An improved inequality related to Vizing's conjecture

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Abstract

Vizing conjectured in 1963 that $\gamma(G\square H) \geqslant \gamma(G)\gamma(H)$ for any graphs G and H. A graph G is said to satisfy Vizing's conjecture if the conjectured inequality holds for G and any graph H. Vizing's conjecture has been proved for $\gamma(G) \leqslant 3$, and it is known to hold for other classes of graphs. Clark and Suen in 2000 showed that $\gamma(G\square H) \geqslant \frac{1}{2}\gamma(G)\gamma(H)$ for any graphs G and H. We give a slight improvement of this inequality by tightening their arguments.

Keywords. Graph domination, Cartesian product, Vizing's conjecture

We use $V(G), E(G), \gamma(G)$, respectively, to denote the vertex set, edge set and domination number of the (simple) graph G. A γ-set of a graph G is a dominating set of G with minimum cardinality. For graphs G and H, the Cartesian product $G\square H$ is the graph with vertex set $V(G) \times V(H)$ and two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. In 1963, V. G. Vizing [4] conjectured that for any graphs G and H,

$$\gamma(G\square H) \geqslant \gamma(G)\gamma(H).$$

The reader is referred to Hartnell and Rall [3] and Brešar et al. [1] for a summary of the history and recent progress on Vizing's conjecture. Clark and Suen [2] in 2000 showed that for any graphs G and H,

$$\gamma(G\square H) \geqslant \frac{1}{2}\gamma(G)\gamma(H).$$

The following theorem is a slight improvement of this inequality.

Theorem 1. For any graphs G and H, $\gamma(G\square H) \geqslant \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$.

Proof. Let G and H be arbitrary graphs, and let D be a γ-set of the Cartesian product $G\Box H$. Let $\{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be a γ-set of G. We partition $V(G)$ into $\gamma(G)$ sets $\Pi_{1\boldsymbol{.}}, \Pi_{2\boldsymbol{.}}, \ldots, \Pi_{\gamma(G)\boldsymbol{.}}$, where $u_i \in \Pi_{i\boldsymbol{.}}$ for all $i = 1, 2, \ldots, \gamma(G)$ and if $u \in \Pi_{i\boldsymbol{.}}$ then $u = u_i$ or $\{u, u_i\} \in E(G)$.

Let $P_{i\boldsymbol{.}}$ denote the projection of $(\Pi_{i\boldsymbol{.}} \times V(H)) \cap D$ onto H. That is,

$$P_{i\boldsymbol{.}} = \{v \in V(H) \mid (u, v) \in D \text{ for some } u \in \Pi_{i\boldsymbol{.}}\}.$$

Define $C_{i\boldsymbol{.}} = V(H) - N_H[P_{i\boldsymbol{.}}]$ as the complement of $N_H[P_{i\boldsymbol{.}}]$, where $N_H[X]$ is the set of closed neighbors of X in graph H. As $P_{i\boldsymbol{.}} \cup C_{i\boldsymbol{.}}$ is a dominating set of H, we have

$$|P_{i\boldsymbol{.}}| + |C_{i\boldsymbol{.}}| \geqslant \gamma(H), \qquad i = 1, 2, \ldots, \gamma(G). \tag{1}$$

For $v \in V(H)$, let

$$D_{\boldsymbol{.}v} = \{u \mid (u, v) \in D\} \quad \text{and} \quad S_{\boldsymbol{.}v} = \{i \mid v \in C_{i\boldsymbol{.}}\}.$$

Observe that if $i \in S_{\boldsymbol{.}v}$ then the vertices in $\Pi_{i\boldsymbol{.}} \times \{v\}$ are dominated "horizontally" by vertices in $D_{\boldsymbol{.}v} \times \{v\}$. Let S_H be the number of pairs (i, v) where $i = 1, 2, \ldots, \gamma(G)$ and $v \in C_{i\boldsymbol{.}}$. Then obviously

$$S_H = \sum_{v \in V(H)} |S_{\boldsymbol{.}v}| = \sum_{i=1}^{\gamma(G)} |C_{i\boldsymbol{.}}|.$$

Since $D_{\boldsymbol{.}v} \cup \{u_i \mid i \notin S_{\boldsymbol{.}v}\}$ is a dominating set of G, we have

$$|D_{\boldsymbol{.}v}| + (\gamma(G) - |S_{\boldsymbol{.}v}|) \geqslant \gamma(G),$$

giving that

$$|S_{\boldsymbol{.}v}| \leqslant |D_{\boldsymbol{.}v}|. \tag{2}$$

Summing over $v \in V(H)$, we have

$$S_H \leqslant |D|. \tag{3}$$

We now consider two cases based on (1).

Case 1. Assume $|P_{i\boldsymbol{.}}| + |C_{i\boldsymbol{.}}| > \gamma(H)$ for all $i = 1, \ldots, \gamma(G)$. Then as $|(\Pi_{i\boldsymbol{.}} \times V(H)) \cap D| \geqslant |P_{i\boldsymbol{.}}|$, we have

$$\sum_{i=1}^{\gamma(G)} (|C_{i\boldsymbol{.}}| + |(\Pi_{i\boldsymbol{.}} \times V(H)) \cap D|) \geqslant \sum_{i=1}^{\gamma(G)} (\gamma(H) + 1),$$

which implies that

$$S_H + |D| \geqslant \gamma(G)\gamma(H) + \gamma(G). \tag{4}$$

Combining (3) and (4) gives that

$$\gamma(G\Box H) = |D| \geqslant \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(G). \tag{5}$$

Case 2. Assume $|P_{i\cdot}| + |C_{i\cdot}| = \gamma(H)$ for some $i = 1, \ldots, \gamma(G)$. Note that $P_{i\cdot} \cup C_{i\cdot}$ is a γ-set of H. We now use this γ-set of H to partition $V(H)$ in the same way as $V(G)$ is partitioned above. That is, label the vertices in $P_{i\cdot} \cup C_{i\cdot}$ as $v_1, v_2, \ldots, v_{\gamma(H)}$, and let $\{\Pi_{\cdot j} \mid 1 \leqslant j \leqslant \gamma(H)\}$ be a partition of H such that for all $j = 1, \ldots, \gamma(H)$, $v_j \in \Pi_{\cdot j}$ and if $v \in \Pi_{\cdot j}$, either $v = v_j$ or $\{v, v_j\} \in E(H)$. We next define the sets $P_{\cdot j}, C_{\cdot j}, S_{u\cdot}$ and $D_{u\cdot}$ in the same way $P_{i\cdot}, C_{i\cdot}, S_{\cdot v}$ and $D_{\cdot v}$ are defined above. To be specific, for $1 \leqslant j \leqslant \gamma(H)$, let

$$P_{\cdot j} = \{u \in V(G) \mid (u, v) \in D \text{ for some } v \in \Pi_{\cdot j}\}, \quad \text{and} \quad C_{\cdot j} = V(G) - N_G[P_{\cdot j}],$$

and for $u \in V(G)$, let

$$D_{u\cdot} = \{v \mid (u, v) \in D\} \quad \text{and} \quad S_{u\cdot} = \{j \mid u \in C_{\cdot j}\}.$$

Similarly, we have

$$S_G = \sum_{u \in V(G)} |S_{u\cdot}| = \sum_{j=1}^{\gamma(H)} C_{\cdot j}.$$

For $u \in V(G)$, let $\hat{D}_{u\cdot} = \{v_j \mid (u, v_j) \in D_{u\cdot}, 1 \leqslant j \leqslant \gamma(H)\}$. We claim that

$$|S_{u\cdot}| \leqslant |D_{u\cdot}| - |\hat{D}_{u\cdot}|. \tag{6}$$

This is because $D_{u\cdot} \cup \{v_j \mid j \notin S_{u\cdot}\}$ is a dominating set of H, with

$$D_{u\cdot} \cap \{v_j \mid j \notin S_{u\cdot}\} = \hat{D}_{u\cdot},$$

and the argument for proving (6) follows in the same way as (2) is proved. To make use of the claim, we note that when we partition the vertices of H, we have at least $\gamma(H)$ vertices in D that are of the form (u, v_k). Indeed, for each $k = 1, 2, \ldots, \gamma(H)$, either $v_k \in P_{i\cdot}$, which implies $(u, v_k) \in D$ for some $u \in \Pi_{i\cdot}$, or $v_k \in C_{i\cdot}$, which implies that the vertices in $\Pi_{i\cdot} \times \{v_k\}$ are dominated "horizontally" by some vertices $(u', v_k) \in D$. It therefore follows that

$$\sum_{u \in V(G)} |\hat{D}_{u\cdot}| \geqslant \gamma(H),$$

and hence summming both sides of (6)

$$\sum_{u \in V(G)} |S_{u\cdot}| \leqslant \sum_{u \in V(G)} (|D_{u\cdot}| - |\hat{D}_{u\cdot}|)$$

gives that

$$S_G \leqslant |D| - \gamma(H). \tag{7}$$

To complete the proof, we note that similar to (1), we have

$$|P_{\cdot j}| + |C_{\cdot j}| \geqslant \gamma(G), \qquad j = 1, 2, \ldots, \gamma(H),$$

and summing over j gives that

$$|D| + S_G \geqslant \gamma(G)\gamma(H). \tag{8}$$

Combining (7) and (8), we obtain

$$\gamma(G \Box H) \geqslant \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(H). \tag{9}$$

As either (5) or (9) holds, it follows that

$$\gamma(G \Box H) \geqslant \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}. \quad \Box$$

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