. K. Vaidya and N. H. Shah, Some new families of prime cordial graphs, J. Math. Research 3(4) (2011) 21-30.
- [2663] S. K. Vaidya and N. H. Shah, Some new odd harmonious graphs, Internat. J. Math. and Soft Comput., 1(1) (2011) 9-16.
- [2664] S. K. Vaidya and N. H. Shah, Odd harmonious labeling of some graphs, International J. Math. Combin., (3) (2012) 105-112.
- [2665] S. K. Vaidya and N. H. Shah, Prime cordial labeling of some graphs, Open J. Discrete Math., 2 (2012) 11-16. http://www.scirp.org/journal/ PaperInformation.aspx?paperID=17154
- [2666] S. K. Vaidya and N. H. Shah, 3-Equitable labeling for some tar and bistar related graphs, Internat. J. Math. and Sci. Comput., 2(1) (2012) 3-8.
- [2667] S. K. Vaidya and N. H. Shah, Graceful and odd graceful labeling of some graphs, Internat. J. of Math. Soft Computing, 3(1) (2013) 61-68.
- [2668] S. K. Vaidya and N. H. Shah, Some star and bistar related divisor cordial graphs, Annals Pure Appl. Math., 3(1) (2013) 67-77. www.researchmathsci.org
- [2669] S. K. Vaidya and N. H. Shah, Prime cordial labeling of some wheel related graphs, Malaya J. Mate., 4(1) (2013) 148-156.

- [2670] S. K. Vaidya and N. H. Shah, Further results on divisor cordial labeling, Annals Pure Appl. Math., 4(2) (2013) 150-159.
- [2671] S. K. Vaidya and N. H. Shah, Some new results on prime cordial labeling, ISRN Combin., 2014 Article ID 607018, 9 pages, 2014. http://www.hindawi.com/ journals/isrn.combinatorics/2014/607018/
- [2672] S. K. Vaidya and N. H. Shah, Cordial labeling for some bistar related graphs, Internat. J. Mathematics Soft Comput., 4(2) (2014) 33-39.
- [2673] S. K. Vaidya and N. H. Shah, Cordial labeling of snakes, Internat. J. Math. Appl., 2(3) (2014) 17-27.
- [2674] S. K. Vaidya and N. H. Shah, On square divisor cordial graphs, J. Scientific Research, 6(3) (2014) 445-455. http://www.banglajol.info/index.php/JSR/ article/view/16412
- [2675] S. K. Vaidya and N. H. Shah, Cordial labeling of degree splitting graphs, J. Graph Labeling, 1(2) (2015) 95-110.
- [2676] S. K. Vaidya, S. Srivastav, G. Ghodasara, and V. Kaneria, Cordial labeling for cycle with one chord and its related graphs, *Indian J. Math. Sciences*, 4 (2008) 145-156.
- [2677] S. K. Vaidya, S. Srivastav, V. J. Kaneria, and G. V. Ghodasara, Cordial and 3equitable labeling of cycle with twin-chords, *Proceed. First International Conference on Emerging Technologies and Applications in Engineering, Technology and Sciences*, (2008) 905-907.
- [2678] S. K. Vaidya, S. Srivastav, V. J. Kaneria, and K. K. Kanani, Some cycle related cordial graphs in the context of vertex switching, *Proceed. International Conf. Discrete Math. - 2008 RMS Lecturer Note Series*, No. **13** (2010) 243-252.
- [2679] S. K. Vaidya and P. L. Vihol, Total product cordial graphs induced by some graph operations on cycle related graphs, *Internat. J. Inform. Sci. and Comput. Math.*, 1(2) (2010) 113-126.
- [2680] S. K. Vaidya and P. L. Vihol, Cordial labeling for middle graph of some graphs, *Elixir Discrete Math.*, **34C** (2011) 2468-2475.
- [2681] S. K. Vaidya and P. L. Vihol, Embedding and NP-complete problems for 3equitable graphs, Inter. J. Contemporary Advanced Math., 2(1) (2011) 1-7.
- [2682] S. K. Vaidya and P. L. Vihol, Prime cordial labeling for some graphs, Modern Applied Science, 4(8) (2010) 119-126.
- [2683] S. K. Vaidya and P. L. Vihol, Prime cordial labeling for some cycle related graphs, Internat. J. Open Problems in Comput. Sci. Math., 3(5) (2010) 223-232.

- [2684] S. K. Vaidya, P. L. Vihol, and C. M. Barasara, 3-equitable labeling in the context of some graph operations, J. Applied Computer Science & Math., 11(5) (2011) 69-75.
- [2685] S. K. Vaidya and P. L. Vihol, Fibonacci and super Fibonacci graceful labeling of some graphs, *Studies Math. Sciences*, 2(2) (2011) 24-35.
- [2686] S. K. Vaidya and P. L. Vihol, Radio labeling for some cycle related graphs, Interna. J. Math. Soft Comput., 2(2) (2012) 11-24.
- [2687] S. K. Vaidya and N. B. Vyas, Product cordial labeling in the context of tensor product of graphs, J. Math. Research, 3(3) (2011) 83-88.
- [2688] S. K. Vaidya and N. B. Vyas, E-cordial labeling of some mirror graphs, Internat. Contemp. Adv. Math., 2(1) (2011) 22-27.
- [2689] S. K. Vaidya and N. B. Vyas, E-cordial labeling for Cartesian product of some graphs, *Studies Math. Sci.*, 3(2) (2011) 11-15.
- [2690] S. K. Vaidya and N. B. Vyas, Further results on E-cordial labeling, Internat. J. Informa. Sci. Comput. Math., 5(1) (2012) 9-18. http://www.pphmj.com/abstract/ 6905.htm
- [2691] S. K. Vaidya and N. B. Vyas, Antimagic labeling in the context of switching of a vertex, Annals Pure Appl. Math., 2(1) (2012) 33-39.
- [2692] S. K. Vaidya and N. B. Vyas, Some results on E-cordial Llabeling, Internat. J. Math. Sci. Comput., 2(1) (2012) 9-13.
- [2693] S. K. Vaidya and N. B. Vyas, E-cordial labeling in the context of switching of a vertex, Internat. J. Adv. Comput. Math. Sci., 3(4) (2012) 446-452. http:// bipublication.com/files/IJCMS-V3I4-2012-9.pdf
- [2694] S. K. Vaidya and N. B. Vyas, Even mean labeling for path and bistar related graphs, *Internat. J. Graph Theory*, 1(4) (2013) 122-130.
- [2695] S. K. Vaidya and N. B. Vyas, Antimagic labeling of some path and cycle related graphs, Annals Pure Appl. Math., 3(2) (2013) 119-128.
- [2696] S. K. Vaidya and N. B. Vyas, Product cordial labeling for alternate snake graphs, Malaya Journal of Mathematik, 2(3) (2014) 188-196. http://www. malayajournal.org/articles/MJM082.pdf
- [2697] S. K. Vaidya and N. B. Vyas, Product cordial labeling for some bistar related graphs, Adv. Modeling Optim., 16(2) (2014) 295-301. http://camo.ici. ro/journal/vol16/v16b4.pdf

- [2698] K. Valentin, Polychrome labelings of trees and cycles, Ars Combin., 52 (1999) 272-284.
- [2699] F. Van Bussel, Relaxed graceful labellings of trees, *Electronic J. Combin.*, 9 (2002) #R4. (12 pages).
- [2700] V. Vanitha, A Study on Graph Labeling of Directed Graphs, PhD thesis, Mother Teresa Women's University, 2012.
- [2701] S. К. Balasubramanian, Some Venkatesh and results generaton graceful trees, Internat. *J*. Engin. Tech., 7(4.10)(2018)570-572. ing https://www.researchgate.net/publication/332682658\_Some\_Results\_ on\_Generating\_Graceful\_Trees
- [2702] M. Varadhan and S. Guruswamy, Generating graceful trees from caterpillars by recursive attachment, *Electronic Notes Disc. Math.*, **53** (2016) 133-147.
- [2703] R. Varatharajan, S. Navanaeethakrishnan, and K. Nagarajan, Divisor cordial graphs, Internat. J. Math. Combin., 4 (2011) 15-25.
- [2704] R. Varatharajan, S. Navanaeethakrishnan, K. Nagarajan, Special classes of divisor cordial graphs, Int. Math. Forum, 7(33-36) (2012) 1737-1749.
- [2705] M. Vargheese and R. Arun, Even harmonious labelings of triangular book graphs, Caribbean J. Sci., 53(2) (2019) 2111-2117. http://www.caribjsci.com/gallery/ is2.159.pdf
- [2706] T. K. Mathew Varkey and B. S. Sunoj, Some new families of odd graceful graphs, Internat. J. Math. Combin., 3 (2016) 158-161.
- [2707] R. Vasuki, S. Arockiaraj, On super mean graphs, Util. Math., 103 (2017) 335-351.
- [2708] R. Vasuki and A. Nagarajan, Meanness of the graphs  $P_{a,b}$  and  $P_a^b$ , Internat. J. Appl. Math., **22**(4) (2009) 663-675.
- [2709] R. Vasuki and A. Nagarajan, Some results on super mean graphs, Internat. J. Math. Combin. 3 (2009) 82-96. https://zenodo.org/record/9459# .X61bqMhKiUk
- [2710] R. Vasuki and A. Nagarajan, Further results on mean graphs, Scientia Magna, 6(3) (2010) 1-14.
- [2711] R. Vasuki and A. Nagarajan, Odd mean labeling of the graphs  $P_{a,b}$ ,  $P_a^b$  and  $P_{\langle 2a \rangle}^b$ , Kragujevac J. Math. **36**(1) (2012) 141-150.
- [2712] R. Vasuki, P. Sugirtha, and J. Venkateswari, Super mean labeling of some subdivision graphs, *Kragujevac J. Math.*, 41(2) (2017) 179-201.

- [2713] R. Vasuki, L. Velmurugan, and P. Sugirtha, Some results on super pair sum graphs, J. Discrete Math. Sci. Cryptogr., 20(2) (2017) 463-476.
- [2714] D. Vickrey, k-equitable labelings of complete bipartite and multipartite graphs, Ars Combin., 54 (2000) 65-85.
- [2715] A. Vietri, Graceful labellings for an infinite class of generalised Petersen graphs, Ars. Combin., 81 (2006) 247-255. See also A. Vietri, Erratum: Ars Combin., 83 (2007) 381.
- [2716] A. Vietri, On the graceful polynomials of a graph, Austraasian J. Combin., 73(2) (2019) 261-273.
- [2717] U. Vaghela and D. Parmar, Difference perfect square cordial labeling, J. Gujarat Res. Soc., 21(15) (2019) 758-770. https://www.researchgate.net/ publication/338503111\_Difference\_Perfect\_Square\_Cordial\_Labeling
- [2718] U. Vaghela and D. Parmar, Difference perfect square cordial labeling of some graphs, J. Xidian Univer., 14(2) (2020) 87-98. https: //www.researchgate.net/publication/339676138\_Difference\_Perfect\_ Square\_Cordial\_Labeling\_of\_Some\_Graphs
- [2719] A. Villar, On the product cordial labeling for some crown graphs, U. Southeastern Philippines, Math. Stat. Depart., Davao City, Philippines, (2013) 19-86.
- [2720] R. Vimal Kumar and D. Vijayalakshmi, Vertex magic total labeling of middle and total graph of cycle, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., 68(2) (2019) 1335-1340.
- [2721] S. Venkatesh, Open cyclic grid graphs are graceful, IOSR J. Math., 10(2) (2014) 65-67. http://iosrjournals.org/iosr-jm/papers/Vol10-issue2/Version-6/ J010266567.pdf
- [2722] S. Venkatesh and S. Bharathi, On generating graceful trees, Int. J. Pure Appl. Math., (IJPAM) 118(9) 899-904.
- [2723] S. Venkatesh, B. Mahalakshmi and N. Amirthavahini, New results on some vertex labeling of graphs, *Inter. J. Pure Appl. Math.*, **118**(9) (2018) 891-898.
- [2724] S. Venkatesh, J. Salah, and G. Sethuraman, Some results on E-cordial graphs, Internat. J. Math. Trendsand Technology, 7(2) (2014) 121-125. http://www. ijmttjournal.org/archive/294-ijmtt-v7p516
- [2725] S. Venkatesh and S. Sivagurunathan, On the gracefulness of cycle related graphs, Int. J. Pure Appl. Math., (IJPAM) 117(15) (2017) 589-598.
- [2726] A. Vietri, A new infinite family of graceful generalised Petersen graphs, via "graceful collages" again, Austral. J. Comb., 41 (2008) 273-282.

- [2727] A. Vietri, Real-graceful labellings: a generalisation of graceful labellings, Ars Combin., 102 (2011) 359-364.
- [2728] A. Vietri, Necessary conditions on graceful labels: a study case on trees and other examples, Util. Math., 89 (2012) 275-287.
- [2729] A. Vietri, Sailing towards, and then against, the Graceful Tree Conjecture: some promiscuous results, *I.C.A. Bulletin*, **53** (2008) 31-46.
- [2730] G. R. Vijayakumar, Arithmetic labelings and geometric labelings of finite graphs, J. Combin. Math. Combin. Comput., 75 (2010) 229-231.
- [2731] G. R. Vijayakumar, Arithmetic labelings and geometric labelings of countable graphs, Discuss. Math. Graph Theory, 30 (2010) 539-544.
- [2732] G. R. Vijayakumar, A note on set graceful labeling, arXiv:1101.2729v1 [math.co] 14 Jan 2011.
- [2733] K. Vilfred and T. Nicholas, The integral sum graph, Graph Theory Notes N. Y., 57 (2009) 43-47.
- [2734] V. Vilfred, Families of graceful banana trees, Internat. J. Management and Systems, to appear.
- [2735] V. Vilfred, Sigma partitions and sigma labeled graphs, J. Decision Math. Sci., 10(1-3) (2005) 1-9.
- [2736] V. Vilfred, Perfectly regular graphs or cyclic regular graphs and ∑-labeling and partitions, Srinivasa Ramanujan Centenary Celebrating International Conference in Mathematics, Anna University, Madras, India, Abstract A23 (1987).
- [2737] V. Vilfred, ∑-labeled Graphs and Circulant Graphs, Ph.D., thesis University of Kerala, Trivandrum, India, (1994).
- [2738] V. Vilfred and M. Jinnah, On sigma labeled graphs, unpublished.
- [2739] V. Vilfred and L. M. Florida, Partition technique in the study of (a, d)-antimagic graphs, unpublished.
- [2740] V. Vilfred and L. M. Florida, Odd antimagic and odd (a, d)-antimagic graphs, unpublished.
- [2741] V. Vilfred and L. M. Florida, Relation between sum, integer sum, chromatic and edge chromatic numbers of few graphs, unpublished.
- [2742] V. Vilfred and L. M. Florida, Two problems on integral sum graphs, unpublished.
- [2743] V. Vilfred and L. M. Florida, Anti-sum and anti-integral sum graphs, unpublished.

- [2744] V. Vilfred and L. M. Florida, Sum number and exclusiveness of graphs  $C_4, L_n$  and  $P_3 \times P_3$ , unpublished.
- [2745] V. Vilfred and T. Nicholas, On integral sum graphs G with  $\Delta(G) = |V(G)| 1$ , unpublished.
- [2746] V. Vilfred and T. Nicholas, Banana trees and unions of stars are integral sum graphs, Ars Combin., 102 (2011) 79-85.
- [2747] V. Vilfred and T. Nicholas, Amalgamation of integral sum graphs, fans and Dutch *M*-windmills are integral sum graphs, National Seminar on Algebra and Discrete Mathematics held at Kerala Univ., Trivandrum, India, 2005, personal communication.
- [2748] Le Anh Vinh, Divisor graphs have arbitrary order and size, arXiv:0606483v1 [math.CO] 20 Jun 2006.
- [2749] V. Vilfred, S. Somasundaram, and T. Nicholas, Classes of prime graphs, International J. Management and Systems, 18(2) (2002).
- [2750] V. Vishnupriya, K. Manimekalai, and J. Baskar Babujee, Edge bimagic total labeling for some trees, Acta Cienc. Indica Math., 35 (2009) 569-573.
- [2751] C. D. Wallace, Mod Sum Numbers of Complete Bipartite Graphs, M.S. Thesis, East Tennessee State University, 1999.
- [2752] W. D. Wallis, Magic Graphs, Birkhäuser, Boston, 2001.
- [2753] W. D. Wallis, Vertex magic labelings of multiple graphs, Congr. Numer., 152 (2001) 81-83.
- [2754] W. D. Wallis, Two results of Kotzig on magic labelings, Bull. Inst. Combin. Appl., 36 (2002) 23-28.
- [2755] W. D. Wallis, Totally magic labellings and graphs, http://lagrange.math.siu. edu/mcsorley/papers/NtmgFIN.pdf
- [2756] W. D. Wallis, E. T. Baskoro, M. Miller, and Slamin, Edge-magic total labelings, Australas. J. Combin., 22 (2000) 177-190.
- [2757] G. Wang, On the number of graceful trees, Internat. J. Math. Combin., 1 (2008) 91-101.
- [2758] J.-G. Wang, D. J. Jin, X.-G Lu, and D. Zhang, The gracefulness of a class of lobster trees, *Math. Comput. Modelling*, **20** (1994) 105-110.
- [2759] H. Wang, The sum numbers and the integral sum numbers of the graph  $K_{n+1} \setminus E(K_{1,r})$ , Discrete Math., **309** (2009) 4137-4143.

- [2760] H. Wang and J. Gao, The sum numbers and the integral sum numbers of  $\overline{P_n}$  and  $\overline{F_n}$ , Ars Combin., **96** (2010) 9-31.
- [2761] H. Wang and J. Gao, The sum numbers and the integral sum numbers of  $\overline{C_n}$  and  $\overline{W_n}$ , Ars Combin., **96** (2010) 479-488.
- [2762] H. Wang and P. Li, Some results on exclusive sum graphs, J. Appl. Math. Compu., 34 (2010) 343-351.
- [2763] H. Wang and C. Li, New results on sum graph theory, Ars Combin., 106 (2012) 509-519.
- [2764] H. Wang, C. Li, and B. Wei, Some results on integral sum graphs with no saturated vertices, Util. Math., 97 (2015) 287-294.
- [2765] H. Wang, Y. Ji, and P. Li, The sum numbers and integral sum numbers of the graph  $K_n \setminus E(C_{n-1})$ , Ars Comb., **101** (2011) 15-26.
- [2766] H. Wang, J. Xu, M. Ma, and H. Zhang, A new type of graphical passwords based on odd-elegant labelled graphs, *Security and Communication Networks*, 2018, Article ID 9482345, 11 pages, doi:org/10.1155/2018/9482345
- [2767] H. Wang, J. Xu, and B. Yao. Twin odd-graceful trees towards information security, *Procedia Comput. Sci.*, **107** (2017) 15-20. doi:10.1016/j.procs.2017.03.050
- [2768] H. Wang, J. Xu, and B. Yao, Exploring new cryptographical construction of complex network data, IEEE First International Conference on Data Science in Cyberspace, *IEEE Comput. Soc.*, (2016) 155-160.
- [2769] H. Wang, J. Xu, and B. Yao, On generalized total graceful labelling of graphs, Ars Combin., 140 (2018) 359-371.
- [2770] H. Wang, X. Xu, Y. Yang, B. Zhang, M. Luo, and G. Wang, Radio number of ladder graphs, Int. J. Comput. Math., 88 (2011) 2026-2034.
- [2771] H. Wang, B. Yao, and M. Yao, Generalized edge-magic total labellings of models from researching networks, *Inform. Sci.*, **279** (2014) 460-467.
- [2772] J.-W. Wang, Vertex ranking of graphs, Master Thesis, Department of Applied Math., National Chiao-Tung University, June 1997.
- [2773] T.-L. Wong and X. Zhu, Antimagic labelling of vertex weighted graphs, J. Graph Theory, 70(3) (2012) 348-359.
- [2774] T.-M. Wang, Toroidal grids are anti-magic, Computing and combinatorics, Lecture Notes in Comput. Sci., 3595, Springer, Berlin (2005) 671-679.
- [2775] T.-M. Wang and C.-C. Hsiao, On anti-magic labeling for graph products, *Discrete Math.*, **308** (2008) 3624-3633.

- [2776] T.-M. Wang, C.-C. Hsiao, S.-M. Lee, A note on edge-graceful spectra of the square of paths, *Discrete Math.*, **308** (2008) 5878-5885
- [2777] T.-M. Wang, C.-C. Hsiao and S.-M. Lee, On anti-magic labeling of graphs, *Discrete Math.*, **308** (16) 3624-3633.
- [2778] T.-M. Wang and S. M. Lee, On edge-graceful spectrum of bicyclic graphs, Congr. Numer., 205 (2010) 161-173.
- [2779] T.-M. Wang, C.-M. Lin, and S.-M. Lee, On edge-balance index sets of regular graphs, The 26th Workshop on Combinatorial Mathematics and Computation Theory, 218-223.
- [2780] T.-M. Wang, C.-M. Lin, and S.-M. Lee, Edge-balance index sets of prisms and Möbius ladders, unpublished.
- [2781] T. Wang and D. M. Li, Gracefulness of some special graph classes, J. Wuhan Univ. Natur. Sci. Ed., 58(5) (2012) 437-440.
- [2782] T. Wang, D. Li, and Q. Wang, some classes of antimagic graphs with regular subgraphs, Ars Combin., 111 (2013) 241-250.
- [2783] T. Wang, H. S. Liu, and D. M. Li, Gracefulness of graphs related to wheels, Acta Sci. Natur. Univ. Sunyatseni 50(6) (2011) 16-19.
- [2784] T. Wang, M. J. Liu, and D. M. Li, Some classes of disconnected antimagic graphs and their joins, Wuhan Univ. J. Nat. Sci., 17(3) (2012) 195-199.
- [2785] T. Wang, M. J. Liu, and D. M. Li, A class of antimagic join graphs, Acta Math. Sin. (Engl. Ser.), 29(5) (2013) 1019-1026.
- [2786] T. Wang, W. Miao, and D. Li, Deming, Antimagic graphs with even factors, Wuhan Univ. J. Nat. Sci., 20(3) (2015) 193-196.
- [2787] T. Wang, Q. Wang, and D. M. Li, Gracefulness of unconnected graphs  $(P_3 + \overline{K_m} \cup G)$ and  $(C_3 + \overline{K_m} \cup G)$ , Natur. Univ. Sunyatseni, **51**(5) (2012) 54-57, 62.
- [2788] T. Wang, Q. Wang, and D. M. Li, Gracefulness of disconnected graph  $W_m^{(k)} \cup G$ , J. Hefei Univ. Technol. Nat. Sci., **35**(7) (2012) 987-990.
- [2789] T. Wang, S.-F. Yan, D. M. Li, Graceful labeling of the union of some graphs, Math. Pract. Theory, 42(16) (2012) 207-212.
- [2790] T.-M. Wang, C.-C. Yang, L.-H. Hsu, and E. Cheng, Infinitely many equivalent versions of the graceful tree conjecture, Appl. Anal. Discrete Math., 9(1) (2015) 1-12.

- [2791] T.-M. Wang and G.-H. Zhang, Note on *E*-super vertex magic graphs, *Discrete Appl. Math.*, **178** (2014) 160-162.
- [2792] T.-M. Wang and G.-H. Zhang, On antimagic labeling of odd regular graphs, *Lecture Notes in Comput. Sci.*, S. Arumugam and B. Smyth (Eds.): IWOCA 2012, LNCS 7643 162-168, Springer-Verlag Berlin Heidelberg 2012.
- [2793] T.-M. Wang and G.-H. Zhang, On vertex magic total labeling of disjoint union of sun graphs, Util. Math., 103 (2017) 289-298.
- [2794] T.-M. Wang and G.-H. Zhang, On edge-graceful regular graphs with particular 3-factors, *Elect. Notes Discr. Math.*, 63 (2017) 33-40.
- [2795] T.-M. Wang and G.-H. Zhang, On edge-graceful labeling and deficiency for regular graphs, AKCE Int. J. Graphs Comb., 15(1) (2018) 105-111.
- [2796] W. Wang, M. Zhang, H. Yu, and D. Shi, The mod sum number of even fans and symmetric complete bipartite graphs, Ars Combin., 94 (2010) 445-457.
- [2797] X. Wang and Y. Chang, Further results on (v, 4, 1)-perfect difference families, Discrete Math., **310** (2010) 1995-2006.
- [2798] Y. Wang and B. Liu, The sum number and integral sum number of complete bipartite graphs, *Discrete Math.*, **239** (2001) 69-82.
- [2799] W. Wannasit and S. El-Zanati, On free α-labelings of cubic bipartite graphs, 24th MCCCC. J. Combin. Math. Combin. Comput., 82 (2012) 269-293.
- [2800] M. E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, J. Combin. Theory, 6 (1969) 152-164.
- [2801] L. X. Wei and Z. Z. Jia, The gracefulness of unconnected graphs  $G_1 \cup G_2$  and  $G_1 \cup G_2 \cup K_2$ , Acta Math. Appl. Sin., **28** (2005) 689-694.
- [2802] L. X. Wei, S. F. Yan, and K. L. Zhang, The researches on gracefulness of two kinds of unconnected graphs, J. Shandong Univ. Nat. Sci. 43 (2008) 7 pp.
- [2803] L. X. Wei and K. L. Zhang, Researches on graceful graphs  $(P_1^{(1)} \vee P_n) \cup (P_1^{(2)} \vee P_{2n})$ and  $(P_2 \vee \overline{K}_n) \cup G_{n-1}$ . J. Hefei Univ. Tech., **31** (2008) 276-279.
- [2804] J. Wei, T, Wang, and C. Sun, Gracefulness of several unconnected graphs with wheel, Wuhan Univ. J. Nat. Sci., 21(3) (2016) 199-203.
- [2805] L. X. Wei and K. L. Zhang, Graceful graphs on several kinds of union graphs, Acta Sci. Natur. Univ. Sunyatseni, 47 (2008) 10-13.
- [2806] Y.-H. Wen, Necessary and sufficient condition for the class of (P, P + 1)-graph to be balanced, J. Lanzhou Univ. Nat. Sci., 43 (2007) 104-106.

- [2807] Y.-H. Wen, Friendliness of generalized wheel, J. Lanzhou Univ. Nat. Sci., 44 (2008) 103-108.
- [2808] Y. Wen and S. M. Lee, On Eulerian graphs of odd sizes which are fully magic, unpublished.
- [2809] Y. Wen, S.-M. Lee, H.-H. Su, On  $Z_2 \oplus Z_2$ -magic graphs, Congr. Numer., 199 (2009) 13-31.
- [2810] Y. Wen, S. M. Lee, and H. Sun, On the supermagic edge-splitting extension of graphs, Ars Combin., 79 (2006) 115-128.
- [2811] R. W. Whitty, Rook polynomials on 2-dimensional surfaces and graceful labellings of graphs, *Discrete Math.*, 308 (2008) 674-683. http://myweb.lsbu.ac. uk/~ruthercg/MathsStudyGroup/SlidesAndNotes/WhittyBCC.pdf
- [2812] T. Wichianpaisarn and U. Mato, Star-supermagic decompositions of the complete bipartite graph minus a one-factor, *Internat. J. Math. Math. Sci.*, (2017) Article ID 5104701, 4 pp.
- [2813] K. Wijaya and E. Baskoro, Edge-magic labelings of a product of two graphs (in Indonesian), Proc. Seminar MIPA, ITB Bandung Indonesia, October 2000.
- [2814] K. Wijaya, Slamin, Surahmat, and S. Jendrol, Total vertex irregular labeling of complete bipartite graphs, J. Combin. Math. Combin. Comput., 55 129-136.
- [2815] S. Wilson and A. Riskin, Edge-graceful labellings of odd cycles and their products, Bulletin of the ICA, 24 (1998) 57-64.
- [2816] S. Winters, personal communication.
- [2817] J. Wojciechowski, Long Induced Cycles in the Hypercube and Colourings of Graphs, Ph. D. Thesis, Cambridge University, England, 1990.
- [2818] J. Wojciechowski, Equitable labelings of cycles, J. Graph Theory, 17 (1993) 531-547.
- [2819] T. Wong and X. Zhu, Antimagic labelling of vertex weighted graphs, J. Graph Theory, 70(3) (2012) 348-350.
- [2820] D. Wood, On vertex-magic and edge-magic total injections of graphs, Australas. J. Combin., 26 (2002) 49-63.
- [2821] J. Wu, J. Mao, and D. Li, New types of integral sum graphs, Discrete Math., 260 (2003) 163-176.
- [2822] S.-L. Wu, Graceful labelings of graphs associated with vertex-saturated graphs, Ars Combin., 62 (2002) 109-120.

- [2823] S.-L. Wu, New families of sequential graphs, Ars Combin., 69 (2003) 9-17.
- [2824] S.-L. Wu, New graceful families on bipartite graphs, Ars Combin., 73 (2004) 79-87.
- [2825] S.-L. Wu, A necessary condition for the existence of an  $\alpha$ -labeling, unpublished.
- [2826] S.-L. Wu, Graceful labelings of vertex-saturated graphs and related graphs, unpublished.
- [2827] S.-L. Wu and H.-C. Lu, On the constructions of new families of graceful graphs, Ars Combin., 106 (2012) 235-246.
- [2828] C. Wyels and M. Tomova, Radio Labeling Cartesian graph products, 9th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, Cologne, Germany, May 25-27, 2010. Extended Abstracts (2010) 163167.
- [2829] Y. Xi, Y. Yang, Mominul, and L. Wang, Super vertex-magic total labelings of  $W_{3,n}$ , Ars Combin., 86 (2008) 121-128.
- [2830] Y. Xi, Y. Yang, and L. Wang, On harmonious labeling of the double triangular snake, *Indian J. Pure Appl. Math.*, **39** (2008) 177-184.
- [2831] L. T. Xie and G. Z. Liu, A survey of the problem of graceful trees, Qufu Shiyuan Xuebao, (1984) 8-15.
- [2832] B. Xu, On integral sum graphs, *Discrete Math.*, **194** (1999) 285-294.
- [2833] S. D. Xu, Cycles with a chord are harmonious, Mathematica Applicata, 8 (1995) 31-37.
- [2834] S. D. Xu, Harmonicity of triangular snakes, J. Math. Res. Exposition, 15 (1995) 475-476.
- [2835] X. Xu, Jirimutu, L. Wang, and Z. Min, On the gracefulness of the digraphs  $n-C_m$ , Ars Combin., **92** (2009) 419-428.
- [2836] X. Xu, J.-M. Xu, M. Lü, Z. Baosheng, and C. Nan, On (a, d)-antimagic labelings of generalized Petersen graphs, Ars Combin., 90 (2009) 411-421.
- [2837] X. Xu, Y. Yang, H. Li, and Y. Xi, The graphs  $C_{11}^{(t)}$  are graceful for  $t \equiv 0, 1 \pmod{4}$ . Ars Combin., 88 (2008) 429-435.
- [2838] X. Xu, Y. Yang, L. Han, and H. Li, The graphs  $C_{13}^{(t)}$  are graceful for  $t \equiv 0, 1 \pmod{4}$ . Ars Combin., **90** (2009) 25-32.
- [2839] X. Xu, Y. Yang, Y. Xi, K. M. M. Haque, and L. Shen, Super edge-magic labelings of generalized Petersen graphs P(n,3), Ars Combin., 85 (2007) 19-31.

- [2840] X. Xu, Y. Yang, Y. Xi, and H. Li, On (a, d)-antimagic labelings of generalized Petersen graphs P(n, 3), Ars Combin., 86 (2008) 23-31.
- [2841] L. Yahyaei and S. A. Katre, Representation number of a caterpillar, J. Combin. Math. Combin. Comput., 101 (2017) 47-57.
- [2842] Q. T. Yan, A proof of a conjecture related to of windmill graphs, Math. Practice Theory, 34 (2004) 116-117.
- [2843] Q. T. Yan, The graceful labelings of  $P_{2r,2m}$ , J. Systems Sci. Math. Sci., **26**(5) (2006) 513-517.
- [2844] Q. T. Yan, Odd gracefulness and odd strongly harmoniousness of the product graphs  $P_n \times P_m$ , J. Systems Sci. Math. Sci., **30** (2010) 341-348.
- [2845] W. Yan and B. Liu, Some results on integral sum graphs, Discrete Math., 240 (2001) 219-229.
- [2846] W. Yan and B. Liu, The sum number and integral sum number of complete bipartite graphs, *Discrete Math.*, 239(1-3) (2001) 69-82.
- [2847] H. Yang, M. K. Siddiqui, M. Ibrahim, S. Ahmad, and A. Ahmad, Computing the irregularity strength of planar graphs, *Mathematics*, (2018) 6, 150.
- [2848] D. Yang, J. Carlson, A. Owens, K. E. Perry, I. Singgih, Z.-X. Song, F. Zhang, X. Zhang, Antimagic orientations of graphs with large maximum degree, preprint.
- [2849] D. Yang, A note on antimagic orientations of even regular graphs, Discrete Appl. Math., 267 (2019) 224-228. https://doi.org/10.1016/j.dam.2019.04.017)
- [2850] X.-W Yang and W. Pan, Gracefulness of the graph  $\bigcup_{i=1}^{n} F_{m_i,4}$ , J. Jilin Univ. Sci., **41** (2003) 466-469.
- [2851] Y. Yang, X. Lin, C. Yu, The graphs  $C_5^{(t)}$  are graceful for  $t \equiv 0, 3 \pmod{4}$ , Ars Combin. 74 (2005) 239-244.
- [2852] Y. Yang, W. Lu, and Q. Zeng, Harmonious graphs  $C_{2k} \cup C_{2j+1}$ , Util. Math., 62 (2002) 191-198.
- [2853] Y. S. Yang, Q. Rong, and X. R. Xu, A class of graceful graphs, J. Math. Research and Exposition, 24 (2004) 520-524.
- [2854] Y. C. Yang and X. G. Wang, On the gracefulness of the product  $C_n \times P_2$ , J. Math. Research and Exposition, 1 (1992) 143-148.
- [2855] Y. C. Yang and X. G. Wang, On the gracefulness of the union of two stars and three stars, *Combinatorics, Graph Theory, Algorithms and Applications (Beijing,* 1993), 417-424, World Sci. Publishing, River Edge, NJ, 1994.

- [2856] Y. C. Yang and X. G. Wang, On the gracefulness of product graph  $C_{4n+2} \times P_{4m+3}$ , Combinatorics, Graph Theory, Algorithms and Applications (Beijing, 1993), 425-431, World Sci. Publishing, River Edge, NJ, 1994.
- [2857] Y. Yang, X. Xu, Y. Xi, H. Li, and K. Haque, The graphs  $C_7^{(t)}$  are graceful for  $t \equiv 0, 1 \pmod{4}$ , Ars Combin., **79** (2006) 295-301.
- [2858] Y. Yang, X. Xu, Y. Xi, and H. Huijun, The graphs  $C_9^{(t)}$  are graceful for  $t \equiv 0, 3 \pmod{4}$ , Ars Combin., 85 (2007) 361-368.
- [2859] Y. Yang, X. Xu, Y. Xi, and J. Qian, On harmonious labelings of the balanced quadruple shells. Ars Combin., 75 (2005) 289-296.
- [2860] B. Yao, X. Chen, M. Yao, and H. Cheng, On  $(k, \lambda)$ -magically total labeling of graphs, *JCMCC*, 87 (2013) 237-253.
- [2861] B. Yao, H. Cheng, M. Yao, and M. Zhao, Meimei, A note on strongly graceful trees, Ars Combin., 92 (2009) 155-169.
- [2862] B. Yao, H. Cheng, Z. Zhongfu, and M. Yao, Labellings of trees with larger maximal degrees, unpublished.
- [2863] B. Yao, X. Liu and M. Yao, Connections between labellings of trees, Bull. Iranian Math. Soc., 43(2) (2017) 275-283.
- [2864] B. Yao, M. Yao, and H. Cheng, On gracefulness of directed trees with short diameters, Bull. Malays. Math. Sci. Soc., (2) 35(1) (2012) 133-146.
- [2865] V. Yegnanarayanan, On some additive analogues of graceful theme: cycle-related graphs, Southeast Asian Bull. Math., 23 (1999) 317-333.
- [2866] V. Yegnanarayanan, On magic graphs. Util. Math., 59 (2001) 181-204.
- [2867] V. Yegnanarayanan and P. Vaidhyanathan, On nice (1, 1) edge-magic graphs, *Electronic Notes Discr. Math.*, **33** (2009) 115-122.
- [2868] Z. B. Yilma, Antimagic properties of graphs with large maximum degree, J. Graph Th., 72(4) (2013) 367-373.
- [2869] R. Yilmaz and I. Cahit, E-cordial graphs, Ars Combin., 46 (1997) 251-266.
- [2870] L. You, J. Yang and Z. You, The maximal total irregularity of unicyclic graphs, Ars Combin., 114 (2014) 153-160.
- [2871] M. Z. Youssef, On Graceful, Harmonious and Prime Labelings of graphs, Ph. D. Thesis, Department of Mathematics, Ain Shams University, 2000.
- [2872] M. Z. Youssef, New families of graceful graphs, Ars Combin., 67 (2003) 303-311.

- [2873] M. Z. Youssef, Two general results on harmonious labelings, Ars Combin., 68 (2003) 225-230.
- [2874] M. Z. Youssef, A necessary condition on k-equitable labelings, Util. Math., 64 (2003) 193-195.
- [2875] M. Z. Youssef, Pseudograceful labelings of graphs, Ars Combin., 76 (2005) 241-247.
- [2876] M. Z. Youssef, On Skolem-graceful and cordial graphs, Ars Combin., 78 (2006) 167-177.
- [2877] M. Z. Youssef, On k-cordial labelling, Australas. J. Combin., 43 (2009) 31-37.
- [2878] M. Z. Youssef, Graph operations and cordiality, Ars Combin., 97 (2010) 161-174.
- [2879] M. Z. Youssef, On  $E_k$ -cordial labeling, Ars Combin., 104 (2012) 271-279.
- [2880] M. Z. Youssef, On k-equitable and k-balanced labeling of graphs, Ars Combin., 108 (2013) 51-64.
- [2881] M. Z. Youssef, On  $\alpha$ -labelling of disconnected graphs, Ars Comb., **123** (2015) 329-338.
- [2882] M. Z. Youssef, Note on even harmonious labeling of cycles, Util. Math., 100 (2016) 137-142.
- [2883] M. Z. Youssef, personal communication.
- [2884] M. Z. Youssef, On even harmonious labelings, Util. Math., 110 (2019) 243-253.
- [2885] M. Z. Youssef and N. A. Al-Kuleab, Some new results on k-cordial labeling, Ars Combin., 126 (2016) 311-322.
- [2886] M. Z. Youssef and N. A. Al-Kuleab, 4-cordiality of some regular graphs and the complete 4-partite graph, J. Combin. Math. Combin. Comput., 76 (2011) 131-136.
- [2887] M. Z. Youssef and N. A. Al-Kuleab, 3-equitable labelings of ladders and graphs related to cycles, Utilitas Math., 92 (2013) 261-274.
- [2888] M. Z. Youssef and E.A. El Sakhawi, Some properties of prime graphs, Ars Combin., 84 (2007) 129-140.
- [2889] P. Yu, A proof of a conjecture about arithmetic graphs, J. Math. Res. Exposition, 16 (1996) 594-598.
- [2890] P. Yu, Strongly arithmetic graphs, Chinese Quart. J. Math., 15 (2000) 22-27.

- [2891] T.-K. Yu, D. T. Lee, and Y.-X. Chen, Graceful and harmonious labelings on 2cube, 3-cube and 4-cube snakes, unpublished.
- [2892] Y.-H. Yu, W.-X. Wang, and L.-x. Song, The gracefulness of a kind of unconnected graphs, Adv. Intelligent Soft Comput., 128 (2012) 525-531.
- [2893] X. Yue, Y. Yuan-sheng, and M. Xin-hong, Skolem-gracefulness of k-stars, Ars Combin., 98 (2011) 149-160.
- [2894] J. Yuan and W. Zhu, Some results on harmonious labelings of graphs, J. Zhengzhou Univ. Nat. Sci. Ed., 30 (1998) 7-12.
- [2895] Y. Yuansheng, X. Yue, X. Xirong, and M. Xinhong, Super edge magic labelings of book graphs B<sub>n</sub>, unpublished. Abstract at https://www.researchgate.net/ publication/268023455\_Super\_edge-magic\_labelings\_of\_book-\_graphs\_B\_ n
- [2896] X. Yue, Y. Yuansheng, and W. Liping, One edge union of k shell graphs is cordial, Ars Combin., 86 (2008) 403-408.
- [2897] A. Zak, Harmonious orders of graphs, *Discrete Math.*, **309** (2009) 6055-6064.
- [2898] M. R. Zeen El Deen, Edge even graceful labeling of some graphs, J. Egyptian Math. Soc., 27(20) (2019) 1-15.
- [2899] M. R. Zeen El Deen and N. A. Omar, Further results on edge even graceful labeling of the join of two graphs, J. Egyptian Math. Soc., 28, article 21, (2020) https: //doi.org/10.1186/s42787-020-00077-5
- [2900] D. Zhang, Y-S. Ho, S. M. Lee, and Y. Wen, On the balance index sets of trees with diameter at most four, unpublished.
- [2901] D. Zhang, S. M. Lee, and L. Wen, On the balancedness of galaxies with at most four stars, J. Combin. Math. Combin. Comput., 50 (2004) 3-15.
- [2902] F. Zhang, S. Nazeer, M. Habib, T. J. Zia, and Z. Ren, Radio number for generalized Petersen graphs P(n, 2), *IEEE Access* (2019) PP(99):1-9.
- [2903] G.-H. Zhang and T.-M. Wang, E-super vertex magic regular graphs of odd degree, The Eighth International Workshop on Graph Labelings (IWOGL 2014), Electron. Notes Discrete Math., 48 (2015) 49-56.
- [2904] P. Zhang, Radio labelings of cycles, Ars Combin., 65 (2002) 21-32.
- [2905] X. Zhang, H. Sun, and B. Yao, Graph theory towards module-K odd-elegant labelling of graphical passwords, MATEC Web of Conferences 139, 00206 (2017) *ICMITE* 2017 doi:10.1051/matecconf/201713900206

- [2906] Z. Zhang, Q. Zhang, and C. Wang, On (a, b; n)-graceful labeling of path  $P_n$ , Ars Combin., **132** (2017) 110-119.
- [2907] L. Zhao, W. Feng, and Jirimutu On the gracefulness of the digraphs  $n \vec{C}_m$ , Util. Math., 82 (2010) 129-134.
- [2908] L. Zhao and Jirimutu, On the gracefulness of the digraph  $n \overline{C}_m$  for m = 5, 7, 9, 11, 13, J. Prime Res. Math., **12**(4)Classes of prime graphs, (2008) 118-126.
- [2909] L. Zhao, Siqintuya, and Jirimutu, On the gracefulness of the digraphs  $n \vec{C}_m$ , Ars Combin., **99** (2011) 421-428.
- [2910] S. L. Zhao, All trees of diameter four are graceful, Graph Theory and its Applications: East and West (Jinan, 1986), 700-706, Ann. New York Acad. Sci., 576, New York Acad. Sci., New York, 1989.
- [2911] X. Y. Zhao, F. Ma, Yao, B. Yao, A class of trees having strongly graceful labellings, J. Jilin Univ. Sci., 54(2) (2016) 222-228.
- [2912] Y. Zhao, personal communication.
- [2913] G. Zhenbin, The labelings of a variation of banana trees, Ars Combin., 94 (2010) 175-181.
- [2914] G. Zhenbin, (a, d)-edge-antimagic total labelings of cycle, Ars Combin., **101** (2011) 217-223.
- [2915] G. Zhenbin and F. Chongjin, Some discussions on super edge-magic labelings of  $St(a_1, \ldots, a_n)$ , Ars Combin. 108 (2013) 187-192.
- [2916] L. Zhihe, The balanced properties of bipartite graphs with applications, Ars Combin., 48 (1998) 283-288.
- [2917] S. C. Zhou, Gracefulness of the graph  $K_m \cup K_n$ , J. Lanzhou Railway Inst., 12 (1993) 70-72.
- [2918] S. Zhou, Unifying approaches for constructing labeled graphs from known ones, J. Combin., Inform. System Sci., 20 (1995) 305-319.
- [2919] S. Zhou and J. Yuan, On constructions of sequential graphs, Mathematica Applicata, 6 (Supplement) (1993) 104-108.
- [2920] X. Zhou, B. Yao and X. Chen, Every lobster is odd-elegant, Infor. Process. Letters, 113 (2013) 30-33.
- [2921] X. Zhou, B. Yao, X. Chen, and H. Tao, A proof to the odd-gracefulness of all lobsters, Ars Combin., 103 (2012) 13-18.

[2922] Z. Zhu, and C. Liu, Characterizations and structure of sequential graphs, Ars Combin., 116 (2014) 279-288.

## Index

 $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial, 279  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial graph, 279  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial labeling, 279 (a, d)-F-antimagic, 207 (a, d)-1-vertex-antimagic vertex, 209 (a, d)-distance antimagic, 203 (a, d) - D antimagic, 208 (a, r)-geometric, 324 (k, d)-Heronian mean, 311 (k, d)-Skolem graceful, 76 (k, d)-graceful labeling, 73 (k, d)-hooked Skolem graceful, 36 (m, n)-gon star, 244  $< K_{m_1,n_1}, \ldots, K_{m_t,n_t} >, 32$ A-antimagic, 198A-cordial graph, 91 A-magic, 186 B(n, r, m), 23 $B_{n,n}^*, 276$  $B_m, 21$  $B_{n,n}, 276$  $C(G_1, G_2, \ldots, G_n), 32$  $C(n \cdot G), 32$  $C_n, 110$  $P_n, 94$  $C_m * C_n, 332$  $C_{n,k}^+, 15$  $C_n^{(t)}, 16$ D-distance, 203 D-distance antimagic, 203 D-distance magic, 208 D-weight, 208  $DSt_n$ , 33  $D_2(G), 78, 103$  $D_m(G), 79, 122$ E-super vertex magic, 166  $E_k$ -cordial, 91  $E_k$ -regular, 167  $Pl_n, 34$ F-geometric mean, 305 R-ring-magic, 189  $F_n, 45$ 

 $Fl_n$ , 256, 281  $G \odot H$ , 18  $G \otimes H, 74$  $G^*, 87, 95$ G', 32 $G_1 \oplus G_2, 49$  $G_1[G_2], 22$ H-E-super magic, 177 H-E-super magic decomposable, 177 H-cordial, 90 *H*-covering, 142, 213, 321 H-decomposable, 176 *H*-graph of  $P_n$ , 295 H-graph of a path, 301 H-magic, 172 H-supermagic strength, 176 H-union, 104 H-V-super magic decomposable, 176  $H_n, 295$  $H_n$ -cordial, 90  $H_n$ -graph, 110  $JF_n, 332$  $J_n, 332$ KP(r, s, l), 63 $K_n^{(m)}, 22$ M(G), 35, 87 $M_m(G), 122$  $M_n, 21$ P(G, f), 35P(n,k), 29 $P(n \cdot G), 32$  $P_{n}^{t}, 33$  $P_n^t(tn \cdot H), 33$  $K_{1.n}, 112$  $P_n^k, 28$  $P_t(G), 88$  $P_t(u, v), 88$  $P_{a,b}, 30, 292$ 

 $R_{\rm m}(G), 35$  $S(G_1, G_2, \ldots, G_n), 33$  $S(n \cdot G), 33$  $S_m, 21$  $S_n, 211$  $S_{m,n}, 105$  $Spl_{m}(G), 122$ St(n), 26 $St(n_1, n_2, \ldots, n_k), 75$ T(G), 99 $T(P_n), 35$  $T_p$ -tree, 73 W(t, n), 13 $\Gamma$ -distance magic, 182  $\alpha$ -labeling eventually, 53 free, 57 near, 58 strong, 56 weakly, 56, 66  $\alpha$ -deficit, 53  $\alpha$ -labeling, 16, 46, 66, 77  $\alpha$ -mean labeling, 293  $\alpha$ -size, 55  $\alpha$ -valuation, 46  $\beta$ -valuation, 5  $\delta$ -optimal, 241  $\delta$ -optimal summable, 241  $\gamma$ -labeling, 64  $\hat{\rho}$ -labelings, 60  $\rho$ -labeling, 61  $\rho$ -valuation, 61  $\rho^+$ -labeling, 63  $\rho^{\star}, 62$  $\theta$ -labeling, 63  $\tilde{\rho}$ -labelings, 66 a-vertex consecutive bimagic labeling, 193 a-vertex consecutive magic labeling, 192 a-vertex multiple magic, 131 b-edge consecutive magic labeling, 192 b-edge multiple magic, 132 d-antimagic, 201 d-graceful, 55

f-permutation graph, 35 k-cordial labeling, 91 k-difference cordial, 286 k-even mean labeling, 300 k-even mean graph, 300 k-even sequential harmonious, 112 *k*-fold, 156 k-graceful, 70 k-graceful digraph, 75 k-magic, 133 k-mean graph, 295 k-multilevel corona, 128 k-prime, 248 k-prime cordial, 290 k-product cordial, 277 k-ranking, 328 minimal, 328 k-remainder cordial, 94 k-super mean, 295 k-total product cordial, 277 k-totally magic cordial, 190 k-ubiquitously graceful, 9 k-vertex amalgamation, 51  $kC_n$ -snake, 18, 61 linear, 18 m-gracefulness, 66 m-mirror graph, 122 m-shadow graph, 122 m-splitting graph, 122 mG, 24*n*-cone, 13 n-cube, 21, 46 n-point suspension, 13 nth quadrilateral snake, 333  $n \cdot C_m, 37$ r-distant irregular, 323 r-distant irregularity strength, 323  $rn \star (f), 267$ s(G), 314 $s_a(G), 320$ *t*-fold, 51 t-join (m, n) kite, 339 t-ply graph, 88

ts(G), 321w-graph, 140 w-tree, 140 y-tree, 10 0-magic, 190 1-vertex bimagic, 184 2-link fence, 51 3-equitable prime cordial, 291 3-product cordial, 275 3-total super sum cordial graph, 278 3-total super sum cordial labeling, 278 hexagonal snake, 17 abbreviated double tree of T, 134 absolutely harmonious graph, 111 additively (a, r)-geometric, 324 adjacency matrix, 62 almost graceful labeling, 61 almost-bipartite graph, 64 alpha-number, 153 alternate hexagonal snake, 17 alternate quadrilateral snake, 248, 273 alternate quadrilateral snake, 284 alternate shell, 87 alternate triangular snake, 248, 273, 284 amalgamation, 175 analytic mean graph, 310 antimagic orientation, 200 antiprism, 176, 207, 228, 320 apex, 15, 103 arank number, 329 arbitrarily distance antimagic, 203 arbitrarily graceful, 70 arbitrary supersubdivision, 30, 87 arithmetic, 113 armed crown, 283 balance index set, 100 balanced cordial, 95 balanced distance graphs, 183 bamboo tree, 8, 78 banana tree, 11, 68, 78 barycentric subdivision, 33

bent ladder, 329 beta combination graph, 340 beta-number, 65 bi-odd sequential, 109 bicomposition, 63 bigraceful graph, 36 bipartite labeling, 55 bistar, 144, 149 block, 17, 152 block graph, 342 block-cut-vertex graph, 152 block-cutpoint, 49 block-cutpoint graph, 17 book, 6, 16, 21, 139 generalized, 246 stacked, 21 boundary value, 52 bow graph, 15 broom, 129  $C_n, 186$ cactus k-angular, 81 triangular, 17 Cartesian product, 19, 256 caterpillar, 8, 46, 58, 68, 107, 145 caterpillar cycle, 328 cells, 49 chain graph, 49, 153 chain of cycles, 15 chain tree, 50 chord, 14 chordal ring, 166, 208 circulant graph, 130 circular lobster, 329 closed helm, 13 coalescence, 49 cocktail party graph, 122, 166, 239 coconut tree, 252 comb tree, 328combination graph, 338 combs, 32complete

n-partite graph, 83, 234 bipartite graph, 17, 22 graph, 22 tripartite graph, 22 complete mixed generalized sausage graph, 196complete star, 317 component, 249 composition, 22, 81, 256 conjunction, 272 consecutive radio labeling, 268 consecutively super edge-magic, 148 consecutively super edge-magic deficiency, 148 contraharmonic mean, 305 convex polytope, 177, 227 cordial graph, 83 cordial labeling, 81 corona, 18, 139 covering, 213 critical number, 52 crown, 18, 105, 107, 236, 266 cube, 20, 36 cube divisor cordial, 281 cubic graph, 155 cycle, 5, 239 cycle of a graph, 189 cycle of graphs, 32, 294 cycle with a  $P_k$ -chord, 14 cycle with parallel  $C_k$  - chord, 15 cycle with parallel  $P_k$  chords, 14 cyclic G-decomposition, 58 cyclic decomposition, 62 cylinders, 177 decomposition, 5, 46, 58, 60, 64 deficiency edge-magic, 151 super edge-magic, 151 degree splitting graph, 286 degree-magic, 128 difference cordial labeling, 284 difference graph, 335

disjoint union, 24 distance k-antimagic, 202 distance antimagic, 202 distance magic labeling, 180 divisor cordial, 279 divisor graph, 342 dodecahedron, 36 double alternate hexagonal snake, 17 double alternate quadrilateral snake, 248, 273double alternate quadrilateral snake, 284 double alternate triangular snake, 248, 273, 284double coconut trees, 252 double cone, 13 double fans, 32 double graph of G, 129 double hexagonal snake, 17 double path union, 74 double quadrilateral snake, 248, 273, 284 double star, 134 double step grid graph, 33 double tree, 134 double triangular snake, 248, 273, 284, 295 dragon, 15 duplication of a vertex, 32, 273 duplication of an edge, 32, 87, 273 Dutch t-windmill, 16 Dutch windmill, 132  $\operatorname{EBI}(G), 101$ edge H-irregularity strength, 321 edge amalgamation, 246 edge bimagic total, 184

direct product, 181

directed edge-graceful, 264

directed graceful graph, 38

edge amalgamation, 246 edge bimagic total , 184 edge even graceful labeling, 67 edge irregular total labeling, 315 edge irregularity strength, 322 edge linked cyclic snake, 292 edge magic graceful, 142 edge magic strength, 132 edge pair sum, 312 edge parity, 56 edge product cordial labeling, 282 edge reduced integral sum number, 238 sum number, 238 edge trimagic total labeling, 155 edge-antimagic graceful, 209 edge-antimagic total, 199 edge-balance index, 101 edge-covering, 321 edge-decomposition, 58 edge-friendly index, 99 edge-graceful deficiency, 257 edge-graceful spectrum, 258 edge-magic index, 156 edge-magic injection, 144 edge-odd graceful, 78, 80 edge-prime graph, 251 ehs(G, H), 321elegant, 116 elegant labeling, 116 elem. parallel transformation, 73 elementary transformation, 48, 141 envelope graph, 101 EP-cordial graph, 277 EP-cordial labeling, 276 Eulerian graph, 102 even 2a-sequential, 126 even 1-vertex bimagic, 184 even graceful, 51 even mean labeling, 300 even vertex equitable even, 334 even vertex magic total, 169 even vertex odd mean, 301 even-even, 80 exclusive sum labeling, 239 exclusive sum number, 239 extended w-tree, 140 extended edge vertex cordial labeling, 97 extended jewel graph, 309

face, 177, 227

face irregular total k-labeling, 321 fan, 45, 116, 127, 137, 139, 150, 166, 177, 243fence, 51 FI(G), 97Fibonacci graceful, 69 firecracker, 11 flag, 84, 112, 264 flower, 13, 164, 243 forest, 151 free  $\alpha$ -labeling, 57 friendly index set, 97 friendship graph, 16, 81, 150, 164, 166, 177, 240full r-ary tree, 10 full edge-friendly index, 99 full friendly index set, 101 full hexagonal caterpillars, 51 full product-cordial index, 275 fully magic, 187 fully product-cordial, 275 functional extension, 134 gamma-number, 37 gear graph, 13 generalized book, 246 bundle, 89 fan, 89 wheel, 89 generalized  $kC_n$ -snake, 292 generalized antiprism, 220 generalized caterpillar, 31 generalized edge linked cyclic snake, 292 generalized helm, 164, 317 generalized Jahangir graph, 164 generalized prisms, 267 generalized sausage graph, 196 generalized shackle, 212 generalized spider, 31 generalized web, 13, 164 geometric mean 3-equitable, 314 geometric mean cordial, 314
Golomb ruler, 24 graceful almost super Fibonacci, 69 graceful center, 57 graceful graph, 5 gracesize, 55 gracious k-labeling, 58 gracious labeling, 57 graph, 132, 285, 299  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial, 279  $(\omega, k)$ -antimagic, 201 (a, d)-F-antimagic, 207 (a, d)-antimagic, 205 (a, d)-distance antimagic, 203 (a, r)-geometric, 324 (k+1)-equitable mean, 314  $(k, \lambda)$ -magically total labeling, 185 (k, d)-Heronian mean, 311 (k, d)-Skolem graceful, 76 (k, d)-arithmetic, 113 (k, d)-balanced, 74 A-cordial, 91 C-geometric, 306 D-distance, 203 D-distance antimagic, 203 E-cordial, 263 E-super vertex magic, 166  $E_k$ -cordial, 91 G-distance magic, 182 G-snake, 17 H-cordial, 90 H-elegant, 117 H-groupmagic, 131 H-harmonious, 117  $H_k$ -cordial, 90  $H_n$ -cordial, 90  $V_k$ -super vertex magic, 167  $\Delta$ -optimum summable, 240  $\Gamma$  irregular, 320  $\theta$ -Petersen, 259 a-vertex multiple magic, 131 b-edge multiple magic, 132 d-graceful, 55

f-permutation, 35 q-graph, 238 k-antimagic, 201 k-balanced, 96, 104 k-difference cordial, 286 k-edge-magic, 133 k-even edge-graceful, 258 k-magic, 133 k-modular multiplicative, 326 k-multilevel corona, 128 k-prime cordial, 290 k-product cordial, 277 k-ubiquitously, 9 m-level wheel, 260 m-mirror, 122 m-shadow, 122 m-splitting, 122 n-uniform, 317 n-uniform cactus chain, 317 t-uniform homeomorph, 88 w-graph, 140 w-tree, 140 (1,0,0)-F-face magic mean, 307 3-equitable prime cordial, 291 3-product cordial, 275 3-total super sum cordial labeling, 278 F-root square mean, 297 absolutely harmonious, 111 additively (a, r)-geometric, 324 additively (a, r)\*-geometric, 324 almost-bipartite, 64 alternate hexagonal snake, 17 alternate quadrilateral snake, 248, 273, 284, 340 alternate shell, 87 alternate triangular snake, 248, 273, 284, 340 analytic mean, 310 analytic odd mean, 310 antimagic, 195 arbitrarily graceful, 70 arithmetic, 113, 324 armed crown, 283

armed helms, 81 balanced distance, 183 balloon, 110 barbell, 315 bent ladder, 329 beta combination, 340 bi-odd sequential, 109 bicomposition, 63 bicyclic, 259 bigraceful, 36 block, 342 bow, 15, 280 braided star, 249, 285 broken wheel, 99 broom, 129 butterfly, 112, 259, 264 cactus, 317 calendula, 173 caterpillar cycle, 328 centered triangular difference mean, 304 centered triangular mean, 304 centroidal mean, 296 chain, 176, 322 chordal ring, 166, 208 circulant, 130 circular lobster, 329 closed helm, 13 cocktail party, 122, 166, 239 coconut tree, 252 comb tree, 328combs, 32complete, 22complete mixed generalized sausage graph, 196 composition, 22 conservative, 128 contraharmonic mean, 305 cordial, 150 countable infinite, 138 cycle butterfly, 289 cycle with parallel chords, 25 decomposable, 147 degree-magic, 128

diamond, 71 difference, 335 difference cordial, 284 difference perfect square cordial, 337 directed, 6 directed  $\Gamma$ -distance magic, 183 directed edge-graceful, 264 disconnected, 24 distance k-antimagic, 202 distance antimagic, 202 divisor, 342 double alternate hexagonal snake, 17 double alternate quadrilateral snake, 248, 273, 284 double alternate trirangular snake, 248, 273, 284 double arrow, 189 double coconut tree, 252 double fans, 32 double graph of G, 129 double hexagonal snake, 17 double quadrilateral snake, 248, 273, 284double step grid, 33 double triangular, 17 double triangular snake, 248, 273, 284 double wheels, 21 dumbbell, 112, 259 edge corona path, 174, 207 edge linked cyclic snake, 292 edge magic graceful, 142 edge pair sum, 313 edge product cordial, 283 edge vertex prime, 252 edge-friendly, 96 edge-magic, 155 edge-prime, 251 EP-cordial, 277 even 2a-sequential, 126 even edge-graceful, 261 even vertex odd mean, 301 even-multiple subdivision, 85 extended w-tree, 140

extended jewel, 309 extended vertex edge additive cordial, 97 extra Skolem difference mean, 302 fan, 45 festoon, 112, 264 Fibonacci graceful, 69 firecracker, 123 flower, 256, 281 flower snark, 289 friendship, 16 fully product-cordial, 275 generalize shacke, 212 generalized caterpillar, 31 generalized edge linked cyclic snake, 292 generalized helm, 164, 315, 317 generalized Jahangir, 152, 164 generalized sausage, 196 generalized spider, 31 generalized web, 13, 164 generalized wheel, 274 globe, 98 graceful, 5 gracefulness, 66 graph-block chain, 31 grid-like, 48 group  $S_3$  cordial prime, 264 Halin, 133 Hamming-graceful, 106 handicap distance d-antimagic, 197 Harary, 218 harmonic mean, 304 harmonious, 6 hexagonal snake, 17 highly vertex prime, 249 holiday star, 285 hybrid quadrilateral snake, 120 hyper strongly multiplicative, 325 ideal magic, 145 indexable, 114 integral sum, 235 irregular quadrilateral snake, 284 irregular triangular snake, 284

Jahangir, 83 jelly fish, 142, 340 jewel, 294, 332 join, 27 join sum, 32 kayak paddle, 16 kite, 15, 149 Knödel, 166, 245 komodo dragon with many tails, 34 komodo dragons, 34 Kusadama, 285 ladder, 19 line-graceful, 266 linear cactus, 119 lollipop, 315 lotus, 71 Lucas divisor cordial, 281 middle, 87 minimally k-equitable, 105 mirror, 35 mixed generalized sausage, 196 modular multiplicative, 326 multiple shell, 104 node-graceful, 75 odd (a, d)-antimagic, 208 odd antimagic, 208 odd sum, 110 odd vertex equitable even, 333 odd vertex magic, 129 one modulo N graceful, 68 one modulo three square mean, 309 ordered, 199 orientable *Gamma*-distance magic, 183 pair mean, 314 pair sum, 312 para-squares cactus, 317 para-squares cactus chain, 317 parity combination cordial, 291 path-block chain, 31 pentagonal sum, 341 perfect, 342 perfect super edge-magic, 143 Perrin graceful, 70

plus, 34, 123 polar grid, 51 prime, 244, 249 prime graceful, 37 pseudo-magic, 131 pyramid, 71, 123 radio mean, 308 reduction, 329 relaxed mean, 296 remainder cordial, 94, 282 replicated, 35 restricted k-mean, 295 restricted triangular difference mean, 310 rigid ladders, 293 SD-prime cordial, 248 semi Jahangir, 189 semi-edge-prime, 251 semi-magic, 127 semismooth graceful, 73 set graceful, 329 set sequential, 329 shacke, 212 shackle, 175 shadow, 78, 103 sharp, 199 shell-butterfly, 15 shell-type, 15 shipping, 337 simply sequential, 327 Skolem difference Lucas mean, 302 Skolem difference mean, 302 Skolem even difference mean, 303 Skolem labeled, 76 Skolem-graceful, 75 slanting ladder, 110, 189 smooth graceful, 34 sparkler, 258 sparklers, 112 splitting, 31 square difference, 337 square sum, 299 SSG(n), 78, 122

star, 21 star extension, 117 star of, 87, 95 step grid, 33, 294 step ladder, 122 strong edge-graceful, 257 strong magic, 145 strong sum, 235 strong super edge-magic, 143 strongly c-elegant, 119 strongly k-indexable, 150 strongly 1-harmonious, 150 strongly felicitous, 119 strongly harmonious, 107 strongly indexable, 114 strongly multiplicative, 325 subdivided shell, 68, 78, 122 sum divisor cordial, 280 sun, 211 sunflower, 84 super (a, d)-F-antimagic, 207 super edge magic graceful, 142 super edge-graceful, 53 super graceful, 66 super Lehmer-3 mean, 305 super pair sum, 312 super root square mean, 298 super subdivision, 333 super vertex mean, 295 supermagic, 127 supersubdivision, 29 swastik, 34 tadpoles, 15 theta, 116 theta graph, 31 Toeplitz, 221 torch, 31 total, 35, 272 total mean cordial, 307 total mean labeling, 301 total mixed, 323 total prime, 249 totally antimagic total, 199

totally magic, 169 triangular difference mean, 304 triangular ladders, 276 triangular snake, 17 triangular tree, 46 twisted cylinder, 101, 275 TIASL signed graph, 331 umbrella, 98, 142 unicyclic, 13 uniform bow, 15 uniformly balanced, 96 uniformly cordial, 95 universal *alpha*-graceful, 57 universal graceful, 57 vertex even mean, 299 vertex odd divisor cordial, 282 vertex odd graceful, 81 vertex product, 278 vertex switching, 32, 69, 87, 195 vertex-edge neighborhood prime, 251 weak antimagic, 198 weak magic, 145 weak sum, 240 weighted-k-antimagic, 202 zero-sum A-magic, 186 zig-zag triangle, 138 graph labeling, 5 graph-block chain, 31 graphs dragon, 296 grid, 19, 72 grid-like graph, 48 group irregularity strength, 320 Halin graph, 133 Hamming-graceful graph, 106 handicap distance antimagic graphs, 197 handicap incomplete tournament, 197 harmonic mean, 304 harmonious graph, 6 harmonious number, 112 harmonious order, 36 Heawood graph, 36, 57

helm, 13, 243 closed, 84 generalized, 84 Herschel graph, 36, 205 hexagonal lattice, 177 holey  $\alpha$ -labeling, 61 homeomorph, 100 honeycomb graph, 229 hooked Skolem sequence, 76 host graph, 53 hybrid quadrilateral snake, 120 hypercycle, 238 strong, 238 hypergraph, 130, 156, 201, 237 hyperwheel, 238 IC-coloring, 327 IC-index, 327 icicle graph, 329 icosahedron, 36 index of cordiality, 88 index of product cordiality, 278 integer-antimagic spectrum, 199 integer-magic spectrum, 133, 187 integral radius, 237 integral sum number, 236 tree, 235 irregular crown, 143 irregular labeling, 314 irregular quadrilateral snake, 284 irregular triangle snake, 284 irregularity strength, 314 irregulat fense, 51 jewel graph, 294 join product, 152 join sum, 32 kayak paddle, 16, 63

kite, 15, 53, 149, 168 Kotzig's Conjecture, 62

L-cordial, 95 labeling  $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ -cordial, 279  $(\omega, k)$ -antimagic, 201 (a, b)-consecutive, 258 (a, b; n)-graceful, 36 (a, d)- vertex-antimagic edge, 205 (a, d)-H-antimagic total labeling, 213 (a, d)-1-vertex-antimagic vertex, 209 (a, d)-distance antimagic, 203 (a, d)-edge-antimagic total, 214 (a, d)-edge-antimagic vertex, 214 (a, d)-face antimagic, 227 (a, d)-indexable, 214 (a, d)-vertex-antimagic total, 211 (a, r)-geometric, 324  $(k, \lambda)$ -magically total labeling, 185 (k, d)-Heronian mean, 311 (k, d)-Skolem, 76 (k, d)-arithmetic, 113 (k, d)-even mean, 300 (k, d)-graceful, 73 (k, d)-hooked Skolem graceful, 36 (k, d)-odd mean, 299 (k, d)-super mean, 304 A-magic, 186 C-geometric mean, 306 E-cordial, 263 F-centroidal mean, 296 F-face mean, 312 F-geometric, 305 G-distance magic, 182 H-E-super magic, 177 H-groupmagic, 131 *H*-irregular k, 321 H-magic, 172 P(a)Q(1)-super vertex-graceful, 262 Q(a)P(b)-super edge-graceful, 262 R-ring-magic, 189  $V_K$ -super vertex magic, 167  $\Delta$ -exclusive sum labeling, 240  $\Gamma$  irregular, 320  $\Theta$ -graceful, 70  $\alpha$ -, 46  $\alpha$ -mean, 293

 $\rho^{\star}, 62$  $\sigma$ -, 63 a-vertex consec. edge bimagic, 193 a-vertex-consecutive magic, 169 d-antimagic, 201 d-antimagic of type (1, 1, 1), 227 d-graceful, 55 k-antimagic, 201 k-balanced, 96 k-cordial, 91 k-edge graceful, 258 k-edge-magic, 133 k-equitable, 102, 105 k-even edge-graceful, 258 k-even mean, 300 k-even sequential harmonious, 112 k-graceful, 75 k-indexable, 114 *k*-mean, 295 k-odd mean, 299 k-prime, 248 k-prime cordial, 290 k-product cordial, 277 k-remainder cordial, 94 k-sequential, 326 k-sequentially additive, 334 k-super harmonic, 311 k-super mean, 295, 304 k-total product cordial, 277 k-totally magic cordial, 190 t-harmonious, 36 w-sum, 240 (1,0,0)-F-face magic, 306 1-vertex bimagic, 184 1-vertex magic, 180 1-vertex magic vertex, 191 3-product cordial, 275 3-total super sum cordial labeling, 278 (k, d)-indexable, 114 absolutely harmonious, 111 additively (a, r)-geometric, 324 additively (a, r)\*-geometric, 324 additively (k, d)-sequential, 335

additively graceful, 113 almost graceful, 61 almost magic, 184 analytic mean, 310 analytic odd mean, 311 antimagic, 195, 200 arbitrarily graceful, 70 balanced, 46, 95 balanced cordial, 95 beta combination, 340 bi-odd sequential, 109 bigraceful, 58 binary magic total, 190 bipartite, 55 centered triangular difference mean, 304 centered triangular mean, 303 complete k-equitable, 105 consecutive, 119 consecutive radio, 268 coprime, 252 cordial, 81 cordial edge deficiency, 95 cordial vertex deficiency, 95 cube divisor cordial, 281 difference cordial, 284 difference perfect aquare cordial, 337 directed  $\Gamma$ -distance magic, 183 directed edge-graceful, 264 distance k-antimagic, 202 distance magic, 180, 191 divisor cordial, 279 edge bimagic, 184 edge bimagic total, 193 edge even graceful, 67 edge irregular k-labeling, 322 edge irregular total, 315 edge pair sum, 312 edge product cordial, 282 edge trimagic total, 155 edge vertex prime, 252 edge-antimagic graceful, 209 edge-friendly, 97 edge-graceful, 255

edge-magic, 136, 155 edge-magic total, 136 edge-odd graceful, 78, 80 edge-prime, 251 elegant, 116 EP-cordial, 276 equitable, 184 even 2a-sequential, 126 even 1-vertex bimagic, 184 even falicitous, 119 even mean labeling, 300 even sequential harmonious, 111 even vertex equitable even, 334 even vertex magic total, 169 even vertex odd mean, 301 even-even, 80 extended edge vertex cordial labeling, 97 face irregular total k-labeling, 321 felicitous, 118 Fibonacci graceful, 69 friendly, 95 geometric mean, 310 geometric mean 3-equitable, 314 geometric mean cordial, 314 graceful data structure, 61 graceful difference, 38 gracefully consistent, 52 gracious, 57 group  $S_3$ , 264 handicap distance *d*-antimagic, 197 harmonious numbering, 112 highly vertex prime, 250 in-magic total, 147 indexable, 114 interlaced, 46 irregular, 314 l, 81 L-cordial, 95 line-graceful, 266 local antimagic, 209 Lucas divisor cordial, 281 magic, 127, 132

consecutive, 177 of type (0,1,1), 177 of type (1,0,0), 178 of type (1,1,0), 177of type (1,1,1), 177 magic valuation, 136 mean, 291 mean cordial, 307 minium coprime, 252 near-elegant, 116 nearly distance magic, 183 nearly graceful, 60 neighborhood-prime, 250 nice (1,1) edge-magic, 144 numbering, 154 odd 1-vertex bimagic, 184 odd harmonious, 120, 124 odd mean, 298 odd sum, 110 odd vertex equitable, 333 odd-elegant, 117 odd-even, 74 odd-graceful, 61, 77 one modulo N graceful, 68 one modulo three graceful, 68 one modulo three mean, 309 one modulo three root square mean, 309 optimal k-equitable, 105 optimal sum graph, 234 ordered, 199 orientable *Gamma*-distance magic, 183 pair mean, 313 pair sum, 312 parity combination cordial, 291 partial vertex, 96 partitional, 108 Pell graceful, 70 pentagonal sum, 341 perfect super edge-magic, 143 Perrin graceful, 70 polychrome, 118 prime, 243 prime cordial, 287

prime-magic, 130 product antimagic, 232 product cordial labeling, 271 product edge-antimagic, 233 product edge-magic, 233 product magic, 232 product-irregular, 323 properly even harmonious, 125 pseudo  $\alpha$ , 65 pseudograceful, 64 radio  $\star$ , 267 radio antipodal, 270 radio graceful, 270 radio mean, 308 radio mean D-distance, 308 range-relaxed graceful, 67 real-graceful, 37 relaxed mean, 296 remainder cordial, 94, 282 restricted k-mean, 295 restricted triangular difference mean, 310 reverse edge-trimagic, 156 reverse super edge-trimagic, 156 rosy, 61SD-prime cordial, 248 semi-elegant, 116 sequential, 107 set-ordered odd-graceful, 79 sharp ordered, 199 shifted antimagic, 222 sigma, 180 simply sequential, 326 Skolem difference Lucas mean, 302 Skolem difference mean, 301 Skolem even difference mean, 303 Skolem even vertex odd difference mean, 303 Skolem odd difference mean, 302 Skolem-graceful, 75 square difference, 337 square divisor cordial, 281 square sum, 299, 335

strength sum, 149 strong edge-graceful, 257 strong super edge-magic, 143 strongly (k, d)-indexable, 114 strongly c-harmonious, 107 strongly k-elegant, 116 strongly balanced, 96 strongly edge-magic, 145 strongly even harmonious, 125 strongly graceful, 46, 56 strongly harmonious, 29, 107, 110 strongly indexable, 114 strongly odd harmonious, 120 strongly square sum, 335 strongly super edge-graceful, 262 strongly vertex-magic total, 167 sum divisor cordial, 280 sum graph, 234 sum perfect square, 337 super (a, d)-F-antimagic, 207 super (a, d)-edge-antimagic graceful, 209super (a, d)-vertex-antimagic total, 211 super edge bimagic cordial, 185 super edge-antimagic total, 215 super edge-graceful, 259 super edge-magic, 145 super edge-magic total, 137 super Fibonacci graceful, 69 super geometric mean, 306 super graceful, 66 super Lehmer-3 mean, 305 super mean, 294 super pair sum, 312 super root mean, 298 super vertex mean, 295 super vertex-graceful, 262 super vertex-magic total, 165 supermagic, 127, 151 total, 199 total *H*-irregular  $\alpha$ , 320 total edge product cordial, 283 total irregular total k, 321

total magic cordial, 189 total mean, 301 total mean cordial, 307 total prime, 249 total product cordial labeling, 276 totally antimagic total, 199 totally magic, 169 totally magic cordial, 192 totally vertex-magic cordial, 191 triangular difference mean, 304 triangular graceful, 67 triangular sum, 341 universal antimagic, 203 vertex balanced cordial, 95 vertex equitable, 331 vertex even mean, 299 vertex irregular total, 315 vertex magic total, 131 vertex odd divisor cordial, 282 vertex odd mean, 299 vertex prime, 249 vertex product cordial, 278 vertex-bimagic, 184 vertex-edge neighborhood prime, 251 vertex-friendly, 101 vertex-graceful, 261 vertex-magic total, 162 vertex-relaxed graceful, 67 weak antimagic, 198 zero-sum A-magic, 186 labeling number, 53 labelings odd-even, 80 total neighborhood prime, 250 lableing 3-equitable prime cordial, 291 ladder, 19, 107, 177, 178 Langford sequence, 142 level joined planar grid, 115 lexicographic product, 132 linear cyclic snake, 18 lobster, 10, 62 lotus inside a circle, 178

Lucas divisor cordial, 281 Möbius grid, 221 Möbius ladder, 21, 108, 127, 131, 177, 244, 257magic b-edge consecutive, 169 magic constant, 136, 183 magic square, 127 magic strength, 132, 144 magic sum index, 131 mean cordial, 307 mean graph, 291 mean number, 307 middle graph, 87 minimum coprime number, 252 mirror graph, 35 mixed generalized sausage graph, 196 mod difference digraph, 335 mod integral sum graph, 239 mod integral sum number, 239 mod sum graph, 238 mod sum number, 239  $mod sum^* graph, 241$  $mod sum^* number, 241$ Mongolian tent, 20, 72 Mongolian village, 20, 72 MSG, 238 multigraph, 151, 156 multiple shell, 15 mutation, 168 mutual duplication, 293 near  $\alpha$ -labeling, 58 nearly distance magic, 183 nearly graceful labeling, 60 neighborhood-prime, 250 nullset, 131 numbering, 154 Oberwolfash Problem, 36 odd 1-vertex bimagic, 184 odd harmonious, 120, 124 odd mean graph, 298 odd mean labeling, 298

odd-elegant, 117 odd-even, 74, 80 odd-graceful labeling, 61, 77 olive tree, 8 one modulo N graceful, 68 one modulo three graceful labeling, 68 one-point union, 16, 22, 47, 77, 81, 118 open star of G, 188 optimal sum graph, 234 pair mean, 313 pair mean graph, 314 pair sum, 312 pair sum graph, 312 parachutes, 205 parallel chord, 99 path, 14, 116 path union, 32, 89 path-block chain, 31 pendent edge, 54 pentagonal number, 341 pentagonal sum labeling, 341 perfect Golomb ruler, 24 perfect system of difference sets, 73 permutation graph, 338 Perrin sequence, 70 Petersen graph, 36 generalized, 29, 82, 139, 150, 163, 205, 211, 214 planar bipyramid, 177 planar graph, 177, 227 Platonic family, 177 plus graph, 34, 123 polargrid, 51 polyminoes, 72 polyominoes, 51 prime cordial strongly, 290 prime cordial labeling, 287 prime graceful, 37 prime graph, 244, 249 prime labeling, 243 prism, 20, 21, 177, 211, 227

product cordial, 271 product cordial labeling, 271 product graph, 234 product irregularity strength, 323 product-cordial index, 275 product-cordial set, 275 properly even harmonious, 125 pseudo  $\alpha$ -labeling, 65 pseudo-magic graph, 131 pseudograceful labeling, 64 quadrilateral snakes, 17 radio k-chromatic number, 269 radio k-coloring, 269 radio star-number, 267 radio antipodal labeling, 270 radio antipodal number, 270 radio graceful, 268, 270 radio labeling, 266 radio mean *D*-distance number, 308 radio mean labeling, 308 radio mean number, 308 radio number, 266 range-relaxed graceful labeling, 67 rank number, 328 real sum graph, 234 regular graph, 127, 130, 139, 164, 191 regular tree, 50 relaxed mean graph, 296 remainder cordial, 94 replicated graph, 35 representation, 270 representation number, 270 restricted triangular difference mean, 310 rigid ladders, 293 Ringel-Kotzig, 8 root, 84 root-union, 99 saturated vertex, 235 SD-prime cordial, 248 semi-edge-prime graph, 251

separating set, 329 sequential join, 54 sequential number, 153 set-ordered odd-graceful, 79 shackle, 212 shadow graph, 78, 103 shell, 15, 84, 86, 103 multiple, 15 shell graph, 93 Skolem labeled graph, 76 Skolem sequence, 10, 25 Skolem-graceful labelings, 75 smooth graceful, 34 snake, 17, 49 *n*-polygonal, 68 double triangular, 17 edge linked cyclic, 292 generalized edge linked cyclic, 292 quadrilateral, 48 triangular, 17, 61 snake polyomono, 50 sparse semi-magic square, 170 special super edge-magic, 146 spider, 8 split graph, 196 splitting graph, 31, 78, 285 spum, 234 square difference graph, 337 square divisor cordial, 281 square sum labeling, 335 SSG(n), 78SSG(n), 122 stable set, 30, 35, 49 star, 26, 28, 164, 266 star of G, 276 star of a G, 87, 95 star of graphs, 33 star super edge-magic deficiency, 138 step grid graph, 33, 294 step ladder, 122 straight simple polyominal caterpillars, 51 strength edge magic, 132

semismooth graceful, 73

magic, 132, 144 maximum magic, 145 strength sum, 149 strong product of graphs, 181 strong A-magic, 131 strong k-combination graph, 339 strong k-permutation graph, 339 strong beta-number, 65 strong edge-graceful, 257 strong gamma-number, 37 strong harmonious number, 112 strong product, 322 strong sequential number, 153 strong sum graph, 235 strong supersubdivision, 30 arbitrary, 30 strong vertex-graceful, 261 strongly c-harmonious, 107 strongly \*-graph, 340 strongly antimagic, 201 strongly even harmonious, 125 strongly graceful labeling, 56 strongly harmonious, 29, 107 strongly odd harmonious, 120 strongly prime cordial, 290 strongly square sum labeling, 335 strsG, 149 stunted tree, 62 subdivided shell graph, 68, 78, 122 subdivision, 9, 19, 77, 178 sum graph, 234 mod, 238 mod integral, 239 real, 234 sum number, 234 sum perfect square, 336  $sum^*$  graph, 241  $sum^*$  number, 241 sunflower, 84, 256 super (a, d)-F-antimagic, 207 super (a, d)-H-antimagic total labeling, 213 super (a, d)-edge-antimagic graceful, 209 super d-antimagic, 202

super edge magic graceful, 142 super edge-magic deficiency, 138 super Fibonacci graceful, 69 super geometric mean, 306 super graceful, 66 super labeling, 199 super magic frame, 129 super magic strength, 132, 149 super mean, 294 super mean number, 298 super subdivision, 333 super vertex mean, 295 super vertex-magic total, 165 super weak sumgraph, 240 superdivision arbitrary, 30 supersubdivision, 29, 89 arbitrary, 30 swastik graph, 34 switching invariant, 247 symmetric product, 22, 49 tadpoles, 15 tensor product, 74, 79, 96, 278 tes(G), 315theta graph, 116 theta graphs, 31 toroidal polyhex, 220 torus grid, 20 total *H*-irregularity strength, 320 total edge (vertex) irregular strength, 315 total edge irregularity strength, 315 total edge product cordial labeling, 283 total graph, 35, 99, 272 total labeling, 199 total mean cordial, 307 total mixed, 323 total negative, 323 total negative edge, 323 total positive edge, 323 total product cordial, 276 total product cordial labeling, 276 total stable, 323

total stable edge, 323 zero-sum A-magic, 186 totally magic cordial, 192 zero-sum h-magic, 131 totally magic cordial deficiency, 191 totally vertex-magic cordial labeling, 191 tree, 5, 197, 239 binary, 139 path-like, 141 symmetrical, 8 triangular graceful labeling, 67 triangular snake, 17 tvs(G), 315umbrella, 142 unicyclic graph, 15 uniform-distant tree, 10 union, 23, 136, 149, 151, 164, 235, 244, 249 universal antimagic, 203 unlabeled vertices, 97 vertex H-irregularity strenght, 322 vertex balance index set, 101 vertex balanced cordial, 95 vertex equitable, 331 vertex irregular total labeling, 315 vertex parity, 56 vertex prime labeling, 249 vertex switching, 32, 69, 87, 195, 247 vertex-antimagic total, 199 vertex-graceful, 261 vertex-relaxed graceful labeling, 67 vhs(G, H), 322weak sum graph, 240 weak tensor product, 54, 58 weakly  $\alpha$ -labeling, 56 web, 13 generalized, 145 weight, 227 weighted-k-antimagic, 202 wheel, 13, 107, 127, 136, 166, 177, 195, 214 windmill, 22, 84 working vertex, 239 wreath product, 118 Young tableau, 20, 72