

# A Dynamic Survey of Graph Labeling

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## Abstract

A vertex *labeling* of a graph  $G$  is an assignment  $f$  of labels to the vertices of  $G$  that induces for each edge  $xy$  a label depending on the vertex labels  $f(x)$  and  $f(y)$ . The two best known labeling methods are called graceful and harmonious labelings. A function  $f$  is called a *graceful* labeling of a graph  $G$  with  $q$  edges if  $f$  is an injection from the vertices of  $G$  to the set  $\{0, 1, \dots, q\}$  such that, when each edge  $xy$  is assigned the label  $|f(x) - f(y)|$ , the resulting edge labels are distinct. A function  $f$  is called a *harmonious* labeling of a graph  $G$  with  $q$  edges if it is an injection from the vertices of  $G$  to the group of integers modulo  $q$  such that when each edge  $xy$  is assigned the label  $f(x) + f(y) \pmod{q}$ , the resulting edge labels are distinct. When  $G$  is a tree, exactly one label may be used on two vertices. Over the past three decades many variations of graceful and harmonious labelings have evolved and about 300 papers have been written on the subject of graph labeling. In this article we survey what is known about the various methods.

## 1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [Ro1] in 1967, or one given by Graham and Sloane [GS] in 1980. Rosa [Ro1] called a function  $f$  a  $\beta$ -*valuation* of a graph  $G$  with  $q$  edges if  $f$  is an injection from the vertices of  $G$  to the set  $\{0, 1, \dots, q\}$  such that, when each edge  $xy$  is assigned the label  $|f(x) - f(y)|$ , the resulting edge labels are distinct. Golomb [Go] subsequently called such labelings *graceful* and this is now the popular term. Rosa introduced  $\beta$ -valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs.

In particular,  $\beta$ -valuations originated as a means of attacking the conjecture of Ringel [Ri] that  $K_{2n+1}$  can be decomposed into  $2n + 1$  subgraphs that are all isomorphic to a given tree with  $n$  edges. Although an unpublished result of Erdős says that most graphs are not graceful (cf. [GS]), most graphs that have some sort of regularity of structure are graceful. Sheppard [Sh] has shown that there are exactly  $q!$  gracefully labeled graphs with  $q$  edges. Balakrishnan and Sampathkumar [BS] have shown that every graph is a subgraph of a graceful graph. Rosa [Ro1] has identified essentially three reasons why a graph fails to be graceful: (1)  $G$  has “too many vertices” and “not enough edges”, (2)  $G$  “has too many edges”, and (3)  $G$  “has the wrong parity”. An infinite class of graphs that are not graceful for the second reason is given in [BG]. As an example of the third condition Rosa [Ro1] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles  $C_{4n+1}$  and  $C_{4n+2}$  are not graceful.

Harmonious graphs naturally arose in the study by Graham and Sloane [GS] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph  $G$  with  $q$  edges to be *harmonious* if there is an injection  $f$  from the vertices of  $G$  to the group of integers modulo  $q$  such that when each edge  $xy$  is assigned the label  $f(x) + f(y) \pmod{q}$ , the resulting edge labels are distinct. When  $G$  is a tree, exactly one label may be used on two vertices. Analogous to the “parity” necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number  $q$  of edges and the degree of every vertex is divisible by  $2^k$  then  $q$  is divisible by  $2^{k+1}$ . Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph  $K_{1,7}$  and a path of length 1) is not harmonious. Liu and Zhang [LZ2] have generalized this condition as follows: if a harmonious graph with  $q$  edges has degree sequence  $d_1, d_2, \dots, d_p$  then  $\gcd(d_1, d_2, \dots, d_p, q)$  divides  $q(q - 1)/2$ . They have also proved that every graph is a subgraph of a harmonious graph.

Over the past three decades approximately 300 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes have been done by several authors. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two methods, include [Be], [Bl], [KRT2], [Ga2], and [Ga4]. The extension of graceful labelings to directed graphs arose in the characterization of finite neofields by Hsu and Keedwell [HK1], [HK2]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings and neofields is discussed in [BH1], [BH2]. The connection between graceful labelings and perfect systems of difference sets is given in [BKT]. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography,

radar, astronomy, circuit design and communication network addressing—see [BG1] and [BG2] for details. Terms and notation not defined below follow that used in [CL] and [Ga2].

## 2 Graceful and Harmonious Labelings

### 2.1 Trees

The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Among the trees known to be graceful are: caterpillars [Ro1] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [HKR], [Zha] and [JMWG]; trees with diameter at most 4 [Zha]; trees with at most 27 vertices [AIM]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [BeS]; and olive trees [PR] and [AB2] (a rooted tree consisting of  $k$  branches, where the  $i$ th branch is a path of length  $i$ ). Stanton and Zarnke [SZ] and Koh, Rogers and Tan [KRT3] gave methods for combining graceful trees to yield larger graceful trees. Burzio and Ferrarese [BurF] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. In 1979 Bermond [Be] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a caterpillar). Special cases of this conjecture have been done by Ng [N1] and by Wang, Jin, Lu and Zhang [WJLZ]. Whether or not lobsters are harmonious seems to have attracted no attention thus far. Chen, Lü and Yeh [CLY] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex  $v$  to one leaf of each of any number of stars ( $v$  is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Various kinds of bananas trees have been shown to be graceful by Bhat-Nayak and Deshmukh [BD2], by Murugan and Arumugam [MA2], [MA4] and by Vilfred [Vil]. Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. Aldred and McKay [AIM] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [GS]. More specialized results about trees are contained in [Be], [Bl], [KRT2], [LL], [C4] and [JLLLLZ].

### 2.2 Cycle-Related Graphs

Cycle-related graphs have been the major focus of attention. Rosa [Ro1] showed that the  $n$ -cycle  $C_n$  is graceful if and only if  $n \equiv 0$  or  $3 \pmod{4}$  and Graham and Sloane [GS] proved that  $C_n$  is harmonious if and only if  $n \equiv 1$  or  $3 \pmod{4}$ . Wheels  $W_n = C_n + K_1$  are both graceful and harmonious – [F1], [HK] and [GS]. Notice that a subgraph of a graceful

(harmonious) graph need not be graceful (harmonious). The  $n$ -cone (also called the  $n$ -point suspension of  $C_m$ )  $C_m + \overline{K_n}$  has been shown to be graceful when  $m \equiv 0$  or  $3 \pmod{12}$  by Bhat-Nayak and Selvam [BNS]. When  $n$  is even and  $m$  is  $2, 6$  or  $10 \pmod{12}$   $C_m + \overline{K_n}$  violates the parity condition for a graceful graph. Bhat-Nayak and Selvam [BNS] also prove that the following cones are graceful:  $C_4 + \overline{K_n}$ ,  $C_5 + \overline{K_2}$ ,  $C_7 + \overline{K_n}$ ,  $C_9 + \overline{K_2}$ ,  $C_{11} + \overline{K_n}$  and  $C_{19} + \overline{K_n}$ . The *helm*  $H_n$  is the graph obtained from a wheel by attaching a pendant edge at each vertex of the  $n$ -cycle. Helms have been shown to be graceful [AF] and harmonious [Gn], [LiuY3], [LiuY4] (see also [LZ2], [SY1], [LiuB2], [DL3] and [RP1]). Koh, et al. [KRTY] define a *web* graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. They ask whether such graphs are graceful. This was proved by Kang, Liang, Gao and Yang [KLGy]. Yang has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendant points to the new cycle. In his notation,  $W(2, n)$  is the web graph whereas  $W(t, n)$  is the generalized web with  $t$   $n$ -cycles. Yang has shown that  $W(3, n)$  and  $W(4, n)$  are graceful (see [KLGy]) and Abhyanker and Bhat-Nayak [AB1] have done  $W(5, n)$ . Gnanajothi [Gn] has shown that webs with odd cycles are harmonious. Seoud and Youssef [SY1] define a *closed helm* as the graph obtained from a helm by joining each pendant vertex to form a cycle and a *flower* as the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A *gear graph* is obtained from the wheel  $W_n$  by adding a vertex between every pair of adjacent vertices of the  $n$ -cycle. Ma and Feng [MF2] have proved all gears are graceful. Liu [LiuY3] has shown that if two or more vertices are inserted between every pair of vertices of the  $n$ -cycle of the wheel  $W_n$ , the resulting graph is graceful. Liu [LiuY1] has also proved that the graph obtain from a gear graph by attaching one or more pendant points to each vertex between the cycle vertices is graceful.

Delorme, et al. [DMTKT] and Ma and Feng [MF1] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher and Wegner [BSW2], who proved various special cases. Koh and Yap [KY] generalized this by defining a *cycle with a  $P_k$ -chord* to be a cycle with the path  $P_k$  joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when  $k = 3$  and conjectured that all cycles with a  $P_k$ -chord are graceful. This was proved for  $k \geq 4$  by Punnim and Pabhapote in 1987 [PP]. Chen [CheZ2], [CheZ3] obtained the same result except for three cases which were then handled by Gao [Gu2]. Xu [X2] proved that all cycles with a chord are harmonious except for  $C_6$  in the case where the distance in  $C_6$  between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords have also been investigated. For  $3 \leq p \leq n - r$ , let  $C_n(p, r)$  denote the  $n$ -cycle with consecutive vertices  $v_1, v_2, \dots, v_n$  to which the  $r$  chords  $v_1v_p, v_1v_{p+1}, \dots, v_1v_{p+r-1}$  have been added. Koh and others, [KRTY] and [KP], have handled the cases  $r = 2, 3$  and  $n - 3$  where  $n$  is the length of the cycle. Goh and Lim [GoL] then proved that all remaining cases are graceful. Moreover, Ma [Ma] has shown that  $C_n(p, n - p)$  is graceful when  $p \equiv 0, 3$

(mod 4) and Ma, Liu and Liu [MLL] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph  $C_n(3, n-3)$  any number  $k_i$  of paths of length 2 from the vertex  $v_1$  to the vertex  $v_i$  for  $i = 2, \dots, n$ , the resulting graph is graceful. Chen [CheZ1] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. This generalizes the result that a cycle plus a chord is graceful. Liu [LiuR] has shown that the  $n$ -cycle with consecutive vertices  $v_1, v_2, \dots, v_n$  to which the chords  $v_1v_k$  and  $v_1v_{k+2}$  ( $2 \leq k \leq n-3$ ) are adjoined is graceful.

In [DL2] Deb and Limaye use the notation  $C(n, k)$  to denote the cycle  $C_n$  with  $k$  cords sharing a common endpoint. For certain choices of  $n$  and  $k$  there is a unique  $C(n, k)$  graph and for other choices there is more than one graph possible. They call these *shell-type* graphs and they call the unique graph  $C(n, n-3)$  a *shell*. Notice that the shell  $C(n, n-3)$  is the same as the fan  $f_{n-1}$ . Deb and Limaye define a *multiple shell* to be a collection of edge disjoint shells that have their apex in common. They show that a variety of multiple shells are harmonious and they conjecture that all multiple shells are harmonious.

Sethuraman and Dharamani [SD1] use  $H(n, t)$  to denote the graph obtained from the cycle  $C_n$  by adding  $t$  consecutive chords incident with a common vertex. If the common vertex is  $u$  and  $v$  is adjacent to  $u$ , then for  $k \geq 1$ ,  $n \geq 4$  and  $1 \leq t \leq n-3$ , Sethuraman and Dharamani denote by  $G(n, t, k)$  the graph obtained by taking the union of  $k$  copies of  $H(n, k)$  with the edge  $uv$  identified. They conjecture that every graph  $G(n, t, k)$  is graceful. They prove the conjecture for the case that  $t = n-3$ .

Truszczyński [T] studied unicyclic graphs (i.e., graphs with a unique cycle) and proved several classes of such graphs are graceful. Among these are what he calls dragons. A *dragon* is formed by joining the end point of a path to a cycle (Koh, et al. [KRTY] call these *tadpoles*). This work led Truszczyński to conjecture that all unicyclic graphs except  $C_n$ , where  $n \equiv 1$  or  $2 \pmod{4}$ , are graceful. Guo [Gu] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 (mod 4). In his Master's thesis, Doma [Do] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Because of the immense diversity of unicyclic graphs, a proof of Truszczyński's conjecture seems out of reach in the near future.

Cycles that share a common edge or a vertex have received some attention. Murugan and Arumugan [MA1] have shown that books with  $n$  pentagonal pages (i.e.,  $nC_5$  with an edge in common) are graceful when  $n$  is even and not graceful when  $n$  is odd. Let  $C_n^{(t)}$  denote the one-point union of  $t$  cycles of length  $n$ . Bermond and others ([BBG] and [BKT]) proved that  $C_3^{(t)}$  (that is, the friendship graph or Dutch  $t$ -windmill) is graceful if and only if  $t \equiv 0$  or  $1 \pmod{4}$  while Graham and Sloane [GS] proved  $C_3^{(t)}$  is harmonious if and only if  $t \not\equiv 2 \pmod{4}$ . Koh et al. [KRLT] conjecture that  $C_n^{(t)}$  is graceful if and only if  $nt \equiv 0$  or  $3 \pmod{4}$ . Qian [Q] verifies this conjecture for the case that  $t = 2$  and  $n$  is even. Bodendiek, Schumacher and Wegner [BSW1] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [S2] has proved that  $C_4^{(t)}$

is graceful for all  $t$ .

Another class of cycle-related graphs is that of triangular cacti. A *triangular cactus* is a connected graph all of whose blocks are triangles. A *triangular snake* is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path  $v_1, v_2, \dots, v_n$  by joining  $v_i$  and  $v_{i+1}$  to a new vertex  $w_i$  for  $i = 1, 2, \dots, n - 1$ ). Rosa [Ro2] conjectured that all triangular cacti with  $t \equiv 0$  or  $1 \pmod{4}$  blocks are graceful (the cases where  $t \equiv 2$  or  $3 \pmod{4}$  fail to be graceful because of the parity condition.) Moulton [Mo] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular cacti) seems hopelessly difficult. Liu and Zhang [LZ2] gave an incorrect proof that triangular snakes with an odd number of triangles are harmonious while triangular snakes with  $n \equiv 2 \pmod{4}$  triangles are not harmonious. Xu [X2] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to  $2 \pmod{4}$ .

Defining  $K_4$ -snakes analogous to triangular snakes, Grace [Gr3] showed that these are harmonious. Rosa [Ro2] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases. Gnanajothi [Gn, pp. 31-34] has shown that quadrilateral snakes are graceful.

Several people have studied cycles with pendant edges attached. Frucht [F1] proved that any cycle with a pendant edge attached at each vertex (i.e., a “crown”) is graceful. Bu, Zhang and He [BZH] have shown that any cycle with a fixed number of pendant edges adjoined to each vertex is graceful. Grace [Gr2] showed that an odd cycle with one or more pendant edges at each vertex is harmonious and conjectured that an even cycle with one pendant edge attached at each vertex is harmonious. This conjecture has been proved by Liu and Zhang [LZ1], Liu [LiuY3] and [LiuY4], Huang [Hua] and Bu [Bu2]. For any  $n \geq 3$  and any  $t$  with  $1 \leq t \leq n$ , let  $C_n^{+t}$  denote the class of graphs formed by adding a single pendant edge to  $t$  vertices of a cycle of length  $n$ . Ropp [Rop] proved that for every  $n$  and  $t$  the class  $C_n^{+t}$  contains a graceful graph. Gallian and Ropp [Ga2] conjecture that for all  $n$  and  $t$ , all members of  $C_n^{+t}$  are graceful. This was proved by Qian [Q] and by Kang, Liang, Gao and Yang [KLGY]. Of course, this is just a special case of the aforementioned conjecture of Truszczyński that all unicyclic graphs except  $C_n$  for  $n \equiv 1$  or  $2 \pmod{4}$  are graceful.

### 2.3 Product Related Graphs.

Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids,  $P_m \times P_n$ , are graceful was proved by Acharya and Gill [AG] in 1978 although the much simpler labeling scheme given by Maheo [Mah] in 1980 for  $P_m \times P_2$  readily extends to all grids. In 1980 Graham and Sloane [GS] proved ladders,  $P_m \times P_2$ , are harmonious when  $m > 2$  and in 1992 Jungreis and Reid [JR] showed that the grids  $P_m \times P_n$  are harmonious when  $(m, n) \neq (2, 2)$ . A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [Kat1] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful

and that graphs obtained by appending an edge to each vertex of a ladder are graceful [Kat2]. Acharya [A2] has shown that certain subgraphs of grid graphs are graceful. Lee [L1] defines a *Mongolian tent* as a graph obtained from  $P_m \times P_n$ ,  $n$  odd, by adding one extra vertex above the grid and joining every other vertex of the top row of  $P_m \times P_n$  to the new vertex. A *Mongolian village* is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A *Young tableau* is a subgraph of  $P_m \times P_n$  obtained by retaining the first two rows of  $P_m \times P_n$  and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and K. C. Ng [LNK] have proved that all Young tableaux are graceful. Lee [L1] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful.

*Prisms* are graphs of the form  $C_m \times P_n$ . These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher and Wegner [BSW2] proved that  $C_m \times P_2$  is graceful when  $m \equiv 0 \pmod{4}$ . According to the survey by Bermond [Be], T. Gangopadhyay and S. P. Rao Hebbare did the case that  $m$  is even about the same time. In a 1979 paper, Frucht [F1] stated without proof that he had done all  $m$ . A complete proof of all cases and some related results were given by Frucht and Gallian [FG] in 1988. In 1992 Jungreis and Reid [JR] proved that all  $C_m \times P_n$  are graceful when  $m$  and  $n$  are even or when  $m \equiv 0 \pmod{4}$ . Yang and Wang [YW1] have shown that the prisms  $C_{4n+2} \times P_{4m+3}$  are graceful. Singh [Sin1] proved that  $C_3 \times P_n$  is graceful for all  $n$ . In their 1980 paper Graham and Sloane [GS] proved that  $C_m \times P_n$  is harmonious when  $n$  is odd and they used a computer to show  $C_4 \times P_2$ , the cube, is not harmonious. In 1992 Gallian, Prout and Winters [GPW] proved that  $C_m \times P_2$  is harmonious when  $m \neq 4$ . In 1992, Jungreis and Reid [JR] showed that  $C_4 \times P_n$  is harmonious when  $n \geq 3$ . Huang and Skiena [HuS] have shown that  $C_m \times P_n$  is graceful for all  $n$  when  $m$  is even and for all  $n$  with  $3 \leq n \leq 12$  when  $m$  is odd.

Torus grids are graphs of the form  $C_m \times C_n$  ( $m > 2, n > 2$ ). Very little success has been achieved with these graphs. The graceful parity condition is violated for  $C_m \times C_n$  when  $m$  and  $n$  are odd and the harmonious parity condition [GS, Theorem 11] is violated for  $C_m \times C_n$  when  $m \equiv 1, 2, 3 \pmod{4}$  and  $n$  is odd. The only result I'm aware of was done in 1992 by Jungreis and Reid [JR] who showed that  $C_m \times C_n$  is graceful when  $m \equiv 0 \pmod{4}$  and  $n$  is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout and Winters [GPW] proved that all prisms  $C_m \times P_2$  with a single vertex deleted or single edge deleted are graceful and harmonious. The *Möbius ladder*  $M_n$  is the graph obtained from the ladder  $P_n \times P_2$  by joining the opposite end points of the two copies of  $P_n$ . In 1989 Gallian [Ga1] showed that all Möbius ladders are graceful and all but  $M_3$  are harmonious. Ropp [Rop] has examined two classes of prisms with pendant edges attached. He proved that all  $C_m \times P_2$  with a single pendant edge at each vertex are graceful and all  $C_m \times P_2$  with

a single pendant edge at each vertex of one of the cycles are graceful.

Another class of cartesian products that has been studied is that of books and “stacked” books. The *book*  $B_m$  is the graph  $S_m \times P_2$  where  $S_m$  is the star with  $m + 1$  vertices. In 1980 Maheo [Mah] proved that the books of the form  $B_{2m}$  are graceful and conjectured that the books  $B_{4m+1}$  were also graceful. (The books  $B_{4m+3}$  do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [D] in 1980. Maheo [Mah] also proved that  $L_n \times P_2$  and  $B_{2m} \times P_2$  are graceful. Both Grace [Gr1] and Reid (see [GJ]) have given harmonious labelings for  $B_{2m}$ . The books  $B_{4m+3}$  do not satisfy the harmonious parity condition [GS, Theorem 11]. Gallian and Jungreis [GJ] conjectured that the books  $B_{4m+1}$  are harmonious. Gnanajothi [Gn] has verified this conjecture by showing  $B_{4m+1}$  has an even stronger form of labeling – see Section 4.1. Liang [Li] also proved the conjecture. In their 1988 paper Gallian and Jungreis [GJ] defined a *stacked book* as a graph of the form  $S_m \times P_n$ . They proved that the stacked books of the form  $S_{2m} \times P_n$  are graceful and posed the case  $S_{2m+1} \times P_n$  as an open question. The *n-cube*  $K_2 \times K_2 \times \cdots \times K_2$  ( $n$  copies) was shown to be graceful by Kotzig [K3]—see also [Mah]. In 1986 Reid [Re] found a harmonious labeling for  $K_4 \times P_n$ .

The composition  $G_1[G_2]$  is the graph having vertex set  $V(G_1) \times V(G_2)$  and edge set  $\{(x_1, y_1), (x_2, y_2) \mid x_1x_2 \in E(G_1) \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(G_2)\}$ . The symmetric product  $G_1 \oplus G_2$  of graphs  $G_1$  and  $G_2$  is the graph with vertex set  $V(G_1) \times V(G_2)$  and edge set  $\{(x_1, y_1), (x_2, y_2) \mid x_1x_2 \in E(G_1) \text{ or } y_1y_2 \in E(G_2) \text{ but not both}\}$ . Seoud and Youssef [SY4] have proved that  $P_n \oplus \overline{K_2}$  is graceful when  $n > 1$  and  $P_n[P_2]$  is harmonious for all  $n$ . They also observe that the graphs  $C_m \oplus C_n$  and  $C_m[C_n]$  violates parity conditions for graceful and harmonious graphs when  $m$  and  $n$  are odd.

## 2.4 Complete Graphs

The questions of the gracefulness and harmoniousness of the complete graphs  $K_n$  have been answered. In each case the answer is positive if and only if  $n \leq 4$  ([Go], [Si], [GS]). Both Rosa [Ro1] and Golomb [Go] proved that the complete bipartite graphs  $K_{m,n}$  are graceful while Graham and Sloane [GS] showed they are harmonious if and only if  $m$  or  $n = 1$ . Aravamudhan and Murugan [AM] have shown that the complete tripartite graph  $K_{1,m,n}$  is both graceful and harmonious while Gnanajothi [Gn,pp.25-31] has shown that  $K_{1,1,m,n}$  is both graceful and harmonious and  $K_{2,m,n}$  is graceful. Some of the same results have been obtained by Seoud and Youssef [SY3] who also observed that when  $m, n$  and  $p$  are congruent to 2 (mod 4),  $K_{m,n,p}$  violates the parity conditions for harmonious graphs.

Define the *windmill* graphs  $K_n^{(m)}$  ( $n > 3$ ) to be the family of graphs consisting of  $m$  copies of  $K_n$  with a vertex in common. A necessary condition for  $K_n^{(m)}$  to be graceful is that  $n \leq 5$  – see [KRTY]. Bermond [Be] has conjectured that  $K_4^{(m)}$  is graceful for all  $m \geq 4$ . This is known to be true for  $m \leq 22$  [HuS]. Bermond, Kotzig and Turgeon [BKT] proved that  $K_n^{(m)}$  is not graceful when  $n = 4$  and  $m = 2$  or  $3$  and when  $m = 2$  and  $n = 5$ . In 1982 Hsu [Hs] proved that  $K_4^{(m)}$  is harmonious for all  $m$ . Graham and Sloane [GS] conjectured that  $K_n^{(2)}$  is harmonious if and only if  $n = 4$ . They verified this



conjecture for the cases that  $n$  is odd or  $n = 6$ . Liu [LiuB2] has shown that  $K_n^{(2)}$  is not harmonious if  $n = 2^a p_1^{a_1} \cdots p_s^{a_s}$  where  $a, a_1, \dots, a_s$  are positive integers and  $p_1, \dots, p_s$  are distinct odd primes and there is a  $j$  for which  $p_j \equiv 3 \pmod{4}$  and  $a_j$  is odd. He also shows that  $K_n^{(3)}$  is not harmonious when  $n \equiv 0 \pmod{4}$  and  $3n = 4^e(8k + 7)$  or  $n \equiv 5 \pmod{8}$ . Koh et al. [KRLT] and Rajasingh and Pushpam [RP2] have shown that  $K_{m,n}^{(t)}$ , the one-point union of  $t$  copies of  $K_{m,n}$ , is graceful. Sethuraman and Selvaraju [SS3] have proved that the one-point union of graphs of the form  $K_{2,m_i}$  for  $i = 1, 2, \dots, n$  where the union is taken at a vertex from the partite set with 2 vertices is graceful if at most two of the  $m_i$  are equal. They conjecture that the restriction that at most two of the  $m_i$  are equal is not necessary. Koh et al. [KRTY] introduced the notation  $B(n, r, m)$  for the graph consisting of  $m$  copies of  $K_n$  with a  $K_r$  in common. (Guo [Gu2] has used the notation  $B(n, r, m)$  to denote three independent paths of lengths  $n, r$  and  $m$  joining two vertices.) Bermond [Be] raised the question: "For which  $m, n$  and  $r$  is  $B(n, r, m)$  graceful?" Of course, the case  $r = 1$  is the same as  $K_n^{(m)}$ . For  $r > 1$ ,  $B(n, r, m)$  is graceful in the following cases:  $n = 3, r = 2, m \geq 1$  [KRL];  $n = 4, r = 2, m \geq 1$  [D];  $n = 4, r = 3, m \geq 1$  (see [Be]), [KRL]. Seoud and Youssef [SY3] have proved  $B(3, 2, m)$  and  $B(4, 3, m)$  are harmonious. Liu [LiuB1] has shown that if there is a prime  $p$  such that  $p \equiv 3 \pmod{4}$  and  $p$  divides both  $n$  and  $n - 2$  and the highest power of  $p$  that divides  $n$  and  $n - 2$  is odd, then  $B(n, 2, 2)$  is not graceful. More generally, Bermond and Farhi [BF] have considered the class of graphs consisting of  $m$  copies of  $K_n$  having exactly  $k$  copies of  $K_r$  in common. They proved such graphs are not graceful for  $n$  sufficiently large compared to  $r$ .

In [SK] Sethuraman and Kishore determine the graceful graphs that are the union of  $n$  copies of  $K_4$  with  $i$  edges deleted for  $1 \leq i \leq 5$  with one edge in common. The only cases that are not graceful are those graphs where the members of the union are  $C_4$  for  $n \equiv 3 \pmod{4}$  and where the members of the union are  $P_2$ . They conjecture that these two cases are the only instances of edge induced subgraphs of the union of  $n$  copies of  $K_4$  with one edge in common that are not graceful.

## 2.5 Disconnected Graphs

There have been many papers dealing with graphs that are not connected. In 1975 Kotzig [K2] considered graphs that are the disjoint union of  $r$  cycles of length  $s$ , denoted by  $rC_s$ . When  $rs \equiv 1$  or  $2 \pmod{4}$ , these graphs violate the parity condition and so are not graceful. Kotzig proved that when  $r = 3$  and  $s = 4k > 4$ , then  $rC_s$  has a stronger form of graceful labeling called  $\alpha$ -labeling (see §3.1) whereas when  $r \geq 2$  and  $s = 3$  or  $5$ ,  $rC_s$  is not graceful. In 1984 Kotzig [K4] once again investigated the gracefulness of  $rC_s$  as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of  $rC_s$ . He [He] has shown that graphs of the form  $2C_{2m}$  and graphs obtained by connecting two copies of  $C_{2m}$  with an edge are graceful. Cahit [C7] has shown that  $rC_s$  is harmonious when  $r$  and  $s$  are

odd and Seoud, Abdel Maqsood and Sheen [SAS1] noted that when  $r$  or  $s$  is even,  $rC_s$  is not harmonious. Seoud, Abdel Maqsood and Sheen [SAS1] proved that  $C_n \cup C_{n+1}$  is harmonious if and only if  $n \geq 4$ . They conjecture that  $C_3 \cup C_{2n}$  is harmonious when  $k \geq 3$ . In 1978 Kotzig and Turgeon [KT] proved that  $mK_n$  (i.e., the union of  $m$  disjoint copies of  $K_n$ ) is graceful if and only if  $m = 1$  and  $n \leq 4$ . Liu and Zhang [LZ2] have shown that  $mK_n$  is not harmonious for  $n$  odd and  $m \equiv 2 \pmod{4}$  and is harmonious for  $n = 3$  and  $m$  odd. They conjecture that  $mK_3$  is not harmonious when  $m \equiv 0 \pmod{4}$ . Bu and Cao [BC1] give some sufficient conditions for the gracefulness of graphs of the form  $K_{m,n} \cup G$  and they prove that  $K_{m,n} \cup P_t$  and the disjoint union of complete bipartite graphs are graceful under some conditions.

A *Skolem sequence* of order  $n$  is a sequence  $s_1, s_2, \dots, s_{2n}$  of  $2n$  terms such that, for each  $k \in \{1, 2, \dots, n\}$ , there exist exactly two subscripts  $i(k)$  and  $j(k)$  with  $s_{i(k)} = s_{j(k)} = k$  and  $|i(k) - j(k)| = k$ . A Skolem sequence of order  $n$  exists if and only if  $n \equiv 0$  or  $1 \pmod{4}$ . Abrham [Ab2] has proved that any graceful 2-regular graph of order  $n \equiv 0 \pmod{4}$  in which all the component cycles are even or of order  $n \equiv 3 \pmod{4}$ , with exactly one component an odd cycle, can be used to construct a Skolem sequence of order  $n + 1$ . Also, he showed that certain special Skolem sequences of order  $n$  can be used to generate graceful labelings on certain 2-regular graphs.

In 1985 Frucht and Salinas [FS] conjectured that  $C_s \cup P_n$  is graceful if and only if  $s + n \geq 7$  and they proved the conjecture for the case that  $s = 4$ . Frucht [F3] did the case the  $s = 3$  and the case that  $s = 2n + 1$ . Bhat-Nayak and Deshmukh [BD5] also did the case  $s = 3$  and they have done the cases of the form  $C_{2x+1} \cup P_{x-2\theta}$  where  $1 \leq \theta \leq \lfloor (x-2)/2 \rfloor$  [BD1]. Choudum and Kishore [CK2] have done the cases where  $s \geq 5$  and  $n \geq (s+5)/2$  and Kishore [Kis] did the case  $s = 5$ . Gao and Liang [GaL] have done the following cases:  $s > 4, n = 2$  (see also [Gao]);  $s = 4k, n = k + 2, n = k + 3, n = 2k + 2; s = 4k + 1, n = 2k, n = 3k - 1, n = 4k - 1; s = 4k + 2, n = 3k, n = 3k + 1, n = 4k + 1; s = 4k + 3, n = 2k + 1, n = 3k, n = 4k$ . Seoud, Abdel Maqsood and Sheehan [SAS2] did the case that  $s = 2k$  ( $k \geq 3$ ) and  $n \geq k + 1$  as well as the cases where  $s = 6, 8, 10, 12$  and  $n \geq 2$ . Seoud and Youssef [SY2] have shown that  $K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q}$  ( $m, n, p, q \geq 2$ ),  $K_{m,n} \cup K_{p,q} \cup K_{r,s}$  ( $m, n, p, q, r, s \geq 2, (p, q) \neq (2, 2)$ ), and  $pK_{m,n}$  ( $m, n \geq 2, (m, n) \neq (2, 2)$ ) are graceful. They also prove that  $C_4 \cup K_{1,n}$  ( $n \neq 2$ ) is not graceful whereas Choudum and Kishore [CK4], [Kis] have proved that  $C_s \cup K_{1,n}$  is graceful for every  $s \geq 7$  and  $n \geq 1$ . Lee, Quach and Wang [LQW] established the gracefulness of  $P_s \cup K_{1,n}$ . Seoud and Wilson [SW] have shown that  $C_3 \cup K_4, C_3 \cup C_3 \cup K_4$  and certain graphs of the form  $C_3 \cup P_n$  and  $C_3 \cup C_3 \cup P_n$  are not graceful. Abrham and Kotzig [AK4] proved that  $C_p \cup C_q$  is graceful if and only if  $p + q \equiv 0$  or  $3 \pmod{4}$ . Zhou [Zho] proved that  $K_m \cup K_n$  is graceful if and only if  $\{m, n\} = \{4, 2\}$  or  $\{5, 2\}$ . Shee [S1] has shown that graphs of the form  $P_2 \cup C_{2k+1}$  ( $k > 1$ ),  $P_3 \cup C_{2k+1}$ ,  $P_n \cup C_3$  and  $S_n \cup C_{2k+1}$  all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [BD3] have shown that  $C_{4t} \cup K_{1,4t-1}$  and  $C_{4t+3} \cup K_{1,4t+2}$  are graceful. Yang and Wang [YW2] proved that  $S_m \cup S_n$  is graceful if and only if  $m$  or  $n$  is odd and that  $S_m \cup S_n \cup S_k$  is graceful if and only if

at least one of  $m, n$  or  $k$  is odd ( $m > 1, n > 1, k > 1$ ).

Seoud and Youssef [SY5] investigated the gracefulfulness of specific families of the form  $G \cup K_{m,n}$ . They obtained the following results:

$C_3 \cup K_{m,n}$  is graceful if and only if  $m \geq 2$  and  $n \geq 2$ ;

$C_4 \cup K_{m,n}$  is graceful if and only if  $m \geq 2$  and  $n \geq 2$  or  $\{m, n\} = \{1, 2\}$ ;

$C_7 \cup K_{m,n}$  and  $C_8 \cup K_{m,n}$  are graceful for all  $m$  and  $n$ ;

$mK_3 \cup nK_{1,r}$  is not graceful for all  $m, n$  and  $r$ ;

$K_i \cup K_{m,n}$  is graceful for  $i \leq 4$  and  $m \geq 2, n \geq 2$  except for  $i = 2$  and  $(m, n) = (2, 2)$ ;

$K_5 \cup K_{1,n}$  is graceful for all  $n$ ;

$K_6 \cup K_{1,n}$  is graceful if and only if  $n = 1$  or  $3$ .

## 2.6 Joins of Graphs

A few classes of graphs that are the join of graphs have been shown to be graceful and harmonious. Among these are fans  $P_n + K_1$  [GS] and double fans  $P_n + \overline{K_2}$  [GS]. More generally, Reid [Re] proved that  $P_n + \overline{K_t}$  is harmonious and Grace showed [Gr2] that if  $T$  is any graceful tree, then  $T + \overline{K_t}$  is also graceful. Fu and Wu [FW] proved that if  $T$  is a graceful tree, then  $T + S_k$  is graceful. Of course, wheels are of the form  $C_n + K_1$  and are graceful and harmonious. Hebbare [H] showed that  $S_m + K_1$  is graceful for all  $m$ . Shee [S] has proved  $K_{m,n} + K_1$  is harmonious and observed that various cases of  $K_{m,n} + K_t$  violate the harmonious parity condition in [GS]. Liu and Zhang [LZ2] have proved that  $K_2 + K_2 + \dots + K_2$  is harmonious. Yuan and Zhu [YZ] proved that  $K_{m,n} + K_2$  is graceful and harmonious. Gnanajothi [Ga, pp.80-127] obtained the following:  $C_n + \overline{K_2}$  is harmonious when  $n$  is odd and not harmonious when  $n \equiv 2, 4, 6 \pmod{8}$ ;  $S_n + \overline{K_t}$  is harmonious; and  $P_n + \overline{K_t}$  is harmonious. Balakrishnan and Kumar [BK2] have proved that the join of  $\overline{K_n}$  and two disjoint copies of  $K_2$  is harmonious if and only if  $n$  is even. Bu [Bu2] obtained partial results for the gracefulfulness of  $K_n + \overline{K_m}$ . Ramírez-Alfonsín [Ra] has proved that if  $G$  is graceful and  $|V(G)| = |E(G)| = e$  and either 1 or  $e$  is not a vertex label then  $G + \overline{K_t}$  is graceful for all  $t$ .

Seoud and Youssef [SY4] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and  $C_n + \overline{K_t}$  is harmonious for every  $t$  when  $n$  is odd. They also prove that if any edge is added to  $K_{m,n}$  the resulting graph is harmonious if  $m$  or  $n$  is at least 2. Deng [De] has shown certain cases of  $C_n + \overline{K_t}$  are harmonious. Seoud and Youssef [SY7] proved: the graph obtained by appending any number of edges from the two vertices of degree  $n \geq 2$  in  $K_{2,n}$  is not harmonious; dragons  $D_{m,n}$  (i.e.,  $P_m$  is appended to  $C_n$ ) are not harmonious when  $m + n$  is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has odd order.

## 2.7 Miscellaneous Results

It is easy to see that  $P_n^2$  is harmonious [Gr2] while a proof that  $P_n^2$  is graceful has been given by Kang, Liang, Gao and Yang [KLGy]. ( $P_n^k$ , the  $k$ th power of  $P_n$ , is the graph obtained from  $P_n$  by adding edges that join all vertices  $u$  and  $v$  with  $d(u, v) = k$ .) This latter result proved a conjecture of Grace [Gr2]. Seoud, Abdel Maqsood and Sheeham [SAS1] proved that  $P_n^3$  is harmonious and conjecture that  $P_n^k$  is not harmonious when  $k > 3$ . However, Youssef [Yo] has observed that  $P_8^4$  is harmonious. Gnanajothi [Gn, p.50] has shown that the graph that consists of  $n$  copies of  $C_6$  that have exactly  $P_4$  in common is graceful if and only if  $n$  is even. For a fixed  $n$ , let  $v_{i1}, v_{i2}, v_{i3}$  and  $v_{i4}$  ( $1 \leq i \leq n$ ) be consecutive vertices of  $n$  4-cycles. Gnanajothi [Gn, p.35] also proves that the graph obtained by joining each  $v_{i1}$  to  $v_{i+1,3}$  is graceful for all  $n$  and the generalized Petersen graph  $P(n, k)$  is harmonious in all cases (see also [LSS]). ( $P(n, k)$ , where  $n \geq 5$  and  $1 \leq k \leq n$ , has vertex set  $\{a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}\}$  and edge set  $\{a_i a_{i+1} \mid i = 0, 1, \dots, n-1\} \cup \{a_i b_i \mid i = 0, 1, \dots, n-1\} \cup \{b_i b_{i+k} \mid i = 0, 1, \dots, n-1\}$  where all subscripts are taken modulo  $n$  [Wat]. The standard Petersen graph is  $P(5, 2)$ .) The gracefulness of the generalized Petersen graphs appears to be an open problem.

Yuan and Zhu [YZ] define a generalization of  $P_n^2$  as follows:  $P_n(2k)$  is the graph obtained from the path  $P_n$  by adding edges that join all vertices  $x$  and  $y$  with  $d(x, y) = 2k$ . They proved that  $P_n(2k)$  is harmonious when  $1 \leq k \leq \frac{n-1}{2}$  and that  $P_n(2k)$  has a stronger form of harmonious labeling (see Section 4.1) when  $2k-1 \leq n \leq 4k-1$ . Cahit [C7] defines a  $p$ -star as the graph obtained by joining  $p$  disjoint paths of a fixed length  $k$  to single vertex. He proves all such graphs are harmonious when  $p$  is odd and when  $k = 2$  and  $p$  is even.

Sethuraman and Selvaraju [SS1] define a graph  $H$  to be a *super-subdivision* of a graph  $G$ , if every edge  $uv$  of  $G$  is replaced by  $K_{2,m}$  ( $m$  may vary for each edge) by identifying  $u$  and  $v$  with the two vertices in  $K_{2,m}$  that form one of the two partite sets. Sethuraman and Selvaraju prove that every super-subdivision of a path is graceful. They conjecture that every super-subdivision of a star is graceful and that paths and stars are the only graphs for which every super-subdivision is graceful. In [SS2] Sethuraman and Selvaraju prove that every connected graph has some super-subdivision that is graceful. They pose the question as to whether this result is valid for disconnected graphs. They also ask if there is any other graph  $F$  other than  $K_{2,m}$  that can be used to replace an edge of a connected graph to obtain a super-subdivision that is graceful.

Kathiresan [Kat4] uses the notation  $P_{a,b}$  to denote the graph obtained by identifying the end points of  $b$  internally disjoint paths each of length  $a$ . He conjectures that  $P_{a,b}$  is graceful except when  $a$  is odd and  $b \equiv 2 \pmod{4}$ . He proves the conjecture for the case that  $a$  is even and  $b$  is odd. Kathiresan also shows that the graph obtained by identifying a vertex of  $K_n$  with any noncenter vertex of the star with  $2^{n-1} - n(n-1)/2$  edges is graceful.

The total graph  $T(P_n)$  has vertex set  $V(P_n) \cup E(P_n)$  with two vertices adjacent whenever they are neighbors in  $P_n$ . Balakrishnan, Selvam and Yegnanarayanan [BSY1] have proved that  $T(P_n)$  is harmonious.

For any graph  $G$  with vertices  $v_1, \dots, v_n$  and a vector  $\mathbf{m} = (m_1, \dots, m_n)$  of positive integers the corresponding *replicated graph*,  $R_{\mathbf{m}}(G)$ , of  $G$  is defined as follows. For each  $v_i$  form a stable set  $S_i$  consisting of  $m_i$  new vertices  $i = 1, 2, \dots, n$  (recall a *stable set*  $S$  consists of a set of vertices such that there is not an edge  $v_i v_j$  for all pairs  $v_i, v_j$  in  $S$ ); two stable sets  $S_i, S_j, i \neq j$ , form a complete bipartite graph if each  $v_i v_j$  is an edge in  $G$  and otherwise there are no edges between  $S_i$  and  $S_j$ . Ramírez-Alfoncín [Ra] has proved that  $R_{\mathbf{m}}(P_n)$  is graceful for all  $\mathbf{m}$  and all  $n > 1$  and that  $R_{(m,1,\dots,1)}(C_{4n}), R_{(2,1,\dots,1)}(C_n)$  ( $n \geq 8$ ) and,  $R_{(2,2,1,\dots,1)}(C_{4n})$  ( $n \geq 12$ ) are graceful.

For any permutation  $f$  on  $1, \dots, n$ , the  $f$ -permutation graph on a graph  $G, P(G, f)$ , consists of two disjoint copies of  $G, G_1$  and  $G_2$ , each of which has vertices labeled  $v_1, v_2, \dots, v_n$  with  $n$  edges obtained by joining each  $v_i$  in  $G_1$  to  $v_{f(i)}$  in  $G_2$ . In 1983 Lee (see [LWK]) conjectured that for all  $n > 1$  and all permutations on  $1, 2, \dots, n$ , the permutation graph  $P(P_n, f)$  is graceful. Lee, Wang and Kiang [LWK] proved that  $P(P_{2k}, f)$  is graceful when  $f = (12)(34) \cdots (k, k+1) \cdots (2k-1, 2k)$ . They conjectured that if  $G$  is a graceful nonbipartite graph with  $n$  vertices then for any permutation  $f$  on  $1, 2, \dots, n$ , the permutation graph  $P(G, f)$  is graceful. Some families of graceful permutation graphs are given in [LLWK].

Gnanajothi [Gn, p.51] calls a graph  $G$  *bigraceful* if both  $G$  and its line graph are graceful. She shows the following are bigraceful:  $P_m; P_m \times P_n; C_n$  if and only if  $n \equiv 0, 3 \pmod{4}$ ;  $S_n; K_n$  if and only if  $n \leq 3$ ; and  $B_n$  if and only if  $n \equiv 3 \pmod{4}$ . She also shows that  $K_{m,n}$  is not bigraceful when  $n \equiv 3 \pmod{4}$ . (Gangopadhyay and Hebbare [GH] used the term bigraceful to mean a bipartite graceful graph.) Murugan and Arumugan [MA3] have shown that graphs obtained from  $C_4$  by attaching two disjoint paths of equal length to two adjacent vertices are bigraceful.

Several well-known isolated graphs have been examined. Graceful labelings of the Petersen graph, the cube, the icosahedron and the dodecahedron can be found in [Go] and [Gar]. On the other hand, Graham and Sloane [GS] showed that all of these except the cube are harmonious. Winters [Wi] verified that the Grötzsch graph (see [BoM], p. 118), the Heawood graph (see [BoM], p. 236) and the Herschel graph (see [BoM], p. 53) are graceful.

## 2.8 Summary

The results and conjectures discussed above are summarized in the tables following. The letter G after a class of graphs indicates that the graphs in that class are known to be graceful; a question mark indicates that the gracefulness of the graphs in the class is an open problem; we put a “G” next to a question mark if the graphs have been conjectured to be graceful. The analogous notation with the letter H is used to indicate the status of the graphs with regard to being harmonious. The tables impart at a glimpse what has been done and what needs to be done to close out a particular class of graphs. Of course, there is an unlimited number of graphs one could consider. One wishes for some general results that would handle several broad classes at once but the experience of

many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture alone has withstood the efforts of scores of people over the past three decades. Analogous sweeping conjectures are probably true but appear hopelessly difficult to prove.

Table 1: **Summary of Graceful Results**

<i>Graph</i>	<i>Graceful</i>
Trees	G if $\leq 27$ vertices [AlM] G if symmetrical [BeS] G if at most 4 end-vertices [HKR] ?G Ringel-Kotzig
Cycles $C_n$	G iff $n \equiv 0, 3 \pmod{4}$ [Ro1]
Wheels $W_n$	G [F1], [HK]
Helms (see §2.2)	G [AF]
Webs (see §2.2)	G [KLGy]
Gears (see §2.2)	G [MF2]
Cycles with $P_k$ -chord (see §2.2)	G [DMTKT], [MF1], [KY], [PP]
$C_n$ with $k$ consecutive chords (see §2.2)	G if $k = 2, 3, n - 3$ [KP], [KRTY]
Unicyclic graphs	?G iff $G \neq C_n, n \equiv 1, 2 \pmod{4}$ [T]

Table 1: **continued**

<i>Graph</i>	<i>Graceful</i>
$C_n^{(t)}$ (see §2.2)	$n = 3$ G iff $t \equiv 0, 1 \pmod{4}$ [BBG], [BKT] ?G if $nt \equiv 0, 3 \pmod{4}$ [KRLT] G if $n = 6, t$ even [KRLT] G if $n = 4, t > 1$ [S2] G if $t = 2, n \not\equiv 1 \pmod{4}$ [Q], [BSW1]
Triangular snakes (see §2.2)	G iff number of blocks $\equiv 0, 1 \pmod{4}$ [Mo]
$K_4$ -snakes (see §2.2)	?
Quadilateral snakes (see §2.2)	G [Gn], [Q]
Crowns $C_n \odot K_1$	G [F1]
Grids $P_m \times P_n$	G [AG]
Prisms $C_m \times P_n$	G if $n = 2$ [FG] G if $m$ even [HuS], G if $m$ odd, $3 \leq n \leq 12$ [HuS] G if $m = 3$ [Sin1]
Torus grids $C_m \times C_n$	G if $m \equiv 0 \pmod{4}, n$ even [JR] not G if $m, n$ odd (parity condition)
Vertex-deleted $C_m \times P_n$	G if $n = 2$ [GPW]
Edge-deleted $C_m \times P_n$	G if $n = 2$ [GPW]
Möbius ladders $M_n$ (see §2.3)	G [Ga1]
Stacked books $S_m \times P_n$ (see §2.3)	$n = 2$ , G iff $m \not\equiv 3 \pmod{4}$ [Mah], [D], [GJ] G if $m$ even [GJ]

Table 1: **continued**

<i>Graph</i>	<i>Graceful</i>
$n$ -cube $K_2 \times K_2 \times \cdots \times K_2$	G [K3]
$K_4 \times P_n$	?
$K_n$	G iff $n \leq 4$ [Go], [Si]
$K_{m,n}$	G [Ro1], [Go]
$K_{1,m,n}$	G [AM]
$K_{1,1,m,n}$	G [Gn]
Windmills $K_n^{(m)}$ ( $n > 3$ ) (see §2.4)	G if $n = 4, m \leq 22$ [HuS] ?G if $n = 4, m \geq 4$ [Be] G if $n = 4, 4 \leq m \leq 22$ [HuS] not G if $n = 4, m = 2, 3$ [Be] not G if $(m, n) = (2, 5)$ [BKT] not G if $n > 5$ [KRTY]
$B(n, r, m)$ $r > 1$ (see §2.4)	G if $(n, r) = (3, 2), (4, 3)$ [KRL], $(4, 2)$ [D]
$mK_n$ (see §2.5)	G iff $m = 1, n \leq 4$ [KT]
$C_s \cup P_n$	? G iff $s + n \geq 7$ [FS] G if $s = 3$ [F3], $s = 4$ [FS], $s = 5$ [Kis] G if $s > 4, n = 2$ [GaL] G if $s = 2n + 1$ [F3] G if $s = 2k, n \geq k + 1$ [SAS2]
$C_p \cup C_q$	? G iff $p + q \equiv 0, 3 \pmod{4}$ [FS] G if $s = 2n + 1$ [F3], $s \geq 5$ and $n \geq (s + 5)/2$ [CK2]



Table 1: **continued**

<i>Graph</i>	<i>Graceful</i>
Fans $F_n = P_n + K_1$	G [GS]
Double fans $P_n + \overline{K_2}$	G [GS]
$t$ -point suspension $P_n + \overline{K_t}$ of $P_n$	G [Gr2]
$S_m + K_1$	G [H]
$t$ -point suspension of $C_n + \overline{K_t}$	G if $n \equiv 0$ or $3 \pmod{12}$ [BNS] not G if $t$ is even and $n \equiv 2, 6$ or $10 \pmod{12}$ G if $n = 4, 7, 11$ or $19$ [BNS] G if $n = 5$ or $9$ and $t = 2$ [BNS]
$P_n^2$ (see §2.7)	G [LK]
Petersen $P(n, k)$ (see §2.7)	?
Caterpillars	G [Ro1]
Lobsters	?G [Be]

Table 2: Summary of Harmonious Results

<i>Graph</i>	<i>Harmonious</i>
Trees	H if $\leq 26$ vertices [AIM] ?H [GS]
Cycles $C_n$	H iff $n \equiv 1, 3 \pmod{4}$ [GS]
Wheels $W_n$	H [GS]
Helms (see §2.2)	H [Gn], [LiuY]
Webs (see §2.2)	H if cycle is odd
Gears (see §2.2)	?
Cycles with $P_k$ -chord (see §2.2)	?
$C_n$ with $k$ consecutive chords (see §2.2)	?
Unicyclic graphs	?
$C_n^{(t)}$ (see §2.2)	$n = 3$ H iff $t \not\equiv 2 \pmod{4}$ [GS] H if $n = 4, t > 1$ [S2]
Triangular snakes (see §2.2)	H if number of blocks is odd [X2] not H if number of blocks $\equiv 2 \pmod{4}$ [X2]
$K_4$ -snakes (see §2.2)	H [Gr3]
Quadrilateral snakes (see §2.2)	?
Crowns $C_n \odot K_1$	H [Gr2], [LZ1]
Grids $P_m \times P_n$	H iff $(m, n) \neq (2, 2)$ [JR]
Prisms $C_m \times P_n$	H if $n = 2, m \neq 4$ [GPW] H if $n$ odd [GS] H if $m = 4$ and $n \geq 3$ [JR]

Table 2: **continued**

<i>Graph</i>	<i>Harmonious</i>
Torus grids $C_m \times C_n$ ,	H if $m = 4, n > 1$ [JR] not H if $m \not\equiv 0 \pmod{4}$ and $n$ odd [JR]
Vertex-deleted $C_m \times P_n$	H if $n = 2$ [GPW]
Edge-deleted $C_m \times P_n$	H if $n = 2$ [GPW]
Möbius ladders $M_n$ (see §2.3)	H iff $n \neq 3$ [Ga]
Stacked books $S_m \times P_n$ (see §2.3)	$n = 2$ , H if $m$ even [Gr1], [Re] not H $m \equiv 3 \pmod{4}, n = 2$ , (parity condition) H if $m \equiv 1 \pmod{4}, n = 2$ [Gn]
$n$ -cube $K_2 \times K_2 \times \cdots \times K_2$	not H if $n = 2, 3$ [GS]
$K_4 \times P_n$	H [Re]
$K_n$	H iff $n \leq 4$ [GS]
$K_{m,n}$	H iff $m$ or $n = 1$ [GS]
$K_{1,m,n}$	H [AM]
$K_{1,1,m,n}$	H [Gn]
Windmills $K_n^{(m)}$ ( $n > 3$ ) (see §2.4)	H if $n = 4$ [Hs] $m = 2$ , ?H iff $n = 4$ [GS] not H if $m = 2, n$ odd or 6 [GS] not H for some cases $m = 3$ [LiuB2]
$B(n, r, m)$ $r > 1$ (see §2.4)	$(n, r) = (3, 2), (4, 3)$ [SY3]
$mK_n$ (see §2.5)	H $n = 3, m$ odd [LZ2] not H for $n$ odd, $m \equiv 2 \pmod{4}$ [LZ2]

Table 2: **continued**

<i>Graph</i>	<i>Harmonious</i>
$C_s \cup P_n$	?
Fans $F_n = P_n + K_1$	H [GS]
Double fans $P_n + \overline{K_2}$	H [GS]
$t$ -point suspension $P_n + \overline{K_t}$ of $P_n$	H [Re]
$S_m + K_1$	H [Gn], [CHR]
$t$ -point suspension $C_n + \overline{K_t}$ of $C_n$	H if $n$ odd and $t = 2$ [Re], [Gn] not H if $n \equiv 2, 4, 6 \pmod{8}$ and $t = 2$ [Gn]
$P_n^2$ (see §2.7)	H [Gr2], [LZ1]
Petersen $P(n, k)$ (see §2.7)	H [Gn], [LSS]
Caterpillars	H [GS]
Lobsters	?

### 3 Variations of Graceful Labelings

#### 3.1 $\alpha$ -labelings

In [Ro1] Rosa defined an  $\alpha$ -labeling to be a graceful labeling with the additional property that there exists an integer  $k$  so that for each edge  $xy$  either  $f(x) \leq k < f(y)$  or  $f(y) \leq k < f(x)$ . (Other names for such labelings are *balanced* and *interlaced*.) It follows that such a  $k$  must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an  $\alpha$ -labeling is necessarily bipartite and therefore can not contain a cycle of odd length. Wu [Wu1] has shown that a necessary condition for a bipartite graph with  $n$  edges and degree sequence  $d_1, d_2, \dots, d_p$  to have an  $\alpha$ -labeling is that the  $\gcd(d_1, d_2, \dots, d_p, n)$  divides  $n(n-1)/2$ .

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an  $\alpha$ -labeling is a typical approach. (See [CHR], [Gr2], [CLY] and [JR].) Moreover, Jungreis and Reid [JR] showed how sequential labelings of graphs (see

Section 4.1) can often be obtained by modifying  $\alpha$ -labelings of the graphs.

Graphs with  $\alpha$ -labelings have proved useful in the development of the theory of graph decompositions. Rosa [Ro1], for instance, has shown that if  $G$  is a graph with  $q$  edges and has an  $\alpha$ -labeling, then for every natural number  $p$ , the complete graph  $K_{2qp+1}$  can be decomposed into copies of  $G$  in such a way that the automorphism group of the decomposition itself contains the cyclic group of order  $n$ . In the same vein El-Zanati and Vanden Eynden [EV1] proved that if  $G$  has  $q$  edges and admits an  $\alpha$ -labeling then  $K_{qm,qn}$  can be partitioned into subgraphs isomorphic to  $G$  for all positive integers  $m$  and  $n$ . Although a proof of Ringel's conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have  $\alpha$ -labelings are easy to construct (see [Ro1]).

As to which graphs have  $\alpha$ -labelings, Rosa [Ro1] observed that the  $n$ -cycle has an  $\alpha$ -labeling if and only if  $n \equiv 0 \pmod{4}$  while  $P_n$  always has an  $\alpha$ -labeling. Other familiar graphs that have  $\alpha$ -labelings include caterpillars [Ro1], the  $n$ -cube [K1],  $B_{4n+1}$  (i.e., books with  $4n + 1$  pages) [GJ],  $C_{2m} \cup C_{2m}$ , and  $C_{4m} \cup C_{4m} \cup C_{4m}$  for all  $m > 1$  [K2],  $P_n \times Q_n$  [Mah],  $K_{1,2k} \times Q_n$  [Mah],  $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$  [LV],  $C_{4m} \cup C_{4n+2} \cup C_{4r+2}$ ,  $C_{4m} \cup C_{4n} \cup C_{4r}$  when  $m+n \leq r$  [AK4],  $C_{4m} \cup C_{4n} \cup C_{4r} \cup C_{4s}$  when  $m \geq n+r+s$  [ACE],  $C_{4m} \cup C_{4n} \cup C_{4r+2} \cup C_{4s+2}$  when  $m \geq n+r+s+1$  [ACE],  $((m+1)^2 + 1)C_4$  for all  $m$  [Zhi],  $k^2C_4$  for all  $k$  [Zhi], and  $(k^2 + k)C_4$  for all  $k$  [Zhi]. Abrham and Kotzig [AK2] have shown that  $kC_4$  has an  $\alpha$ -labeling for  $1 \leq k \leq 10$  and that if  $kC_4$  has an  $\alpha$ -labeling then so does  $(4k+1)C_4$ ,  $(5k+1)C_4$  and  $(9k+1)C_4$ .

Zhile [Zhi] uses  $C_m(n)$  to denote the connected graph all of whose blocks are  $C_m$  and whose block-cutpoint-graph is a path. He proves that for all positive integers  $m$  and  $n$ ,  $C_{4m}(n)$  has an  $\alpha$ -labeling but  $C_m(n)$  does not have an  $\alpha$ -labeling when  $m$  is odd.

Abrham and Kotzig [AK4] have proved that  $C_m \cup C_n$  has an  $\alpha$ -labeling if and only if both  $m$  and  $n$  are even and  $m+n \equiv 0 \pmod{4}$ . Kotzig [K2] has also shown that  $C_4 \cup C_4 \cup C_4$  does not have an  $\alpha$ -labeling. He asked if  $n=3$  is the only integer such that the disjoint union of  $n$  copies of  $C_4$  does not have an  $\alpha$ -labeling. This was confirmed by Abrham and Kotzig in [AK3]. Eshghi [Es] proved that every 2-regular bipartite graph with 3 components has an  $\alpha$ -labeling if and only if the number of edges is a multiple of four except for  $C_4 \cup C_4 \cup C_4$ .

Jungreis and Reid [JR] investigated the existence of  $\alpha$ -labelings for graphs of the form  $P_m \times P_n$ ,  $C_m \times P_n$ , and  $C_m \times C_n$  (see also [Ga4]). Of course, the cases involving  $C_m$  with  $m$  odd are not bipartite, so there is no  $\alpha$ -labeling. The only unresolved cases among these three families are  $C_{4m+2} \times P_{2n+1}$  and  $C_{4m+2} \times C_{4n+2}$ . All other cases result in  $\alpha$ -labelings. Balakrishman [Ba] uses the notation  $Q_n(G)$  to denote the graph  $P_2 \times P_2 \times \cdots \times P_2 \times G$  where  $P_2$  occurs  $n-1$  times. Snevily [Sn1] has shown that the graphs  $Q_n(C_{4m})$  and the cycles  $C_{4m}$  with the path  $P_n$  adjoined at each vertex have  $\alpha$ -labelings. He also has shown [Sn2] that compositions of the form  $G[\overline{K_n}]$  have an  $\alpha$ -labeling whenever  $G$  does (see §2.3 for the definition of composition). Balakrishman and Kumar [BK1] have shown that all graphs of the form  $Q_n(G)$  where  $G$  is  $K_{3,3}$ ,  $K_{4,4}$ , or  $P_m$  have an  $\alpha$ -labeling. Balakrishman [Ba] poses the following two problems. For

which graphs  $G$  does  $Q_n(G)$  have an  $\alpha$ -labeling? For which graphs  $G$  does  $Q_n(G)$  have a graceful labeling? Rosa [Ro1] has shown that  $K_{m,n}$  has an  $\alpha$ -labeling. Qian [Q] has proved that quadrilateral snakes have  $\alpha$ -labelings. Fu and Wu [FW] showed that if  $T$  is a tree that has an  $\alpha$ -labeling with partite sets  $V_1$  and  $V_2$  then the graph obtained from  $T$  by joining new vertices  $w_1, w_2, \dots, w_k$  to every vertex of  $V_1$  has an  $\alpha$ -labeling. Similarly, they prove that the graph obtained from  $T$  by joining new vertices  $w_1, w_2, \dots, w_k$  to the vertices of  $V_1$  and new vertices  $u_1, u_2, \dots, u_t$  to every vertex of  $V_2$  has an  $\alpha$ -labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of  $V_1$  or the vertices of both  $V_1$  and  $V_2$ , the resulting graphs have  $\alpha$ -labelings. Fu and Wu [FW] further show that if  $T$  is a tree with an  $\alpha$ -labeling and the sizes of the two partite sets of  $T$  differ at by at most 1, then  $T \times P_m$  has an  $\alpha$ -labeling.

Wu ([Wu2] and [Wu3]) has given a number of methods for constructing larger graceful graphs from graceful graphs. Let  $G_1, G_2, \dots, G_p$  be disjoint connected graphs. Let  $w_i$  be in  $G_i$  for  $1 \leq i \leq p$ . Let  $w$  be a new vertex not in any  $G_i$ . Form a new graph  $\oplus_w(G_1, G_2, \dots, G_p)$  by adjoining to the graph  $G_1 \cup G_2 \cup \dots \cup G_p$  the edges  $ww_1, ww_2, \dots, ww_p$ . In the case where each of  $G_1, G_2, \dots, G_p$  is isomorphic to a graph  $G$  which has an  $\alpha$ -labeling and each  $w_i$  is the isomorphic image of the same vertex in  $G_i$ , Wu shows that the resulting graph is graceful. If  $f$  is an  $\alpha$ -labeling of a graph, the integer  $k$  with the property that for any edge  $uv$  either  $f(u) \leq k < f(v)$  or  $f(v) \leq k < f(u)$  is called the *boundary value* or *critical number* of  $f$ . Wu [Wu2] has also shown that if  $G_1, G_2, \dots, G_p$  are graphs of the same order and have  $\alpha$ -labelings where the labelings for each pair of graphs  $G_i$  and  $G_{p-i+1}$  have the same boundary value for  $1 \leq i \leq n/2$ , then  $\oplus_w(G_1, G_2, \dots, G_p)$  is graceful.

Snevely [Sn2] says that a graph  $G$  eventually has an  $\alpha$ -labeling provided that there is a graph  $H$ , called a *host* of  $G$ , which has an  $\alpha$ -labeling and that the edge set of  $H$  can be partitioned into subgraphs isomorphic to  $G$ . He defines the  $\alpha$ -labeling number of  $G$  to be  $G_\alpha = \min\{t : \text{there is a host } H \text{ of } G \text{ with } |E(H)| = t|G|\}$ . Snevely proved that even cycles have  $\alpha$ -labeling number at most 2 and he conjectured that every bipartite graph has an  $\alpha$ -labeling number. This conjecture was proved by El-Zanati, Fu and Shiue [EFS]. There are no known examples of a graph  $G$  with  $G_\alpha > 2$ .

Given two bipartite graphs  $G_1$  and  $G_2$  with partite sets  $H_1$  and  $L_1$  and  $H_2$  and  $L_2$ , respectively, Snevely [Sn1] defines their *weak tensor product*  $G_1 \overline{\otimes} G_2$  as the bipartite graph with vertex set  $(H_1 \times H_2, L_1 \times L_2)$  and with edge  $(h_1, h_2)(l_1, l_2)$  if  $h_1 l_1 \in E(G_1)$  and  $h_2 l_2 \in E(G_2)$ . He proves that if  $G_1$  and  $G_2$  have  $\alpha$ -labelings then so does  $G_1 \overline{\otimes} G_2$ . This result considerably enlarges the class of graphs known to have  $\alpha$ -labelings.

The sequential join of graphs  $G_1, G_2, \dots, G_n$  is formed from  $G_1 \cup G_2 \cup \dots \cup G_n$  by adding edges joining each vertex of  $G_i$  with each vertex of  $G_{i+1}$  for  $1 \leq i \leq n-1$ . Lee and Wang [LWa] have shown that for all  $n \geq 2$  and any positive integers  $a_1, a_2, \dots, a_n$  the sequential join of the graphs  $\overline{K}_{a_1}, \overline{K}_{a_2}, \dots, \overline{K}_{a_n}$  has an  $\alpha$ -labeling.

In [Ga2] Gallian and Ropp conjectured that every graph obtained by adding a single pendant edge to one or more vertices of a cycle is graceful. Qian [Q] has proved this

conjecture and in the case that the cycle is even he shows the graphs have an  $\alpha$ -labeling. He further proves that for  $n$  even any graph obtained from an  $n$ -cycle by adding one or more pendant edges at some vertices has an  $\alpha$ -labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

For any tree  $T(V, E)$  whose vertices are properly 2-colored Rosa and Širáň [RS] define a *bipartite labeling* of  $T$  as a bijection  $f : V \rightarrow \{0, 1, 2, \dots, |E|\}$  for which there is a  $k$  such that whenever  $f(w) \leq k \leq f(v)$ , then  $u$  and  $v$  have different colors. They define the  $\alpha$ -size of a tree  $T$  as the maximum number of distinct values of the induced edge labels  $|f(u) - f(v)|$ ,  $uv \in E$ , taken over all bipartite labelings  $f$  of  $T$ . They prove that the  $\alpha$ -size of any tree with  $n$  edges is at least  $5(n+1)/7$  and that there exist trees whose  $\alpha$ -size is at most  $(5n+9)/6$ . They conjectured that minimum of the  $\alpha$ -sizes over all trees with  $n$  edges is asymptotically  $5n/6$ . This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [BoS]. Heinrich and Hell [HeH] defined the *gracesize* of a graph  $G$  with  $n$  vertices as the maximum, over all bijections  $f: V(G) \rightarrow \{1, 2, \dots, n\}$ , of the number of distinct values  $|f(u) - f(v)|$  over all edges  $uv$  of  $G$ . So, from Rosa and Širáň's result, the gracesize of any tree with  $n$  edges is at least  $5(n+1)/7$ .

In [GPW] Gallian weakened the condition for an  $\alpha$ -labeling somewhat by defining a *weakly  $\alpha$ -labeling* as a graceful labeling for which there is an integer  $k$  so that for each edge  $xy$  either  $f(x) \leq k \leq f(y)$  or  $f(y) \leq k \leq f(x)$ . Unlike  $\alpha$ -labelings, this condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph; namely, that the vertex with the label  $k$  must be on every odd cycle. Gallian, Prout and Winters [GPW] showed that the prisms  $C_n \times P_2$  with a vertex deleted have  $\alpha$ -labelings. The same paper reveals that  $C_n \times P_2$  with an edge deleted from a cycle has an  $\alpha$ -labeling when  $n$  is even and a weakly  $\alpha$ -labeling when  $n > 3$ .

A special case of  $\alpha$ -labeling called strongly graceful was introduced by Maheo [Mah] in 1980. A graceful labeling  $f$  of a graph  $G$  is called *strongly graceful* if  $G$  is bipartite with two partite sets  $A$  and  $B$  of the same order  $s$ , the number of edges is  $2t + s$ , there is an integer  $k$  with  $t - s \leq k \leq t + s - 1$  such that if  $a \in A$ ,  $f(a) \leq k$ , and if  $b \in B$ ,  $f(b) > k$ , and there is an involution  $\pi$  which is an automorphism of  $G$  such that:  $\pi$  exchanges  $A$  and  $B$  and the  $s$  edges  $a\pi(a)$  where  $a \in A$  have as labels the integers between  $t + 1$  and  $t + s$ . Maheo's main result is that if  $G$  is strongly graceful then so is  $G \times Q_n$ . In particular, she proved that  $(P_n \times Q_n) \times K_2$ ,  $B_{2n}$  and  $B_{2n} \times Q_n$  have strongly graceful labelings. El-Zanati and Vanden Eynden [EV2] call a strongly graceful labeling a *strong  $\alpha$ -labeling*. They show that if  $G$  has a strong  $\alpha$ -labeling, then  $G \times P_n$  has an  $\alpha$ -labeling. They show that  $K_{m,2} \times K_2$  has a strong  $\alpha$ -labeling and that  $K_{m,2} \times P_n$  has an  $\alpha$ -labeling. They also show that if  $G$  is a bipartite graph with one more vertex than the number of edges, and if  $G$  has an  $\alpha$ -labeling such that the cardinalities of the sets of the corresponding bipartition of the vertices differ by at most 1, then  $G \times K_2$  has a strong  $\alpha$ -labeling and  $G \times P_n$  has an  $\alpha$ -labeling. El-Zanati and Vanden Eynden [EV2] also note that  $K_{3,3} \times K_2$ ,  $K_{3,4} \times K_2$ ,  $K_{4,4} \times K_2$ , and  $C_{4k} \times K_2$  all have strong  $\alpha$ -labelings. El-Zanati and Vanden Eynden proved that  $K_{m,2} \times Q_n$  has a strong  $\alpha$ -valuation and that

$K_{m,2} \times P_n$  has an  $\alpha$ -labeling for all  $n$ . They also prove that if  $G$  is a connected bipartite graph with partite sets of odd order such that in each partite set each vertex has the same degree, then  $G \times K_2$  does not have a strong  $\alpha$ -valuation. As a corollary they have that  $K_{m,n} \times K_2$  does not have a strong  $\alpha$ -valuation when  $m$  and  $n$  are odd.

An  $\alpha$ -labeling  $f$  of a graph  $G$  is called *free* in [EV3] if the critical number  $k$  (in the definition of  $\alpha$ -labeling) is greater than 2 and if neither 1 nor  $k-1$  is used in the labeling. Their main result is that the union of graphs with free  $\alpha$ -labelings has an  $\alpha$ -labeling. In particular, they show that  $K_{m,n}$ ,  $m > 1$ ,  $n > 2$ , has a free  $\alpha$ -labeling. They also show that  $Q_n$ ,  $n \geq 3$ , and  $K_{m,2} \times Q_n$ ,  $m > 1$ ,  $n \geq 1$ , have free  $\alpha$ -labelings. El-Zanati [personal communication] has shown that the Heawood graph has a free  $\alpha$ -labeling.

Another kind of labelings for trees was introduced by Ringel, Llado and Serra [RLS] in an approach to proving their conjecture  $K_{n,n}$  is edge-decomposable into  $n$  copies of any given tree with  $n$  edges. If  $T$  is a tree with  $n$  edges and partite sets  $A$  and  $B$ , they define a labeling  $f$  from the set of vertices to  $\{1, 2, \dots, n\}$  to be a *bigraceful* labeling of  $T$  if  $f$  restricted to  $A$  is injective,  $f$  restricted to  $B$  is injective, and the edge labels given by  $f(y) - f(x)$  where  $yx$  is an edge with  $y$  in  $B$  and  $x$  in  $A$  is the set  $\{0, 1, 2, \dots, n-1\}$ . (Notice that this terminology conflicts with that given in Section 2.7 In particular, the Ringel, Llado and Serra bigraceful does not imply the usual graceful.) Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars  $S_{k,m}$  with  $k$  spokes of paths of length  $m$ , and complete  $d$ -ary trees for  $d$  odd. They also prove that if  $T$  is a tree then there is a vertex  $v$  and a nonnegative integer  $m$  such that the addition of  $m$  leaves to  $v$  results in a bigraceful tree. They conjecture that all trees are bigraceful.

### 3.2 $k$ -graceful Labelings

A natural generalization of graceful graphs is the notion of  $k$ -graceful graphs introduced independently by Slater [Sl2] in 1982 and by Maheo and Thuillier [MT] in 1982. A graph  $G$  with  $q$  edges is  *$k$ -graceful* if there is labeling  $f$  from the vertices of  $G$  to  $\{0, 1, 2, \dots, q+k-1\}$  such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is  $\{k, k+1, \dots, q+k-1\}$ . Obviously, 1-graceful is graceful and it is readily shown that any graph that has an  $\alpha$ -labeling is  $k$ -graceful for all  $k$ . Graphs that are  $k$ -graceful for all  $k$  are sometimes called *arbitrarily graceful*. Ng [N2] has shown that there are graphs that are  $k$ -graceful for all  $k$  but do not have an  $\alpha$ -labeling.

Results of Maheo and Thuillier [MT] together with those of Slater [Sl2] show that:  $C_n$  is  $k$ -graceful if and only if either  $n \equiv 0$  or  $1 \pmod{4}$  with  $k$  even and  $k \leq (n-1)/2$ , or  $n \equiv 3 \pmod{4}$  with  $k$  odd and  $k \leq (n^2-1)/2$ . Maheo and Thuillier [MT] also proved that the wheel  $W_{2k+1}$  is  $k$ -graceful and conjectured that  $W_{2k}$  is  $k$ -graceful when  $k \neq 3$  or  $k \neq 4$ . This conjecture was proved by Liang, Sun and Xu [LSX]. Kang [Ka] proved that  $P_m \times C_{4n}$  is  $k$ -graceful for all  $k$ . Lee and Wang [LW] showed that all pyramids, lotuses and diamonds are  $k$ -graceful and Liang and Liu [LL] have shown that  $K_{m,n}$  is  $k$ -graceful. Bu, Gao and Zhang [BGZ] have proved that  $P_n \times P_2$  and  $(P_n \times P_2) \cup (P_n \times P_2)$



are  $k$ -graceful for all  $k$ . Acharya (see [A2]) has shown that a  $k$ -graceful Eulerian graph with  $q$  edges must satisfy one of the following conditions:  $q \equiv 0 \pmod{4}$ ,  $q \equiv 1 \pmod{4}$  if  $k$  is even, or  $q \equiv 3 \pmod{4}$  if  $k$  is odd. Bu, Zhang and He [BZH] have shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is  $k$ -graceful.

Several authors have investigated the  $k$ -gracefulness of various classes of subgraphs of grid graphs. Acharya [A1] proved that all 2-dimensional polyminoes that are convex and Eulerian are  $k$ -graceful for all  $k$ ; Lee [L1] showed that Mongolian tents and Mongolian villages are  $k$ -graceful for all  $k$  (see Section 2.3 for definitions); Lee and K. C. Ng [LNK] proved that all Young tableaux are  $k$ -graceful for all  $k$  (see Section 2.3 for definitions). Lee and H. K. Ng [LNH] subsequently generalized these results on Young tableaux to a wider class of planar graphs.

Let  $c, m, p_1, p_2, \dots, p_m$  be positive integers. For  $i = 1, 2, \dots, m$ , let  $S_i$  be a set of  $p_i + 1$  integers and let  $D_i$  be the set of positive differences of the pairs of elements of  $S_i$ . If all these differences are different then the system  $D_1, D_2, \dots, D_m$  is called a *perfect system of difference sets starting at  $c$*  if the union of all the sets  $D_i$  is  $c, c + 1, \dots, c - 1 + \sum_{i=1}^m \binom{p_i + 1}{2}$ . There is a relationship between  $k$ -graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with  $c$  describes a  $c$ -graceful labeling of a graph which is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in [Ab1].

Acharya and Hegde [AH2] generalized  $k$ -graceful to  $(k, d)$ -graceful labelings by permitting the vertex labels to belong to  $\{0, 1, 2, \dots, k + (q - 1)d\}$  and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be  $\{k, k + d, k + 2d, \dots, k + (q - 1)d\}$ . They also introduce an analog of  $\alpha$ -labelings in the obvious way. Notice that a (1,1)-graceful labeling is a graceful labeling and a  $(k, 1)$ -graceful labeling is a  $k$ -graceful labeling. Bu and Zhang [BZ] have shown that  $K_{m,n}$  is  $(k, d)$ -graceful for all  $k$  and  $d$ ; for  $n > 2$ ,  $K_n$  is  $(k, d)$ -graceful if and only if  $k = d$  and  $n \leq 4$ ; if  $m_i, n_i \geq 2$  and  $\max\{m_i, n_i\} \geq 3$ , then  $K_{m_1, n_1} \cup K_{m_2, n_2} \cup \dots \cup K_{m_r, n_r}$  is  $(k, d)$ -graceful for all  $k, d$ , and  $r$ ; if  $G$  has an  $\alpha$ -labeling, then  $G$  is  $(k, d)$ -graceful for all  $k$  and  $d$ ; a  $k$ -graceful graph is a  $(kd, d)$ -graceful graph; a  $(kd, d)$ -graceful connected graph is  $k$ -graceful; and a  $(k, d)$ -graceful graph with  $q$  edges that is not bipartite must have  $k \leq (q - 2)d$ .

Slater [Sl5] has extended the definition of  $k$ -graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices and the countably infinite Dutch windmill is  $k$ -graceful for all  $k$ .

More specialized results on  $k$ -graceful labelings can be found in [L1], [LNK], [LNH], [Sl2], [BuF], [BH], [BGZ] and [ChJ].

### 3.3 Skolem-Graceful

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [LSh]) calls a graph  $G$  with  $p$  vertices and  $q$  edges *Skolem-graceful* if there is an injection from the set of vertices of  $G$  to  $\{1, 2, \dots, p\}$  such that the edge labels induced by  $|f(x) - f(y)|$  for each edge  $xy$  are  $1, 2, \dots, q$ . A necessary condition for a graph to be Skolem-graceful is that  $p \geq q + 1$ . Lee and Wui [LWu] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. Although the disjoint union of trees can not be graceful, they can be Skolem-graceful. Lee and Wui [LWu] prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. In [CK3] Choudum and Kishore show that the disjoint union of  $k$  copies of the star  $K_{1,2p}$  is Skolem graceful if  $k \leq 4p + 1$  and the disjoint union of any number of copies of  $K_{1,2}$  is Skolem graceful. For  $k \geq 2$ , let  $St(n_1, n_2, \dots, n_k)$  denote the disjoint union of  $k$  stars with  $n_1, n_2, \dots, n_k$  edges. Lee, Wang and Wui [LWW] showed that the 4-star  $St(n_1, n_2, n_3, n_4)$  is Skolem-graceful for some special cases and conjectured that all 4-stars are Skolem-graceful. Denham, Leu and Liu [DLL] proved this conjecture. Kishore [Kis] has shown that a necessary condition for  $St(n_1, n_2, \dots, n_k)$  to be Skolem graceful is that some  $n_i$  is even or  $k \equiv 0$  or  $1 \pmod{4}$ . He conjectures that each one of these conditions is sufficient. Choudum and Kishore [CK1] proved that all 5-stars are Skolem graceful. Lee, Quach and Wang [LQW] showed that the disjoint union of the path  $P_n$  and the star of size  $m$  is Skolem-graceful if and only if  $n = 2$  and  $m$  is even or  $n \geq 3$  and  $m \geq 1$ . It follows from the work of Skolem [Sk] that  $nP_2$ , the disjoint union of  $n$  copies of  $P_2$ , is Skolem-graceful if and only if  $n \equiv 0$  or  $1 \pmod{4}$ . Harary and Hsu [HH] studied Skolem-graceful graphs under the name *node-graceful*. Frucht [F3] has shown that  $P_m \cup P_n$  is Skolem-graceful when  $m + n \geq 5$ . Bhat-Nayak and Deshmukh [BD4] have shown that  $P_{n_1} \cup P_{n_2} \cup P_{n_3}$  is Skolem-graceful when  $n_1 < n_2 \leq n_3$ ,  $n_2 = t(n_1 + 2) + 1$  and  $n_1$  is even and when  $n_1 < n_2 \leq n_3$ ,  $n_2 = t(n_1 + 3) + 1$  and  $n_1$  is odd. They also prove that the graphs of the form  $P_{n_1} \cup P_{n_2} \cup \dots \cup P_{n_i}$  where  $i \geq 4$  are Skolem-graceful under certain conditions.

### 3.4 Odd Graceful Labelings

Gnanajothi [Gn, p. 182] defined a graph  $G$  with  $q$  edges to be *odd graceful* if there is an injection  $f$  from  $V(G)$  to  $\{0, 1, 2, \dots, 2q - 1\}$  such that, when each edge  $xy$  is assigned the label  $|f(x) - f(y)|$ , the resulting edge labels are  $\{1, 3, \dots, 2q - 1\}$ . She proved that the class of odd graceful graphs lies between the class of graphs with  $\alpha$ -labelings and the class of bipartite graphs by showing that every graph with an  $\alpha$ -labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful. She also proved the following graphs are odd graceful:  $P_n$ ;  $C_n$  if and only if  $n$  is even;  $K_{m,n}$ ; combs  $P_n \odot K_1$  (graphs obtained by joining a single pendant edge to each vertex of  $P_n$ ); books; crowns  $C_n \odot K_1$  (graphs obtained by joining a single pendant edge to each vertex of  $C_n$ ) if and only if  $n$  is even; the disjoint union of copies of  $C_4$ ; the one-point union of copies of  $C_4$ ;  $C_n \times K_2$  if and only if  $n$  is even; caterpillars; rooted trees of height 2; the

graphs obtained from  $P_n$  ( $n \geq 3$ ) by adding exactly two leaves at each vertex of degree 2 of  $P_n$ ; the graphs consisting of vertices  $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n$  with edges  $a_i a_{i+1}, b_i b_{i+1}$  for  $i = 0, \dots, n-1$  and  $a_i b_i$  for  $i = 1, \dots, n-1$ ; the graphs obtained from a star by adjoining to each end vertex the path  $P_3$  or by adjoining to each end vertex the path  $P_4$ . She conjectures that all trees are odd graceful and proves the conjecture for all trees with order up to 10. Eldergill [E] generalized Gnanajothi's result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd graceful. He also proved that the one-point union of any number of copies of  $C_6$  is odd graceful. Kathiresan [Kat3] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd graceful.

Seoud, Diab and Elsakhawi [SDE] have shown that a connected  $r$ -partite graph is odd graceful if and only if  $r = 2$  and that the join of any two connected graphs is not odd graceful.

### 3.5 Graceful-like Labelings

As a means of attacking graph decomposition problems, Rosa [Ro1] invented another analogue of graceful labelings by permitting the vertices of a graph with  $q$  edges to assume labels from the set  $\{0, 1, \dots, q+1\}$ , while the edge labels induced by the absolute value of the difference of the vertex labels are  $\{1, 2, \dots, q-1, q\}$  or  $\{1, 2, \dots, q-1, q+1\}$ . He calls these  $\hat{\rho}$ -labelings. Frucht [F3] used the term *nearly graceful labeling* instead of  $\hat{\rho}$ -labelings. Frucht [F3] has shown that the following graphs have nearly graceful labelings with edge labels from  $\{1, 2, \dots, q-1, q+1\}$ :  $P_m \cup P_n$ ;  $S_m \cup S_n$ ;  $S_m \cup P_n$ ;  $G \cup K_2$  where  $G$  is graceful; and  $C_3 \cup K_2 \cup S_m$  where  $m$  is even or  $m \equiv 3 \pmod{14}$ . Seoud and Elsakhawi [SE1] have shown that all cycles are nearly graceful. Barrientos [Bar2] proved that  $C_n$  is nearly graceful with edge labels  $1, 2, \dots, n-1, n+1$  if and only if  $n \equiv 1$  or  $2 \pmod{4}$ . Rosa [Ro2] conjectured that triangular snakes with  $t \equiv 0$  or  $1 \pmod{4}$  blocks are graceful and those with  $t \equiv 2$  or  $3 \pmod{4}$  blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [Mo] proved Rosa's conjecture while introducing the slightly stronger concept of *almost graceful* by permitting the vertex labels to come from  $\{0, 1, 2, \dots, q-1, q+1\}$  while the edge labels are  $\{1, 2, \dots, q-1, q\}$ , or  $\{1, 2, \dots, q-1, q+1\}$ . Seoud and Elsakhawi [SE1] have shown that the following graphs are almost graceful:  $C_n$ ;  $P_n + \overline{K_m}$ ;  $P_n + K_{1,m}$ ;  $K_{m,n}$ ;  $K_{1,m,n}$ ;  $K_{2,2,m}$ ;  $K_{1,1,m,n}$ ; ladders; and  $P_n \times P_3$  ( $n \geq 3$ ).

Barrientos [Bar2] calls a graph a  $kC_n$ -snake if it is a connected graph with  $k$  blocks whose block-cutpoint graph is path and each of the  $k$  blocks is isomorphic to  $C_n$ . (When  $n > 3$  and  $k > 3$  there is more than one  $kC_n$ -snake.) A  $kC_n$ -snake where the path of minimum length that contains all the cut-vertices of the graph has the property that the distance between any two consecutive cut-vertices is  $\lfloor n/2 \rfloor$  is called *linear*. Barrientos proves that  $kC_4$ -snakes are graceful and that the linear  $kC_6$ -snakes are graceful when  $k$  is even. When  $k$  is odd he proves that the linear  $kC_6$ -snake is nearly graceful. Barrientos further proves that  $kC_8$ -snakes and  $kC_{12}$ -snakes are graceful in the cases where the

distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph are all even and that certain cases of  $kC_{4n}$ -snakes and  $kC_{5n}$ -snakes are graceful (depending on the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph).

Yet another kind of labeling introduced by Rosa in his 1967 paper [Ro1] is a  $\rho$ -valuation. A  $\rho$ -valuation of a graph is an injection from the vertices of the graph with  $q$  edges to the set  $\{0, 1, \dots, 2q\}$ , where if the edge labels induced by the absolute value of the difference of the vertex labels are  $a_1, a_2, \dots, a_q$ , then  $a_i = i$  or  $a_i = 2q + 1 - i$ . Rosa [Ro1] proved that a cyclic decomposition of the edge set of the complete graph  $K_{2q+1}$  into subgraphs isomorphic to a given graph  $G$  with  $q$  edges exists if and only if  $G$  has a  $\rho$ -valuation. (A decomposition of  $K_n$  into copies of  $G$  is called *cyclic* if the automorphism group of the decomposition itself contains the cyclic group of order  $n$ .) Dufour [Duf] and Eldergill [E] have some results on the decomposition of the complete graph using labeling methods. Balakrishnan and Sampathkumar [BS] showed that for each positive integer  $n$  the graph  $\overline{K_n} + 2K_2$  admits a  $\rho$ -valuation. Balakrishnan [Ba] asks if it is true that  $\overline{K_n} + mK_2$  admits a  $\rho$ -valuation for all  $n$  and  $m$ . Balakrishnan and Sampathkumar ask for which  $m \geq 3$  is the graph  $\overline{K_n} + mK_2$  graceful for all  $n$ . Bhat-Nayak and Gokhale [BG] have proved that  $\overline{K_n} + 2K_2$  is not graceful.

For graphs with the property  $p = q + 1$ , Frucht [F3] has introduced a stronger version of almost graceful graphs by permitting as vertex labels  $\{0, 1, \dots, q - 1, q + 1\}$  and as edge labels  $\{1, 2, \dots, q\}$ . He calls such a labeling *pseudograceful*. Frucht proved that  $P_n$  ( $n \geq 3$ ), combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star),  $C_3 \cup P_n$  ( $n \neq 3$ ), and  $C_4 \cup P_n$  ( $n \neq 1$ ) are pseudograceful while  $K_{1,n}$  ( $n \geq 3$ ) is not. Kishore [Kis] proved that  $C_s \cup P_n$  is pseudograceful when  $s \geq 5$  and  $n \geq (s + 7)/2$  and that  $C_s \cup S_n$  is pseudograceful when  $s = 3, s = 4$ , and  $s \geq 7$ . Seoud and Youssef [SY2] and [SY5] extended the definition of pseudograceful to all graphs with  $p \leq q + 1$ . They proved that  $K_m$  is pseudograceful if and only if  $m = 1, 3$  or  $4$  [SY5];  $K_{m,n}$  is pseudograceful when  $n \geq 2$  and  $P_m + \overline{K_n}$  ( $m \geq 2$ ) [SY2] is pseudograceful. They also proved that if  $G$  is pseudograceful, then  $G \cup K_{m,n}$  is graceful for  $m \geq 2$  and  $n \geq 2$  and  $G \cup K_{m,n}$  is pseudograceful for  $m \geq 2, n \geq 2$  and  $(m, n) \neq (2, 2)$  [SY5]. They ask if  $G \cup K_{2,2}$  is pseudograceful whenever  $G$  is.

McTavish [Mc] has investigated labelings where the vertex and edge labels are from  $\{0, \dots, q, q + 1\}$ . She calls these  $\tilde{\rho}$ -labelings. Graphs that have  $\tilde{\rho}$ -labelings include cycles and the disjoint union of  $P_n$  or  $S_n$  with any graceful graph.

Frucht [F3] has made an observation about graceful labelings that yields nearly graceful analogs of  $\alpha$ -labelings and weakly  $\alpha$ -labelings in a natural way. Suppose  $G(V, E)$  is a graceful graph with the vertex labeling  $f$ . For each edge  $xy$  in  $E$ , let  $[f(x), f(y)]$  (where  $f(x) \leq f(y)$ ) denote the interval of real numbers  $r$  with  $f(x) \leq r \leq f(y)$ . Then the intersection  $\cap [f(x), f(y)]$  over all edges  $xy \in E$  is a unit interval, a single point, or empty. Indeed, if  $f$  is an  $\alpha$ -labeling of  $G$  then the intersection is a unit interval; if  $f$  is a weakly  $\alpha$ -labeling, but not an  $\alpha$ -labeling, then the intersection is a point; and, if  $f$  is graceful but not a weakly  $\alpha$ -labeling, then the intersection is empty. For nearly graceful

labelings, the intersection also gives three distinct classes.

### 3.6 Cordial Labelings

Cahit [C2] has introduced a variation of both graceful and harmonious labelings. Let  $f$  be a function from the vertices of  $G$  to  $\{0, 1\}$  and for each edge  $xy$  assign the label  $|f(x) - f(y)|$ . Call  $f$  a *cordial labeling* of  $G$  if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [C3] proved the following: every tree is cordial;  $K_n$  is cordial if and only if  $n \leq 3$ ;  $K_{m,n}$  is cordial for all  $m$  and  $n$ ; the friendship graph  $C_3^{(t)}$  (i.e., the one-point union of  $t$  3-cycles) is cordial if and only if  $t \not\equiv 2 \pmod{4}$ ; all fans are cordial; the wheel  $W_n$  is cordial if and only if  $n \not\equiv 3 \pmod{4}$  (see also [Du]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang and Kwong [KCK] determine all  $m$  and  $n$  for which  $mK_n$  is cordial. A *k-angular cactus* is a connected graph all of whose blocks are cycles with  $k$  vertices. In [C3] Cahit proved that a  $k$ -angular cactus with  $t$  cycles is cordial if and only if  $kt \not\equiv 2 \pmod{4}$ . This was improved by Kirchherr [Ki1] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [Ki2] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho, Lee and Shee [HLS2] proved:  $P_n \times C_{4m}$  is cordial for all  $m$  and all odd  $n$ ; the composition  $G[H]$  is cordial if  $G$  is cordial and  $H$  is cordial and has odd order and even size (see §2.3 for definition of composition); for  $n \geq 4$  the composition  $C_n[K_2]$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ ; the cartesian product of two cordial graphs of even size is cordial. The same authors [HLS1] showed that a unicyclic graph is cordial unless it is  $C_{4k+2}$  and that the generalized Petersen graph (see §2.7 for definition)  $P(n, k)$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ . Du [Du] determines the maximal number of edges in a cordial graph of order  $n$  and gives a necessary condition for a  $k$ -regular graph to be cordial.

Seoud and Abdel Maqusoud [SA2] proved that if  $G$  is a graph with  $n$  vertices and  $m$  edges and every vertex has odd degree then  $G$  is not cordial when  $m + n \equiv 2 \pmod{4}$ . They also prove the following: for  $m \geq 2$ ,  $C_n \times P_m$  is cordial except for the case  $C_{4k+2} \times P_2$ ;  $P_n^2$  is cordial for all  $n$ ;  $P_n^3$  is cordial if and only if  $n \neq 4$ ; and  $P_n^4$  is cordial if and only if  $n \neq 4, 5$  or  $6$ . Seoud, Diab and Elsakhawi [SDE] have proved the following graphs are cordial:  $P_n + P_m$  for all  $m$  and  $n$  except  $(m, n) = (2, 2)$ ;  $C_m + C_n$  if  $m \not\equiv 0 \pmod{4}$  and  $n \not\equiv 2 \pmod{4}$ ;  $P_n + K_{1,m}$  for all  $n$  and odd  $m$ ;  $C_n + K_{1,m}$  for  $n \not\equiv 3 \pmod{4}$  and odd  $m$  except  $(n, m) = (3, 1)$ ;  $C_n + \overline{K_m}$  when  $n$  is odd and when  $n$  is even and  $m$  is odd;  $K_{1,m,n}$ ;  $K_{2,2,m}$ ; the  $n$ -cube; books  $B_n$  if and only if  $n \not\equiv 3 \pmod{4}$ ;  $B(3, 2, m)$  for all  $m$ ;  $B(4, 3, m)$  if and only if  $m$  is even; and  $B(5, 3, m)$  if and only if  $m \not\equiv 1 \pmod{4}$  (see §2.4 for the notation  $B(n, r, m)$ ).

Lee, Lee and Chang [LLC] prove the following graphs are cordial: the complete  $n$ -partite graph if and only if at most three of its partite sets have odd cardinality (see also [Du]); the Cartesian product of an arbitrary number of paths; the Cartesian product of

two cycles if and only if at least one of them is even; and the Cartesian product of an arbitrary number of cycles if at least one of them has length a multiple of 4 or at least two of them are even.

Shee and Ho [SH1] have investigated the cordiality of the one-point union of  $n$  copies of various graphs. For  $C_m^{(n)}$ , the one-point union of  $n$  copies of  $C_m$ , they proved:

- (i) If  $m \equiv 0 \pmod{4}$ , then  $C_m^{(n)}$  is cordial for all  $n$ ;
- (ii) If  $m \equiv 1$  or  $3 \pmod{4}$ , then  $C_m^{(n)}$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ ;
- (iii) If  $m \equiv 2 \pmod{4}$ , then  $C_m^{(n)}$  is cordial if and only if  $n$  is even.

For  $K_m^{(n)}$ , the one-point union of  $n$  copies of  $K_m$ , Shee and Ho [SH1] prove:

- (i) If  $m \equiv 0 \pmod{8}$ , then  $K_m^{(n)}$  is not cordial for  $n \equiv 3 \pmod{4}$ ;
- (ii) If  $m \equiv 4 \pmod{8}$ , then  $K_m^{(n)}$  is not cordial for  $n \equiv 1 \pmod{4}$ ;
- (iii) If  $m \equiv 5 \pmod{8}$ , then  $K_m^{(n)}$  is not cordial for all odd  $n$ ;
- (iv)  $K_4^{(n)}$  is cordial if and only if  $n \not\equiv 1 \pmod{4}$ ;
- (v)  $K_5^{(n)}$  is cordial if and only if  $n$  is even;
- (vi)  $K_6^{(n)}$  is cordial if and only if  $n > 2$ ;
- (vii)  $K_7^{(n)}$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ ;
- (viii)  $K_n^{(2)}$  is cordial if and only if  $n$  has the form  $p^2$  or  $p^2 + 1$ .

Benson and Lee [BL] have investigated the regular windmill graphs  $K_m^{(n)}$  and determined precisely which ones are cordial for  $m < 14$ .

For  $W_m^{(n)}$ , the one-point union of  $n$  copies of the wheel  $W_m$  with the common vertex being the center, Shee and Ho [SH1] show:

- (i) If  $m \equiv 0$  or  $2 \pmod{4}$ , then  $W_m^{(n)}$  is cordial for all  $n$ ;
- (ii) If  $m \equiv 3 \pmod{4}$ , then  $W_m^{(n)}$  is cordial if  $n \not\equiv 1 \pmod{4}$ ;
- (iii) If  $m \equiv 1 \pmod{4}$ , then  $W_m^{(n)}$  is cordial if  $n \not\equiv 3 \pmod{4}$ .

For all  $n$  and all  $m > 1$  Shee and Ho [SH1] prove  $F_m^{(n)}$ , the one-point union of  $n$  copies of the fan  $F_m = P_m + K_1$  with the common point of the fans being the center, is cordial. The *flag*  $Fl_m$  is obtained by joining one vertex of  $C_m$  to an extra vertex called the *root*. Shee and Ho [SH1] show all  $Fl_m^{(n)}$ , the one-point union of  $n$  copies of  $Fl_m$  with the common point being the root, are cordial.

Andar, Boxwala and Limaye [ABL1] and [ABL2] have proved the following graphs are cordial: helms; closed helms; generalized helms obtained by taking a web and attaching pendent vertices to all the vertices of the outermost cycle in the case that the number cycles is even; flowers, which are obtained by joining the vertices of degree one of a helm to the central vertex; sunflower graphs, which are obtained by taking a wheel with the central vertex  $v_0$  and the cycle  $v_1, v_2, \dots, v_n, v_1$ , and additional vertices  $w_1, w_2, \dots, w_n$  where  $w_i$  is joined by edges to  $v_i, v_{i+1}$ , where  $i+1$  is taken modulo  $n$ , and multiple shells (see §2.2).

For graphs  $G_1, G_2, \dots, G_n$  ( $n \geq 2$ ) which are all copies of a fixed graph  $G$ , Shee and Ho [SH2] call a graph obtained by adding an edge from  $G_i$  to  $G_{i+1}$  for  $i = 1, \dots, n-1$  a

*path-union* of  $G$  (the resulting graph may depend on how the edges are chosen). Among their results they show the following graphs are cordial: path-unions of cycles; path-unions of  $n$  copies of  $K_m$  when  $m = 4, 6$  or  $7$ ; path-unions of three or more copies of  $K_5$ ; and path-unions of two copies of  $K_m$  if and only if  $m - 2, m$  or  $m + 2$  is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees and various compositions.

Lee and Liu [LeL] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let  $H$  be a graph with an even number of edges and a cordial labeling such that the vertices of  $H$  can be divided into  $t$  parts  $H_1, H_2, \dots, H_t$  each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let  $G$  be any graph and  $G_1, G_2, \dots, G_t$  be any  $t$  subsets of the vertices of  $G$ . Let  $(G, H)$  be the graph which is the disjoint union of  $G$  and  $H$  augmented by edges joining every vertex in  $G_i$  to every vertex in  $H_i$  for all  $i$ . Then  $G$  is cordial if and only if  $(G, H)$  is. From this it follows that: all generalized fans  $F_{m,n} = \overline{K_m} + P_n$  are cordial; the generalized bundle  $B_{m,n}$  is cordial if and only if  $m$  is even or  $n \not\equiv 2 \pmod{4}$  ( $B_{m,n}$  consists of  $2n$  vertices  $v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n$  with an edge from  $v_i$  to  $u_i$  and  $2m$  vertices  $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m$  with  $x_i$  joined to  $v_i$  and  $y_i$  joined to  $u_i$ ); if  $m$  is odd a generalized wheel  $W_{m,n} = \overline{K_m} + C_n$  is cordial if and only if  $n \not\equiv 3 \pmod{4}$ . If  $m$  is even,  $W_{m,n}$  is cordial if and only if  $n \not\equiv 2 \pmod{4}$ ; a complete  $k$ -partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Cahit [C8] calls a graph  $H$ -cordial if it is possible to label the edges with the numbers from the set  $\{+1, -1\}$  in such a way that, for some  $k$ , at each vertex  $v$  the algebraic sum of the labels on the edges incident with  $v$  is either  $+k$  or  $-k$  and the inequalities  $|v(+k) - v(-k)| \leq 1$  and  $|e(+1) - e(-1)| \leq 1$  are also satisfied, where  $v(i)$  and  $e(j)$  are, respectively, the number of vertices labeled with  $i$  and the number of edges labeled with  $j$ . He proves:  $K_n$  is  $H$ -cordial if and only if  $n \equiv 0 \pmod{4}$ ;  $K_{n,n}$  is  $H$ -cordial if and only if  $n > 2$  and  $n$  is even;  $K_{m,n}, m \neq n$ , is  $H$ -cordial if and only if  $n \equiv 0 \pmod{4}$ ,  $m$  is even and  $m > 2, n > 2$ ;  $W_n$  is  $H$ -cordial if and only if  $n \equiv 1 \pmod{4}$ . By allowing 0 as the possible induced vertex label of an  $H$ -cordial labeling he studies semi- $H$ -cordiality of trees. He also generalizes  $H$ -cordial labelings.

Cahit and Yilmaz [CY] call a graph  $E_k$ -cordial if it is possible to label the edges with the numbers from the set  $\{0, 1, 2, \dots, k - 1\}$  in such a way that, at each vertex  $v$ , the sum modulo  $k$  of the labels on the edges incident with  $v$  satisfies the inequalities  $|v(i) - v(j)| \leq 1$  and  $|e(i) - e(j)| \leq 1$ , where  $v(s)$  and  $e(t)$  are, respectively, the number of vertices labeled with  $s$  and the number of edges labeled with  $t$ . Obviously,  $E_2$ -cordial is the same as cordial. Cahit and Yilmaz prove the following graphs are  $E_3$ -cordial:  $P_n$  ( $n \geq 3$ ), stars  $S_n$  if and only if  $n \not\equiv 1 \pmod{3}$ ,  $K_n$  ( $n \geq 3$ ),  $C_n$  ( $n \geq 3$ ), friendship graphs  $F_n$ , fans  $f_n$  ( $n \geq 3$ ). They also prove that  $S_n$  ( $n \geq 2$ ) is  $E_k$ -cordial if and only if  $n \not\equiv 1 \pmod{k}$  when  $k$  is odd or  $n \not\equiv 1 \pmod{2k}$  when  $k$  is even and  $k \neq 2$ .

Hovey [Ho] has introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group  $A$  (under addition) and graph  $G(V, E)$  he defines  $G$  to be  $A$ -cordial if there is a labeling of  $V$  with elements of  $A$  so that for all  $a$  and  $b$  in  $A$

when the edge  $ab$  is labeled with  $f(a) + f(b)$ , the number of vertices labeled with  $a$  and the number of vertices labeled  $b$  differ by at most one and the number of edges labeled with  $a$  and the number labeled with  $b$  differ by at most one. In the case where  $A$  is the cyclic group of order  $k$ , the labeling is called  $k$ -cordial. With this definition we have:  $G(V, E)$  is harmonious if and only if  $G$  is  $|E|$ -cordial;  $G$  is cordial if and only if  $G$  is 2-cordial.

Hovey has obtained the following: caterpillars are  $k$ -cordial for all  $k$ ; all trees are  $k$ -cordial for  $k = 3, 4$  and  $5$ ; odd cycles with pendant edges attached are  $k$ -cordial for all  $k$ ; cycles are  $k$ -cordial for all odd  $k$ ; for  $k$  even,  $C_{2mk+j}$  is  $k$ -cordial when  $0 \leq j \leq \frac{k}{2} + 2$  and when  $k < j < 2k$ ;  $C_{(2m+1)k}$  is not  $k$ -cordial;  $K_m$  is 3-cordial; and, for  $k$  even,  $K_{mk}$  is  $k$ -cordial if and only if  $m = 1$ .

Hovey advances the following conjectures: all trees are  $k$ -cordial for all  $k$ ; all connected graphs are 3-cordial; and  $C_{2mk+j}$  is  $k$ -cordial if and only if  $j \neq k$ , where  $k$  and  $j$  are even and  $0 \leq j < 2k$ . The last conjecture was verified by Tao [Ta]. This result combined with those of Hovey show that for all positive integers  $k$  the  $n$ -cycle is  $k$ -cordial with the exception that  $k$  is even and  $n = 2mk + k$ . Tao also proved that the crown with  $2mk + j$  vertices is  $k$ -cordial unless  $j = k$  is even and for  $4 \leq n \leq k$  and the wheel  $W_n$  is  $k$ -cordial unless  $k \equiv 5 \pmod{8}$  and  $n = (k + 1)/2$ .

Cairnie and Edwards [CE] have determined the computational complexity of cordial and  $k$ -cordial labelings. They prove a conjecture of Kirchherr [Ki2] that deciding whether a graph admits a cordial labeling is NP-complete. As a corollary, this result implies that the same problem for  $k$ -cordial labelings is NP-complete. They remark that even the restricted problem of deciding whether connected graphs of diameter 2 have a cordial labeling is also NP-complete.

### 3.7 $k$ -equitable Labelings

In 1990 Cahit [C4] proposed the idea of distributing the vertex and edge labels among  $\{0, 1, \dots, k - 1\}$  as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph  $G(V, E)$  and any positive integer  $k$ , assign vertex labels from  $\{0, 1, \dots, k - 1\}$  so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with  $i$  and the number of vertices labeled with  $j$  differ by at most one and the number of edges labeled with  $i$  and the number of edges labeled with  $j$  differ by at most one. Cahit has called a graph with such an assignment of labels  $k$ -equitable. Note that  $G(V, E)$  is graceful if and only if it is  $|E| + 1$ -equitable and  $G(V, E)$  is cordial if and only if it is 2-equitable. Cahit [C3] has shown the following:  $C_n$  is 3-equitable if and only if  $n \not\equiv 3 \pmod{6}$ ; a triangular snake with  $n$  blocks is 3-equitable if and only if  $n$  is even; the friendship graph  $C_3^{(n)}$  is 3-equitable if and only if  $n$  is even;  $W_n$  is 3-equitable if and only if  $n \not\equiv 3 \pmod{6}$ ; an Eulerian graph with  $q \equiv 3 \pmod{6}$  edges is not 3-equitable; and all caterpillars are 3-equitable [C3]. Cahit conjectures [C3] that a triangular cactus with  $n$  blocks is 3-equitable if and only if  $n$  is even. In [C4] Cahit proves that every tree with fewer than



five end vertices has a 3-equitable labeling. He conjectures that all trees are  $k$ -equitable [C5]. In 1999 Speyer and Szaniszló [SpS] proved Cahit's conjecture for  $k = 3$ .

In [SA1] Seoud and Abdel Maqsood prove: a graph with  $n$  vertices and  $q$  edges in which every vertex has odd degree is not 3-equitable if  $n \equiv 0 \pmod{3}$  and  $q \equiv 3 \pmod{6}$ ; all fans except  $P_2 + \overline{K_1}$  are 3-equitable; all double fans except  $P_4 + \overline{K_2}$  are 3-equitable;  $P_n^2$  is 3-equitable for all  $n$  except 3;  $K_{1,1,n}$  is 3-equitable if and only if  $n \equiv 0$  or  $2 \pmod{3}$ ;  $K_{1,2,n}$ ,  $n \geq 2$ , is 3-equitable if and only if  $n \equiv 2 \pmod{3}$ ;  $K_{m,n}$ ,  $3 \leq m \leq n$ , is 3-equitable if and only if  $(m, n) = (4, 4)$ ;  $K_{1,m,n}$ ,  $3 \leq m \leq n$ , is 3-equitable if and only if  $(m, n) = (3, 4)$ .

Szaniszló [Sz] has proved the following:  $P_n$  is  $k$ -equitable for all  $k$ ;  $K_n$  is 2-equitable if and only if  $n = 1, 2$  or  $3$ ;  $K_n$  is not  $k$ -equitable for  $3 \leq k < n$ ;  $S_n$  is  $k$ -equitable for all  $k$ ;  $K_{2,n}$  is  $k$ -equitable if and only if  $n \equiv k - 1 \pmod{k}$ , or  $n \equiv 0, 1, 2, \dots, \lfloor k/2 \rfloor - 1 \pmod{k}$ , or  $n = \lfloor k/2 \rfloor$  and  $k$  is odd. She also proves that  $C_n$  is  $k$ -equitable if and only if  $k$  meets all of the following conditions:  $n \neq k$ ; if  $k \equiv 2, 3 \pmod{4}$ , then  $n \neq k - 1$ ; if  $k \equiv 2, 3 \pmod{4}$  then  $n \not\equiv k \pmod{2k}$ .

Vickrey [Vi] has determined the  $k$ -equitability of complete multipartite graphs. He shows that for  $m \geq 3$  and  $k \geq 3$ ,  $K_{m,n}$  is  $k$ -equitable if and only if  $K_{m,n}$  is one of the following graphs:  $K_{4,4}$  for  $k = 3$ ;  $K_{3,k-1}$  for all  $k$ ; or  $K_{m,n}$  for  $k > mn$ . He also shows that when  $k$  is less than or equal to the number of edges in the graph and at least 3, the only complete multipartite graphs that are  $k$ -equitable are  $K_{kn+k-1,2,1}$  and  $K_{kn+k-1,1,1}$ .

As a corollary of the result of Cairnie and Edwards [CE] on the computational complexity of cordially labeling graphs, it follows that the problem of finding  $k$ -equitable labelings of graphs is NP-complete as well.

Seoud and Abdel Maqsood [SA2] call a graph  $k$ -balanced if the vertex labels can be selected from  $\{0, 1, \dots, k - 1\}$  so that the number of edges labeled  $i$  and the number of edges labeled  $j$  induced by the absolute value of the differences of the vertex labels differ by at most 1. They prove that  $P_n^2$  is 3-balanced if and only if  $n = 2, 3, 4$  or  $6$ ; for  $k \geq 4$ ,  $P_n^2$  is not  $k$ -balanced if  $k \leq n - 2$  or  $n + 1 \leq k \leq 2n - 3$ ; for  $k \geq 4$ ,  $P_n^2$  is  $k$ -balanced if  $k \geq 2n - 2$ ; for  $k, m, n \geq 3$ ,  $K_{m,n}$  is  $k$ -balanced if and only if  $k \geq mn$ ; for  $m \leq n$ ,  $K_{1,m,n}$  is  $k$ -balanced if and only if

- (i)  $m = 1, n = 1$  or  $2$ , and  $k = 3$ ;
- (ii)  $m = 1$  and  $k = n + 1$  or  $n + 2$ ; or
- (iii)  $k \geq (m + 1)(n + 1)$ .

Bloom has used the term  $k$ -equitable to describe another kind of labeling (see [W1], [W2] and [BR]). He calls a graph  $k$ -equitable if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label induced occurs exactly  $k$  times. A graph of order  $n$  is called *minimally  $k$ -equitable* if the vertex labels are  $1, 2, \dots, n$  and it is  $k$ -equitable. Both Bloom and Wojciechowski [W1], [W2] proved that  $C_n$  is minimally  $k$ -equitable if and only if  $k$  is a proper divisor of  $n$ . Barrientos, Dejter and Hevia [BDH] have shown that forests of even size are 2-equitable (in the sense of Bloom). They also prove that for  $k = 3$  or  $k = 4$  a forest  $F$  of size  $k w$  is  $k$ -equitable if and only if the degree of  $F$  is at most  $2w$  and that if 3

divides the size of the double star  $S_{m,n}$  ( $1 \leq m \leq n$ ), then  $S_{m,n}$  is 3-equitable if and only if  $q/3 \leq m \leq \lfloor (q-1)/2 \rfloor$ . ( $S_{m,n}$  is  $K_2$  with  $m$  pendant edges attached at one end and  $n$  pendant edges attached at the other end.) They discuss the  $k$ -equitability of forests for  $k \geq 5$  and characterize all caterpillars of diameter 2 that are  $k$ -equitable for all possible values of  $k$ . In [Bar1] Barrientos proves that the one-point union of a cycle and a path (dragons) and the disjoint union of a cycle and a path are  $k$ -equitable for all  $k$  that divide the size of the graph. Bhat-Nayak and M. Acharya [BA] have proved the following: the graphs  $C_{2n} \odot K_1$  are minimally 2-equitable, minimally  $2n$ -equitable, minimally 4-equitable, and minimally  $n$ -equitable; the graphs  $C_{3n} \odot K_1$  are minimally 3-equitable, minimally  $3n$ -equitable, minimally  $n$ -equitable, and minimally 6-equitable; the graphs  $C_{5n} \odot K_1$  are minimally 5-equitable, minimally  $5n$ -equitable, minimally  $n$ -equitable, and minimally 10-equitable; the graphs  $C_{2n+1} \odot K_1$  are minimally  $2n+1$  equitable; and that the graphs  $P_{kn+1}$  are  $k$ -equitable.

In [Bar3] Barrientos calls a  $k$ -equitable labeling *optimal* if the vertex labels are consecutive integers and *complete* if the induced edge labels are  $1, 2, \dots, w$  where  $w$  is the number of distinct edge labels. He proves that  $C_m \odot nK_1$  (that is, an  $m$ -cycle with  $n$  pendant edges attached at each vertex) is optimal 2-equitable when  $m$  is even,  $C_3 \odot nK_1$  is complete 2-equitable when  $n$  is odd and that  $C_3 \odot nK_1$  is complete 3-equitable for all  $n$ . He also shows that  $C_n \odot K_1$  is  $k$ -equitable for every proper divisor  $k$  of the size  $2n$ .

### 3.8 Hamming-graceful Labelings

Mollard, Payan and Shixin [MPS] introduced a generalization of graceful graphs called Hamming-graceful. A graph  $G = (V, E)$  is called *Hamming-graceful* if there exists an injective labeling  $g$  from  $V$  to the set of binary  $|E|$ -tuples such that  $\{d(g(v), g(u)) | uv \in E\} = \{1, 2, \dots, |E|\}$ . Shixin and Yu [ShY] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful;  $C_n$  is Hamming-graceful if and only if  $n \equiv 0$  or  $3 \pmod{4}$ ; if  $K_n$  is Hamming-graceful, then  $n$  has the form  $k^2$  or  $k^2 + 2$ ;  $K_n$  is Hamming-graceful for  $n = 2, 3, 4, 6, 9, 11, 16$ , and  $18$ . They conjecture that  $K_n$  is Hamming-graceful for  $n$  of the form  $k^2$  and  $k^2 + 2$  for  $k \geq 5$ .

## 4 Variations of Harmonious Labelings

### 4.1 Sequential and Strongly $c$ -harmonious Labelings

Chang, Hsu and Rogers [CHR] and Grace [Gr1, Gr2] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling  $f$  of a graph  $G$  with  $q$  vertices to be *strongly  $c$ -harmonious* if the vertex labels are from  $\{0, 1, \dots, q-1\}$  and the edge labels induced by  $f(x) + f(y)$  for each edge  $xy$  are  $c, \dots, c+q-1$ . Grace called such a labeling *sequential*. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices while Grace allows the vertex labels to range from 0 to  $q$  with no vertex label used twice. By taking

the edge labels of a sequentially labeled graph with  $q$  edges modulo  $q$ , we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [Gr2] proved that caterpillars, caterpillars with a pendant edge, odd cycles with zero or more pendant edges, trees with  $\alpha$ -labelings, wheels  $W_{2n+1}$ , and  $P_n^2$  are sequential. Liu and Zhang [LZ1] finished off the crowns  $C_{2n} \odot K_1$ . (The case  $C_{2n+1} \odot K_1$  was a special case of Grace's results. Liu [LiuY3] proved crowns are harmonious.) Bu [Bu2] also proved that crowns are sequential as are all even cycles with  $m$  pendant edges attached at each vertex. Singh has proved the following:  $C_n \odot K_2$  is sequential for all odd  $n > 1$  [Sin2];  $C_n \odot P_3$  is sequential for all odd  $n$  [Sin4];  $K_2 \odot C_n$  (each vertex of the cycle is joined by edges to the end points of a copy of  $K_2$ ) is sequential for all odd  $n$  [Sin4]; helms  $H_n$  are sequential when  $n$  is even [Sin4]; and  $K_{1,n} + K_2, K_{1,n} + \overline{K}_2$ , and ladders are sequential [Sin5]. Both Grace [Gr1] and Reid (see [GJ]) have found sequential labelings for the books  $B_{2n}$ . Jungreis and Reid [JR] have shown the following graphs are sequential:  $P_m \times P_n$  ( $m, n \neq (2, 2)$ );  $C_{4m} \times P_n$  ( $m, n \neq (1, 2)$ );  $C_{4m+2} \times P_{2n}$ ;  $C_{2m+1} \times P_n$ ; and,  $C_4 \times C_{2n}$  ( $n > 1$ ). The graphs  $C_{4m+2} \times C_{2n+1}$  and  $C_{2m+1} \times C_{2n+1}$  fail to satisfy a necessary parity condition given by Graham and Sloane [GS]. The remaining cases of  $C_m \times P_n$  and  $C_m \times C_n$  are open. Gallian, Prout and Winters [GPW] proved that all graphs  $C_n \times P_2$  with a vertex or edge deleted are sequential.

Gnanajothi [Gn, pp.68-78] has shown the following graphs are sequential:  $K_{1,m,n}$ ;  $mC_n$ , the disjoint union of  $m$  copies of  $C_n$ , if and only if  $m$  and  $n$  are odd; books with triangular pages or pentagonal pages; and books of the form  $B_{4n+1}$ , thereby answering a question and proving a conjecture of Gallian and Jungreis [GJ]. Sun [Su] has also proved that  $B_n$  is sequential if and only if  $n \not\equiv 3 \pmod{4}$ .

Yuan and Zhu [YZ] have shown that  $mC_n$  is sequential when  $m$  and  $n$  are odd. Although Graham and Sloane [GS] proved that the Möbius  $M_3$  is not harmonious, Gallian [Ga1] established that all other Möbius ladders are sequential (see §2.3 for the definition). Chung, Hsu and Rogers [CHR] have shown that  $K_{m,n} + K_1$ , which includes  $S_m + K_1$ , is sequential. Seoud and Youssef [SY7] proved that if  $G$  is sequential and has the same number of edges as vertices, then  $G + \overline{K}_n$  is sequential for all  $n$ . Hegde [Heg2] proved that every graph can be embedded as an induced subgraph of a sequential graph.

Among the strongly 1-harmonious (also called *strongly harmonious*) are: fans  $F_n$  with  $n \geq 2$  [CHR]; wheels  $W_n$  with  $n \not\equiv 2 \pmod{3}$  [CHR];  $K_{m,n} + K_1$  [CHR]; French windmills  $K_4^{(t)}$  [Hs], [KZ]; the friendship graphs  $C_3^{(n)}$  if and only if  $n \equiv 0$  or  $1 \pmod{4}$  [Hs], [KZ];  $C_{4k}^{(t)}$  [SW]; and helms [RP1].

Seoud, Diab and Elsakhawi [SDE] have shown that the following graphs are strongly harmonious:  $K_{m,n}$  with an edge joining two vertices in the same partite set;  $K_{1,m,n}$ ;  $K_{1,1,m}$ ; the composition  $P_n[P_2]$  (see §2.3 for definition);  $B(3, 2, m)$  and  $B(4, 3, m)$  for all  $m$  (see §2.4 for notation);  $P_n^2$  ( $n \geq 3$ ); and  $P_n^3$  ( $n \geq 3$ ). Seoud et al. [SDE] have also proved:  $B_{2n}$  is strongly  $2n$ -harmonious;  $P_n$  is strongly  $\lfloor n/2 \rfloor$ -harmonious; ladders  $L_{2k+1}$  are strongly  $(k+1)$ -harmonious; and that if  $G$  is strongly  $c$ -harmonious and has an equal

number of vertices and edges, then  $G + \overline{K_n}$  is also strongly  $c$ -harmonious.

Acharya and Hegde [AH2] have generalized sequential labelings as follows. Let  $G$  be a graph with  $q$  edges and let  $k$  and  $d$  be positive integers. A labeling  $f$  of  $G$  is said to be  $(k, d)$ -arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by  $f(x) + f(y)$  for each edge  $xy$  are  $k, k+d, k+2d, \dots, k+(q-1)d$ . They obtained a number of necessary conditions for various kinds of graphs to have a  $(k, d)$ -arithmetic labeling. The case where  $k = 1$  and  $d = 1$  was called *additively graceful* by Hegde [Heg1]. Hegde showed that  $K_n$  is additively graceful if and only if  $n = 2, 3$  or  $4$ ; every additively graceful graph except  $K_2$  or  $K_{1,2}$  contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendant edge attached. Jinnah and Singh [JS] noted that  $P_n^2$  is additively graceful. Bu and Shi [BuS1] proved that  $K_n$  is not  $(k, d)$ -arithmetic for  $n \geq 5$  and that  $K_{m,n}$  is  $(k, d)$ -arithmetic when  $k$  is not of the form  $id$  for  $1 \leq i \leq n - 1$ . Yu [Yu] proved that a necessary condition for  $C_{4t+1}$  to be  $(k, d)$ -arithmetic is that  $k = 2dt + r$  for some  $r \geq 0$  and a necessary condition for  $C_{4t+3}$  to be  $(k, d)$ -arithmetic is that  $k = (2t + 1)d + 2r$  for some  $r \geq 0$ . These conditions were conjectured by Acharya and Hegde [AH2]. Singh [Sin3] proved that the graph obtained by subdividing every edge of the ladder  $L_n$  is  $(5, 2)$ -arithmetic.

A graph is called *arithmetic* if it is  $(k, d)$ -arithmetic for some  $k$  and  $d$ . Singh has proved that  $P_m \times C_n$  is arithmetic for odd  $n$  [Sin5]. Jinnah and Singh [JS] ask if the disjoint union of two arithmetic graphs is arithmetic. Singh and Vilfred [SV] showed that various classes of trees are arithmetic.

Acharya and Hegde [AH2] introduced a stronger form of sequential labeling by calling a  $(p, q)$ -graph  $(V, E)$  *strongly  $k$ -indexable* if there is an injective function from  $V$  to  $\{0, 1, 2, \dots, p - 1\}$  such that the set of edge labels induced by adding the vertex labels is  $\{k, k + 1, k + 2, \dots, k + q - 1\}$ . Strongly 1-indexable graphs are simply called *strongly indexable*. Notice that for trees and unicyclic graphs the notions of sequential labelings and strongly  $k$ -indexable labelings coincide. Acharya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are  $K_2, K_3$  and  $K_2 \times K_3$  and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [AH2] call a graph with  $p$  vertices *indexable* if there is an injective labeling of the vertices with labels from  $\{0, 1, 2, \dots, p - 1\}$  such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [ArG] who also proved that all trees are indexable. Bu and Shi [BuS2] also proved that all trees are indexable and that all unicyclic graphs with the cycle  $C_3$  are indexable.

## 4.2 Elegant Labelings

An *elegant labeling*  $f$  of a graph  $G$  with  $q$  edges is an injective function from the vertices of  $G$  to the set  $\{0, 1, \dots, q\}$  such that when each edge  $xy$  is assigned the label  $f(x) + f(y) \pmod{q+1}$  the resulting edge labels are distinct and nonzero. This notion was introduced

by Chang, Hsu and Rogers in 1981 [CHR]. Note that in contrast to the definition of a harmonious labeling, it is not necessary to make an exception for trees. While the cycle  $C_n$  is harmonious if and only if  $n$  is odd, Chang et al. [CHR] proved that  $C_n$  is elegant when  $n \equiv 0$  or  $3 \pmod{4}$  and not elegant when  $n \equiv 1 \pmod{4}$ . Chang et al. further showed that all fans are elegant and the paths  $P_n$  are elegant for  $n \not\equiv 0 \pmod{4}$ . Cahit [C1] then showed that  $P_4$  is the only path that is not elegant. Balakrishnan, Selvam and Yegnanarayanan [BSY2] have proved numerous graphs are elegant. Among them are  $K_{m,n}$  and the  $m$ th-subdivision graph of  $K_{1,2n}$ . They prove that the bistar  $B_{n,n}$  ( $K_2$  with  $n$  pendant edges at each endpoint) is elegant if and only if  $n$  is even. They also prove that every simple graph is a subgraph of an elegant graph and that several families of graphs are not elegant. Deb and Limaye [DL1] have shown that triangular snakes are elegant if and only if the number of triangles is not equal to  $3 \pmod{4}$ . In the case where the number of triangles is  $3 \pmod{4}$  they show the triangular snakes satisfy a weaker condition they call *semi-elegant* whereby the edge label 0 is permitted. In [DL3] Deb and Limaye define a graph  $G$  with  $q$  edges to be *near-elegant* if there is an injective function  $f$  from the vertices of  $G$  to the set  $\{0, 1, \dots, q\}$  such that when each edge  $xy$  is assigned the label  $f(x) + f(y) \pmod{q+1}$  the resulting edge labels are distinct and not equal to  $q$ . Thus, in a near-elegant labeling, instead of 0 being the missing value in the edge labels,  $q$  is the missing value. Deb and Limaye show that triangular snakes where the number of triangles is  $3 \pmod{4}$  are near-elegant. For any positive integers  $\alpha \leq \beta \leq \gamma$  where  $\beta$  is at least 2 the *theta graph*  $\theta_{\alpha,\beta,\gamma}$  consists of three edge disjoint paths of lengths  $\alpha, \beta$  and  $\gamma$  having the same end points. Deb and Limaye [DL3] provide elegant and near-elegant labelings for some theta graphs where  $\alpha = 1, 2$  or  $3$ . Seoud and Elsakawi [SE1] have proved that the following graphs are elegant:  $K_{1,m,n}$ ;  $K_{1,1,m,n}$ ;  $K_2 + \overline{K_m}$ ;  $K_3 + \overline{K_m}$ ; and  $K_{m,n}$  with an edge joining two vertices of the same partite set.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let  $G$  be a graph with  $q$  edges and  $H$  a finite Abelian group (under addition) of order  $q$ . Define  $G$  to be *H-harmonious* if there is an injection  $f$  from the vertices of  $G$  to  $H$  such that when each edge  $xy$  is assigned the label  $f(x) + f(y)$  the resulting edge labels are distinct. When  $G$  is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley and Jungreis [BGHJ] have shown that if  $H$  is a finite Abelian group of order  $n > 1$  then  $C_n$  is *H-harmonious* if and only if  $H$  has a non-cyclic or trivial Sylow 2-subgroup and  $H$  is not of the form  $Z_2 \times Z_2 \times \dots \times Z_2$ . Thus, for example,  $C_{12}$  is not  $Z_{12}$ -harmonious but is  $(Z_2 \times Z_2 \times Z_3)$ -harmonious. Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let  $G$  be a graph with  $q$  edges and  $H$  a finite Abelian group (under addition) with  $q+1$  elements. We say  $G$  is *H-elegant* if there is an injection  $f$  from the vertices of  $G$  to  $H$  such that when each edge  $xy$  is assigned the label  $f(x) + f(y)$  the resulting set of edge labels is the non-identity elements of  $H$ . Beals et al. [BGHJ] proved that if  $H$  is a finite Abelian group of order  $n$  with  $n \neq 1$  and  $n \neq 3$ , then  $C_{n-1}$  is *H-elegant* using only the non-identity elements of  $H$  as vertex labels if and only if  $H$  has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu

and Rogers [CHR] by showing that  $C_n$  is elegant when  $n \equiv 2 \pmod{4}$ . Mollard and Payan [MP] also proved that  $C_n$  is elegant when  $n \equiv 2 \pmod{4}$  and gave another proof that  $P_n$  is elegant when  $n \neq 4$ .

For a graph  $G(V, E)$  and an Abelian group  $H$  Valentin [V] defines a *polychrome* labeling of  $G$  by  $H$  to be a bijection  $f$  from  $V$  to  $H$  such that the edge labels induced by  $f(uv) = f(v) + f(u)$  are distinct. Valentin investigates the existence of polychrome labelings for paths and cycles by various Abelian groups.

### 4.3 Felicitous Labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function  $f$  from the vertices of a graph  $G$  with  $q$  edges to the set  $\{0, 1, \dots, q\}$  is called *felicitous* if the edge labels induced by  $f(x) + f(y) \pmod{q}$  for each edge  $xy$  are distinct. This definition first appeared in a paper by Lee, Schmeichel and Shee in [LSS] and is attributed to E. Choo. Balakrishnan and Kumar [BK2] proved the conjecture of Lee, Schmeichel and Shee [LSS] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are:  $C_n$  except when  $n \equiv 2 \pmod{4}$  [LSS];  $K_{m,n}$  when  $m, n > 1$  [LSS];  $P_2 \cup C_{2n+1}$  [LSS];  $P_2 \cup C_{2n}$  [Tel];  $P_3 \cup C_{2n+1}$  [LSS];  $S_m \cup C_{2n+1}$  [LSS];  $K_n$  if and only if  $n \leq 4$  [SE1];  $P_n + \overline{K_m}$  [SE1]; the friendship graph  $C_3^{(n)}$  for  $n$  odd [LSS];  $P_n \cup C_3$  [ShL];  $P_n \cup C_{n+3}$  [Tel]; and the one-point union of an odd cycle and a caterpillar [ShL]. Shee [S1] conjectured that  $P_m \cup C_n$  is felicitous when  $n > 2$  and  $m > 3$ . Lee, Schmeichel and Shee [LSS] ask for which  $m$  and  $n$  is the one-point union of  $n$  copies of  $C_m$  felicitous. They showed that the case where  $mn$  is twice an odd integer is not felicitous. In contrast to the situation for felicitous labelings, we remark that  $C_{4k}$  and  $K_{m,n}$  where  $m, n > 1$  are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel and Shee [LSS] conjecture that the  $n$ -cube is felicitous. This is known to be true for  $n = 2, 3$  and  $4$  ([LSS] and [BK2]).

Balakrishnan, Selvam and Yegnanarayanan [BSY1] obtained numerous results on felicitous labelings. The *wreath product*,  $G * H$ , of graphs  $G$  and  $H$  has vertex set  $V(G) \times V(H)$  and  $(g_1, h_1)$  is adjacent to  $(g_2, h_2)$  whenever  $g_1 g_2 \in E(G)$  or  $g_1 = g_2$  and  $h_1 h_2 \in E(H)$ . They define  $H_{n,n}$  as the graph with vertex set  $\{u_1, \dots, u_n; v_1, \dots, v_n\}$  and edge set  $\{u_i v_j \mid 1 \leq i \leq j \leq n\}$ . They let  $\langle K_{1,n} : m \rangle$  denote the graph obtained by taking  $m$  disjoint copies of  $K_{1,n}$ , and joining a new vertex to the roots of the  $m$  copies of  $K_{1,n}$ . They prove the following are felicitous:  $H_{n,n}$ ;  $P_n * \overline{K_2}$ ;  $\langle K_{1,m} : m \rangle$ ;  $\langle K_{1,2} : m \rangle$  when  $m \not\equiv 0 \pmod{3}$  or  $m \equiv 3 \pmod{6}$  or  $m \equiv 6 \pmod{12}$ ;  $\langle K_{1,2n} : m \rangle$  for all  $m$  and  $n \geq 2$ ;  $\langle K_{1,2t+1} : 2n+1 \rangle$  when  $n \geq t$ ;  $P_n^k$  when  $k = n-1$  and  $n \not\equiv 2 \pmod{4}$  or  $k = 2t$  and  $n \geq 3$  and  $k < n-1$ ; the join of a star and  $\overline{K_n}$ ; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [Y] conjectures that the graphs obtained from an even cycle by attaching  $n$  new vertices to each vertex of the cycle is felicitous.

Chang, Hsu and Rogers [CHR] have given a sequential counterpart to felicitous

labelings. They call a graph *strongly  $c$ -elegant* if the vertex labels are from  $\{0, 1, \dots, q\}$  and the edge labels induced by addition are  $\{c, c+1, \dots, c+q-1\}$ . (A strongly 1-elegant labeling has also been called a *consecutive* labeling.) Notice that every strongly  $c$ -elegant graph is felicitous and that strongly  $c$ -elegant is the same as  $(c, 1)$ -arithmetic in the case where the vertex labels are from  $\{0, 1, \dots, q\}$ . Results on strongly  $c$ -elegant graphs are meager. Chang et al. [CHR] have shown:  $K_n$  is strongly 1-elegant if and only if  $n = 2, 3, 4$ ;  $C_n$  is strongly 1-elegant if and only if  $n = 3$ ; and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [S2] has proved that  $K_{m,n}$  is strongly  $c$ -elegant for a particular value of  $c$  and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

Seoud and Elsakhawi [SE2] have shown:  $K_{m,n}$  with an edge joining two vertices of the same partite set is strongly  $c$ -elegant for  $c = 1, 3, 5, \dots, \max\{2m+1, 2n+1\} + 1$ ;  $K_{1,m,n}$  is strongly  $c$ -elegant for  $c = 1, 3, 5, \dots, 2m$  when  $m = n$ , and for  $c = 1, 3, 5, \dots, m+n+1$  when  $m \neq n$ ;  $K_{1,1,m,m}$  is strongly  $c$ -elegant for  $c = 1, 3, 5, \dots, 2m+1$ ;  $P_n + \overline{K_m}$  is strongly  $c$ -elegant for  $c = \lfloor n/2 \rfloor$ -elegant;  $C_m + \overline{K_n}$  is strongly  $c$ -elegant for odd  $m$  and all  $n$  for  $c = (m-1)/2, (m-1)/2+2, \dots, 2m$  when  $(m-1)/2$  is even and for  $c = (m-1)/2, (m-1)/2+2, \dots, 2m - (m-1)/2$  when  $(m-1)/2$  is odd; ladders  $L_{2k+1}$  ( $k > 1$ ) are strongly  $(k+1)$ -elegant; and  $B(3, 2, m)$  and  $B(4, 3, m)$  (see §2.4 for notation) are strongly 1-elegant and strongly 3-elegant for all  $m$ ; the composition  $P_n[P_2]$  is strongly  $c$ -elegant for  $1, 3, 5, \dots, 5n-6$  when  $n$  is odd and for  $c = 1, 3, 5, \dots, 5n-5$  when  $n$  is even;  $P_n$  is strongly  $\lfloor n/2 \rfloor$ -elegant;  $P_n^2$  is strongly  $c$ -elegant for  $c = 1, 3, 5, \dots, q$  where  $q$  is the number of edges of  $P_n^2$ ; and  $P_n^3$  ( $n > 3$ ) is strongly  $c$ -elegant for  $c = 1, 3, 5, \dots, 6k-1$  when  $n = 4k$ ,  $c = 1, 3, 5, \dots, 6k+1$  when  $n = 4k+1$ ,  $c = 1, 3, 5, \dots, 6k+3$  when  $n = 4k+2$ ,  $c = 1, 3, 5, \dots, 6k+5$  when  $n = 4k+3$ .

## 5 Miscellaneous Labelings

### 5.1 Magic, Edge-magic, Vertex-magic and Antimagic Labelings

Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedláček [Se1] in 1963. Responding to a problem raised by Sedláček, Stewart ([St1] and [St2]) studied various ways to label the edges of a graph in the mid 60s. Stewart calls a connected graph *semi-magic* if there is a labeling of the edges with integers such that for each vertex  $v$  the sum of the labels of all edges incident with  $v$  is the same for all  $v$ . A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic* labeling. Stewart calls a magic labeling *supermagic* if the set of edge labels consists of consecutive integers. The classic concept of an  $n \times n$  magic square in number theory corresponds to a supermagic labeling of  $K_{n,n}$ . Stewart [St1] proved the following:  $K_n$  is magic for  $n = 2$  and all  $n \geq 5$ ;  $K_{n,n}$  is magic for all  $n \geq 3$ ; fans  $F_n$  are magic if and only if  $n$  is odd and  $n \geq 3$ ; wheels  $W_n$  are magic for  $n \geq 4$ ;  $W_n$  with one spoke deleted is magic for  $n = 4$  and for  $n \geq 6$ . Stewart [St1] also proved that  $K_{m,n}$  is

semi-magic if and only if  $m = n$ . In [St2] Stewart proved that  $K_n$  is supermagic for  $n \geq 5$  if and only if  $n > 5$  and  $n \not\equiv 0 \pmod{4}$ . Sedláček [Se2] showed that Möbius ladders  $M_n$  are supermagic when  $n \geq 3$  and  $n$  is odd and that  $C_n \times P_2$  is magic, but not supermagic, when  $n \geq 4$  and  $n$  is even. Shiu, Lam and Lee [SLL1] have proved that the composition of  $C_m$  and  $\overline{K}_n$  is supermagic when  $m \geq 3$  and  $n \geq 2$ . Bača, Holländer and Lih [BHL] have found two families of 4-regular supermagic graphs. Trenkler [Tre1] has proved that a connected magic  $(p, q)$ -graph other than  $P_2$  exists if and only if  $5p/4 < q \leq p(p-1)/2$ . Trenkler [Tre2] extended the definition of supermagic graphs to include hypergraphs and proved that the complete  $k$ -uniform  $n$ -partite hypergraph is supermagic if  $n \neq 2$  or  $6$  and  $k \geq 2$ . Sedláček [Se2] also proves that graphs obtained from an odd cycle with at least 5 vertices in which every vertex  $v$  of the cycle has two chords joining  $v$  to the two vertices at greatest distance from  $v$  are magic. (He calls these *Möbius ladders*.) Trenkler and Vetchý [TV] have shown that if  $G$  has order at least 5 then  $G^i$  is magic for all  $i \geq 3$  and  $G^2$  is magic if and only if  $G$  is not  $P_5$  and  $G$  does not have a 1-factor whose every edge is incident with an end-vertex of  $G$ . Seoud and Abdel Maqsood [SA1] proved that  $K_{1,m,n}$  is magic for all  $m$  and  $n$  and that  $P_n^2$  is magic for all  $n$ . Characterizations of regular magic graphs were given by Dood [Do] and necessary and sufficient conditions for a graph to be magic were given in [Je] and [JT].

In 1976 Sedláček [Se2] defined a connected graph with at least two edges to be *pseudo-magic* if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when  $n \geq 4$  and  $n$  is even, the Möbius ladder  $M_n$  is not pseudo-magic and when  $m \geq 3$  and  $m$  is odd,  $C_m \times P_2$  is not pseudo-magic.

For any magic graph  $G$ , Kong, Lee and Sun [KLS] let  $M(G)$  denote the set of all magic labelings of  $G$ . For any  $L$  in  $M(G)$ , they let  $s(L) = \max\{L(e) : e \text{ in } E\}$  and define the *magic strength* of  $G$  as  $m(G) = \min\{s(L) : L \text{ in } M(G)\}$ . In [KLS], Kong et al. determine the magic strengths of several classes of graphs and introduce some constructions of magic graphs. They also show that every connected graph is an induced subgraph of a magic graph (see also [ELNR] and [FIM1]). In [LSS] Lee, Saba and Sun show that the magic strength of  $P_n^k$  is 0 when  $k$  and  $n$  are both odd. Sun and Lee [SuL] show that the Cartesian, conjunctive, normal, lexicographic, and disjunctive products of two magic graphs are magic and the sum of two magic graphs is magic. They also determine the magic strength of the constructed graphs. Avadayappan, Jeyanthi and Vasuki [AJV] use the notation  $\langle K_{1,n} : 2 \rangle$  for the tree obtained from the bistar  $B_{n,n}$  by subdividing the edge joining the two stars. They prove:  $m(P_{2n}) = 5n+1$ ;  $m(P_{2n+1}) = 5n+3$ ;  $m(B_{n,n}) = 5n+6$ ;  $m(\langle K_{1,n} : 2 \rangle) = 4n+9$ ;  $m((2n+1)P_2) = 9n+6$ ;  $m(C_{2n+1}) = 5n+4$ ;  $m(C_{2n}) = 5n+2$ ;  $m(K_{1,n}) = 2n+4$ ;  $m(P^2) = 3n$ ; and  $m(K_{n,m}) \leq (m+2)(n+1)$  where  $n \leq m$ .

In 1970 Kotzig and Rosa [KR1] defined a magic labeling of a graph  $G(V, E)$  as a bijection  $f$  from  $V \cup E$  to  $\{1, 2, \dots, |V \cup E|\}$  such that for all edges  $xy$ ,  $f(x) + f(y) + f(xy)$  is constant. To distinguish between this usage from that of Stewart we will call this labeling an *edge-magic total* labeling. Kotzig and Rosa proved:  $K_{m,n}$  has an edge-magic



total labeling for all  $m$  and  $n$ ;  $C_n$  has an edge-magic total labeling for all  $n \geq 3$  (see also [GoS]); and the disjoint union of  $n$  copies of  $P_2$  has an edge-magic total labeling if and only if  $n$  is odd. They further state that  $K_m$  has an edge-magic total labeling if and only if  $n = 1, 2, 3, 5$  or  $6$  (see [KR2], [CT] and [ELNR]) and ask whether all trees have edge-magic total labelings. Wallis et al. [WBMS] enumerate every edge-magic total labeling of complete graphs. They also prove that the following graphs are edge-magic total: paths, crowns, complete bipartite graphs, and cycles with a single edge attached to one vertex. Enomoto, Llado, Nakamigana and Ringel [ELNR] prove that all complete bipartite graphs and all even cycles are edge-magic total. They also show that wheels  $W_n$  are not edge-magic total when  $n \equiv 3 \pmod{4}$  and conjectured that all other wheels are edge-magic total. This conjecture has been proved when  $n \equiv 0, 1 \pmod{4}$  by Phillips, Rees and Wallis [PRW1]. Ringel and Llado [RL] prove that a  $(p, q)$ -graph is not edge-magic total if  $q$  is even and  $p + q \equiv 2 \pmod{4}$  and each vertex has odd degree. Ringel and Llado conjecture that trees are edge-magic total. Balakrishnan and Kumar [BK2] proved that the join of  $\overline{K_n}$  and two disjoint copies of  $K_2$  is edge-magic total if and only if  $n = 3$ . Craft and Tesar [CT] proved that an  $r$ -regular graph with  $r$  odd and  $p \equiv 4 \pmod{8}$  vertices can not be edge-magic total.

Figueroa-Centeno, Ichishima and Muntaner-Batle and Ichishima ([FIM1], [FIM2] and [FMI]) have proved the following graphs are edge-magic total:  $P_4 \cup nK_2$  for  $n$  odd;  $P_3 \cup nK_2$ ;  $P_5 \cup nK_2$ ;  $nP_i$  for  $n$  odd and  $i = 3, 4, 5$ ;  $2P_n$ ;  $P_1 \cup P_2 \cup \dots \cup P_n$ ;  $mK_{1,n}$ ;  $K_{1,n} \cup K_{1,n+1}$ ;  $C_m \odot nK_1$ ;  $K_1 \odot nK_2$  for  $n$  even;  $W_{2n}$ ;  $K_2 \times \overline{K_n}$ ,  $nK_3$  for  $n$  odd; binary trees, generalized Petersen graphs, ladders, books, fans, and odd cycles with pendant edges attached to one vertex. Enomoto et al. [ELNR] conjecture that if  $G$  is a graph of order  $n + m$  that contains  $K_n$  then  $G$  is not edge-magic total for  $n \gg m$ .

Enomoto et al. call an edge-magic labeling total *super edge-magic* if the set of vertex labels is  $\{1, 2, \dots, |V|\}$ . They prove that  $C_n$  is super edge-magic if and only if  $n$  is odd, caterpillars are super edge-magic,  $K_{m,n}$  is super edge-magic if and only if  $m = 1$  or  $n = 1$ , and  $K_n$  is super edge-magic if and only if  $n = 1, 2$  or  $3$ . They prove that if a  $(p, q)$ -graph is super edge-magic then  $q \leq 2p - 3$ . Enomoto et al. conjecture that every tree is super edge-magic. K. Masuda (see [ELNR]) has proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph.

Figueroa-Centeno, Muntaner and Ichishima [FMI] prove that a graph is super edge-magic if and only if it is strongly 1-harmonious and that a super edge-magic graph is cordial. They also prove that  $P_n^2$  and  $K_2 \times C_{2n+1}$  are super edge-magic. In [FIM2] Figueroa-Centeno et al. show that the following graphs are super edge-magic:  $P_3 \cup kP_2$  for all  $k$ ;  $kP_n$  when  $k$  is odd; and  $k(P_2 \cup P_n)$  when  $k$  is odd and  $n = 3$  or  $n = 4$ ; fans  $f_n$  if and only if  $n \leq 6$ . They conjecture that  $kP_2$  is not super edge-magic when  $k$  is even. This conjecture has been proved by Z. Chen [Che3] who showed that  $kP_2$  is super edge-magic if and only if  $k$  is odd. Figueroa-Centeno et al. provide a strong necessary condition for a book to have a super edge-magic labeling and conjecture that for  $n \geq 5$  the book  $B_n$  is super edge-magic if and only if  $n$  is even or  $n \equiv 5 \pmod{8}$ . They prove that every tree with an  $\alpha$ -labeling is super edge-magic. Yokomura (see [ELNR]) has

shown that  $P_{2m+1} \times P_2$  and  $C_{2m+1} \times P_m$  are super edge-magic (see also [FIM1]).

In 1970 Kotzig and Rosa [KR1] defined the *edge-magic deficiency*,  $\mu(G)$ , of a graph  $G$  as the minimum  $n$  such that  $G \cup nK_1$  is edge-magic total. If no such  $n$  exists they define  $\mu(G) = +\infty$ . In 1999 Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM4] extended this notion to *super edge-magic deficiency*,  $\mu_s(G)$ , in the analogous way. They prove the following:  $\mu_s(nK_2) = \mu(nK_2) = n - 1 \pmod{2}$ ;  $\mu_s(C_n) = 0$  if  $n$  is odd;  $\mu_s(C_n) = 1$  if  $n \equiv 0 \pmod{4}$ ;  $\mu_s(C_n) = +\infty$  if  $n \equiv 2 \pmod{4}$ ;  $\mu_s(K_n) = +\infty$  if and only if  $n \geq 5$ ;  $\mu_s(K_{m,n}) \leq (m-1)(n-1)$ ;  $\mu_s(K_{2,n}) = n - 1$ ; and  $\mu_s(F)$  is finite for all forests  $F$ . They also prove that if a graph  $G$  has  $q$  edges with  $q/2$  odd and every vertex even then  $\mu_s(G) = +\infty$ .

Z. Chin [Che3] has proven: the join of  $K_1$  with any subgraph of a star is super edge-magic; the join of two nontrivial graphs is super edge-magic if and only if at least one of them has exactly two vertices and their union has exactly one edge; and if a  $k$ -regular graph is super edge-magic, then  $k \leq 3$ . Chin also obtained the following conditions: there is a connected super edge-magic graph with  $p$  vertices and  $q$  edges if and only if  $p-1 \leq q \leq 2p-3$ ; there is a connected 3-regular super edge-magic graph with  $p$  vertices if and only if  $p \equiv 2 \pmod{4}$ ; if  $G$  is a  $k$ -regular edge-magic total graph with  $p$  vertices and  $q$  edges then  $(p+q)(1+p+q) \equiv 0 \pmod{2d}$  where  $d = \gcd(k-1, q)$ . As a corollary of the last result Chen observes that  $nK_2 + nK_2$  is not edge-magic total.

Another labeling that has been called “edge-magic” was introduced by Lee, Seah and Tan in 1992 [LST]. They defined a graph  $G = (V, E)$  to be *edge-magic* if there exists a bijection  $f: E \rightarrow \{1, 2, \dots, |E|\}$  such that the induced mapping  $f^+: V \rightarrow N$  defined by  $f^+(u) = \sum_{(u,v) \in E} f(u, v) \pmod{|V|}$  is a constant map. Lee conjectured that a cubic  $(p, q)$ -graph is edge-magic if and only if  $p \equiv 2 \pmod{4}$ . Lee, Pigg and Cox [LPC] verified this conjecture for several classes of cubic graphs. A necessary condition for a  $(p, q)$ -multigraph to be edge-magic is that  $p$  divides  $q(q+1)$  [LST]. Lee, Seah and Tan [LST] proved:  $K_{n,n}$  ( $n \geq 3$ ) is edge-magic,  $K_n$  is edge-magic for  $n \equiv 1, 2 \pmod{4}$  and for  $n \equiv 3 \pmod{4}$  ( $n \geq 7$ ). Lee, Seah and Tan also proved that following graphs are not edge-magic: all trees except  $P_2$ , all unicyclic graphs, and  $K_n$  where  $n \equiv 0 \pmod{4}$ . Schaffer and Lee [ScL] have proved that  $C_m \times C_n$  is always edge-magic. Lee, Tong and Seah [LTS] have conjectured that the total graph of a  $(p, p)$ -graph is edge-magic if and only if  $p$  is odd. They prove this conjecture for cycles.

For any graph  $G$  and any positive integer  $k$  the graph  $G[k]$ , called the  *$k$ -fold  $G$* , is the hypergraph obtained from  $G$  by replacing each edge of  $G$  with  $k$  parallel edges. Lee, Seah and Tan [LST] proved that for any  $(p, q)$ -graph  $G$ ,  $G[2p]$  is edge-magic and, if  $p$  is odd,  $G[p]$  is edge-magic. Shiu, Lam and Lee [SLL3] show that if  $G$  is an  $(n+1, n)$ -multigraph, then  $G$  is edge-magic if and only if  $n$  is odd and  $G$  is isomorphic to the disjoint union of  $K_2$  and  $(n-1)/2$  copies of  $K_2[2]$ . They also prove that if  $G$  is a  $(2m+1, 2m)$ -multigraph and  $k$  is at least 2, then  $G[k]$  is edge-magic if and only if  $2m+1$  divides  $k(k-1)$ . For a  $(2m, 2m-1)$ -multigraph  $G$  and  $k$  at least 2, they show that  $G[k]$  is edge-magic if  $4m$  divides  $(2m-1)k((2m-1)k+1)$  or if  $4m$  divides  $(2m+k-1)k$ . In [SLL2] Shiu, Lam and Lee characterize the  $(p, p)$ -multigraphs that are edge-magic as  $mK_2[2]$  or the disjoint

union of  $mK_2[2]$  and two particular multigraphs or the disjoint union of  $K_2, mK_2[2]$  and 4 particular multigraphs. They also show for every  $(2m+1, 2m+1)$ -multigraph  $G$ ,  $G[k]$  is edge-magic for all  $k$  at least 2. Lee, Seah and Tan [LST] prove that the multigraph  $C_n[k]$  is edge-magic for  $k \geq 2$ .

MacDougall, Miller, Slamin and Wallis [MMSW] introduced the notion of a vertex-magic total labeling in 1999. For a graph  $G(V, E)$  an injective mapping  $f$  from  $V \cup E$  to the set  $\{1, 2, \dots, |V| + |E|\}$  is a *vertex-magic total labeling* if there is a constant  $k$  so that for every vertex  $v$ ,  $f(v) + \sum f(vu) = k$  where the sum is over all vertices  $u$  adjacent to  $v$ . They prove that the following graphs have vertex-magic total labelings:  $C_n$  for  $n > 2$ ;  $P_n$  for  $n > 2$ ;  $K_{m,m}$  for  $m > 1$ ;  $K_{m,m} - e$  for  $m > 2$ ; and  $K_n$  for  $n$  odd. They also prove that when  $n > m + 1$ ,  $K_{m,n}$  does not have a vertex-magic total labeling. They conjecture that  $K_{m,m+1}$  has a vertex-magic total labeling for all  $m$  and that  $K_n$  has vertex-magic total labeling for all  $n \geq 3$ . Lin and Miller [LN] have shown that  $K_{m,m}$  is vertex-magic total for all  $m > 1$  and that  $K_n$  is vertex-magic total for all  $n \equiv 2 \pmod{4}$ ,  $n > 2$ . Phillips, Rees and Wallis [PRW2] generalized the Lin and Miller result by proving that  $K_{m,n}$  is vertex-magic total if and only if  $m$  and  $n$  differ by at most 1. Miller, Bača, and MacDougall [MBM] have proved that the generalized Petersen graphs  $P(n, k)$  (see Section 2.7 for the definition) are vertex-magic total when  $n$  is even and  $k \leq n/2 - 1$ . They conjecture that all  $P(n, k)$  are vertex-magic total when  $k \leq (n-1)/2$  and all prisms  $C_n \times P_2$  are vertex-magic total. In [SIM] Slamin and Miller prove the first of these conjectures when  $k$  and  $n$  are relatively prime and the second of the conjectures for all cases.

Bača, et al. [BBMMSS] introduced the notion of a  $(a, d)$ -vertex-antimagic total labeling in 2000. For a graph  $G(V, E)$ , an injective mapping  $f$  from  $V \cup E$  to the set  $\{1, 2, \dots, |V| + |E|\}$  is a  $(a, d)$ -*vertex-antimagic total labeling* if the set  $\{f(v) + \sum f(vu)\}$  where the sum is over all vertices  $u$  adjacent to  $v$  for all  $v$  in  $G$  is  $\{a, a+d, a+2d, \dots, a+|V|d\}$ . They prove that every super-magic graph has an  $(a, 1)$ -vertex-antimagic total labeling and show that paths and cycles have  $(a, d)$ -vertex-antimagic total labelings for a wide variety for  $a$  and  $d$ .

A magic-type method for labeling the vertices, edges and faces of a planar graph was introduced by Lih [Lih] in 1983. Lih defines a *magic labeling of type (1,1,0)* of a planar graph  $G(V, E)$  as an injective function from  $\{1, 2, \dots, |V| + |E|\}$  to  $V \cup E$  with the property that for each interior face the sum of the labels of the vertices and the edges surrounding that face is some fixed value. Similarly, Lih defines a *magic labeling of type (1, 1, 1)* of a planar graph  $G(V, E)$  with face set  $F$  as an injective function from  $\{1, 2, \dots, |V| + |E| + |F|\}$  to  $V \cup E \cup F$  with the property that for each interior face the sum of the labels of the face and the vertices and the edges surrounding that face is some fixed value. Lih calls a labeling involving the faces of a plane graph *consecutive* if for every integer  $s$  the weights of all  $s$ -sided faces constitute a set of consecutive integers. Lih gave consecutive magic labelings of type  $(1, 1, 0)$  for wheels, friendship graphs, prisms and some members of the Platonic family. In [Bac14] Bača shows that the cylinders  $C_n \times P_m$  have magic labelings of type  $(1, 1, 0)$  when  $m \geq 2, n \geq 3, n \neq 4$ . Bača has

given magic labelings of type  $(1, 1, 1)$  for fans [Bac1], ladders [Bac1], planar bipyramids (2-point suspension of a path) [Bac1], grids [Bac4], hexagonal lattices [Bac3], Möbius ladders [Bac8] and  $P_n \times P_3$  [Bac12]. Bača [Bac2], [Bac10], [Bac11], [Bac12], [Bac13] and Bača and Holländer [BHo2] give magic labelings of type  $(1, 1, 1)$  and type  $(1, 1, 0)$  for certain classes of convex polytopes. Bača [Bac9] also provides consecutive and magic labelings of type  $(0, 1, 1)$  (that is, an injective function from  $\{1, 2, \dots, |E| + |F|\}$  to  $E \cup F$  with the property that for each interior face the sum of the labels of the face and the edges surrounding that face is some fixed value) and a consecutive labeling of type  $(1, 1, 1)$  for a kind of planar graph with hexagonal faces.

Hartsfield and Ringel [HR] introduced antimagic graphs in 1990. A graph with  $q$  edges is called *antimagic* if its edges can be labeled with  $1, 2, \dots, q$  so that the sums of the labels of the edges incident to each vertex are distinct. Among the antimagic graphs are [HR]:  $P_n$  ( $n \geq 3$ ), cycles, wheels, and  $K_n$  ( $n \geq 3$ ). Hartsfield and Ringel conjecture that every tree except  $P_2$  is antimagic and, moreover, every connected graph except  $P_2$  is antimagic.

The concept of an  $(a, d)$ -antimagic labelings was introduced by Wagner and Bodendiek [WB] in 1993. A connected graph  $G = (V, E)$  is said to be  $(a, d)$ -antimagic if there exist positive integers  $a, d$  and a bijection  $f: E \rightarrow \{1, 2, \dots, |E|\}$  such that the induced mapping  $g_f: V \rightarrow N$ , defined by  $g_f(v) = \sum \{f(u, v) : (u, v) \in E(G)\}$ , is injective and  $g_f(V) = \{a, a + d, \dots, a + (|V| - 1)d\}$ . They prove ([BW1] and [BW3]) the Herschel graph is not  $(a, d)$ -antimagic and certain cases of graphs called *parachutes*  $P_{g,b}$  are antimagic. ( $P_{g,b}$  is the graph obtained from the wheel  $W_{g+p}$  by deleting  $p$  consecutive spokes.) In [BHo1] Bača and Holländer prove that necessary conditions for  $C_n \times P_2$  to be  $(a, d)$ -antimagic are  $d = 1, a = (7n + 4)/2$  or  $d = 3, a = (3n + 6)/2$  when  $n$  is even and  $d = 2, a = (5n + 5)/2$  or  $d = 4, a = (n + 7)/2$  when  $n$  is odd. Bodendiek and Walther [BW2] conjectured that  $C_n \times P_2$  ( $n \geq 3$ ) is  $((7n + 4)/2, 1)$ -antimagic when  $n$  is even and is  $((5n + 5)/2, 2)$ -antimagic when  $n$  is odd. These conjectures were verified by Bača and Holländer [BHo1] who further proved that  $C_n \times P_2$  ( $n \geq 3$ ) is  $(3n + 6)/2, 3$ -antimagic when  $n$  is even. Bača and Holländer [BHo1] conjecture that  $C_n \times P_2$  is  $((n + 7)/2, 4)$ -antimagic when  $n$  is odd and at least 7. Bodendiek and Walther [BW3] also conjectured that  $C_n \times P_2$  ( $n \geq 7$ ) is  $((n + 7)/2, 4)$ -antimagic. Bodendiek and Walther [BW5] proved that the following graphs are not  $(a, d)$ -antimagic: even cycles; paths of even order; stars;  $C_3^{(k)}$ ;  $C_4^{(k)}$ ; trees of odd order at least 5 that have a vertex that is adjacent to three or more end vertices;  $n$ -ary trees with at least two layers when  $d = 1$ ;  $K_{3,3}$ ; the Petersen graph; and  $K_4$ . They also prove:  $P_{2k+1}$  is  $(k, 1)$ -antimagic;  $C_{2k+1}$  is  $(k + 2, 1)$ -antimagic; if a tree of odd order  $2k + 1$  ( $k > 1$ ) is  $(a, d)$ -antimagic, then  $d = 1$  and  $a = k$ ; if  $K_{4k}$  ( $k \geq 2$ ) is  $(a, d)$ -antimagic then  $d$  is odd and  $d \leq 2k(4k - 3) + 1$ ; if  $K_{4k+2}$  is  $(a, d)$ -antimagic then  $d$  is even and  $d \leq (2k + 1)(4k - 1) + 1$ ; and if  $K_{2k+1}$  ( $k \geq 2$ ) is  $(a, d)$ -antimagic then  $d \leq (2k + 1)(k - 1)$ .

The *antiprism* on  $2n$  vertices has vertex set  $\{x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{2,n}\}$  and edge set  $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$  (subscripts are taken modulo  $n$ ). For  $n \geq 3$  and  $n \not\equiv 2 \pmod{4}$  Bača [Bac5] gives  $(6n + 3, 2)$ -antimagic labelings and  $(4n + 4, 4)$ -

antimagic labelings for the antiprism on  $2n$  vertices. He conjectures that the antiprism on  $2n$  vertices has a  $(2n + 5, 6)$ -antimagic labeling for  $n \geq 4$ , and a  $(6n + 3, 2)$ -antimagic labeling and a  $(4n + 4, 4)$ -antimagic labeling for  $n \not\equiv 2 \pmod{4}$  with  $n \geq 6$ .

Bača [Bac6] defines a connected plane graph  $G$  with edge set  $E$  and face set  $F$  to be  $(a, d)$ -face antimagic if there exist positive integers  $a$  and  $d$  and a bijection  $g: E \rightarrow \{1, 2, \dots, |E|\}$  such that the induced mapping  $\psi_g: F \rightarrow \{a, a+d, \dots, a+(|F(G)|-1)d\}$  is also a bijection where for a face  $f$ ,  $\psi_g(f)$  is the sum of all  $g(e)$  for all edges  $e$  surrounding  $f$ . Bača [Bac6] and Bača and Miller [BaM] describe  $(a, d)$ -face antimagic labelings for a certain classes of convex polytopes. In [Bac7] Bača proves that for  $n$  even and at least 4 the prism  $C_n \times P_2$  is  $(6n + 3, 2)$ -face antimagic and  $(4n + 4, 4)$ -face antimagic. He also conjectures that  $C_n \times P_2$  is  $(2n + 5, 6)$ -face antimagic.

## 5.2 Prime and Vertex Prime Labelings

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy and Howalla (see [LWY]). A graph with vertex set  $V$  is said to have a *prime labeling* if its vertices are labeled with distinct integers  $1, 2, \dots, |V|$  such that for each edge  $xy$  the labels assigned to  $x$  and  $y$  are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. So far, there has been little progress towards proving this conjecture. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with a one vertex of degree at least 3 and with all other vertices with degree at most 2) and all trees of order less than 16 (see [FH]). Other graphs with prime labelings include all cycles and the disjoint union of  $C_{2k}$  and  $C_n$  [DLM]. The complete graph  $K_n$  does not have a prime labeling for  $n \geq 4$  and  $W_n$  is prime if and only if  $n$  is even (see [LWY]).

Seoud, Diab and Elsakhawi [SDE] have shown the following graphs are prime: fans; helms; flowers; stars;  $K_{2,n}$ ; and  $K_{3,n}$  unless  $n = 3$  or  $7$ . They also shown that  $P_n + \overline{K_m}$  ( $m \geq 3$ ) is not prime.

For  $m$  and  $n$  at least 3, Seoud and Youssef [SY] define  $S_n^{(m)}$ , the  $(m, n)$ -gon star, as the graph obtained from the cycle  $C_n$  by joining the two end vertices of the path  $P_{m-2}$  to every pair of consecutive vertices of the cycle such that each of the end vertices of the path is connected to exactly one vertex of the cycle. Seoud and Youssef [SY6] have proved the following graphs have prime labelings: books,  $S_n^{(m)}$ ,  $C_n \odot P_m$ ,  $P_n + \overline{K_2}$  if and only if  $n = 2$  or  $n$  is odd, and  $C_n \odot K_1$  with a complete binary tree of order  $2^k - 1$ ,  $k \geq 2$  attached at each pendant vertex. They also prove that every spanning subgraph of a prime graph is prime and every graph is a subgraph of a prime graph. They conjecture that all unicycle graphs have prime labelings. Seoud and Youssef [SY] prove the following graphs are not prime:  $C_m + C_n$ ,  $C_n^2$  for  $n \geq 4$ ,  $P_n^2$  for  $n = 6$  and for  $n \geq 8$ , and Möbius ladders  $M_n$  for  $n$  even. They also give an exact formula for the maximum number of edges in a prime graph of order  $n$  and an upper bound for the chromatic number of a prime graph. Salmasian [Sa] has shown that every tree with  $n$  vertices ( $n \geq 50$ ) can be labeled with  $n$  integers between 1 and  $4n$  so that every two

adjacent vertices have relatively prime labels.

Given a collection of graphs  $G_1, \dots, G_n$  and some fixed vertex  $v_i$  from each  $G_i$ , Lee, Wui and Yeh [LWY] define  $Amal\{(G_i, v_i)\}$ , the amalgamation of  $\{(G_i, v_i) \mid i = 1, \dots, n\}$ , as the graph obtained by taking the union of the  $G_i$  and identifying  $v_1, v_2, \dots, v_n$ . Lee et al. [LWY] have shown  $Amal\{(G_i, v_i)\}$  has a prime labeling when  $G_i$  are paths and when  $G_i$  are cycles. They also showed that the amalgamation of any number of copies of  $W_n$ ,  $n$  odd, with a common vertex is not prime. They conjecture that for any tree  $T$  and  $v$  from  $T$ , the amalgamation of two or more copies of  $T$  with  $v$  in common is prime. They further conjecture that the amalgamation of two or more copies of  $W_n$  that share a common point is prime when  $n$  is even ( $n \neq 4$ ).

A dual of prime labelings has been introduced by Deretsky, Lee and Mitchem [DLM]. They say a graph with edge set  $E$  has a *vertex prime labeling* if its edges can be labeled with distinct integers  $1, \dots, |E|$  such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs;  $C_{2k} \cup C_n$ ;  $C_{2m} \cup C_{2n} \cup C_{2k+1}$ ;  $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$ ; and  $5C_{2m}$ . They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Let  $G = \bigcup_{i=1}^t C_{2n_i}$  and  $N = \sum_{i=1}^t n_i$ . In [BHH] Borosh, Hensley and Hobbs proved that there is a positive constant  $n_0$  such that the conjecture of Deretsky et al. is true for the cases that (i)  $G$  is the disjoint union of at most seven cycles, or (ii)  $G$  is a union of cycles all of the same even length  $2n$  if  $n \leq 150\,000$  or if  $n \geq n_0$ , or (iii)  $n_i \geq (\log N)^{4 \log \log \log n}$  for all  $i = 1, \dots, t$ , or (iv) each  $C_{2n_i}$  is repeated at most  $n_i$  times. They end their paper with a discussion of graphs whose components are all even cycles, and of graphs with some components that are not cycles and some components that are odd cycles.

Figuroa-Centeno, Ichishima and Muntaner-Batle [FIM3] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph  $G$  of size  $q$  to be *product magic* if there is a labeling  $f$  from  $E(G)$  onto  $\{1, 2, \dots, q\}$  such that, at each vertex  $v$ , the product of the labels on the edges incident with  $v$  is the same. They call a graph  $G$  of size  $q$  *product antimagic* if there is a labeling  $f$  from  $E(G)$  onto  $\{1, 2, \dots, q\}$  such that the products of the labels on the edges incident at each vertex  $v$  are distinct. They prove that a graph of size  $q$  is product magic if and only if  $q \leq 1$ ,  $P_n$  ( $n \geq 4$ ) is product antimagic, every 2-regular graph is product antimagic and if  $G$  is product antimagic, then so are  $G + K_1$  and  $G \cdot \overline{K}_n$ . They conjecture that a connected graph of size  $q$  is product antimagic if and only if  $q \geq 3$ . They also define a  $(p, q)$ -graph  $G$  to be *product edge-magic* if there is a labeling  $f$  from  $V(G) \cup E(G)$  onto  $\{1, 2, \dots, p + q\}$  such that  $f(u) \cdot f(v) \cdot f(uv)$  is a constant for all edges  $uv$  and *product edge-antimagic* if there is a labeling  $f$  from  $V(G) \cup E(G)$  onto  $\{1, 2, \dots, p + q\}$  such that for all edges  $uv$  the products  $f(u) \cdot f(v) \cdot f(uv)$  are distinct. They prove  $K_2 \cup \overline{K}_n$  is product edge-magic, a graph of size  $q$  without isolated vertices is product edge-magic if and only if  $q \leq 1$  and

that every graph other than  $K_2$  and  $K_2 \cup \overline{K}_n$  is product edge-antimagic.

### 5.3 Edge-graceful Labelings

In 1985, Lo [Lo] introduced the notion of edge-graceful graphs. A graph  $G(V, E)$  is said to be *edge-graceful* if there exists a bijection  $f$  from  $E$  to  $\{1, 2, \dots, |E|\}$  so that the induced mapping  $f^+$  from  $V$  to  $\{0, 1, \dots, |V| - 1\}$  given by  $f^+(x) = \sum\{f(xy) | xy \in E\} \pmod{|V|}$  is a bijection. A necessary condition for a  $(p, q)$ -graph to be edge-graceful is that  $q(q+1) \equiv p(p+1)/2 \pmod{p}$ . Lee [L3] notes that this necessary condition extends to any  $(p, q)$ -multigraph. Lee, Lee and Murthy [LLM] proved that  $K_n$  is edge-graceful if and only if  $n \not\equiv 2 \pmod{4}$ . (An edge-graceful labeling for  $K_n$  for  $n \not\equiv 2 \pmod{4}$  in [Lo] was incorrect.) Lee [L3] notes that a multigraph with  $p \equiv 2 \pmod{4}$  vertices is not edge-graceful and conjectures that this condition is sufficient for the edge-gracefulness of connected graphs. Lo proved that all odd cycles are edge-graceful and Wilson and Riskin [WR] proved the Cartesian product of any number of odd cycles is edge-graceful. Lee [L2] has conjectured that all trees of odd order are edge-graceful. Small [Sm] has proved that spiders (see §5.2 for the definition) of odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [KS] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low and Mitchem [CLM] have shown that regular spiders of odd order are edge-graceful. Lee and Seah [LSe2] have shown that  $K_{n,n,\dots,n}$  is edge-graceful if and only if  $n$  is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [LSe1] have also proved that  $C_n^k$  (the  $k$ th power of  $C_n$ ) is edge-graceful for  $k < \lfloor n/2 \rfloor$  if and only if  $n$  is odd and  $C_n^k$  is edge-graceful for  $k \geq \lfloor n/2 \rfloor$  if and only if  $n$  is a multiple of 4 or  $n$  is odd (see also [CLM]). Lee, Seah and Wang [LSW] gave a complete characterization of edge-graceful  $P_n^k$  graphs. Shiu, Lam and Cheng [SLC] proved that the composition of the path  $P_3$  and any null graph of odd order is edge-graceful. Shiu, Lee and Schaffer [SLS] investigated the edge-gracefulness of multigraphs derived from paths, combs and spiders obtained by replacing each edge by  $k$  parallel edges. Lee and Seah [LSe3] have also investigated edge-gracefulness of various multigraphs.

Lee and Seah (see [L3]) define a *sunflower* graph  $SF(n)$  as the graph obtained by starting with an  $n$ -cycle with consecutive vertices  $v_1, v_2, \dots, v_n$  and creating new vertices  $w_1, w_2, \dots, w_n$  with  $w_i$  connected to  $v_i$  and  $v_{i+1}$  ( $v_{n+1}$  is  $v_1$ ). They prove that  $SF(n)$  is edge-graceful if and only if  $n$  is even. Lee and Seah [LSe4] prove that for  $k \leq n/2$ ,  $C_n^k$  is edge-graceful if and only if  $n$  is odd and, for  $k \geq n/2$ ,  $C_n^k$  is edge-graceful if and only if  $n \not\equiv 2 \pmod{4}$ . Lee, Seah and Lo (see [L3]) have proved that for  $n$  odd,  $C_{2n} \cup C_{2n+1}$ ,  $C_n \cup C_{2n+2}$  and  $C_n \cup C_{4n}$  are edge-graceful. They also show that for odd  $k$  and odd  $n$ ,  $kC_n$  is edge-graceful. Lee and Seah (see [L3]) prove that the generalized Petersen graph  $P(n, k)$  (see Section 2.7) is edge-graceful if and only if  $n$  is even and  $k < n/2$ . In particular,  $P(n, 1) = C_n \times P_2$  is edge-graceful if and only if  $n$  is even. Lee and Schaffer (see [L3]) proved that  $C_m \times C_n$  ( $m > 2, n > 2$ ) is edge-graceful if

and only if  $m$  and  $n$  are odd. Kendrick and Lee (see [L3]) proved that there are only finitely many  $n$  for which  $K_{m,n}$  is edge-graceful and they completely solve the problem for  $m = 2$  and  $m = 3$ . Ho, Lee and Seah [HLSe] use  $S(n; a_1, a_2, \dots, a_k)$  where  $n$  is odd and  $1 \leq a_1 \leq a_2 \leq \dots \leq a_k < n/2$  to denote the  $(n, nk)$ -multigraph with vertices  $v_0, v_1, \dots, v_{n-1}$  and edge set  $\{v_i v_j \mid i \neq j, i - j \equiv a_t \pmod{n} \text{ for } t = 1, 2, \dots, k\}$ . They prove that all such multigraphs are edge-graceful. Lee and Pritikin (see [L3]) prove that the Möbius ladders of order  $4n$  are edge-graceful. Lee, Tong and Seah [LTS] have conjectured that the total graph of a  $(p, p)$ -graph is edge-graceful if and only if  $p$  is even. They have proved this conjecture for cycles.

In 1997 Yilmaz and Cahit [YC] introduced a weaker version of edge-graceful called  $E$ -cordial. Let  $G$  be a graph with vertex set  $V$  and edge set  $E$  and let  $f$  a function from  $E$  to  $\{0, 1\}$ . Define  $f$  on  $V$  by  $f(v) = \sum \{f(uv) \mid uv \in E\} \pmod{2}$ . The function  $f$  is called an  $E$ -cordial labeling of  $G$  if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an  $E$ -cordial labeling is called  $E$ -cordial. Yilmaz and Cahit prove the following graphs are  $E$ -cordial: trees with  $n$  vertices if and only if  $n \not\equiv 2 \pmod{4}$ ;  $K_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ;  $K_{m,n}$  if and only if  $m+n \not\equiv 2 \pmod{4}$ ;  $C_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ; regular graphs of degree 1 on  $2n$  vertices if and only if  $n$  is even; friendship graphs  $C_3^{(n)}$  for all  $n$ ; fans  $F_n$  if and only if  $n \not\equiv 1 \pmod{4}$ ; and wheels  $W_n$  if and only if  $n \not\equiv 1 \pmod{4}$ . They observe that graphs with  $n \equiv 2 \pmod{4}$  vertices can not be  $E$ -cordial. They generalize  $E$ -cordial labelings to  $E_k$ -cordial ( $k > 1$ ) labelings by replacing  $\{0, 1\}$  by  $\{0, 1, 2, \dots, k-1\}$ . Of course,  $E_2$ -cordial is the same as  $E$ -cordial.

## 5.4 Line-graceful Labelings

Gnanajothi [Gn] has defined a concept similar to edge-graceful. She calls a graph with  $n$  vertices *line-graceful* if it is possible to label its edges with  $0, 1, 2, \dots, n$  so that when each vertex is assigned the sum modulo  $n$  of all the edge labels incident with that vertex the resulting vertex labels are  $0, 1, \dots, n-1$ . A necessary condition for the line-gracefulness of a graph is that its order is not congruent to  $2 \pmod{4}$ . Among line-graceful graphs are (see [Gn, pp. 132-181])  $P_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ;  $C_n$  if and only if  $n \not\equiv 2 \pmod{4}$ ;  $K_{1,n}$  if and only if  $n \not\equiv 1 \pmod{4}$ ;  $P_n \odot K_1$  (combs) if and only if  $n$  is even;  $(P_n \odot K_1) \odot K_1$  if and only if  $n \not\equiv 2 \pmod{4}$ ; (in general, if  $G$  has order  $n$ ,  $G \odot H$  is the graph obtained by taking one copy of  $G$  and  $n$  copies of  $H$  and joining the  $i$ th vertex of  $G$  with an edge to every vertex in the  $i$ th copy of  $H$ );  $mC_n$  when  $mn$  is odd;  $C_n \odot K_1$  (crowns) if and only if  $n$  is even;  $mC_4$  for all  $m$ ; complete  $n$ -ary trees when  $n$  is even;  $K_{1,n} \cup K_{1,n}$  if and only if  $n$  is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete  $n$ -ary tree when  $n$  is even; trees for which exactly one vertex has even degree. She conjectures that all trees with  $p \not\equiv 2 \pmod{4}$  vertices are line-graceful and



proved this conjecture for  $p \leq 9$ .

Gnanajothi [Gn] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from  $K_{1,4}$  by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has  $p \not\equiv 2 \pmod{4}$  vertices; the graph obtained from  $K_{1,n}$  by replacing each spoke with a path of length  $m$  (counting the center vertex) is line-graceful in the following cases:  $n = 2$ ;  $n = 3$  and  $m \not\equiv 3 \pmod{4}$ ;  $m$  is even and  $mn + 1 \equiv 0 \pmod{4}$ .

Gnanajothi studied graphs obtained by joining disjoint graphs  $G$  and  $H$  with an edge. She proved such graphs are line-graceful in the following circumstances:  $G = H$ ;  $G = P_n, H = P_m$  and  $m+n \not\equiv 0 \pmod{4}$ ; and  $G = P_n \odot K_1, H = P_m \odot K_1$  and  $m+n \not\equiv 0 \pmod{4}$ .

## 5.5 Sum Graphs

In 1990, Harary [Ha1] introduced the notion of a sum graph. A graph  $G(V, E)$  is called a *sum graph* if there is a bijective labeling  $f$  from  $V$  to a set of positive integers  $S$  such that  $xy \in E$  if and only if  $f(x) + f(y) \in S$ . Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. For a connected graph  $G$ , let  $s(G)$ , the *sum number of  $G$* , denote the minimum number of isolated vertices that must be added to  $G$  so that the resulting graph is a sum graph (some authors use  $\sigma(G)$  for the sum number of  $G$ ). A labeling that makes  $G$  together with  $s(G)$  isolated points a sum graph is called an *optimal sum graph labeling*. Ellingham [El] proved the conjecture of Harary [Ha1] that  $s(T) = 1$  for every tree  $T \neq K_1$ . Bergstrand et al. [BHHJKW] proved that  $s(K_n) = 2n - 3$ . Hartsfield and Smyth [HaS] claimed to have proved that  $s(K_{m,n}) = \lceil 3m + n - 3 \rceil / 2$  when  $n \geq m$  but Yan and Liu [YL1] found counterexamples to this assertion when  $m \neq n$ . Miller et al. [MRSS] proved that  $s(W_n) = \frac{n}{2} + 2$  for  $n$  even and  $s(W_n) = n$  for  $n \geq 5$  and  $n$  odd. Miller, Ryan and Smyth [MRSm] prove that the complete  $n$ -partite graph on  $n$  sets of 2 nonadjacent vertices has sum number  $4n - 5$  and obtain upper and lower bounds on the complete  $n$ -partite graph on  $n$  sets of  $m$  nonadjacent vertices. Teichert [Tei] determined the sum number of the  $d$ -partite complete hypergraph  $K_{n_1, \dots, n_d}^d$ . Gould and Rödl [GR] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [GBGGGJ] proved that  $s(K_n - \text{edge}) \leq 2n - 4$ . The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the *spum*. They proved spum of  $K_n$  is  $4n - 6$  and the spum of  $C_n$  is at most  $4n - 10$ .

At a conference in 2000 Miller [Mi] posed the following problems. Given any graph  $G$ , does there exist an optimal sum graph labeling that uses the label 1? Does the disjoint union of two or more graphs with sum number 1 have sum number 1? Find a class of graphs  $G$  that have sum number of the order  $|V(G)|^s$  for  $s > 1$ . (Such graphs were shown to exist for  $s = 2$  by Gould and Rödl in [GR]).

In 1994 Harary [Ha2] generalized sum graphs by permitting  $S$  to be any set of integers. He calls these graphs *integral sum graphs*. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [Sha] has shown that  $C_n$  and  $W_n$  are integral sum graphs for all  $n \neq 4$ . Chen [Che2] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. Chen also gives methods for constructing new connected integral sum graphs from given integral sum graphs by identification. B. Xu [XuB] has shown that the following are integral sum graphs: the union of any three stars;  $T \cup K_{1,n}$  for all trees  $T$ ;  $mK_3$  for all  $m$ ; the union of any number of integral sum trees; and all caterpillars with exactly one leaf attached to an end point of the spine. Xu also proved that if  $2G$  and  $3G$  are integral sum graphs, then so is  $mG$  for all  $m > 1$ . Xu poses the question as to whether all disconnected forests are integral sum graphs. Liaw, Kuo and Chang [LKC] proved that all caterpillars and all cycles except  $C_4$  are integral sum graphs and they conjecture that every tree is an integral sum graph.

The *integral sum number*,  $\zeta(G)$ , of  $G$ , is the minimum number of isolated vertices that must be added to  $G$  so that the resulting graph is an integral sum graph. Thus, by definition,  $G$  is a integral sum graph if and only if  $\zeta(G) = 0$ . Harary [Ha2] conjectured that for  $n \geq 4$  the integral sum number  $\zeta(K_n) = 2n - 3$ . This conjecture was verified by Chen [Che1], by Sharary [Sha] and by B. Xu [XuB]. Yan and Liu proved:  $\zeta(K_n - E(K_r)) = n - 1$  when  $n \geq 6$ ,  $n \equiv 0 \pmod{3}$  and  $r = 2n/3 - 1$  [YL2];  $\zeta(K_{m,m}) = 2m - 1$  for  $m \geq 2$  [YL2];  $\zeta(K_n - \text{edge}) = 2n - 4$  for  $n \geq 4$  [YL2], [XuB]; if  $n \geq 5$  and  $n - 3 \geq r$ , then  $\zeta(K_n - E(K_r)) \geq n - 1$  [YL2]; if  $\lceil 2n/3 \rceil - 1 > r \geq 2$ , then  $\zeta(K_n - E(K_r)) \geq 2n - r - 2$  [YL2]; and if  $2 \leq m < n$ , and  $n = (i + 1)(im - i + 2)/2$ , then  $s(K_{m,n}) = \zeta(K_{m,n}) = (m - 1)(i + 1) + 1$  while if  $(i + 1)(im - i + 2)/2 < n < (i + 2)[(i + 1)m - i + 1]/2$ , then  $s(K_{m,n}) = \zeta(K_{m,n}) = \lceil ((m - 1)(i + 1)(i + 2) + 2n)/(2i + 2) \rceil$  [YL2].

Nagamochi, Miller and Slamin [NMS] have determined upper and lower bounds on the sum number a graph. For most graphs  $G(V, E)$  they show that  $s(G) = \Omega(|E|)$ . He et al. [HYM] investigated  $\zeta(K_n - E(K_r))$  where  $n \geq 5$  and  $r \geq 2$ . They proved that  $\zeta(K_n - E(K_r)) = 0$  when  $r = n$  or  $n - 1$ ;  $\zeta(K_n - E(K_r)) = n - 1$  when  $n - 2 \geq r \geq \lceil 2n/3 \rceil - 1$ ;  $\zeta(K_n - E(K_r)) = 3n - 2r - 4$  when  $\lceil 2n/3 \rceil - 1 > r \geq n/2$ ;  $\zeta(K_n - E(K_r)) = 2n - 4$  when  $\lceil 2n/3 \rceil - 1 > n/2 \geq r$ . Moreover, they prove that if  $r \neq n - 1$ , then  $s(K_n - E(K_r)) = \zeta(K_n - E(K_r))$ . Lian, Kuo and Chang [LKC] show that for  $2 \leq m \leq n$ ,  $s(K_{m,n}) = \zeta(K_{m,n}) = \lceil \frac{n}{p} + \frac{(p+1)(m-1)}{2} \rceil$  where  $p = \lceil \sqrt{\frac{2n}{m-1} + \frac{1}{4}} - \frac{1}{2} \rceil$  is the unique integer such that  $\frac{(p-1)p(m-1)}{2} < n \leq \frac{(p+1)p(m-1)}{2}$ .

Alon and Scheinermann [AS] generalized sum graphs by replacing the condition  $f(x) + f(y) \in S$  with  $g(f(x), f(y)) \in S$  where  $g$  is an arbitrary symmetric polynomial. They called a graph with this property a *g-graph* and proved that for a given symmetric polynomial  $g$  not all graphs are *g-graphs*. On the other hand, for every symmetric polynomial  $g$  and every graph  $G$  there is some vertex labeling so that  $G$  together with at most  $|E(G)|$  isolated vertices is a *g-graph*.

Boland, Laskar, Turner, and Domke [BLTD] investigated a modular version of sum graphs. They call a graph  $G(V, E)$  a *mod sum graph* (MSG) if there exists a positive integer  $n$  and an injective labeling from  $V$  to  $\{1, 2, \dots, n-1\}$  such that  $xy \in E$  if and only if  $f(x) + f(y) \pmod{n} = f(z)$  for some vertex  $z$ . Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [BLTD] have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and all  $K_{2,n}$ . They further proved that  $K_p$  ( $p \geq 2$ ) is not MSG (see also [GLPF]) and that  $W_4$  is MSG. They conjecture that  $W_p$  is MSG for  $p \geq 4$ . This conjecture was refuted by Sutton, Miller, Ryan and Slamin [SMRS] who proved that for  $n \neq 4$ ,  $W_n$  is not MSG (the case where  $n$  is prime had been proved in 1994 by Ghoshal et al. [GLPF]). In the same paper Sutton et al. also showed that for  $n \geq 3$ ,  $K_{n,n}$  is not MSG. Ghoshal, Laskar, Pillone and Fricke [GLPF] proved that every connected graph is an induced subgraph of a connected MSG graph and any graph with  $n$  vertices and at least two vertices of degree  $n-1$  is not MSG. Sutton et al. define the *mod sum number*,  $\rho(G)$ , of a connected graph  $G$  to be the least integer  $r$  such that  $G + \overline{K_r}$  is MSG. Sutton and Miller [SuM] define the cocktail party graph  $H_{m,n}$ ,  $m, n \geq 2$ , as the graph with a vertex set  $V = \{v_1, v_2, v_3, \dots, v_{mn}\}$  partitioned into  $n$  independent sets  $V = \{I_1, I_2, \dots, I_n\}$  each of size  $m$  such that  $v_i v_j \in E$  for all  $i, j \in \{1, \dots, mn\}$  where  $i \in I_p, j \in I_q, p \neq q$ . They prove that  $\rho(K_n) = n$  for  $n \geq 4$  and  $H_{m,n}$  is not MSG for  $m \geq 3$  and  $m > n$ . In [SDM] Sutton, Draganova and Miller prove that for  $n$  odd and  $n \geq 5$ ,  $\rho(W_n) = n$  and when  $n$  is even  $\rho(W_n) = 2$ . Draganova [Dr] has shown that for  $n \geq 5$  and  $n$  odd,  $\rho(F_n) = n$ . She poses as an open problem the determination of the mod sum number of the  $t$ -point suspension of  $C_n$ . Wallace [Wa] has proved that  $K_{m,n}$  is MSG when  $n$  is even and  $n \geq 2m$  or when  $n$  is odd and  $n \geq 3m - 3$  and that  $\rho(K_{m,n}) = m$  when  $3 \leq m \leq n < 2m$ . He also proves that the complete  $m$ -partite  $K_{n_1, n_2, \dots, n_m}$  is not MSG when there exist  $n_i$  and  $n_j$  such that  $n_i < n_j < 2n_i$ . Miller [Mi] has asked if it is possible for the mod sum number of a graph  $G$  be of the order  $|V(G)|^2$ ?

Grimaldi [Gr] has investigated labeling the vertices of a graph  $G(V, E)$  with  $n$  vertices with distinct elements of the ring  $Z_n$  so that  $xy \in E$  whenever  $(x + y)^{-1}$  exists in  $Z_n$

## 5.6 Binary Labelings

In 1996 Caccetta and Jia [CJ] introduced binary labelings of graphs. Let  $G = (V, E)$  be a graph. A mapping  $f : E \mapsto \{0, 1\}^m$  is called an *M-coding* if the induced mapping  $g : V \mapsto \{0, 1\}^m$ , defined as  $g(v) = \sum_{u \in V, uv \in E} f(uv)$  is injective, where the summation is mod 2. An M-coding is called *positive* if the zero vector is not assigned to an edge and a vertex of  $G$ . Caccetta and Jia show that the minimal  $m$  for a positive M-coding equals  $k + 1$  if  $|V| \in \{2^k, 2^k - 2, 2^k - 3\}$  and  $k$  otherwise, where  $k = \lceil \log_2 |V| \rceil$ .

## 5.7 Average Labelings

In 1997 Harminc [Har] introduced a new kind of labeling in an effort to characterize forests and graphs without edges. Let  $G = (V, E)$  be a graph. A mapping  $f : V \mapsto N$  is called *average labeling* if for any  $(v_1, v_2), (v_2, v_3) \in E$  one has  $f(v_2) = (f(v_1) + f(v_3))/2$ . A labeling is called *nontrivial* if any connected component of  $G$  (excluding isolated vertices) has at least two differently labeled vertices. Harminc provides three results towards the characterization of hereditary graphs properties in terms of average labelings. In particular, all maximal connected subgraphs of  $G$  are exactly paths (i.e.,  $G$  is a linear forest) if and only if there exists a nontrivial average labeling of  $G$ . He also characterizes forests and graphs without edges by introducing a bit more complicated average-type labelings.

## 5.8 $k$ -sequential Labelings

In 1981 Bange, Barkauskas and Slater [BBS1] defined a  *$k$ -sequential* labeling  $f$  of a graph  $G(V, E)$  as one for which  $f$  is a bijection from  $V \cup E$  to  $\{k, k+1, \dots, |V \cup E| + k - 1\}$  such that for each edge  $xy$  in  $E$ ,  $f(xy) = |f(x) - f(y)|$ . This generalized the notion of *simply sequential* where  $k = 1$  introduced by Slater. Bange, Barkauskas and Slater showed that cycles are 1-sequential and if  $G$  is 1-sequential then  $G + K_1$  is graceful. In [Sl1], Slater proved:  $K_n$  is 1-sequential if and only if  $n \leq 3$ ; for  $n \geq 2$ ,  $K_n$  is not  $k$ -sequential for any  $k \geq 2$ ; and  $K_{1,n}$  is  $k$ -sequential if and only if  $k$  divides  $n$ . Acharya and Hegde [AH1] proved:  $P_{2n}$  is  $n$ -sequential for all  $n$  and  $P_{2n+1}$  is both  $n$ -sequential and  $(n+1)$ -sequential for all  $n$ ;  $K_{m,n}$  is  $k$ -sequential for  $k = 1, m$  and  $n$ ;  $K_{m,n,1}$  is 1-sequential; and the join of any caterpillar and  $\overline{K}_t$  is 1-sequential. Acharya [A1] showed that if  $G(E, V)$  is an odd graph with  $|E| + |V| \equiv 1$  or  $2 \pmod{4}$  when  $k$  is odd or  $|E| + |V| \equiv 2$  or  $3 \pmod{4}$  when  $k$  is even, then  $G$  is not  $k$ -sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig and Turgeon [BKT] we have:  $mK_4$  is not  $k$ -sequential for any  $k$  when  $m$  is odd and  $mK_2$  is not  $k$ -sequential for any odd  $k$  when  $m \equiv 2$  or  $3 \pmod{4}$  or for any even  $k$  when  $m \equiv 1$  or  $2 \pmod{4}$ . He further noted that  $K_{m,n}$  is not  $k$ -sequential when  $k$  is even and  $m$  and  $n$  are odd, while  $K_{m,k}$  is  $k$ -sequential for all  $k$ . Acharya [A1] points out that the following result of Slater's [Sl2] for  $k = 1$  linking  $k$ -graceful graphs and  $k$ -sequential graphs holds in general: A graph is  $k$ -sequential if and only if  $G + v$  has a  $k$ -graceful labeling  $f$  with  $f(v) = 0$ . Slater [Sl1] also proved that a  $k$ -sequential graph with  $p$  vertices and  $q > 0$  edges must satisfy  $k \leq p - 1$ .

## 5.9 Sequentially Additive Graphs

Bange, Barkauskas and Slater [BBS2] defined a  *$k$ -sequentially additive labeling*  $f$  of a graph  $G(V, E)$  to be a bijection from  $V \cup E$  to  $\{k, \dots, k + |V \cup E| - 1\}$  such that for each edge  $xy$ ,  $f(xy) = f(x) + f(y)$ . They proved:  $K_n$  is 1-sequentially additive if and only if  $n \leq 3$ ;  $C_{3n+1}$  is not  $k$ -sequentially additive for  $k \equiv 0$  or  $2 \pmod{3}$ ;  $C_{3n+2}$  is not  $k$ -sequentially additive for  $k \equiv 1$  or  $2 \pmod{3}$ ;  $C_n$  is 1-sequentially additive if and only

if  $n \equiv 0$  or  $1 \pmod{3}$ ; and  $P_n$  is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive.

Acharya and Hegde [AH2] have generalized  $k$ -sequentially additive labelings by allowing the image of the bijection to be  $\{k, k + d, \dots, (k + |V \cup E| - 1)d\}$ . They call such a labeling *additively  $(k, d)$ -sequential*.

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