Bartholdi Zeta Functions for Hypergraphs

Iwao SATO

Oyama National College of Technology,

Oyama, Tochigi 323-0806, JAPAN

e-mail: isato@oyama-ct.ac.jp

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Abstract

Recently, Storm [8] defined the Ihara-Selberg zeta function of a hypergraph, and gave two determinant expressions of it. We define the Bartholdi zeta function of a hypergraph, and present a determinant expression of it. Furthermore, we give a determinant expression for the Bartholdi zeta function of semiregular bipartite graph. As a corollary, we obtain a decomposition formula for the Bartholdi zeta function of some regular hypergraph.

1 Introduction

Graphs and digraphs treated here are finite. Let G be a connected graph and D the symmetric digraph corresponding to G. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path* P *of length* n in D(or G) is a sequence $P = (e_1, \cdots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leq i \leq n-1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \cdots, n$, then we write $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$. Set $\mid P \mid= n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$*-path*. We say that a path $P = (e_1, \cdots, e_n)$ has a *backtracking* or a *bump* at $t(e_i)$ if $e^{-1}_{i+1} = e_i$ for some $i(1 \leq i \leq n-1)$. A (v, w)-path is called a v*-cycle* (or v*-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \cdots, e_n)$ is the cycle $C^{-1} = (e^{-1}_n, \cdots, e^{-1}_1)$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called *equivalent* if $f_j = e_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

Let G be a connected graph. Then the *cyclic bump count* $cbc(\pi)$ of a cycle $\pi = (\pi_1, \cdots, \pi_n)$ is

$$cbc(\pi) = \mid \{ i = 1, \cdots, n \mid \pi_i = \pi_{i+1}^{-1} \} \mid,$$

where $\pi_{n+1} = \pi_1$.

Bartholdi [1] introduced the Bartholdi zeta function of a graph. The *Bartholdi zeta function* of G is defined by

$$\zeta(G, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G, and u, t are complex variables with $\mid u \mid$, $\mid t \mid$ sufficiently small.

If $u = 0$, then, since $0^0 = 1$, the Bartholdi zeta function of G is the (Ihara) zeta function of G(see [5]):

$$\zeta(G, 0, t) = \mathbf{Z}(G, t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G. Ihara [5] defined zeta functions of graphs, and showed that the reciprocals of zeta functions of regular graphs are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [9,10]. Hashimoto [4] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph G, and showed that the reciprocal of the zeta function of G is given by

$$\mathbf{Z}(G, t)^{-1} = (1 - t^2)^{r-1} \det(\mathbf{I} - t\mathbf{A}(G) + t^2(\mathbf{D}_G - \mathbf{I})),$$

where r is the Betti number of G, and $\mathbf{D}_G = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg_G v_i$ $(V(G) = \{v_1, \cdots, v_n\})$. Stark and Terras [7] gave an elementary proof of this formula, and discussed three different zeta functions of any graph. Various proofs of Bass' Theorem were given by Kotani and Sunada [6], and Foata and Zeilberger [3].

Bartholdi [1] gave a determinant expression of the Bartholdi zeta function of a graph.

Theorem 1 (Bartholdi) *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\zeta(G, u, t)^{-1} = (1 - (1-u)^2 t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + (1-u)(\mathbf{D}_G - (1-u)\mathbf{I})t^2).$$

Storm [8] defined the Ihara-Selberg zeta function of a hypergraph. A hypergraph $H = (V(H), E(H))$ is a pair of a set of *hypervertices* $V(H)$ and a set of *hyperedges* $E(H)$, which the union of all hyperedges is $V(H)$. A hypervertex v is *incident* to a hyperedge e if $v \in e$. For a hypergraph H, its dual H^* is the hypergraph obtained by letting its hypervertex set be indexed by $E(H)$ and its hyperedge set by $V(H)$.

A bipartite graph B_H associated with a hypergraph H is defined as follows: $V(B_H) = V(H) \cup E(H)$ and $v \in V(H)$ and $e \in E(H)$ are *adjacent* in B_H if v is incident to e. Let $V(H) = \{v_1, \ldots, v_n\}$. Then an *adjacency matrix* $\mathbf{A}(H)$ of H is defined as a mtarix whose rows and columns are parameterized by $V(H)$, and (i, j)-entry is the number of directed paths in B_H from v_i to v_j of length 2 with no backtracking.

Let H be a hypergraph. A *path P of length n* in H is a sequence $P = (v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1})$ of $n+1$ hypervertices and n hyperedges such that $v_i \in V(H)$, $e_j \in E(H)$, $v_1 \in e_1$, $v_{n+1} \in e_n$ and $v_i \in e_i, e_{i-1}$ for $i = 2, \ldots, n-1$. Set $\mid P \mid= n$, $o(P) = v_1$ and $t(P) = v_{n+1}$. Also, P is called an *$(o(P), t(P))$-path*. We say that a path P has a *hyperedge backtracking* if there is a subsequence of P of the form (e, v, e), where $e \in E(H)$, $v \in V(H)$. A (v, w)-path is called a *v-cycle* (or *v-closed path*) if $v = w$.

We introduce an equivalence relation between cycles. Such two cycles $C_1 = (v_1, e_1, v_2, \cdots, e_m, v_1)$ and $C_2 = (w_1, f_1, w_2, \cdots, f_m, w_1)$ are called *equivalent* if $w_j = v_{j+k}$ and $f_j = e_{j+k}$ for all j. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no hyperedge backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle.

The *Ihara-Selberg zeta function* of H is defined by

$$\zeta_H(t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of H, and t is a complex variable with $\mid t \mid$ sufficiently small(see [8]).

Let H be a hypergraph with $E(H) = \{e_1, \ldots, e_m\}$, and let $\{c_1, \ldots, c_m\}$ be a set of *m colors*, where $c(e_i) = c_i$. Then an *edge-colored graph GH_c* is defined as a graph with vertex set $V(H)$ and edge set $\{vw \mid v, w \in V(H); v, w \in e \in E(H)\}$, where an edge vw is colored c_i if $v, w \in e_i$.

Let GH_c^o be the symmetric digraph corresponding to the edge-clored graph GH_c. Then the *oriented line graph* $H_L^o = (V_L, E_L^o)$ associated with GH_c^o by

$$V_L = D(GH_c^o), \text{ and } E_L^o = \{(e_i, e_j) \in D(GH_c^o) \times D(GH_c^o) \mid c(e_i) \neq c(e_j), t(e_i) = o(e_j)\},$$

where $c(e_i)$ is the color assigned to the oriented edge $e_i \in D(GH_c^o)$. The *Perron-Frobenius operator* $T : C(V_L) \longrightarrow C(V_L)$ is given by

$$(Tf)(x) = \sum_{e \in E_o(x)} f(t(e)),$$

where $E_o(x) = \{e \in E_L^o \mid o(e) = x\}$ is the set of all oriented edges with x as their origin vertex, and $C(V_L)$ is the set of functions from V_L to the complex number field $\mathbf{C}$.

Storm [8] gave two nice determinant expressions of the Ihara-Selberg zeta function of a hypergraph by using the results of Kotani and Sunada [6], and Bass [2].

Theorem 2 (Storm) *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Then*

$$\zeta_H(t)^{-1} = \det(\mathbf{I} - tT) = (1-t)^{m-n} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + t\mathbf{Q}_{B_H}),$$

where $n =\mid V(B_H) \mid$, $m =\mid E(B_H) \mid$ *and* $\mathbf{Q}_{B_H} = \mathbf{D}_{B_H} - \mathbf{I}$.

Furthermore, Storm [8] presented the Ihara-Selberg zeta function of a (d,r)-regular hypergraph by using the results of Hashimoto [4].

In Section 2, we define the Bartholdi zeta function of a hypergraph, and present a determinant expression of it. In Section 3, we give a decomposition formula (Theorem 4) for the Bartholdi zeta function of semiregular bipartite graph. As a corollary, we obtain a decomposition formula for the Bartholdi zeta function of some regular hypergraph. In Section 4, we prove Theorem 4 by using an analogue of Hashimoto's method [4].

2 Bartholdi zeta function of a hypergraph

Let H be a hypergraph. Then a path $P = (v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1})$ has a *(broad) backtracking* or *(broad) bump* at e or v if there is a subsequence of P of the form (e, v, e) or (v, e, v), where $e \in E(H)$, $v \in V(H)$. Furthermore, the *cyclic bump count* $cbc(C)$ of a cycle $C = (v_1, e_1, v_2, e_2, \cdots, e_n, v_1)$ is

$$cbc(C) =\mid \{i = 1, \cdots, n \mid v_i = v_{i+1}\} \mid + \mid \{i = 1, \cdots, n \mid e_i = e_{i+1}\} \mid,$$

where $v_{n+1} = v_1$ and $e_{n+1} = e_1$.

The *Bartholdi zeta function* of H is defined by

$$\zeta(H, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of H, and u, t are complex variables with $\mid u \mid$, $\mid t \mid$ sufficiently small.

If $u = 0$, then the Bartholdi zeta function of H is the Ihara-Selberg zeta function of H.

A determinant expression of the Bartholdi zeta function of a hypergraph is given as follows:

Theorem 3 *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Then*

$$\zeta(H, u, t) = \zeta(B_H, u, \sqrt{t}) = (1-(1-u)^2 t)^{-(m-n)} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + (1-u)t(\mathbf{D}_{B_H} - (1-u)\mathbf{I}))^{-1}$$

where $n =\mid V(B_H) \mid$ *and* $m =\mid E(B_H) \mid$.

Proof. The argument is an analogue of Storm's method [8].

At first, we show that there exists a one-to-one correspondence between equivalence classes of prime cycles of length l in H and those of prime cycles of length $2l$ in B_H, and $cbc(C) = cbc(\tilde{C})$ for any prime cycle C in H and the corresponding cycle $\tilde{C}$ in B_H.

Let $C = (v_1, e_1, v_2, \ldots, v_l, e_l, v_1)$ be a prime cycle of length l in H. Then a cycle $\tilde{C} = (v_1, v_1e_1, e_1, \ldots, v_l, v_le_l, e_l, e_lv_1, v_1)$ is a prime cycle of length $2l$ in B_H. Thus, there exists a one-to-one correspondence between equivalence classes of prime cycles of length l in H and those of prime cycles of length $2l$ in B_H.

Let C a prime cycle in H and $\tilde{C}$ a prime cycle corresponding to C in B_H. Then there exists a subsequence (v, e, v) (or (e, v, e)) in C if and only if there exists a subsequence (v, ve, e, ev, v) (or (e, ev, v, ve, e)) in $\tilde{C}$. Thus, we have $cbc(C) = cbc(\tilde{C})$.

Therefore, it follows that

$$\zeta(H, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1} = \prod_{[\tilde{C}]} (1 - u^{cbc(\tilde{C})} t^{|\tilde{C}|/2})^{-1} = \zeta(B_H, u, \sqrt{t}),$$

where $[C]$ and $[\tilde{C}]$ runs over all equivalence classes of prime cycles in H and B_H, respectively.

By Theorem 1, we have

$$\zeta(H, u, t) = (1 - (1-u)^2 t)^{-(m-n)} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + (1-u)t(\mathbf{D}_{B_H} - (1-u)\mathbf{I}))^{-1},$$

where $n =| V(H) |$ and $m =| E(H) |$. $\Box$

If $u = 0$, then Theorem 3 implies Theorem 2.

Corollary 1 *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Then*

$$\zeta(H, u, t) = \zeta(H^*, u, t).$$

Proof. By the fact that $B_H = B_{H^*}$. $\Box$

3 Bartholdi zeta functions of (d, r)-regular hypergraphs

At first, we state a decomposition formula for the Bartholdi zeta function of a semiregular bipartite graph. Hashimoto [4] presented a determinant expression for the Ihara zeta function of a semiregular bipartite graph. We generalize Hashimoto's result on the Ihara zeta function to the Bartholdi zeta function.

A graph G is called *bipartite*, denoted by $G = (V_1, V_2)$ if there exists a partition $V(G) = V_1 \cup V_2$ of $V(G)$ such that the vertices in V_i are mutually nonadjacent for $i = 1, 2$. A bipartite graph $G = (V_1, V_2)$ is called (q_1+1, q_2+1)-*semiregular* if $\deg_G v = q_i+1$ for each $v \in V_i (i = 1, 2)$. For a $(q_1 + 1, q_2 + 1)$-semiregular bipartite graph $G = (V_1, V_2)$, let $G^{[i]}$ be the graph with vertex set V_i and edge set $\{P : \text{ reduced path } | \mid P \mid= 2; o(P), t(P) \in V_i\}$ for $i = 1, 2$. Then $G^{[1]}$ is $(q_1 + 1)q_2$-regular, and $G^{[2]}$ is $(q_2 + 1)q_1$-regular.

A determinant expression for the Bartholdi zeta function of a semiregular bipartite graph is given as follows. For a graph G, let $Spec(G)$ be the set of all eigenvalues of the adjacency matrix of G.

Theorem 4 *Let $G=(V_1,V_2)$ be a connected (q_1+1,q_2+1)-semiregular bipartite graph with ν vertices and ϵ edges. Set $\mid V_1 \mid = n$ and $\mid V_2 \mid = m (n \leq m)$. Then*

$$\begin{aligned}
&\zeta(G,u,t)^{-1} = (1-(1-u)^2t^2)^{\epsilon-\nu}(1+(1-u)(q_2+u)t^2)^{m-n} \\
\times & \prod_{j=1}^{n}(1-(\lambda_j^2-(1-u)(q_1+q_2+2u))t^2+(1-u)^2(q_1+u)(q_2+u)t^4) \\
= & (1-(1-u)^2t^2)^{\epsilon-\nu}(1+(1-u)(q_2+u)t^2)^{m-n}\det(\mathbf{I}_n-(\mathbf{A}^{[1]}-((q_2-1) \\
& +(q_1+q_2-2)u+2u^2)\mathbf{I}_n)t^2+(1-u)^2(q_1+u)(q_2+u)t^4\mathbf{I}_n) \\
= & (1-(1-u)^2t^2)^{\epsilon-\nu}(1+(1-u)(q_1+u)t^2)^{n-m}\det(\mathbf{I}_m-(\mathbf{A}^{[2]}-((q_1-1) \\
& +(q_1+q_2-2)u+2u^2)\mathbf{I}_m)t^2+(1-u)^2(q_1+u)(q_2+u)t^4\mathbf{I}_m),
\end{aligned}$$

where $Spec(G)=\{\pm\lambda_1,\cdots,\pm\lambda_n,0,\cdots,0\}$ and $\mathbf{A}^{[i]}=\mathbf{A}(G^{[i]})(i=1,2)$.

The proof of Theorem 4 is given in section 4.

A hypergraph H is a (d,r)-*regular* if every hypervertex is incident to d hyperedges, and every hyperedge contains r hypervertices. If H is a (d,r)-regular hypergraph, then the associated bipartite graph B_H is (d,r)-semiregular. Let $V_1=V(H)$, $V_2=E(H)$ and $d \geq r$. Set $n = \mid V_1 \mid$ and $m = \mid V_2 \mid$. Then we have $\mathbf{A}^{[1]}=\mathbf{A}(H)$ and $\mathbf{A}^{[2]}=\mathbf{A}(H^*)$. By Theorems 3 and 4, we obtain the following result. Let *Spec*$(\mathbf{B})$ be the set of all eigenvalues of the square matrix $\mathbf{B}$.

Theorem 5 *Let H be a finite, connected (d,r)-regular hypergraph with $d \geq r$. Set $n = \mid V(H) \mid$ and $m = \mid E(H) \mid$. Then*

$$\begin{aligned}
&\zeta(H,u,t)^{-1} = (1-(1-u)^2t)^{\epsilon-\nu}(1+(1-u)(r-1+u)t)^{m-n} \\
\times & \prod_{j=1}^{n}(1-(\lambda_j^2-(1-u)(d+r-2+2u))t+(1-u)^2(d-1+u)(r-1+u)t^2) \\
= & (1-(1-u)^2t)^{\epsilon-\nu}(1+(1-u)(r-1+u)t)^{m-n}\det(\mathbf{I}_n-(\mathbf{A}(H)-(r-2 \\
& +(d+r-4)u+2u^2)\mathbf{I}_n)t+(1-u)^2(d-1+u)(r-1+u)t^2\mathbf{I}_n) \\
= & (1-(1-u)^2t)^{\epsilon-\nu}(1+(1-u)(d-1+u)t)^{n-m}\det(\mathbf{I}_m-(\mathbf{A}(H^*)-(d-2 \\
& +(d+r-4)u+2u^2)\mathbf{I}_m)t+(1-u)^2(d-1+u)(r-1+u)t^2\mathbf{I}_m),
\end{aligned}$$

where $\epsilon=nd=mr$, $\nu=n+m$ and $Spec(\mathbf{A}(H))=\{\pm\lambda_1,\cdots,\pm\lambda_n,0,\cdots,0\}$.

In the case of $u=0$, we obtain Theorem 16 in [8].

Corollary 2 (Storm) *Let H be a finite, connected (d,r)-regular hypergraph with $d \geq r$. Set $n = \mid V(H) \mid$, $m = \mid E(H) \mid$ and $q=(d-1)(r-1)$. Then*

$$\begin{aligned}\zeta_H(t)^{-1} &= (1-t)^{\epsilon-\nu}(1+(r-1)t)^{m-n}\det(\mathbf{I}_n-(\mathbf{A}(H)-r+2)t+qt^2) \\ &= (1-t)^{\epsilon-\nu}(1+(d-1)t)^{n-m}\det(\mathbf{I}_m-(\mathbf{A}(H^*)-d+2)t+qt^2),\end{aligned}$$

where $\epsilon = nd = mr$ and $\nu = n+m$.

4 A proof of Theorem 4

The argument is an analogue of Hashimoto's method [4].

By Theorem 1, we have

$$\zeta(G,u,t)^{-1} = (1-(1-u)^2t^2)^{\epsilon-\nu}\det(\mathbf{I}_\nu - t\mathbf{A}+(1-u)t^2(\mathbf{Q}_G+u\mathbf{I}_\nu)).$$

Let $V_1=\{u_1,\cdots,u_n\}$ and $V_2=\{v_1,\cdots,v_m\}$. Arrange vertices of G in $n+m$ blocks: $u_1,\cdots,u_n;v_1,\cdots,v_m$. We consider the matrix $\mathbf{A}=\mathbf{A}(G)$ under this order. Then, let

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ {}^t\mathbf{E} & \mathbf{0} \end{bmatrix},$$

where ${}^t\mathbf{E}$ is the transpose of $\mathbf{E}$.

Since $\mathbf{A}$ is symmetric, there exists a orthogonal matrix $\mathbf{W} \in O(m)$ such that

$$\mathbf{EW} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mu_1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ \star & & \mu_n & 0 & \cdots & 0 \end{bmatrix}.$$

Now, let

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{bmatrix}.$$

Then we have

$${}^t\mathbf{PAP} = \begin{bmatrix} \mathbf{0} & \mathbf{F} & \mathbf{0} \\ {}^t\mathbf{F} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Furthermore, we have

$${}^t\mathbf{P}(\mathbf{Q}_G+u\mathbf{I}_\nu)\mathbf{P} = \mathbf{Q}_G+u\mathbf{I}_\nu.$$

Thus,

$$\begin{aligned}\zeta(G,u,t)^{-1} &= (1-(1-u)^2t^2)^{\epsilon-\nu}(1+(1-u)(q_2+u)t^2)^{m-n}\det\begin{bmatrix} a\mathbf{I}_n & -t\mathbf{F} \\ -t\,{}^t\mathbf{F} & b\mathbf{I}_n \end{bmatrix} \\ &= (1-(1-u)^2t^2)^{\epsilon-\nu}(1+(1-u)(q_2+u)t^2)^{m-n}\det\begin{bmatrix} a\mathbf{I}_n & \mathbf{0} \\ -t\,{}^t\mathbf{F} & b\mathbf{I}_n - a^{-1}t^2\,{}^t\mathbf{FF} \end{bmatrix} \\ &= (1-(1-u)^2t^2)^{\epsilon-\nu}(1+(1-u)(q_2+u)t^2)^{m-n}\det(ab\mathbf{I}_n - t^2\,{}^t\mathbf{FF}),\end{aligned}$$

where $a = 1 + (1-u)(q_1+u)t^2$ and $b = 1 + (1-u)(q_2+u)t^2$.

Since $\mathbf{A}$ is symmetric, ${}^t\mathbf{F}\mathbf{F}$ is symmetric and positive semi-definite, i.e., the eigenvalues of ${}^t\mathbf{F}\mathbf{F}$ are of form:

$$\lambda_1^2, \cdots, \lambda_n^2 (\lambda_1, \cdots, \lambda_n \geq 0).$$

Therefore it follows that

$$\zeta(G,u,t)^{-1} = (1-(1-u)^2t^2)^{\epsilon-\nu}(1+(1-u)(q_2+u)t^2)^{m-n} \prod_{j=1}^{n} (ab - \lambda_j^2 t^2).$$

But, we have

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^{m-n} \det(\lambda^2 \mathbf{I} - {}^t\mathbf{F}\mathbf{F}),$$

and so

$$Spec(\mathbf{A}) = \{\pm\lambda_1, \cdots, \pm\lambda_n, 0, \cdots, 0\}.$$

Thus, there exists a orthogonal matrix $\mathbf{S}$ such that

$${}^t\mathbf{S}\mathbf{A}^2\mathbf{S} = \begin{bmatrix} \lambda_1^2 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & \lambda_n^2 & & & & & & \\ & & & \lambda_1^2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_n^2 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 0 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix},$$

where $\mathbf{S}_1$ is an $n \times n$ matrix. Furthermore, we have

$$\mathbf{A}^2 = \mathbf{A}_2 + (\mathbf{Q}_G + \mathbf{I}_\nu),$$

where $\mathbf{A}_2 = ((\mathbf{A}_2)_{uv})_{u,v \in V(G)}$:

$$(\mathbf{A}_2)_{uv} = \text{the number of reduced } (u,v)-\text{paths with length } 2.$$

By the definition of the graphs $G^{[i]} (i = 1, 2)$,

$$\mathbf{A}^2 = \begin{bmatrix} \mathbf{A}^{[1]} + (q_1+1)\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{[2]} + (q_2+1)\mathbf{I}_m \end{bmatrix}.$$

Thus,

$${}^t\mathbf{S}\mathbf{A}^2\mathbf{S} = \begin{bmatrix} \mathbf{S}_1^{-1}\mathbf{A}^{[1]}\mathbf{S}_1 + (q_1+1)\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix}.$$

Therefore, it follows that

$$\mathbf{S}_1^{-1}\mathbf{A}^{[1]}\mathbf{S}_1 = \begin{bmatrix} \lambda_1^2 - (q_1+1) & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 - (q_1+1) \end{bmatrix}.$$

Hence

$$\det(ab\mathbf{I}_n - (\mathbf{A}^{[1]} + (q_1+1)\mathbf{I}_n)t^2) = \prod_{j=1}^{n}(ab - \lambda_j^2 t^2).$$

Thus, the second equation follows.

Similarly to the proof of the second equation, the third equation is obtained. □

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