Algebraically grid-like graphs have large tree-width

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Abstract

By the Grid Minor Theorem of Robertson and Seymour, every graph of sufficiently large tree-width contains a large grid as a minor. Tree-width may therefore be regarded as a measure of ‘grid-likeness’ of a graph.

The grid contains a long cycle on the perimeter, which is the $\mathbb{F}_2$-sum of the rectangles inside. Moreover, the grid distorts the metric of the cycle only by a factor of two. We prove that every graph that resembles the grid in this algebraic sense has large tree-width:

Let $k, p$ be integers, $\gamma$ a real number and $G$ a graph. Suppose that $G$ contains a cycle of length at least $2\gamma pk$ which is the $\mathbb{F}_2$-sum of cycles of length at most $p$ and whose metric is distorted by a factor of at most $\gamma$. Then $G$ has tree-width at least $k$.

Mathematics Subject Classifications: 05C83, 05C12, 05C38

1 Introduction

The edge space of a graph $G = (V, E)$ is the vector space over the 2-element field $\mathbb{F}_2$ consisting of all maps $E \to \mathbb{F}_2$, with addition defined pointwise as $(f \oplus g)(e) = f(e) + g(e)$. Each such map $f : E \to \mathbb{F}_2$ may be identified with the set of all $e \in E$ for which $f(e) = 1$. Addition of maps then corresponds to symmetric difference of sets of edges, and we call $F_1 \oplus F_2 := F_1 \Delta F_2$ the $\mathbb{F}_2$-sum of $F_1, F_2 \subseteq E$.

The cycle space of $G$ is the subspace of the edge space which is generated by the (edge sets of) cycles of $G$. In this context, we do not formally distinguish between a cycle $C \subseteq G$ and its set of edges $E(C) \subseteq E$ and simply speak of the $\mathbb{F}_2$-sum of cycles. See the book by Diestel [5] for more background on the cycle space.

For a positive integer $n$, the $(n \times n)$-grid is the graph $G_n$ whose vertices are all pairs $(i, j)$ with $1 \leq i, j \leq n$, where two points are adjacent when they are at Euclidean
distance 1. The cycle $C_n$, which bounds the outer face in the natural drawing of $G_n$ in the plane, has length $4(n - 1)$ and is the $\mathbb{F}_2$-sum of the rectangles bounding the inner faces. This is by itself not a distinctive feature of graphs with large tree-width: The situation is similar for the $n$-wheel $W_n$, the graph consisting of a cycle $D_n$ of length $n$ and a vertex $x \notin D_n$ which is adjacent to every vertex of $D_n$. There, $D_n$ is the $\mathbb{F}_2$-sum of all triangles $xyz$ for $yz \in E(D_n)$. Still, $W_n$ only has tree-width 3.

The key difference is the fact that in the wheel, the metric of the cycle is heavily distorted: any two vertices of $D_n$ are at distance at most two within $W_n$, even if they are far apart within $D_n$. In the grid, however, the distance between two vertices of $C_n$ within $G_n$ is at least half of their distance within $C_n$.

In order to incorporate this factor of two and to allow for more flexibility, we equip the edges of our graphs with lengths. For a graph $G$, a length-function on $G$ is simply a map $\ell : E(G) \to \mathbb{R}_{>0}$. We then define the $\ell$-length $\ell(H)$ of a subgraph $H \subseteq G$ as the sum of the lengths of all edges of $H$. This naturally induces a notion of distance between two vertices of $G$, where we define $d_G^\ell$ as the minimum $\ell$-length of a path containing both. A subgraph $H \subseteq G$ is $\ell$-geodesic if it contains a path of length $d_G^\ell(a, b)$ between any two vertices $a, b \in V(H)$.

When no length-function is specified, the notions of length, distance and geodesity are to be read with respect to $\ell \equiv 1$ constant.

On the grid-graph $G_n$, consider the length-function $\ell$ which is equal to 1 on $E(C_n)$ and assumes the value 2 elsewhere. Then $C_n$ is $\ell$-geodesic of length $\ell(C_n) = 4(n - 1)$ and the sum of cycles of $\ell$-length at most 8. We show that any graph which shares this algebraic feature has large tree-width.

**Theorem 1.** Let $k$ be a positive integer and $r > 0$. Let $G$ be a graph with rational-valued length-function $\ell$. Suppose $G$ contains an $\ell$-geodesic cycle $C$ with $\ell(C) \geq 2rk$, which is the $\mathbb{F}_2$-sum of cycles of $\ell$-length at most $r$. Then the tree-width of $G$ is at least $k$.

In Section 3, we prove a qualitative converse to Theorem 1, showing that graphs of large tree-width are ‘algebraically grid-like’.

The starting point of Theorem 1 was a similar result of Matthias Hamann and the author [2]. There, it is assumed that not only the fixed cycle $C$, but the whole cycle space of $G$ is generated by short cycles.

**Theorem 2** ([2, Corollary 3]). Let $k, p$ be positive integers. Let $G$ be a graph whose cycle space is generated by cycles of length at most $p$. If $G$ contains a geodesic cycle of length at least $kp$, then the tree-width of $G$ is at least $k$.

It should be noted that Theorem 2 is not implied by Theorem 1, as the constant factors are different. In fact, the proofs are also quite different, although Lemma 5 below was inspired by a similar parity-argument in [2].

It is tempting to think that, conversely, Theorem 1 could be deduced from Theorem 2 by adequate manipulation of the graph $G$, but we have not been successful with such attempts.
2 Proof of Theorem 1

The relation to tree-width is established via a well-known separation property of graphs of bounded tree-width, due to Robertson and Seymour [3].

Lemma 3 ([3]). Let \( k \) be a positive integer, \( G \) a graph and \( A \subseteq V(G) \). If the tree-width of \( G \) is less than \( k \), then there exists \( X \subseteq V(G) \) with \( |X| \leq k \) such that every component of \( G - X \) contains at most \( \frac{1}{2}|A \setminus X| \) vertices of \( A \).

It is not hard to see that Theorem 1 can be reduced to the case where \( \ell \equiv 1 \). This case is treated in the next theorem.

Theorem 4. Let \( k, p \) be positive integers. Let \( G \) be a graph containing a geodesic cycle \( C \) of length at least \( 4\lfloor p/2 \rfloor k \), which is the \( \mathbb{F}_2 \)-sum of cycles of length at most \( p \). Then for every \( X \subseteq V(G) \) of order at most \( k \), some component of \( G - X \) contains at least half the vertices of \( C \).

Proof of Theorem 1, assuming Theorem 4. Let \( \mathcal{D} \) be a set of cycles of \( \ell \)-length at most \( r \) with \( C = \bigoplus \mathcal{D} \). Since \( \ell \) is rational-valued and the premise also holds for \( r' \) the maximum \( \ell \)-length of a cycle in \( \mathcal{D} \), we may assume that \( r \in \mathbb{Q} \). Take an integer \( M \) so that \( rM \) and \( \ell'(e) \) are natural numbers for every \( e \in E(G) \).

Obtain the subdivision \( G' \) of \( G \) by replacing every \( e \in E(G) \) by a path of length \( \ell'(e) \). Denote by \( C', D' \) the subdivisions of \( C \) and \( D \) for every \( D \in \mathcal{D} \), respectively. Then \( C' = \bigoplus_{D \in \mathcal{D}} D' \) and \( |C'| = \ell(C) \geq 2(Mr)k \), while \( |D'| = \ell(D) \leq Mr \) for every \( D \in \mathcal{D} \).

Assume for a contradiction that the tree-width of \( G \) was less than \( k \). Since tree-width is invariant under subdivision, the tree-width of \( G' \) is also less than \( k \). By Lemma 3, there exists a set \( X \subseteq V(G') \) with \( |X| \leq k \) such that every component of \( G' - X \) contains at most \( \frac{1}{2}|V(C') \setminus X| \) vertices of \( V(C') \). Since \( C' \) is a connected subgraph of \( G' \), \( X \) must contain at least one vertex of \( C' \). By Theorem 4, there is a component \( K \) of \( G' - X \) that contains at least half the vertices of \( C' \). But then

\[
\frac{1}{2}|C'| \leq |K \cap V(C')| \leq \frac{1}{2}|V(C') \setminus X| < \frac{1}{2}|C'|,
\]

which is a contradiction. \( \square \)

Our goal is now to prove Theorem 4. The proof consists of two separate lemmas. The first lemma involves separators and \( \mathbb{F}_2 \)-sums of cycles.

Lemma 5. Let \( G \) be a graph, \( C \subseteq G \) a cycle and \( \mathcal{D} \) a set of cycles in \( G \) such that \( C = \bigoplus \mathcal{D} \). Let \( \mathcal{R} \) be a set of disjoint vertex-sets of \( G \) such that for every \( R \in \mathcal{R} \), \( R \cap V(C) \) is either empty or induces a connected subgraph of \( C \). Then either some \( D \in \mathcal{D} \) meets two distinct \( R, R' \in \mathcal{R} \) or there is a component \( Q \) of \( G - \bigcup \mathcal{R} \) with \( V(C) \subseteq V(Q) \cup \bigcup \mathcal{R} \).

Proof. Suppose that no \( D \in \mathcal{D} \) meets two distinct \( R, R' \in \mathcal{R} \). Then \( C \) has no edges between the sets in \( \mathcal{R} \), since any such edge would have to lie in at least one \( D \in \mathcal{D} \). Let \( Y := \bigcup \mathcal{R} \) and let \( Q \) be the set of components of \( G - Y \).
We claim that for all $Q \in \mathcal{Q}, R \in \mathcal{R}$ and $D \in \mathcal{D}$, the number of edges of $D$ between $Q$ and $R$ is even. This is trivial if $D$ has no edges between $Q$ and $R$. Otherwise, $D$ meets $R$ and thus cannot meet $Y \setminus R$. Therefore, all edges of $D$ between $Q$ and $V(G) \setminus Q$ must join $Q$ to $R$. As $D$ is a cycle, it has an even number of edges between $Q$ and $V(G) \setminus Q$ and thus between $Q$ and $R$. This finishes the proof of the claim.

Now, as $C = \bigoplus \mathcal{D}$, we find that for all $Q \in \mathcal{Q}, R \in \mathcal{R}$, the number of edges of $C$ between $Q$ and $R$ is

$$e_C(Q, R) \equiv \sum_{D \in \mathcal{D}} e_D(Q, R) \equiv 0 \mod 2.$$ .

For every $R \in \mathcal{R}$, there exists a family $\mathcal{D}$ with $\bigcup \mathcal{D} \supseteq X$ to deduce that some component of $R$. Then there exists a family $\mathcal{D}$ with $\bigcup \mathcal{D} \supseteq X$ to deduce that some component of $C$. Here, $\mathcal{D}$ consists of cycles of length at most $\ell$, so if the sets in $\mathcal{R}$ are at pairwise distance $\geq \lfloor \ell/2 \rfloor$, then no $D \in \mathcal{D}$ can pass through two of them. The next lemma ensures that we can find such a family $\mathcal{R}$ with a bound on $|\bigcup \mathcal{R}|$, when the cycle $C$ is geodesic.

**Lemma 6.** Let $d$ be a positive integer, $G$ a graph, $X \subseteq V(G)$ and $C \subseteq G$ a geodesic cycle. Then there exists a family $\mathcal{R}$ of disjoint sets of vertices of $G$ with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|V(C) \cap \bigcup \mathcal{R}| \leq 2d|X|$ such that for each $R \in \mathcal{R}$, the set $V(C) \cap R$ induces a (possibly empty) connected subgraph of $C$ and the distance between any two sets in $\mathcal{R}$ is greater than $d$.

**Proof.** Let $Y \subseteq V(G)$ and $y \in Y$. For $j \geq 0$, let $B^j_Y(y)$ be the set of all $z \in Y$ at distance at most $jd$ from $y$. Since $|B^0_Y(y)| = 1$, there is a maximum number $j$ for which $|B^j_Y(y)| \geq 1 + j$, and we call this $j = j_Y(y)$ the range of $y$ in $Y$. Observe that every $z \in Y \setminus B^{j_Y(y)}_Y$ has distance greater than $(j_Y(y) + 1)d$ from $y$.

Starting with $X_1 := X$, repeat the following procedure for $k \geq 1$. If $X_k$ does not meet $C$, terminate the process. Otherwise, pick an $x_k \in X_k \cap V(C)$ of maximum range in $X_k$. Let $j_k := j_{X_k}(x_k)$ and $B_k := B^{j_k}_{X_k}(x_k)$. Define $X_{k+1} := X_k \setminus B_k$ and repeat.

Since the size of $X_k$ decreases in each step, there is a smallest integer $m$ for which $X_{m+1} \cap V(C)$ is empty, at which point the process terminates. By construction, the distance between $B_k$ and $X_{k+1}$ is greater than $d$ for each $k \leq m$. For each $k$ satisfying $1 \leq k \leq m$, there are two edge-disjoint paths $P^1_k, P^2_k \subseteq C$, starting at $x_k$ and each of length at most $j_kd$, so that $B_k \cap V(C) \subseteq S_k := P^1_k \cup P^2_k$, because $C$ is geodesic. Choose these paths minimal, so that the endvertices of $S_k$ lie in $B_k$. Note that every vertex of $S_k$ has distance at most $j_kd$ from $x_k$. Therefore, the distance between $R_k := B_k \cup S_k$ and $X_{k+1}$ is greater than $d$. 

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We claim that the distance between \( R_k \) and \( R_{k'} \) is greater than \( d \) for any \( k < k' \). Since \( B_{k'} \subseteq X_{k+1} \), it is clear that every vertex of \( B_{k'} \) has distance greater than \( d \) from \( R_k \). Take a vertex \( q \in S_{k'} \setminus B_{k'} \) and assume for a contradiction that its distance to \( R_k \) was at most \( d \). Then the distance between \( x_k \) and \( q \) is at most \((j_k + 1)d\). Let \( a, b \in B_{k'} \) be the endvertices of \( S_{k'} \). If \( x_k \notin S_{k'} \), then one of \( a \) and \( b \) lies on the shortest path from \( x_k \) to \( q \) within \( C \). Since \( C \) is geodesic, that endvertex then has distance at most \((j_k + 1)d\) from \( x_k \). But then, since \( j_k \) is the range of \( x_k \) in \( X_k \), that vertex would already lie in \( B_k \), a contradiction. Suppose now that \( x_k \in S_{k'} \). Then \( x_k \) lies on one of \( P^1_k \) or \( P^2_k \), so the distance between \( x_k \) and \( x_{k'} \) is at most \( j_kd \). Since \( x_{k'} \in X_k \cap V(C) \), it follows from our choice of \( x_k \) that

\[ j_k = jX_k(x_k) \geq jX_{k'}(x_{k'}) \geq jX_{k'}(x_{k'}) = j_{k'}, \]

where the second inequality follows from the fact that \( X_{k'} \subseteq X_k \) and \( jY(y) \geq jY'(y) \) whenever \( Y \supseteq Y' \). But then \( x_{k'} \in B_k \), a contradiction. This finishes the proof of the claim.

Finally, let \( \mathcal{R} := \{ R_k : 1 \leq k \leq m \} \cup \{ X_{m+1} \} \). The distance between any two sets in \( \mathcal{R} \) is greater than \( d \). For \( k < m \), \( R_k \cap V(C) = S_k \) is a connected subgraph of \( C \), while \( X_{m+1} \cap V(C) \) is empty. Moreover,

\[
|V(C) \cap \bigcup \mathcal{R}| = \sum_{k=1}^{m} |S_k| \leq \sum_{k=1}^{m} (1 + 2j_kd) \leq \sum_{k=1}^{m} (1 + 2(|B_k| - 1)d) \\
\leq \sum_{k=1}^{m} 2|B_k|d \leq 2d|X|.
\]

\( \square \)

**Proof of Theorem 4.** Let \( X \subseteq V(G) \) be of order at most \( k \) and let \( d := \lfloor p/2 \rfloor \). By Lemma 6, there exists a family \( \mathcal{R} \) of disjoint sets of vertices of \( G \) with \( X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C) \) and \( |V(C) \cap \bigcup \mathcal{R}| \leq 2dk \) so that for each \( R \in \mathcal{R} \), the set \( R \cap V(C) \) induces a (possibly empty) connected subgraph of \( C \) and the distance between any two sets in \( \mathcal{R} \) is greater than \( d \).

Let \( \mathcal{D} \) be a set of cycles of length at most \( p \) with \( C = \bigoplus \mathcal{D} \). Then no \( D \in \mathcal{D} \) can meet two distinct \( R, R' \in \mathcal{R} \), since the diameter of \( D \) is at most \( d \). By Lemma 5, there is a component \( Q \) of \( G - \bigcup \mathcal{R} \) which contains every vertex of \( C \setminus \bigcup \mathcal{R} \). This component is connected in \( G - X \) and therefore contained in some component \( Q' \) of \( G - X \), which then satisfies

\[
|Q' \cap V(C)| \geq |C| - |V(C) \cap \bigcup \mathcal{R}| \geq |C| - 2dk.
\]

Since \( |C| \geq 4dk \), the claim follows. \( \square \)

### 3 Remarks

We have described the content of Theorem 1 as an **algebraic** criterion for a graph to have large tree-width. The reader might object that the cycle \( C \) being \( \ell \)-geodesic is a metric property and not an algebraic one. Karl Heuer has pointed out to us, however,
that geodecity of a cycle can be expressed as an algebraic property after all. This is a consequence of a more general lemma of Gollin and Heuer [1], which allowed them to introduce a meaningful notion of geodecity for cuts.

**Proposition 7** ([1]). Let \( G \) be a graph with length-function \( \ell \) and \( C \subseteq G \) a cycle. Then \( C \) is \( \ell \)-geodesic if and only if there do not exist cycles \( D_1, D_2 \) with \( \ell(D_1), \ell(D_2) < \ell(C) \) such that \( C = D_1 \oplus D_2 \).

Finally, we would like to point out that Theorem 1 does not only offer a ‘one-way criterion’ for large tree-width, but that it has a qualitative converse. First, we recall the Grid Minor Theorem of Robertson and Seymour [4], phrased in terms of walls. For a positive integer \( t \), an elementary \( t \)-wall is the graph obtained from the \( 2t \times t \)-grid as follows. Delete all edges with endpoints \( (i, j), (i, j + 1) \) when \( i \) and \( j \) have the same parity. Delete the two resulting vertices of degree one. A \( t \)-wall is any subdivision of an elementary \( t \)-wall. Note that the \((2t \times 2t)\)-grid has a subgraph isomorphic to a \( t \)-wall.

**Theorem 8** (Grid Minor Theorem [4]). For every \( t \in \mathbb{N} \) there exists a \( k \in \mathbb{N} \) such that every graph of tree-width at least \( k \) contains a \( t \)-wall.

Here, then, is our qualitative converse to Theorem 1, showing that the algebraic condition in the premise of Theorem 1 in fact captures tree-width.

**Corollary 9.** For every \( L \in \mathbb{N} \) there exists a \( k \in \mathbb{N} \) such that for every graph \( G \) the following holds. If \( G \) has tree-width at least \( k \), then there exists a rational-valued length-function on \( G \) such that \( G \) contains an \( \ell \)-geodesic cycle \( C \) with \( \ell(C) \geq L \) which is the \( \mathbb{F}_2 \)-sum of cycles of \( \ell \)-length at most 1.

**Proof.** Let \( s := 6L \). By the Grid Minor Theorem, there exists an integer \( k \) such that every graph of tree-width at least \( k \) contains an \( s \)-wall. Suppose \( G \) is a graph of tree-width at least \( k \). Let \( W \) be an elementary \( s \)-wall so that \( G \) contains some subdivision \( W' \) of \( W \), where each \( e \in E(W) \) has been replaced by some path \( P^e \subseteq G \) of length \( m(e) \). The outer cycle \( C \) of \( W \) satisfies \( d_C(u, v) \leq 3d_W(u, v) \) for all \( u, v \in V(C) \). Moreover, \( C \) is the \( \mathbb{F}_2 \)-sum of cycles of length at most six.

Define a length-function \( \ell \) on \( G \) as follows. Let \( e \in E(G) \). If \( e \in P^f \) for \( f \in E(C) \), let \( \ell(e) := 1/m(f) \). Then \( \ell(P^f) = 1 \) for every \( f \in E(C) \). If \( e \in P^f \) for \( f \in E(W) \setminus E(C) \), let \( \ell(e) := 3/m(f) \). Then \( \ell(P^f) = 3 \) for every \( f \in E(W) \setminus E(C) \). If \( e \notin E(W') \), choose \( \ell(e) \) large enough so that \( \ell(e) > \ell(W') \), for example take \( \ell(e) := 10s^3 \).

It is easy to see that the subdivision \( C' \subseteq G \) of \( C \) is \( \ell \)-geodesic in \( G \). It has length \( \ell(C') = |C| \geq 3s \) and is the \( \mathbb{F}_2 \)-sum of the subdivisions of 6-cycles of \( W \). Each of these has \( \ell \)-length 18. Rescaling all lengths by a factor of 1/18 yields the desired result. \( \square \)

**References**


