A Note on Restricted Online Ramsey Numbers of Matchings

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Abstract

Consider the following game between Builder and Painter. We take some families of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_t$ and an integer n such that $n \geqslant R(\mathcal{G}_1, \ldots, \mathcal{G}_t)$. In each turn, Builder picks an edge of initially uncoloured K_n and Painter colours that edge with some colour $i \in \{1, \ldots, t\}$ of her choice. The game ends when a graph G_i in colour i for some $G_i \in \mathcal{G}_i$ and some i is created. The restricted online Ramsey number $\tilde{R}(\mathcal{G}_1, \ldots, \mathcal{G}_t; n)$ is the minimum number of turns that Builder needs to guarantee the game to end.

In a recent paper, Briggs and Cox studied the restricted online Ramsey numbers of matchings and determined a general upper bound for them. They proved that for $n = 3r-1 = R_2(rK_2)$ we have $\tilde{R}_2(rK_2; n) \leqslant n-1$ and asked whether this was tight. In this short note, we provide a general lower bound for these Ramsey numbers. As a corollary, we answer this question of Briggs and Cox, and confirm that for $n = 3r-1$ we have $\tilde{R}_2(rK_2; n) = n-1$. We also show that for $n' = 4r-2 = R_3(rK_2)$ we have $\tilde{R}_3(rK_2; n') = 5r-4$.

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1 Introduction

For families of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_t$, the *Ramsey number* $R(\mathcal{G}_1, \ldots, \mathcal{G}_t)$ is the smallest integer n such that any colouring of edges of K_n with colours $1, \ldots, t$ contains a graph G_i in colour i for some $G_i \in \mathcal{G}_i$ and some $i \in \{1, \ldots, t\}$. When each family $\mathcal{G}_i$ contains a single graph G_i, we instead use the notation $R(G_1, \ldots, G_t)$ for the corresponding Ramsey number. When moreover we have $G_1 = \cdots = G_t$, we use the notation $R_t(G_1)$ for $R(G_1, \ldots, G_t)$.

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The Ramsey numbers of graphs have been studied extensively, see for instance a survey of Conlon, Fox and Sudakov [6].

Many variants of the Ramsey numbers have been considered. One of them are the so-called *online Ramsey numbers*, introduced by Beck [1] and later independently by Kurek and Ruciński [12]. For families of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_t$, the online Ramsey number $\tilde{R}(\mathcal{G}_1, \ldots, \mathcal{G}_t)$ is the smallest integer k for which Builder can always guarantee a win within the first k moves of the following game between Builder and Painter. Initially, we are given an infinite set of vertices, with every two vertices connected by an uncoloured edge. In each turn, Builder picks an edge between some two vertices in our set and Painter chooses any colour out of $1, \ldots, t$ and colours the edge with this colour. Builder wins once there is a graph G_i in colour i for some $G_i \in \mathcal{G}_i$ and some $i \in \{1, \ldots, t\}$. For various results about online Ramsey numbers, see [4, 5, 8, 9, 11, 13, 14].

In 2008, Prałat [15] also introduced the *restricted online Ramsey numbers* (under different name, the name restricted online Ramsey numbers was first used for these by Conlon, Fox, Grinshpun and He [5]). These correspond to the same game as the online Ramsey numbers, but this game is now instead played on a finite board. To define this formally, for families of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_t$ and an integer n such that $n \geqslant R(\mathcal{G}_1, \ldots, \mathcal{G}_t)$, the restricted online Ramsey number $\tilde{R}(\mathcal{G}_1, \ldots, \mathcal{G}_t; n)$ is the smallest integer l for which Builder can always guarantee a win within the first l moves of the following game between Builder and Painter. In each turn, Builder picks an edge of initially uncoloured K_n and Painter chooses any colour out of $1, \ldots, t$ to colour this edge. Builder wins once there appears a graph G_i in colour i for some $G_i \in \mathcal{G}_i$ and some $i \in \{1, \ldots, t\}$. We note that the definitions of $\tilde{R}(\mathcal{G}_1, \ldots, \mathcal{G}_t; n)$ differ slightly between the previous papers on this topic [2, 5, 7, 10, 15], but it is easy to see that all are equivalent.

Analogously to the usual Ramsey numbers, when each family $\mathcal{G}_i$ contains a single graph G_i, we use the notation $\tilde{R}(G_1, \ldots, G_t; n)$ for the corresponding restricted online Ramsey number. And when we further have $G_1 = \cdots = G_t$, we use the notation $\tilde{R}_t(G_1; n)$ for $\tilde{R}(G_1, \ldots, G_t; n)$.

Briggs and Cox [2] studied the restricted online Ramsey numbers of matchings and trees. Before stating their results, recall the following well-known result of Cockayne and Lorimer [3] about the Ramsey numbers of matchings.

Theorem 1. *For any $t \geqslant 2$ and positive integers $r_1, \ldots, r_t$, we have*

$$R(r_1K_2, \ldots, r_tK_2) = \max_i r_i + 1 + \sum_{i=1}^{t}(r_i - 1).$$

Hence in particular, $R_t(rK_2) = r + 1 + t(r-1)$.

When r is fixed, we will denote by n_t for $t \geqslant 2$ the number $R_t(rK_2) = r+1+t(r-1)$. So in particular we have $n_2 = 3r-1$, $n_3 = 4r-2$ and $n_4 = 5r-3$. Now we are ready to state the result of Briggs and Cox [2].

Theorem 2. *Fix $t \geqslant 2$ and positive integers $r_1, \ldots, r_t$. If $n \geqslant R(r_1K_2, \ldots, r_tK_2)$, then*

$$\tilde{R}(r_1K_2,\ldots,r_tK_2;n) \leqslant \frac{2t-1+(t-3)\log_2(t-2)}{t+1}n$$

with the convention that $\log_2 0 = 0$.

Moreover, if we fix $r \geqslant 1$, *then* $\tilde{R}_2(rK_2;n_2) \leqslant 3r-2 = n_2-1$, $\tilde{R}_3(rK_2;n_3) \leqslant 5r-4$ *and* $\tilde{R}_4(rK_2;n_4) \leqslant 7r-5$.

They ask whether we have $\tilde{R}_2(rK_2;n_2) = n_2-1$. The aim of this short note is to verify that this indeed holds. We also show that the bound $\tilde{R}_3(rK_2;n_3) \leqslant 5r-4$ is tight and that the bound $\tilde{R}_4(rK_2;n_4) \leqslant 7r-5$ is tight except possibly for the exact value of the additive constant.

By describing a suitable strategy of Painter, we prove the following more general lower bound and a corollary about restricted online Ramsey numbers of matchings with few colours. This in particular answers the question of Briggs and Cox [2].

Theorem 3. *Fix* $t \geqslant 2$ *and positive integers* $r_1,\ldots,r_t$. *If* $n \geqslant R(r_1K_2,\ldots,r_tK_2)$, *then* $\tilde{R}(r_1K_2,\ldots,r_tK_2;n) \geqslant 3(\sum_{i=1}^t r_i - t+1) - n$.

Hence if we fix $r \geqslant 1$, *then* $\tilde{R}_2(rK_2;n_2) = 3r-2 = n_2-1$, $\tilde{R}_3(rK_2;n_3) = 5r-4$ *and* $\tilde{R}_4(rK_2;n_4) \in \{7r-6, 7r-5\}$.

It remains unclear whether for t and r large, the magnitude of $\tilde{R}_t(rK_2;n_t)$ is closer to the upper bound from Theorem 2 or to the lower bound from Theorem 3.

2 Proof of Theorem 3

Consider the game played with t colours on the edges of an initially uncoloured K_n. To prove Theorem 3, we will describe a strategy of Painter that ensures that after $T = 3(\sum_{i=1}^t r_i - t+1) - n - 1$ moves (where by a move we mean Builder choosing some still uncoloured edge and Painter colouring it), there is no r_iK_2 of colour i for $i = 1,\ldots,t$.

While taking her turns (and to help her with her colouring decisions), Painter will moreover assign the following states to the coloured edges of K_n and to all the vertices of K_n. Coloured edges are either *free*, or *rooted*. Every rooted edge is characterized by its *root*, which is one of its endpoints. Painter will assign (and update) the states of the coloured edges according to the strategy described below.

Vertices are of three types, characterized in the following way.

- If a vertex v is a root of at least one coloured edge, it is of type I.
- If a vertex v is not of type I, but there is at least one free edge with endpoint v, it is of type II.
- If a vertex v is neither of type I nor of type II, it is of type III.

In particular, note that initially all the vertices are of type III, since no edges are coloured at the start of the game.

For $0 \leqslant j \leqslant \binom{n}{2}$ and $i = 1, \dots, t$, let $A_j(i)$ be a number of type I vertices that are roots to at least one edge of colour i after j moves and let $B_j(i)$ be a number of free edges of colour i after j moves. Let $A_j = \sum_{i=1}^{t} A_j(i)$ and $B_j = \sum_{i=1}^{t} B_j(i)$.

Assume Builder chooses the edge ab in $(k+1)$st turn of his (where $0 \leqslant k \leqslant \binom{n}{2} - 1$). Without loss of generality (as we could otherwise switch a and b), we can assume that if b is of type I, then a is also of type I; and if b is of type II, then a is of type I or of type II. Painter chooses the colour of an edge and updates the states of the coloured edges as follows.

(i) If a is a vertex of type I, we declare the edge ab to be rooted at a. By definition, there exists at least one other edge rooted at a, of some colour c_1 (if there are more edges rooted at a, pick one arbitrarily). We colour ab by colour c_1.

(ii) If a is a vertex of type II, there exists by definition a free edge ac for some c, of some colour c_2 (if there are more free edges with endpoint a, pick one arbitrarily). We declare both edges ab, ac to be rooted at a and colour ab in c_2.

(iii) If a is a vertex of type III, then the edge ab is declared to be free. It is coloured in any colour c_3 such that $A_k(c_3) + B_k(c_3) \leqslant r_{c_3} - 2$ if at least one such colour exists, and if not in an arbitrary colour.

The next two observations are straightforward.

Observation 4. *The number of vertices of type III:*

- *stays the same during move (i)*
- *increases by* 1 *or stays the same during move (ii)*
- *decreases by* 2 *during move (iii)*

Observation 5. *If move j was (i) or (ii), we have $A_j(i) + B_j(i) = A_{j-1}(i) + B_{j-1}(i)$ for $i = 1, \dots, t$. If move j was (iii) and Painter used colour c, we have $A_j(c) + B_j(c) = A_{j-1}(c) + B_{j-1}(c) + 1$ and for any $c' \neq c$ we have $A_j(c') + B_j(c') = A_{j-1}(c') + B_{j-1}(c')$.*

Using Observation 4 and Observation 5, we prove the key lemma.

Lemma 6. *We have $A_T + B_T \leqslant \sum_{i=1}^{t} r_i - t$.*

Proof. Let C_2 be the number of moves (ii) up to time T, and let C_3 be the number of moves (iii) up to time T. At time T, by Observation 4 we have at most $n + C_2 - 2C_3$ vertices of type III. That implies $n + C_2 - 2C_3 \geqslant 0$. Since we further have $C_2 + C_3 \leqslant T$, we must have $C_3 \leqslant \frac{n+T}{3}$.

Now by Observation 5, $A_T + B_T \leqslant C_3 \leqslant \frac{n+T}{3} = \sum_{i=1}^{t} r_i - t + \frac{2}{3}$, and since $A_T + B_T$ is an integer, we have $A_T + B_T \leqslant \sum_{i=1}^{t} r_i - t$ as required. □

Continuing the proof of Theorem 3, we are now ready to show that after T moves, there is no r_iK_2 of colour i for $i = 1, \dots, t$.

Note that the existence of r_mK_2 of colour m would in particular imply that $A_T(m) + B_T(m) \geqslant r_m$. Because of the strategy of Painter and Observation 5, that would imply that $A_T(i) + B_T(i) \geqslant r_i - 1$ for $i = 1, \dots, t$. Hence we would have $A_T + B_T \geqslant (r_1 - 1) + \dots + r_m + \dots + (r_t - 1) = \sum_{i=1}^{t} r_i - t + 1$, contradicting Lemma 6. Thus the proof of Theorem 3 is finished.

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