On optimal linear codes over $\mathbb{F}_8$

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Abstract

Let $n_q(k,d)$ be the smallest integer n for which there exists an $[n,k,d]_q$ code for given q,k,d. It is known that $n_8(4,d) = \sum_{i=0}^{3} \lceil d/8^i \rceil$ for all $d \geq 833$. As a continuation of Jones et al. [Electronic J. Combinatorics 13 (2006), #R43], we determine $n_8(4,d)$ for 117 values of d with $113 \leq d \leq 832$ and give upper and lower bounds on $n_8(4,d)$ for other d using geometric methods and some extension theorems for linear codes.

1 Introduction

We denote by $\mathbb{F}_q^n$ the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. A *q-ary linear code* $\mathcal{C}$ of *length* n and *dimension* k (an $[n,k]_q$ code) is a k-dimensional subspace of $\mathbb{F}_q^n$. The Hamming distance $d(\boldsymbol{x}, \boldsymbol{y})$ between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_q^n$ is the number of nonzero coordinate positions in $\boldsymbol{x} - \boldsymbol{y}$. The *minimum distance* of a linear code $\mathcal{C}$ is defined by $d(\mathcal{C}) = \min\{d(\boldsymbol{x},\boldsymbol{y}) \mid \boldsymbol{x},\boldsymbol{y} \in \mathcal{C}, \boldsymbol{x} \neq \boldsymbol{y}\}$ which is equal to the minimum weight of $\mathcal{C}$ defined by $wt(\mathcal{C}) = \min\{wt(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{C},\ \boldsymbol{x} \neq \boldsymbol{0}\}$, where $\boldsymbol{0}$ is the all-0-vector and $wt(\boldsymbol{x}) = d(\boldsymbol{x}, \boldsymbol{0})$ is the *weight* of $\boldsymbol{x}$. A q-ary linear code of length n, dimension k and minimum distance d is referred to as an $[n,k,d]_q$ code. The *weight distribution* of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. The weight distribution (w.d. for brevity) with $(A_0, A_d, ...) = (1, \alpha, ...)$ is also expressed as $0^1 d^\alpha \cdots$. A $k \times n$ matrix having as rows the vectors of a basis of $\mathcal{C}$ is called a *generator matrix* of $\mathcal{C}$.

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A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists ([5]). An $[n,k,d]_q$ code is called *optimal* if $n = n_q(k,d)$. The Griesmer bound (see [11]) gives a lower bound on $n_q(k,d)$:

$$n_q(k,d) \geq g_q(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. An $[n,k,d]_q$ code $\mathcal{C}$ is called *Griesmer* if it attains the Griesmer bound, i.e. $n = g_q(k,d)$. The values of $n_8(k,d)$ are determined for all d only for $k \leq 3$, see [19]. See [3] and [13] for the known results on optimal $[n,4,d]_8$ codes for $d \leq 112$. It is known from Theorem 2.12 of [5] that $n_8(4,d) = g_8(4,d)$ for all $d \geq 833$. So, we concentrate on finding optimal linear codes over $\mathbb{F}_8$ of dimension 4 with minimum distance $113 \leq d \leq 832$. Our results are summarized to the following theorem, see also Table 3.

Theorem 1.1. (1) *$n_8(4,d) = g_8(4,d)$ for $d \in$* {257, 258, 265-272, 385-392, 441-568, 577-580, 705-728, 769-784}.
(2) *$n_8(4,d) = g_8(4,d)+1$ for $d \in$* {113-120, 286-288, 377, 378, 399, 400, 407, 408, 414-440, 702-704, 750-752, 757-768, 813-816, 820-832}.
(3) *$n_8(4,d) \leq g_8(4,d)+1$ for $d \in$* {129-132, 193, 259-264, 273-285, 321-328, 393-398, 401-406, 409-413, 569-576, 581-632, 641-701, 729-749, 753-756, 785-812, 817-819}.
(4) *$g_8(4,d)+1 \leq n_8(4,d) \leq g_8(4,d)+2$ for $d \in$* {121-128, 177, 185-192, 225-227, 233-236, 241-246, 249-256, 289-316, 337-376, 379-384, 639, 640}.
(5) *$n_8(4,d) \leq g_8(4,d)+2$ for $d \in$* {133-176, 194-206, 209-220, 329-336, 633-638}.
(6) *$g_8(4,d)+1 \leq n_8(4,d) \leq g_8(4,d)+3$ for $d \in$* {178-184, 221-224, 228-232, 237-240, 247, 248, 317-320}.
(7) *$n_8(4,d) \leq g_8(4,d)+3$ for $d = 207, 208$.*

We also give a new construction of a $[g_q(4,d),4,d]_q$ code for $d = 2q^3 - 5q^2 + 3q$, $q \geq 5$ (Proposition 3.6) and prove $n_q(4,d) \geq g_q(4,d)+1$ for $q^3/2 - q^2 - q + 1 \leq d \leq q^3/2 - q^2$ for even $q \geq 4$ (Theorem 5.23).

2 Preliminary results

In this section, we give the geometric method and some known results which will be used in the later sections. We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A *j-flat* is a projective subspace of dimension j in $\mathrm{PG}(r,q)$. 0-flats, 1-flats, 2-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, secundums* and *hyperplanes*, respectively. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r,q)$ and denote by θ_j the number of points in a j-flat, i.e. $\theta_j = (q^{j+1}-1)/(q-1)$. We set $\theta_j = 0$ for $j < 0$. Let $\mathcal{C}$ be an $[n,k,d]_q$ code which does not have any coordinate position in which all the codewords have a zero entry. We always consider such codes throughout this paper. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1, q)$ denoted also by $\mathcal{C}$. We see linear codes from this geometrical point of view. An *i-point* is a point of Σ which has multiplicity i in $\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{C}$ and let C_i be the set of i-points in Σ, $0 \le i \le \gamma_0$. For any subset S of Σ we define *the multiplicity of S with respect to $\mathcal{C}$*, denoted by $m_{\mathcal{C}}(S)$, as

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,$$

where $|T|$ denotes the number of points in a set T in Σ. When the code is *projective*, i.e. when $\gamma_0 = 1$, the multiset $\mathcal{C}$ forms an n-set in Σ and the above $m_{\mathcal{C}}(S)$ is equal to $|\mathcal{C} \cap S|$. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane* and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that

$$\begin{aligned} n &= m_{\mathcal{C}}(\Sigma), \\ n - d &= \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \end{aligned}$$

Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n, k, d]_q$ code in the natural manner. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. It holds that $\gamma_{k-2} = n - d,\ \gamma_{k-1} = n$.

Lemma 2.1. *For two distinct t-flats δ_1 and δ_2 in a fixed $(t+1)$-flat Δ in Σ, $1 \le t \le k-2$, it holds that $m_{\mathcal{C}}(\delta_1) + m_{\mathcal{C}}(\delta_2) \ge m_{\mathcal{C}}(\Delta) - (q-1)\gamma_t + q \cdot m_{\mathcal{C}}(\delta_1 \cap \delta_2)$.*

Proof. Considering the t-flats in Δ through $\delta_1 \cap \delta_2$, we have

$$m_{\mathcal{C}}(\Delta) \le m_{\mathcal{C}}(\delta_1) + m_{\mathcal{C}}(\delta_2) - m_{\mathcal{C}}(\delta_1 \cap \delta_2) + (\gamma_t - m_{\mathcal{C}}(\delta_1 \cap \delta_2))(q-1). \qquad \square$$

Setting $t = k-2$, $a = m_{\mathcal{C}}(\delta_1)$, $b = m_{\mathcal{C}}(\delta_2)$, $c = m_{\mathcal{C}}(\delta_1 \cap \delta_2)$ in Lemma 2.1, we get

$$a + b \ge (q-1)d - (q-2)n + qc. \tag{2.1}$$

When $\mathcal{C}$ is Griesmer, γ_j's are uniquely determined as follows.

Lemma 2.2 ([18])**.** *For a Griesmer $[n, k, d]_q$ code, it holds for $0 \le j \le k-1$ that*

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil.$$

By Lemma 2.2, every Griesmer $[n, k, d]_q$ code is projective if $d \le q^{k-1}$. Denote by a_i the number of hyperplanes Π of Σ with $m_{\mathcal{C}}(\Pi) = i$ and by λ_s the number of s-points in Σ. When $\gamma_0 = 2$, we have $\lambda_0 + \lambda_1 + \lambda_2 = \theta_{k-1}$ and $\lambda_1 + 2\lambda_2 = n$, hence

$$\lambda_2 = \lambda_0 + n - \theta_{k-1}. \tag{2.2}$$

The list of a_i's is called the *spectrum* of $\mathcal{C}$. Note that $a_i = A_{n-i}/(q-1)$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following three equalities.

$$\sum_{i=0}^{n-d} a_i = \theta_{k-1}. \tag{2.3}$$

$$\sum_{i=1}^{n-d} i a_i = n\theta_{k-2}. \tag{2.4}$$

$$\sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s. \tag{2.5}$$

(2.3) and (2.4) yield the following:

$$\sum_{i=0}^{n-d-1} (n-d-i)a_i = nq^{k-1} - d\theta_{k-1}. \tag{2.6}$$

Furthermore, when $\gamma_0 \le 2$, we get the following from (2.3)-(2.5):

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \tag{2.7}$$

Lemma 2.3 ([22])**.** *Let Π be an i-hyperplane through a t-secundum δ. Then*
(1) $t \le \gamma_{k-2} - n - i/q = (i + q\gamma_{k-2} - n)/q$.
(2) $a_i = 0$ *if an* $[i, k-1, d_0]_q$ *code with* $d_0 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.
(3) $\gamma_{k-3}(\Pi) = \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ *if an* $[i, k-1, d_1]_q$ *code with* $d_1 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor + 1$ *does not exist.*

(4) *Let c_j be the number of j-hyperplanes through δ other than Π. Then the following equality holds:*

$$\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \tag{2.8}$$

(5) *For a γ_{k-2}-hyperplane Π_0 with spectrum $(\tau_0, \cdots, \tau_{\gamma_{k-3}})$, $\tau_t > 0$ holds if* $i + q\gamma_{k-2} - n - qt < q$.

The code obtained by deleting the same coordinate from each codeword of $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1, k, d+1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable* and $\mathcal{C}'$ is an *extension* of $\mathcal{C}$. We use the following extension theorems in Sections 4 and 5.

Theorem 2.4 ([6], [7]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with gcd$(d,q)=1$, $k \geq 3$. Then $\mathcal{C}$ is extendable if $A_i = 0$ for all $i \not\equiv 0,\ d \pmod q$.*

Theorem 2.5 ([24]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with $q \geq 5$, $d \equiv -2 \pmod q$, $k \geq 3$. Then $\mathcal{C}$ is extendable if $A_i = 0$ for all $i \not\equiv 0,-1,-2 \pmod q$.*

Theorem 2.6 ([21]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with gcd$(d,q)=1$ and assume that $\sum_{i \not\equiv n, n-d \pmod q} a_i < q^{k-2}$. Then $\sum_{i \not\equiv n, n-d \pmod q} a_i = 0$ and $\mathcal{C}$ is extendable.*

An $[n,k,d]_q$ code is called *m-divisible* if all codewords have weights divisible by an integer $m > 1$. The following theorem gives a restriction on the weights of a Griesmer $[n,k,d]_8$ code with $d \equiv 0 \pmod 8$.

Lemma 2.7 ([23]). *Let $\mathcal{C}$ be a Griesmer $[n,k,d]_8$ code. If 8 divides d, then $\mathcal{C}$ is 2-divisible.*

In the remainder of this section, we give some known results on $n_q(4,d)$.

Theorem 2.8 ([17]). *$n_q(4,d) = g_q(4,d)$ for all q for $q^3 - 2q^2 + 1 \leq d \leq q^3 - 2q^2 + q$ and for $q^3 - q^2 - q + 1 \leq d \leq q^3 + q^2 - q$.*

Theorem 2.9 ([17],[20]). *For $q \geq 4$, $n_q(4,d) = g_q(4,d) + 1$ for $q^3 - q^2 - 2q + 1 \leq d \leq q^3 - q^2 - q$ and for $2q^3 - 3q^2 - q + 1 \leq d \leq 2q^3 - 3q^2$.*

Theorem 2.10 ([17]). *$n_q(4,d) \geq g_q(4,d) + 1$ for*
(1) *$2q^2 - 2q + 1 \leq d \leq 2q^2$ for $q \geq 4$,*
(2) *$(\nu - 1)q^2 - 3q + 1 \leq d \leq (\nu - 1)q^2$ for $4 \leq \nu < q$ with ν not dividing q,*
(3) *$2q^3 - rq^2 - q + 1 \leq d \leq 2q^3 - rq^2$ for $q > r, r = 3,4$ and for $q > 2(r-1), r \geq 5$.*

Corollary 2.11. (1) *$n_8(4,d) = g_8(4,d)$ for $d \in \{$*385-392, 441-568*$\}$,*
(2) *$n_8(4,d) = g_8(4,d) + 1$ for $d \in \{$*433-440, 825-832*$\}$,*
(3) *$n_8(4,d) \geq g_8(4,d) + 1$ for $d \in \{$*113-128, 233-256, 297-320, 361-384, 761-768*$\}$.*

Theorem 2.12 ([20]). *There exist no $[n, 4, n + s - q^2]_q$ codes for $q^3 - s\theta_1 - q + 1 \leq n \leq q^3 - s\theta_1$ for $s = 2$, $q \geq 4$ and for $s = 3$, $q \geq 7$, $q \neq 9$.*

Corollary 2.13. *$n_8(4,d) \geq g_8(4,d) + 1$ for $417 \leq d \leq 432$.*

3 Upper bounds on $n_8(4, d)$

Recall that the existence of an $[n, k, d]_q$ code implies the existence of an $[n-1, k, d-1]_q$ code. So, from (1) and (2) of Corollary 2.11, it suffices to prove the following proposition in order to give the upper bounds on $n_8(4, d)$ in Theorem 1.1.

Proposition 3.1. (1) *There exist* $[g_8(4, d)+2, 4, d]_8$ *codes for* $d \in \{128, 136, 144, 152, 160, 168, 176, 177, 192, 200, 206, 216, 220, 227, 236, 246, 296, 304, 312, 316, 336, 344, 352, 360, 368, 640\}$.
(2) *There exist* $[g_8(4, d)+1, 4, d]_8$ *codes for* $d \in \{120, 132, 193, 280, 288, 328, 378, 400, 408, 416, 424, 432, 576, 584, 592, 600, 608, 616, 624, 632, 648, 656, 664, 672, 680, 688, 696, 704, 736, 744, 752, 760, 768, 792, 800, 808, 816, 824\}$.
(3) *There exist* $[g_8(4, d), 4, d]_8$ *codes for* $d \in \{258, 272, 580, 712, 720, 728, 776, 784\}$.

As a method to construct good codes, we first introduce the projective dual.

Lemma 3.2 ([15])**.** (1) *There exists a* $[39, 4, 32]_8$ *code with w.d.* $0^1 32^{1911} 36^{2184}$.
(2) *There exists a* $[121, 4, 104]_8$ *code with w.d.* $0^1 104^{3136} 112^{945} 120^{14}$.

Lemma 3.3 ([22])**.** *Let $\mathcal{C}$ be an m-divisible $[n, k, d]_q$ code with $q = p^h$, p prime, whose spectrum is*

$$(a_{n-d-(w-1)m}, a_{n-d-(w-2)m}, \cdots, a_{n-d-m}, a_{n-d}) = (\alpha_{w-1}, \alpha_{w-2}, \cdots, \alpha_1, \alpha_0),$$

where $m = p^r$ for some $1 \le r < h(k-2)$ satisfying $\lambda_0 > 0$. Then there exists a t-divisible $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with $t = p^{h(k-2)-r}$, $n^* = \sum_{j=0}^{w-1} j\alpha_j = ntq - \frac{d}{m}\theta_{k-1}$, $d^* = n^* - nt + \frac{d}{m}\theta_{k-2} = ((n-d)q - n)t$ whose spectrum is

$$(a_{n^*-d^*-\gamma_0 t}, a_{n^*-d^*-(\gamma_0-1)t}, \cdots, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_{\gamma_0}, \lambda_{\gamma_0-1}, \cdots, \lambda_1, \lambda_0).$$

$\mathcal{C}^*$ is called the *projective dual* of $\mathcal{C}$. Applying Lemma 3.3 to the codes in Lemma 3.2, we obtain the following codes.

Corollary 3.4. (1) *There exists a* $[312, 4, 272]_8$ *code with w.d.* $0^1 272^{3822} 288^{273}$.
(2) *There exists a* $[139, 4, 120]_8$ *code with w.d.* $0^1 120^{3297} 128^{749} 136^{49}$.

An f-set F in $\mathrm{PG}(r, q)$ with $m = \min\{|F \cap \pi| \mid \pi \in \mathcal{F}_{r-1}\}$ is called an $\{f, m; r, q\}$-*minihyper.* When an $[n, k, d]_q$ code is projective (i.e. $\gamma_0 = 1$), the set of 0-points C_0 forms a $\{\theta_{k-1} - n, \theta_{k-2} - (n-d); k-1, q\}$-minihyper, and vice versa, see [4].

Lemma 3.5. (1) *There exists a* $[\theta_3 - x\theta_1, 4, q^3 - xq]_q$ *code for* $0 \le x \le q^2 - 1$.
(2) *There exists a* $[2q^3 - x\theta_1, 4, 2q^3 - 2q^2 - xq]_q$ *code for* $0 \le x \le q^2$.

Proof. (1) Take x skew lines of $\mathrm{PG}(3,q)$ as the corresponding minihyper.
(2) Let δ_1 and δ_2 be planes meeting in a line l in $\mathrm{PG}(3,q)$ and take skew x lines $l_1,\cdots,l_x$ not intersecting l. Deleting δ_1, δ_2 and the skew x lines from two copies of $\mathrm{PG}(3,q)$, that is, setting $C_0=(\delta_1\cap\delta_2)\cup(\cup_{i=1}^{2}\cup_{1\le j\le x}(\delta_i\cap l_j))$, $C_1=(\delta_1\cup\delta_2\cup l_1\cup\cdots\cup l_x)\setminus C_0$ and $C_2=\mathrm{PG}(3,q)\setminus(C_0\cup C_1)$, we get the partition of $\mathrm{PG}(3,q)$ giving a generator matrix of the desired code. □

We get $[g_8(4,d)+2,4,d]_8$ codes for $d=336$, 344, 352, 360, 368 and $[g_8(4,d)+1,4,d]_8$ codes for $d=400$, 408, 416, 424, 432 from (1) of Lemma 3.5. We also get $[g_8(4,d)+1,4,d]_8$ codes $d=808$, 816, 824 from (2) of Lemma 3.5.
A (q^2+2q+1)-set $\mathcal{H}$ in $\mathrm{PG}(3,q)$ which is projectively equivalent to the set

$$\{\mathbf{P}(x_0,x_1,x_2,x_3)\in\mathrm{PG}(3,q)\mid x_0x_1+x_2x_3=0\}$$

is called a *hyperbolic quadric* in $\mathrm{PG}(3,q)$, see [9]. $\mathcal{H}$ contains a set of $q+1$ skew lines called a *regulus*. $\mathcal{H}$ consists of $(q+1)^2$ points, which are all the points on a pair of reguli. Using this property, we give a new construction of a non-projective Griesmer code as follows, which yields a $[833,4,728]_8$ code.

Table 1: Codes obtained by Lemmas 3.7 and 3.8.

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}$	Lemma
$[139,4,120]_8$	$[10,3,8]_8$	$[149,4,128]_8$	3.7
$[139,4,120]_8$	$[15,3,12]_8$	$[154,4,132]_8$	3.7
$[94,4,80]_8$	$[65,4,56]_8$	$[159,4,136]_8$	3.8
$[103,4,88]_8$	$[65,4,56]_8$	$[168,4,144]_8$	3.8
$[112,4,96]_8$	$[65,4,56]_8$	$[177,4,152]_8$	3.8
$[121,4,104]_8$	$[65,4,56]_8$	$[186,4,160]_8$	3.8
$[130,4,112]_8$	$[65,4,56]_8$	$[195,4,168]_8$	3.8
$[139,4,120]_8$	$[65,4,56]_8$	$[204,4,176]_8$	3.8
$[312,4,272]_8$	$[65,4,56]_8$	$[377,4,328]_8$	3.8
$[650,4,568]_8$	$[10,3,8]_8$	$[660,4,576]_8$	3.7
$[585,4,512]_8$	$[80,4,68]_8$	$[665,4,580]_8$	3.8
$[605,4,528]_8$	$[65,4,56]_8$	$[670,4,584]_8$	3.8
$[614,4,536]_8$	$[65,4,56]_8$	$[679,4,592]_8$	3.8
$[623,4,544]_8$	$[65,4,56]_8$	$[688,4,600]_8$	3.8
$[632,4,552]_8$	$[65,4,56]_8$	$[697,4,608]_8$	3.8
$[641,4,560]_8$	$[65,4,56]_8$	$[706,4,616]_8$	3.8
$[650,4,568]_8$	$[65,4,56]_8$	$[715,4,624]_8$	3.8
$[585,4,512]_8$	$[139,4,120]_8$	$[724,4,632]_8$	3.8
$[585,4,512]_8$	$[149,4,128]_8$	$[734,4,640]_8$	3.8
$[449,4,392]_8$	$[312,4,272]_8$	$[761,4,664]_8$	3.8
$[724,4,632]_8$	$[64,3,56]_8$	$[788,4,688]_8$	3.7
$[724,4,632]_8$	$[73,3,64]_8$	$[797,4,696]_8$	3.7
$[512,4,448]_8$	$[312,4,272]_8$	$[824,4,720]_8$	3.8
$[531,4,464]_8$	$[312,4,272]_8$	$[843,4,736]_8$	3.8
$[540,4,472]_8$	$[312,4,272]_8$	$[852,4,744]_8$	3.8
$[549,4,480]_8$	$[312,4,272]_8$	$[861,4,752]_8$	3.8
$[558,4,488]_8$	$[312,4,272]_8$	$[870,4,760]_8$	3.8
$[567,4,496]_8$	$[312,4,272]_8$	$[879,4,768]_8$	3.8
$[576,4,504]_8$	$[312,4,272]_8$	$[888,4,776]_8$	3.8
$[585,4,512]_8$	$[312,4,272]_8$	$[897,4,784]_8$	3.8
$[585,4,512]_8$	$[322,4,280]_8$	$[907,4,792]_8$	3.8
$[585,4,512]_8$	$[331,4,288]_8$	$[916,4,800]_8$	3.8

Proposition 3.6. *There exists a $[g_q(4,d),4,d]_q$ code for $d = 2q^3 - 5q^2 + 3q$, $q \geq 5$.*

Proof. Let $\mathcal{H}$ be a hyperbolic quadric in PG$(3,q)$ and let l_1 and l_2 be two skew lines contained in $\mathcal{H}$. We further take two skew lines l_3 and l_4 contained in $\mathcal{H}$ meeting l_1 and l_2 and four points $P_1, \cdots, P_4$ of $\mathcal{H}$ so that $l_1 \cap l_3 = P_1$, $l_1 \cap l_4 = P_2$, $l_2 \cap l_3 = P_3$, $l_2 \cap l_3 = P_4$. Let l_5 be the line $\langle P_1, P_4 \rangle$ and let l_6 be the line $\langle P_2, P_3 \rangle$. We set $C_0 = l_1 \cup l_2 \cup \cdots \cup l_6$, $C_1 = (\langle l_1, l_3 \rangle \cup \langle l_1, l_4 \rangle \cup \langle l_2, l_3 \rangle \cup \langle l_2, l_4 \rangle \cup \mathcal{H}) \setminus C_0$ and $C_2 = \mathrm{PG}(3,q) \setminus (C_0 \cup C_1)$, where $\langle l_i, l_j \rangle$ stands for the plane containing l_i and l_j. Taking the points of C_i as the columns of a generator matrix i times, we get the desired $[2q^3 - 3q^2 + 1, 4, d]_q$ code, which is Griesmer for $q \geq 5$. □

The next two lemmas are well-known to construct good codes from old ones, see Table 1 for the resulting codes.

Lemma 3.7 ([8]). *Let $\mathcal{C}_1$ be an $[n_1, k, d_1]_q$ code and $\mathcal{C}_2$ be an $[n_2, k-1, d_2]_q$ code. Assume that $\mathcal{C}_1$ has a codeword c_1 with $wt(c_1) \geq d_1 + d_2$. Then an $[n_1 + n_2, k, d_1 + d_2]_q$ code $\mathcal{C}$ exists.*

Lemma 3.8 ([8]). *If there exist an $[n_1, k, d_1]_q$ code $\mathcal{C}_1$ and an $[n_2, k, d_2]_q$ code $\mathcal{C}_2$, then so does an $[n_1 + n_2, k, d_1 + d_2]_q$ code $\mathcal{C}$.*

We also constructed linear codes with parameters $[206,4,177]_8$, $[222,4,192]_8$, $[224,4,193]_8$, $[232,4,200]_8$, $[239,4,206]_8$, $[250,4,216]_8$, $[255,4,220]_8$, $[263,4,227]_8$, $[273,4,236]_8$, $[284,4,246]_8$, $[297,4,258]_8$, $[322,4,280]_8$, $[331,4,288]_8$, $[341,4,296]_8$, $[350,4,304]_8$, $[359,4,312]_8$, $[364,4,316]_8$, $[434,4,378]_8$, $[743,4,648]_8$, $[752,4,656]_8$, $[770,4,672]_8$, $[779,4,680]_8$, $[806,4,704]_8$, $[815,4,712]_8$, by puncturing or lengthening some good codes with the aid of a computer.

4 The spectra of some $[n,3,d]_8$ codes

As a preliminary for Section 5, we give the needed results on the spectra of $[n,3,d]_8$ codes in this section. Table 2 can be obtained from the known results. Note that a Griesmer $[64-e, 3, 56-e]_8$ code corresponds to a $\{\theta_1 + e, 1; 2, 8\}$-minihyper, which necessarily contains a line in PG$(2,8)$ if $e \leq 3$, see Chap. 13 in [10].

Lemma 4.1 ([3]). *$n_8(3,d) = g_8(3,d) + 1$ for $d \in \{$13-16, 29-32, 37-40, 43-48$\}$ and $n_8(3,d) = g_8(3,d)$ for other d.*

The following three lemmas give the characterization of $[119,3,104]_8$, $[118,3,103]_8$ and $[117,3,102]_8$ codes for $q = 8$.

Lemma 4.2. *The spectrum of a $[2q^2-q-1, 3, 2q^2-3q]_q$ code with $q \geq 5$ is $(a_{q-1}, a_{2q-1}) = (3, \theta_2 - 3)$. A $(2q^2-q-1)$-plane is obtained from two copies of PG$(2,q)$ with three non-concurrent lines deleted.*

Proof. Let $\mathcal{C}$ be an $[n = 2q^2 - q - 1, 3, 2q^2 - 3q]_q$ code with $q \geq 5$. By Lemma 2.2, $\gamma_0 = 2$ and $\gamma_1 = 2q - 1$. Since $(\gamma_1 - \gamma_0)\theta_1 + \gamma_0 = n$, any line through a 2-point is a γ_1-line. Hence $a_i = 0$ for $\theta_1 + 1 \leq i \leq \gamma_1 - 1$. Let l be a t-line containing a 1-point P. Considering the lines through P, we get $n \leq (\gamma_1 - 1)q + t$, so $t \geq q - 1$. Hence $a_i = 0$ for $1 \leq i \leq q - 2$. Suppose $a_0 > 0$. Considering the lines through a fixed point of a 0-line, we have $n = q\gamma_1 + 0 - 1$, which implies $a_{\gamma_1 - 1} > 0$, a contradiction. Hence $a_0 = 0$. One can prove $a_{\theta_1 - 1} = a_{\theta_1} = 0$ similarly for $q \geq 5$ considering the lines though a fixed 1-point. Hence $a_i = 0$ for all $i \notin \{q - 1, 2q - 1\}$. The spectrum of $\mathcal{C}$ follows from (2.6) and (2.3). Since $2(q - 1) + (q - 1)\gamma_1 = n$, the three $(q - 1)$-lines are not concurrent. □

Table 2: The spectra of some $[n, 3, d]_8$ codes.

parameters	possible spectra	reference
$[6, 3, 4]_8$	$(a_0, a_1, a_2) = (34, 24, 15)$	[13]
$[7, 3, 5]_8$	$(a_0, a_1, a_2) = (31, 21, 21)$	[13]
$[8, 3, 6]_8$	$(a_0, a_1, a_2) = (29, 16, 28)$	[13]
$[9, 3, 7]_8$	$(a_0, a_1, a_2) = (28, 9, 36)$	[13]
$[10, 3, 8]_8$	$(a_0, a_2) = (28, 45)$	[13]
$[33, 3, 28]_8$	$(a_0, a_3, a_5) = (9, 16, 48)$	[2]
	$(a_0, a_1, a_4, a_5) = (4, 5, 28, 36)$	
	$(a_0, a_3, a_4, a_5) = (6, 10, 18, 39)$	
$[42, 3, 36]_8$	$(a_0, a_4, a_5, a_6) = (4, 6, 24, 39)$	[1]
	$(a_0, a_3, a_5, a_6) = (3, 7, 21, 42)$	
	$(a_0, a_4, a_6) = (3, 21, 49)$	
	$(a_0, a_2, a_4, a_6) = (2, 3, 18, 50)$	
$[60, 3, 52]_8$	$(a_4, a_6, a_8) = (3, 16, 54)$	[12]
	$(a_0, a_4, a_7, a_8) = (1, 1, 32, 39)$	
	$(a_0, a_5, a_6, a_7, a_8) = (1, 1, 3, 27, 41)$	
	$(a_0, a_6, a_7, a_8) = (1, 6, 24, 42)$	
$[61, 3, 53]_8$	$(a_0, a_6, a_7, a_8) = (1, 3, 21, 48)$	[10]
	$(a_0, a_5, a_7, a_8) = (1, 1, 24, 47)$	
$[62, 3, 54]_8$	$(a_0, a_6, a_7, a_8) = (1, 1, 16, 55)$	[10]
$[63, 3, 55]_8$	$(a_0, a_7, a_8) = (1, 9, 63)$	[4]
$[64, 3, 56]_8$	$(a_0, a_8) = (1, 72)$	[4]
$[70, 3, 61]_8$	$(a_6, a_8, a_9) = (1, 24, 48)$	[4]
	$(a_7, a_8, a_9) = (3, 21, 49)$	
$[71, 3, 62]_8$	$(a_7, a_8, a_9) = (1, 16, 56)$	[4]
$[72, 3, 63]_8$	$(a_8, a_9) = (9, 64)$	[4]
$[73, 3, 64]_8$	$a_9 = 73$	[4]
$[92, 3, 80]_8$	$(a_0, a_8, a_{12}) = (1, 9, 63)$	[16]
	$(a_4, a_{12}) = (6, 67)$	
	$(a_4, a_8, a_{12}) = (1, 10, 62)$	
	$(a_8, a_{12}) = (12, 61)$	
$[101, 3, 88]_8$	$(a_5, a_{13}) = (5, 68)$	[16]
	$(a_9, a_{13}) = (10, 63)$	

Lemma 4.3. *A $[2q^2 - q - 2, 3, 2q^2 - 3q - 1]_q$ code with $q \geq 7$ is extendable and its spectrum is $(a_{q-2}, a_{q-1}, a_{2q-2}, a_{2q-1}) = (1, 2, q, q^2 - 2)$ or $(a_{q-1}, a_{2q-2}, a_{2q-1}) = (3, q + 1, q^2 - 3)$.*

Proof. Let $\mathcal{C}$ be an $[n = 2q^2 - q - 2, 3, 2q^2 - 3q - 1]_q$ code with $q \geq 7$. By Lemma 2.2, $\gamma_0 = 2$ and $\gamma_1 = 2q - 1$. Since $(\gamma_1 - \gamma_0)\theta_1 + \gamma_0 - 1 = n$, the lines though a fixed 2-point is one $(\gamma_1 - 1)$-line and q γ_1-lines, and $a_i = 0$ for $\theta_1 + 1 \leq i \leq \gamma_1 - 2$. Let l be a t-line containing a 1-point P. Considering the lines through P, we get $n \leq (\gamma_1 - 1)q + t$, so $q - 2 \leq t$. Hence $a_i = 0$ for $1 \leq i \leq q - 3$.

Suppose $a_{\theta_1} > 0$. Let l be a θ_1-line. Since $n = (\gamma_1 - 1)q + \theta_1 - 3$, the lines ($\neq l$) through a fixed 1-point of l are three $(\gamma_1 - 1)$-lines and $q - 3$ γ_1-lines, for $\gamma_1 - 3 > \theta_1$. Hence $\sum_{i \not\equiv n,n-d \pmod q} a_i = a_{\theta_1} = 1$, which contradicts Theorem 2.6. Hence $a_{\theta_1} = 0$. One can prove $a_0 = a_q = 0$ similarly applying Theorem 2.6. Therefore $a_i = 0$ for all $i \notin \{q-2, q-1, 2q-2, 2q-1\}$. Applying Theorem 2.4, $\mathcal{C}$ is extendable. Hence $\mathcal{C}$ can be obtained from a $[2q^2 - q - 1, 3, 2q^2 - 3q]_q$ code $\mathcal{C}'$ by removing one coordinate. Let P be the point corresponding to the coordinate. There are two possible spectra as stated according to the cases that P is a 1-point or a 2-point, respectively. □

Lemma 4.4. *A $[2q^2-q-3, 3, 2q^2-3q-2]_q$ code with $q \geq 7$ is extendable and its spectrum is one of the followings:*
(a) $(a_{q-3}, a_{q-1}, a_{2q-2}, a_{2q-1}) = (1, 2, 2q, q^2 - q - 2)$,
(b) $(a_{q-2}, a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (2, 1, 1, 2q - 2, q^2 - q - 1)$,
(c) $(a_{q-2}, a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (1, 2, 1, 2q - 1, q^2 - q - 2)$,
(d) $(a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (3, 1, 2q, q^2 - q - 3)$,
(e) $(a_{q-1}, a_{2q-3}, a_{2q-1}) = (3, q + 1, q^2 - 3)$.

Proof. Let $\mathcal{C}$ be a $[2q^2-q-3, 3, 2q^2-3q-2]_q$ code with $q \geq 7$. By Lemma 2.2, $\gamma_0 = 2$ and $\gamma_1 = 2q - 1$. Since $(\gamma_1 - \gamma_0)\theta_1 + \gamma_0 - 2 = n$, a j-line though a 2-point satisfies $j \geq \gamma_1 - 2$, and $a_i = 0$ for $\theta_1 + 1 \leq i \leq \gamma_1 - 3$. Every t-line through a 1-point satisfies $n \leq (\gamma_1 - 1)q + t$, so $q - 3 \leq t$. Hence $a_i = 0$ for $1 \leq i \leq q - 4$.
Suppose $a_0 > 0$ and let l be a 0-line. Then $\lambda_2 = q^2 - 2q - 4 + \lambda_0 \geq q^2 - q - 3$ from (2.2), for $\lambda_0 \geq |l| = \theta_1$. It follows from (2.1) that $a_0 = 1$ and $a_i = 0$ for $1 \leq i \leq 2q - 5$, for $\theta_1 \leq 2q - 4$. Calculating $(2.6) - 2 \cdot (2.7)$, we get $a_{2q-2} = 2q^3 - 3q^2 - 2q + 3 - 2q\lambda_2 \geq 0$. Hence $\lambda_2 \leq q^2 - 3q - 1$, a contradiction. Thus $a_0 = 0$.
Suppose $a_{\theta_1} > 0$. Let l be a θ_1-line. From (2.8) with $i = \theta_1$ and $t = 1$, we have $a_{\theta_1} = 1$ and $a_i = 0$ for $i \notin \{q+1, 2q-3, 2q-2, 2q-1\}$, for $q - 2 > 4$. Calculating $(2.6) - 2 \cdot (2.7)$, we get $a_{2q-2} = 2q^3 - 6q^2 + 8 - 2q\lambda_2 \geq 0$, whence $\lambda_2 \leq q^2 - 3q$, contradicting (2.2). Hence $a_{\theta_1} = 0$.
Suppose $a_q > 0$. Let l be a q-line and let Q be the 0-point of l. Since $n = (\gamma_1 - 1)q + q - 3$ and $\gamma_1 - 3 > \theta_1$, every j-line ($\neq l$) through a fixed 1-point of l satisfies $j \geq \gamma_1 - 2 = 2q - 3$. From (2.8) with $i = q$ and $t = 0$, we have

$$c_{q-3} + c_{q-2} + c_{q-1} + c_q \leq 1.$$

Assume $c_{q-3} = 1$. Then $a_{q-3} = a_q = 1$ and $a_i = 0$ for $i \notin \{q-3, q, 2q-3, 2q-2, 2q-1\}$. We have $\lambda_2 \geq q^2 - 2q$ from (2.2) since a $(q-3)$-line contains four 0-points. Calculating $(2.6) - 2 \cdot (2.7)$, we get $a_{2q-2} = 2q^3 - 5q^2 + 4q + 3 - 2q\lambda_2 \geq 0$, whence $\lambda_2 \leq q^2 - 5q/2 + 2$, a contradiction. One can get a contradiction similarly for the other cases $c_{q-2} = 1$; $c_{q-1} = 1$; $c_q = 1$; $c_{q-3} = c_{q-2} = c_{q-1} = c_q = 0$. Hence $a_q = 0$.
Thus $a_i = 0$ for all $i \notin \{q-3, q-2, q-1, 2q-3, 2q-2, 2q-1\}$. Applying Theorem 2.5, $\mathcal{C}$ is extendable. Hence by the previous lemmas, $\mathcal{C}$ can be obtained from a $[2q^2 - q - 1, 3, 2q^2 - 3q]_q$ code $\mathcal{C}'$ by removing two coordinates. Let P and Q be the point corresponding to the coordinates. There are five possible spectra (a)-(e) as stated, according to the cases (a)

P and Q are 1-points on the same $(q-1)$-line, (b) P and Q are 1-points from different $(q-1)$-lines, (c) P is a 1-point and Q is a 2-point, (d) P and Q are distinct 2-points, (e) P and Q are the same 2-points, respectively. $\square$

The following three lemmas give the characterization of $[110, 3, 96]_8$, $[109, 3, 95]_8$ and $[108, 3, 94]_8$ codes for $q = 8$. The proofs are quite similar to the proofs of Lemmas 4.2-4.4 and hence we omit here, see [14].

Lemma 4.5. *The spectrum of a* $[2q^2-2q-2, 3, 2q^2-4q]_q$ *code with* $q \geq 7$ *is* $(a_{q-2}, a_{2q-2}) = (4, \theta_2 - 4)$. *A* $(2q^2 - 2q - 2)$*-plane is obtained from two copies of* $PG(2, q)$ *with* 4*-arc of lines deleted.*

Lemma 4.6. *A* $[2q^2-2q-3, 3, 2q^2-4q-1]_q$ *code with* $q \geq 8$ *is extendable and its spectrum is* $(a_{q-3}, a_{q-2}, a_{2q-3}, a_{2q-2}) = (1, 3, q, q^2 - 3)$ *or* $(a_{q-2}, a_{2q-3}, a_{2q-2}) = (4, q + 1, q^2 - 4)$.

Lemma 4.7. *A* $[2q^2 - 2q - 4, 3, 2q^2 - 4q - 2]_q$ *code with* $q \geq 8$ *is extendable and its spectrum is one of the followings:*
(a) $(a_{q-4}, a_{q-2}, a_{2q-3}, a_{2q-2}) = (1, 3, 2q, q^2 - q - 3)$,
(b) $(a_{q-3}, a_{q-2}, a_{2q-4}, a_{2q-3}, a_{2q-2}) = (2, 2, 1, 2q - 2, q^2 - q - 2)$,
(c) $(a_{q-3}, a_{q-2}, a_{2q-4}, a_{2q-3}, a_{2q-2}) = (1, 3, 1, 2q - 1, q^2 - q - 3)$,
(d) $(a_{q-2}, a_{2q-4}, a_{2q-3}, a_{2q-2}) = (4, 1, 2q, q^2 - q - 4)$,
(e) $(a_{q-2}, a_{2q-4}, a_{2q-2}) = (4, q + 1, q^2 - 4)$.

Lemma 4.8. (1) *The spectrum of a* $[59, 3, 51]_8$ *code satisfies* $a_1 = a_2 = 0$.
(2) *The spectrum of a* $[58, 3, 50]_8$ *code satisfies* $a_1 = 0$.

Proof. (1) Let $\mathcal{C}$ be a $[59, 3, 51]_8$ code. By Lemma 2.2, $\gamma_0 = 1$ and $\gamma_1 = 8$. Let l be a t-line containing a 1-point P. Considering the lines through P, we get $59 \leq (8 - 1)8 + t$, so $3 \leq t$. Hence $a_i = 0$ for $1 \leq i \leq 2$. One can prove (2) similarly. $\square$

For q even, there exists a $(q(q - 1)/2, q/2)$-arc K in PG$(2, q)$ corresponding to a Griesmer $[q(q - 1)/2, 3, q(q - 2)/2]_q$ code. Since $(q/2 - 1)\theta_1 + 1 = q(q - 1)/2$, every line meeting K is a $q/2$-line. Hence, from (2.6) and (2.3), we get the spectrum of K as (1) in the following lemma. Note that the 0-lines form a $(q + 2)$-arc of lines, that is, no three of which are concurrent.

Lemma 4.9. *Assume* $q \geq 4$ *is even.*
(1) *The spectrum of a* $[q(q - 1)/2, 3, q(q - 2)/2]_q$ *code is* $(a_0, a_{q/2}) = (q + 2, q^2 - 1)$.
(2) *The spectrum of a* $[q(q - 1)/2 - 1, 3, q(q - 2)/2 - 1]_q$ *code is* $(a_0, a_{q/2-1}, a_{q/2}) = (q + 2, q + 1, q^2 - q - 2)$.

Proof. (2) It is easy to see that the possible lines are 0-, $(q/2 - 1)$- and $q/2$-lines. Hence the spectrum follows from (2.3), (2.6) and (2.7) with $\lambda_2 = 0$. $\square$

Lemma 4.10. (1) *The spectrum of a* $[12,3,9]_8$ *code is* $(a_0,a_1,a_2,a_3) = (v, 69-3v, 3v-27, 31-v)$ *with* $9 \le v \le 23$.
(2) *The spectrum of a* $[13,3,10]_8$ *code is* $(a_0,a_1,a_2,a_3) = (v, 63-3v, 3v-24, 34-v)$ *with* $8 \le v \le 21$.
(3) *The spectrum of a* $[14,3,11]_8$ *code is* $(a_0,a_1,a_2,a_3) = (v, 58-3v, 3v-23, 38-v)$ *with* $8 \le v \le 19$.
(4) *The spectrum of a* $[15,3,12]_8$ *code is* $(a_0,a_1,a_2,a_3) = (v, 54-3v, 3v-24, 43-v)$ *with* $8 \le v \le 18$.

Proof. (4) Let $\mathcal{C}$ be a $[15,3,12]_8$ code. We have $\gamma_0 = 1$ and $\gamma_1 = 3$ by Lemma 2.2. Hence, from (2.3), (2.6) and (2.7) with $\lambda_2 = 0$, we get the spectrum as stated, where we have $8 \le v \le 18$ from $a_1 \ge 0$ and $a_2 \ge 0$. (1)-(3) are proved similarly. □

5 Lower bounds on $n_8(4,d)$

Known results on $n_8(3,d)$ implies that $n_8(4,d) \ge g_8(4,d)+1$ for $225 \le d \le 232$, $289 \le d \le 296$ and $337 \le d \le 360$, for the residual code (see [11]) of each $[g_8(4,d),4,d]_8$ code with respect to a codeword with weight d cannot exist. It follows from Corollaries 2.11 and 2.13 that in order to give the lower bounds on $n_8(4,d)$ in Theorem 1.1, it suffices to prove the nonexistence of $[g_8(4,d),4,d]_8$ codes for $d \in \{177, 185, 221, 286, 399, 407, 414, 639, 702, 750, 757, 813, 820\}$.

Lemma 5.1. *There exists no* $[938,4,820]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[938,4,820]_8$ code. Let δ be an i-plane through a t-line. Then $t \le (i+6)/8$ by Lemma 2.3. The spectrum of a γ_2-plane $\delta_0 (\ne \delta)$ is (a) $(\tau_6,\tau_7,\tau_{14},\tau_{15}) = (1,2,8,62)$ or (b) $(\tau_7,\tau_{14},\tau_{15}) = (3,9,61)$ by Lemma 4.3. So, $t \in \{6,7,14,15\}$ and $i \ge 6 \cdot 8 - 6 = 42$. Hence $a_i = 0$ for all $i \notin \{42$, 106-110, 114-118$\}$ by Lemmas 2.3 and 4.1. The equality (2.7) gives

$$2850a_{42} + 66a_{106} + 55a_{107} + 45a_{108} + 36a_{109} + 28a_{110} + 6a_{114} + 3a_{115} + a_{116} = 64\lambda_2 - 18126. \tag{5.1}$$

Setting $i = \gamma_2 = 118$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.1) are $(c_{42}, c_{118}) = (1,7)$ for $t = 6$; $(c_{106}, c_{110}, c_{118}) = (5,1,2)$ for $t = 7$; $(c_{106}, c_{118}) = (1,7)$ for $t = 14$; $(c_{114}, c_{118}) = (1,7)$ for $t = 15$. Estimating the LHS of (5.1) for each of the two possible spectra of δ, we get

$$64\lambda_2 - 18126 \le 0 + 2850\tau_6 + 358\tau_7 + 66\tau_{14} + 6\tau_{15} \le 4466.$$

Hence $\lambda_2 \le 353$ and the equality holds only if $a_{42} > 0$ and δ has spectrum (a). On the other hand, (2.2) gives $\lambda_2 = 353 + \lambda_0 \ge 353$, and we have $\lambda_2 \ge 384$ when $a_{42} > 0$ since a 42-plane has 31 0-points, a contradiction. This completes the proof. □

Lemma 5.2. *There exists no* $[930,4,813]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[930, 4, 813]_8$ code. By Lemma 4.4, the spectrum of a γ_2-plane is $(\tau_5, \tau_7, \tau_{14}, \tau_{15}) = (1, 2, 16, 54)$, $(\tau_6, \tau_7, \tau_{13}, \tau_{14}, \tau_{15}) = (2, 1, 1, 14, 55)$, $(\tau_6, \tau_7, \tau_{13}, \tau_{14}, \tau_{15}) = (1, 2, 1, 15, 54)$, $(\tau_7, \tau_{13}, \tau_{14}, \tau_{15}) = (3, 1, 16, 53)$ or $(\tau_7, \tau_{13}, \tau_{15}) = (3, 9, 61)$, so there is no i-plane for all $i < 34$ by Lemma 2.3. Hence $a_i = 0$ for all $i \notin \{42,98\text{-}101,106\text{-}110,114\text{-}117\}$ by Lemmas 2.3 and 4.1. It follows from (2.7) that

$$\begin{aligned}2775a_{42} + 171a_{98} + 153a_{99} + 136a_{100} + 120a_{101} + 55a_{106} + 45a_{107}\\ +36a_{108} + 28a_{109} + 21a_{110} + 3a_{114} + a_{115} = 64\lambda_2 - 17565.\end{aligned} \tag{5.2}$$

Let δ be an i-plane with spectrum τ_j's.

Setting $i = 42$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.2) are $(c_{98}, c_{107}, c_{117}) = (2, 1, 5)$ for $t = 0$; $(c_{101}, c_{117}) = (2, 6)$ for $t = 2$; $(c_{98}, c_{114}, c_{115}, c_{117}) = (1, 1, 1, 5)$ for $t = 3$; $(c_{101}, c_{117}) = (1, 7)$ for $t = 4$; $(c_{109}, c_{117}) = (1, 7)$ for $t = 5$; $c_{117} = 8$ for $t = 6$. Estimating the LHS of (5.2) for each of the four possible spectra of δ (see Table 2), we get

$$64\lambda_2 - 17565 \leq 2775 + 387\tau_0 + 240\tau_2 + 175\tau_3 + 120\tau_4 + 28\tau_5 + 0 \cdot \tau_6 \leq 6456,$$

whence $\lambda_2 \leq 375$. Since δ has at least 31 0-points, it follows from (2.2) that $\lambda_2 = 345 + \lambda_0 \geq 376$, a contradiction. Hence $a_{42} = 0$.
Setting $i = 117$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.2) are $(c_{98}, c_{110}, c_{117}) = (4, 1, 3)$ for $t = 5$; $(c_{98}, c_{99}, c_{117}) = (3, 1, 3)$ for $t = 6$; $(c_{98}, c_{107}, c_{117}) = (3, 1, 4)$ for $t = 7$; $(c_{98}, c_{117}) = (1, 7)$ for $t = 13$; $(c_{106}, c_{117}) = (1, 7)$ for $t = 14$; $(c_{114}, c_{117}) = (1, 7)$ for $t = 15$. Estimating the LHS of (5.2) according to each of the five possible spectra of δ, we get

$$64\lambda_2 - 17565 \leq 0 + 705\tau_5 + 666\tau_6 + 558\tau_7 + 171\tau_{13} + 55\tau_{14} + 3\tau_{15} \leq 3396.$$

Hence $\lambda_2 \leq 327$, which contradicts that $\lambda_2 = 345 + \lambda_0 \geq 345$. □

The following Lemma can be proved similarly as in the proof of Lemma 5.2 using Lemmas 2.3, 4.1, Table 2 for the possible spectra of a 42-plane, Lemma 4.6 for the possible spectra of a γ_2-plane and estimating the LHS of (2.7), see [14].

Lemma 5.3. *There exists no* $[866, 4, 757]_8$ *code.*

Lemma 5.4. *There exists no* $[860, 4, 752]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[860, 4, 752]_8$ code. By Lemma 4.7, the spectrum of a γ_2-plane δ is $(\tau_4, \tau_6, \tau_{13}, \tau_{14}) = (1, 3, 16, 53)$, $(\tau_5, \tau_6, \tau_{12}, \tau_{13}, \tau_{14}) = (2, 2, 1, 14, 54)$, $(\tau_5, \tau_6, \tau_{12}, \tau_{13}, \tau_{14}) = (1, 3, 1, 15, 53)$, $(\tau_6, \tau_{12}, \tau_{13}, \tau_{14}) = (4, 1, 16, 52)$ or $(\tau_6, \tau_{12}, \tau_{14}) = (4, 9, 60)$, so there is no i-plane for all $i < 28$ by Lemma 2.3. Hence $a_i = 0$ for all $i \notin \{28, 92, 100, 108\}$ by Lemmas 2.3, 4.1 and 2.7. It follows from (2.3)-(2.5) that

$$45a_{28} + a_{92} = \lambda_2 - 225. \tag{5.3}$$

Recall that a 28-plane has a 0-line by Lemma 4.9. Setting $i = 28$ and $t = 0$, (2.8) has no solution since a 108-plane has no 0-line. Hence $a_{28} = 0$.
Setting $i = 108$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.3) are $(c_{92}, c_{108}) = (5, 3)$ for $t = 4$; $(c_{92}, c_{100}, c_{108}) = (4, 1, 3)$ for $t = 5$; $(c_{92}, c_{108}) = (4, 4)$ for $t = 6$; $(c_{92}, c_{108}) = (1, 7)$ for $t = 12$; $(c_{100}, c_{108}) = (1, 7)$ for $t = 13$; $c_{108} = 8$ for $t = 14$. Estimating the LHS of (5.3) for each of the five spectra for δ, we get $\lambda_2 - 225 \leq 0 + 5\tau_4 + 4\tau_5 + 4\tau_6 + \tau_{14} \leq 25$. Hence $\lambda_2 \leq 250$. On the other hand, from (2.2), we get $\lambda_2 \geq 275$, a contradiction. This completes the proof. □

Lemma 5.5. *There exists no* $[859, 4, 751]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[859, 4, 751]_8$ code. Note that γ_j is the same with that for a putative $[860, 4, 752]_8$ code for $0 \leq j \leq 2$. Hence there are five possible spectra for a γ_2-plane δ, so there is no i-plane for all $i < 27$ by Lemma 2.3. Hence $a_i = 0$ for all $i \notin \{27, 28, 91, 92, 99\text{-}101, 107, 108\}$ by Lemmas 2.3 and 4.1. (2.7) gives

$$\begin{aligned} 3240a_{27} + 3160a_{28} + 136a_{91} + 120a_{92} + 36a_{99} + 28a_{100} + 21a_{101} \\ = 64\lambda_2 - 12920. \end{aligned} \tag{5.4}$$

Setting $i = 101$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.4) are $(c_{91}, c_{92}, c_{108}) = (2, 2, 4)$ for $t = 5$; $(c_{91}, c_{108}) = (2, 6)$ for $t = 9$; $(c_{107}, c_{108}) = (2, 6)$ for $t = 13$. Estimating the LHS of (5.4) according to each of the two spectra of a 101-plane in Table 2, we get

$$64\lambda_2 - 12920 \leq 21 + 512\tau_5 + 272\tau_9 + 0 \cdot \tau_{13} \leq 2741,$$

whence $\lambda_2 \leq 244$. From (2.2), we get $\lambda_2 = 274 + \lambda_0 \geq 274$ a contradiction. Hence $a_{101} = 0$. Applying Theorem 2.4, $\mathcal{C}$ is extendable, which contradicts Lemma 5.4. □

Lemma 5.6. *There exists no* $[858, 4, 750]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[858, 4, 750]_8$ code. From the possible five spectra for a γ_2-plane, we have $a_i = 0$ for all $i \notin \{26\text{-}28, 42, 90\text{-}92, 98\text{-}101, 106\text{-}108\}$ by Lemmas 2.3 and 4.1. It holds from (2.7) that

$$\begin{aligned} 3321a_{26} + 3240a_{27} + 3160a_{28} + 2145a_{42} + 153a_{90} + 136a_{91} + 120a_{92} \\ +45a_{98} + 36a_{99} + 28a_{100} + 21a_{101} + a_{106} = 64\lambda_2 - 12831. \end{aligned} \tag{5.5}$$

Setting $i = 42$, (2.8) has no solution satisfying $c_{42} > 0$ for all t. Hence $a_{42} \leq 1$.
Setting $i = 101$, the maximum possible contributions of c_j's in (2.8) to the LHS of (2.3) are $(c_{42}, c_{107}, c_{108}) = (1, 1, 6)$ if $c_{42} > 0$ and $(c_{90}, c_{91}, c_{92}, c_{108}) = (1, 1, 2, 4)$ if $c_{42} = 0$ for $t = 5$; $(c_{90}, c_{91}, c_{108}) = (1, 1, 6)$ for $t = 9$; $(c_{106}, c_{107}, c_{108}) = (1, 1, 6)$ for $t = 13$. Since $a_{42} \leq 1$, estimating the LHS of (5.5) for the two possible spectra of a 101-plane in Table 2, we get two inequalities:

(a) $64\lambda_2 - 12831 \leq 21 + (2145 \cdot 1 + 529(\tau_5 - 1)) + 1 \cdot \tau_{13} = 4350,$

(b) $64\lambda_2 - 12831 \le 21 + 289\tau_9 + 1\tau_{13} = 2974$.

Hence $\lambda_2 \le 268$. On the other hand, from (2.2), we have $\lambda_2 = 273 + \lambda_0 \ge 273$, a contradiction. Hence $a_{101} = 0$. Applying Theorem 2.5, $\mathcal{C}$ is extendable, which contradicts Lemma 5.5. This completes the proof. □

Lemma 5.7. *There exists no* $[805, 4, 704]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[805, 4, 704]_8$ code. By Table 2, the spectrum of a γ_2-plane δ is $(\tau_5, \tau_{13}) = (5, 68)$ or $(\tau_9, \tau_{13}) = (10, 63)$, so there is no i-plane for all $i < 37$ by Lemma 2.3. Hence $a_i = 0$ for all $i \notin \{69, 71, 73, 101\}$ by Lemmas 2.3, 4.1 and 2.7. Setting $i = 101$, the solutions of (2.8) are $(c_{69}, c_{101}) = (2, 6)$ for $t = 5$; $(c_{69}, c_{101}) = (1, 7)$ for $t = 9$; $c_{101} = 8$ for $t = 13$. Hence $a_{71} = a_{73} = 0$. It follows from (2.6) and (2.3) that $(a_{69}, a_{101}) = (10, 575)$, $\lambda_2 = 230$ and

$$\lambda_0 = 10. \tag{5.6}$$

Assume that the spectrum of δ is $(\tau_5, \tau_{13}) = (5, 68)$. Then δ has exactly ten 0-points from (2.5). For a 5-line l on δ, there are six 101-planes through l other then δ. Since l has four 0-points, we get $\lambda_0 \ge 10 + (10 - 4)6$, contradicting (5.6). Hence all 101-planes have spectrum $(\tau_9, \tau_{13}) = (10, 63)$ containing no 0-point from (2.5). For a 13-line l' on δ, all planes through l' are 101-planes of this type, whence $\lambda_0 = 0$, contradicting (5.6) again. This completes the proof. □

Lemma 5.8. *There exists no* $[804, 4, 703]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[804, 4, 703]_8$ code. It follows from the possible spectra for a γ_2-plane and from Lemmas 2.3 and 4.1 that $a_i = 0$ for all $i \notin \{$68-73,100,101$\}$. Then (2.7) gives

$$528a_{68} + 496a_{69} + 465a_{70} + 435a_{71} + 406a_{72} + 378a_{73} = 64\lambda_2 - 9696. \tag{5.7}$$

Suppose $a_{73} > 0$ and let δ be a 73-plane. Recall from Table 2 that δ has 9-lines only. Setting $i = 73$, the solution of (2.8) is $(c_{100}, c_{101}) = (5, 3)$ for $t = 9$. Hence $a_j = 0$ for all $j \notin \{73, 100, 101\}$ and $a_{73} = 1$. Then, (5.7) gives $\lambda_2 = 5037/32$, a contradiction. Hence $a_{73} = 0$. Setting $i = 72$ and $t = 8$, (2.8) has no solution. Hence $a_{72} = 0$. We can see $a_{71} = a_{70} = 0$ similarly. Thus we have $a_i = 0$ for all $i \notin \{68, 69, 100, 101\}$.
Applying Theorem 2.4, $\mathcal{C}$ is extendable, which contradicts Lemma 5.7. □

Lemma 5.9. *There exists no* $[803, 4, 702]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[803, 4, 702]_8$ code. It follows from the possible spectra for a γ_2-plane and from Lemmas 2.3 and 4.1 that $a_i = 0$ for all $i \notin \{67\text{-}73, 99\text{-}101\}$, and (2.7) yields

$$561a_{67} + 528a_{68} + 496a_{69} + 465a_{70} + 435a_{71} + 406a_{72} + 378a_{73} + a_{99} = 64\lambda_2 - 9623. \tag{5.8}$$

Setting $i = 73$, the maximum possible contribution of c_j's in (2.8) to the LHS of (5.8) is $(c_{99}, c_{101}) = (3, 5)$ for $t = 9$. Estimating the LHS of (5.8) for the spectrum of a 73-plane $\tau_9 = 73$, we get $64\lambda_2 - 9623 \le 378 + 3\tau_9 = 597$, whence $\lambda_2 \le 159$. On the other hand, from (2.2), we get $\lambda_2 \ge 218$, a contradiction. Hence $a_{73} = 0$. We can prove $a_{72} = 0$ similarly. We also get $a_{71} = a_{70} = 0$ since (2.8) has no solution for $(i, t) = (71, 7)$ and $(i, t) = (70, 6)$. Thus we have proved that $a_i = 0$ for all $i \notin \{67\text{-}69, 99\text{-}101\}$. Applying Theorem 2.5, $\mathcal{C}$ is extendable, which contradicts Lemma 5.8. This completes the proof. □

Lemma 5.10. *There exists no* $[732, 4, 640]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[732, 4, 640]_8$ code. From Table 2, the spectrum of a γ_2-plane δ is $(\tau_0, \tau_8, \tau_{12}) = (1, 9, 63)$, $(\tau_4, \tau_{12}) = (6, 67)$ or $(\tau_4, \tau_8, \tau_{12}) = (1, 10, 62)$. Since δ has no i-line for $i = 1\text{-}3, 5\text{-}7, 9\text{-}11$, $a_i = 0$ for all $i \notin \{0,28,60,62,64,92\}$ by Lemmas 2.3, 4.1 and 2.7. It also holds that $a_0 = a_{28} = a_{60} = a_{62} = a_{64} = 0$ since (2.8) has no solution for $(i, t) \in \{(0, 0), (28, 0), (60, 6), (60, 7), (62, 7), (64, 8)\}$, see Table 2 and Lemma 4.9 for the possible spectra for an i-plane. Hence $a_{92} = 585$, which contradicts (2.4). □

For a putative $[731, 4, 639]_8$ code, we have $a_i = 0$ for all $i \notin \{0, 27, 28, 59\text{-}64, 91, 92\}$ by Lemmas 2.3, 4.1. One can rule out the possibility of i-planes for $i = 0$, 61-64 using Theorem 2.6 and (2.8). Hence, applying Theorem 2.4, we get a contradiction. Thus the following holds.

Lemma 5.11. *There exists no* $[731, 4, 639]_8$ *code.*

Lemma 5.12. *There exists no* $[476, 4, 416]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[476, 4, 416]_8$ code. Then $\gamma_0 = 1$ by Theorem 2.2 and $\gamma_2 = n - d = 60$. Let δ be a γ_2-plane. From Table 2, there are four possible spectra for δ. Since δ has no i-line for $i = 1, 2, 3$, $a_i = 0$ for all $i \notin \{0, 28, 60\}$ by Lemmas 2.3, 4.1. It follows from (2.3)-(2.5) that $105a_0 = -336$, a contradiction. □

Lemma 5.13. *There exists no* $[475, 4, 415]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[475, 4, 415]_8$ code. It follows from the possible spectra for a γ_2-plane and from Lemmas 2.3 and 4.1 that $a_i = 0$ for all $i \notin \{0, 27, 28, 59, 60\}$. Suppose $a_0 > 0$. Setting $i = t = 0$, the possible solution for (2.8) is $(c_{59}, c_{60}) = (5, 3)$. Hence $\sum_{i \not\equiv n, n-d \pmod q} a_i = a_0 = 1$, which contradicts Theorem 2.6 since $\mathcal{C}$ is not extendable by Lemma 5.12. Hence $a_0 = 0$. Then, we can apply Theorem 2.4 so that $\mathcal{C}$ is extendable, a contradiction again. □

Lemma 5.14. *There exists no* $[474, 4, 414]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[474, 4, 414]_8$ code. From the possible spectra for a γ_2-plane and from Lemmas 2.3, 4.1, we have $a_i = 0$ for all $i \notin \{0, 26\text{-}28, 42, 58\text{-}60\}$. Then, (2.7) with $\lambda_2 = 0$ gives

$$1770a_0 + 561a_{26} + 528a_{27} + 496a_{28} + 153a_{42} + a_{58} = 2841. \tag{5.9}$$

Setting $i = t = 0$, the possible solutions for (2.8) are $(c_{58}, c_{60}) = (3, 5)$, $(c_{58}, c_{59}, c_{60}) = (2, 2, 4)$, $(c_{58}, c_{59}, c_{60}) = (1, 4, 3)$ or $(c_{59}, c_{60}) = (6, 2)$. Hence $a_0 = 1$, $a_{26} = a_{27} = a_{28} = a_{42} = 0$ and $a_{58} \le 3\cdot 73 = 219$, which implies that the LHS of (5.9) is at most $1770+219 = 1989$, a contradiction. Hence $a_0 = 0$. Applying Theorem 2.5, $\mathcal{C}$ is extendable, which contradicts Lemma 5.13. This completes the proof. □

Lemma 5.15. *There exists no* $[467, 4, 408]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[467, 4, 408]_8$ code. By Lemma 4.8, the spectrum of a γ_2-plane satisfies $\tau_1 = \tau_2 = 0$. Hence $a_i = 0$ for all $i \notin \{27, 59\}$ by Lemmas 2.3, 4.1 and 2.7. It follows from (2.6) that $a_{27} = 53/4$, a contradiction. □

Lemma 5.16. *There exists no* $[466, 4, 407]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[466, 4, 407]_8$ code. We can see $a_i = 0$ for all $i \notin \{0, 26, 27, 28, 42, 58, 59\}$ similarly with the case for a putative $[467, 4, 408]_8$ code. It follows from (2.7) with $\lambda_2 = 0$ that

$$1711a_0 + 528a_{26} + 496a_{27} + 465a_{28} + 136a_{42} = 2996. \tag{5.10}$$

Suppose $a_0 > 0$. From the possible solutions for (2.8) with $i = t = 0$, we have $a_0 = 1$ and $a_{26} = a_{27} = a_{28} = a_{42} = 0$, contradicting (5.10). Hence $a_0 = 0$.
Suppose $a_{28} > 0$. Let δ be a 28-plane. Then δ has spectrum $(\tau_0, \tau_4) = (10, 63)$ by Lemma 4.9. Setting $i = 28$, the minimum possible contributions of c_j's in (2.8) to the LHS of (5.10) are $(c_{42}, c_{59}) = (2, 6)$ for $t = 0$; $(c_{58}, c_{59}) = (2, 6)$ for $t = 4$. Estimating the LHS of (5.10) with the spectrum of δ, we get

$$2996 \ge 465 + 272\tau_0 + 0 \cdot \tau_4 = 3185,$$

a contradiction. Hence $a_{28} = 0$.
Applying Theorem 2.4, $\mathcal{C}$ is extendable, which contradicts Lemma 5.15. □

Lemma 5.17. *There exists no* $[458, 4, 400]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[458, 4, 400]_8$ code. We have $\gamma_0 = 1$ and $a_i = 0$ for all $i \notin \{0, 10, 26, 28, 42, 58\}$ by Lemmas 2.2, 2.3, 4.1, 2.7. It follows from (2.3)-(2.5) that $609a_0 + 384a_{10} + 128a_{26} + 105a_{28} = -288$, a contradiction. □

Lemma 5.18. *There exists no* $[457, 4, 399]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[457, 4, 399]_8$ code. By Lemma 4.8, the spectrum of a γ_2-plane satisfies $\tau_1 = 0$. Hence $a_i = 0$ for all $i \notin \{0, 9, 10, 25\text{-}28, 33, 41, 42, 49, 57, 58\}$ by Lemmas 2.3 and 4.1. It follows from (2.7) with $\lambda_2 = 0$ that

$$1653a_0 + 1176a_9 + 1128a_{10} + 528a_{25} + 496a_{26} + 465a_{27} + 435a_{28} + 300a_{33} + 136a_{41} + 120a_{42} + 36a_{49} = 3192. \quad (5.11)$$

Setting $i = t = 0$, the possible solution for (2.8) is $(c_{57}, c_{58}) = (7, 1)$. Hence $a_0 = 1$ and $a_j = 0$ for $1 \le j \le 56$, contradicting (5.11). Hence $a_0 = 0$.
Suppose $a_{27} > 0$. Setting $i = 27$ in (2.8), $c_{27} + c_{28} \le 1$ for $t = 0$ and $c_{27} + c_{28} = 0$ for $t = 3, 4$. Hence $\sum_{i \not\equiv n, n-d \pmod q} a_i = a_{27} + a_{28} \le 1 + 1 \cdot 10 = 11$, which implies that $\mathcal{C}$ is extendable by Theorem 2.6, a contradiction. Thus $a_{27} = 0$. One can prove $a_{28} = 0$ similarly. Applying Theorem 2.4, $\mathcal{C}$ is extendable, which contradicts Lemma 5.17. This completes the proof. $\square$

Lemma 5.19. *There exists no* $[330, 4, 288]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[330, 4, 288]_8$ code. From Table 2, there are four possible spectra for a γ_2-plane. Since δ has no i-line for $i = 1$, It follows from Lemmas 2.3, 4.1 and 2.7 that $a_i = 0$ for all $i \notin \{0, 10, 26, 28, 42\}$. Hence, from (2.3)-(2.5), we get

$$147a_0 + 72a_{10} + 4a_{26} = 360. \quad (5.12)$$

Setting $i = t = 0$, (2.8) has no solution. Hence $a_0 = 0$.
Suppose $a_{10} > 0$ and let δ be a 10-plane. Recall from Table 2 that δ has spectrum $(\tau_0, \tau_2) = (28, 45)$. For $i = 10$, the solutions for (2.8) are $(c_{26}, c_{42}) = (1, 7)$ for $t = 0$; $c_{42} = 8$ for $t = 2$. Hence $a_{10} = 1$, $a_{28} = 0$ and $a_{26} = \tau_0 = 28$, which implies that the LHS of (5.12) is equal to $72 + 4 \cdot 28 = 184$, a contradiction. Hence $a_{10} = 0$. Then, from (5.12) and (2.6), we get $a_{28} = -480/7$, a contradiction. $\square$

Lemma 5.20. *There exists no* $[329, 4, 287]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[329, 4, 287]_8$ code. Since a γ_2-plane has no 1-line, $a_i = 0$ for all $i \notin \{0, 9, 10, 25\text{-}28, 33, 41, 42\}$ by Lemmas 2.3 and 4.1.
Suppose $a_0 > 0$. Setting $i = t = 0$, the solution for (2.8) is $(c_{41}, c_{42}) = (7, 1)$. Hence $a_0 = 1$ and $a_j = 0$ for $1 \le j \le 40$. Then $\sum_{i \not\equiv n, n-d \pmod q} a_i = a_0 = 1$, which contradicts Theorem 2.6. Hence $a_0 = 0$.
Suppose $a_{28} > 0$. A 28-plane has spectrum $(\tau_0, \tau_4) = (10, 63)$ by Lemma 4.9. Setting $i = 28$, the solutions for (2.8) satisfies $c_{27} + c_{28} \le 2$ for $t = 0$ and $c_{27} + c_{28} = 0$ for $t = 4$. Hence $\sum_{i \not\equiv n, n-d \pmod q} a_i = a_{27} + a_{28} \le 1 + 2\tau_0 = 21$, which contradicts Theorem 2.6. Thus $a_{28} = 0$. One can prove $a_{27} = 0$ similarly.
Applying Theorem 2.4, $\mathcal{C}$ is extendable, which contradicts Lemma 5.19. $\square$

Lemma 5.21. *There exists no* $[328, 4, 286]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[328, 4, 286]_8$ code. Since a γ_2-plane has no 1-line, $a_i = 0$ for all $i \notin \{0, 8\text{-}10, 24\text{-}28, 32, 33, 40\text{-}42\}$ by Lemmas 2.3 and 4.1. It follows from (2.7) with $\lambda_2 = 0$ that

$$\begin{aligned} 861a_0 + 561a_8 + 528a_9 + 496a_{10} + 153a_{24} + 136a_{25} + 120a_{26} \\ +105a_{27} + 91a_{28} + 45a_{32} + 36a_{33} + a_{40} = 4633. \end{aligned} \tag{5.13}$$

We first note that $a_j \le 1$ holds for all $j \le 10$ from (2.1).
Suppose $a_0 > 0$. Setting $i = t = 0$, the maximum possible contribution of c_j's in (2.8) to the LHS of (5.13) is $(c_{40}, c_{42}) = (4, 4)$. Hence $a_0 = 1$, $a_j = 0$ for $1 \le j \le 39$, and $a_{40} \le 4 \cdot 73 = 292$, which implies that the LHS of (5.13) is at most $861 + 292 = 1153$, a contradiction. Thus $a_0 = 0$.
Suppose $a_{28} > 0$. A 28-plane has spectrum $(\tau_0, \tau_4) = (10, 63)$ by Lemma 4.9. For $i = 28$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.13) are $(c_8, c_{40}, c_{42}) = (1, 1, 6)$ if $c_8 > 0$, $(c_9, c_{40}, c_{41}, c_{42}) = (1, 1, 1, 5)$ if $c_9 > 0$, $(c_{10}, c_{40}, c_{42}) = (1, 2, 5)$ if $c_{10} > 0$ and $(c_{24}, c_{42}) = (2, 6)$ if $c_8 = c_9 = c_{10} = 0$ for $t = 0$; $(c_{30}, c_{42}) = (2, 6)$ for $t = 4$. Since a_8, a_9, a_{10} are at most 1, estimating the LHS of (5.13), we get

$$4633 \le 91 + 561 \cdot 1 + 529 \cdot 1 + 498 \cdot 1 + 306(\tau_0 - 3) + 2\tau_4 = 3947,$$

a contradiction. Hence $a_{28} = 0$. One can prove $a_{27} = 0$ similarly. Applying Theorem 2.5, $\mathcal{C}$ is extendable, which contradicts Lemma 5.20. This completes the proof. □

Lemma 5.22. *There exists no* $[254, 4, 221]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[254, 4, 221]_8$ code. Recall from Table 2 that the spectrum of a γ_2-plane δ is $(\tau_0, \tau_3, \tau_5) = (9, 16, 48)$, $(\tau_0, \tau_1, \tau_4, \tau_5) = (4, 5, 28, 36)$ or $(\tau_0, \tau_3, \tau_4, \tau_5) = (6, 10, 18, 39)$. Since δ has no 2-line, $a_i = 0$ for all $i \notin \{0, 1, 14, 15, 22\text{-}28, 30\text{-}33\}$ by Lemmas 2.3 and 4.1. It follows from (2.7) with $\lambda_2 = 0$ that

$$\begin{aligned} 528a_0 + 496a_1 + 171a_{14} + 153a_{15} + 55a_{22} + 45a_{23} + 36a_{24} + 28a_{25} \\ +21a_{26} + 15a_{27} + 10a_{28} + 3a_{30} + a_{31} = 4715. \end{aligned} \tag{5.14}$$

Suppose $a_0 > 0$. Setting $a = c = 0$ in (2.1) we have $b \ge 23$. Hence $a_0 = 1$ and $a_j = 0$ for $1 \le j \le 22$. Calculating $5 \cdot (2.6) - (5.14)$ gives $5a_{23} + 9a_{24} + 12a_{25} + 14a_{26} + 15a_{27} + 15a_{28} + 12a_{30} + 9a_{31} + 5a_{32} = -537$, a contradiction. Hence $a_0 = 0$. We can prove $a_1 = 0$ similarly.
Setting $i = 14$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.14) are $(c_{14}, c_{28}, c_{33}) = (1, 1, 6)$ for $t = 0$; $(c_{22}, c_{30}, c_{31}, c_{33}) = (1, 1, 1, 5)$ for $t = 1$; $(c_{25}, c_{33}) = (1, 7)$ for $t = 2$; $c_{33} = 8$ for $t = 3$. Estimating the LHS of (5.14) with the spectrum of (3) in Lemma 4.10, we get

$$171 + 181\tau_0 + 59\tau_1 + 28\tau_2 + 0 \cdot \tau_3 = 2949 + 88v \le 2949 + 88 \cdot 19 = 4533,$$

a contradiction. Hence $a_{14} = 0$. One can see $a_{15} = 0$ similarly.

Now, calculating $5 \cdot (2.6) - (5.14)$ again gives $5a_{23} + 9a_{24} + 12a_{25} + 14a_{26} + 15a_{27} + 15a_{28} + 12a_{30} + 9a_{31} + 5a_{32} = -900$, a contradiction. $\square$

The following theorem gives the nonexistence of $[213, 4, 185]_8$ codes.

Theorem 5.23. *There exists no* $[(q^3 - q^2 - 3q + 2)/2, 4, q^3/2 - q^2 - q + 1]_q$ *code for even* $q \geq 4$.

Proof. For a putative Griesmer $[(q^3 - q^2 - 3q + 2)/2, 4, q^3/2 - q^2 - q + 1]_q$ code, the spectrum of a γ_2-plane is $(\tau_0, \tau_{q/2}) = (q + 2, q^2 - 1)$ by Lemma 4.9. Hence we have $a_i = 0$ for all $i \notin \{0, (q^2 - 3q + 2)/2, \cdots, (q^2 - q)/2\}$ by Lemma 2.3 and the Griesmer bound. It follows from (2.7) with $\lambda_2 = 0$ that

$$\binom{\gamma_2}{2} a_0 + \sum_{j=0}^{q-3} \binom{q-1-j}{2} a_{s+j} = (q^5 + q^4 - 12q^2 - 4q + 8)/8, \tag{5.15}$$

where $s = (q^2 - 3q + 2)/2$. We can see $a_0 \leq 1$ from (2.1). Calculating $4(q-2)\cdot(2.6) - 8\cdot(5.15)$ gives $0 \leq \sum_{j=0}^{q-2} 4j(q-1-j)a_{s+j} = -(q^5 - 5q^4 + 8q^3 - 4q^2) + (q^4 - 4q^3 + 5q^2 - 2q)a_0 \leq -q(q^4 - 6q^3 + 12q^2 - 9q + 2) < 0$, a contradiction. $\square$

Lemma 5.24. *There exists no* $[204, 4, 177]_8$ *code.*

Proof. For a putative $[204, 4, 177]_8$ code, the spectrum of a γ_2-plane δ is $(\tau_0, \tau_3, \tau_4) = (10, 9, 54)$ by Lemma 4.9. Since δ has no t-line for $t = 1, 2$, we have $a_i = 0$ for all $i \notin \{0, 12\text{-}15, 20\text{-}27\}$ by Lemmas 2.3 and 4.1. Then, (2.7) gives

$$\begin{aligned} 351a_0 + 105a_{12} + 91a_{13} + 78a_{14} + 66a_{15} + 21a_{20} \\ +15a_{21} + 10a_{22} + 6a_{23} + 3a_{24} + a_{25} = 4497. \end{aligned} \tag{5.16}$$

It is easy to see that $a_0 \leq 1$ from (2.1).
Suppose $a_{15} > 0$. Setting $i = 15$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.16) are $(c_0, c_{27}) = (1, 7)$ if $c_0 > 0$ and $(c_{12}, c_{15}, c_{27}) = (1, 1, 6)$ if $c_0 = 0$ for $t = 0$; $(c_{15}, c_{26}) = (1, 7)$ for $t = 1$ since $c_{27} = 0$; $(c_{23}, c_{26}) = (1, 7)$ for $t = 2$ since $c_{27} = 0$; $(c_{24}, c_{27}) = (1, 7)$ for $t = 3$. Since $a_0 \leq 1$, estimating the LHS of (5.16) with the spectrum of (4) in Lemma 4.10, we get

$$66 + 351 \cdot 1 + 171(\tau_0 - 1) + 66\tau_1 + 6\tau_2 + 3\tau_3 = 3795 - 12v \leq 3795 - 12 \cdot 8 = 3699,$$

a contradiction. Hence $a_{15} = 0$.
Suppose $a_0 > 0$. Then, we have $a_0 = 1$ and $a_{12} = a_{13} = a_{14} = 0$ from (2.1), and calculating $3 \cdot (2.6) - (5.16)$ gives $3a_{21} + 5a_{22} + 6a_{23} + 6a_{24} + 5a_{25} + 3a_{26} = -1518$, a contradiction. Hence $a_0 = 0$.
Suppose $a_{14} > 0$. For $i = 14$, the maximum possible contributions of c_j's in (2.8) to the LHS of (5.16) are $(c_{14}, c_{27}) = (2, 6)$ for $t = 0$; $(c_{20}, c_{22}, c_{26}) = (1, 1, 6)$ for $t = 1$ since

$c_{27} = 0$; $(c_{24}, c_{26}) = (1, 7)$ for $t = 2$ since $c_{27} = 0$; $(c_{25}, c_{27}) = (1, 7)$ for $t = 3$. Estimating the LHS of (5.16) with the spectrum of (3) in Lemma 4.10, we get

$$78 + 156\tau_0 + 31\tau_1 + 3\tau_2 + 1\tau_3 = 1835 + 71v \le 1835 + 71 \cdot 19 = 3184,$$

a contradiction. Hence $a_{14} = 0$. One can prove $a_{13} = a_{12} = 0$ similarly.
Now, calculating $3 \cdot (2.6) - (5.16)$ gives $3a_{21} + 5a_{22} + 6a_{23} + 6a_{24} + 5a_{25} + 3a_{26} = -1518$, a contradiction. This completes the proof. □

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Table 3: Values and bounds for $n_8(4,d)$ for $d \leq 832$.

d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$
1	4	4	61	71	72	121	140	141-142
2	5	5	62	72	73	122	141	142-143
3	6	6	63	73	74	123	142	143-144
4	7	7	64	74	75	124	143	144-145
5	8	8	65	77	77	125	144	145-146
6	9	9	66	78	78	126	145	146-147
7	10	11	67	79	79	127	146	147-148
8	11	12	68	80	80	128	147	148-149
9	13	13	69	81	81-82	129	150	150-151
10	14	14	70	82	82-83	130	151	151-152
11	15	15	71	83	83-84	131	152	152-153
12	16	16	72	84	84-85	132	153	153-154
13	17	17-18	73	86	86-87	133	154	154-156
14	18	18-19	74	87	87-88	134	155	155-157
15	19	20	75	88	88-89	135	156	156-158
16	20	21	76	89	89-90	136	157	157-159
17	22	22	77	90	90-91	137	159	159-161
18	23	23	78	91	91-92	138	160	160-162
19	24	24	79	92	92-93	139	161	161-163
20	25	25	80	93	94	140	162	162-164
21	26	26-27	81	95	95-96	141	163	163-165
22	27	28	82	96	96-97	142	164	164-166
23	28	29	83	97	97-98	143	165	165-167
24	29	30	84	98	98-99	144	166	166-168
25	31	31-32	85	99	99-100	145	168	168-170
26	32	32-33	86	100	100-101	146	169	169-171
27	33	33-34	87	101	102	147	170	170-172
28	34	34-35	88	102	103	148	171	171-173
29	35	35-36	89	104	104-105	149	172	172-174
30	36	37	90	105	105-106	150	173	173-175
31	37	38	91	106	106-107	151	174	174-176
32	38	39	92	107	107-108	152	175	175-177
33	40	40-41	93	108	108-109	153	177	177-179
34	41	41-42	94	109	109-110	154	178	178-180
35	42	42-43	95	110	111	155	179	179-181
36	43	43-44	96	111	112	156	180	180-182
37	44	44-45	97	113	114	157	181	181-183
38	45	46-47	98	114	115	158	182	182-184
39	46	47-48	99	115	116	159	183	183-185
40	47	48-49	100	116	117	160	184	184-186
41	49	49-50	101	117	118	161	186	186-188
42	50	50-51	102	118	119	162	187	187-189
43	51	51-52	103	119	120	163	188	188-190
44	52	52-53	104	120	121	164	189	189-191
45	53	53-54	105	122	123	165	190	190-192
46	54	55	106	123	124	166	191	191-193
47	55	56	107	124	125	167	192	192-194
48	56	57	108	125	126	168	193	193-195
49	58	58	109	126	127	169	195	195-197
50	59	59	110	127	128	170	196	196-198
51	60	60	111	128	129	171	197	197-199
52	61	61	112	129	130	172	198	198-200
53	62	62	113	131	132	173	199	199-201
54	63	63	114	132	133	174	200	200-202
55	64	64	115	133	134	175	201	201-203
56	65	65	116	134	135	176	202	202-204
57	67	68	117	135	136	177	204	205-206
58	68	69	118	136	137	178	205	206-208
59	69	70	119	137	138	179	206	207-209
60	70	71	120	138	139	180	207	208-210

Table 3: Continued.

d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$
181	208	209-211	241	277	278-279	301	345	346-347
182	209	210-212	242	278	279-280	302	346	347-348
183	210	211-213	243	279	280-281	303	347	348-349
184	211	212-214	244	280	281-282	304	348	349-350
185	213	214-215	245	281	282-283	305	350	351-352
186	214	215-216	246	282	283-284	306	351	352-353
187	215	216-217	247	283	284-286	307	352	353-354
188	216	217-218	248	284	285-287	308	353	354-355
189	217	218-219	249	286	287-288	309	354	355-356
190	218	219-220	250	287	288-289	310	355	356-357
191	219	220-221	251	288	289-290	311	356	357-358
192	220	221-222	252	289	290-291	312	357	358-359
193	223	223-224	253	290	291-292	313	359	360-361
194	224	224-226	254	291	292-293	314	360	361-362
195	225	225-227	255	292	293-294	315	361	362-363
196	226	226-228	256	293	294-295	316	362	363-364
197	227	227-229	257	296	296	317	363	364-366
198	228	228-230	258	297	297	318	364	365-367
199	229	229-231	259	298	298-299	319	365	366-368
200	230	230-232	260	299	299-300	320	366	367-369
201	232	232-234	261	300	300-301	321	369	369-370
202	233	233-235	262	301	301-302	322	370	370-371
203	234	234-236	263	302	302-303	323	371	371-372
204	235	235-237	264	303	303-304	324	372	372-373
205	236	236-238	265	305	305	325	373	373-374
206	237	237-239	266	306	306	326	374	374-375
207	238	238-241	267	307	307	327	375	375-376
208	239	239-242	268	308	308	328	376	376-377
209	241	241-243	269	309	309	329	378	378-380
210	242	242-244	270	310	310	330	379	379-381
211	243	243-245	271	311	311	331	380	380-382
212	244	244-246	272	312	312	332	381	381-383
213	245	245-247	273	314	314-315	333	382	382-384
214	246	246-248	274	315	315-316	334	383	383-385
215	247	247-249	275	316	316-317	335	384	384-386
216	248	248-250	276	317	317-318	336	385	385-387
217	250	250-252	277	318	318-319	337	387	388-389
218	251	251-253	278	319	319-320	338	388	389-390
219	252	252-254	279	320	320-321	339	389	390-391
220	253	253-255	280	321	321-322	340	390	391-392
221	254	255-257	281	323	323-324	341	391	392-393
222	255	256-258	282	324	324-325	342	392	393-394
223	256	257-259	283	325	325-326	343	393	394-395
224	257	258-260	284	326	326-327	344	394	395-396
225	259	260-261	285	327	327-328	345	396	397-398
226	260	261-262	286	328	329	346	397	398-399
227	261	262-263	287	329	330	347	398	399-400
228	262	263-265	288	330	331	348	399	400-401
229	263	264-266	289	332	333-334	349	400	401-402
230	264	265-267	290	333	334-335	350	401	402-403
231	265	266-268	291	334	335-336	351	402	403-404
232	266	267-269	292	335	336-337	352	403	404-405
233	268	269-270	293	336	337-338	353	405	406-407
234	269	270-271	294	337	338-339	354	406	407-408
235	270	271-272	295	338	339-340	355	407	408-409
236	271	272-273	296	339	340-341	356	408	409-410
237	272	273-275	297	341	342-343	357	409	410-411
238	273	274-276	298	342	343-344	358	410	411-412
239	274	275-277	299	343	344-345	359	411	412-413
240	275	276-278	300	344	345-346	360	412	413-414

Table 3: Continued.

d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$
361	414	415-416	421	482	483	481	551	551
362	415	416-417	422	483	484	482	552	552
363	416	417-418	423	484	485	483	553	553
364	417	418-419	424	485	486	484	554	554
365	418	419-420	425	487	488	485	555	555
366	419	420-421	426	488	489	486	556	556
367	420	421-422	427	489	490	487	557	557
368	421	422-423	428	490	491	488	558	558
369	423	424-425	429	491	492	489	560	560
370	424	425-426	430	492	493	490	561	561
371	425	426-427	431	493	494	491	562	562
372	426	427-428	432	494	495	492	563	563
373	427	428-429	433	496	497	493	564	564
374	428	429-430	434	497	498	494	565	565
375	429	430-431	435	498	499	495	566	566
376	430	431-432	436	499	500	496	567	567
377	432	433	437	500	501	497	569	569
378	433	434	438	501	502	498	570	570
379	434	435-436	439	502	503	499	571	571
380	435	436-437	440	503	504	500	572	572
381	436	437-438	441	505	505	501	573	573
382	437	438-439	442	506	506	502	574	574
383	438	439-440	443	507	507	503	575	575
384	439	440-441	444	508	508	504	576	576
385	442	442	445	509	509	505	578	578
386	443	443	446	510	510	506	579	579
387	444	444	447	511	511	507	580	580
388	445	445	448	512	512	508	581	581
389	446	446	449	515	515	509	582	582
390	447	447	450	516	516	510	583	583
391	448	448	451	517	517	511	584	584
392	449	449	452	518	518	512	585	585
393	451	451-452	453	519	519	513	589	589
394	452	452-453	454	520	520	514	590	590
395	453	453-454	455	521	521	515	591	591
396	454	454-455	456	522	522	516	592	592
397	455	455-456	457	524	524	517	593	593
398	456	456-457	458	525	525	518	594	594
399	457	458	459	526	526	519	595	595
400	458	459	460	527	527	520	596	596
401	460	460-461	461	528	528	521	598	598
402	461	461-462	462	529	529	522	599	599
403	462	462-463	463	530	530	523	600	600
404	463	463-464	464	531	531	524	601	601
405	464	464-465	465	533	533	525	602	602
406	465	465-466	466	534	534	526	603	603
407	466	467	467	535	535	527	604	604
408	467	468	468	536	536	528	605	605
409	469	469-470	469	537	537	529	607	607
410	470	470-471	470	538	538	530	608	608
411	471	471-472	471	539	539	531	609	609
412	472	472-473	472	540	540	532	610	610
413	473	473-474	473	542	542	533	611	611
414	474	475	474	543	543	534	612	612
415	475	476	475	544	544	535	613	613
416	476	477	476	545	545	536	614	614
417	478	479	477	546	546	537	616	616
418	479	480	478	547	547	538	617	617
419	480	481	479	548	548	539	618	618
420	481	482	480	549	549	540	619	619

Table 3: Continued.

d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$
541	620	620	601	689	689-690	661	757	757-758
542	621	621	602	690	690-691	662	758	758-759
543	622	622	603	691	691-692	663	759	759-760
544	623	623	604	692	692-693	664	760	760-761
545	625	625	605	693	693-694	665	762	762-763
546	626	626	606	694	694-695	666	763	763-764
547	627	627	607	695	695-696	667	764	764-765
548	628	628	608	696	696-697	668	765	765-766
549	629	629	609	698	698-699	669	766	766-767
550	630	630	610	699	699-700	670	767	767-768
551	631	631	611	700	700-701	671	768	768-769
552	632	632	612	701	701-702	672	769	769-770
553	634	634	613	702	702-703	673	771	771-772
554	635	635	614	703	703-704	674	772	772-773
555	636	636	615	704	704-705	675	773	773-774
556	637	637	616	705	705-706	676	774	774-775
557	638	638	617	707	707-708	677	775	775-776
558	639	639	618	708	708-709	678	776	776-777
559	640	640	619	709	709-710	679	777	777-778
560	641	641	620	710	710-711	680	778	778-779
561	643	643	621	711	711-712	681	780	780-781
562	644	644	622	712	712-713	682	781	781-782
563	645	645	623	713	713-714	683	782	782-783
564	646	646	624	714	714-715	684	783	783-784
565	647	647	625	716	716-717	685	784	784-785
566	648	648	626	717	717-718	686	785	785-786
567	649	649	627	718	718-719	687	786	786-787
568	650	650	628	719	719-720	688	787	787-788
569	652	652-653	629	720	720-721	689	789	789-790
570	653	653-654	630	721	721-722	690	790	790-791
571	654	654-655	631	722	722-723	691	791	791-792
572	655	655-656	632	723	723-724	692	792	792-793
573	656	656-657	633	725	725-727	693	793	793-794
574	657	657-658	634	726	726-728	694	794	794-795
575	658	658-659	635	727	727-729	695	795	795-796
576	659	659-660	636	728	728-730	696	796	796-797
577	662	662	637	729	729-731	697	798	798-799
578	663	663	638	730	730-732	698	799	799-800
579	664	664	639	731	732-733	699	800	800-801
580	665	665	640	732	733-734	700	801	801-802
581	666	666-667	641	735	735-736	701	802	802-803
582	667	667-668	642	736	736-737	702	803	804
583	668	668-669	643	737	737-738	703	804	805
584	669	669-670	644	738	738-739	704	805	806
585	671	671-672	645	739	739-740	705	808	808
586	672	672-673	646	740	740-741	706	809	809
587	673	673-674	647	741	741-742	707	810	810
588	674	674-675	648	742	742-743	708	811	811
589	675	675-676	649	744	744-745	709	812	812
590	676	676-677	650	745	745-746	710	813	813
591	677	677-678	651	746	746-747	711	814	814
592	678	678-679	652	747	747-748	712	815	815
593	680	680-681	653	748	748-749	713	817	817
594	681	681-682	654	749	749-750	714	818	818
595	682	682-683	655	750	750-751	715	819	819
596	683	683-684	656	751	751-752	716	820	820
597	684	684-685	657	753	753-754	717	821	821
598	685	685-686	658	754	754-755	718	822	822
599	686	686-687	659	755	755-756	719	823	823
600	687	687-688	660	756	756-757	720	824	824

Table 3: Continued.

d	$g_8(4,d)$	$n_8(4,d)$	d	$g_8(4,d)$	$n_8(4,d)$
721	826	826	781	894	894
722	827	827	782	895	895
723	828	828	783	896	896
724	829	829	784	897	897
725	830	830	785	899	899-900
726	831	831	786	900	900-901
727	832	832	787	901	901-902
728	833	833	788	902	902-903
729	835	835-836	789	903	903-904
730	836	836-837	790	904	904-905
731	837	837-838	791	905	905-906
732	838	838-839	792	906	906-907
733	839	839-840	793	908	908-909
734	840	840-841	794	909	909-910
735	841	841-842	795	910	910-911
736	842	842-843	796	911	911-912
737	844	844-845	797	912	912-913
738	845	845-846	798	913	913-914
739	846	846-847	799	914	914-915
740	847	847-848	800	915	915-916
741	848	848-849	801	917	917-918
742	849	849-850	802	918	918-919
743	850	850-851	803	919	919-920
744	851	851-852	804	920	920-921
745	853	853-854	805	921	921-922
746	854	854-855	806	922	922-923
747	855	855-856	807	923	923-924
748	856	856-857	808	924	924-925
749	857	857-858	809	926	926-927
750	858	859	810	927	927-928
751	859	860	811	928	928-929
752	860	861	812	929	929-930
753	862	862-863	813	930	931
754	863	863-864	814	931	932
755	864	864-865	815	932	933
756	865	865-866	816	933	934
757	866	867	817	935	935-936
758	867	868	818	936	936-937
759	868	869	819	937	937-938
760	869	870	820	938	939
761	871	872	821	939	940
762	872	873	822	940	941
763	873	874	823	941	942
764	874	875	824	942	943
765	875	876	825	944	945
766	876	877	826	945	946
767	877	878	827	946	947
768	878	879	828	947	948
769	881	881	829	948	949
770	882	882	830	949	950
771	883	883	831	950	951
772	884	884	832	951	952
773	885	885			
774	886	886			
775	887	887			
776	888	888			
777	890	890			
778	891	891			
779	892	892			
780	893	893			