The maximum number of copies of $K_{r,s}$ in graphs without long cycles or paths

Changhong Lu* Long-Tu Yuan†

School of Mathematical Sciences, Shanghai Key Laboratory of PMMP
East China Normal University
Shanghai 200241, China

{chlu, ltyuan}@math.ecnu.edu.cn

Ping Zhang‡

College of Science
University of Shanghai for Science and Technology
Shanghai 200093, China

mathzhangping@126.com

Submitted: Jan 19, 2021; Accepted: Aug 17, 2021; Published: Oct 8, 2021

Abstract

The circumference of a graph is the length of a longest cycle of it. We determine the maximum number of copies of $K_{r,s}$, the complete bipartite graph with classes sizes r and s, in a 2-connected graph with circumference less than k. As corollaries of our main result, we determine the maximum number of copies of $K_{r,s}$ in n-vertex P_k-free and M_k-free graphs for all values of n, where P_k is a path on k vertices and M_k is a matching on k edges.

Mathematics Subject Classifications: 05C35, 05C38

1 Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. Let $e(G)$ denote the number of edges in G. For two graphs G_1 and G_2, $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For positive integer

*Supported by NSFC grant 11871222, and Science and Technology Commission of Shanghai Municipality grants 18dz2271000 and 19jc1420100.

†Supported by NSFC grant 11901554, and Science and Technology Commission of Shanghai Municipality grants 18dz2271000 and 19jc1420100.

‡Corresponding author.

α, let $[\alpha] := \{1, \ldots, \alpha\}$. If $G_i \cong G_1$ for $i \in [\alpha]$, then we use αG_1 to denote $\bigcup_{i=1}^{\alpha} G_i$. For a given graph H, we use $N(H, G)$ to denote the number of (not necessarily induced) copies of H in G. If there is no copy of H in G, we say that G is *H-free*. For a family of graphs $\mathcal{F}$, if there is no copy of any member of $\mathcal{F}$ in G, we say that G is *$\mathcal{F}$-free*. For a subgraph H of G, we use $G - H$ to denote the graph obtained from G by deleting the vertices of H and the edges incident with at least one vertex in H. The *length* of a cycle or a path is the number of edges in them. The *circumference* of G is the length of the longest cycle in G. For positive integers r and s, we use $K_{r,s}$ to denote the complete bipartite graph with two parts of size r and s, respectively. We use P_k and M_k to denote the path on k vertices and a matching with k edges, respectively. By $\mathcal{C}_{\geqslant k}$, we mean the set of all cycles of length at least k. Let G be a graph and v be a vertex of G. The *neighborhood* of v in G, denoted by $N(v)$, is the set of vertices in $V(G)$ which are adjacent to v.

For a graph T and a family of graphs $\mathcal{F}$, the maximum number of copies of T in an $\mathcal{F}$-free graph of order n is called *the generalized Turán number*, denoted by $\mathrm{ex}(n, T, \mathcal{F})$. When $T \cong K_2$, it reduces to the classical Turán number $\mathrm{ex}(n, \mathcal{F})$, which is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. Similarly, we use $\mathrm{ex}_{\mathrm{con}}(n, T, \mathcal{F})$ and $\mathrm{ex}_{2\text{-con}}(n, T, \mathcal{F})$ to denote the maximum number of copies of T in an $\mathcal{F}$-free n-vertex connected graph and $\mathcal{F}$-free n-vertex 2-connected graph, respectively. When $\mathcal{F}$ contains a single graph F, we write $\mathrm{ex}(n, T, F)$, $\mathrm{ex}_{\mathrm{con}}(n, T, F)$, $\mathrm{ex}_{2\text{-con}}(n, T, F)$ instead of $\mathrm{ex}(n, T, \{F\})$, $\mathrm{ex}_{\mathrm{con}}(n, T, \{F\})$, $\mathrm{ex}_{2\text{-con}}(n, T, \{F\})$, respectively. Recently, the generalized Turán problem has received a lot of attention, see [1, 5, 6, 7, 8, 9, 10, 14, 15, 18].

The following are the famous theorems of Erdős and Gallai [4]. They first studied the maximum number of edges in P_k-free graphs and $\mathcal{C}_{\geqslant k}$-free graphs on n vertices and characterized the extremal graphs for some values of n.

Theorem 1 (Erdős and Gallai [4]). *Let $n \geqslant k$. Then* $\mathrm{ex}(n, P_k) \leqslant (k-2)n/2$.

Theorem 2 (Erdős and Gallai [4]). *Let $n \geqslant k$. Then* $\mathrm{ex}(n, \mathcal{C}_{\geqslant} k) \leqslant (k-1)(n-1)/2$.

In [12], Kopylov extended the above results to 2-connected graphs. He showed the maximum number of edges in 2-connected n-vertex graphs with circumference less than k and characterized the extremal graphs. We first give the definition of a graph class.

For $n \geqslant k \geqslant 4$ and $k/2 > a \geqslant 1$, define the n-vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C with $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$. The edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$ (see Figure 1). Note that when $a \geqslant 2$, $H_{n,k,a}$ is 2-connected, has no cycle with k or more vertices.

The number of copies of K_s in $H_{n,k,a}$, denoted by $f_s(n, k, a)$, is $\binom{k-a}{s} + (n-k+a)\binom{a}{s-1}$.

Theorem 3 (Kopylov [12]). *Let $n \geqslant k \geqslant 5$ and $t = \lfloor (k-1)/2 \rfloor$. Then* $\mathrm{ex}_{2\text{-con}}(n, K_2, \mathcal{C}_{\geqslant k}) = \max\{f_2(n, k, 2), f_2(n, k, t)\}$.

For connected graphs, Kopylov[12] and Balister, Győri, Lehel and Schelp [2] independently proved the following theorem.

Theorem 4 (Kopylov[12] and Balister, Győri, Lehel, Schelp [2]). *Let $n \geqslant k$ and $t = \lfloor (k-2)/2 \rfloor$. Then* $\mathrm{ex}_{\mathrm{con}}(n, K_2, P_k) = \max\{f_2(n, k-1, 1), f_2(n, k-1, t)\}$.

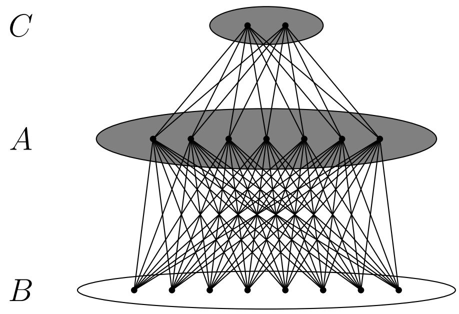

Figure 1: $H_{n,k,a}$

Luo [13] generalized the above results to s-cliques. She showed that the same extremal examples that maximize the number of edges in 2-connected n-vertex graphs with circumference less than k also maximize the number of copies of clique with given size.

Theorem 5 (Luo [13]). *Let $n \geqslant k \geqslant 5$ and $t = \lfloor (k-1)/2 \rfloor$. Then* $\mathrm{ex}_{2\text{-con}}(n, K_s, \mathcal{C}_{\geqslant k}) = \max\{f_s(n, k, 2), f_s(n, k, t)\}$.

Applying the shifting method, Wang [17] considered the maximum number of copies of $K^*_{r,s}$ in M_{k+1}-free graphs, where $K^*_{r,s}$ is obtained by taking a copy of $K_{r,s}$ and joining each pair of vertices inside the part with size r.

Theorem 6 (Wang [17]). *Let $r \geqslant 1, s \geqslant 2$ and $n \geqslant 2k+1$. Then* $\mathrm{ex}(n, K^*_{r,s}, M_{k+1}) = \max\left\{ \binom{2k+1}{s+r}\binom{s+r}{s}, \binom{k}{r}\binom{n-s}{s} + (n-k)\binom{k}{s+r-1}\binom{s+r-1}{s} \right\}$.

Denote by S_r and C_r the star and cycle on r vertices respectively. Győri, Salia, Tompkins, Zamora [11] considered the maximum number of S_r and C_4 copies in P_k-free graphs with sufficiently larger order.

Theorem 7 (Győri, Salia, Tompkins, Zamora [11]). *For $k \geqslant 3, r \geqslant 3, t = \lfloor (k-2)/2 \rfloor$ and sufficiently large n,* $\mathrm{ex}(n, S_r, P_k) = g_{1,r-1}(n, k-1, t)$. *Moreover, the only extremal graph is $H_{n,k-1,t}$, unless k is even and $t \leqslant r-2$ in which the only extremal graphs are $H_{n,k-1,t}$ and $H_{n,k-2,t}$.*

Theorem 8 (Győri, Salia, Tompkins, Zamora [11]). *For $k \geqslant 5, t = \lfloor (k-2)/2 \rfloor$ and sufficiently large n,* $\mathrm{ex}(n, C_4, P_k) = g_{2,2}(n, k-1, t)$. *Moreover, the only extremal graph is $H_{n,k-1,t}$.*

In this paper, we show that the maximum number of copies of $K_{r,s}$ in 2-connected $\mathcal{C}_{\geqslant k}$-free n-vertex graphs and connected P_k-free n-vertex graphs. As corollaries of our main results, we also show the maximum number of copies of $K_{r,s}$ in P_k-free n-vertex graphs. Moreover, since C_4 and S_r are complete bipartite graphs, we can improve Theorem 7 and Theorem 8 to all values of n. Also, the shifting method used in [17] seems not work for the case $K_{r,s}$. We can also determine the maximum number of copies of $K_{r,s}$ in M_k-free n-vertex graphs.

2 Main Results

We first consider the maximum number of copies of $K_{r,s}$ in 2-connected $\mathcal{C}_{\geqslant k}$-free graphs.

For $n \geqslant k \geqslant 4$ and $k/2 > a \geqslant 1$, let

$$g_{r,s}(n,k,a) = \begin{cases} \sum_{i=1}^{n-k+a} \binom{a}{r}\binom{n-r-i}{r-1} + \frac{1}{2}\binom{k-a}{2r}\binom{2r}{r}, & r = s; \\ \sum_{i=1}^{n-k+a} \left(\binom{a}{r}\binom{n-r-i}{s-1} + \binom{a}{s}\binom{n-s-i}{r-1}\right) + \binom{k-a}{r+s}\binom{r+s}{r}, & r \neq s. \end{cases}$$

For $B \subset V(H_{n,k,a})$, let $B = \{b_1, b_2, \ldots, b_{n-k+a}\}$. Note that for $i \in [n-k+a]$, the number of copies of $K_{r,s}$ containing b_i and not containing $b_1, \ldots, b_{i-1}$ is $\binom{a}{r}\binom{n-r-i}{r-1}$ when $r = s$ and $\binom{a}{r}\binom{n-r-i}{s-1} + \binom{a}{s}\binom{n-s-i}{r-1}$ when $r \neq s$. Hence, the number of copies of $K_{r,s}$ in $H_{n,k,a}$ is $g_{r,s}(n,k,a)$.

Theorem 9. *Let $n \geqslant k \geqslant 5$ and $t = \lfloor (k-1)/2 \rfloor$. Then*

$$\mathrm{ex}_{\text{2-con}}(n, K_{r,s}, \mathcal{C}_{\geqslant k}) = \max\{g_{r,s}(n,k,2), g_{r,s}(n,k,t)\}.$$

In connected P_k-free graphs, we have the following result.

Theorem 10. *Let $n \geqslant k \geqslant 4$ and $t = \lfloor (k-2)/2 \rfloor$. Then*

$$\mathrm{ex}_{\text{con}}(n, K_{r,s}, P_k) = \max\{g_{r,s}(n,k-1,1), g_{r,s}(n,k-1,t)\}.$$

Very recently, Chakraborti and Chen [3] determined extremal graphs for the maximum number of cliques in n-vertex P_k-free graphs for all values of n. Their proof based on some convexity inequalities along with a recent generalized extremal result on maximizing the number of cliques in a graph with a given maximum degree. By Theorem 10, we have the following corollaries determining the maximum number of $K_{r,s}$ in n-vertex P_k-free and M_k-free graphs for all values of n. Let $n = \alpha(k-1) + \beta$ with $0 \leqslant \beta \leqslant k-2$. Define $f_{r,s}(n,k-1) = N(K_{r,s}, \alpha K_{k-1} \cup K_\beta)$.

Corollary 11. *Let $n \geqslant k \geqslant 4$, $t = \lfloor (k-2)/2 \rfloor$ and $s \geqslant r \geqslant 2$. Then*

$$\mathrm{ex}(n, K_{r,s}, P_k) = \max\{g_{r,s}(n,k-1,t), f_{r,s}(n,k-1)\}.$$

Corollary 12. *Let $n \geqslant 2k \geqslant 4$ and $s \geqslant r \geqslant 1$. Then*

$$\mathrm{ex}(n, K_{r,s}, M_k) = \max\{N(K_{r,s}, K_{2k-1}), g_{r,s}(n, 2k-1, k-1)\}.$$

3 Proof of Main Results

We need the following lemmas, proposition and definition.

Lemma 13 (Pósa [16])**.** *Let G be a 2-connected n-vertex graph and P be a path on m vertices with endpoints x and y. For $v \in V(G)$, let $d_P(v) = |N(v) \cap V(P)|$. Then G contains a cycle of length at least* $\min\{m, d_P(x) + d_P(y)\}$.

The following two lemmas are well-known.

Lemma 14. *For positive integer r, $\binom{a}{r}$ is a convex function of a.*

Lemma 15. *Let f_1, f_2 be convex functions and g an affine function of a. Then $f_1 + f_2, f_1 * f_2, f_1(g)$ are also convex functions of a.*

Proposition 16. *$g_{r,s}(n, k, a)$ is a convex function of a.*

Proof. Note that the number of copies of $K_{r,r}$ inside $A \cup C$ and not inside $A \cup C$ are $\frac{1}{2}\binom{k-a}{2r}\binom{2r}{r}$ and $\binom{a}{r}\left(\binom{n-r}{r} - \binom{k-a-r}{r}\right)$ respectively. Also note that the number of copies of $K_{r,s}$ inside $A \cup C$ and not inside $A \cup C$ are $\binom{k-a}{s+r}\binom{s+r}{r}$ and $\binom{a}{s}\left(\binom{n-s}{r} - \binom{k-a-s}{r}\right) + \binom{a}{r}\left(\binom{n-r}{s} - \binom{k-a-r}{s}\right)$ respectively. It can be checked that

$$g_{r,s}(n,k,a) = \begin{cases} \frac{1}{2}\binom{k-a}{2r}\binom{2r}{r} + \binom{a}{r}\left(\binom{n-r}{r} - \binom{k-a-r}{r}\right), & r = s; \\ \binom{k-a}{s+r}\binom{s+r}{r} + \binom{a}{s}\left(\binom{n-s}{r} - \binom{k-a-s}{r}\right) + \binom{a}{r}\left(\binom{n-r}{s} - \binom{k-a-r}{s}\right), & r \neq s. \end{cases}$$

By Lemmas 14 and 15, $g_{r,s}(n, k, a)$ is a convex function of a. □

Definition 17 (Kopylov [12])**.** *Let G be a graph and α be a natural number. Delete all vertices of degree at most α from G; for the resulting graph G', again delete all vertices of degree at most α from it. We keep running this progress until the minimum degree of the resulting graph is at least $\alpha + 1$. The resulting graph, denoted by $H(G, \alpha)$, is called the $(\alpha + 1)$-core of G.*

Now we begin the proof of Theorem 9.

Proof of Theorem 9:

Proof. Let $n \geqslant k \geqslant 5$, $t = \lfloor (k-1)/2 \rfloor$. Let G be an edge-maximal counter-example, i.e., adding any additional edge to G creates a cycle of length at least k and

$$N(K_{r,s}, G) > \max\{g_{r,s}(n, k, 2), g_{r,s}(n, k, t)\}. \tag{1}$$

Thus for each pair of nonadjacent vertices u and v of G, there is a path on at least k vertices starting from u and ending at v. We have that

Claim 1. $H(G, t)$ is not empty.

Proof. Suppose $H(G, t)$ is empty. For convenience, we divide the proof into the following two cases.

Case 1. $\boldsymbol{r = s}$**.** In the process of getting $H(G, t)$, for the first $n - t$ vertices, once the i-th vertex of degree at most t is deleted, we delete at most $\binom{t}{r}\binom{n-r-i}{r-1}$ copies of $K_{r,r}$;

for all of the last t vertices, we delete at most $\frac{1}{2}\binom{t}{2r}\binom{2r}{r}$ copies of $K_{r,r}$. Thus we have the following upper bound on $N(K_{r,r}, G)$:

$$\begin{aligned} N(K_{r,r}, G) &\leqslant \sum_{i=1}^{n-t} \binom{t}{r}\binom{n-r-i}{r-1} + \frac{1}{2}\binom{t}{2r}\binom{2r}{r} \\ &= g_{r,r}(n, k, t), \end{aligned}$$

a contradiction to (1). Thus $H(G, t)$ is not empty.

Case 2. $\boldsymbol{r \neq s}$**.** In the process of getting $H(G, t)$, for the first $n - t$ vertices, once the i-th vertex of degree at most t is deleted, we delete at most $\binom{t}{r}\binom{n-r-i}{s-1} + \binom{t}{s}\binom{n-s-i}{r-1}$ copies of $K_{s,r}$; for all of the last t vertices, we delete at most $\binom{t}{s+r}\binom{s+r}{r}$ copies of $K_{s,r}$. Thus we have the following upper bound on $N(K_{r,s}, G)$:

$$\begin{aligned} N(K_{r,s}, G) &\leqslant \sum_{i=1}^{n-t} \left(\binom{t}{r}\binom{n-r-i}{s-1} + \binom{t}{s}\binom{n-s-i}{r-1} \right) + \binom{t}{s+r}\binom{s+r}{r} \\ &= g_{r,s}(n, k, t), \end{aligned}$$

a contradiction to (1). Hence $H(G, t)$ is not empty. □

Claim 2. $H(G, t)$ is a clique.

Proof. Suppose there are two nonadjacent vertices in $H(G, t)$, then there is a path of length at least $k - 1$ between these two vertices. Among all these nonadjacent pairs of vertices in $H(G, t)$, we choose $x, y \in H(G, t)$ such that the path between them is the longest. Then by the maximality of P, all neighbors of x in $H(G, t)$ lie in P. Similarly, all neighbors of y in $H(G, t)$ lie in P. Hence by Lemma 13, G has a cycle of length at least $\min\{k, d_P(x) + d_P(y)\} = \min\{k, 2(t + 1)\} = k$, a contradiction. □

Claim 3. Let $\ell = |V(H(G, t))|$. Then $2 \leqslant k - \ell \leqslant t$.

Proof. Since each vertex of $H(G, t)$ has degree at least $t + 1$, we have $\ell \geqslant t + 2$. Note that G is 2-connected and $H(G, t)$ is a clique. If $\ell \geqslant k - 1$, then since G is 2-connected, there is a cycle of length at least $\ell + 1 \geqslant k$, a contradiction. Hence $t + 2 \leqslant \ell \leqslant k - 2$, i.e., $2 \leqslant k - \ell \leqslant t$. □

Claim 4. $H(G, t) \neq H(G, k - \ell)$.

Proof. Suppose $H(G, t) = H(G, k - \ell)$. As in the proof of Claim 1, we divide the proof into the following two cases:

Case 1. $\boldsymbol{r = s}$**.** Then the number of copies of $K_{r,r}$ can be estimated as follows.

$$\begin{aligned}N(K_{r,r},G) &\leqslant \sum_{i=1}^{n-\ell}\binom{k-\ell}{r}\binom{n-r-i}{r-1}+\frac{1}{2}\binom{\ell}{2r}\binom{2r}{r}\\ &= g_{r,r}(n,k,k-\ell)\\ &\leqslant \max\{g_{r,r}(n,k,2),g_{r,r}(n,k,t)\},\end{aligned}$$

where the last inequality is obtained from Proposition 16, a contradiction to (1). Thus we have $H(G,t)\neq H(G,k-\ell)$.

Case 2. $\boldsymbol{r \neq s}$**.** We count the number of copies of $K_{r,s}$ as follows.

$$\begin{aligned}N(K_{r,s},G) &\leqslant \sum_{i=1}^{n-\ell}\left(\binom{k-\ell}{r}\binom{n-r-i}{s-1}+\binom{k-\ell}{s}\binom{n-s-i}{r-1}\right)+\binom{\ell}{s+r}\binom{s+r}{r}\\ &= g_{r,s}(n,k,k-\ell)\\ &\leqslant \max\{g_{r,s}(n,k,2),g_{r,s}(n,k,t)\},\end{aligned}$$

where the last inequality is obtained from Proposition 16, a contradiction to (1). Hence we have $H(G,t)\neq H(G,k-\ell)$. □

Claim 5. G contains a cycle of length at least k.

Proof. Note that $H(G,t)\subseteq H(G,k-\ell)$. By Claim 4, $H(G,t)$ is a proper subgraph of $H(G,k-\ell)$ and there must be a vertex in $H(G,t)$ and a vertex in $H(G,k-\ell)$ that are nonadjacent. Among all such pairs of vertices, we choose $x\in V(H(G,t))$ and $y\in V(H(G,K-\ell))$ such that there is a longest path P between them. Then P contains at least k vertices, and all neighbors of x in $H(G,t)$ and all neighbors of y in $H(G,k-\ell)$ lie in P. Then by Lemma 13, G contains a cycle of length at least $\min\{k,d_P(x)+d_P(y)\}=\min\{k,\ell-1+k-\ell+1\}=k$. This finishes the proof of Claim 5. □

Claim 5 contradicts our assumption. Hence $N(K_{r,s},G)\leqslant \max\{g_{r,s}(n,k,2),g_{r,s}(n,k,t)\}$. The proof is complete. □

Proof of Theorem 10:

Proof. Let $n\geqslant k\geqslant 4$ and $t=\lfloor (k-2)/2\rfloor$. Suppose for contradiction that

$$N(K_{r,s},G)>\max\{g_{s,r}(n,k-1,1),g_{s,r}(n,k-1,t)\}. \tag{2}$$

Let G_0 be the graph obtained from G by adding a dominating vertex v_0 adjacent to all vertices of G. Then G_0 is 2-connected, has $n+1$ vertices and contains no cycle of length $k+1$ or more. Let G' be the $(k+1)$-closure of G_0, i.e., add edges to G_0 until any additional edge creates a cycle of length at least $k+1$. Let $G^*=G'-\{v_0\}$. Thus

$N(K_{r,s},G^*) \geqslant N(K_{r,s},G_0-\{v_0\}) \geqslant N(K_{r,s},G)$. Now we show that $H(G',t+1)$ is not empty. Suppose $H(G',t+1)$ is empty. We divide the proof into the following two cases.

Case 1. $\boldsymbol{r=s}$**.** Note that v_0 is adjacent to each vertex of G^* and $k\in\{2t+2,2t+3\}$. We have

$$\begin{aligned}N(K_{r,r},G^*) &\leqslant \sum_{i=1}^{n-t}\binom{t}{r}\binom{n-r-i}{r-1}+\frac{1}{2}\binom{t}{2r}\binom{2r}{r}\\ &\leqslant g_{r,r}(n,k-1,t),\end{aligned}$$

a contradiction to (2).

Case 2. $\boldsymbol{r\neq s}$**.** Similarly, we have

$$\begin{aligned}N(K_{r,s},G^*) &\leqslant \sum_{i=1}^{n-t}\left(\binom{t}{r}\binom{n-r-i}{s-1}+\binom{t}{s}\binom{n-s-i}{r-1}\right)+\binom{t}{s+r}\binom{s+r}{r}\\ &\leqslant g_{r,s}(n,k-1,t),\end{aligned}$$

a contradiction to (2). Thus $H(G',t+1)$ is not empty.

The same argument as in the proof of Theorem 9 also shows that $H(G',t+1)$ is a clique, otherwise there would be a cycle of length at least $2(t+2)\geqslant (k-1)+2$ in G', a contradiction. Note that v_0 must be contained in $H(G',t+1)$ as it is adjacent to all other vertices of G'. Let $|V(H(G',t+1))|=\ell$, where $t+3\leqslant \ell\leqslant k-1$. And so $k-\ell\geqslant 1$. In particular, $k+1-\ell\leqslant t+1$. If $H(G',t+1)\neq H(G',k+1-\ell)$, then again we can find a cycle of length at least $\ell-1+k+2-\ell=k+1$ in G', a contradiction. Otherwise, suppose $H(G',t+1)=H(G',k+1-\ell)$. We will finish our proof in the following two cases.

Case 1. $\boldsymbol{r=s}$**.** In $H(G',t+1)$, the number of $K_{r,r}$ that do not include v_0 is $\frac{1}{2}\binom{\ell-1}{2r}\binom{2r}{r}$. In $G'-H(G',k+\ell-1)$, every vertex had at most $k-\ell$ neighbors that were not v_0 at the time of its deletion. We have

$$\begin{aligned}N(K_{r,r},G^*) &\leqslant \sum_{i=1}^{n+1-\ell}\binom{k-\ell}{r}\binom{n-r-i}{r-1}+\frac{1}{2}\binom{\ell-1}{2r}\binom{2r}{r}\\ &= g_{r,r}(n,k-1,k-\ell)\\ &\leqslant \max\{g_{r,r}(n,k-1,1),g_{r,r}(n,k-1,t)\},\end{aligned}$$

where the last inequality is obtained from Proposition 16, a contradiction.

Case 2. $\boldsymbol{r\neq s}$**.** Similarly, we have

$$\begin{aligned}N(K_{r,s},G^*) &\leqslant \sum_{i=1}^{n+1-\ell}\left(\binom{k-\ell}{r}\binom{n-r-i}{s-1}+\binom{k-\ell}{s}\binom{n-s-i}{r-1}\right)\\ &\quad+\binom{\ell-1}{s+r}\binom{s+r}{r}\\ &= g_{r,s}(n,k-1,k-\ell)\\ &\leqslant \max\{g_{r,s}(n,k-1,1),g_{s,r}(n,k-1,t)\},\end{aligned}$$

where the last inequality is obtained from Proposition 16, a contradiction. This completes the proof of Theorem 10. $\square$

To prove Corollary 11, we need the following lemmas.

Lemma 18. *Let a, b, k and m be positive integers with $a \leqslant b \leqslant k-1$. Then $\binom{a}{m} + \binom{b}{m} \leqslant \binom{k-1}{m} + \binom{a+b-k+1}{m}$.*

Proof. We only need to prove $\binom{a}{m} - \binom{a+b-k+1}{m} \leqslant \binom{k-1}{m} - \binom{b}{m}$. If $k-1 < m$ or $b = k-1$, then the inequality holds trivially by $a \leqslant b \leqslant k-1$. So assume $k-1 \geqslant m$ and $b < k-1$. For positive integer x, let $f(x) = \binom{x+k-1-b}{m} - \binom{x}{m}$. Then

$$\begin{aligned} f(x+1) - f(x) &= \binom{x+1+k-1-b}{m} - \binom{x+1}{m} - \binom{x+k-1-b}{m} + \binom{x}{m} \\ &= \binom{x+k-1-b}{m-1} - \binom{x}{m-1} \\ &\geqslant 0. \end{aligned}$$

So $f(x+1) \geqslant f(x)$ and $f(x)$ is nondecreasing of x. So $f(a+b-k+1) \leqslant f(b)$. We complete the proof of Lemma 18. $\square$

Lemma 19. *Let $t = \lfloor (k-2)/2 \rfloor$. There is a positive integer n_0 such that*

$$\mathrm{ex}_{\mathrm{con}}(n, K_{r,s}, P_k) = \begin{cases} g_{r,s}(n, k-1, t), & n \geqslant n_0; \\ g_{r,s}(n, k-1, 1), & n < n_0. \end{cases}$$

Proof. It is enough to prove that, for positive integer n, if $g_{r,s}(n, k-1, t) \geqslant g_{r,s}(n, k-1, 1)$, then $g_{r,s}(n+1, k-1, t) \geqslant g_{r,s}(n+1, k-1, 1)$. For $r = s$, it follows from the definition of $g_{r,r}$ that

$$g_{r,r}(n+1, k-1, t) - g_{r,r}(n, k-1, t) = \binom{t}{r}\binom{n-r}{r-1},$$

and

$$g_{r,r}(n+1, k-1, 1) - g_{r,r}(n, k-1, 1) = \binom{1}{r}\binom{n-r}{r-1}.$$

So we have $g_{r,r}(n+1, k-1, t) - g_{r,r}(n, k-1, t) \geqslant g_{r,r}(n+1, k-1, 1) - g_{r,r}(n, k-1, 1)$. Thus $g_{r,r}(n+1, k-1, t) \geqslant g_{r,r}(n+1, k-1, 1)$ provided $g_{r,r}(n, k-1, t) \geqslant g_{r,r}(n, k-1, 1)$. If $r < s$, then the proof is similar and is omitted. $\square$

Lemma 20. *Let $t = \lfloor (k-2)/2 \rfloor$. For two positive integers n_1, n_2, we have $g_{r,s}(n_1, k-1, t) + g_{r,s}(n_2, k-1, t) \leqslant g_{r,s}(n_1 + n_2, k-1, t)$.*

Proof. It can be easily checked. So we omit the proof here. $\square$

Lemma 21. *For positive integers x, y, a, with $a \geqslant 2$ and $x \geqslant y$, we have $\binom{x}{a} + \binom{y}{a} \leqslant \binom{x+y-1}{a}$.*

Proof. If $y < a$, then the inequality holds. So we suppose $y \geqslant a$. We prove the lemma by induct on y. If $y = a$, then the inequality holds. Assume the lemma holds for $y = y_1$, i.e., $\binom{x}{a} + \binom{y_1}{a} \leqslant \binom{x+y_1-1}{a}$. Since $\binom{y_1+1}{a} - \binom{y_1}{a} \leqslant \binom{x+y_1}{a} - \binom{x+y_1-1}{a}$, we have $\binom{x}{a} + \binom{y_1+1}{a} \leqslant \binom{x+y_1}{a}$ by Lemma 18. So the inequality holds for $y = y_1 + 1$. We complete the proof of Lemma 21. □

Proof of Corollary 11:

Proof. Let $n \geqslant k \geqslant 4$, $t = \lfloor (k-2)/2 \rfloor$ and $s \geqslant r \geqslant 2$. Let G be a P_k-free n-vertex graph and n_0 be the positive integer determined by Lemma 19. We divide the proof into the following two cases basing on the value of n.

Case 1. $n < n_0$.

Let $n = \alpha(k-1) + \beta$ and $0 \leqslant \beta \leqslant k-2$. We show that $N(K_{r,s}, G) \leqslant N(K_{r,s}, \alpha K_{k-1} \cup K_\beta)$. We induct on the number of components in G. If G is connected, then $n < k$ and

$$N(K_{r,s}, G) = N(K_{r,s}, K_n) \leqslant N(K_{r,s}, \alpha K_{k-1} \cup K_\beta),$$

or $n \geqslant k$ and

$$N(K_{r,s}, G) \leqslant g_{r,s}(n, k-1, 1)$$

by Theorem 10 and Lemma 19. Note that when $s \geqslant r \geqslant 2$, we have

$$g_{r,s}(n, k-1, 1) = N(K_{r,s}, K_{k-2}) \leqslant N(K_{r,s}, K_{k-1}) \leqslant N(K_{r,s}, \alpha K_{k-1} \cup K_\beta).$$

Otherwise if G is not connected, let C_1 be a component of G with n_1 vertices, $n_1 = \alpha_1(k-1) + \beta_1, 0 \leqslant \beta_1 \leqslant k-2$, $n - n_1 = \alpha_2(k-1) + \beta_2, 0 \leqslant \beta_2 \leqslant k-2$. Then

$$\begin{aligned} N(K_{r,s}, G) &= N(K_{r,s}, C_1) + N(K_{r,s}, G - C_1) \\ &\leqslant N(K_{r,s}, \alpha_1 K_{k-1} \cup K_{\beta_1}) + N(K_{r,s}, \alpha_2 K_{k-1} \cup K_{\beta_2}) \\ &\leqslant N(K_{r,s}, \alpha K_{k-1} \cup K_\beta), \end{aligned}$$

where the last inequality is obtained by Lemma 18.

Case 2. $n \geqslant n_0$.

If G is connected, then we have $\mathrm{ex}_{\mathrm{con}}(n, K_{r,s}, P_k) = g_{r,s}(n, k-1, t)$ by Theorem 10 and Lemma 19. Suppose G is not connected and $C_i, i \in [p]$, are the components of G with n_i vertices, respectively. If $n_i < n_0$ for all $i \in [p]$, then

$$N(K_{r,s}, G) \leqslant N(K_{r,s}, \alpha K_{k-1} \cup K_\beta)$$

by Case 1 and Lemma 18. If $n_i \geqslant n_0$ for all $i \in [p]$, then

$$N(K_{r,s}, G) \leqslant \sum_{i=1}^{p} g_{r,s}(n_i, k-1, t) \leqslant g_{r,s}(n, k-1, t)$$

by Lemma 20. If $n_i \geqslant n_0$ for $i \in [q]$ and $n_i < n_0$ for $i \in [p] \setminus [q]$, then by Lemma 20, we may assume $q = 1$. Then by Case 1,

$$N(K_{r,s}, G) \leqslant N(K_{r,s}, C_1) + N(K_{r,s}, G - C_1) \leqslant g_{r,s}(n_1, k-1, t) + f_{r,s}(n - n_1, k-1).$$

We now prove that $g_{r,s}(n_1, k-1, t) + f_{r,s}(n-n_1, k-1) \leqslant \max\{g_{r,s}(n, k-1, t), f_{r,s}(n, k-1)\}$ for any $k \leqslant n_1 \leqslant n$. Assume that $r = s$. For the case $r \neq s$, the proof is similar and is omitted.

If $g_{r,s}(n_1, k-1, t) \leqslant f_{r,s}(n_1, k-1)$, then by Lemma 18,

$$g_{r,s}(n_1, k-1, t) + f_{r,s}(n - n_1, k-1) \leqslant f_{r,s}(n_1, k-1) + f_{r,s}(n - n_1, k-1) \leqslant f_{r,s}(n, k-1).$$

So suppose that

$$g_{r,s}(n_1, k-1, t) > f_{r,s}(n_1, k-1). \tag{3}$$

Let $n - n_1 = \alpha'(k-1) + \beta'$ where $\alpha' \geqslant 0, 0 \leqslant \beta' \leqslant k-2$ and $n_1 = \alpha''(k-1) + \beta''$ where $1 \leqslant \alpha'', 1 \leqslant \beta'' \leqslant k-1$. Let $H = H(n_1, k-1, t)$ with the vertex sets A, B, C as its definition. We order the vertices of H in A, B, C with $v_1, \ldots, v_t, v_{t+1}, \ldots, v_{k-1-t}, v_{k-t}, \ldots, v_{n_1}$ successively. Let S_0 be the vertex set of the first β'' vertices in H and H_0 be the subgraph of H induced by S_0. Then $H_0 \subseteq K_{\beta''}$ and

$$N(K_{r,r}, H_0) \leqslant N(K_{r,r}, K_{\beta''}). \tag{4}$$

We divide the remaining vertices of H into α'' sets of size $k-1$ as its order. Let S_i be the vertex set of the i-th $k-1$ vertices of $H - H_0$ and H_i be the subgraph induced by $\bigcup_{j=0}^{i} S_j$. Let $N_i = N(K_{r,r}, H_i) - N(K_{r,r}, H_{i-1})$ for $i = 1, \ldots, \alpha''$. Let X_i be the number of $K_{r,r}$ in H containing v_i but not containing $v_{i+1}, \ldots, v_{n_1}$ for $i = 1, \ldots, n_1$. It can be checked that if k is even, then $t = (k-2)/2$, $|B| = 1$ and X_i is nondecreasing. Hence if $\alpha'' \geqslant 2$, then

$$N_{i+1} \geqslant N_i \text{ for } i = 1, \ldots, \alpha'' - 1. \tag{5}$$

If k is odd, then $t = (k-3)/2$, $|B| = 2$. It can be checked that $X_{k-1} \geqslant X_{t+2}$ and $X_{i+1} \geqslant X_i$ for $i \in [n_1 - 1] \setminus \{t+2\}$. So if $\alpha'' \geqslant 2$, (5) also holds. By (3) and (4), we have that

$$\sum_{i=1}^{\alpha''} N_i > N(K_{r,r}, \alpha'' K_{k-1}) = \alpha'' N(K_{r,r}, K_{k-1}).$$

By (5), we have $N_{\alpha''} > N(K_{r,r}, K_{k-1})$. Furthermore, adding additional $k-1$ vertices (if any) to C of H and joining the new $k-1$ vertices with A of H will produce more number of $K_{r,r}$ than $N_{\alpha''}$. Hence

$$g_{r,r}(n_1, k-1, t) + f_{r,r}(n - n_1, k-1) \leqslant g_{r,r}(n - \beta', k-1, t) + N(K_{r,r}, K_{\beta'}).$$

If $\beta' < 2r$, then we are done. So suppose that $2r \leqslant \beta' < k-1$.

Note that

$$N(K_{r,r}, K_{\beta'}) \leqslant \frac{k-1}{\beta'} N(K_{r,r}, K_{k-1})$$

and

$$N(K_{r,r}, H(n, k-1, t)) - N(K_{r,r}, H(n-\beta', k-1, t)) \geqslant \frac{k-1}{\beta'} N_{\alpha''}.$$

Hence

$$g_{r,r}(n-\beta', k-1, t) + N(K_{r,r}, K_{\beta'}) \leqslant g_{r,r}(n, k-1, t).$$

We complete the proof of the corollary. □

Corollary 22. *Let $n \geqslant 2k \geqslant 4$. Then*

$$\mathrm{ex}_{\mathrm{con}}(n, K_{r,s}, M_k) = \max\{g_{r,s}(n, 2k-1, 1), g_{r,s}(n, 2k-1, k-1)\}.$$

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the classes of connected M_k-free graphs and connected P_{2k}-free graphs on n vertices, respectively. We know that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. So $\mathrm{ex}_{\mathrm{con}}(n, K_{r,s}, M_k) \leqslant \mathrm{ex}_{\mathrm{con}}(n, K_{r,s}, P_{2k})$. Note that $H(n, 2k-1, 1), H(n, 2k-1, t)$ are M_k-free and equality holds if G is $H(n, 2k-1, 1)$ or $H(n, 2k-1, k-1)$. So $\mathrm{ex}_{\mathrm{con}}(n, K_{r,s}, M_k) = \max\{g_{r,s}(n, 2k-1, 1), g_{r,s}(n, 2k-1, k-1)\}$. □

Proof of Corollary 12: Let $n \geqslant 2k \geqslant 4$ and $s \geqslant r \geqslant 1$. Let G be an extremal graph for M_k on n vertices.

Assume G is not connected. Let $i = 1, \ldots, p$ and C_i be the components of G with n_i vertices and $k_i > 0$ matchings, respectively. We may suppose $\sum_{i=1}^{p} k_i = k-1$.

We partition the components of G into two classes.

- **Class I.** Components C_i such that $n_i \leqslant 2k_i + 1$, that is $n_i = 2k_i$ or $2k_i + 1$.
- **Class II.** Components C_i such that $n_i \geqslant 2k_i + 2$.

First we show that if there is a component of Class I, then there will not be any component of Class II. Assume C_1 and C_2 are components of Class I and Class II, respectively. Then $n_1 = 2k_1$ or $2k_1 + 1$ and $N(K_{r,s}, C_1) \leqslant N(K_{r,s}, K_{n_1})$.

$$\begin{aligned} N(K_{r,s}, C_2) &\leqslant \max\{g_{r,s}(n_2, 2k_2+1, 1), g_{r,s}(n, 2k_2+1, k_2)\} \\ &= \max\{N(K_{r,s}, H(n_2, 2k_2+1, 1)), N(K_{r,s}, H(n_2, 2k_2+1, k_2))\}. \end{aligned}$$

Let A, B, C and A', B', C' be the vertex sets of $H(n_2, 2k_2+1, 1)$ and $H(n_2, 2k_2+1, k_2)$, respectively. Let H_1 (H_2) be the graph obtained from C_1 and $H(n_2, 2k_2+1, 1)$ ($H(n_2, 2k_2+1, k_2)$) by adding edges between C_1 and A (A'). Then H_1, H_2 are $M_{k_1+k_2+1}$-free and

$$\begin{aligned} N(K_{r,s}, C_1 \cup C_2) &\leqslant \max\{N(K_{r,s}, H_1), N(K_{r,s}, H_2)\} \\ &< \mathrm{ex}_{\mathrm{con}}(n_1+n_2, K_{r,s}, M_{k_1+k_2+1}), \end{aligned}$$

contradicting the maximality of G.

Assume all components belong to Class I. Then $N(K_{r,s}, \bigcup_{i=1}^{p} C_i) \leqslant N(K_{r,s}, \bigcup_{i=1}^{p} K_{n_i})$. If there is one such components, such as C_1, have even number of vertices, then let C' be the graph obtained from C_1, C_2 by adding the edges between C_1 and C_2. C' is a $M_{k_1+k_2+1}$-free graph with n_1+n_2 vertices. $N(K_{r,s}, C_1 \cup C_2) \leqslant N(K_{r,s}, C')$. So we may suppose that all the components have odd number of vertices, that is, $n_i = 2k_i + 1$. Then

$$N\left(K_{r,s}, \bigcup_{i=1}^{p} C_i\right) \leqslant N\left(K_{r,s}, \bigcup_{i=1}^{p} K_{2k_i+1}\right) < N(K_{r,s}, K_{2k-1}) \tag{6}$$

The last inequality is obtained by Lemma 21.

Assume all components belong to Class II. Suppose $\mathrm{ex}(n_i, K_{r,s}, M_{k_i+1}) = g_{r,s}(n_i, 2k_i + 1, 1) = N(K_{r,s}, H(n_i, 2k_i + 1, 1))$ for $i \in [q]$ ($q = 0$ means that such i does not exist) and $\mathrm{ex}(n_i, K_{r,s}, M_{k_i+1}) = g_{r,s}(n_i, 2k_i + 1, k_i) = N(K_{r,s}, H(n_i, 2k_i + 1, k_i))$ for $i \in [p] \setminus [q]$. Let

$$G_i = \begin{cases} H(n_i, 2k_i + 1, 1), & i \in [q]; \\ H(n_i, 2k_i + 1, k_i), & i \in [p] \setminus [q]. \end{cases}$$

Then $N(K_{r,s}, G) \leqslant N(K_{r,s}, \bigcup_{i=1}^{p} G_i)$. Let A_i, B_i and C_i be the vertex sets of G_i as the definition of $H(n, k, a)$, respectively. Let G' be the graph obtained from $\bigcup_{i=1}^{p} G_i$ by adding the edges between A_i and C_{i+1} for $i = 1, \dots, p-1$. Note that G' is also M_k-free and the number of copies of $K_{r,s}$ does not decrease. So we may suppose G is connected. By Corollary 22, we have

$$N(K_{r,s}, G) \leqslant \max\{g_{r,s}(n, 2k-1, 1), g_{r,s}(n, 2k-1, k-1)\}. \tag{7}$$

Hence combining (6) and (7), we have

$$N(K_{r,s}, G) \leqslant \max\{N(K_{r,s}, K_{2k-1}), g_{r,s}(n, 2k-1, 1), g_{r,s}(n, 2k-1, k-1)\}.$$

We now prove that $g_{r,s}(n, 2k-1, 1) \leqslant \max\{N(K_{r,s}, K_{2k-1}), g_{r,s}(n, 2k-1, k-1)\}$. If $k = 2$, then we are done. If $s \geqslant r \geqslant 2$, then

$$g_{r,s}(n, 2k-1, 1) = N(K_{r,s}, H(n, 2k-1, 1)) = N(K_{r,s}, K_{2k-2}) \leqslant N(K_{r,s}, K_{2k-1}),$$

the result follows.

Thus we may assume that $s \geqslant 2$, $r = 1$ and $k \geqslant 3$. Note that both K_{2k-1} and $H_{n,2k-1,1}$ contain K_{2k-2} as a subgraph. Counting the number of copies of $K_{1,s}$ with given center, we have

$$N(K_{r,s}, K_{2k-1}) = (2k-1)\binom{2k-2}{s},$$

$$g_{r,s}(n, 2k-1, 1) = \binom{n-1}{s} + (2k-3)\binom{2k-3}{s}$$

and

$$g_{r,s}(n, 2k-1, k-1) = (k-1)\binom{n-1}{s} + (n-k+1)\binom{k-1}{s}.$$

Suppose that $g_{r,s}(n, 2k-1, 1) \geqslant N(K_{r,s}, K_{2k-1})$, that is $\binom{n-1}{s} \geqslant (2k-1)\binom{2k-2}{s} - (2k-3)\binom{2k-3}{s} = (s+1)\binom{2k-2}{s} + \binom{2k-3}{s}$. Then, we have

$$\begin{aligned}
& g_{r,s}(n, 2k-1, k-1) - g_{r,s}(n, 2k-1, 1) \\
= \; & (k-1)\binom{n-1}{s} + (n-k+1)\binom{k-1}{s} - \binom{n-1}{s} - (2k-3)\binom{2k-3}{s} \\
= \; & (k-2)\binom{n-1}{s} + (n-k+1)\binom{k-1}{s} - (2k-3)\binom{2k-3}{s} \\
\geqslant \; & (k-2)(s+1)\binom{2k-2}{s} - (k-1)\binom{2k-3}{s} + (n-k+1)\binom{k-1}{s} \\
\geqslant \; & 0,
\end{aligned}$$

where the last inequality holds by $k \geqslant 3$ and $s \geqslant 2$. The proof is complete. □

4 Concluding Remarks

In [4], Erdős and Gallai showed the extremal graphs for the maximum number of edges in n-vertex P_k-free graphs are the $n/(k-1)$ disjoint unions of cliques of size $k-1$, where $k-1$ divides n. In [13], Luo showed these graphs are also the extremal examples for the maximum number of cliques in n-vertex P_k-free graphs. But as we show in Corollary 11, if n is sufficiently large, the extremal graphs for the maximum number of complete bipartite graphs in n-vertex P_k-free graphs are not the same as the above two cases.

Acknowledgements

The authors would like to thank the two anonymous referees for their valuable suggestions which helped us a lot to improve the presentation of this paper.

References

[1] N. Alon and C. Shikhelman. Many T copies in H-free graphs. *J. Combin. Theory Ser. B*, 121:146–172, 2016.

[2] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp. Connected graphs without long paths. *Discrete Math.*, 308(19):4487–4494, 2008.

[3] D. Chakraborti and D. Q. Chen. Exact results on generalized Erdős-Gallai problems. arXiv:2006.04681v2, 2020.

[4] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Mathematica Hungarica*, 10(3-4):337–356, 1959.

[5] D. Gerbner. Generalized Turán problems for small graphs. arXiv:2006.16150v1, 2020.

[6] D. Gerbner, E. Győri, A. Methuku and M. Vizer. Generalized Turán numbers for even cycles. *J. Combin. Theory Ser. B*, 145:169–213, 2020.

[7] D. Gerbner, A. Methuku, and M. Vizer. Generalized Turán problems for disjoint copies of graphs. *Discrete Math.*, 342(11): 3130–3141, 2019.

[8] D. Gerbner and C. Palmer. Counting copies of a fixed subgraph in F-free graphs. *European J. Combin.*, 82:103001, 2019.

[9] L. Gishboliner and A. Shapira. A generalized Turán problem and its applications. *Int. Math. Res. Not.*, 2020(11):3417–3452, 2020.

[10] E. Győri and H. Li. The maximum number of triangles in C_{2k+1}-free graph. *J. Combin. Theory Ser. B*, 102:1061–1066, 2012.

[11] E. Győri, N. Salia, C. Tompkins and O. Zamora. The maximum number of P_ℓ copies in P_k-free graphs. *Discrete Math. Theor. Comput. Sci.*, 21 (2019), no. 1, Paper No. 14, 21 pp, 2019.

[12] G. N. Kopylov. Maximal paths and cycles in a graph. *Dokl. Akad. Nauk SSSR*, 234:19–21, 1977; English translation: *Sov. Math., Dokl.*, 18(3):593–596, 1977.

[13] R. Luo. The maximum number of cliques in graphs without long cycles. *J. Combin. Theory Ser. B*, 128:219–226, 2017.

[14] J. Ma and Y. Qiu. Some sharp results on the generalized Turán numbers. *European J. Combin.*, 84:103026, 2020.

[15] B. Ning and X. Peng. Extensions of Erdős-Gallai Theorem and Luo's Theorem with applications. *Comb. Probab. Comput.*, 29(1):128–136, 2020.

[16] L. Pósa. A theorem concerning Hamilton lines. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7:225–226, 1962.

[17] J. Wang. The shifting method and generalized Turán number of matching. *European J. Combin.*, 85:103057, 2020.

[18] L. Zhang, L. Wang, and J. Zhou. The generalized Turán number of spanning linear forests. arXiv:2009.00181v1, 2020.